

# A Little Charity Guarantees Almost Envy-Freeness

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## Abstract

Fair division of indivisible goods is a very well-studied problem. The goal of this problem is to distribute  $m$  goods to  $n$  agents in a “fair” manner, where every agent has a valuation for each subset of goods. We assume general valuations.

Envy-freeness is the most extensively studied notion of fairness. However, envy-free allocations do not always exist when goods are indivisible. The notion of fairness we consider here is “envy-freeness up to any good” (EFX) where no agent envies another agent after the removal of any single good from the other agent’s bundle. It is not known if such an allocation always exists even when  $n = 3$ .

We show there is always a partition of the set of goods into  $n + 1$  subsets  $(X_1, \dots, X_n, P)$  where for  $i \in [n]$ ,  $X_i$  is the bundle allocated to agent  $i$  and the set  $P$  is unallocated (or donated to charity) such that we have:

- (1) envy-freeness up to any good,
- (2) no agent values  $P$  higher than her own bundle, and
- (3) fewer than  $n$  goods go to charity, i.e.,  $|P| < n$  (typically  $m \gg n$ ).

Our proof is constructive. When agents have additive valuations and  $|P|$  is large (i.e., when  $|P|$  is close to  $n$ ), our allocation also has a good maximin share (MMS) guarantee. Moreover, a minor variant of our algorithm also shows the existence of an allocation which is  $4/7$  *groupwise* maximin share (GMMS): this is a notion of fairness stronger than MMS. This improves upon the current best bound of  $1/2$  known for an approximate GMMS allocation.

## 1 Introduction

Fair division of goods among competing agents is a fundamental problem in Economics and Computer Science. There is a set  $M$  of  $m$  goods and the goal is to *allocate* goods among  $n$  agents in a *fair* way. An allocation is a partition of  $M$  into disjoint subsets  $X_1, \dots, X_n$  where  $X_i$  is the set of goods given to agent  $i$ . When can an

allocation be considered “fair”? One of the most well-studied notions of fairness is *Envy-freeness*. Every agent has a *value* associated with each subset of  $M$  and agent  $i$  *envies* agent  $j$  if  $i$  values  $X_j$  more than  $X_i$ . An allocation is *envy-free* if no agent envies another. An envy-free allocation can be regarded as a fair and desirable partition of  $M$  among the  $n$  agents since no agent envies another; as mentioned in [27], such a mechanism of partitioning land dates back to the Bible.

Unlike land which is divisible, goods in our setting are *indivisible* and an envy-free allocation of the given set of goods need not exist. Consider the following simple example with two agents and a single good: one of the agents has to receive this good and the other agent envies her. Since envy-free allocations need not exist, several relaxations have been considered.

**Relaxations.** Budish [12] introduced the notion of *EF1*: this is an allocation of goods that is “envy-free up to *one* good”. In an EF1 allocation, agent  $i$  may envy agent  $j$ , however this envy would vanish as soon as *some* good is removed from  $X_j$ . Note that no good is really removed from  $X_j$ : this is simply a way of assessing how much  $i$  values  $X_j$  more than  $X_i$ . That is, if  $i$  values  $X_j$  more than  $X_i$ , then there exists some  $g \in X_j$  such that  $i$  values  $X_i$  at least as much as  $X_j \setminus \{g\}$ . Going back to the example of two agents and a single good, the allocation where one agent receives this good is EF1. It is known that EF1 allocations always exist; as shown by Lipton et al. [26], such an allocation can be efficiently computed.

Caragiannis et al. [14] introduced a notion of envy-freeness called *EFX* that is stronger than EF1. An EFX allocation is one that is “envy-free up to *any* good”. In an EFX allocation, agent  $i$  may envy agent  $j$ , however this envy would vanish as soon as *any* good is removed from  $X_j$ . Thus every EFX allocation is also EF1 but not every EF1 allocation is EFX.

	$a$	$b$	$c$
Agent 1	1	1	2
Agent 2	1	1	2

Consider the simple example given above: there are three goods  $a, b, c$  and two agents with *additive*

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valuations (defined in Section 1.1) as described below. Both agents value  $c$  twice as much as  $a$  or  $b$ . The allocation where agent 1 gets  $\{a\}$  and agent 2 gets  $\{b, c\}$  is EF1 but not EFX. On the other hand, the allocation where agent 1 gets  $\{a, b\}$  and agent 2 gets  $\{c\}$  is EFX. Indeed, the latter allocation seems fairer than the former allocation. As said in [13]: “*Arguably, EFX is the best fairness analog of envy-freeness of indivisible items*”. While it is known that EF1 allocations always exist, the question of whether EFX allocations always exist is still an open problem (despite significant effort, according to [14]).

Plaut and Roughgarden [27] showed that EFX allocations always exist (i) when there are only two agents or (ii) when all  $n$  agents have the same valuations. Moreover, it was shown in [27] that exponentially many value queries may be needed to determine EFX allocations even in the restricted setting where there are only two agents with identical *submodular* valuation functions<sup>1</sup>. It is not known if an EFX allocation always exists even when there are only three agents with additive valuations. It was remarked in [27]: “*We suspect that at least for general valuations, there exist instances where no EFX allocation exists*”.

**A relaxation of EFX.** Very recently, Caragiannis et al. [13] introduced a more relaxed notion of EFX called *EFX-with-charity*. This is a partial allocation that is EFX, i.e., the entire set of goods need not be distributed among the agents. So some goods may be left *unallocated* and it is assumed that these unallocated goods are donated to charity. There is a very simple allocation that is EFX-with-charity where *no* good is assigned to any agent—thus all goods are donated to charity. Obviously, this is not an interesting allocation and one seeks allocations with better guarantees and one such allocation was shown in [13].

Let  $X^* = \langle X_1^*, \dots, X_n^* \rangle$  be an optimal *Nash social welfare* allocation<sup>2</sup> on the entire set of goods. It was shown in [13] that there always exists an EFX-with-charity allocation  $X = \langle X_1, \dots, X_n \rangle$  where every agent receives at least half the value of her allocation in  $X^*$ . Interestingly, as shown in [13],  $X_i \subseteq X_i^*$  for all  $i$ . Unfortunately, there are no upper bounds on how *complete* this allocation is (wrt bounding the number of unallocated goods) or on the value any agent has for the set of goods donated to charity.

We believe these are important questions to ask. The ideal allocation is one that is EFX and *complete*;

<sup>1</sup>These are valuation functions with decreasing marginal values.

<sup>2</sup>This is an allocation that maximizes  $\prod_{i=1}^n v_i(X_i^*)$ , where  $v_i$  is agent  $i$ 's valuation function.

so we would like a guarantee that a large number of goods have been allocated to agents. Moreover, since EFX allocations guarantee envy-freeness once any good is removed from another agent's set, it is in the same spirit that we seek an EFX (partial) allocation where nobody envies the set of unallocated goods. The allocation in [13] gives no guarantee either on the number of unallocated goods or on whether any agent values the set of unallocated goods more than her own bundle. Here we consider the notion of *EFX-with-bounded-charity*. That is, we seek EFX-with-charity allocations with bounds on the set given to charity, i.e., a bound on the *size* and a bound on the *value* of the set of goods donated to charity.

**1.1 Our Results** Let  $N = [n]$  be the set of agents. Every agent  $i \in [n]$  has a valuation function  $v_i : 2^M \rightarrow \mathbb{R}_{\geq 0}$ , where  $M$  is the set of  $m$  goods. We show our main existence result for *general* valuation functions, i.e., the only assumptions we make on any valuation function  $v_i$  is that (i) it is *normalized*, i.e.,  $v_i(\emptyset) = 0$ , and (ii) it is *monotone*, i.e.,  $S \subseteq T$  implies  $v_i(S) \leq v_i(T)$ . In contrast, the EFX-with-charity allocation in [13] works only for *additive valuations*, i.e.,  $v_i(S) = \sum_{g \in S} v_i(\{g\})$ .

We show there always exists an allocation<sup>3</sup>  $X = \langle X_1, \dots, X_n \rangle$  that satisfies the following properties:

1.  $X$  is EFX, i.e., for any two agents  $i, j$ :  

$$v_i(X_i) \geq v_i(X_j \setminus \{g\})$$
 for any  $g \in X_j$ ;
2.  $v_i(X_i) \geq v_i(P)$  for all agents  $i$ , where  

$$P = M \setminus \cup_{i=1}^n X_i$$
 is the set of unallocated goods;
3.  $|P| < n$  (recall that  $n$  is the number of agents).

Note that our result implies that among the  $n$  agents, if there is just *one* agent who has no preferences (say, for some  $i$ , we have  $v_i(S) = 0$  for all  $S \subseteq M$ ) then a complete EFX allocation always exists for general valuations.

Our proof is constructive. We start with no goods being allocated to the agents and find the claimed allocation by at most  $nmV/\Delta$  applications of three simple update rules. Here,  $n$  is the number of agents,  $m$  is the number of goods,  $V = \max_i v_i(M)$  is the maximum valuation of any agent, and  $\Delta = \min_i \min\{|v_i(T) - v_i(S)| : S, T \subseteq M \text{ and } v_i(S) \neq v_i(T)\}$  is the minimum difference between distinct valuations.

