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Time delay system control and application in electricity market

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by

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Abstract

In many real-world engineering systems, the rate of variation in the system state depends on the past states, which characteristic is called delay or a time delay. As time delay will degrade the system dynamic performance and even destroy the system stability, the stability analysis of a system with a time delay has been investigated in many real control systems such as load frequency control scheme of power systems, genetic regulatory network, digital filter, electricity market and economic dispatch of power system. Several improved stability criteria of time delay systems are proposed and applied in power system control with time delay such as load frequency control and electricity market.

In this thesis, the Lyapunov-Krasovskii functional and the Wirtinger inequality have been investigated to establish several new stability criterion at first. The Wirtinger-based inequality used in a linear system with two additive time-varying delays are investigated and applied in load frequency control system in chapter 2. In chapter 3, an improve stability criterion based on Wirtinger-type double integral inequality was developed and applied in genetic regulatory networks. Then, in chapter4, the proposed method hsa been extended to discrete system and applied in digital filters. Chapter 5 presents a further improve a stability criterion using Wirtinger-type three integral inequality and applied in the power system, including load frequency control system and economical dispatch system. In chapter 6, the stability analysis of an electricity market with a time-varying delay are studied. The dynamic model of an electricity market is presented to minimize the operation cost. The impact of a time delay on this dynamic model is analysed via stability criterion developed in Chapter 5. The main contribution of the thesis is not only propses several stability criterion based on the Lyapunov-Krasovskii functional and

Wirtinger inequality but also applying the methods in power systems, genetic regulatory network and digital filter. Effectiveness of those new stability criterion have been verified via simulation studies based on numerical examples and those industry time-delay system.

Declaration

The author hereby declares that this thesis is a record of work carried out in the Department of Electrical Engineering and Electronics at the University of Liverpool during the period from October 2016 to September 2019. The thesis is original in content except where otherwise indicated.

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List of Abbreviations

Abbreviations

LKF Lyapunov-Krasovskii funtional
LMI Linear Matrix Inequality
FWM Free weighting matrix
LFC Load frequency control
GRNs Genetic Regulatory Networks
NoVs number of variable
WBDII Wirtinger-based double integral inequality
JBDII Jensen-based double integral inequality
WTDII Wirtinger-type double integral inequality
MADBs Maximal admissible delay bounds
ISS Input-to-state stability
IOSS Input/output-to-state stability
ED Economic Dispatch
GenCo Generating company
ConCo Consumer company
ISO Independent System Operator
KKT Karush Kuhn Tucker

Chapter 1

Introduction

1.1 Time delay system

In many real-world engineering systems, the rate of variation in the system state depends on the past states, which characteristic is called delay or a time delay [1]. Time delays exist in many engineering systems such as biological systems, mechanical transmissions, fluid transmissions, metallurgical processes, and networked control systems. Time delay will usually deteriorate the dynamic performance of the whole system or may even destroy the whole system stability [2, 3]. The significant and widespread occurrence of time delay makes time delay system attracted lots of researchers' attention [1]. Theoretical and practical development in the field of time delay system that explore new directions have been generally launched from a consideration of stability analysis and robust control.

1.1.1 Stability analysis of a time delay system

In control theory, system stability is a fundamental objective to be achieved, which has been extensively analysed [1]. Research on the stability of time delay systems started at 1950s. The objective is to find the maximum time delay of the system, under which delay the system stability can still be ensured. The maximum time delay is defined as the stability delay margin. For the control design prospective, a proper controller is desired be such that the expected time delay robustness can be achieved.

A typical following linear system with a delay can be represented as:

$$\begin{cases} \dot{x}(t) &= Ax(t) + A_d x(t-h), \\ x(t) &= \phi(t), t \in [-h, 0], \end{cases} \tag{1.1.1}$$

where $x(t) \in R^n$ is the state vector; $h > 0$ is a delay in the system state, that is, it is a discrete delay; $\phi(t)$ is the initial condition; and $A \in R^{n\times n}$ and $A_d \in R^{n\times n}$ are the system matrices. It can be seen that the future evolution of this system depends not only on its present state, but also on its history. The stability analysis of time delay system usually adopt two main types of stability analysis method: frequency-domain method and time-domain method respectively [3].

Frequency-domain method

Frequency-domain method is the most classical analysis method in control system theory and provides the most sophisticated approach to analyzing the stability of a system without delay $(h = 0)$. The necessary and sufficient condition for the stability of such system is $s(A + A_d) < 0$. When $h > 0$, frequency-domain method yields the result that system (1.1.1) is stable if and only if all the roots of its characteristic function,

$$f(s) = det(sI - A - A_d e^{-hs}) = 0 \tag{1.1.2}$$

have negative real parts. However, the equation is transcendental and hard to be solved, which limits the application of frequency-domain method. Moreover, frequency-domain method is good at daling with constant time delay only.

Time-domain method

Time-domain method is primarily based on two famous theorems: the Lyapunov-Krasovskii stability theorem and the Razumikhin stability theorem. The main idea is to obtain a sufficient condition for the stability of system (1.1.1) via constructing an appropriate Lyapunov-Krasovskii functional (LKF) or an appropriate Lyapunov function. The Linear Matrix Inequality (LMI) can then be used to construct LKFs and Lyapunov functions. The general LKF is:

$$V_1(x_t) = x^T(t)Px(t) + \int_{t-h}^{t} x^T(s)Qx(s)\,ds \tag{1.1.3}$$

where $P > 0$ and $Q > 0$ are the Lyapunov matrices to be determined; x_t denotes the translation operator acting on the trajectory: $x_t(\theta) = x(t+\theta)$ for some (non-zero) interval $[-h, 0]$ ($\theta \in [-h, 0]$). Calculating the derivative of $V_1(x_t)$ along the solutions of system (1.1.1) and restricting it to less than zero yield the delay-independent stability condition of the system:

$$\begin{bmatrix} PA + A^TP + Q & PA_d \\ * & -Q \end{bmatrix} < 0 \tag{1.1.4}$$

Since the inequality (1.1.4) is linear with respect to the matrix variables P and Q, it is called an LMI. If the LMI toolbox of Matlab yields solutions to LMI (1.4) for these variables, then according to the Lyapunov-Krasovskii stability theorem, system (1.1.1) is asymptotically stable for all $h \leq 0$. Furthermore, an appropriate LKF can be obtained.

In 1990s, the main approach to study delay-dependent stability involved an addition of a quadratic double-integral term to the LKF (1.1.3), which is shown in equation (1.1.5).

$$V(x_t) = V_1(x_t) + V_2(x_t) \tag{1.1.5}$$

where

$$V_2(x_t) = \int_{-h}^{o} \int_{t+\theta}^{t} x^T(s)Zx(s)\,dsd\theta$$

The derivative of $V_2(x_t)$ is

$$\dot{V}_2(x_t) = hx^T(t)Zx(t) - \int_{t-h}^{t} x^T(s)Zx(s)\,ds \tag{1.1.6}$$

Delay-dependent conditions can be obtained from the Lyapunov-Krasovskii stability theorem. However, how to solve the integral term on the right side of (1.1.6) is a challenge.

1.1.2 Time-domain delay dependent stability analysis of a time delay system

The stability criterion and controller design conditions obtained by above time-domain methods are both sufficient, and exist conservatism. The main research motivation is to design a method so that conservatism of these criteria can be reduced. Conservativeness of the results obtained by various methods is also the main criterion to evaluate the superiority of the corresponding methods. In order to achieve this objective for reducing conservatism, the existing research is often implemented from two aspects, including functional structure and the processing of its derivatives. The following part is a brief review of the current research on these two aspects.

1) Functional structure

In the Lyapunov functional analysis method, selecting or constructing a suitable functional is a crucial step. For the best knowledge of author, the constructed functional in the existing research can be roughly divided into general type, complete type, discrete type, augmented type and time-delay decomposition type [1].

The general LKF is mentioned in (1.1.3). Some quadratic terms which considers time delay information are added on the basis of the classical quadratic function [4]. The advantages of simple structure involve clear physical meaning, less matrix variables and simple calculation. A complete functional is constructed in [5]. Obviously, this kind of functional can obtain sufficient and necessary conditions for the asymptotic stability of the system. However, this kind of functional is generally infinite dimensional and difficult to verify. In order to solve the problem caused by infinite dimension of complete functional, discrete functional is proposed in [6, 7], but it is difficult to integrate for a time-varying systems. In recent years, based on the idea of simple functional and discretization, many researchers have proposed a functional construction method based on the delay decomposition approach to effectively reduce the conservatism of analysis and design [8–11], but with the increase of the number of segments, the computational complexity will increase rapidly. In addition, He et al. improved simple functional from another perspective and proposed that augmented functional could extend some items of simple functional to consider more system information and time-delay information [12], thus reducing the conservatism of analysis and design.

2)Estimation on functional derivatives

After selecting the functional, the next problem becomes to reasonably estimate its derivative along the system. The processing objective is to express the derivative as a negative qualitative form that can be determined by one or more matrix inequalities. The processing principle is to estimate the derivative of the functional without any amplification or to minimize the amplification degree, including the amplification times and the amplification amplitude, in order to obtain the lowest conservative result [1].

Because the time delay in real systems cannot be infinite, it is often desirable to obtain time delay correlation conditions in analysis and design. For this case, the difficulty in processing functional derivatives is how to reasonably estimate the quadratic integral terms contained in them. According to different treatment methods, the existing research can be divided into three categories:

Before 2004, almost all the research results were based on model transformation combined with cross term definition technology. It uses Newton-Leibniz formula to redescribe the time delay system and uses various inequalities such as basic inequalities, Park inequalities [13], Moon inequalities [14] to enlarge the cross term appearing in the derivatives, so as to offset the quadratic integral term in the derivative. However, the process of model transformation and inequality enlarge the cross term will bring conservatism.

In 2004, by analyzing the essence of the model transformation method, He et al. [15,16] proposed a new Free weighting matrix (FWM) method, which used FWM to express the relations of terms in Newton-Leibniz formula. Then, the optimal values of these matrices are obtained by LMIs. This method avoids the conservatism caused by model transformation and inequality definition. Later, He et al. [17–19] further improved the FWM and made it one of the most important methods for the study of time-delay systems.

Zhang et al. [20] proposed an integral inequality method based on the FWM method. It estimates the quadratic integrals of functional derivatives directly and obtains the delay dependent conditions based on matrix inequality. Then, based on this idea, some new and more effective integral inequalities, such as Jensen's inequality [21] and Wirtinger's inequality [22], were proposed.

Plenty of important results in stability analysis of time delay system are investigated and published. Following the rapid development, the conservatism of stability analysis of time delay system is reduced. Theoretical results are much better than past following the rise of complicated computation. However, the complicated computation makes many advanced results hard to be applied in a real system. The application of stability criteria becomes a popular topic in this area.

How to choose a good application background becomes an important problem. The application must have a non-negligible delay. The time delay is difficult to determine as a constant and will affect the stability of the system after the controller is added. Firstly, in power system, load frequency control requires transmitting measurement form remote terminal units to control center and control signals from the control center to plant side. The open communication networks will introduce time-varying delays. Those delays would degrade the dynamic performance of LFC and in the worst case, cause instability. The delay margin which allows an LFC scheme embedded with controllers to retain stable is an important parameter to design a controller with time-delay robustness. Secondly, a delay system needs to be used more widely to prove the applicability of the delay system such as GRNs. The messenger RNA and proteins may be synthesized at different locations, an important issue in modeling GRNs is that the slow processes of transcription, translation, and translocation results in sizable delays. Time delays arising in the GRNs may lead to incorrect prediction of dynamic behaviors. Thirdly, the limitation of register length reduces the wordlength of quantization. therefore, nonlinearities are unvoidable and in turn lead to undesirable behaviors. Finally, in electricity market, generators and consumers are competed though price signal. The time delay arising by discrete price signals and communication has long been neglected by scholars. However, the delay in electricity market increases the cost and even unstable the market. This motivates the PhD research on investigating the stability analysis of a time delay system and its application in electricity market.

1.2 Electricity markets

For more than one decade, the electricity markets, as a part of electricity industry restructuring, have experienced huge transformations in the United States (US) and all around the world. This restructuring has a series of reforms, which mainly include the building of spot market and the separation of electricity market into generation, transmission, distribution and load. The reason for restructuring is because the conventional electricity supply market, where generation, transmission and distribution were regarded as a whole electricity supply, was vertically integrated in the electricity markets as a monopoly utility prior to electricity industry restructuring. However, with the occurrence of the electricity supply industry competition in restructuring, electricity markets will no longer be charged by a single electricity supply industry [23]. The restructuring for generation market and the load market is deregulation. As for the transmission and distribution networks, they will remain regulated, but they must be open to all customers. Therefore, in order to study electricity markets, modelling electricity markets is a key step.

These reforms have started since 1970. Peru tried to reform the monopoly nature of electricity market, many countries such as the US, the UK and most of European countries have changed their policies to transit from vertically integrated regulated monopoly companies to a competitive market. For example, electricity supply in Britain was a monopoly activity until the start of the privatization of electricity industries in November 1990 [24]. Full competition was introduced in 1999. Since then domestic and non-domestic consumers have been able to choose and change their electricity suppliers freely [24].

Electricity market is a nature monopoly market because the essential features of electrical energy are different with other commodities. The purpose of electricity industry rstruan is deregulated the authority from government of oligrchic company to price. As the purpose of electricity industry restructuring is deregulated power from government or oligarchic company to electricity price. As shown in Fig. 1.1, in a conventional electricity market, the price is set by an organization who controls the power grid and is monopoly. Oppositely, in the deregulated electricity market, the electricity pricing is the key factor [25].

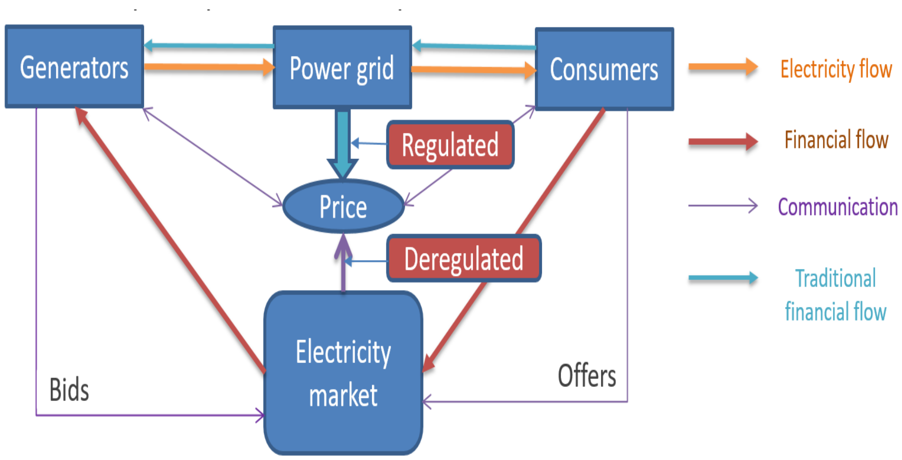

Figure 1.1: The electricity market model

1.2.1 Structure of electricity markets

Although the structure of electricity markets reform varies from country to country, there are many commonalities that can be used to introduce the general structure of electricity markets. As mentioned above, the competition is introduced in the generation market and load market. Two new markets are formed as the electricity wholesale market and electricity retail market accordingly. In the electricity retail market, suppliers sell the electricity that they have purchased from generators in the electricity wholesale market to end consumers [26]. The electricity wholesale market is the market for the sale and purchase of electricity between suppliers (to meet the demands of their customers) and generators [27]. Competition is achieved through unrestricted bilateral contract trading [27].

Since the characteristics of wholesale market are more representative of the power market, the general research will focus on the electricity wholesale market. The wholesale market can be separated as long-term forward contract, day-ahead market and real-time market. The day-ahead market is a forward market in which hourly clearing prices are calculated for each node in the system, for each hour of the next operating day. The real-time market is a spot market in which market clearing prices are computed for each node in the system, based on actual system operating conditions on near real-time basis. Fig. 1.2 shows the operation principle of this two market.

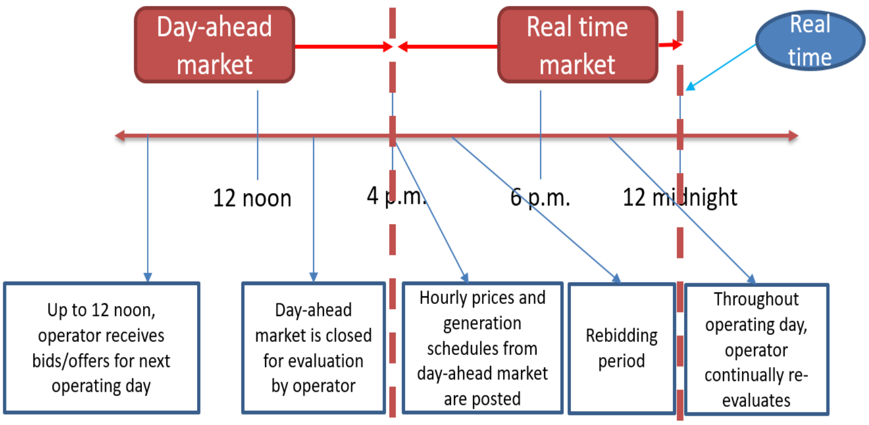

Figure 1.2: The structure of day ahead market and real time market

There are two types of wholesale market: Pool and Bilateral trading which are designed by the UK and the US separately. The Pool trading is a compulsory day-ahead market for bulk physical trading between generators and purchasers (suppliers and large-scale consumers) [27]. However, the two dominant generators at that time (PowerGen and National Power) had too much market power, which resulted in a series of problems including non-sufficient competition (manipulation of the pool selling price), unnecessary increase of system marginal price, etc. The Pool trading is replaced by Bilateral trading in 2001. Then, system operators are set to operate the power grid without benefit from Bilateral trading in 2005.

The current electricity trading arrangements in the wholesale market are described as follows: Electricity is generated, transmitted, distributed and consumed continuously in real-time and supply must always match demand. Although the generation, transmission, distribution and consumption of electricity is continuous, for the purposes of trading and settlement electricity is considered to be generated, transmitted, distributed and consumed in chunks called Settlement Periods. The length of each Settlement Period in the GB wholesale market is currently half an hour.

1.2.2 Model of electricity market

The model of electricity market is popular topic to research the electricity market. However, the deregulated electricity market has been separated as several different markets. In this part, the general market model and electricity market model is introduced here.

Markets

All markets consist of a supply and a demand side. On the supply side, economic actors (people or companies) who are willing to provide some amount of a product or a service in exchange for money. The demand side consists of consumers, who are willing to buy some amount of the product or service using money. Whenever the suppliers are ready to sell for a lower (or equal) price than what the consumers are prepared to pay, a mutually beneficial opportunity for trade presents itself, which both parties can take advantage of. In other words, how to find these mutually beneficial trading opportunities requires some exploration. The parties must learn about and find each other, then work out the details of the trade (a contract) so that no one gets cheated, and must also be able to enforce the agreement afterwards. Organized markets are created to make sure that transactions go as smoothly as possible and the costs associated with making trades are minimized.

Supply The supply side of a market consists of producers or services providers whose purpose is to fulfill the needs and wants of consumers, and to do so profitably. In this section, we will first look at different ways of describing the costs associated with the production process, and then analyze the optimal supply decision of a company that faces a fixed product price on its output market.

Cost categories The cost must be produced when supply provides a product or service. In large category, total production costs consist of a fixed cost and a variable cost. Fixed costs must be paid regardless of the level of production, whereas variable costs add up as producing more and more units which entails additional expenses (incremental or marginal costs).

The per unit cost (average cost) of production comes together from two elements: the marginal cost of producing each unit and the fixed cost distributed onto each unit of production. Average costs are usually decreasing with the level of production at first, as the fixed costs get distributed to more and more units. After a while, however, the decrease in average fixed costs becomes insignificant.

At the same time, the marginal costs will usually start to increase at some point. The overall effect is that average costs decrease for a while as the company expands its production, but will hit a minimum level and then start increasing afterwards. The production level at which the company can supply the market at the lowest per unit cost is called the efficient scale of production.

Optimal production level of a company With mainly the aforementioned inspirations, the optimal production decision of a competitive supplier can be discussed. Before the discussion, an assumption which called price taking assumption must be given. In a perfect competition (the company believes that it cannot act in a strategic way regarding the market price), in other words, the company takes the market price that it can get for the product as given, and does not believe that its own production decisions influence this price at all. This is the so-called price taking assumption, which will also be employed to describe the demand side of the market as well.

If a company sells one more unit of a product in the market, its total revenue increases by the market price, whereas its total cost increases by the marginal cost of producing the additional unit. Since the profit is the difference between total revenues and total costs, the company's profit will increase with the market price minus the marginal cost of production. If this amount is positive, the company gains profit by increasing its output. Conversely, if it is negative (marginal cost is larger than the market price), the company gains profit by reducing its output.

Demand Consumer demand is the relationship between the price of a given product and the quantity that consumers would like to purchase of it at that price. Graphically, it is usually represented by a downward-sloping curve, where price is on the vertical and quantity is on the horizontal axis. Each point on the curve shows the maximum quantity demanded for a given price and at the same time it also shows that the maximum unit price that consumers are willing to pay for the given quantity (reservation price). The negative slope indicates that consumers would like to buy less of the product when the price is higher, as it can be naturally expected.

Another useful concept is the (price) elasticity of demand. This measure shows the percentage change in the quantity demanded that occurs in response to one percent rise in the price. For example, if the elasticity of demand is -0.8, then when the price of a product increases by one percent, the demanded amount will decrease by 0.8 percent. The demand is called elastic when the elasticitys absolute value is more than 1, otherwise it is inelastic. It is also usual to make the distinction between more and less elastic demand, the former denoting a point on the demand function where the absolute value of the demand elasticity is larger.

Market equilibrium With mainly the aforementioned inspirations, the short-run determination of market prices can be discussed. Obviously, the price must be such that the quantity supplied by the companies and the quantity demanded by the consumers exactly equal each other (the market clears). This point is shown by the intersection of the market-level supply and demand curves in Fig. 1.3. The brown line

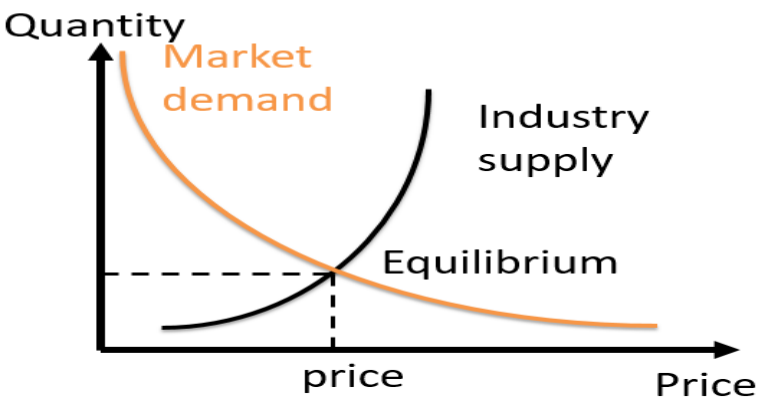

Figure 1.3: Short-run market equilibrium in competitive markets

is market demand, the price of which increases with the amount decreases; while the black line is the power supply, the price of which increases with the amount increases.