The update rules use a *minimum-size-valuable-set-oracle*: given  $S \subseteq M$ , agent  $i$ , and  $\alpha \in \mathbb{R}$  such that  $v_i(S) > \alpha$ , find a minimum cardinality subset  $Z \subseteq S$  such that  $v_i(Z) > \alpha$ . If this oracle can be

<sup>3</sup>Henceforth, allocations are *partial* and we will use “complete allocation” to refer to one where all goods are allocated.

implemented in pseudo-polynomial time, then our proof yields a pseudo-polynomial time algorithm to compute the above allocation.

It also follows from our proof that when all agents have the same valuation function, our allocation is *complete*. That is,  $|P| = 0$ . This is an alternate proof to the existence of complete EFX allocations for identical (general) valuations, originally shown in [27].

Our next result is a pseudo-polynomial time algorithm to find an allocation that obeys properties 1-3 given above. For this, we assume that all agents have *gross substitute valuations* (defined in Section 3). For gross substitute valuations, the minimum-size-valuable-set-oracle can be realized by a simple greedy algorithm that uses only value queries. Every additive valuation is a gross substitute valuation and gross substitute valuations form a subclass of the set of submodular valuations [23].

- Suppose all agents have gross substitute valuations. Then an EFX allocation with properties 1-3 can be computed with  $\text{poly}(n, m, V, 1/\Delta)$  value queries.

**1.1.1 Additive valuations** The most well-understood class of valuation functions is the set of *additive* valuations. We consider the case when all agents have additive valuations and show that our allocation or very minor variants of our allocation can guarantee several other notions of fairness.

**Ensuring high Nash social welfare.** We show that modifying the starting step of our algorithm ensures that our allocation  $X$ , which satisfies properties 1-3 stated above, also has a high Nash social welfare. That is,  $v_i(X_i) \geq \frac{1}{2} \cdot v_i(X_i^*)$  as promised in [13], where  $X^* = \langle X_1^*, \dots, X_n^* \rangle$  is an optimal Nash social welfare allocation. Here we use the allocation computed in [13] as a black box in our starting step, thus our result can be regarded as an extension of the result in [13].

**Number of Unallocated Goods and MMS Guarantee.** Another interesting and well-studied notion of fairness is *maximin share*. Suppose agent  $i$  has to partition  $M$  into  $n$  *bundles* (or sets) knowing that she would receive the worst bundle with respect to her valuation. Then  $i$  will choose a partition of  $M$  that maximizes the valuation of the worst bundle (wrt her valuation). The value of this worst bundle is the maximin share of agent  $i$ . An important question here is: does there always exist an allocation of  $M$  where every agent gets a bundle worth at least her maximin share?

Formally, let  $N$  and  $M$  be the sets of  $n$  agents and  $m$  goods, respectively. We define the maximin share of an agent (say,  $i$ ) as follows: (here  $\mathcal{X}$  is the set of all

complete allocations)

$$MMS_i(n, M) = \max_{\langle X_1, \dots, X_n \rangle \in \mathcal{X}} \min_{j \in [n]} v_i(X_j).$$

The goal is to determine an allocation  $\langle X_1, X_2, \dots, X_n \rangle$  of  $M$  such that for every  $i$  we have  $v_i(X_i) \geq MMS_i(n, M)$ . This question was first posed by Budish [12]. Procaccia and Wang [28] showed that such an allocation need not exist, even in the restricted setting of only three agents! Thereafter, *approximate-MMS* allocations were studied [28, 19, 21, 20] and we know polynomial time algorithms to find allocations where for all  $i$ , agent  $i$  gets a bundle of value at least  $\alpha \cdot MMS_i(n, M)$ ; the current best guarantee for  $\alpha$  is  $3/4 - \epsilon$  by Ghodsi et al. [21] (for any  $\epsilon > 0$ ) and this was very recently improved to  $3/4$  by Garg and Taki [20].

Amanatidis et al. [2] showed that any complete EFX allocation is also a  $\frac{4}{7}$ -MMS allocation. We show that our allocation promises better MMS guarantees when the number of unallocated goods is large. Let  $X = \langle X_1, \dots, X_n \rangle$  be our allocation as described by properties 1-3 above and  $P$  be the set of unallocated goods. For any agent  $i \in [n]$ , we have:

$$v_i(X_i) \geq \frac{1}{2 - |P|/n} MMS_i(n, M).$$

Hence, larger the number of unallocated goods, better are the guarantees that we get on MMS. The extreme values are  $|P| = 0$  and  $|P| = n - 1$ . When  $|P| = 0$ , we have a complete EFX allocation and when  $|P| = n - 1$ , we have an EFX allocation that is an *almost-MMS* allocation:  $v_i(X_i) \geq (1 - 1/n) \cdot MMS_i(n, M)$  for all  $i$ .

**Improved Guarantees for Groupwise MMS.** Barman et al. [7] recently introduced a notion of fairness called *groupwise maximin share* (GMMS) which is stronger than MMS. An allocation is said to be GMMS if the MMS condition is satisfied for *every subgroup* of agents and the union of the sets of goods allocated to them. Formally, a complete allocation  $X = \langle X_1, X_2, \dots, X_n \rangle$  is  $\alpha$ -GMMS if for any  $N' \subseteq N$ , we have  $v_i(X_i) \geq \alpha \cdot MMS_i(n', \bigcup_{i \in N'} X_i)$  where  $n' = |N'|$ . Every GMMS allocation, i.e.  $\alpha = 1$ , is also a complete EFX allocation [7].

It is known [7] that GMMS strictly generalizes MMS. In particular, in the same paper, it was shown that GMMS allocations rule out some very unsatisfactory allocations that have MMS guarantees. For example, consider an instance with  $n$  agents with additive valuations and a set  $M$  of  $n - 1$  goods and every agent has a valuation of one for each good. Since the number of goods is less than the number of agents, we have  $MMS_i(n, M) = 0$  for every agent  $i$ . So any allocation has MMS guarantees. It is not hard to see that the

only allocation with a GMMS guarantee is where  $n - 1$  agents get one good each and one agent is left without any goods. See subsection 2.1 in [7] for more discussion. Naturally, it is a harder problem to approximate GMMS than MMS. While  $\frac{3}{4}$ -MMS allocations always exist, the largest  $\alpha$  for which  $\alpha$ -GMMS allocations are known to exist is  $\frac{1}{2}$  [7]. We extend the result of Amanatidis et al. [2] for MMS to show the following:

- A  $\frac{4}{7}$ -GMMS allocation always exists and can be computed in pseudo-polynomial time.

In particular, we show that modifying the last step of our algorithm results in a complete allocation that is  $\frac{4}{7}$ -GMMS. Very recently and independently, Amanatidis et al. [4] showed the same approximation.

**1.2 Our Techniques** We now give an overview of the main ideas used to find our EFX allocation. We first recall the algorithm of Lipton et al. [26] for finding an EF1 allocation<sup>4</sup>. They use the notion of an *envy-graph*: here each vertex corresponds to an agent and there is an edge  $(i, j)$  iff  $i$  envies  $j$ . The invariant maintained is that the envy-graph is a DAG: a cycle corresponds to a cycle of envy and by swapping bundles along a cycle, every agent becomes better-off and the number of envy edges decreases. More precisely, if  $i_0 \rightarrow i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_{\ell-1} \rightarrow i_0$  is a cycle in the envy graph, then reassigning  $X_{i_{j+1}}$  to agent  $i_j$  for  $0 \leq j < \ell$  (indices are to be read modulo  $\ell$ ) will increase the valuation of every agent in the cycle. Also if there was an edge from  $s$  to some  $i_k$  where  $s$  is not a part of the cycle, this edge just gets directed now from  $s$  to  $i_{k+1}$  after we exchange bundles along the cycle. Thus the number of envy edges in the graph does not increase and the valuations of the agents in the cycle goes up. Thus cycles can be eliminated.

The algorithm in [26] runs in rounds and always maintains an allocation that is also EF1. At the beginning of every round, an unenvied agent  $s$  (this is a source vertex in this DAG) is identified and an unallocated good  $g$  is allocated to  $s$ . The new allocation is also EF1, as nobody will envy the bundle of  $s$  after removing the good  $g$ .

**The Reallocation Operation.** We now highlight a key difference between an EF1 allocation and an EFX allocation. From the algorithm of Lipton et al. [26], it is clear that given an EF1 allocation on a set  $M_0$  of goods, one can determine an EF1 allocation on  $M_0 \cup M_1$ , for any  $M_1 \subseteq M \setminus M_0$ , by simply adding goods from  $M_1$  one-by-one to the existing bundles and changing the owners (if

necessary) in a clever way. Intuitively, we never need to cut or merge the bundles formed in any EF1 allocation. We can just append the unallocated goods appropriately to the current bundles.

The above strategy is very far from true for EFX. Consider the example illustrated below with three agents with additive valuations and four goods  $a, b, c$ , and  $d$ .

	$a$	$b$	$c$	$d$
Agent 1	0	1	1	2
Agent 2	1	0	1	2
Agent 3	1	1	0	2

An EFX allocation for the first three goods has to give exactly one of  $a, b, c$  to each of the three agents. However an EFX allocation for all the four goods has to allocate the singleton set  $\{d\}$  to some agent (say, agent 1) and say,  $\{a\}$  to agent 2 and  $\{b, c\}$  to agent 3. Thus the allocation needs to be *cut and merged*. When there are many agents - each with her own valuation, figuring out the cut-and-merge operations is the difficult step. Here we implement our global reallocation operation as follows.

**Improving Social Welfare.** Suppose we have an EFX allocation  $X = \langle X_1, \dots, X_n \rangle$  on some subset  $M_0 \subset M$ . We would now like to add a good  $g \in M \setminus M_0$ . However we will not be able to guarantee an EFX allocation on  $M_0 \cup \{g\}$ . What we will ensure is that either case (i) or case (ii) occurs:

- We have an EFX allocation  $X' = \langle X'_1, \dots, X'_n \rangle$  on a subset of  $M_0 \cup \{g\}$  such that  $v_i(X'_i) \geq v_i(X_i)$  for all  $i$  and for at least one agent  $j$  we have  $v_j(X'_j) > v_j(X_j)$ . Thus  $\sum_{i \in [n]} v_i(X'_i) > \sum_{i \in [n]} v_i(X_i)$ ; in other words, *social welfare* strictly improves.
- We have an EFX allocation on  $M_0 \cup \{g\}$  and the social welfare does not decrease.

Hence in each step of our algorithm, we either increase the number of allocated goods or we increase social welfare—thus we always make progress. This is similar to the approach used by Plaut and Roughgarden [27] to guarantee the existence of  $\frac{1}{2}$ -EFX<sup>5</sup> when agents have subadditive valuations. We now outline how we ensure one of the cases (i) and (ii) has to happen:

For simplicity of exposition, we assume the envy-graph corresponding to our starting EFX allocation  $X$  has a single source  $s$ . Add  $g$  to  $s$ 's bundle: if nobody

<sup>4</sup>The algorithm in [26] was published in 2004 with a different property and EF1 was proposed in 2011.

<sup>5</sup>An allocation  $X = (X_1, \dots, X_n)$  is  $\frac{1}{2}$ -EFX if for any two agents  $i, j$ :  $v_i(X_i) \geq \frac{1}{2} \cdot v_i(X_j \setminus \{g\})$  for all  $g \in X_j$ .



envies  $s$  up to any good then we are in an easy case as we have an EFX allocation on  $M_0 \cup \{g\}$ . In this case, we “decycle” the envy-graph (if cycles are created) and continue. Observe that swapping bundles along a cycle in the envy-graph increases social welfare.

**Most Envious Agent.** So assume there are one or more agents who envy  $s$  up to any good after  $g$  is allocated to  $s$ . To resolve this, we introduce the concept of a *most envious agent*. Let  $i$  be an agent who envies  $s$  up to any good, so  $v_i(X_i) < v_i(S')$  for some  $S' \subset X_s \cup \{g\}$ . Let  $S_i$  be the subset of  $X_s \cup \{g\}$  with the *minimum cardinality* such that  $v_i(X_i) < v_i(S_i)$ . So for any  $T_i \subset S_i$ , we have  $v_i(X_i) \geq v_i(T_i)$ . The agent  $i$  with the smallest value of  $|S_i|$  (break ties arbitrarily) is the most envious agent of  $s$ . Call this agent  $t$ .

The crucial observation is that no agent envies  $S_t$  up to any good—otherwise it would contradict  $t$  being the agent with the smallest value of  $|S_t|$ . Recall the assumption that  $s$  is the only source, so there is a path  $s \rightarrow i_1 \rightarrow \dots \rightarrow i_{k-1} \rightarrow t$  in the envy-graph. We do a leftwise shift of bundles along this path: so  $s$  gets  $i_1$ ’s bundle, and for  $1 \leq r \leq k-1$ :  $i_r$  gets  $i_{r+1}$ ’s bundle (where  $i_k = t$ ), and finally  $t$  gets  $S_t$ . The goods in  $X_s \cup \{g\} \setminus S_t$  are thrown back into the pool of unallocated goods.

Observe that every agent in this path is strictly better-off now than in the allocation  $X$  and nobody is worse-off. Moreover, by the definition of  $S_t$ , there are no agents envying any agent up to any good. Thus we have a desired EFX allocation  $X'$ . When there are multiple sources, we can adapt this technique *provided* there are enough unallocated goods; in particular, the number of unallocated goods must be at least the number of sources in the envy-graph. We describe this in detail in Section 2.

We would like to contrast the above approach with other EFX algorithms [27, 13]. The  $\frac{1}{2}$ -EFX algorithm by Plaut and Roughgarden [27] either merges  $g$  (the new good) with an existing bundle or allocates the singleton set  $\{g\}$  to an agent while the EFX-with-charity algorithm by Caragiannis et al. [13] takes an allocation of maximum Nash social welfare as input and then permanently removes some goods from the instance. We regard the notion of “most envious agent” that shows a natural way of breaking up a bundle to preserve envy-freeness up to any good as one of the innovative contributions of our work.

**Our Other Results.** Regarding our result with approximate MMS guarantee, if the number of unallocated goods in our EFX allocation is large, then the number of sources also has to be large: these are unenvied agents. Moreover, no agent envies the set of unallocated goods.

Suppose for now that  $|P| = n - 1$ . This means every agent is a source. So no agent envies the bundle of any other agent and also the set of unallocated goods. Thus for each agent  $i$ , we have:

$$\begin{aligned} v_i(X_i) &\geq \frac{v_i(M)}{n+1} && \geq (1 + 1/n)^{-1} \cdot \frac{v_i(M)}{n} \\ &&& \geq (1 - 1/n) \cdot MMS_i(n, M), \end{aligned}$$

where the constraint that  $v_i(M)/n \geq MMS_i(n, M)$  holds for additive valuations. We show our result for approximate-MMS allocation and our improved bound for approximate-GMMS allocation in Section 4.

**1.3 Related Work** Fair division of divisible resources is a classical and well-studied subject starting from the 1940’s [29]. Fair division of indivisible goods among competing agents is a young and exciting topic with recent work on EF1 and EFX allocations [14, 9, 27, 10, 13], approximate maximin share allocations [12, 11, 3, 8, 24, 21, 19], and approximation algorithms for maximizing Nash social welfare and generalizations [17, 16, 15, 5, 18, 6]. As mentioned earlier, Caragiannis et al. [14] introduced the notion of EFX: whether such allocations always exist is an enigmatic open problem. It was shown in [14] that there always exists an EF1 allocation that is also Pareto-optimal<sup>6</sup> and Barman et al. in [9] showed a weakly-polynomial time algorithm to compute such an allocation.

**Applications.** Fair division of goods or resources occurs in many real-world scenarios and this is demonstrated by the popularity of the website Spliddit (<http://www.spliddit.org>) that implements mechanisms for fair division where users can log in, define what needs to be divided, and enter their valuations. This website guarantees an EF1 allocation that is also Pareto-optimal and since its launch in 2014, it has been used tens of thousands of times [14]. We refer to [22, 27] for details on the diverse applications for which Spliddit has been used: these range from rent division and taxi fare division to credit assignment for an academic paper or group project. Another such website is Fair Outcomes, Inc. (<http://www.fairoutcomes.com>). Another interesting application is *Course Allocate* used at Wharton School that guarantees certain fairness properties to allocate courses among students [27].

<sup>6</sup>An allocation  $X = \langle X_1, \dots, X_n \rangle$  is Pareto-optimal if there is no allocation  $Y = \langle Y_1, \dots, Y_n \rangle$  where  $v_i(Y_i) \geq v_i(X_i)$  for all  $i \in [n]$  and  $v_i(Y_j) > v_i(X_j)$  for some  $j$ .

## 2 Existence of an EFX-Allocation with Bounded Charity

We prove our main result on EFX-with-bounded-charity allocations in this section. We will define three *update rules*. Each update rule takes a pair  $(X, P)$  consisting of an allocation  $X$  and a set  $P$  of unallocated goods (we will call  $P$  the *pool*) and returns a modified pair  $(X', P')$ .

Each application of an update rule will ensure that either (i) the *social welfare*  $\phi(X) = \sum_{i \in [n]} v_i(X_i)$  of the current allocation increases or (ii) the size of the pool decreases and the social welfare does not decrease, so  $|P'| < |P|$  in this case. Hence the update process will terminate. The overall structure of the algorithm is given in Algorithm 1.

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**Algorithm 1** Algorithm for Computing an EFX-Allocation

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**Postcondition:**  $X$  is EFX,  $|P| < n$  and  $v_i(P) \leq v_i(X_i)$  for all  $i \in [n]$ .