Competition and welfare Free market competition is generally a good way of organizing the provision of the most products and services that people in a society would like to consume.

In economic definition, social welfare encapsulates the benefits of a market to the society. It is reasonable to increase this benefit as much as possible. Social welfare describes the aggregate well-being of consumers and producers in a given market. It is consisted by two main ingredients which are consumer surplus and producer profits.

Company profits, which are expressed by deducting the total costs from total revenues, is the amount that shareholders of a company earn as capital income. The higher profit indicates the higher income of the shareholders.

Consumer surplus, which measures by the difference between the price consumers are willing to pay for a product and the price they actually have to pay for it. Willingness to pay is measured by the demand curve. The market price is what people have to pay is in the nature of things. Therefore, consumer surplus is nothing but the aggregated difference between the reservation prices of different people who buy the product or service and the market price they pay for it. The shadow areas shown in Fig. 1.4 described the consumer surplus.

$$Welfare = Consumersurplus + Producerprofits \tag{1.2.1}$$

$$Welfare = Reservation - totalcost \tag{1.2.2}$$

Efficiency and welfare maximization in a market require the lowest willingness to pay for a product (among those who purchase) to equal the (marginal) cost of producing another unit of the product in the industry.

Efficiency of the competitive market equilibrium In a competitive market, where both buyers and sellers are price takers, the market equilibrium implies the following

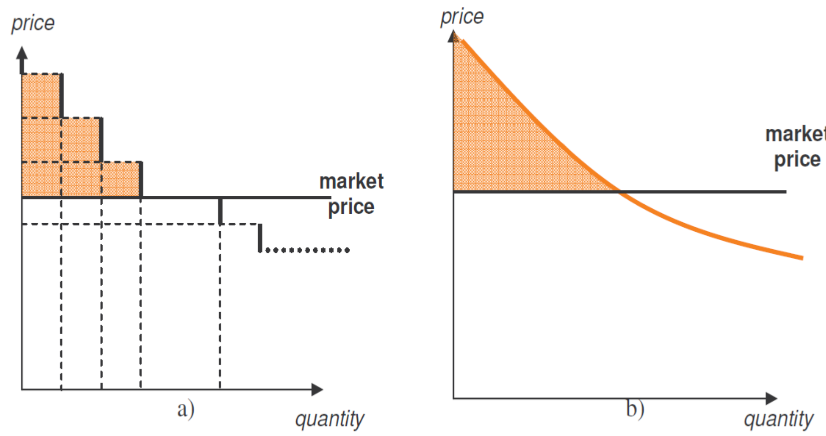

Figure 1.4: Consumer surplus with small (a) and large (b) quantities

two relationships.

- The market price equals to the marginal cost of producing the next unit of output (as a result of producer optimization).

- The market price equals to the lowest willingness to pay for the product among those people, who purchase (as a result of consumer sovereignty).

These two conditions taken together imply that the competitive market equilibrium situation is both efficient and welfare maximizing.

The electricity is a special market as its character. The model of electricity market can be described as follow. First, each generating company (GenCo) submits the bidding stacks of each of its units to the pool. Similarly, each consumer (ConCo) submits the bidding stacks of each of its demands to the pool. Then the ISO clears the market using an appropriate market-clearing procedure resulting in prices and production and consumption schedules. In what follows, each of the components (GenCo, ConCo, and ISO) is modeled together with their constraints and the optimization goal.

1.2.3 Dynamic characteristic of electricity markets

Experience in the restructured power market shows that power trading can cause large price fluctuations. Electric energy is a liquid commodity that cannot be stored.

In order to achieve real-time balance, the formation of day-ahead market price needs to be supplemented by continuous trading or settlement to meet the requirements of real-time operation. Since electricity cannot be stored, the electricity market is more complex than the traditional commodity market. Therefore, the existing commodity market price formation model does not apply to the electricity market. In addition, the high concentration of the industry may inevitably generate strategic behaviour. Given these characteristics, economic theoretical analysis is often based on highly stylized models that, although sometimes criticized by power engineers for not taking into account electrical properties, such as circulation and reactive power, can be useful for regulatory policy.

Up until now, in the most model of the electricity market, the supplier and the consumer interact continuously and instantaneously to set the price of power, the price evolves continuously, and price signals are transmitted instantaneously. However, in real markets, the participants receive discrete price signals at intervals equal to the market-clearing time, which ranges from a few minutes to as much as an hour. The price signals can be sent through e-mail, special-purpose wireless networks, Internet-based middle-ware, pagers, or any number of other commercial communication networks and devices, all of which act to delay the price signals arrival.

1.2.4 Electricity market with time delay

The communication infrastructure, a necessary intermediate for conveying price information to a real-time market and hence a stable operation of the grid, introduces certain challenges. The main challenge is the introduction of constant or time varying delays due to the presence of Smart Meters, Smart Devices, and related information processing and communication lags. The presence of smart devices are not real-time because of delays in data collection, computation, and communication. These delays can range in size from 10 seconds to 1 minute. However, the realization of real-time data processing needs a huge computing cost.This is an unacceptable fixed cost for the average retailer. Such problems also exist in the management of the electricity market. The only difference is that the cost is affordable. But it faces far more complex and massive data. There are more time delays from 1 minute to 5 minutes existing here. For generators, another problem arises. Large thermal power generation equipment is difficult to adjust power, while small new energy generation equipment is frequent insufficient to make up for real-time power changes caused by price. This causes the market to have to adjust over a constant time which now generally sets at five minutes. These delays in turn can endanger market operation and stability of the electricity grid.

The effect of delays in electricity markets has begun to be explored in [28–32]. Reference [28], is one of the earliest papers to discuss a dynamic market model. In [29], an upper bound on the market clearing time and price signal delay was computed beyond which a time-discretized version of the single supplier and single-consumer power market model used in [28] became unstable. The results in [29] indicated that the impact of the power market could be significant and should be anticipated by proper design of balancing mechanisms and market regulations. In [30], the authors continued the research in the direction of [29] by investigating the limitations imposed by delays on large-scale ancillary service market for real-time balancing originating from a hierarchical tree-based communication topology. A dynamic model was proposed for the wholesale market in [31], whose stability was evaluated in the presence of a delay due to the presence of smart meter and other communicating devices. In [31, 32], dynamic market models that included transparent connections to Local Marginal Prices were developed, outlining a clear relationship between stability and delays. In this article, a similar analysis is carried out to [31, 32], and adopt a discrete time framework to derive the underlying dynamics, thereby providing a direct connection to the actual market practices that exist today.

In the conventional study of time-varying delay system, the Lyapunov-Krasovskii functional and Jason-based inequality together methods has been used commonly. However, the Wirtinger-based inequality which leads much better results than Jason-based inequality are proposed recently.

1.3 Motivations and objectives

1.3.1 Motivations

The time delays appear in all real control system and it can not be ignored in many areas, such as LFC, GRNs and digital filters. However, most researchers find that these results are not optimal when using the existing delay upper bound were calculated by existing stability criteria. A great deal of conservatism exists in these results. Meanwhile, in the simulation of some simple models, it can be found that the existing computational stability delay upper bound is not the actual delay upper bound. There is even a big gap. It motivates the researchers to further investigate and reduce the conservatism.

In additions, the application of the proposed stability criteria in different real industry systems with time delays is hard work. It motivates the researchers to seek a suitable and necessary application for the proposed methods.

Firstly, LFC is designed for maintaining the frequency at its required value, should transmit the frequency derivation from the remote power plant to the control centre and send to the calculated power reference signal from the control centre to the power generation plant. The time delays arising in the feedback measurement channel and the forward control channel are combined by one time delay in most research. There are few types of research to consider the two additive time delays in the load frequency control system and it motivates the author to study.

Secondary, GRNs have been becoming a new research area of biological sciences. Mathematical modelling provides a useful tool for studying gene regulation processes in living organisms. Among them, the nonlinear differential equation model provides a more detailed understanding and insights into the nonlinear dynamical behaviour exhibited by GRNs. Since mRNAs and proteins in the GRNs may be synthesized at different locations, an important issue in modelling GRNs is that the slow processes of transcription, translation, and translocation results in sizeable delays. Time delays arising in the GRNs may lead to an incorrect prediction of dynamic behaviours which may result in very serious consequences. Stability is essential for designing or controlling GRNs. There is a great significance to study the influence of delays on the stability of the GRNs.

Thirdly, the digital filter is a necessary element of everyday electronics. It is an effective device that produces the desired discrete-time output signal from the original input signal, which will involve undesired information. The analysis of digital filters is helpful for their implementation. Because of the limitation of register length, the quantization and overflow correction mechanisms are commonly required to reduce the word length. Therefore, nonlinearities are unavoidable. Time delay is frequently encountered in many systems and exists in digital filters. However, there are few kinds of research focusing on a digital filter which considers both nonlinearities and time delays. Therefore, the analysis of digital filters is helpful for their implementation.

Finally, the electricity market has been deregulated in vast countries for two decades. Instead of government or oligopoly, price becomes the key decision-making variable in the electricity market. Under the high coupling of electricity markets and power systems, how to further reduce the extra loss in the market becomes a popular research topic. Therefore, the time delay which decreases the stability of the market and increases the cost of the market becomes e a very important direction. It motivates the researchers to further investigate and combine with proposed methods.

1.3.2 Objectives

The popular investigation framework is combining the LKF and LMI, and the previous results are all obtained in this framework. It is well known that the effort is to reduce the conservatism of the obtained criteria from the viewpoints of construction of the LKFs and estimation of their derivatives. Up until now, many LKFs with more general forms and integral inequalities with smaller estimation errors were proposed for this task. However, there still exists room for further investigation and reduction of the conservatism.

Firstly, The objective of this thesis is to present four novel stability criterion of the system to reduce the conservatism of the previous criteria. secondly, the practical application of related research has not kept up with the progress of the research on time delay system. Thus, the second objective is applying the proposed methods in a different system with time delay. Thirdly, stability analysis of the dynamic electricity market model with two communication delays is proposed to investigate.

1.4 Main contributions

This thesis focus on the stability analysis of time delay system and its application on the electricity market. The first part develops the new stability criterion for time delay system, as the results presented in Chapter 2 to Chapter 5. And the second part investigates the delay dependent stability of electricity market with time delays and load frequency control (LFC) with delays, respectively.

- the stability analysis of linear systems with two additive time-varying delays are studied. A novel stability criteria has been used for the linear system with additive delays. A new LKF with delay-product-type terms is constructed, and then the Wirtinger-based inequality, together with the reciprocally convex combination technique, is applied to estimate the derivative of the LKF. As a result, a less conservative stability criterion is established. A numerical example is used to demonstrate the advantage of the proposed criterion. An application of the proposed criterion to analyze the stability of the LFC scheme of power systems is also studied.
- The Wiringter-type double integral inequality method(WTDII) are investigated for establishing to estimate the double integral term. The theoretically advantage can be proved by comparing with the widely used JBDII and the recently developed WTDII, the presented WTDII. Two stability criteria of the GRNs are respectively established by applying the proposed WTDII to estimate the double integral terms appearing in the derivative of the LKFs.
- Stability analysis problem of digital filters with generalized overflow nonlinearity and a time-varying delay is further investigated. The main contribution is that a new delay and nonlinearity bounds dependent stability criterion with

less conservatism are developed, which can judge the stability of digital filters more accurately. Firstly, several augmented terms are introduced into the Lyapunov function and can provide extra freedom for the feasibility of the obtained criterion. Secondly, several new methods, including the Wirtinger-based summation inequality, extended reciprocally convex matrix inequality, and two zero-value equations, are applied to estimate the forward difference of the function as accurate as possible.

- Chapter 5 proposed a novel inequalities of control system with time-varying delay. The delay decomposition approach which combines with relaxed three integral inequalities has been studied. Those techniques have led to a stability criterion with less conservatism in comparison with the existing criteria. Then the effect of the delays on system stability can be assessed accurately by using the proposed stability criterion. A numerical example have been used to demonstrate the advantages of proposed method. Further more, the application of LFC and ED are researched to show the benefit of proposed method.

  The dynamic model of electricity market has been introduced. Then, the stability of this model with communication delay has been analyzed. A stability criterion, which using the Wirtinger-based inequality with less conservatism in comparing to the existing criteria is applied in the electricity market. Then,the effect of the delays on system stability can be assessed accurately by using the proposed stability criterion.

1.5 Publication list

1. H.T. Xu, C.K. Zhang, L. Jiang, and J. Smith. Stability analysis of linear systems with two additive time-varying delays via delay-product-type Lyapunov functional. *Applied Mathematical Modelling* vol.45, pp.955-964, 2017.
2. F.D. Li, Q. Zhu, H.T. Xu and L. Jiang. Stability analysis of delayed genetic regulatory networks via a relaxed double integral inequality. *Math-*

ematical Problems in Engineering vol. 2017, Article ID 4157256, 16 pages, 2017.

3. F. Li, H.T. Xu, J. Chen, L. Zhang, Q. Wang and L. Jiang. Improved Delay-Dependent Stability Analysis of Fixed-Point State-Space Digital Filters with Time-Varying Delay and Generalized Overflow Arithmetic. *IEEE Access* 2019.(2rd revision, Accept with minor revision)

1.6 Outline of thesis

The stability analysis of time delay system and it's application in electricity market are investigated in this thesis. Chapter 2, 3, 4 and 5 investigates delay-dependent stability analysis of time delay system from different aspects. Chapters 5 and 6 present electricity market with time delay by using criterion developed previous chapter.

Chapters 2 Linear systems with two additive time-varying delays via new Lyapunov functional

This chapter is concerned with the stability analysis of continuous linear systems with two additive time-varying delays in the LKF framework. Two novel delay-product-type terms are introduced into LKF candidate. The Wirtinger-based inequality, together with the reciprocally convex combination technique, is applied to estimate the integral terms arising in the derivative of the LKF. As a result, a new delay- and its-change-rate-dependent stability criterion is established. Its advantage of less conservatism than some existing criteria is demonstrated through a numerical example. Finally, the stability criterion is applied to analyze the stability of the LFC scheme of power systems.

Chapters 3 Stability analysis of delayed GRNs via a relaxed double integral inequality The delay-dependent stability of the GRNs by developing a more effective inequality to estimate the double integral term is investigated. By applying the WT-DII to the stability analysis of a delayed GRN, together with the usage of useful information of regulatory functions, several delay-range- and delay-rate-dependent (or delay-rate-independent) criteria are derived in terms of LMIs. An example is achieved to verify the effectiveness of the proposed method and also show the advantages of the established stability criteria through the comparison with some literature.

Chapters 4 Improved delay-dependent stability analysis of digital filters with time-varying delay and generalized overflow arithmetic

This chapter is concerned with the stability analysis of fixed-point state-space digital filters with generalized overflow arithmetic and a time-varying delay. In order to assess the influence of the time delay on the stability of digital filters more precisely, this chapter aims to derive a delay and nonlinear function bound dependent asymptotical stability criterion with less conservatism in comparison to the previous criteria. Firstly, a new Lyapunov functional with several augmented terms is constructed. Then, the Wirtinger-based summation inequality is introduced and several zero-value terms are applied to add more cross terms. As a result, a stability criterion in the form of LMIs is established and its conservatism is smaller than the previous ones due to the usage of more flexible Lyapunov functional and more accurate estimation techniques. Finally, several numerical examples are given to illustrate the advantages of the proposed method.

Chapters 5 Delay dependent stability analysis via relaxed three integral inequality method and its application to optimal economic dispatch In this chapter, the stability analysis via relaxed three integral inequality was investigated. A Wirtinger-based double integral inequality can reduce the conservatism than Wirtinger-based inequality. There is no research to investigate the effect on the stability analysis by the number of integral. The relaxed three integral inequality is established to estimate the three integral term appearing in the derivative of LKF with a triple integral term. The delay decomposition approach which compared the above inequalities has been used. A new stability criterion with less conservatism in comparison with the existing criteria are proposed. A numerical example have been used to demonstrate the advantages of proposed method. The application on the LFC and economic dispatch has been investigated to demonstrate the merit of the proposed method.

Chapters 6 Dynamic Stability of Electricity Market with Time Delay In a deregulated electricity market, the electricity pricing is the key factor. The interaction between an electricity market and prices makes communication delays more important. However, the studies in this area are limited. Therefore, Chapter 6 presents the stability analysis of an electricity market with a time-varying delay. The dynamic model of an electricity market is presented to minimize the operation cost. The impact of a time delay on this dynamic model is analysed by previous research in chapter 4. A relaxed three integral inequality is applied to estimate the integral terms arising in the derivative of the LKF. Finally, simulations are provided to demonstrate the benefit of propose method.

Chapters 7 Conclusions

The thesis has concluded with a summary of the results and several suggestions for future work. the suggestions for future work will highlight the unsolved problems that remained.

Chapter 2

Linear Systems with Two Additive Time-Varying Delays via New Lyapunov Functional

2.1 Introduction

As an increasing number of closed-loop control systems are being implemented using communication networks, time delays inevitably arise in these communication channels, which may degrade the system dynamic performance and even destroy the system stability. Therefore, the effect of those delays on the system stability has been becoming an important topic in the last few decades and significant research has been devoted to this topic [2, 3, 33]. In the literature, many researchers have studied the systems with one time-varying delay, where the time-varying delay is a combination of all the delays appearing in the total communication networks of the control system. For some systems, such as remote control systems and networked control systems, the measured signals transmitted from the sensors to the control center and the control signals sent from the control center may experience different segments of networks and the time delays arising may have different properties due to variable network transmission conditions [34]. Therefore, it is also an important issue to assess the effects on system stability from different parts of delays.

In [34], the system with two additive time-varying delay components has been firstly proposed to model different properties of delays for different channels and the free-weighting-matrix (FWM) approach [15] was used to develop a stability criterion. After that, many results for the analysis and design of this model were reported. Robust stability analysis was discussed via taking into account the system uncertainties in [35]. Stability criteria with less conservatism were developed via the improved FWM approach [36], Jensen inequality [37–39] and other integral inequalities [40], respectively. The comparison of several stability criteria was investigated in [41]. In [42], the idea of additive delays modelling was extended to the singular system and several stability criteria were reported. The case that two additive delays of linear systems were constant and had overlapping ranges was studied in [43]. In recent years, the reciprocally convex combination lemma was widely used to develop new stability criteria for systems with additive delays [38, 39]. Stability analysis and stabilization design for the systems with additive delays were discussed via the delay-partitioning-based LKF [44], Jensen inequality and connected component labeling algorithm [45] and a relaxed LKF [46]. Robust control design of the systems with both additive delays and parameter uncertainties was studied in [47].

The time delays appearing in the control loops are usually time-varying. For the case of time-varying delays, the popular investigation framework is combining the LKF and LMI, and the aforementioned results are all obtained in this framework. It is well known that the objective is to reduce the conservatism of the obtained criteria from the viewpoints of construction of the LKFs and estimation of their derivatives. Up until now, many LKFs with more general forms and integral inequalities with smaller estimation errors were proposed for this objective, such as the augmented-based LKFs [38], the delay-partitioning-based LKFs [44] and simple Wirtinger-based inequality [47]. However, further investigation and reduction of the conservatism can still be further researched. On the one hand, The delay-product-type LKFs were developed and found to be helpful for improving the results. On the other hand, a tighter Wirtinger-based inequality was proposed in [48], while it has not been used for the systems with additive delays. Therefore, it is expected that the stability criterion of system with additive delays will be further improved by combining those two new techniques. This motivates the research introduced in this chapter.

This chapter provides further study on the stability analysis of linear systems with two additive time-varying delays. A new LKF with delay-product-type terms is constructed and then the Wirtinger-based inequality together with the reciprocally convex combination technique is applied to estimate the derivative of the new LKF. As a result, a less conservative stability criterion is established. A numerical example is used to demonstrate the advantage of the proposed criterion. An application of the proposed criterion to analyze the stability of the LFC scheme of a power system is also studied.

The reminder of this chapter is organized as follows. Section 2 explains the problem formulation. In Section 3, a new stability criterion is developed through a delay-product-type LKF and the Wirtinger-based inequality. Section 4 illustrates the advantages of the proposed method via a numerical example and an application example on the LFC of a power system is also studied in this section. Finally, conclusion is presented in Section 5.

Notations: Throughout this chapter, the superscripts T and -1 mean the transpose and the inverse of a matrix, respectively; $\mathcal{R}^n$ denotes the n-dimensional Euclidean space; $\|\cdot\|$ refers to the Euclidean vector norm; $P > 0$ (≥ 0) means P is a real symmetric and positive-definite (semi-positive-definite) matrix; I and 0 stand for the identity matrix and the zero-matrix, respectively; $diag\{\cdot\}$ denotes the block-diagonal matrix; the symmetric term in the symmetric matrix is denoted by $*$; and $He\{X\} = X + X^T$.

2.2 Modeling and Wirtinger-based Integral Inequality

Consider the following continuous linear closed-loop system with two additive time-varying delays

$$\dot{x}(t) = Ax(t) + BKx(t - d_1(t) - d_2(t)) \tag{2.2.1}$$

where $x(t) \in \mathcal{R}^n$ is the state, A and B are the known real constant matrices, K is the state feedback control gain, $d_1(t)$ and $d_2(t)$ are time delays arising during the measured signal transmitted from sensor to the controller and the control signal sent from the controller to the actuator, respectively and they satisfy the following conditions

$$0 \leq d_1(t) \leq h_1, \ \ 0 \leq d_2(t) \leq h_2 \tag{2.2.2}$$

$$|\dot{d}_1(t)| \leq \mu_1, \ \ |\dot{d}_2(t)| \leq \mu_2 \tag{2.2.3}$$

where h_i and μ_i, $i = 1, 2$ are constant. Let $d(t) = d_1(t) + d_2(t)$ and $h = h_1 + h_2$.

This chapter is concerned with the stability problem of system (2.2.1) and understanding the effect of time delays therein on the stability. In order to accurately assess the system stability, this chapter aims to develop a new stability criterion with as small conservatism as possible.

The following lemmas were used for developing the main results.