- 1:  $X_i \leftarrow \emptyset$  for  $i \in [n]$ ;  $P \leftarrow M$ ;
  - 2: **while** one of the update rules shown in Algorithm 2 is applicable **do**
  - Invariant:**  $X$  is EFX and the envy-graph  $G_X$  is acyclic
  - 3: Let  $U_\ell$  be an applicable update rule;
  - 4:  $(X, P) \leftarrow U_\ell(X, P)$ ;
  - 5: Decycle the envy-graph;
  - 6: **end while**
- 

In order to define our update rules, we need the concepts of *envy-graph* and the *most envious agent* for a bundle of goods. These were discussed in Section 1.2 and we formally define them below.

**DEFINITION 2.1.** *The envy-graph  $G_X$  for an allocation  $X = \langle X_1, X_2, \dots, X_n \rangle$  has the set of agents as vertices and there is a directed edge from agent  $i$  to agent  $j$  if and only if  $v_i(X_i) < v_i(X_j)$ .*

The notion of envy-graph was introduced in [26] and it is well-known that cycles can be removed from the envy-graph without destroying desirable properties (see Lemma 2.1). Thus we can maintain  $G_X$  as a DAG. For an agent  $s$ , the *reachability component*  $C(s)$  consists of all agents reachable from  $s$  in the envy-graph. The *sources* in the envy-graph are the vertices with indegree zero.

For ease of notation, we will use  $B \setminus g$  and  $B \cup g$  to denote  $B \setminus \{g\}$  and  $B \cup \{g\}$ , respectively.

**LEMMA 2.1.** *Let  $i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_{k-1} \rightarrow i_0$  be a cycle in the envy-graph. Consider the allocation  $X'$  where*

*$X'_\ell = X_{i_{\ell+1}}$  (indices are modulo  $k$ ) for  $\ell \in \{0, \dots, k-1\}$  and  $X'_j = X_j$  for  $j \notin \{i_0, \dots, i_{k-1}\}$ . If  $X$  is EFX, then  $X'$  is also EFX. Moreover,  $\phi(X') > \phi(X)$ .*

*Proof.* Consider any agent  $i$ . We have  $v_i(X'_i) \geq v_i(X_i)$  with strict inequality if  $i$  lies on the cycle. So  $\sum_{i \in [n]} v_i(X'_i) > \sum_{i \in [n]} v_i(X_i)$ . Thus  $\phi(X') > \phi(X)$ .

Since  $X'$  is just a permutation of  $X$ , for any agent  $j$  there exists some agent  $j'$  such that  $X'_j = X_{j'}$ . Therefore, since  $X$  is EFX, for any good  $g \in X_{j'}$  (or equivalently  $X'_j$ ) we have  $v_i(X'_j \setminus g) = v_i(X_{j'} \setminus g) \leq v_i(X_i) \leq v_i(X'_i)$ . Thus  $X'$  is also EFX.  $\square$

**DEFINITION 2.2.** *Let  $X$  be an allocation and  $S \subseteq M$ . For an agent  $i$  with  $v_i(S) > v_i(X_i)$ , let  $\kappa_X(i, S)$  be the minimum  $k$  such that there is a set  $Z \subseteq S$  of size  $k$  with  $v_i(Z) > v_i(X_i)$ . We define  $\kappa_X(S) = \min_{i \in [n]} \kappa_X(i, S)$ .*

The following definition formalizes the notion of “most envious agents”. Let  $S \subseteq M$ , then we define

$$A_X(S) = \{i \in [n] : \kappa_X(i, S) = \kappa_X(S)\}.$$

If there is no  $i$  with  $v_i(S) > v_i(X_i)$ , then  $A_X(S)$  is empty. We make a simple observation.

**LEMMA 2.2.** *Consider an agent  $i \in A_X(S)$  and let  $Z \subseteq S$  be such that  $|Z| = \kappa_X(S)$  and  $v_i(Z) > v_i(X_i)$ . Then for any agent  $j$  (incl.  $i$ ) we have that  $v_j(Z \setminus g) \leq v_j(X_j)$  for all  $g \in Z$ .*

*Proof.* Let  $j$  be any agent. There are two cases: either  $v_j(X_j) \geq v_j(S)$  or  $v_j(X_j) < v_j(S)$ . In the former case, we have  $v_j(X_j) \geq v_j(Z \setminus g)$  by monotonicity.

In the latter case,  $v_j(X_j) \geq v_j(Z')$  for all sets  $Z' \subseteq S$  of size at most  $\kappa_X(j, S) - 1$  (by definition of  $\kappa_X(j, S)$ ). Note that the set  $Z \setminus g$  has size  $\kappa_X(S) - 1 \leq \kappa_X(j, S) - 1$  since  $\kappa_X(S) \leq \kappa_X(j, S)$  (by definition of  $\kappa_X(S)$ ). Thus  $v_j(Z \setminus g) \leq v_j(X_j)$ .  $\square$

We are now ready to present our three update rules  $U_0$ ,  $U_1$ , and  $U_2$ , see Algorithm 2.

**RULE  $U_0$ :** Rule  $U_0$  is the easiest of the update rules. It is applicable whenever adding a good from the pool to some source of  $G_X$  does not destroy the EFX-property (see Algorithm 2).

**LEMMA 2.3.** (RULE  $U_0$ )

- (a) *Rule  $U_0$  returns an EFX allocation. An application of the rule does not decrease social welfare and decreases the size of the pool.*
- (b) *If rule  $U_0$  is not applicable then for any source  $i$  of  $G_X$  and good  $g \in P$ , there will be an agent  $j \neq i$  such that  $v_j(X_i \cup g) > v_j(X_j)$ . In particular,  $A_X(X_i \cup g)$  will not be empty, and  $\kappa_X(j, X_i \cup g) \leq |X_i|$  for all  $j \in A_X(X_i \cup g)$ .*

*Proof.* The first part of a) follows directly from the precondition of the rule. The second part holds since the valuations are monotone and because  $|P'| = |P| - 1$ .

The first two sentences in part b) are obvious. We come to the third sentence. Since adding  $g$  to  $X_i$  destroys the EFX-property, there must be some  $g' \in X_i \cup g$  and some  $j \in [n]$  such that  $v_j(X_i \cup g \setminus g') > v_j(X_j)$ . Thus  $\kappa_X(j, X_i \cup g) \leq |X_i|$ .  $\square$

**Rule  $U_1$**  : Rule  $U_1$  is applicable whenever there is an agent that values the pool higher than her current bundle (see Algorithm 2).

LEMMA 2.4. (RULE  $U_1$ ) *Rule  $U_1$  increases the social welfare and returns an EFX allocation.*

*Proof.* Since there is an agent that values the pool higher than her own bundle,  $A_X(P)$  is non-empty. Choose  $i$  from  $A_X(P)$  arbitrarily. Let  $X'$  be the allocation defined in Algorithm 2, line 7. Then  $v_i(X'_i) > v_i(X_i)$  and  $v_j(X'_j) = v_j(X_j)$  for  $j \neq i$ . Thus  $\phi(X') > \phi(X)$ . It remains to show that the allocation  $X'$  is EFX, i.e., for every pair of agents  $j$  and  $k$  and any good  $g \in X'_k$ , we have  $v_j(X'_k \setminus g) \leq v_j(X'_j)$ .

Since  $X$  is EFX, this is obvious if neither  $j$  nor  $k$  is equal to  $i$ . If  $j = i$ , then  $v_i(X'_i) > v_i(X_i) \geq v_i(X_k \setminus g) = v_i(X'_k \setminus g)$  for all  $g \in X'_k$  (or equivalently  $g \in X_k$ ). Finally, we consider  $k = i$ . Since  $k = i \in A_X(P)$ , we have  $v_j(X'_j) = v_j(X_j) \geq v_j(Z \setminus g) = v_j(X'_i \setminus g)$  for any  $g \in Z$  (where  $Z$  is defined in Algorithm 2, line 6) by Lemma 2.2.  $\square$

**Rule  $U_2$**  : Rule  $U_2$  is our most complex rule. It is applicable if for some  $\ell \geq 1$ , there are distinct goods  $g_0, g_1, \dots, g_{\ell-1}$  in  $P$ , distinct sources  $s_0, s_1, \dots, s_{\ell-1}$  of  $G_X$  and distinct agents  $t_1, t_2, \dots, t_\ell$  (indices are to be interpreted modulo  $\ell$ ) such that for each  $i$ : (1)  $t_i$  is a most envious agent when  $g_{i-1}$  is added to  $s_{i-1}$  and (2)  $t_i$  is reachable from  $s_i$ . We first show that rule  $U_2$  is applicable if rule  $U_0$  is not applicable and the pool contains at least  $n$  goods.

LEMMA 2.5. *If  $|P| \geq n$  and rule  $U_0$  is not applicable then there is an  $\ell \geq 1$ , distinct goods  $g_0, g_1, \dots, g_{\ell-1}$  in  $P$ , distinct sources  $s_0, s_1, \dots, s_{\ell-1}$  of  $G_X$ , and distinct agents  $t_1, t_2, \dots, t_\ell$  such that  $t_i \in C(s_i) \cap A_X(X_{s_{i-1}} \cup g_{i-1})$  for  $i \in \{0, \dots, \ell-1\}$  (indices are modulo  $\ell$ ).*

*Proof.* Since rule  $U_0$  is not applicable,  $A_X(X_s \cup g)$  is non-empty for every source  $s$  of  $G_X$  and every good  $g \in P$ . Construct a sequence of triples  $(s_i, g_i, t_{i+1})$ ,  $i \geq 0$  defined as follows. Let  $s_0$  be an arbitrary source of  $G_X$  and  $g_0$  be an arbitrary good in  $P$ . Assume we have defined  $s_{i-1}$  and  $g_{i-1}$ . Let  $t_i \in A_X(X_{s_{i-1}} \cup g_{i-1})$ .

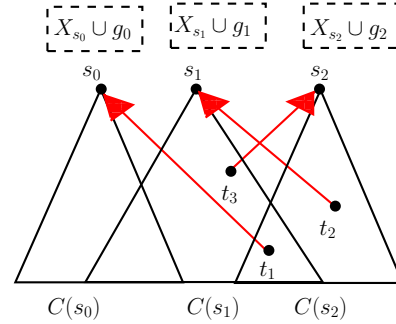


Figure 1: We have  $t_i \in A_X(X_{s_{i-1}} \cup g_{i-1})$ . Moreover,  $t_i \notin C(s_0) \cup \dots \cup C(s_{i-1})$  for  $i = 1, 2$  and  $t_3 \in C(s_0) \cup \dots \cup C(s_2)$ .  $j = 1$  is largest such that  $t_3 \in C(s_j)$ . The cycle is defined by  $s_1, s_2, g_1, g_2, t_2$  and  $t_3$ .

be arbitrary. If  $t_i \in C(s_0) \cup \dots \cup C(s_{i-1})$  then stop the construction of the sequence and let  $j$  be maximum such that  $t_i \in C(s_j)$ . Set  $\ell = i - j$  and return  $s_j, \dots, s_{i-1}, g_j, \dots, g_{i-1}$  and  $t_{j+1}, \dots, t_i$ ; see Figure 1 for an illustration. If  $t_i \notin C(s_0) \cup \dots \cup C(s_{i-1})$ , let  $s_i$  be such that  $t_i \in C(s_i)$ . Also, let  $g_i$  be a good in  $P$  distinct from  $g_0$  to  $g_{i-1}$ .

The construction is well-defined since  $|P| \geq n$  and hence we cannot run out of goods. The sources and goods are pairwise distinct by construction. The agents  $t_1$  to  $t_{i-1}$  are distinct by construction. The agent  $t_i$  is distinct from  $t_{j+1}$  to  $t_{i-1}$  since  $t_k \in C(s_k) \setminus (C(s_0) \cup \dots \cup C(s_{k-1}))$  for  $k < i$  and  $j$  is maximum such that  $t_i \in C(s_j)$ .  $\square$

For each  $i$ , let  $u_0^i \rightarrow u_1^i \rightarrow \dots \rightarrow u_{m_i}^i$  be the path of length  $m_i$  from  $s_i = u_0^i$  to  $t_i = u_{m_i}^i$  in  $C(s_i)$ . Rule  $U_2$  assigns (i)  $X'_{u_k^i} = X_{u_{k+1}^i}$  for all  $k \in \{0, \dots, m_i - 1\}$  and all  $i \in \{0, \dots, \ell - 1\}$  and (ii)  $X'_{t_i} = Z_{i-1}$  for all  $i \in \{1, \dots, \ell\}$ , where  $Z_i$  is defined in Algorithm 2 (see line 12). For all other  $j$ , we have  $X'_j = X_j$ .

LEMMA 2.6. (RULE  $U_2$ ) *Rule  $U_2$  increases social welfare and returns an EFX allocation.*

*Proof.* We first observe that the valuations of the agents for their bundles have either increased or remained the same (since either the agents are left with their old bundles or assigned bundles that they envied). In particular, the valuations of all the agents in  $\bigcup_{i=0}^{\ell-1} \bigcup_{k=0}^{m_i} \{u_k^i\}$  are strictly larger, where the vertices  $u_k^i$  are defined above. Thus  $\phi(X') > \phi(X)$ .

It remains to show that the allocation  $X'$  is EFX, i.e., for every pair of agents  $j$  and  $k$  and any good  $g \in X'_k$  we have  $v_j(X'_k \setminus g) \leq v_j(X'_j)$ . Let  $T = \{t_1, t_2, \dots, t_\ell\}$ . For every agent  $r \notin T$  we have  $X'_r = X_r$  for some  $r'$ . Now consider two cases depending on  $k$ :

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**Algorithm 2** The Update Rules

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- 1: **function**  $U_0$ (allocation  $X$ , pool  $P$ )  
    **Precondition:** There is a good  $g \in P$  and an agent  $i$  such that allocating  $g$  to  $i$  results in an EFX allocation.
  - 2:     Allocate  $g$  to  $i$ , i.e.,  $X'_i \leftarrow X_i \cup g$ ,  $P' \leftarrow P \setminus g$ , and  $X'_j = X_j$  for  $j \neq i$ .
  - 3:     **return**  $(X', P')$ .
  - 4: **end function**
  
  - 5: **function**  $U_1$ (allocation  $X$ , pool  $P$ )  
    **Precondition:** There is an agent  $i$  such that  $v_i(P) > v_i(X_i)$ .
  - 6:     Let  $i$  be an agent in  $A_X(P)$  and  $Z \subseteq P$  be a set of size  $\kappa_X(P)$  such that  $v_i(Z) > v_i(X_i)$ .
  - 7:     Set  $X'_i = Z$  and  $X'_j = X_j$  for  $j \neq i$ .
  - 8:     Set  $P' = X_i \cup (P \setminus Z)$ .
  - 9:     **return**  $(X', P')$ .
  - 10: **end function**
  
  - 11: **function**  $U_2$ (allocation  $X$ , pool  $P$ )  
    **Precondition:** There is an  $\ell \geq 1$ , distinct goods  $g_0, g_1, \dots, g_{\ell-1}$  in  $P$ , distinct sources  $s_0, s_1, \dots, s_{\ell-1}$  of  $G_X$  and distinct agents  $t_1, t_2, \dots, t_\ell$  such that  $t_i \in C(s_i) \cap A_X(X_{s_{i-1}} \cup g_{i-1})$  for  $0 \leq i \leq \ell-1$  (indices are to be interpreted modulo  $\ell$ ).
  - 12:     Let  $Z_i \subseteq X_{s_i} \cup g_i$  of size  $\kappa_X(X_{s_i} \cup g_i)$  be such that  $v_{t_{i+1}}(Z_i) > v_{t_{i+1}}(X_{t_{i+1}})$  for  $0 \leq i \leq \ell-1$ .
  - 13:     Set  $P' = (P \setminus \bigcup_{i=0}^{\ell-1} \{g_i\}) \bigcup_{i=0}^{\ell-1} ((X_{s_i} \cup \{g_i\}) \setminus Z_i)$ .
  - 14:     Let  $u_0^i \rightarrow \dots \rightarrow u_{m_i}^i$  be the path of length  $m_i$  from  $s_i = u_0^i$  to  $t_i = u_{m_i}^i$  in  $C(s_i)$  for  $0 \leq i \leq \ell-1$ .
  - 15:     Set  $X'_{u_k^i} = X_{u_{k+1}^i}$  for all  $k \in \{0, \dots, m_i-1\}$  and all  $i \in \{0, \dots, \ell-1\}$ .
  - 16:     Set  $X'_{t_i} = Z_{i-1}$  for all  $i \in \{1, \dots, \ell\}$ .
  - 17:     Set  $X'_j = X_j$  for all other  $j$ .
  - 18:     **return**  $(X', P')$ .
  - 19: **end function**
- 

- $k \notin T$ : Note that valuations of the agents for their current bundles (in  $X'$ ) is at least as good as their old bundles (in  $X$ ). We have  $v_j(X'_j) \geq v_j(X_j) \geq v_j(X_{k'} \setminus g) = v_j(X'_k \setminus g)$  for any  $g \in X'_k$  (or equivalently,  $g \in X_{k'}$ ).
- $k \in T$ : Let  $k = t_i$ . We have  $v_j(X'_j) \geq v_j(X_j) \geq v_j(Z_{i-1} \setminus g)$  for any  $g \in Z_{i-1}$  (by Lemma 2.2) and  $v_j(Z_{i-1} \setminus g) = v_j(X'_{t_i} \setminus g) = v_j(X'_k \setminus g)$  for any  $g \in X'_k$ .  $\square$

We can now summarize. Let  $V = \max_i v_i(M)$  be the maximum valuation of any agent and  $\Delta = \min_i \min\{|v_i(T) - v_i(S)| : S, T \subseteq M \text{ and } v_i(S) \neq v_i(T)\}$  be the minimum difference between distinct valuations. Each application of rule  $U_1$  or rule  $U_2$  increases the social welfare by at least  $\Delta$  and hence there can be no more than  $nV/\Delta$  applications of these rules. Each application of rule  $U_0$  decreases the size of the pool by one and hence there cannot be more than  $m$  successive applications of this rule. We conclude that the number of iterations is at most  $nmV/\Delta$ . Thus we have shown the following theorem.