Lemma 1. *(Wirtinger-based integral inequality [48]) For a symmetric matrix $R > 0$, scalars a and b with $a < b$ and vector ω such that the integration concerned are well defined, the following inequality holds*

$$(b-a)\int_a^b \dot{\omega}^T(s)R\dot{\omega}(s)ds \geq \chi_1^T R\chi_1 + 3\chi_2^T R\chi_2 \tag{2.2.4}$$

where

$$\begin{aligned} \chi_1 &= \omega(b) - \omega(a) \\ \chi_2 &= \omega(b) + \omega(a) - \frac{2}{b-a}\int_a^b \omega(s)ds \end{aligned}$$

Lemma 2. *(Reciprocally convex combination lemma [49]) For given positive integers n and m, a given scalar α in the interval $(0,1)$, a given $n \times n$-matrix $R > 0$, two matrices $W_1, W_2 \in \mathcal{R}^{n\times m}$. For all vector $\xi \in \mathcal{R}^m$, define the function $\Theta(\alpha, R)$ with the following form*

$$\Theta(\alpha, R) \ = \ \frac{1}{\alpha}\xi^T W_1^T R W_1 \xi + \frac{1}{1-\alpha}\xi^T W_2^T R W_2 \xi$$

if there exists a matrix $X \in \mathcal{R}^{n\times n}$ satisfying $\begin{bmatrix} R & X \\ * & R \end{bmatrix} > 0$, then the following inequality holds

$$\min_{\alpha\in(0,1)} \Theta(\alpha, R) \ \geq \ \begin{bmatrix} W_1\xi \\ W_2\xi \end{bmatrix}^T \begin{bmatrix} R & X \\ * & R \end{bmatrix} \begin{bmatrix} W_1\xi \\ W_2\xi \end{bmatrix} \tag{2.2.5}$$

2.3 New criterion via Wirtinger-based Integral Inequality

In this section, a new LKF with delay-product-type terms, together with Lemmas 1 and 2, is applied to develop a novel stability criterion, which is shown as follows.

Theorem 1. *For given scalars K, h_1, h_2, μ_1, and μ_2, system (2.2.1) with the time-varying delay satisfying (2.2.2) and (2.2.3) is asymptotically stable if there exist a $5n \times 5n$-matrix $P = P^T \geqslant 0$, $n \times n$-matrices $Q_i = Q_i^T (i = 1, 2, \cdots, 5)$, $Z_1 = Z_1^T > 0$, and $Z_2 = Z_2^T > 0$, and $2n \times 2n$-matrices $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, $P_3 = P_3^T > 0$, and $P_4 = P_4^T > 0$, and any $2n \times 2n$-matrices X and Y such that the following conditions hold*

$$\Omega_0 \ = \ \Phi - \frac{1}{h_1} G_a^T \Omega_1 G_a - \frac{1}{h_1 + h_2} G_b^T \Omega_2 G_b < 0 \tag{2.3.1}$$

$$\Omega_1 \ = \ \begin{bmatrix} \tilde{Z}_1 & X \\ * & \tilde{Z}_1 \end{bmatrix} > 0 \tag{2.3.2}$$

$$\Omega_2 \ = \ \begin{bmatrix} \tilde{Z}_2 & Y \\ * & \tilde{Z}_2 \end{bmatrix} > 0 \tag{2.3.3}$$

where

$$
\begin{aligned}
\Phi \;=\;& He(F_1^T P F_a + F_2^T P_1 F_b + F_3^T P_2 F_c + F_4^T P_3 F_d + F_5^T P_4 F_e) \\
& - e_3^T(Q_2 - Q_5)e_3 - e_2^T(Q_1 - Q_2 - Q_3)e_2 + e_1^T Q_1 e_1 + e_4^T(Q_4 - Q_3)e_4 \\
& - e_5^T(Q_4 + Q5)e_5 + (Ae_1 + BKe_3)^T(h_1 Z_1 + hZ_2)(Ae_1 + BKe_3) \\
& + \dot{d}_1(t)(F_2^T P_1 F_2 - F_3^T P_2 F_3 + F_4^T P_3 F_4 - F_5^T P_4 F_5 + e_3^T(Q_2 - Q_5)e_3 \\
& + e_2^T(Q_1 - Q_2 - Q_3)e_2) + \dot{d}_2(t)(F_4^T P_3 F_4 - F_5^T P_4 F_5 \\
& + e_3^T(Q_2 - Q_5)e_3) \qquad (2.3.4)
\end{aligned}
$$

$$
F_1 = \begin{bmatrix} e_1 \\ d_1(t)e_6 \\ [d_1(t) + d_2(t)]e_8 \\ [h_1 - d_1(t)]e_7 + d_1(t)e_6 \\ d(t)e_8 + [h - d(t)]e_9 - d_1(t)e_6 - [h_1 - d_1(t)]e_7 \end{bmatrix} \qquad (2.3.5)
$$

$$
F_a = \begin{bmatrix} Ae_1 + BKe_3 \\ e_1 - (1 - \dot{d}_1(t))e_2 \\ e_1 - (1 - \dot{d}(t))e_3 \\ e_1 - e_4 \\ e_4 - e_5 \end{bmatrix} \qquad (2.3.6)
$$

$$
F_2 = \begin{bmatrix} e_1 \\ e_6 \end{bmatrix}, \quad F_b = \begin{bmatrix} d_1(t)(Ae_1 + BKe_3) \\ e_1 - (1 - \dot{d}_1(t))e_2 - \dot{d}_1(t)e_6 \end{bmatrix} \qquad (2.3.7)
$$

$$
F_3 = \begin{bmatrix} e_1 \\ e_7 \end{bmatrix}, \quad F_c = \begin{bmatrix} (h_1 - d_1(t))(Ae_1 + BKe_3) \\ (1 - \dot{d}_1(t))e_2 - e_4 + \dot{d}_1(t)e_7 \end{bmatrix} \qquad (2.3.8)
$$

$$
F_4 = \begin{bmatrix} e_1 \\ e_8 \end{bmatrix}, \quad F_d = \begin{bmatrix} d(t)(Ae_1 + BKe_3) \\ e_1 - (1 - \dot{d}(t))e_3 - \dot{d}(t)e_8 \end{bmatrix} \qquad (2.3.9)
$$

$$
F_5 = \begin{bmatrix} e_1 \\ e_9 \end{bmatrix}, \quad F_e = \begin{bmatrix} (h - d(t))(Ae_1 + BKe_3) \\ (1 - \dot{d}(t))e_3 - e_5 + \dot{d}(t)e_9 \end{bmatrix} \qquad (2.3.10)
$$

$$
G_a = \begin{bmatrix} G_3^T & G_4^T & G_1^T & G_2^T \end{bmatrix}^T, \quad G_b = \begin{bmatrix} G_7^T & G_8^T & G_5^T & G_6^T \end{bmatrix}^T, \qquad (2.3.11)
$$

$$
\tilde{Z}_1 = \begin{bmatrix} Z_1 & 0 \\ * & 3Z_1 \end{bmatrix}, \quad \tilde{Z}_2 = \begin{bmatrix} Z_2 & 0 \\ * & 3Z_2 \end{bmatrix} \qquad (2.3.12)
$$

$$
(2.3.13)
$$

$$
\begin{aligned}
G_1 &= e_2 - e_4,\ \ G_2 = e_2 + e_4 - 2e_7,\ \ G_3 = e_1 - e_2,\ \ G_4 = e_1 + e_2 - 2e_6,\\
G_5 &= e_3 - e_5,\ \ G_6 = e_3 + e_5 - 2e_9,\ \ G_7 = e_1 - e_3,\ \ G_8 = e_1 + e_3 - 2e_8,\\
h &= h_1 + h_2,\quad d(t) = d_1(t) + d_2(t)
\end{aligned}
$$

Proof: Inspired by previous research [50], the following LKF candidate with two delay-product-type terms is constructed

$$
V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t) + V_4(x_t) + V_5(x_t) + V_6(x_t) \tag{2.3.14}
$$

where

$$
\begin{aligned}
V_1(x_t) &= \xi_0^T(t)P\xi_0(t)\\
V_2(x_t) &= d_1(t)\xi_1^T(t)P_1\xi_1(t) + d(t)\xi_2^T(t)P_3\xi_2(t)\\
V_3(x_t) &= [h_1 - d_1(t)]\xi_3^T(t)P_2\xi_3(t) + [h - d(t)]\xi_4^T(t)P_4\xi_4(t)\\
V_4(x_t) &= \int_{t-d_1(t)}^{t} x^T(s)Q_1x(s)ds + \int_{t-d(t)}^{t-d_1(t)} x^T(s)Q_2x(s)ds\\
V_5(x_t) &= \int_{t-h_1}^{t-d_1(t)} x^T(s)Q_3x(s)ds + \int_{t-h}^{t-h_1} x^T(s)Q_4x(s)ds\\
&\quad + \int_{t-h}^{t-d(t)} x^T(s)Q_5x(s)ds\\
V_6(x_t) &= \int_{-h_1}^{0}\int_{t+s}^{t} \dot{x}^T(\alpha)Z_1\dot{x}(\alpha)d\alpha ds + \int_{-h}^{0}\int_{t+s}^{t} \dot{x}^T(\alpha)Z_2\dot{x}(\alpha)d\alpha ds
\end{aligned}
$$

with

$$
\begin{aligned}
\xi_0(t) &= \left[x^T(t),\ \ \int_{t-d_1(t)}^{t} x^T(s)ds,\ \ \int_{t-d(t)}^{t} x^T(s)ds,\ \ \int_{t-h_1}^{t} x^T(s)ds,\ \ \int_{t-h}^{t-h_1} x^T(s)ds\right]^T\\
\xi_1(t) &= \left[x^T(t),\ \ \int_{t-d_1(t)}^{t} \tfrac{x^T(s)}{d_1(t)}ds\right]^T,\ \ \xi_2(t) = \left[x^T(t),\ \ \int_{t-d(t)}^{t} \tfrac{x^T(s)}{d(t)}ds\right]^T\\
\xi_3(t) &= \left[x^T(t),\ \ \int_{t-h_1}^{t-d_1(t)} \tfrac{x^T(s)}{h_1-d_1(t)}ds\right]^T,\ \ \xi_4(t) = \left[x^T(t),\ \ \int_{t-h}^{t-d(t)} \tfrac{x^T(s)}{h-d(t)}ds\right]^T
\end{aligned}
$$

On the one hand, if the matrices in $V(x_t)$ satisfying $P > 0$, $P_1 > 0$, $P_2 > 0$, $P_3 > 0$, $P_4 > 0$, $Q_i = Q_i^T$ $(i = 1, 2, \cdots, 5)$, $Z_1 > 0$ and $Z_2 > 0$, then $V(x_t) \geq \varepsilon_1||x(t)||$ for a scalar $\varepsilon_1 > 0$.

On the other hand, the conditions guaranteeing the negative definite of the derivative of $V(x_t)$ are discussed. At first, for simplifying the representation of the subsequent part, the following notations are defined:

$$\begin{aligned}\zeta(t) &= \Big[x^T(t),\ x^T(t-d_1(t)),\ x^T(t-d(t)),\ x^T(t-h_1),\ x^T(t-h) \\ &\quad \frac{1}{d_1(t)}\int_{t-d_1(t)}^{t} x^T(s)ds,\ \frac{1}{h_1-d_1(t)}\int_{t-h_1}^{t-d_1(t)} x^T(s)ds, \\ &\quad \frac{1}{d(t)}\int_{t-d(t)}^{t} x^T(s)ds,\ \frac{1}{h-d(t)}\int_{t-h}^{t-d(t)} x^T(s)ds\Big] \\ e_j &= \Big[0_{n\times(i-1)n},\quad I_n,\quad 0_{n\times(9-i)n}\Big]\end{aligned}$$

Calculating the derivative of the $V_1(x_t)$ along the solutions of system (2.2.1) leads to

$$\begin{aligned}\dot{V}_1(x_t) &= 2\tilde{x}^T(t)P\dot{\tilde{x}}(t) \qquad (2.3.15)\\ &= 2\begin{bmatrix} x(t) \\ \int_{t-d_1(t)}^{t} x(s)ds \\ \int_{t-d(t)}^{t} x(s)ds \\ \int_{t-h_1}^{t-d_1(t)} x(s)ds + \int_{t-d_1(t)}^{t} x(s)ds \\ \int_{t-d(t)}^{t} x(s)ds + \int_{t-h}^{t-d(t)} x(s)ds - \int_{t-d_1(t)}^{t} x(s)ds - \int_{t-h_1}^{t-d_1(t)} x(s)ds \end{bmatrix} \\ &\quad P\begin{bmatrix} \dot{x}(t) \\ x(t)-[1-\dot{d}_1(t)]x(t-d_1(t)) \\ x(t)-[1-\dot{d}(t)]x(t-d(t)) \\ x(t)-x(t-h_1) \\ x(t-h_1)-x(t-h) \end{bmatrix} \\ &= \zeta^T(t)He\{F_1^TPF_a\}\zeta(t) \qquad (2.3.16)\end{aligned}$$

where F_1 and F_a are defined in (2.3.5) and (2.3.6) separately.

Calculating the derivative of the $V_2(x_t)$ along the solutions of system (2.2.1) leads to

$$\begin{aligned}\dot{V}_2(x_t) &= \dot{d}_1(t)\xi_1^T(t)P_1\xi_1(t) + 2d_1(t)\xi_1^T(t)P_1\dot{\xi}_1(t) \qquad (2.3.17)\\ &\quad +\dot{d}(t)\xi_2^T(t)P_3\xi_2(t) + 2d(t)\xi_2^T(t)P_3\dot{\xi}_2(t)\end{aligned}$$

where

$$\begin{aligned}
\xi_1(t) &= \begin{bmatrix} e_1 \\ e_6 \end{bmatrix} \zeta(t) = F_2\zeta(t), \\
d_1(t)\dot{\xi}_1(t) &= \begin{bmatrix} d_1(t)(Ae_1 + BKe_3) \\ e_1 - (1-\dot{d}_1(t))e_2 - \dot{d}_1(t)e_6 \end{bmatrix} \zeta(t) = F_b\zeta(t) \\
\xi_2(t) &= \begin{bmatrix} e_1 \\ e_8 \end{bmatrix} \zeta(t) = F_4\zeta(t), \\
d(t)\dot{\xi}_2(t) &= \begin{bmatrix} d(t)(Ae_1 + BKe_3) \\ e_1 - (1-\dot{d}(t))e_3 - \dot{d}(t)e_8 \end{bmatrix} \zeta(t) = F_d\zeta(t)
\end{aligned}$$

Thus, $\dot{V}_2(x_t)$ is rewritten as

$$\begin{aligned}
\dot{V}_2(x_t) &= \dot{d}_1(t)\zeta^T(t)F_2^TP_1F_2\zeta(t) + \zeta^T(t)He(F_2^TP_1F_b)\zeta(t) && (2.3.18) \\
&\quad + \dot{d}(t)\zeta^T(t)F_4^TP_3F_4\zeta(t) + \zeta^T(t)He(F_4^TP_3F_d)\zeta(t) \\
&= \zeta^T(t)\Big[\dot{d}_1(t)\big(F_2^TP_1F_2 + F_4^TP_3F_4\big) + \dot{d}_2(t)F_4^TP_3F_4 && (2.3.19) \\
&\quad + He\{F_2^TP_1F_b + F_4^TP_3F_d\}\Big]\zeta(t)
\end{aligned}$$

where F_2, F_b, F_4, and F_d are defined in (2.3.7) and (2.3.9).

Calculating the derivative of the $V_3(x_t)$ along the solutions of system (2.2.1) leads to

$$\begin{aligned}
\dot{V}_3(x_t) &= -\dot{d}_1(t)\xi_3^T(t)P_2\xi_3(t) + 2(h_1 - d_1(t))\xi_3^T(t)P_2\dot{\xi}_3(t) && (2.3.20) \\
&\quad -\dot{d}(t)\xi_4^T(t)P_4\xi_4(t) + 2(h - d(t))\xi_4^T(t)P_4\dot{\xi}_4(t)
\end{aligned}$$

where

$$\begin{aligned}
\xi_3(t) &= \begin{bmatrix} e_1 \\ e_7 \end{bmatrix} \zeta(t) = F_3\zeta(t), \\
&(h_1 - d_1(t))\dot{\xi}_3(t) = \begin{bmatrix} (h_1 - d_1(t))(Ae_1 + BKe_3) \\ (1-\dot{d}_1(t))e_2 - e_4 + \dot{d}_1(t)e_7 \end{bmatrix} \zeta(t) = F_c\zeta(t) \\
\xi_4(t) &= \begin{bmatrix} e_1 \\ e_9 \end{bmatrix} \zeta(t) = F_5\zeta(t), \\
&(h - d(t))\dot{\xi}_4(t) = \begin{bmatrix} (h - d(t))(Ae_1 + BKe_3) \\ (1-\dot{d}(t))e_3 - e_5 + \dot{d}(t)e_9 \end{bmatrix} \zeta(t) = F_e\zeta(t)
\end{aligned}$$

Thus, $\dot{V}_3(x_t)$ is rewritten as

$$
\begin{aligned}
\dot{V}_3(x_t) &= -\dot{d}_1(t)\zeta^T(t)F_3^T P_2 F_3\zeta(t) + \zeta^T(t)He(F_3^T P_2 F_c)\zeta(t) \qquad (2.3.21)\\
&\quad -\dot{d}(t)\zeta^T(t)F_5^T P_4 F_5\zeta(t) + \zeta^T(t)He(F_5^T P_4 F_e)\zeta(t)\\
&= \zeta^T(t)\Big[-\dot{d}_1(t)\big(F_3^T P_2 F_3 + F_5^T P_4 F_5\big) \qquad (2.3.22)\\
&\quad - \dot{d}_2(t)F_5^T P_4 F_5 + He\{F_3^T P_2 F_c + F_5^T P_4 F_e\}\Big]\zeta(t)
\end{aligned}
$$

where F_3, F_c, F_5, and F_e are defined in (2.3.8) and (2.3.10).

Taking the derivative of $V_4(x_t)$ along the solutions of system (2.2.1) yields

$$
\begin{aligned}
\dot{V}_4(x_t) &= x^T(t)Q_1x(t) - (1-\dot{d}_1(t))x^T(t-d_1(t))Q_1x(t-d_1(t))\\
&\quad +(1-\dot{d}_1(t))x^T(t-d_1(t))Q_2x(t-d_1(t))\\
&\quad -(1-\dot{d}(t))x^T(t-d(t))Q_2x(t-d(t)) \qquad (2.3.23)
\end{aligned}
$$

The derivative of $V_5(x_t)$ along the solutions of system (2.2.1) can be obtained as

$$
\begin{aligned}
\dot{V}_5(x_t) &= (1-\dot{d}_1(t))x^T(t-d_1(t))Q_3x(t-d_1(t)) - x^T(t-h_1)Q_3x(t-h_1)\\
&\quad +x^T(t-h_1)Q_4x(t-h_1) - x^T(t-h)Q_4x(t-h) \qquad (2.3.24)\\
&\quad +(1-\dot{d}(t))x^T(t-d(t))Q_5x(t-d(t)) - x^T(t-h)Q_5x(t-h) \qquad (2.3.25)
\end{aligned}
$$

Taking the derivative of $V_6(x_t)$ yields

$$
\begin{aligned}
\dot{V}_6(x_t) &= h_1\dot{x}^T(t)Z_1\dot{x}(t) + h\dot{x}^T(t)Z_2\dot{x}(t) - \int_{t-h_1}^{t}\dot{x}^T(s)Z_1\dot{x}(s)ds\\
&\quad -\int_{t-h}^{t}\dot{x}^T(s)Z_2\dot{x}(s)ds \qquad (2.3.26)
\end{aligned}
$$

Combining (2.3.16), (2.3.20), (2.3.23)-(2.3.26) yields

$$
\begin{aligned}
\dot{V}(x_t) &= \zeta^T(t)\Phi\zeta(t) - \int_{t-h_1}^{t}\dot{x}^T(s)Z_1\dot{x}(s)ds\\
&\quad -\int_{t-h}^{t}\dot{x}^T(s)Z_2\dot{x}(s)ds \qquad (2.3.27)
\end{aligned}
$$

where Φ is defined in (2.3.4).

By applying Lemma 1 and 2 to estimate the Z_1-dependent integral term, the following is obtained

$$
\begin{aligned}
-\int_{t-h_1}^{t}\dot{x}^T(s)Z_1\dot{x}(s)ds &= -\int_{t-d_1(t)}^{t}\dot{x}^T(s)Z_1\dot{x}(s)ds-\int_{t-h_1}^{t-d_1(t)}\dot{x}^T(s)Z_1\dot{x}(s)ds\\
\leq\ & -\frac{1}{d_1(t)}\begin{bmatrix} x(t)-x(t-d_1(t)) \\ x(t)+x(t-d_1(t))-\frac{2}{d_1(t)}\int_{t-d_1(t)}^{t}x(t)ds\end{bmatrix}^T\begin{bmatrix} Z_1 & 0\\ 0 & 3Z_1\end{bmatrix}\\
&\begin{bmatrix} x(t)-x(t-d_1(t)) \\ x(t)+x(t-d_1(t))-\frac{2}{d_1(t)}\int_{t-d_1(t)}^{t}x(t)ds\end{bmatrix}\\
&-\frac{1}{h_1-d_1(t)}\begin{bmatrix} x(t-d_1(t))-x(t-h_1) \\ x(t-d_1(t))+x(t-h_1)-\int_{t-h_1}^{t-d_1(t)}\frac{2x(t)}{h_1-d_1(t)}ds\end{bmatrix}^T\begin{bmatrix} Z_1 & 0\\ 0 & 3Z_1\end{bmatrix}\\
&\begin{bmatrix} x(t-d_1(t))-x(t-h_1) \\ x(t-d_1(t))+x(t-h_1)-\int_{t-h_1}^{t-d_1(t)}\frac{2x(t)}{h_1-d_1(t)}ds\end{bmatrix}\\
=\ & -\zeta^T(t)\left\{\frac{1}{d_1(t)}\begin{bmatrix} e_1-e_2\\ e_1+e_2-2e_6\end{bmatrix}^T\begin{bmatrix} Z_1 & 0\\ * & 3Z_1\end{bmatrix}\begin{bmatrix} e_1-e_2\\ e_1+e_2-2e_6\end{bmatrix}\right.\\
&\left.+\frac{1}{h_1-d_1(t)}\begin{bmatrix} e_2-e_4\\ e_2+e_4-2e_7\end{bmatrix}^T\begin{bmatrix} Z_1 & 0\\ * & 3Z_1\end{bmatrix}\begin{bmatrix} e_2-e_4\\ e_2+e_4-2e_7\end{bmatrix}\right\}\zeta(t)\\
\leq\ & -\frac{1}{h_1}\zeta^T(t)G_a^T\begin{bmatrix}\tilde{Z}_1 & X\\ * & \tilde{Z}_1\end{bmatrix}G_a\zeta(t) \qquad (2.3.28)
\end{aligned}
$$

where G_a and $\tilde{Z}_1$ are defined in (2.3.11) and $\begin{bmatrix}\tilde{Z}_1 & X\\ * & \tilde{Z}_1\end{bmatrix} > 0$.

Similarly, the Z_2-dependent integral term in (2.3.27) estimated through Lemma 1 and 2 leads to

$$
-\int_{t-h}^{t}\dot{x}^T(s)Z_2\dot{x}(s)ds \ \leq\ -\frac{1}{h_1+h_2}\zeta^T(t)G_b^T\begin{bmatrix}\tilde{Z}_2 & Y\\ * & \tilde{Z}_2\end{bmatrix}G_b\zeta(t) \qquad (2.3.29)
$$

where G_b and $\tilde{Z}_2$ are defined in (2.3.11) and $\begin{bmatrix}\tilde{Z}_2 & Y\\ * & \tilde{Z}_2\end{bmatrix} > 0$.