**THEOREM 2.1.** *For normalized and monotone valuations, there is always an allocation  $X$  and a pool  $P$  of*

*unallocated goods such that*

- $X$  is EFX,
- $v_i(X_i) \geq v_i(P)$  for all agents  $i$ , and
- $|P|$  is less than the number of sources in the envy-graph; in particular,  $|P| < n$ .

*Algorithm 1 determines such an allocation in at most  $nmV/\Delta$  iterations.*

**2.1 New Proof of a Result from [27]** For agents with identical (general) valuations, it was shown by Plaut and Roughgarden [27] that an allocation that maximizes the the minimum valuation, then maximizes the size of this bundle, then maximizes the second minimum valuation, then maximizes the size of this bundle, and so on is EFX. We now show that Algorithm 1 gives another proof that when all the agents have identical valuations, a complete allocation that is EFX always exists.

Recall that Algorithm 1 consists of applying 3 update rules:  $U_0, U_1, U_2$  – whichever of these is applicable. Moreover, if a certain precondition is satisfied (see Algorithm 2), then rule  $U_2$  is applicable.





a set of cardinality  $k$  of maximum value, i.e.,  $v(S_k) \geq v(T)$  for every set  $T$  with  $|T| \leq k$ .

*Proof.* Consider any  $k$ . If  $v(g_{k+1}|S_k) = 0$ , then  $v(g \cup S_k) = v(S_k)$  for every  $g \notin S_k$  and hence  $v(S_k) = v(M)$  by the *submodularity*<sup>8</sup> of  $v$  [23]. Thus  $S_k$  is a most valuable set of size  $k$ . This is also true for  $k = m$ . So assume  $k < m$  and  $v(g_{k+1}|S_k) > 0$ . Then  $v(g_k|S_{k-1}) \geq v(g_{k+1}|S_{k-1}) \geq v(g_{k+1}|S_k) > 0$ , where the first inequality holds since  $g_k$  is chosen in round  $k$  and the second inequality follows from the submodularity of  $v$ . Let  $p$  be a price vector with  $p_g = v(g_k|S_{k-1})$  for all  $g \in M$ ; we abuse the notation and use also  $p$  for the common price. We will show that  $S_k \in D(v, p)$ .

CLAIM 1.  $S_k \in D(v, p)$ .

*Proof.* We will use Algorithm 3 to derive that  $S_k \in D(v, p)$ . Observe that the price is the same (this is  $p$ ) for all goods. So in each round, Algorithm 3 chooses a good  $g$  that maximizes  $v(g|S)$ . This is also what Algorithm 4 does. We may assume that ties are broken in the same way and hence both algorithms add goods in the same order. We need to guarantee that Algorithm 3 can return the set  $S_k$ . This holds because  $v(g_k|S_{k-1}) - p = 0$ , so we can add this item and terminate the algorithm in the next round where  $\Delta_{g_{k+1}} \leq 0$ .  $\square$

We are now ready to show that  $v(T) \leq v(S)$  for any set  $T$  of size at most  $k$ . We have

$$v(T) - \sum_{g \in T} p_g \leq v(S_k) - \sum_{g \in S_k} p_g = v(S_k) - kp,$$

for every set  $T \subseteq M$  since  $S_k \in D(v, p)$ . Hence we have:

$$v(T) \leq v(S_k) + (|T| - k)p \leq v(S_k)$$

for every set  $T$  with  $|T| \leq k$ .  $\square$

The minimum-size-valuable-set oracle is now readily realized. We simply run Algorithm 4 on the set  $S$  until a set of value greater  $\alpha$  is obtained. We can conclude the following theorem.

**THEOREM 3.1.** *For gross substitute valuations, the allocation defined in Theorem 2.1 can be determined with  $\text{poly}(n, m, V, 1/\Delta)$  value queries.*

<sup>8</sup>So  $v(S \cup g) - v(S) \geq v(T \cup g) - v(T)$  whenever  $S \subseteq T$ .

**3.1 An FPTAS to Determine an “Almost” Desired Allocation** Our algorithm is pseudo-polynomial, since the increase in individual valuations of the agents when we perform the update rules could be very small. Suppose we just wanted an “almost” EFX property, i.e., for every pair of agents  $i$  and  $j$ , we are happy to ensure that  $(1 + \varepsilon) \cdot v_i(X_i) \geq v_i(X_j)$  and also  $(1 + \varepsilon) \cdot v_i(X_i) \geq v_i(P)$  for every  $i$ . Then we have an algorithm that runs in  $\text{poly}(n, m, \frac{1}{\varepsilon}, \log V)$  time and finds a desired allocation.

**THEOREM 3.2.** *For normalized and gross substitute integral valuations, given any  $\varepsilon > 0$ , in time  $\text{poly}(n, m, \frac{1}{\varepsilon}, \log V)$ , we can determine an allocation  $X = \langle X_1, X_2, \dots, X_n \rangle$  and a pool of unallocated goods  $P$  such that*

- for any pair of agents  $i$  and  $j$  we have  $(1 + \varepsilon) \cdot v_i(X_i) \geq v_i(X_j \setminus g)$  for all  $g \in X_j$ ,
- for any agent  $i$ , we have  $(1 + \varepsilon) \cdot v_i(X_i) \geq v_i(P)$ , and
- $|P| < n$ .

The proof follows in a straightforward manner from the proof of Theorem 2.1 in Section 2. The key idea is that the “almost” EFX property is violated if and only if  $(1 + \varepsilon) \cdot v_i(X_i) < v_i(X_j \setminus g)$  for some  $i, j \in [n]$  or  $(1 + \varepsilon) \cdot v_i(X_i) < v_i(P)$  for some  $i \in [n]$ . So every time we apply the update rules  $U_1$  or  $U_2$  there is a multiplicative improvement (by a factor of  $1 + \varepsilon$ ) in the valuation of some agents. Since these valuations are upper-bounded by  $V$  we get a bound of  $\text{poly}(n, m, \log_{(1+\varepsilon)} V)$  on the number of iterations.

#### 4 Guarantees on Other Notions of Fairness

In this section we assume that all agents have additive valuations. We show that a minor variant of our algorithm finds an allocation with a good guarantee on Nash social welfare and groupwise maximin share (GMMS).

**Guarantee in Terms of Nash Social Welfare.** We claimed in Section 1 that for additive valuations, it can also be ensured that for each  $i$ , we have  $v_i(X_i) \geq \frac{1}{2} \cdot v_i(X_i^*)$  where  $X^* = \langle X_1^*, \dots, X_n^* \rangle$  is an optimal Nash social welfare allocation and  $X$  is the allocation in Theorem 2.1. This is easy to see from Algorithm 1:

- rather than initialize  $X_i = \emptyset$ , we will initialize  $X_i$  to the bundle corresponding to the allocation determined by the algorithm in [13].

So we have  $v_i(X_i) \geq \frac{1}{2} \cdot v_i(X_i^*)$ , to begin with. As the algorithm progresses, our invariant is that  $v_i(X_i)$  never

decreases for any  $i$ . So if  $X' = \langle X'_1, \dots, X'_n \rangle$  is the final allocation computed by our algorithm, then we have  $v_i(X'_i) \geq \frac{1}{2} \cdot v_i(X_i^*)$  for  $i \in [n]$ .

**LEMMA 4.1.** *Given a set  $N$  of agents with additive valuations and a set  $M$  of goods, there exists an allocation  $X = \langle X_1, \dots, X_n \rangle$  and a pool  $P$  of unallocated goods that satisfy all the conditions of Theorem 2.1 and  $v_i(X_i) \geq \frac{1}{2}v_i(X_i^*)$  for all  $i \in N$ , where  $X^* = \langle X_1^*, \dots, X_n^* \rangle$  is an optimal Nash social welfare allocation.*

**4.1 An Approximate MMS Allocation for Large  $|P|$**  We now show that if  $|P|$  (the number of unallocated goods in our allocation) is sufficiently large, then our EFX allocation  $X$  has a very good MMS guarantee. Recall that our algorithm continues till  $|P|$  is smaller than the number of sources in the envy-graph  $G_X$  and recall that sources are *unenvied* agents. In particular, if  $|P| = n - 1$ , then the number of sources in  $G_X$  is  $n$ ; so no agent envies another. That is, for each  $i$ , we have  $v_i(X_i) \geq v_i(X_j)$  for all  $j \in [n]$ . Moreover,  $v_i(X_i) \geq v_i(P)$ . So we have

$$\begin{aligned} v_i(X_i) &\geq \frac{v_i(M)}{n+1} &&\geq \left(1 + \frac{1}{n}\right)^{-1} \cdot \frac{v_i(M)}{n} \\ &&&\geq \left(1 + \frac{1}{n}\right)^{-1} \cdot MMS_i(n, M), \end{aligned}$$

where for every agent  $i$ , the inequality  $MMS_i(n, M) \leq v_i(M)/n$  holds for additive valuations. We formalize the above intuition in Theorem 4.1. The following proposition will be useful.

**PROPOSITION 4.1.** ([19]) *Let  $N$  be a set of  $n$  agents with additive valuations and  $M$  be a set of  $m$  goods. If  $N' \subseteq N$  and  $M' \subseteq M$  are such that  $|N \setminus N'| \geq |M \setminus M'|$  then for any agent  $i \in N'$ , we have  $MMS_i(n', M') \geq MMS_i(n, M)$  where  $n' = |N'|$ .*

**THEOREM 4.1.** *Given a set  $N$  of  $n$  agents with additive valuations and a set  $M$  of  $m$  goods, there exists an allocation  $X = \langle X_1, X_2, \dots, X_n \rangle$  and set  $P$  of unallocated goods that satisfies:*

- the 3 conditions stated in Theorem 2.1;
- $v_i(X_i) \geq \frac{1}{2}v_i(X_i^*)$  for all  $i \in N$ , where  $X^*$  is an optimal Nash social welfare allocation;
- $v_i(X_i) \geq MMS_i(n, M)/(2 - \frac{k}{n})$  for every  $i \in N$ , where  $k = |P|$ .

*Proof.* Let  $(X, P)$  be the allocation guaranteed by Lemma 4.1. Hence the first two conditions given in

the theorem statement are satisfied by  $(X, P)$ . So what we need to show now is that for any agent  $i$ , we have  $v_i(X_i) \geq MMS_i(n, M)/(2 - \frac{k}{n})$ .

Let  $N' \subseteq N$  be the set of agents  $j$  for which either  $v_i(X_j) \leq v_i(X_i)$  or  $|X_j| \geq 2$ . Then  $i \in N'$  and all sources of  $G_X$  belong to  $N'$ . Also,  $|X_j| = 1$  and  $v_i(X_j) > v_i(X_i)$  for  $j \in N \setminus N'$ .

Let  $M'$  be the set of goods allocated to the agents in  $N'$ . The agents in  $N \setminus N'$  are allocated the goods in  $M \setminus (M' \cup P)$ . Observe that every agent in  $N \setminus N'$  is allocated at most one good. So we have  $|N \setminus N'| \geq |M \setminus (M' \cup P)|$ .

Proposition 4.1 tells us that  $MMS_i(n', M' \cup P) \geq MMS_i(n, M)$  where  $n' = |N'|$ . Thus it suffices to show that  $v_i(X_i) \geq MMS_i(n', M' \cup P)/(2 - \frac{k}{n})$ .

Consider any  $j \in N'$  with  $v_i(X_j) > v_i(X_i)$ . So  $|X_j| \geq 2$ . Since valuations are additive and  $v_i(X_i) \geq v_i(X_j \setminus \{g\})$  for all  $g \in X_j$ , we have

$$v_i(X_i) \geq \left(1 - \frac{1}{|X_j|}\right) \cdot v_i(X_j) \geq \frac{1}{2} \cdot v_i(X_j).$$

We know the following inequalities hold:

$$\begin{aligned} (4.1) \quad v_i(X_i) &\geq v_i(P), \\ (4.2) \quad v_i(X_i) &\geq v_i(X_j) \text{ for all } j \text{ that} \\ &\text{were sources in } G_X, \\ (4.3) \quad 2v_i(X_i) &\geq v_i(X_j) \\ &\text{for all other } j \in N'. \end{aligned}$$

Recall that the number of sources is at least  $|P| + 1 = k + 1$ . Summing up all inequalities in (4.1)-(4.3) and using the fact that  $v_i$  is additive, we have  $(2(n' - (k + 1)) + k + 2) \cdot v_i(X_i) \geq v_i(M' \cup P)$ .

Hence we have

$$\begin{aligned} v_i(X_i) &\geq \frac{v_i(M' \cup P)}{2n' - k} \\ &\geq \frac{v_i(M' \cup P)}{n'} \cdot \frac{n'}{2n' - k} \\ &\geq MMS_i(n', M' \cup P) \cdot \frac{n'}{2n' - k} \\ &\quad \text{(since } v_i \text{ is additive)} \\ &= MMS_i(n', M' \cup P) / \left(2 - \frac{k}{n'}\right) \\ &\geq MMS_i(n', M' \cup P) / \left(2 - \frac{k}{n}\right) \\ &\quad \text{(since } n' \leq n). \quad \square \end{aligned}$$

**4.2 An Improved Bound for Approximate-GMMS** As mentioned in Section 1, a new notion of fairness called *groupwise maximin share* (GMMS) was recently introduced by Barman et al. [7]. We formally define a GMMS allocation below.

DEFINITION 4.1. Given a set  $N$  of  $n$  agents and a set  $M$  of  $m$  goods, an allocation  $X = \langle X_1, X_2, \dots, X_n \rangle$  is  $\alpha$ -GMMS if for every  $N' \subseteq N$ , we have  $v_i(X_i) \geq \alpha \cdot MMS_i(n', \bigcup_{i \in N'} X_i)$  where  $n' = |N'|$ .

Observe that a GMMS allocation is also an MMS allocation. Since MMS allocations do not always exist in a given instance [28], GMMS allocations also need not always exist. Interestingly,  $\frac{1}{2}$ -GMMS allocations always exist [7]. We now describe how to modify our allocation so that the resulting allocation is  $\frac{4}{7}$ -GMMS.

Let  $X = \langle X_1, \dots, X_n \rangle$  be the allocation and  $P$  be the pool of unallocated goods that satisfy the conditions of Lemma 4.1. Without loss of generality, assume that agent 1 is a source in the envy-graph  $G_X$ . Define the complete allocation  $Y = \langle Y_1, \dots, Y_n \rangle$  as follows:

$$* Y_1 = X_1 \cup P \text{ and } Y_i = X_i \text{ for all } i \neq 1.$$

Theorem 4.2 shows that  $Y$  is our desired allocation. The proof of Theorem 4.2 is similar to [2, Proposition 3.4]. We also remark that one can use the proof of [2, Proposition 3.4] to show that any EFX allocation is a  $4/7$ -GMMS. However note that  $Y$  is not necessarily an EFX allocation. But it has sufficiently nice properties that we can still show that it is  $4/7$ -GMMS.

THEOREM 4.2. Given a set  $N$  of  $n$  agents with additive valuations and a set  $M$  of  $m$  goods, there exists a complete allocation  $Y = \langle Y_1, Y_2, \dots, Y_n \rangle$  of  $M$  such that

- $Y$  is  $\frac{4}{7}$ -GMMS.
- $v_i(Y_i) \geq \frac{1}{2}v_i(X_i^*)$  for all  $i \in N$  where  $X^*$  is the optimal Nash social welfare allocation.<sup>9</sup>

*Proof.* Observe that the bound on Nash social welfare holds for allocation  $X$  and thus for allocation  $Y$  (since  $v_i(Y_i) \geq v_i(X_i)$  for all  $i \in [n]$ ). So what we need to show now is the guarantee on GMMS. That is, we need to show that for every  $\tilde{N} \subseteq N$  and all  $i \in \tilde{N}$ , we have  $v_i(Y_i) \geq \frac{4}{7}MMS_i(\tilde{n}, \tilde{M})$  where  $\tilde{n} = |\tilde{N}|$  and  $\tilde{M} = \bigcup_{j \in \tilde{N}} Y_j$ .

Fix some  $i \in \tilde{N}$ . Define  $N'$  as the subset of  $\tilde{N}$  that contains  $i$  and all agents that have been allocated at least two goods in  $Y$ , i.e.,  $j \in N'$  if and only if  $j = i$  or  $|Y_j| \geq 2$ . Let  $M' = \bigcup_{j \in N'} Y_j$ .

<sup>9</sup>In private communication we are aware that Jugal Garg and Setareh Taki have obtained independently related results. For **additive** valuations, they can show that there is an EFX-allocation after donating at most  $n-1$  goods to charity. However, there is no bound on the value of the goods donated to charity. Thus they obtain a  $4/7$ -GMMS-allocation after removing  $n-1$  goods from the original set of goods.

Note that  $Y$  allocates all goods in  $\tilde{M} \setminus M'$  to agents in  $\tilde{N} \setminus N'$ . Since every agent in  $\tilde{N} \setminus N'$  has been allocated at most one good, we have  $|\tilde{N} \setminus N'| \geq |\tilde{M} \setminus M'|$ . Proposition 4.1 tells us that  $MMS_i(n', M') \geq MMS_i(\tilde{n}, \tilde{M})$  where  $n' = |N'|$ . Thus it suffices to show  $v_i(Y_i) \geq 4/7 \cdot MMS_i(n', M')$ .