Thus, based on (2.3.27)-(2.3.29), the following is true

$$
\begin{aligned}
\dot{V}(x_t) &< \zeta^T(t)\left(\Phi-\frac{1}{h_1}G_a^T\Omega_1G_a-\frac{1}{h_1+h_2}G_b^T\Omega_2G_b\right)\zeta(t)\\
&= \zeta^T(t)\Omega_0\zeta(t) \qquad (2.3.30)
\end{aligned}
$$

Therefore, $\Omega_0 < 0$ leads to $\dot{V}(x_t) \leq -\epsilon_2||x(t)||^2$ for a sufficient small scalar $\epsilon_2 > 0$. Hence, when (2.3.1)-(2.3.3) hold, system (2.2.1) with the time-varying delay satisfying (2.2.2) and (2.2.3) is asymptotically stable. This completes the proof. ■

Remark 1. *The condition, $\Omega_0 < 0$, of Theorem 1 is dependent on the time-varying delays, $d_1(t)$ and $d_2(t)$, and their change rate, $\dot{d}_1(t)$ and $\dot{d}_2(t)$, and it cannot be directly checked. In fact, this condition can be rewritten as the following form:*

$$\begin{aligned}\Omega_0(d_1(t), d_2(t), \dot{d}_1(t), \dot{d}_2(t)) &= d_1(t)[\Upsilon_1 + \dot{d}_1(t)\Upsilon_2 + \dot{d}_2(t)\Upsilon_3] \\ &\quad + d_2(t)[\Upsilon_4 + \dot{d}_1(t)\Upsilon_5 + \dot{d}_2(t)\Upsilon_6] < 0\end{aligned} \quad (2.3.31)$$

where $\Upsilon_i, i = 1, 2, \cdots, 6$ are time-independent matrix-combinations. By using the convex combination technique [51] and following the same proof procedure in [52] [proof of Theorem 1], the condition $\Omega_0(d_1(t), d_2(t), \dot{d}_1(t), \dot{d}_2(t)) < 0$ holds if the following conditions hold

$$\Omega_0|_{(d_1(t),d_2(t),\dot{d}_1(t),\dot{d}_2(t))\in([0,h_1]\times[0,h_2]\times[-\mu_1,\mu_1]\times[-\mu_2,\mu_2])} < 0 \quad (2.3.32)$$

Remark 2. *The LKF used in this chapter is different from the ones reported in the literature. It not only contains some augmented terms similar to the one used in [38] but also introduces four delay-product-type terms, which are inspired by previous work for discrete-time time delay system [50] and whose simple form has been proved to be helpful to reduce the conservatism of the criteria in [53].*

Remark 3. *During the proof of Theorem 1, the Wirtinger-based integral inequality and the reciprocally convex combination technique are applied to estimate the derivative of the LKF. It is well known that the Wirtinger-based integral inequality is tighter than the Jensen inequality used in [37–39], which means that the conservatism of the proposed criterion can be reduced. Moreover, the usage of the reciprocally convex combination avoids some enlargement treatments, such as $d(t)$ is directly enlarged to its upper bound h [34]. As a result, the conservatism of the proposed criterion is further reduced.*

2.4 Numerical test

This section gives a typical numerical example to show the advantages of the proposed criterion. Moreover, the application to the LFC of a single area power system is also studied.

2.4.1 A numerical example

Consider system (2.2.1) with the following parameters

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad BK = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}.$$

This example is widely used for checking the conservatism of the stability criteria [34–36, 45, 47, 54]. It is assumed that the bounds of the delay change rates are respective 0.1 and 0.8, i.e., $|\dot{d}_1(t)| \leq 0.1$ and $|\dot{d}_2(t)| \leq 0.8$. For given different upper bounds of $d_1(t)$, i.e., $h_1 \in \{1.0, 1.2, 1.5\}$, the upper bounds of $d_2(t)$ guaranteeing the stability of system calculated by Theorem 1 are listed in Table 2.1, where the results reported in other literature are also given for comparison. Note that '—-' indicates that the results for corresponding cases are not reported in other literatures. It can be found that the proposed Theorem 1 can provide less conservative results.

Table 2.1: Upper bounds of $d_2(t)$ for given h_1 and upper bounds of $d_1(t)$ for given h_2

Method	h_2 for given h_1			h_1 for given h_2		
	h_1=1.0	h_1=1.2	h_1=1.5	h_2=0.3	h_2=0.4	h_2=0.5
[34]	0.415	0.376	0.248	1.324	1.039	0.806
[54]	0.512	0.406	0.283	1.453	1.214	1.021
[36]	0.519	0.453	0.378	—-	—-	—-
[45]	0.596	0.463	0.313	1.532	1.313	1.140
[35]	0.872	0.672	0.371	1.572	1.472	1.372
Theorem 1	1.163	0.965	0.669	1.875	1.773	1.671

The results which calculated by Theorem 1 can be found that the sum of h_1 and

h_2 are similar. It leads a question that the limit would be only on the total delay and considering two separate delay terms would become meaningless. It influenced by the two delay change rates are not significant difference. For demonstrated purpose, The delay change rate of second time delay $\dot{d}_2(t)$ are changed to 2. The table 2.2 shows the sum of h_1 and h_2 are changed followed the enlarge between two delay change rates.

Table 2.2: Upper bounds of $d_2(t)$ and $d_1(t)$ for $|\dot{d}_2(t)| \leq 2$

Method	h_2 for given h_1			h_1 for given h_2		
	h_1=1.0	h_1=1.2	h_1=1.5	h_2=0.3	h_2=0.4	h_2=0.5
Theorem 1	1.079	1.039	0.897	2.043	1.758	1.522

2.4.2 Application to the stability analysis of load frequency control

As mentioned in [55–57], LFC is designed for maintaining the frequency at its required value, should transmit the frequency derivation from the remote power plant to the control center and send the calculated power reference signal from the control center to the power generation plant. The time delays arising in the feedback measurement channel and the forward control channel are combined by one time delay in [55–57], while those delays in different channels may be different.

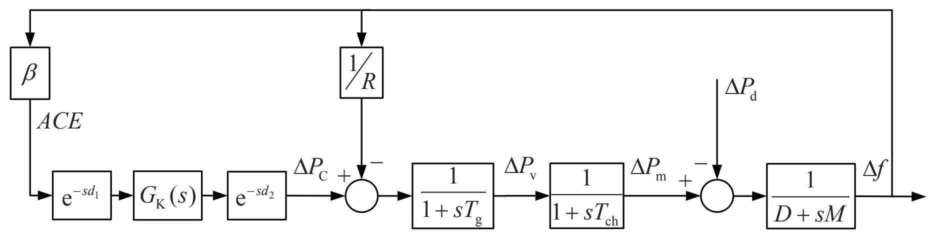

Figure 2.1: Diagram of the LFC for a single area power system

The basic diagram of the simplified LFC of a single area power system is shown in Fig. 2.1, where Δf, ΔP_m, ΔP_v, and ΔP_d are the frequency deviation, the mechanical output change, the valve position change and the load disturbance, respectively; M and D are the moment of inertia of the generator and generator damping coefficient, respectively; T_g and T_{ch} are the time constant of the governor and the turbine, respectively; R is the speed drop; β is the frequency bias factor; K_P and K_I are PI gains of the LFC; $d_1(t)$ and $d_2(t)$ are the time delays in feedback and forward channels, respectively. By modifying the dynamic model in [55], the new model with two additive delays can be obtained as follows

$$\dot{x}(t) = Ax(t) + A_d x(t - d_1(t) - d_2(t)) \tag{2.4.1}$$

where

$$x(t) = \begin{bmatrix} \Delta f \\ \Delta P_m \\ \Delta P_v \end{bmatrix}, \quad A = \begin{bmatrix} -\frac{D}{M} & \frac{1}{M} & 0 & 0 \\ 0 & -\frac{1}{T_{ch}} & \frac{1}{T_{ch}} & 0 \\ -\frac{1}{RT_g} & 0 & -\frac{1}{T_g} & 0 \\ \beta & 0 & 0 & 0 \end{bmatrix},$$

$$A_d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{K_p\beta}{T_g} & 0 & 0 & -\frac{K_I}{T_g} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with the parameters given in [55], $M = 10$, $D = 1$, $T_{ch} = 0.3$, $T_g = 0.1$, $R = 0.05$, $\beta = 21$, $K_I = 0.2$, and $K_p = 0.1$. The results for the case of $|\dot{d}_1(t)| \leq 0.1$ and $|\dot{d}_2(t)| \leq 0.8$ are given in Table 2.3. h_R shows the real upper bound by texted in simulation. The results show the proposed criterion has less conservatism. However, It can be found there is still room to improve and reduce the conservatism for future work.

For the case: an increase step load of 0.1 pu; $d_1(t) = \frac{1.5}{2}\sin\left(\frac{20}{1.5}x(t)\right) + \frac{1.5}{2}$ and $d_2(t) = \frac{5.383}{2}\sin\left(\frac{2.5}{5.383}x(t)\right) + \frac{5.383}{2}$ (satisfying $d_1(t) \leq 1.5, d_2(t) \leq 5.383, |\dot{d}_1(t)| \leq 0.1, |\dot{d}_2(t)| \leq 0.8$), the simulation results are provided in Fig. 2.2. The obtained simulation results show that the LFC is stable, which verifies the effectiveness of the proposed method.

Table 2.3: Upper bounds of $d_2(t)$ for given h_1 and upper bounds of $d_1(t)$ for given h_2

Method	h_2 for given h_1			h_1 for given h_2		
	h_1=1.0	h_1=1.2	h_1=1.5	h_2=2	h_2=3	h_2=4
[55]	4.803	4.603	4.303	3.803	2.803	1.803
Theorem 1	5.882	5.682	5.383	4.892	3.886	2.885
h_R	8.496	8.308	8.035	7.503	6.520	5.514

2.5 Conclusion

This chapter has investigated the stability of linear systems with two additive time-varying delays. A delay-product-type LKF has been developed and its derivative has been estimated through Wirtinger-based inequality. Those techniques have led to a stability criterion with less conservatism in comparison with the existing criteria. Then, the effect of the delays on system stability can be assessed accurately by using the proposed stability criterion. A numerical example and an application of the LFC have been used to demonstrate the effectiveness of the proposed method.

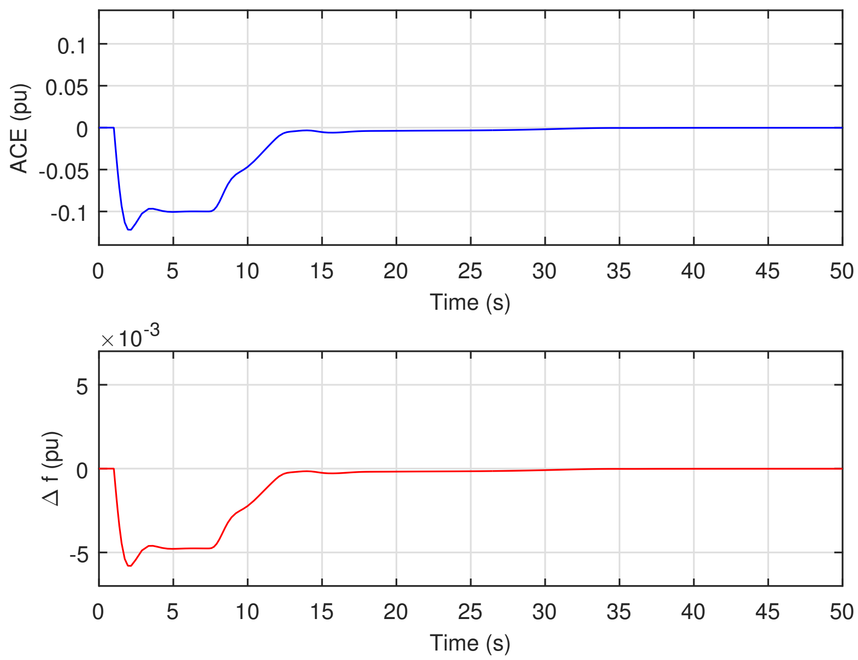

Figure 2.2: Frequency deviation and ACE of the LFC under a step load change (0.1 pu)

Chapter 3

Stability analysis of delayed genetic regulatory networks via a relaxed double integral inequality

3.1 Introduction

In the past few years, genetic regulatory networks, which describe the interactions of many molecules (DNA, RNA, proteins, etc.), have been becoming a new research area of biological and biomedical sciences [58–61]. A genetic regulatory network (GRN) is a collection of molecular regulators that interact with each other and with other substances in the cell to govern the gene expression levels of mRNA and proteins. These play a central role in morphogenesis, the creation of body structures, which in turn is central to evolutionary developmental biology.

Mathematical modeling based on the extracted functional information from the time-series data provides a useful tool for studying gene regulation processes in living organisms [62], and a large variety of formalisms have been proposed to model and simulate GRNs, such as directed graphs, Boolean networks and nonlinear differential equations [63]. Among them, the nonlinear differential equation model provides more detailed understanding and insights into the nonlinear dynamical behavior exhibited by GRNs [64].

Since mRNAs and proteins in the GRNs may be synthesized at different locations, an important issue in modeling GRNs is that the slow processes of transcription, translation, and translocation results in sizable delays [65]. Time delays arising in the GRNs may lead to incorrect prediction of dynamic behaviors [66], which may result in very serious consequences. Stability is essential for designing or controlling GRNs [67], it is of a great significance to study the influence of delays on the stability of the GRNs.

Up to now, a huge number of results on the stability of the delayed GRNs have been reported in the literature (see e.g. [68]). The sufficient and necessary local stability criteria were firstly given for the GRNs with constant delay in [68] and [69]. However, local stability is not enough for understanding nonlinear GRNs, the globally asymptotical stability of GRNs with SUM regulatory functions has been widely investigated [70–72]. Meanwhile, by taking into account the unavoidable uncertainties caused by modelling errors and parameter fluctuations, many scholars paid attentions to the robust stability analysis of the delayed GRNs [73]. Moveover, both the intrinsic noise derived from the random births and deaths of individual molecules and the extrinsic noise due to environment fluctuations make the gene regulation process be an intrinsically noisy process [74]. Thus, many researches aimed at the robust stability analysis of the GRNs in consideration of those noises [75]. Also, some results have considered both the uncertainties and the noises [76]. In addition, based on the definition of convergence rate index, the exponential stability problem was also studied in [77].

On the other hand, no matter what type of stability problems concerned, the analysis methods for finding stability criteria have always been an important topic. To the best of the authors' knowledge, there are mainly two methods that have been used for the delayed GRNs. The first type of method is the M-matrix-based method. For example, the delay- and rate-independent stability criteria were proposed in [71], the delay-independent but rate-dependent criteria were established in [78] and [79], and the delay- and rate-dependent criteria were developed in [80] and [81]. The stability of the GRNs through those M-matrix-based criteria is judged by verifying whether or not a matrix is a nonsingular M-matrix. Although the computational complexity is low, those criteria are just available for slow-varying delay cases [71]. However, the time delays encountered in GRNs may be fast-varying or random changing. The M-matrix-based method is inapplicable for those cases. The second type of method is based on the framework of LKF and LMI. The LKF-based method can be used to handle all time delays aforementioned and it is available for not only stability analysis but also many other problems, such as controller synthesis, state estimation, filter design, passivity analysis, and so on [74]. Meanwhile, the LMI-based criteria can be easily checked through MATLAB/LMI toolbox for determining the system stability. Therefore, the most existing researches for the GRNs are based on this type of method [75].

The stability analysis creteria obtained by using the LKF and the LMI is the criteria obtained have more or less conservatism. Thus, one important issue is to develop new criteria with as small conservatism as possible. The key point of the stability analysis based on such framework is to find an LKF satisfying some requirements for ensuring the globally asymptotical stability of the GRNs. It is predictable that the form of the LKF candidate is tightly related to the conservatism of the obtained criteria.

In most researches, the used LKFs were constructed by introducing delay-based single and/or double integral terms into the typical non-integral quadratic form of Lyapunov function for delay-free systems [75]. Based on a predictable fact that the conservatism-reducing of criteria can be achieved by constructing more general LKF, two types of more general LKFs have been developed to reduce the conservatism. The first one is the delay-partition-based LKFs, which is constructed by dividing the delay interval into several small subintervals and then replacing the original integral terms with multiple new integral terms based on delay subintervals. This type of LKF has been used to investigate the robust stability of various GRNs [82], the exponential stability of switch GRNs [83], and the stochastic stability of jumping GRNs [84]. The other is the augmented LKF constructed by using various state vectors (current, delayed, and/or integrated state vectors etc.) to augment the quadratic terms of original LKFs, it has been used to derive the improved stability criteria of the GRNs [85].

Beside above mentioned two types of improved LKFs, a new LKF including triple integral terms firstly developed in [86] is proved to be very useful to reduce the conservatism. However, only a few researches of the GRNs have applied such type of LKF. The LKF with triple integral terms was used to discuss the asymptotical stability of the GRNs [87]. The following form of double integral term will be introduced into the derivative of the LKF with a triple integral term:

$$-\int_{a}^{b}\int_{s}^{b} y^{T}(u)Zy(u)duds, \quad Z > 0 \tag{3.1.1}$$

As mentioned in [88], the effective estimation of the above term is strongly linked to the conservatism of the criteria. To the best of the authors' knowledge, for the researches referring to the triple integral term in the LKFs, most literature directly applied the Jensen-based double integral inequality (JBDII) (see (3.1.15) for details) to achieve the estimation task [85]. Although an improved integral inequality was developed in [87], it is also derived based Jensen inequality. Very recently, a Wirtinger-based double integral inequality (WBDII) was developed to general linear time-delay system and it was proved to be less conservative than the JBDII [88]. However, such inequality has not been used to discuss the GRNs. Furthermore, the gap between the term (3.1.1) and its estimated value obtained by the WBDII still leads to conservatism. Therefore, it can be expected that the results may be further improved if a new estimation method that brings tighter gap is applied for the term (3.1.1). This is the motivation of the chapter.

This chapter further investigates the delay-dependent stability of the GRNs by developing a more effective inequality to estimate the double integral term (3.1.1). The contributions of the chapter are summarized as follows:

1) A relaxed double integral inequality, i.e. WTDII, is established to estimate the double integral term. Compared with the widely used JBDII and the recently developed WTDII, the presented WTDII is theoretically proved to be the tightest.

2) Two less conservative stability criteria of the GRNs are derived. For the GRNs with time-varying delays satisfying different conditions, two stability criteria

are respectively established by applying the proposed WTDII to estimate the double integral terms appearing in the derivative of the LKFs.

The rest of the chapter is organized as follows. Problem statements and preliminaries are presented in Section II. In Section III, the development and the comparison of the WTDII approach are discussed in detail. Two stability criteria of the GRN with time-varying delay are derived through the WTDII in Section IV. An example is given to show the validity of the obtained results in Section V. Finally, in Section IV, the conclusions are drawn.

Notations: Throughout this chapter, the superscripts T and -1 mean the transpose and the inverse of a matrix, respectively; $\mathcal{R}^n$ denotes the n-dimensional Euclidean space; $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices; $\|\cdot\|$ refers to the Euclidean vector norm; $P > 0$ (≥ 0) means that P is a real symmetric and positive-definite (semi-positive-definite) matrix; I stands for an appropriately dimensioned identity matrix; diag$\{\cdots\}$ denotes a block-diagonal matrix; symmetric term in a symmetric matrix is denoted by $*$; and $\mathrm{Sym}\{X\} = X + X^T$.

This section describes the problem to be investigated and gives some necessary preliminaries.

The following nonlinear differential equations have been used recently to describe the GRNs with time-varying feedback regulation delays and translational delays [73]:

$$\begin{cases} \dot{m}_i(t) &= -a_i m_i(t) + b_i(p_1(t-\sigma(t)), p_2(t-\sigma(t)), \\ & \qquad \cdots, p_n(t-\sigma(t))) \\ \dot{p}_i(t) &= -c_i p_i(t) + d_i m_i(t-\tau(t)) \end{cases} \tag{3.1.2}$$

as shown in Fig. 3.1, where $m_i(t)$ and $p_i(t)$ are the concentrations of the ith mRNA and protein, respectively. $a_i > 0$ and $c_i > 0$ are the positive real numbers that represent the degradation rate of the ith mRNA and protein, respectively. $d_i > 0$ is the positive real number that represents the translating rate from mRNA i to protein i. b_i is the regulatory function of the ith gene. $\sigma(t)$ and $\tau(t)$ are the transcriptional and translational delays, respectively.

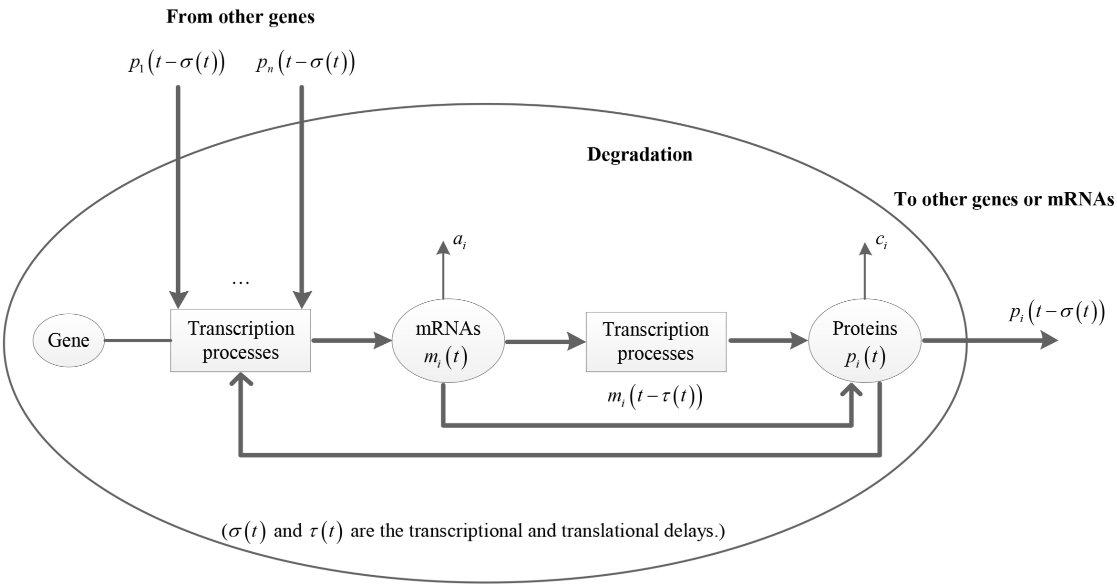

Figure 3.1: GRNs with time-varying feed-back regulation delays and translational delays.

Since each transcription factor acts additively to regulate the gene, it is usually to assume that the regulatory function b_i satisfies the following SUM logic [75]:

$$b_i(p_1(t), p_2(t), \cdots, p_n(t)) = \sum_{j=1}^{n} b_{ij} p_j(t) \tag{3.1.3}$$

and b_{ij} is a monotonic function of the Hill form, that is

$$b_{ij} = \begin{cases} \frac{\alpha_{ij}}{1+(x/\beta_j)^{H_j}}, & \text{if transcription factor } j \text{ represses gene } i \\ \frac{\alpha_{ij}(x/\beta_j)^{H_j}}{1+(x/\beta_j)^{H_j}}, & \text{if transcription factor } j \text{ activates gene } i \end{cases}$$

where α_{ij} is bounded constant that denotes the dimensionless transcriptional rate of transcription factor j to gene i, β_j is a positive scalar, and H_j is the Hill coefficient that represents the degree of cooperativity.

The transcriptional and translational delays, $\sigma(t)$ and $\tau(t)$, are assumed to satisfy the following two different conditions:

Case I: $\tau(t)$ and $\sigma(t)$ satisfy

$$\begin{cases} 0 \leq \tau_1 \leq \tau(t) \leq \tau_2, \ \ 0 \leq \sigma_1 \leq \sigma(t) \leq \sigma_2 \\ \dot{\tau}(t) \leq \tau_d, \ \ \dot{\sigma}(t) \leq \sigma_d \end{cases} \tag{3.1.4}$$

Case II: $\tau(t)$ and $\sigma(t)$ satisfy

$$0 \leq \tau_1 \leq \tau(t) \leq \tau_2, \;\; 0 \leq \sigma_1 \leq \sigma(t) \leq \sigma_2 \tag{3.1.5}$$

Clearly, based on (3.1.3), GRN (3.1.2) can be rewritten as [87]

$$\begin{cases} \dot{m}_i(t) &= -a_i m_i(t) + \sum_{j=1}^{n} w_{ij} g_j(p_j(t-\sigma(t))) + l_i \\ \dot{p}_i(t) &= -c_i p_i(t) + d_i m_i(t-\tau(t)) \end{cases} \tag{3.1.6}$$

where $l_i = \sum_{j\in\mathcal{V}_i} \alpha_{ij}$ with $\mathcal{V}_i$ being the set of all the transcription factor j which is a repressor of gene i; $w_{ij} = \alpha_{ij}$ if transcription factor j activates gene i, $w_{ij} = 0$ if there is no connection between j and i, and $w_{ij} = -\alpha_{ij}$ if transcription factor j represses gene i; and $g_j(x) = \frac{(x/\beta_j)^{H_j}}{1+(x/\beta_j)^{H_j}}$, $x \geq 0$ is a monotonically increasing function satisfying

$$\rho \leq \frac{g_j(s_1) - g_j(s_2)}{s_1 - s_2} \leq \rho_i \tag{3.1.7}$$

with $\rho = \min_{s\geq 0} \dot{g}_j(s) = 0$ and

$$\rho_i = \max_{s\geq 0} \dot{g}_j(s) = \frac{(H_j-1)^{(H_j-1)/H_j}(H_j+1)^{(H_j+1)/H_j}}{4\beta_j H_j} \tag{3.1.8}$$

GRN (3.1.6) can be expressed as the following vector-matrix form:

$$\begin{cases} \dot{m}(t) &= -Am(t) + Wg(p(t-\sigma(t))) + l \\ \dot{p}(t) &= -Cp(t) + Dm(t-\tau(t)) \end{cases} \tag{3.1.9}$$

where $m(t) = [m_1(t), m_2(t), \cdots, m_n(t)]^T$, $p(t) = [p_1(t), p_2(t), \cdots, p_n(t)]^T$, $A = diag\{a_1, a_2, \cdots, a_n\} > 0, C = diag\{c_1, c_2, \cdots, c_n\} > 0, D = diag\{d_1, d_2, \cdots, d_n\} > 0$, $g(p(t)) = [g_1(p_1(t)), g_2(p_2(t)), \cdots, g_n(p_n(t))]^T$, $W = [w_{ij}]_{n\times n}$, and $l = [l_1, l_2, \cdots, l_n]$.

Let (m^*, p^*) be the equilibrium point (steady state) of (3.1.9), that is, $-Am^* + Wg(p^*) + l = 0$ and $-Cp^* + Dm^* = 0$. Using the transformation $x(t) = m(t) - m^*$ and $y(t) = p(t) - p^*$, one can shift the equilibrium point (m^*, p^*) to the origin and rewrite (3.1.9) as the following GRN:

$$\begin{cases} \dot{x}(t) &= -Ax(t) + Wf(y(t-\sigma(t))) \\ \dot{y}(t) &= -Cy(t) + Dx(t-\tau(t)) \end{cases} \tag{3.1.10}$$

where $f(s) = [f_1(s), f_2(s), \cdots, f_n(s)]^T$ and $f_i(y(t)) = g_i(y(t) + p^*) - g_i(p^*)$ with $f_i(0) = 0$. Then,

$$\frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} = \frac{g_i(s_1 + p^*) - g_i(s_2 + p^*)}{s_1 + p^* - (s_2 + p^*)}$$

Thus, it follows from (3.1.7) and $f_i(0) = 0$ that

$$0 \le \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \le \rho_i, \quad s_1 \neq s_2 \tag{3.1.11}$$

$$0 \le \frac{f_i(s)}{s} \le \rho_i, \quad s \neq 0 \tag{3.1.12}$$

This chapter aims to analyze the asymptotical stability of GRN (3.1.2) and to determine the delay bounds, name as maximal admissible delay bounds (MADBs), under which the GRN is asymptotically stable. In order to achieve this aim, this chapter will develop a new double integral inequality (i.e., WTDII) for estimating the double integral term (3.1.1) so as to derive some less conservative stability criteria.

Several lemmas used to obtain main results are given as follows.

For the estimation of single integral term, the most popular technique is Wirtinger-based inequality, shown as lemma 3.

Lemma 3. *(Wirtinger-based inequality [89]) For symmetric positive definite matrix $R \in \mathcal{R}^{n\times n}$, scalars $a < b$, and vector $\omega : [a, b] \mapsto \mathcal{R}^n$ such that the integration concerned are well defined, the following inequality holds*

$$\int_a^b \omega^T(s) R\omega(s) ds \ge \frac{1}{b-a} \begin{bmatrix} \chi_a \\ \chi_b \end{bmatrix}^T \begin{bmatrix} R & 0 \\ 0 & 3R \end{bmatrix} \begin{bmatrix} \chi_a \\ \chi_b \end{bmatrix} \tag{3.1.13}$$

where $\chi_a = \int_a^b \omega(s)ds$ and $\chi_b = \chi_a - \frac{2}{b-a}\int_a^b \int_a^s \omega(u)duds = -\chi_a + \frac{2}{b-a}\int_a^b \int_s^b \omega(u)duds$.

The Auxiliary function-based integral inequality, which encompasses the Wirtinger-based inequality, has been developed recent years.

Lemma 4. *(Auxiliary function-based integral inequality [90]) For symmetric positive definite matrix $R \in \mathcal{R}^{n\times n}$, scalars $a < b$, and vector $\omega : [a, b] \mapsto \mathcal{R}^n$ such that the integration concerned are well defined, the following inequality holds*

$$(b-a)\int_a^b \dot{\omega}^T(s) R\dot{\omega}(s) ds \ge \chi_1^T R\chi_1 + 3\chi_2^T R\chi_2 + 5\chi_3^T R\chi_3 \tag{3.1.14}$$

where $\chi_1 = \omega(b) - \omega(a)$*,* $\chi_2 = \omega(b) + \omega(a) - \frac{2}{b-a}\int_a^b \omega(s)ds$*, and* $\chi_3 = \omega(b) - \omega(a) + \frac{6}{b-a}\int_a^b \omega(s)ds - \frac{12}{(b-a)^2}\int_a^b\int_s^b \omega(u)duds$*.*

For the estimation of double integral term, the JBDII is widely applied in [86], and as its improvement, the WBDII was developed in [88] very recently, respectively shown as lemma 3 and lemma 4.

Lemma 5. *(Jensen-based double inequality (JBDII) [86]) For symmetric positive definite matrix* $Z \in \Re^{n\times n}$*, scalars* $a < b$*, and vector* $\nu : [a, b] \mapsto \Re^n$ *such that the integration concerned are well defined, the following inequality holds*

$$\frac{(b-a)^2}{2}\int_a^b\int_s^b \nu^T(u)Z\nu(u)duds \geq \chi_4^T Z\chi_4 \tag{3.1.15}$$

where $\chi_4 = \int_a^b\int_s^b \nu(u)duds$*.*

Lemma 6. *(Wirtinger-based double inequality (WBDII) [88]) For symmetric positive definite matrix* $Z \in \Re^{n\times n}$*, scalars* $a < b$*, and vector* $\nu : [a, b] \mapsto \Re^n$ *such that the integration concerned are well defined, the following inequality holds*

$$\frac{(b-a)^2}{2}\int_a^b\int_s^b \nu^T(u)Z\nu(u)duds \geq \chi_4^T Z\chi_4 + 2\chi_5^T Z\chi_5 \tag{3.1.16}$$

where $\chi_5 = -\chi_4 + \frac{3}{b-a}\int_a^b\int_s^b\int_\theta^b \nu(u)dud\theta ds$ *with* χ_4 *given in Lemma 5.*

For time-varying delay, when using the integral inequality, the reciprocally convex lemma is needed, its simple form can be reformulated as Lemma 5.

Lemma 7. *(Reciprocally convex combination lemma [91]) For any vectors* β_1 *and* β_2*, symmetric matrix* R*, any matrix* S*, and real scalar* $0 \leq \alpha \leq 1$ *satisfying* $\begin{bmatrix} R & S \\ * & R \end{bmatrix} \geq 0$*, the following inequality holds,*

$$\frac{1}{\alpha}\beta_1^T R\beta_1 + \frac{1}{1-\alpha}\beta_2^T R\beta_2 \geq \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}^T \begin{bmatrix} R & S \\ * & R \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \tag{3.1.17}$$

3.2 A relaxed double integral inequality and its advantages

This section develops a new integral inequality, i.e., the WTDII, to estimate the double integral terms existing. The comparison of the WTDII and the existing double integral inequalities is also given.

Based on the technique of integral in parts, the following WTDII is given.

Lemma 8. *For symmetric positive definite matrix $Z \in \mathcal{R}^{n\times n}$, scalars $a < b$, and vector $\nu : [a, b] \mapsto \mathcal{R}^n$ such that the integration concerned are well defined, the following inequality holds*

$$\frac{(b-a)^2}{2}\int_a^b \int_s^b \nu^T(u)Z\nu(u)duds \geq \chi_4^T Z\chi_4 + 8\chi_5^T Z\chi_5 \quad (3.2.1)$$

where χ_4 and χ_5 are defined in Lemmas 5 and 6.

Proof: For a function $\lambda(u) = k_1 + k_2u$, the calculation through integration by parts leads to

$$\begin{aligned}\int_a^b \int_s^b \lambda(u)\nu(u)duds &= \lambda(a)\int_a^b \int_s^b \nu(u)duds \\ &\quad +2k_2\int_a^b \int_s^b \int_\theta^b x(u)dud\theta ds\end{aligned}$$

By setting $\lambda(a) = -1$, $2k_2 = \frac{3}{b-a}$, i.e., $\lambda(u) = \frac{-a-2b}{2(b-a)} + \frac{3}{2(b-a)}u$, the above equality is rewritten as

$$\int_a^b \int_s^b \lambda(u)\nu(u)duds = \chi_5 \quad (3.2.2)$$

Then the following equality is obtained for any vector χ_0 and any matrix M:

$$\int_a^b \int_s^b \lambda(u)\chi_0^T M\nu(u)duds = \chi_0^T M\chi_5 \quad (3.2.3)$$

Similarly, the following equalities are derived:

$$\int_a^b\int_s^b \chi_0^T L\nu(u)duds = \chi_0^T L\chi_4$$

$$\int_a^b\int_s^b \chi_0^T LR^{-1}L^T\chi_0 duds = \frac{(b-a)^2}{2}\chi_0^T LR^{-1}L^T\chi_0$$

$$\int_a^b\int_s^b \chi_0^T LR^{-1}M^T\lambda(u)\chi_0 duds = 0$$

$$\int_a^b\int_s^b \lambda^2(u)\chi_0^T MR^{-1}M^T\chi_0 duds = \frac{(b-a)^2}{16}\chi_0^T MR^{-1}M^T\chi_0$$

Therefore, using above five equalities and the Schur complement derives the following equality:

$$\begin{aligned}
&\int_a^b\int_s^b \begin{bmatrix}\chi_0\\ \lambda(u)\chi_0\\ \nu(u)\end{bmatrix}^T \begin{bmatrix} LZ^{-1}L^T & LZ^{-1}M^T & L\\ * & MZ^{-1}M^T & M\\ * & * & Z\end{bmatrix}\begin{bmatrix}\chi_0\\ \lambda(u)\chi_0\\ \nu(u)\end{bmatrix} duds\\
=\ &\int_a^b\int_s^b \nu^T(u)R\nu(u)duds + Sym\{\chi_0^T L\chi_4 + \chi_0^T M\chi_5\}\\
&+\frac{(b-a)^2}{2}\chi_0^T\left(\frac{8LZ^{-1}L^T + MZ^{-1}M^T}{8}\right)\chi_0\\
\geq\ & 0
\end{aligned} \tag{3.2.4}$$

By letting $\chi_0^T = [\chi_4^T,\ \chi_5^T]$, $L = -\frac{2}{(b-a)^2}[Z, 0]^T$, and $M = -\frac{16}{(b-a)^2}[0, Z]$, i.e., $\chi_0^T L = -\frac{2}{(b-a)^2}\chi_4^T Z$ and $\chi_0^T M = -\frac{16}{(b-a)^2}\chi_5^T Z$, then (3.2.4) leads to

$$\int_a^b\int_s^b \nu^T(u)Z\nu(u)duds \geq \frac{2}{(b-a)^2}\left(\chi_4^T Z\chi_4 + 8\chi_5^T Z\chi_5\right) \tag{3.2.5}$$

Thus (3.2.1) holds. This completes the proof. ■

Remark 4. *Based on the comparison of the proposed WTDII (3.2.1) with the widely used JBDII (3.1.15) and the recently developed WBDII (3.1.16), it can be found that WTDII (3.2.1) provides the tightest estimation value of the double integral term (3.1.1). More specifically, compared with the widely used JBDII (3.1.15), the extra positive term* $8\chi_5^T Z\chi_5$ *reduces the gap between the original double integral term*

(3.1.1) and its estimated value; and compared with the recently developed WBDII (3.1.16), the extra positive term $6\chi_5^T Z\chi_5$ reduces the estimation gap. As mentioned in [88–90], it is helpful to reduce the conservatism by reducing such estimation gap. Therefore, the proposed WTDII (3.2.1) will lead to less conservative criteria than the ones derived by JBDII (3.1.15) [87] or WBDII (3.1.16).

By setting $\nu(u) = \dot{\omega}(u)$, the following lemma can be directly obtained from Lemma 8.

Lemma 9. *For symmetric positive definite matrix $Z \in \mathcal{R}^{n\times n}$, scalars $a < b$, and vector $\dot{\omega} : [a, b] \mapsto \mathcal{R}^n$ such that the integration concerned are well defined, the following inequality holds*

$$\int_a^b \int_s^b \dot{\omega}^T(u) Z\dot{\omega}(u)duds \geq 2\theta_1^T Z\theta_1 + 16\theta_2^T Z\theta_2 \tag{3.2.6}$$

where $\theta_1 = \frac{1}{b-a}\chi_4|_{\nu(u)=\dot{\omega}(u)} = \omega(b) - \int_a^b \frac{\omega(s)}{b-a}ds$ and $\theta_2 = \frac{1}{b-a}\chi_5|_{\nu(u)=\dot{\omega}(u)} = -\frac{1}{2}\omega(b) - \int_a^b \frac{\omega(s)}{b-a}ds + 3\int_a^b \int_s^b \frac{\omega(u)}{(b-a)^2}duds.$

3.3 Delay-dependent stability analysis of genetic regulatory networks

This section derives delay-dependent stability criteria of GRN (3.1.2) by constructing the LKF with triple integral terms and applying the proposed WTDII (3.2.1) to estimate the double integral terms appearing in its derivative.

The following notations are introduced at first for simplifying the representation of subsequent parts:

$$\begin{aligned}
&\tau_{1\tau}(t) = \tau(t) - \tau_1, && \tau_{2\tau}(t) = \tau_2 - \tau(t)\\
&\sigma_{1\sigma}(t) = \sigma(t) - \sigma_1, && \sigma_{2\sigma}(t) = \sigma_2 - \sigma(t)\\
&x_{\tau_1}(t) = x(t - \tau_1), && y_{\sigma_1}(t) = y(t - \sigma_1)\\
&x_{\tau}(t) = x(t - \tau(t)), && y_{\sigma}(t) = y(t - \sigma(t))\\
&x_{\tau_2}(t) = x(t - \tau_2), && y_{\sigma_2}(t) = y(t - \sigma_2)
\end{aligned}$$

$$
\begin{aligned}
&v_1(t)=\int_{t-\tau_1}^{t}\frac{x(s)}{\tau_1}ds\ , \quad v_4(t)=\int_{t-\tau_1}^{t}\int_{s}^{t}\frac{x(u)}{\tau_1^2}duds\\
&v_2(t)=\int_{t-\tau(t)}^{t-\tau_1}\frac{x(s)}{\tau_{1\tau}(t)}ds\ , \quad v_5(t)=\int_{t-\tau(t)}^{t-\tau_1}\int_{s}^{t-\tau_1}\frac{x(u)}{\tau_{1\tau}^2(t)}duds\\
&v_3(t)=\int_{t-\tau_2}^{t-\tau(t)}\frac{x(s)}{\tau_{2\tau}(t)}ds\ , \quad v_6(t)=\int_{t-\tau_2}^{t-\tau(t)}\int_{s}^{t-\tau(t)}\frac{x(u)}{\tau_{2\tau}^2(t)}duds\\
&v_7(t)=\int_{t-\sigma_1}^{t}\frac{x(s)}{\sigma_1}ds\ , \quad v_{10}(t)=\int_{t-\sigma_1}^{t}\int_{s}^{t}\frac{x(u)}{\sigma_1^2}duds\\
&v_8(t)=\int_{t-\sigma(t)}^{t-\sigma_1}\frac{x(s)}{\sigma_{1\sigma}(t)}ds\ , v_{11}(t)=\int_{t-\sigma(t)}^{t-\sigma_1}\int_{s}^{t-\sigma_1}\frac{x(u)}{\sigma_{1\sigma}^2(t)}duds\\
&v_9(t)=\int_{t-\sigma_2}^{t-\sigma(t)}\frac{x(s)}{\sigma_{2\sigma}(t)}ds\ , v_{12}(t)=\int_{t-\sigma_2}^{t-\sigma(t)}\int_{s}^{t-\sigma(t)}\frac{x(u)}{\sigma_{2\sigma}^2(t)}duds\\
&\zeta(t)=\big[x^T(t),x^T(t-\tau_1),x^T(t-\tau(t)),x^T(t-\tau_2),v_1^T(t),\\
&\qquad v_2^T(t),\cdots,v_6^T(t),y^T(t),y^T(t-\sigma_1),y^T(t-\sigma(t)),\\
&\qquad y^T(t-\sigma_2),v_7^T(t),v_8^T(t),\cdots,v_{12}^T(t),f^T(y(t)),\\
&\qquad f^T(y(t-\sigma_1)),f^T(y(t-\sigma(t))),f^T(y(t-\sigma_2))\big]^T
\end{aligned}
\tag{3.3.1}
$$

$$
\begin{aligned}
&e_x=[-A,\ 0_{n\times 21n},\ W,\ 0_{n\times n}]\\
&e_y=[0_{n\times 2n},\ D,\ 0_{n\times 7n},\ -C,\ 0_{n\times 13n}]\\
&e_0=[0_{n\times 24n}]\\
&e_i=[0_{n\times(i-1)n},\ I_{n\times n},\ 0_{n\times(24-i)n}], i=1,2,\cdots,24\\
&\Sigma=\mathrm{diag}\{\rho_1,\rho_2,\cdots,\rho_n\}
\end{aligned}
\tag{3.3.2}
$$

3.3.1 Stability of genetic regulatory networks with delay satisfying

For GRN (3.1.2) with a delay satisfying (3.1.4), the following stability criterion is derived by using the proposed WTDII (3.2.6), together with Lemmas 3, 4, and 7, to estimate the derivative of the LKF.