We now classify the goods in  $M'$  as good or bad. A good is *good* if it contained in a  $Y_j$  of cardinality at least three or is contained in  $Y_1$  or  $Y_i$ . All other goods are *bad*, i.e., a bad good is contained in a bundle of cardinality two different from  $Y_1$  and  $Y_i$ . A bundle in  $Y$  containing good goods is good.

We will next reduce the problem further. Let  $x$  be the number of bad goods in  $M'$ . Since the bad goods are contained in bundles of cardinality two, the good goods in  $M'$  come from  $n' - x/2$  good bundles of  $Y$ . As long as  $x > n'$ , we will apply a reduction step. Each reduction step will reduce the number of bad goods in  $M'$  by two, the number of agents by one, will not decrease the  $MMS_i$ -value, and will leave the quantity  $n' - x/2$  and set of good goods in  $M'$  unchanged.

Let  $Z = \langle Z_1, Z_2, \dots, Z_{n'} \rangle$  be an optimal MMS partition for agent  $i$  of the set  $M'$  of goods. If there are more than  $n'$  bad goods in  $M'$ , there is a set  $Z_k$  with at least *two* bad goods, say  $g_1$  and  $g_2$ . We distribute the goods in  $Z_k \setminus \{g_1, g_2\}$  arbitrarily among the other sets in  $Z$ . So we have a partition of the set  $M' \setminus \{g_1, g_2\}$  of goods into  $n' - 1$  many bundles. The value for agent  $i$  of any remaining bundles did not decrease. Set  $M'$  to  $M' \setminus \{g_1, g_2\}$  and decrement  $n'$ . Note that we reduced the number of bad goods by two, the number of agents by one, did not decrease  $MMS_i(n', M')$  and the set of good goods in  $M'$  still come from the  $n' - x/2$  good bundles in  $Y$ . We keep repeating this reduction until  $M'$  contains at most  $n'$  bad goods. At this point, we have a set  $M'$  of goods and an integer  $n'$  with the following properties:

- (1) The number  $x$  of bad goods in  $M'$  is at most  $n'$ .
- (2)  $MMS_i(n', M') \geq MMS_i(\tilde{n}, \tilde{M})$ , and
- (3) The set of good goods in  $M'$  has not changed. They come from  $n' - x/2$  goods bundles in  $Y$ .

We will next relate the value of good and bad goods to the value of  $Y_i$  for all  $i$ .

LEMMA 4.2. We have

- a) For any bad good  $g$ ,  $v_i(g) \leq v_i(Y_i)$ .
- b)  $v_i(Y_1) \leq 2v_i(Y_i)$ .
- c) If  $j \neq 1$  and  $Y_j$  is a good bundle then  $v_i(Y_j) \leq 3/2 \cdot v_i(Y_i)$ .

*Proof.* a) Let  $Y_j = \{g, g'\}$  be the bundle containing  $g$ . Since  $j \neq 1$  (by definition of a bad good), we have  $v_i(Y_i) \geq v_i(X_i) \geq v_i(X_j \setminus g') = v_i(Y_j \setminus g') = v_i(g)$ .



- b) Since agent 1 is a source,  $v_i(X_i) \geq v_i(X_1)$ . By Theorem 2.1,  $v_i(X_i) \geq v_i(P)$ . Therefore,  $v_i(Y_1) = v_i(X_1 \cup P) = v_i(X_1) + v_i(P) \leq v_i(X_i) + v_i(X_i) = 2v_i(X_i) = 2v_i(Y_i)$ .
- c) Let  $g \in Y_j$  be such that  $v_i(g)$  is minimal. Then  $v_i(Y_j - g) \leq v_i(Y_i)$  and  $v_i(g) \leq v_i(Y_j)/|Y_j|$ . Thus

$$\begin{aligned} v_i(Y_i) &\geq \left(1 - \frac{1}{|Y_j|}\right) \cdot v_i(Y_j) \geq \left(1 - \frac{1}{3}\right) \cdot v_i(Y_j) \\ &= \frac{2}{3} \cdot v_i(Y_j). \end{aligned}$$

□

Now we are ready to show the bound on GMMS. We have  $x$  bad goods in  $M'$ . The good goods in  $M'$  come from  $n' - x/2$  good bundles. Also  $x \leq n'$ . The total value of the good goods for agent  $i$  is at most

$$\frac{3}{2}(n' - \frac{x}{2} - 2) \cdot v_i(Y_i) + v_i(Y_i) + 2v_i(Y_i) = \frac{3}{2}(n' - \frac{x}{2}) \cdot v_i(Y_i),$$

since there are at most  $n' - x/2 - |\{1, i\}|$  good bundles different from  $Y_1$  and  $Y_i$ : each of value at most  $(3/2 \cdot v_i(Y_i))$ , and  $Y_1$  has value at most  $2v_i(Y_i)$ . Also, the total value of the bad goods for agent  $i$  is at most  $x \cdot v_i(Y_i)$ , since there are  $x$  many bad goods and each bad good is worth at most  $v_i(Y_i)$ . Therefore,

$$\begin{aligned} v_i(M') &= v_i(\text{bad goods in } M') + v_i(\text{good goods in } M') \\ &\leq x \cdot v_i(Y_i) + \frac{3}{2}(n' - \frac{x}{2}) \cdot v_i(Y_i) \\ &= \frac{6n' + x}{4} \cdot v_i(Y_i) \leq \frac{7n'}{4} \cdot v_i(Y_i). \end{aligned}$$

□

## 5 Large Markets with Subadditive Valuations

Our main result (Theorem 2.1) implies that complete allocations that are *approximately* EFX always exist in large markets with subadditive valuations. In large markets, the valuation of any good by an agent is only a small fraction of her valuation for the entire set  $M$ .

The intuition behind this observation is that the number of unallocated goods in Theorem 2.1 is less than the number of agents that nobody envies in the allocation  $X$ . We can therefore assign at most one unallocated good to each of the agents that nobody envies. This does not create much envy from other agents since the valuation any agent has for a single good is not significantly high in large markets. Hence we can get very close to an EFX allocation.

More precisely, a large market with parameter  $\varepsilon$  is one where for any agent  $i \in N$  and good  $g \in M$ , we have  $v_i(g) \leq \varepsilon \cdot v_i(M)/n$  [13]. It is easy to see that

Theorem 2.1 can be used to show that in a large market when agents have subadditive valuations, there always exists an allocation  $X = \langle X_1, \dots, X_n \rangle$  such that for any pair of agents  $i, j$ :

- (1) either  $v_i(X_i) \geq v_i(X_j \setminus g)$  for all  $g \in X_j$  or
- (2)  $v_i(X_i) \geq (1 - \delta) \cdot v_i(X_j)$ , where  $\delta = \varepsilon + \varepsilon/n$ .

## 6 Conclusions and Open Problems

We studied the existence of EFX allocations when agents have general valuations. We showed that we can ensure such an allocation always exists when we donate a small number of goods that nobody envies to charity. The major open problem here is whether EFX allocations always exist. Plaut and Roughgarden [27] remarked that an instance with *no* EFX allocation may be easier to find in the setting of general valuations. Our result on “almost-EFX” allocations for general valuations allows one to hope that EFX allocations always exist, at least for more structured valuations such as additive.

Plaut and Roughgarden [27] showed that an exponential number of “value queries” are required to determine an EFX allocation even for two agents with identical submodular valuations. From our proof it is evident that we can determine a  $(1 - \varepsilon)$  EFX allocation (as in Theorem 3.2) with polynomially many *size-constrained-optimal-valuation* queries (where for a given  $k, S$  and agent  $i$  we need to find the subset of  $S$  of size at most  $k$  that maximizes agent  $i$ 's valuation). Studying the complexity of determining approximate EFX allocations under other queries is a line of direction for future work.

We also showed that we get guarantees in terms of other notions of fairness when agents have additive valuations. To the best of our knowledge, allocations with good guarantees (i.e., constant factor approximation) on Nash social welfare and MMS (as well as GMMS) were not known prior to our work. It would also be interesting to investigate whether these guarantees can be improved or if instances can be constructed where our guarantees are tight. We believe that our work is just the beginning towards determining an allocation that gives good guarantees with respect to several notions of fairness: an allocation that is *universally* fair.

Very recently, Amanatidis et al. [4] have announced an allocation that has good approximation guarantees simultaneously with respect to four notions of fairness when valuation functions are additive: in particular, their allocation is  $(\phi - 1)$ -EFX and  $2/(\phi + 2)$ -GMMS, where  $\phi \approx 1.618$  is the golden ratio. Moreover, a fine-tuned version of their algorithm also achieves  $4/7$ -GMMS which matches our result (see Theorem 4.2). They also show that for additive valuations, when  $m$

is at most  $n + 2$ , where  $m$  is the number of goods and  $n$  is the number of agents, GMMS (and hence, EFX) allocations always exist.

**Acknowledgments.** We thank the reviewers for their helpful comments.

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