Theorem 2. *For given scalars τ_i, $\sigma_i, i=1,2$, τ_d, and σ_d, GRN (3.1.2) with the time delay satisfying (3.1.4) and regulatory function satisfying (3.1.3) is asymptotically stable, if there exist symmetric matrices $P>0, Q_i>0, R_j>0, Z_k>0, i=1,2,\cdots,6, j=1,2,\cdots,5, k=1,2,\cdots,4$; diagonal matrices $\Lambda_1>0, \Lambda_2>0$, $H_j>0, j=1,2,\cdots,4$, $U_{lk}>0, l=1,2,\cdots,4, k=l+1,\cdots,4$; and any* *matrices $S_i, i = 1, 2$, such that the following LMIs hold*

$$\begin{bmatrix} \tilde{R}_{2i+1} & S_i \\ * & \tilde{R}_{2i+1} \end{bmatrix} > 0, i = 1, 2 \tag{3.3.3}$$

$$\Psi_1 = \Xi_{\tau(t)}|_{\tau(t)=\tau_1} + \sum_{i=1}^{8} \Xi_i \leq 0 \tag{3.3.4}$$

$$\Psi_2 = \Xi_{\tau(t)}|_{\tau(t)=\tau_2} + \sum_{i=1}^{8} \Xi_i \leq 0 \tag{3.3.5}$$

where $\tau_{12} = \tau_2 - \tau_1$, $\sigma_{12} = \sigma_2 - \sigma_1$, and

$$\begin{aligned} \Xi_{\tau(t)} &= -\tau_{1\tau}(t)[e_6^T R_2 e_6 + 3(2e_9 - e_6)^T R_2(2e_9 - e_6)] \\ &\quad -\tau_{2\tau}(t)[e_7^T R_2 e_7 + 3(2e_{10} - e_7)^T R_2(2e_{10} - e_7)] \end{aligned} \tag{3.3.6}$$

$$\Xi_1 = \Xi_{11} + \Xi_{11}^T \tag{3.3.7}$$

$$\Xi_{11} = \begin{bmatrix} e_1 \\ e_{11} \end{bmatrix}^T P \begin{bmatrix} e_x \\ e_y \end{bmatrix} + \left[(\Sigma e_{11} - e_{21})^T \Lambda_1 + e_{21}^T \Lambda_2\right] e_y \tag{3.3.8}$$

$$\begin{aligned} \Xi_2 &= e_1^T Q_1 e_1 - e_2^T (Q_1 - Q_2 - Q_3) e_2 \\ &\quad -e_4^T Q_2 e_4 - (1 - \tau_d) e_3^T Q_3 e_3 \end{aligned} \tag{3.3.9}$$

$$\Xi_3 = \Xi_{31} + \Xi_{32} + \Xi_{33} \tag{3.3.10}$$

$$\Xi_{31} = e_x^T (\tau_1^2 R_1 + \tau_{12}^2 R_3) e_x + \tau_{12} e_1^T R_2 e_1 \tag{3.3.11}$$

$$\Xi_{32} = E_1^T \tilde{R}_1 E_1, \quad \tilde{R}_1 = diag\{R_1, 3R_1, 5R_1\} \tag{3.3.12}$$

$$\Xi_{33} = \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} \tilde{R}_3 & S_1 \\ * & \tilde{R}_3 \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}, \tilde{R}_3 = diag\{R_3, 3R_3, 5R_3\} \tag{3.3.13}$$

$$\Xi_4 = \Xi_{41} + \Xi_{42} + \Xi_{43} \tag{3.3.14}$$

$$\Xi_{41} = e_x^T \Big(\frac{\tau_1^2}{2} Z_1 + \frac{\tau_2^2 - \tau_1^2}{2} Z_2 - \tau_1 \tau_{12} Z_2 \Big) e_x \tag{3.3.15}$$

$$\begin{aligned} \Xi_{42} &= -2[e_1 - e_5]^T Z_1 [e_1 - e_5] \\ &\quad -16\Big[3e_8 - \frac{e_1}{2} - e_5\Big]^T Z_1 \Big[3e_8 - \frac{e_1}{2} - e_5\Big] \end{aligned} \tag{3.3.16}$$

$$\begin{aligned} \Xi_{43} &= -2[e_2 - e_6]^T Z_2 [e_2 - e_6] \\ &\quad -16\Big[3e_9 - \frac{e_2}{2} - e_6\Big]^T Z_2 \Big[3e_9 - \frac{e_2}{2} - e_6\Big] \\ &\quad -2[e_3 - e_7]^T Z_2 [e_3 - e_7] \end{aligned} \tag{3.3.17}$$

$$-16\Big[3e_{10}-\frac{e_3}{2}-e_7\Big]^T Z_2\Big[3e_{10}-\frac{e_3}{2}-e_7\Big] \tag{3.3.18}$$

$$\begin{aligned}\Xi_5 \;=& \begin{bmatrix} e_{11}\\ e_{21}\end{bmatrix}^T Q_4\begin{bmatrix} e_{11}\\ e_{21}\end{bmatrix}+\begin{bmatrix} e_{12}\\ e_{22}\end{bmatrix}^T(Q_5+Q_6-Q_4)\begin{bmatrix} e_{12}\\ e_{22}\end{bmatrix}\\ &-\begin{bmatrix} e_{14}\\ e_{24}\end{bmatrix}^T Q_5\begin{bmatrix} e_{14}\\ e_{24}\end{bmatrix}-(1-\sigma_d)\begin{bmatrix} e_{13}\\ e_{23}\end{bmatrix}^T Q_6\begin{bmatrix} e_{13}\\ e_{23}\end{bmatrix}\end{aligned} \tag{3.3.19}$$

$$\Xi_6 \;=\Xi_{61}+\Xi_{62}+\Xi_{63} \tag{3.3.20}$$

$$\Xi_{61}=e_y^T(\sigma_1^2R_4+\sigma_{12}^2R_5)e_y \tag{3.3.21}$$

$$\Xi_{62}=E_4^T\tilde{R}_4E_4,\quad \tilde{R}_4=diag\{R_4,3R_4,5R_4\} \tag{3.3.22}$$

$$\Xi_{63}=\begin{bmatrix} E_5\\ E_6\end{bmatrix}^T\begin{bmatrix} \tilde{R}_5 & S_2\\ * & \tilde{R}_5\end{bmatrix}\begin{bmatrix} E_5\\ E_6\end{bmatrix},\tilde{R}_5=diag\{R_5,3R_5,5R_5\} \tag{3.3.23}$$

$$\Xi_7 \;=\Xi_{71}+\Xi_{72}+\Xi_{73} \tag{3.3.24}$$

$$\Xi_{71}=e_y^T\Big(\frac{\sigma_1^2}{2}Z_3+\frac{\sigma_2^2-\sigma_1^2}{2}Z_4-\sigma_1\sigma_{12}Z_4\Big)e_y \tag{3.3.25}$$

$$\begin{aligned}\Xi_{72}=&-2[e_{11}-e_{15}]^TZ_3[e_{11}-e_{15}]\\ &-16\Big[3e_{18}-\frac{e_{11}}{2}-e_{15}\Big]^TZ_3\Big[3e_{18}-\frac{e_{11}}{2}-e_{15}\Big]\end{aligned} \tag{3.3.26}$$

$$\begin{aligned}\Xi_{73}=&-2[e_{12}-e_{16}]^TZ_4[e_{12}-e_{16}]\\ &-16\Big[3e_{19}-\frac{e_{12}}{2}-e_{16}\Big]^TZ_4\Big[3e_{19}-\frac{e_{12}}{2}-e_{16}\Big]\\ &-2[e_{13}-e_{17}]^TZ_4[e_{13}-e_{17}]\\ &-16\Big[3e_{20}-\frac{e_{13}}{2}-e_{17}\Big]^TZ_4\Big[3e_{20}-\frac{e_{13}}{2}-e_{17}\Big]\end{aligned} \tag{3.3.27}$$

$$\Xi_8 \;=\Xi_{81}+\Xi_{81}^T \tag{3.3.28}$$

$$\begin{aligned}\Xi_{81}=&\sum_{i=1}^{4}\big[(\Sigma e_{1i}-e_{2i})^TH_ie_{2i}\big] \qquad (3.3.29)\\ &+\sum_{i=1}^{4}\sum_{j=i+1}^{4}[\Sigma(e_{1i}-e_{1j})-(e_{2i}-e_{2j})]^TU_{ij}(e_{2i}-e_{2j})\end{aligned}$$

$$E_1=\begin{bmatrix} e_1-e_2\\ e_1+e_2-2e_5\\ e_1-e_2+6e_5-12e_8\end{bmatrix} \tag{3.3.30}$$

$$E_2=\begin{bmatrix} e_2-e_3\\ e_2+e_3-2e_6\\ e_2-e_3+6e_6-12e_9\end{bmatrix} \tag{3.3.31}$$

$$E_3 = \begin{bmatrix} e_3 - e_4 \\ e_3 + e_4 - 2e_7 \\ e_3 - e_4 + 6e_7 - 12e_{10} \end{bmatrix} \quad (3.3.32)$$

$$E_4 = \begin{bmatrix} e_{11} - e_{12} \\ e_{11} + e_{12} - 2e_{15} \\ e_{11} - e_{12} + 6e_{15} - 12e_{18} \end{bmatrix} \quad (3.3.33)$$

$$E_5 = \begin{bmatrix} e_{12} - e_{13} \\ e_{12} + e_{13} - 2e_{16} \\ e_{12} - e_{13} + 6e_{16} - 12e_{19} \end{bmatrix} \quad (3.3.34)$$

$$E_6 = \begin{bmatrix} e_{13} - e_{14} \\ e_{13} + e_{14} - 2e_{17} \\ e_{13} - e_{14} + 6e_{17} - 12e_{20} \end{bmatrix} \quad (3.3.35)$$

Proof: Construct the following LKF candidate:

$$V(t) = \sum_{i=1}^{7} V_i(t) \quad (3.3.36)$$

where

$$\begin{aligned} V_1(t) &= \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}^T P \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \quad (3.3.37) \\ &\quad + \sum_{i=1}^{n} \int_0^{y_i} [\lambda_{1i}(\rho_i s - f_i(s)) + \lambda_{2i} f_i(s)] ds \\ V_2(t) &= \int_{t-\tau_1}^{t} x^T(s) Q_1 x(s) ds + \int_{t-\tau_2}^{t-\tau_1} x^T(s) Q_2 x(s) ds \\ &\quad + \int_{t-\tau(t)}^{t-\tau_1} x^T(s) Q_3 x(s) ds \quad (3.3.38) \\ V_3(t) &= \tau_1 \int_{-\tau_1}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) R_1 \dot{x}(s) ds d\theta \quad (3.3.39) \\ &\quad + \int_{-\tau_2}^{-\tau_1} \int_{t+\theta}^{t} \left[x^T(s) R_2 x(s) + \tau_{12} \dot{x}^T(s) R_3 \dot{x}(s) \right] ds d\theta \end{aligned}$$

$$\begin{aligned} V_4(t) &= \int_{-\tau_1}^{0}\int_{\theta}^{0}\int_{t+s}^{t} \dot{x}^T(u)Z_1\dot{x}(u)dudsd\theta && (3.3.40)\\ &+\int_{-\tau_2}^{-\tau_1}\int_{\theta}^{-\tau_1}\int_{t+s}^{t} \dot{x}^T(u)Z_2\dot{x}(u)dudsd\theta && (3.3.41)\\ V_5(t) &= \int_{t-\sigma_1}^{t}\begin{bmatrix} y(s)\\ f(y(s))\end{bmatrix}^T Q_4 \begin{bmatrix} y(s)\\ f(y(s))\end{bmatrix} ds\\ &+\int_{t-\sigma_2}^{t-\sigma_1}\begin{bmatrix} y(s)\\ f(y(s))\end{bmatrix}^T Q_5 \begin{bmatrix} y(s)\\ f(y(s))\end{bmatrix} ds\\ &+\int_{t-\sigma(t)}^{t-\sigma_1}\begin{bmatrix} y(s)\\ f(y(s))\end{bmatrix}^T Q_6 \begin{bmatrix} y(s)\\ f(y(s))\end{bmatrix} ds && (3.3.42)\\ V_6(t) &= \sigma_1\int_{-\sigma_1}^{0}\int_{t+\theta}^{t}\dot{y}^T(s)R_4\dot{y}(s)dsd\theta\\ &+\sigma_{12}\int_{-\sigma_2}^{-\sigma_1}\int_{t+\theta}^{t}\dot{y}^T(s)R_5\dot{y}(s)dsd\theta && (3.3.43)\\ V_7(t) &= \int_{-\sigma_1}^{0}\int_{\theta}^{0}\int_{t+s}^{t}\dot{y}^T(u)Z_3\dot{y}(u)dudsd\theta && (3.3.44)\\ &+\int_{-\sigma_2}^{-\sigma_1}\int_{\theta}^{-\sigma_1}\int_{t+s}^{t}\dot{y}^T(u)Z_4\dot{y}(u)dudsd\theta && (3.3.45) \end{aligned}$$

and $P > 0, Q_i > 0, R_j > 0, Z_k > 0, i = 1,2,\cdots,6, j = 1,2,\cdots,5, k = 1,2,\cdots,4$ are the symmetric positive definite matrices; and $\Lambda_i = \text{diag}\{\lambda_{i1},\lambda_{i2},\cdots,\lambda_{in}\} > 0, i = 1,2$ are the symmetric positive definite diagonal matrices.

Calculating the derivative of the LKF along the solutions of GRN (3.1.10) yields,

$$\dot{V}(t) = \sum_{i=1}^{7}\dot{V}_i(t) \qquad (3.3.46)$$

where

$$\begin{aligned} \dot{V}_1(t) &= 2\begin{bmatrix} x(t)\\ y(t)\end{bmatrix}^T P \begin{bmatrix} \dot{x}(t)\\ \dot{y}(t)\end{bmatrix}\\ &\quad +2\{[\Sigma y(t)-f(y(t))]^T\Lambda_1+f^T(y(t))\Lambda_2\}\dot{y}(t)\\ &= \zeta^T(t)(\Xi_{11}+\Xi_{11}^T)\zeta(t) && (3.3.47) \end{aligned}$$

$$
\begin{aligned}
\dot{V}_2(t) &= x^T(t)Q_1x(t) + x_{\tau_1}^T(t)(Q_2+Q_3-Q_1)x_{\tau_1}(t) \\
&\quad -x_{\tau_2}^T(t)Q_2x_{\tau_2}(t) - (1-\dot{\tau}(t))x_{\tau}^T(t)Q_3x_{\tau}(t) \\
&\leq x^T(t)Q_1x(t) + x_{\tau_1}^T(t)(Q_2+Q_3-Q_1)x_{\tau_1}(t) \\
&\quad -x_{\tau_2}^T(t)Q_2x_{\tau_2}(t) - (1-\tau_d)x_{\tau}^T(t)Q_3x_{\tau}(t) \\
&= \zeta^T(t)\Xi_2\zeta(t) \qquad (3.3.48)
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_3(t) &= \dot{x}^T(t)(\tau_1^2R_1+\tau_{12}^2R_3)\dot{x}(t) + \tau_{12}x^T(t)R_2x(t) \\
&\quad -\tau_1\int_{t-\tau_1}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds \\
&\quad -\int_{t-\tau_2}^{t-\tau_1}\big(x^T(s)R_2x(s)+\tau_{12}\dot{x}^T(s)R_3\dot{x}(s)\big)ds \qquad (3.3.49)
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_4(t) &= \dot{x}^T(t)\Big(\frac{\tau_1^2}{2}Z_1+\frac{\tau_2^2-\tau_1^2}{2}Z_2-\tau_1\tau_{12}Z_2\Big)\dot{x}(t) \\
&\quad -\int_{t-\tau_1}^{t}\int_{s}^{t}\dot{x}^T(u)Z_1\dot{x}(u)duds \\
&\quad -\int_{t-\tau_2}^{t-\tau_1}\int_{s}^{t-\tau_1}\dot{x}^T(u)Z_2\dot{x}(u)duds \qquad (3.3.50)
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_5(t) &= \begin{bmatrix} y(t) \\ f(y(t)) \end{bmatrix}^T Q_4 \begin{bmatrix} y(t) \\ f(y(t)) \end{bmatrix} \\
&\quad + \begin{bmatrix} y_{\sigma_1}(t) \\ f(y(t-\sigma_1)) \end{bmatrix}^T (Q_5+Q_6-Q_4) \begin{bmatrix} y_{\sigma_1}(t) \\ f(y(t-\sigma_1)) \end{bmatrix} \\
&\quad - \begin{bmatrix} y_{\sigma_2}(t) \\ f(y(t-\sigma_2)) \end{bmatrix}^T Q_5 \begin{bmatrix} y_{\sigma_2}(t) \\ f(y(t-\sigma_2)) \end{bmatrix} \\
&\quad -(1-\dot{\sigma}(t))\begin{bmatrix} y_{\sigma}(t) \\ f(y(t-\sigma(t))) \end{bmatrix}^T Q_6 \begin{bmatrix} y_{\sigma}(t) \\ f(y(t-\sigma(t))) \end{bmatrix} \\
&\leq \zeta^T(t)\Xi_5\zeta(t) \qquad (3.3.51)
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_6(t) &= \dot{y}^T(t)(\sigma_1^2R_4+\sigma_{12}^2R_5)\dot{y}(t)-\sigma_1\int_{t-\sigma_1}^{t}\dot{y}^T(s)R_4\dot{y}(s)ds \\
&\quad -\sigma_{12}\int_{t-\sigma_2}^{t-\sigma_1}\dot{y}^T(s)R_5\dot{y}(s)ds \qquad (3.3.52)
\end{aligned}
$$

$$\begin{aligned}\dot{V}_7(t) &= \dot{y}^T(t)\Big(\frac{\sigma_1^2}{2}Z_3 + \frac{\sigma_2^2-\sigma_1^2}{2}Z_4 - \sigma_1\sigma_{12}Z_4\Big)\dot{y}(t) \\ &\quad - \int_{t-\sigma_1}^{t}\int_{s}^{t}\dot{y}^T(u)Z_3\dot{y}(u)duds \\ &\quad - \int_{t-\sigma_2}^{t-\sigma_1}\int_{s}^{t-\sigma_1}\dot{y}^T(u)Z_4\dot{y}(u)duds \qquad (3.3.53)\end{aligned}$$

where Ξ_{11}, Ξ_2, and Ξ_5 are defined in (3.3.8), (3.3.9), and (3.3.19), respecitvely.

Using Lemma 4 to estimate the R_1-dependent single integral terms in $\dot{V}_3(t)$ yields

$$\begin{aligned}-\tau_1\int_{t-\tau_1}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds &\le -\eta_1^T(t)\tilde{R}_1\eta_1(t) \\ &= \zeta^T(t)\Xi_{32}\zeta(t) \qquad (3.3.54)\end{aligned}$$

where $\tilde{R}_1$ and Ξ_{32} is defined in (3.3.12) and

$$\eta_1(t) = \begin{bmatrix} x(t) - x_{\tau_1}(t) \\ x(t) + x_{\tau_1}(t) - 2v_1(t) \\ x(t) - x_{\tau_1}(t) + 6v_1(t) - 12v_4(t)\end{bmatrix}$$

Using Lemma 3 to estimate the R_2-dependent single integral terms in $\dot{V}_3(t)$ yields

$$\begin{aligned}&-\int_{t-\tau_2}^{t-\tau_1}x^T(s)Rx(s)ds \\ &= -\int_{t-\tau(t)}^{t-\tau_1}x^T(s)Rx(s)ds - \int_{t-\tau_2}^{t-\tau(t)}x^T(s)Rx(s)ds \\ &\le -\tau_{1\tau}(t)[v_2^T(t)R_2v_2(t)+3(2v_5(t)-v_2(t))^TR_2(2v_5(t)-v_2(t))] \\ &\quad -\tau_{2\tau}(t)[v_3^T(t)R_2v_3(t)+3(2v_6(t)-v_3(t))^TR_2(2v_6(t)-v_3(t))] \\ &= \zeta^T(t)\Xi_{\tau(t)}\zeta(t) \qquad (3.3.55)\end{aligned}$$

where $\Xi_{\tau(t)}$ is defined in (3.3.6).

Using Lemmas 4 and 7, together with (3.3.3), to estimate the R_3-dependent single integral terms in $\dot{V}_3(t)$ yields

$$
\begin{aligned}
& -\tau_{12}\int_{t-\tau_2}^{t-\tau_1}\dot{x}^T(s)R_3\dot{x}(s)ds \\
= & -\tau_{12}\int_{t-\tau(t)}^{t-\tau_1}\dot{x}^T(s)R_3\dot{x}(s)ds-\tau_{12}\int_{t-\tau_2}^{t-\tau(t)}\dot{x}^T(s)R_3\dot{x}(s)ds \\
\leq & -\frac{\tau_{12}}{\tau(t)-\tau_1}\Big\{\eta_2^T(t)\tilde{R}_3\eta_2(t)\Big\}-\frac{\tau_{12}}{\tau_2-\tau(t)}\Big\{\eta_3^T(t)\tilde{R}_3\eta_3(t)\Big\} \\
\leq & -\begin{bmatrix}\eta_2(t)\\ \eta_3(t)\end{bmatrix}^T\begin{bmatrix}\tilde{R}_3 & S_1\\ * & \tilde{R}_3\end{bmatrix}\begin{bmatrix}\eta_2(t)\\ \eta_3(t)\end{bmatrix} \\
= & \ \zeta^T(t)\Xi_{33}\zeta(t)
\end{aligned}
\tag{3.3.56}
$$

where $\tilde{R}_3$ and Ξ_{33} is defined in (3.3.13) and

$$
\begin{aligned}
\eta_2(t) &= \begin{bmatrix} x_{\tau_1}(t)-x_\tau(t)\\ x_{\tau_1}(t)+x_\tau(t)-2v_2(t)\\ x_{\tau_1}(t)-x_\tau(t)+6v_2(t)-12v_5(t)\end{bmatrix}\\
\eta_3(t) &= \begin{bmatrix} x_\tau(t)-x_{\tau_2}(t)\\ x_\tau(t)+x_{\tau_2}(t)-2v_3(t)\\ x_\tau(t)-x_{\tau_2}(t)+6v_3(t)-12v_6(t)\end{bmatrix}
\end{aligned}
$$

Using Lemma 9 to estimate the Z_1-dependent double integral terms in $\dot{V}_4(t)$ yields

$$
\begin{aligned}
& -\int_{t-\tau_1}^{t}\int_{s}^{t}\dot{x}^T(u)Z_1\dot{x}(u)duds \\
\leq & -2[x(t)-v_1(t)]^TZ_1[x(t)-v_1(t)] \\
& +16\Big[3v_4(t)-\frac{x(t)}{2}-v_1(t)\Big]^TZ_1\Big[3v_4(t)-\frac{x(t)}{2}-v_1(t)\Big] \\
= & \ \zeta^T(t)\Xi_{42}\zeta(t)
\end{aligned}
\tag{3.3.57}
$$

where Ξ_{42} is defined in (3.3.16).

Using Lemma 9 to estimate the Z_2-dependent double integral terms in $\dot{V}_4(t)$

yields

$$\begin{aligned}
& -\int_{t-\tau_2}^{t-\tau_1}\int_{s}^{t-\tau_1}\dot{x}^T(u)Z_2\dot{x}(u)duds \\
= & -\int_{t-\tau(t)}^{t-\tau_1}\int_{s}^{t-\tau_1}\dot{x}^T(u)Z_2\dot{x}(u)duds \\
& -\int_{t-\tau_2}^{t-\tau(t)}\int_{s}^{t-\tau_1}\dot{x}^T(u)Z_2\dot{x}(u)duds \\
\leq & -\int_{t-\tau(t)}^{t-\tau_1}\int_{s}^{t-\tau_1}\dot{x}^T(u)Z_2\dot{x}(u)duds \\
& -\int_{t-\tau_2}^{t-\tau(t)}\int_{s}^{t-\tau(t)}\dot{x}^T(u)Z_2\dot{x}(u)duds \\
\leq & -2[x_{\tau_1}(t)-v_2(t)]^T Z_2[x_{\tau_1}(t)-v_2(t)] \\
& -16\Big[3v_5(t)-\frac{x_{\tau_1}(t)}{2}-v_2(t)\Big]^T Z_2\Big[3v_5(t)-\frac{x_{\tau_1}(t)}{2}-v_2(t)\Big] \\
& -2[x_{\tau}(t)-v_3(t)]^T Z_2[x_{\tau}(t)-v_3(t)] \\
& -16\Big[3v_6(t)-\frac{x_{\tau}(t)}{2}-v_3(t)\Big]^T Z_2\Big[3v_6(t)-\frac{x_{\tau}(t)}{2}-v_3(t)\Big] \\
= & \ \zeta^T(t)\Xi_{43}\zeta(t)
\end{aligned} \tag{3.3.58}$$

where Ξ_{43} is defined in (3.3.17).

Similarly, Using Lemmas 4, 7, and 9 to estimate the single and double integral terms in $\dot{V}_6(t)$ and $\dot{V}_7(t)$ yields

$$-\sigma_1\int_{t-\sigma_1}^{t}\dot{x}^T(s)R_4\dot{x}(s)ds \leq \zeta^T(t)\Xi_{62}\zeta(t) \tag{3.3.59}$$

$$-\sigma_{12}\int_{t-\sigma_2}^{t-\sigma_1}\dot{x}^T(s)R_5\dot{x}(s)ds \leq \zeta^T(t)\Xi_{63}\zeta(t) \tag{3.3.60}$$

$$-\int_{t-\sigma_1}^{t}\int_{s}^{t}\dot{x}^T(u)Z_3\dot{x}(u)duds \leq \zeta^T(t)\Xi_{72}\zeta(t) \tag{3.3.61}$$

$$-\int_{t-\sigma_2}^{t-\sigma_1}\int_{s}^{t-\sigma_1}\dot{x}^T(u)Z_4\dot{x}(u)duds \leq \zeta^T(t)\Xi_{73}\zeta(t) \tag{3.3.62}$$

where Ξ_{62}, Ξ_{63}, Ξ_{72}, and Ξ_{73} are defined in (3.3.22)-(3.3.27).

Taking into account the assumption of the activation function, (3.1.11) and (3.1.12), the following inequalities hold [92, 93]:

$$\begin{aligned} h_i(s) &= 2\left[\Sigma y(s) - f(y(s))\right]^T H_i f(y(s)) \geq 0 \\ u_{ij}(s_1, s_2) &= 2\left[\Sigma(y(s_1) - y(s_2)) - (f(y(s_1)) - f(y(s_2)))\right]^T U_{ij} \\ &\quad \times (f(y(s_1)) - f(y(s_2))) \geq 0 \end{aligned}$$

where $H_i, i = 1, 2, \cdots, 4$ and $U_{ij}, i = 1, 2, \cdots, 4, j = i+1, \cdots, 4$ are the symmetric diagonal matrices. Thus, the following inequality holds:

$$\begin{aligned} & H(t) + U(t) \\ &= h_1(t) + h_2(t - \sigma_1) + h_3(t - \sigma(t)) + h_4(t - \sigma_2) \\ &\quad + u_{12}(t, t - \sigma_1) + u_{13}(t, t - \sigma(t)) + u_{14}(t, t - \sigma_2) \\ &\quad + u_{23}(t - \sigma_1, t - \sigma(t)) + u_{24}(t - \sigma_1, t - \sigma_2) \\ &\quad + u_{34}(t - \sigma(t), t - \sigma_2) \\ &= \zeta^T(t) \Xi_8 \zeta(t) \geq 0 \end{aligned} \tag{3.3.63}$$

where Ξ_8 is defined in (3.3.28).

Finally, combining (3.3.46) - (3.3.63) yields

$$\dot{V}(t) \leq \zeta^T(t) \left[\Xi_{\tau(t)} + \sum_{i=1}^{8} \Xi_i \right] \zeta(t) \tag{3.3.64}$$

where the related notations are defined in (3.3.4).

Therefore, if LMIs (3.3.4) and (3.3.5) hold, then the following holds for a sufficiently small scalar $\epsilon > 0$ based on convex combination method [91]:

$$\dot{V}(t) \leq -\epsilon(||x(t)||^2 + ||y(t)||^2) \tag{3.3.65}$$

which shows the asymptotical stability of GRN (3.1.2) with time delay satisfying (3.1.4). This completes the proof. ■

For some cases, the change rates of the time-varying delays are unmeasurable, i.e., time delay satisfying (3.1.5). For this case, the following stability criterion can be derived by using the proposed WTDII (3.2.6), together with Lemmas 3, 4, 7, and 9, to estimate the derivative of the LKF.

Theorem 3. *For given scalars τ_i and $\sigma_i, i = 1, 2$, GRN (3.1.2) with the time delay satisfying (3.1.5) and regulatory function satisfying (3.1.3) is asymptotically stable, if there exist symmetric matrices $P > 0, Q_i > 0, R_j > 0, Z_k > 0, i = 1, 2, 4, 5, j = 1, 2, \cdots, 5, k = 1, 2, \cdots, 4$; diagonal matrices $\Lambda_1 > 0, \Lambda_2 > 0$, $H_j > 0, j = 1, 2, 3, 4$, $U_{lk} > 0, l = 1, 2, \cdots, 4, k = l + 1, \cdots, 4$; and any matrices $S_i, i = 1, 2$, such that the following LMIs hold*

$$\begin{bmatrix} \tilde{R}_{2i+1} & S_i \\ * & \tilde{R}_{2i+1} \end{bmatrix} > 0, i = 1, 2 \tag{3.3.66}$$

$$\Psi_3 = \Xi_{\tau(t)}|_{\tau(t)=\tau_1} + \sum_{i=1,3,4,6,7} \Xi_i + \bar{\Xi}_2 + \bar{\Xi}_5 \leq 0 \tag{3.3.67}$$

$$\Psi_4 = \Xi_{\tau(t)}|_{\tau(t)=\tau_2} + \sum_{i=1,3,4,6,7} \Xi_i + \bar{\Xi}_2 + \bar{\Xi}_5 \leq 0 \tag{3.3.68}$$

where $\Xi_i, i = 1, 3, 4, 6, 7$ are defined in Theorem 2, and

$$\begin{aligned} \bar{\Xi}_2 &= e_1^T Q_1 e_1 - e_2^T (Q_1 - Q_2) e_2 - e_4^T Q_2 e_4 \\ \bar{\Xi}_5 &= \begin{bmatrix} e_{11} \\ e_{21} \end{bmatrix}^T Q_4 \begin{bmatrix} e_{11} \\ e_{21} \end{bmatrix} + \begin{bmatrix} e_{12} \\ e_{22} \end{bmatrix}^T (Q_5 - Q_4) \begin{bmatrix} e_{12} \\ e_{22} \end{bmatrix} \\ &\quad - \begin{bmatrix} e_{14} \\ e_{24} \end{bmatrix}^T Q_5 \begin{bmatrix} e_{14} \\ e_{24} \end{bmatrix} \end{aligned}$$

Proof: The above stability criterion can be obtained by setting $Q_3 = 0$ and $Q_6 = 0$ in Theorem 2. ■

This part gives some remarks for the above criteria.

Remark 5. *During the proof of the above two stability criteria, the double integral terms arising in the derivative of the LKFs are estimated by using the proposed WTDII, i.e., Lemma 9. The WTDII is tighter than the widely used JBDII (3.1.15), which was used for the GRN [85, 87], and the recently developed WBDII (3.1.16), which has not been used for the GRN. Thus, the proposed criteria are less conservative than the ones reported in [87] and [85].*

Remark 6. *Compared with the literature, more information of regulatory function has been used during the proof of criteria. Specifically, in the literature, only (3.1.12) is used during the estimation of the derivative of the LKF, while, in this* *chapter, an extra information of regulatory function, (3.1.11), is also used for estimating task. It has been proved in [93] that such additional information is helpful to reduce the conservatism.*

Remark 7. *The conditions given in Theorems 2 and 3 are in the form of LMI. Such LMI conditions can be easily checked by using MATLAB/Toolbox [94]. The one can refer to [56, 95, 96] for more details.*

Remark 8. *Although this chapter has just investigated the asymptotical stability, the proposed method can be extended to the robust stability analysis by taking into account the parameter uncertainties and/or noises of the GRNs. Moreover, the proposed method can also be extended to other problems discussed in introduction, such as controller synthesis, state estimation, filter design, passivity analysis, and so on [74, 97–108].*

3.4 Illustrative example

An example will be presented to illustrate the effectiveness of results. As mentioned in Section I, the important aim of the stability analysis of delayed GRNs is to determine the MADBs. And the stability criterion that provides bigger MADBs has less conservative than the one that gives smaller ones. Therefore, the advantages of the proposed criteria are demonstrated via the comparison of the MADBs calculated by various criteria. Moreover, the index of the number of variable (NoV) is applied to show the complexity of criteria.

For the GRN model which is theoretically predicted and experimentally investigated in $Escherichia\ coli$ in [61], the genetic network is composed of three repressilators ($lacl, tetR, cl$) which forms a cyclic negative feedback loop, each repressor protein inhibits the transcription of its downstream repressor gene, as shown in Fig. 3.2, the protein of $lacl$ represses the gene transcription of $tetR$, and the protein of $tetR$ inhibits the gene transcription of cl simultaneously, finally, the transcription of $lacl$ is inhibited by cl, that completes the cycle.

The kinetics of the genetic network are modelled as the GRN (3.1.2) with the

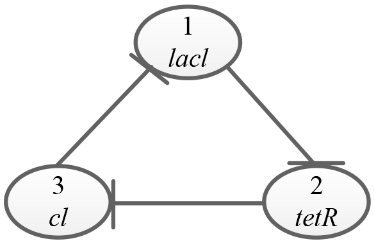

Figure 3.2: The repressilator network.

following parameters [87]:

$$
\begin{aligned}
A &= \text{diag}\{3,3,3\},\ C=\text{diag}\{2.5,2.5,2.5\}\\
W &= \begin{bmatrix} 0 & 0 & -2.5\\ -2.5 & 0 & 0\\ 0 & -2.5 & 0\end{bmatrix},\ D=\begin{bmatrix}0.8\\0.8\\0.8\end{bmatrix}\\
b_i(x) &= \frac{x^2}{1+x^2},\ i=1,2,\cdots,n
\end{aligned}
$$

It follows from (3.1.8) and (3.3.2) that

$$\Sigma = diag\{3\sqrt{3}/8,\ 3\sqrt{3}/8,\ 3\sqrt{3}/8\} \tag{3.4.1}$$

1) Calculation results: The first case study is that the changing rates of the time-varying delays are measurable. The Theorem 3 can directly employ to discuss the stability of the GRN 3.1.10 which satisfy delays satisfy (3.1.4).

Assume that $\sigma_1 = 0.1, \sigma_2 = 0.3, \sigma_d = 0.7, \tau_d = 1.5$ [87],

the MADBs of τ_2 with respect to various τ_1 obtained by the proposed criteria are given in Table 3.1, where the MADBs reported in the literature are also listed for comparison.

The second case study is that the changing rates of the time-varying delays are nonmeasurable. The Theorem 3 can directly employ to discuss the stability of the GRN 3.1.10 which satisfy delays satisfy (3.1.5). Assume that $\sigma_1 = 1, \sigma_2 = 2$, the MADBs of τ_2 with respect to various τ_1 obtained by the proposed criteria, together with the ones provided by the least literature [87], are given in Table 3.2.

Moveover, the NoVs of criteria reported in the least literature [87] and that of criteria established in this paper are also given in tables to compare the computation complexity.

From the results in the tables, it can be easily found that the proposed stability criteria can provide the larger MADBs for two cases than those given in the existing literature. It shows that the proposed criteria are indeed less conservative than the ones reported in the literature. On the other hand, it is found that the NoV of the proposed criteria (Theorem 2 and Corollary 1) is smaller than the one reported in [87], $(40.5n^2 + 16.5n) - (32n^2 + 22n) = 8.5n^2 - 5.5n > 0$ and $(38n^2 + 15n) - (29.5n^2 + 20.5n) = 8.5n^2 - 5.5n > 0$ for any n. Both of those observations show the advantages of the proposed criterion.

Table 3.1: The MADBs of τ_2 for various τ_1 and the NoVs of various criteria.

Criteria	NoVs	τ_2 for given τ_1		
		τ_1= 0.1	τ_1= 0.5	τ_1= 1
[85, 109, 110]	—	< 5.5	< 5.9	< 6.4
Theorem 1 [87]	$40.5n^2 + 16.5n$	5.5	5.91	6.41
Theorem 2	$32n^2 + 22n$	9.2681	9.6682	10.1681

Table 3.2: The MADBs of τ_2 for various τ_1 and the NoVs of various criteria.

Criteria	NoVs	τ_2 for given τ_1		
		τ_1= 0	τ_1= 1	τ_1= 2
Corollary 1 [87]	$38n^2 + 15n$	2.3101	3.3101	4.3102
Corollary 1	$29.5n^2 + 20.5n$	4.1647	5.1647	6.1646

2) Simulation verification: From the given parameters, the equilibrium points of the GRN can be obtained as

$$m^* = [0.7840, 0.7840, 0.7840],\ p^* = [0.2509, 0.2509, 0.2509]$$

Simulation studies for the following two types of time-varying delays are carried out:

- Case I: the initial condition $m(t) = [0.70, 0.85, 0.80]^T$, $t \in [-10.1681, 0]$ and $p(t) = [0.15, 0.20, 0.30]^T$, $t \in [-0.3, 0]$, and the following delays satisfying

$\sigma_1 = 0.1, \sigma_2 = 0.3, \sigma_d = 0.7, \tau_1 = 1, \tau_2 = 10.1681, \tau_d = 1.5$:

$$\begin{cases} \tau(t) = 9.1681 \sin^2(0.1636t) + 1 \\ \sigma(t) = 0.2 \sin^2(3.5t) + 0.1 \end{cases} \tag{3.4.2}$$

- Case II: the initial condition $m(t) = [0.70, 0.85, 0.80]^T$, $t \in [-6.1646, 0]$ and $p(t) = [0.15, 0.20, 0.30]^T$, $t \in [-2, 0]$, and the random delays satisfying $\sigma_1 = 1, \sigma_2 = 2, \tau_1 = 2, \tau_2 = 6.1646$.

Based on Tables 3.1 and 3.2, the GRN with the above delays respectively is stable. The trajectories of the concentrations of mRNA and protein are shown in Figs. 3.3-3.4. The results show that they are stable at their equilibrium points.

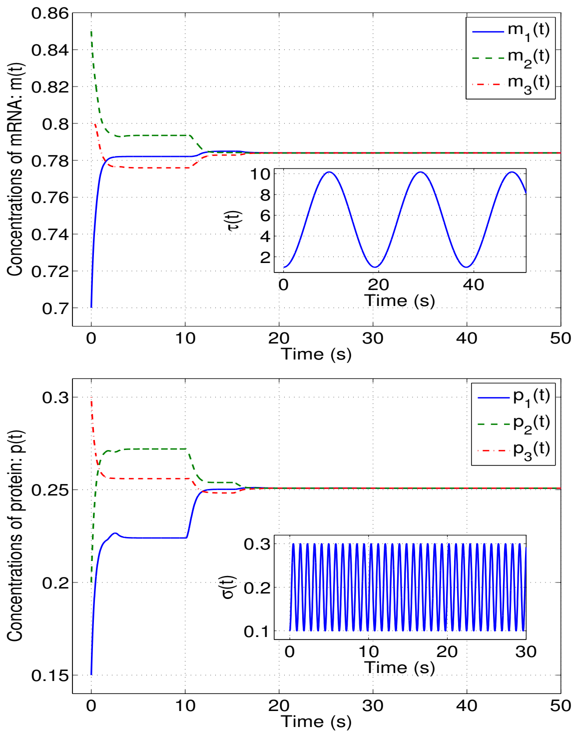

Figure 3.3: The trajectories of concentrations of mRNA and protein for Case I.

3.5 Conclusion

This chapter has investigated the stability of the GRN with time-varying delay, and its contributions have been revealed from two aspects. The novel WTDII has

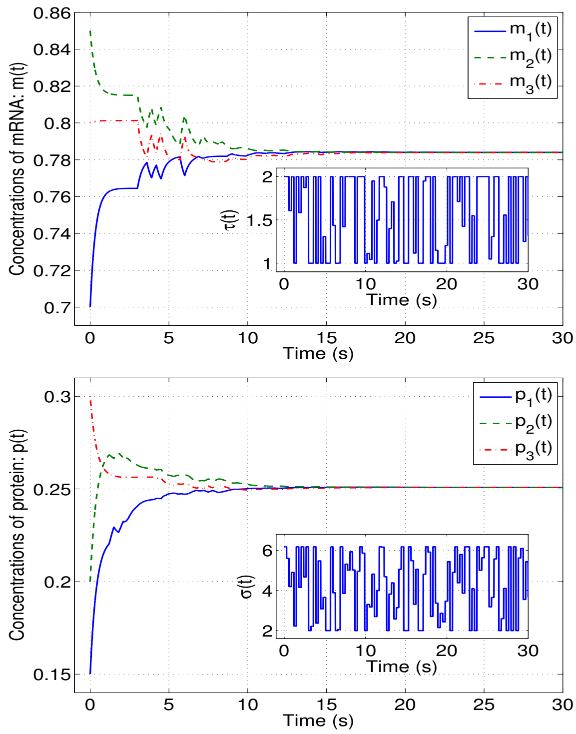

Figure 3.4: The trajectories of concentrations of mRNA and protein for Case II.

been developed for the estimation of the double integral terms, and it has been also proved to be tighter than the widely used JBDII and the recently developed WBDII for the same task. Then, benefit from the WTDII, two LMI-based stability criteria with less conservatism have been derived for checking the stability of the GRN with time delays. Finally, the advantages of the proposed inequality and the established criteria have been verified through an example.

Chapter 4

Improved delay-dependent stability analysis of digital filters with time-varying delay and generalized overflow arithmetic

4.1 Introduction

As an effective device that produces the desired discrete-time output signal from the original input signal will involve undesired information, the digital filter becomes a necessary element of everyday electronics like radios, cell phones, and stereo receivers. Due to its large-scale applications in many areas such as radar, image processing, telecommunications, signal processing and chemical pollution modeling, the analysis of properties and performances of the digital filters has attracted considerable attention in the past few decades (see [111–113] and references therein). Therefore, the analysis of digital filters is helpful for their implementation.

During the practical implementation of a digital filter via hardware using the fixed-point arithmetic, the complex operations within the hardware require increasing wordlength to deal with the signals. On the other side, because of the limitation of register length, the quantization and overflow correction mechanisms are commonly required to reduce the wordlength [111]. Therefore, nonlinearities, including magnitude truncation, roundoff, or value truncation due to quantization and saturation, zeroing, two's complement, or triangular for overflow, are unavoidable [114,115]. Those nonlinearities in turn lead to undesirable behaviors, for example, performance degradation, oscillations and limited cycles [116]. Many scholars have investigated the stability problem of digital filters with the consideration of different nonlinearities. Under the consideration that the influence of quantization and that of overflow can be studied separately if the total number of quantization steps (or internal wordlength) is large sufficiently [117], a stability criterion of fixed-point state-space digital filters with saturation arithmetic was presented in [118] and was improved in [119]. LMI based stability criteria for direct form digital filters utilizing single saturation nonlinearity were developed in [117] and [120]. In [121], the stability analysis of fixed-point state-space digital filters with generalized saturation nonlinearity was discussed. Due to the fact that the hardware implementation of saturation arithmetic is more expensive than that of two's complement arithmetic, the digital filters using two's complement arithmetic were investigated and the stability criteria for such type of filters were also proposed. A global asymptotic stability criterion in the form of LMIs for fixed-point state-space digital filters using two's complement arithmetic was presented in [122] and its simple form and improved form were respectively given in [123] and [124]. Stability criteria for direct-form digital filters utilizing two's complement nonlinearity were proposed in [125] and [126]. By taking into account the possibility of influence of both overflow and quantization, stability criteria for digital filters with different combinations of overflow and quantization nonlinearities were established [115, 127, 128]. Furthermore, in order to analyze the possible effect of external disturbances, different performances of digital filters were successively investigated, for example, the H_∞, $l_2 - l_\infty$, and l_∞ performances [129–132], the input-to-state stability (ISS) and the input/output-to-state stability (IOSS) analysis [133], the dissipativity analysis [134, 135], local stability analysis [136, 137], and etc.

Besides the nonlinearities and external disturbances mentioned above, time delay is frequently encountered in many systems [138,139] and also exists in digital filters. For example, a causal digital filter with a fixed order and cutoff frequency will delay different frequency signals [140]. For the discrete-time systems with quantization/overflow nonlinearities and time delays, a LMI based stability criterion was proposed in [141], in which the criterion was delay-independent and conservative. For the digital filters with time delays, saturation nonlinearity and externally disturbances, the method to check the exponential stability and H_∞ performance was proposed in [113]. For the digital filters with time delays, parameter uncertainties and both the quantization and overflow nonlinearities, the robust stability criterion was derived in [111]. However, the delays concerned in [111] and [113] are all constant. In [142], a delay-dependent criterion was developed by using free-weighting matrix approach for the asymptotic stability of a class of uncertain discrete-time state-delayed systems with the combination of quantization and overflow nonlinearities. For the digital filters with generalized overflow nonlinearity, a stability condition depends not only on the delay bounds but also on the bounds of nonlinear function was reported in [140] with the help of Jensen-based summation inequality. In [143], the extended dissipativity analysis for digital filters with a time-varying delay and Markovian jumping parameters was investigated and a criterion was given by putting forward a general form of nonlinearity functions and employing the reciprocally convex combination approach. While the criteria reported in [140, 142, 143] are all based on a simple Lyapunov functional, which is simple but conservative. In [112], the ISS problem of digital filters in the presence of both external disturbance and time-varying delay was discussed and stability criteria were derived by using simple Lyapunov functionals and Jensen-like summation inequality.

Based on the above discussions, there still remains room for further investigation on the analysis of digital filters with both the nonlinearity and the time-varying delay. On the one hand, from the research on digital filters point of view, there are only a few works on digital filters considering the time-varying delay [112,140,142,143], and the techniques used therein are all conservative in comparison to the ones developed for the time-delay systems. In fact, many more effective methods have been developed for dealing with time-varying delays, such as augmented Lyapunov functionals [144–146], new inequalities [147–150], extended reciprocally convex matrix inequalities [151–153], etc. On the other hand, from the viewpoint of techniques dealing with the discrete-time delay systems, every term in the Lyapunov functionals is usually required to positive in order to guarantee the positive-definiteness of the functionals. Such strict requirement leads to the conservatism.

In this chapter, the stability analysis problem of digital filters with generalized overflow nonlinearity and a time-varying delay is further investigated. The main contribution of the paper is that a new delay and nonlinearity bounds dependent stability criterion with less conservatism is developed,

and the proposed criterion can provide more accurate delay stable region (namely, the allowably maximal delay region such that the stability of the digital filter with any delay belonging to such region is guaranteed).

The advantage of the proposed stability criterion is illustrated based on several numerical examples. The main techniques, different from the previous publications, are summarized as follows.

- The first aspect is on the construction of the Lyapunov functional. Several augmented terms, especially the one with the information of overflow nonlinearity, are introduced into the Lyapunov functional and the condition of positive-definiteness of functional is relaxed by requiring the sum of all terms, instead of each term, be positive. Those treatments can provide extra freedom for the feasibility of the obtained criterion.

- The second aspect relies on the estimation of the forward difference of the functional. A new lemma is developed to introduce new cross terms for constructing the link between the delay states and the overflow nonlinear function. Moreover, several methods (such as the Wirtinger-based summation inequality, the extended reciprocally convex matrix inequality, and zero-value equations), which are not used in the literature on delayed digital filters, are applied to estimate the forward difference of the functional as accurate as possible.

Notations: Throughout this chapter, $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ respectively denote the set of all the n-dimensional vectors and that of all the $m \times n-$dimensional real matrices; $||\cdot||$ denotes the Euclidean norm; the superscripts T and -1 stand for the transpose and the inverse of a matrix, respectively; $\mathrm{diag}\{\cdots\}$ denotes a block-diagonal matrix; $P > 0$ (≥ 0) means that P is a positive-definite (semi-positive-definite) symmetric matrix; I and 0 represent the identity matrix and the zero-matrix with appropriate dimensions, respectively; the symmetric term in a symmetric matrix is denoted by $*$; and $\mathrm{Sym}\{X\} = X + X^T$.

4.2 Model of digital filter and Wirtinger-based inequality

Consider the following digital filter with a time-varying delay:

$$\begin{cases} x(k+1) = f(y(k)), \\ y(k) = Ax(k) + A_d x(k - \tau(k)), \\ x(k) = \phi(k), \ \ k \in \{-\tau_2, \ldots, 0\}, \end{cases} \tag{4.2.1}$$

where $x(k) = [x_1(k), x_2(k), \cdots, x_n(k)]^T \in \mathbb{R}^n$ is the state vector; $\phi(k) = [\phi_1(k), \phi_2(k), \cdots, \phi_n(k)]^T \in \mathbb{R}^n$ is the initial condition with $|\phi_i(k)| \leq 1$, $y(k) = [y_1(k), y_2(k), \cdots, y_n(k)]^T \in \mathbb{R}^n$ is the filter output vector; $\tau(k)$ is the time-varying delay satisfying

$$\tau_1 \leq \tau(k) \leq \tau_2, \tag{4.2.2}$$

with τ_1 and τ_2 being constant; A and A_d are the known interconnection weight matrices; the nonlinearity function $f(\cdot)$: $\mathbb{R}^n \rightarrow \mathbb{R}^n$ is defined as follows [140]

$$\begin{cases} -1 \leq l_i \leq f_i(y_i(k)) \leq l_{1i} \leq 1, & y_i(k) > 1, \\ f_i(y_i(k)) = y_i(k), & -1 \leq y_i(k) \leq 1, \\ -1 \leq -l_{2i} \leq f_i(y_i(k)) \leq -l_i \leq 1, & y_i(k) < -1, \end{cases} \tag{4.2.3}$$

with $i = 1, 2, \ldots, n$, l_i, l_{1i} and l_{2i} being known real scalars.

Remark 9. *As mentioned in [140], the nonlinear relationships shown in* (4.2.3) *include various overflow arithmetics by fixing the values of l_i, l_{1i}, and l_{2i}. For example,* (4.2.3) *gives saturation nonlinearity for $l_i = l_{1i} = l_{2i} = 1$;* (4.2.3) *indicates* *zeroing nonlinearity for* $l_i = l_{1i} = l_{2i} = 0$*; and* (4.2.3) *shows two's complement nonlinearity for* $l_i = -1, l_{1i} = l_{2i} = 1$*. That is, the stability criterion developed in this chapter can be used to check the stability of digital filters with the above three types of overflow nonlinearities.*

In order to analyze the influence of the time-varying delay on the stability of digital filter (4.2.1), this chapter aims to develop a less conservative delay-dependent stability criterion. The following Lemmas which are used for handling time delays are given as follows.

Lemma 10. *(Wirtinger-based inequality [147]). For a given positive definite matrix* R*, integers* $b \geq a$*, any sequence of discrete-time variable* $x : \mathcal{Z}[a, b] \to \mathbb{R}^n$*, the following inequality holds:*

$$(b-a)\sum_{i=a}^{b-1} \triangle x^T(i) R \triangle x(i) \geq \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix}^T \begin{bmatrix} R & 0 \\ 0 & 3R \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix}, \tag{4.2.4}$$

where

$$\begin{aligned} \triangle x(k) &= x(k+1) - x(k), \\ \chi_1 &= x(b) - x(a), \\ \chi_2 &= x(b) + x(a) - \frac{2}{b-a+1}\sum_{i=a}^{b} x(i). \end{aligned}$$

Lemma 11. *(Jensen-based inequality [154]). For a given positive definite matrix* R*, integers* $b \geq a$*, any sequence of discrete-time variable* $x : \mathcal{Z}[a, b] \to \mathbb{R}^n$*, the following inequality holds:*

$$\sum_{i=a}^{b-1} x^T(i) R x(i) \geq \frac{1}{b-a}\left(\sum_{i=a}^{b-1} x(i)\right)^T R \left(\sum_{i=a}^{b-1} x(i)\right). \tag{4.2.5}$$

Lemma 12. *(Extended reciprocally convex matrix inequality [151,152]). For a real scalar* $0 < \alpha < 1$*, positive-definite symmetric matrices* $X, Y \in \mathbb{R}^{n\times n}$*, and any matrix* $N \in \mathbb{R}^{n\times n}$*, the following inequality holds:*

$$\begin{bmatrix} \frac{1}{\alpha}X & 0 \\ 0 & \frac{1}{1-\alpha}Y \end{bmatrix} \geq \begin{bmatrix} X + (1-\alpha)T_1 & N \\ * & Y + \alpha T_2 \end{bmatrix}, \tag{4.2.6}$$

where $T_1 = X - NY^{-1}N^T$ *and* $T_2 = Y - N^T X^{-1} N$*.*

The following Lemmas related to the overflow nonlinearity are given.

Lemma 13. *[140] Let* $\hat{l}_i = \min\{l_i, 0\}$*, then the following inequality holds for nonlinear functions* $f_i(\cdot)$ *satisfying condition (4.2.3):*

$$\Big[y_i(k) - f_i(y_i(k))\Big]\Big[f_i(y_i(k)) - \hat{l}_i y_i(k)\Big] \geq 0. \tag{4.2.7}$$

Lemma 14. *[124] For a given digital filter (4.2.1) satisfying condition (4.2.3), if there exists matrix* $S = \mathrm{diag}(s_1, s_2, \ldots, s_n) > 0$ *and any matrices* $M = [m_{ij}]_{n\times n}$*,* $N = [n_{ij}]_{n\times n}$ *satisfying*

$$s_i \geq \sum_{j=1}^{n} |m_{ji}| + \sum_{j=1}^{n} |n_{ji}|, \quad i = 1, 2, \ldots, n, \tag{4.2.8}$$

then the following inequality holds:

$$\Big[y^T(k)S + x^T(k)M + f^T(y(k))N\Big]\Big[y(k) - f(y(k))\Big] \geq 0. \tag{4.2.9}$$

Lemma 15. *For a given digital filter (4.2.1) satisfying condition (4.2.3), if there exists matrices* $S_1 = \mathrm{diag}(s_{11}, s_{12}, \ldots, s_{1n}) > 0$*,* $S_2 = \mathrm{diag}(s_{21}, s_{22}, \ldots, s_{2n}) > 0$*, and any matrices* $M_1 = [m_{1ij}]_{n\times n}$ *and* $M_2 = [m_{2ij}]_{n\times n}$ *satisfying*

$$s_{1i} \geq \sum_{j=1}^{n} |m_{1ji}|, \quad i = 1, 2, \ldots, n, \tag{4.2.10}$$

$$s_{2i} \geq \sum_{j=1}^{n} |m_{2ji}|, \quad i = 1, 2, \ldots, n, \tag{4.2.11}$$

then the following inequalities hold:

$$\Big[y^T(k)S_1 + x^T(k-\tau(k))M_1\Big]\Big[y(k) - f(y(k))\Big] \geq 0, \tag{4.2.12}$$

$$\Big[y^T(k)S_2 + x^T(k-1)M_2\Big]\Big[y(k) - f(y(k))\Big] \geq 0. \tag{4.2.13}$$

Proof: It is easy to find that (4.2.12) holds if $|y_i(k)| \leq 1$ (i.e., $y_i(k) = f_i(y_i(k))$ based on (4.2.3)). For the case of $|y_i(k)| > 1$, the left-hand side of (4.2.12) can be rewritten as

$$\begin{aligned} & \sum_{i=1}^{n} \Big[y_i(k)s_{1i} + \sum_{j=1}^{n} x_j(k-\tau(k))m_{1ji}\Big]\Big[y_i(k) - f_i(y_i(k))\Big] \\ = \; & \sum_{i=1}^{n} y_i^2(k)\Big[s_{1i} + \sum_{j=1}^{n} \frac{x_j(k-\tau(k))}{y_i(k)} m_{1ji}\Big]\Big[1 - \frac{f_i(y_i(k))}{y_i(k)}\Big]. \end{aligned} \tag{4.2.14}$$

Then, it follows from $|y_i(k)| > 1$, $|f_j(y_j(k))| \leq 1$, $|x_j(k-\tau(k))| \leq 1$ (obtained from (4.2.1)) and (4.2.10) that

$$1-\frac{f_i(y_i(k))}{y_i(k)} > 0, \tag{4.2.15}$$

and

$$\begin{aligned} & s_{1i} + \sum_{j=1}^{n} \frac{x_j(k-\tau(k))}{y_i(k)} m_{1ji} \\ \geq\ & s_{1i} - \sum_{j=1}^{n} \left| \frac{x_j(k-\tau(k))}{y_i(k)} \right| |m_{1ji}| \\ \geq\ & s_{1i} - \sum_{j=1}^{n} |m_{1ji}| \\ \geq\ & 0. \end{aligned} \tag{4.2.16}$$

Combining (4.2.14), (4.2.15), and (4.2.16) leads that (4.2.12) holds for the case of $|y_i(k)| > 1$. Thus, (4.2.12) holds for all $y_i(k)$.

Similar, the holding of (4.2.13) can be proved if (4.2.11) holds. ■

Remark 10. *Compared with (4.2.9) used in [112, 140, 143], in which only delay-free states, $y(k)$ and $x(k)$, are linked with the nonlinear function $f(y(k))$, (4.2.12) and (4.2.13) in Lemma 15 introduce many additional cross terms related to the delayed states, $x(k-\tau(k))$ and $x(k-1)$, and overflow nonlinear function, $f(y(k))$, which constructs the link between delayed states and overflow nonlinear function. In fact, the holding of (4.2.7), (4.2.9), (4.2.12) and (4.2.13) is based on the special feature of nonlinear function caused by overflow correction mechanism. The usage of those information is an important difference in comparison to the linear discrete-time delayed systems [147–150] or traditional Lur'e nonlinear discrete-time delayed systems [155, 156], and it is also the one of important treatments for reducing the conservatism.*

4.3 Stability analysis of digital filter with delay and overflow arithmetic

In this section, a new delay-dependent stability criterion is derived by constructing an augmented Lypunov functional and using several new techniques to estimate the forward difference of the functional.

Before giving the main results, the following notations are defined to simplify the expression of the proof of stability criterion.

$$
\begin{aligned}
\upsilon_1(k) &= \sum_{i=k-\tau_1}^{k-1} x(i), \\
\upsilon_2(k) &= \sum_{i=k-\tau(k)}^{k-\tau_1-1} x(i), \\
\upsilon_3(k) &= \sum_{i=k-\tau_2}^{k-\tau(k)-1} x(i), \\
\upsilon_4(k) &= \sum_{i=k-\tau(k)}^{k-\tau_1} \frac{x(i)}{\tau(k)-\tau_1+1}, \\
\upsilon_5(k) &= \sum_{i=k-\tau_2}^{k-\tau(k)} \frac{x(i)}{\tau_2-\tau(k)+1}, \\
\xi(k) &= \Big[f^T(y(k)), \\
&\quad x^T(k),\ x^T(k-\tau_1),\ x^T(k-\tau(k)),\ x^T(k-\tau_2), \\
&\quad \upsilon_1^T(k),\ \upsilon_2^T(k),\ \upsilon_3^T(k),\ \upsilon_4^T(k),\ \upsilon_5^T(k),\ x^T(k-1) \Big]^T, \\
e_1 &= [I_n, 0_{n\times 10n}], \quad f(y(k)) = e_1\xi(k), \\
e_2 &= [0_{n\times n}, I_n, 0_{n\times 9n}], \quad x(k) = e_2\xi(k), \\
e_3 &= [0_{n\times 2n}, I_n, 0_{n\times 8n}], \quad x(k-\tau_1) = e_3\xi(k), \\
e_4 &= [0_{n\times 3n}, I_n, 0_{n\times 7n}], \quad x(k-\tau(t)) = e_4\xi(k), \\
e_5 &= [0_{n\times 4n}, I_n, 0_{n\times 6n}], \quad x(k-\tau_2) = e_5\xi(k), \\
e_6 &= [0_{n\times 5n}, I_n, 0_{n\times 5n}], \quad \upsilon_1(k) = e_6\xi(k), \\
e_7 &= [0_{n\times 6n}, I_n, 0_{n\times 4n}], \quad \upsilon_2(k) = e_7\xi(k), \\
e_8 &= [0_{n\times 7n}, I_n, 0_{n\times 3n}], \quad \upsilon_3(k) = e_8\xi(k),
\end{aligned}
$$

$$
\begin{aligned}
e_9 &= [0_{n\times 8n}, I_n, 0_{n\times 2n}], \quad \upsilon_4(k) = e_9\xi(k),\\
e_{10} &= [0_{n\times 9n}, I_n, 0_{n\times n}], \quad \upsilon_5(k) = e_{10}\xi(k),\\
e_{11} &= [0_{n\times 10n}, I_n], \quad x(k-1) = e_{11}\xi(k),\\
e_i &= [0_{n\times(i-1)n}, I_n, 0_{n\times(11-i)n}], i = 1, 2, \cdots, 11,\\
\upsilon_6(k) &= \sum_{i=k-\tau_1}^{k-1} x(i),\\
\upsilon_7(k) &= \sum_{i=k-\tau_2}^{k-\tau_1-1} x(i),\\
\bar{\xi}(k) &= \left[x^T(k), \upsilon_6^T(k), \upsilon_7^T(k), x^T(k-\tau_1), x^T(k-\tau_2), x^T(k-1)\right]^T,\\
\bar{e}_1 &= [I_n, 0_{n\times 5n}], \quad x(k) = \bar{e}_1\bar{\xi}(k),\\
\bar{e}_2 &= [0_{n\times n}, I_n, 0_{n\times 4n}], \quad \upsilon_6(k) = \bar{e}_2\bar{\xi}(k),\\
\bar{e}_3 &= [0_{n\times 2n}, I_n, 0_{n\times 3n}], \quad \upsilon_7(k) = \bar{e}_3\bar{\xi}(k),\\
\bar{e}_4 &= [0_{n\times 3n}, I_n, 0_{n\times 2n}], \quad x(k-h_1) = \bar{e}_4\bar{\xi}(k),\\
\bar{e}_5 &= [0_{n\times 4n}, I_n, 0_{n\times n}], \quad x(k-h_2) = \bar{e}_5\bar{\xi}(k),\\
\bar{e}_6 &= [0_{n\times 5n}, I_n], \quad x(k-1) = \bar{e}_6\bar{\xi}(k),\\
\bar{e}_i &= [0_{n\times(i-1)n}, I_n, 0_{n\times(6-i)n}], i = 1, 2, \cdots, 6.
\end{aligned}
$$

The delay-dependent stability criterion derived in this chapter is shown as follows.

Theorem 4. *For given scalars l_i, τ_1 and τ_2, digital filter (4.2.1) with time-varying satisfying (4.2.2) is asymptotically stable if there exists symmetric matrices P_1, P_2, Z, Q_1, Q_2, R_1, R_2, T_1, T_2, positive definite diagonal matrices S, S_1, S_2, D, and any matrices X, U_1, U_2, M, M_1, M_2, N, such that conditions (4.2.8), (4.2.10) and (4.2.11), and the following LMIs are feasible:*

$$R_i > 0, \quad i = 1, 2, \tag{4.3.1}$$

$$\hat{\Phi}_1 = \begin{bmatrix} 0 & 0 \\ 0 & R_2 \end{bmatrix} + Z > 0, \tag{4.3.2}$$

$$\hat{\Phi}_2 = \tau_{12}^2 Z + \begin{bmatrix} Q_1 & 0 \\ 0 & \tau_{12}^2 R_2 + \tau_1 R_1 \end{bmatrix} > 0, \tag{4.3.3}$$

$$\hat{\Phi}_3 = \tau_{12} Z + \begin{bmatrix} Q_2 & 0 \\ 0 & \tau_{12} R_2 \end{bmatrix} > 0, \tag{4.3.4}$$

$$\begin{aligned}\hat{\Phi}_4 = {} & \begin{bmatrix} \bar{e}_2 \\ \bar{e}_1 - \bar{e}_4 \end{bmatrix}^T \frac{\hat{\Phi}_2}{\tau_1} \begin{bmatrix} \bar{e}_2 \\ \bar{e}_1 - \bar{e}_4 \end{bmatrix} + \begin{bmatrix} \bar{e}_3 \\ \bar{e}_4 - \bar{e}_5 \end{bmatrix}^T \frac{\hat{\Phi}_3}{\tau_{12}} \begin{bmatrix} \bar{e}_3 \\ \bar{e}_4 - \bar{e}_5 \end{bmatrix} \\ & + \begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \end{bmatrix}^T P_1 \begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \end{bmatrix} + \begin{bmatrix} \bar{e}_6 \\ \bar{e}_1 \end{bmatrix}^T P_2 \begin{bmatrix} \bar{e}_6 \\ \bar{e}_1 \end{bmatrix} \\ & + \tau_1(\tau_1 - 1)[\bar{e}_1 - \bar{e}_{11}]^T R_1 [\bar{e}_1 - \bar{e}_{11}] \\ > {} & 0, \end{aligned} \tag{4.3.5}$$

$$\begin{bmatrix} \Psi|_{\tau(k)=\tau_1} & E_2^T X \\ * & -\Xi_1 \end{bmatrix} < 0, \tag{4.3.6}$$

$$\begin{bmatrix} \Psi|_{\tau(k)=\tau_2} & E_3^T X^T \\ * & -\Xi_2 \end{bmatrix} < 0, \tag{4.3.7}$$

$$Z + \begin{bmatrix} 0 & T_1 \\ T_1 & T_1 \end{bmatrix} > 0, \tag{4.3.8}$$

$$Z + \begin{bmatrix} 0 & T_2 \\ T_2 & T_2 \end{bmatrix} > 0, \tag{4.3.9}$$

where

$$\begin{aligned} \Psi &= \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5 + \Phi_6 - \Phi_7 - \Phi_8, \\ \Phi_1 &= \Pi_1^T P_1 \Pi_1 - \Pi_2^T P_1 \Pi_2 + \Pi_3^T P_2 \Pi_3 - \Pi_4^T P_2 \Pi_4, \\ \Pi_1 &= \begin{bmatrix} e_1 \\ e_6 + e_2 - e_3 \\ e_7 + e_8 + e_3 - e_5 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} e_2 \\ e_6 \\ e_7 + e_8 \end{bmatrix}, \\ \Pi_3 &= \begin{bmatrix} e_2 \\ e_1 \end{bmatrix}, \quad \Pi_4 = \begin{bmatrix} e_{11} \\ e_2 \end{bmatrix}, \\ \Phi_2 &= e_2^T Q_1 e_2 - e_3^T (Q_1 - Q_2) e_3 - e_5^T Q_2 e_5, \end{aligned}$$

$$
\begin{aligned}
\Phi_3 &= \Pi_5^T(\tau_1^2 R_1 + \tau_{12}^2 R_2)\Pi_5 + \tau_{12}^2 \begin{bmatrix} e_2 \\ \Pi_5 \end{bmatrix}^T Z \begin{bmatrix} e_2 \\ \Pi_5 \end{bmatrix}, \\
\Pi_5 &= e_1 - e_2, \\
\Phi_4 &= \tau_{12}\Big[e_3^T T_1 e_3 - e_4^T(T_1 - T_2)e_4 - e_5^T T_2 e_5\Big], \\
\Phi_5 &= \mathrm{Sym}\Big\{[\Pi_6 - e_1]^T D[e_1 - L\Pi_6]\Big\}, \\
&\quad +\mathrm{Sym}\Big\{[\Pi_6^T S + e_2^T M + e_1^T N][\Pi_6 - e_1]\Big\}, \\
&\quad +\mathrm{Sym}\Big\{[\Pi_6^T S_1 + e_4^T M_1][\Pi_6 - e_1]\Big\}, \\
&\quad +\mathrm{Sym}\Big\{[\Pi_6^T S_2 + e_{11}^T M_2][\Pi_6 - e_1]\Big\}, \\
\Pi_6 &= Ae_2 + A_d e_4,
\end{aligned}
$$

$$
\begin{aligned}
\Phi_6 &= \mathrm{Sym}\Big\{e_g^T U_1[(\tau(k) - \tau_1 + 1)e_9 - e_7 - e_3]\Big\} \\
&\quad +\mathrm{Sym}\Big\{e_g^T U_2[(\tau_2 - \tau(k) + 1)e_{10} - e_8 - e_4]\Big\}, \\
\Phi_7 &= E_1^T \begin{bmatrix} R_1 & 0 \\ 0 & 3R_1 \end{bmatrix} E_1, \\
E_1 &= \begin{bmatrix} e_2 - e_3 \\ e_2 + e_3 - \frac{2}{\tau_1+1}(e_2 + e_6) \end{bmatrix}, \\
\Phi_8 &= \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} \frac{2\tau_2 - \tau(k) - \tau_1}{\tau_{12}}\Xi_1 & X \\ * & \frac{\tau_2 + \tau(k) - 2\tau_1}{\tau_{12}}\Xi_2 \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}, \\
E_2 &= \begin{bmatrix} e_7 \\ e_3 - e_4 \\ e_3 + e_4 - 2e_9 \end{bmatrix}, \\
E_3 &= \begin{bmatrix} e_8 \\ e_4 - e_5 \\ e_4 + e_5 - 2e_{10} \end{bmatrix},
\end{aligned}
$$

$$\Xi_1 = \left[\begin{array}{c|c} Z+\begin{bmatrix} 0 & T_1 \\ T_1 & R_2+T_1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 \end{bmatrix} & 3R_2 \end{array}\right],$$

$$\Xi_2 = \left[\begin{array}{c|c} Z+\begin{bmatrix} 0 & T_2 \\ T_2 & R_2+T_2 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 \end{bmatrix} & 3R_2 \end{array}\right],$$

$$\begin{aligned} e_i &= \left[0_{n\times(i-1)n},\ I_n,\ 0_{n\times(11-i)n}\right]\ (i=1,2,\cdots,11), \\ e_g &= \left[e_3,\ e_4,\ e_7,\ e_8,\ e_9,\ e_{10}\right], \\ \bar{e}_i &= [0_{n\times(i-1)n}, I_n, 0_{n\times(6-i)n}], i=1,2,\cdots,6, \\ L &= \mathrm{diag}\{\hat{l}_1,\hat{l}_2,\cdots,\hat{l}_n\}, \\ \hat{l}_i &= \min\{l_i,0\}. \end{aligned}$$

Proof: Firstly, choose the following functional candidate:

$$V(k)=\sum_{i=1}^{4} V_i(k), \tag{4.3.10}$$

where

$$\begin{aligned} V_1(k) &= \eta_1^T(k)P_1\eta_1(k)+\eta_2^T(k)P_2\eta_2(k), \\ V_2(k) &= \sum_{i=k-\tau_1}^{k-1} x^T(i)Q_1x(i)+\sum_{i=k-\tau_2}^{k-\tau_1-1} x^T(i)Q_2x(i), \\ V_3(k) &= \tau_1\sum_{i=-\tau_1}^{-1}\sum_{j=k+i}^{k-1}\triangle x^T(j)R_1\triangle x(j) \\ &\quad +\tau_{12}\sum_{i=-\tau_2}^{-\tau_1-1}\sum_{j=k+i}^{k-1}\triangle x^T(j)R_2\triangle x(j), \\ V_4(k) &= \tau_{12}\sum_{i=-\tau_2}^{-\tau_1-1}\sum_{j=k+i}^{k-1}\eta_3^T(j)Z\eta_3(j), \end{aligned}$$

with P_1, P_2, Z, Q_1, Q_2, R_1, and R_2 being symmetric matrices, $R_i > 0,\ i = 1, 2$, and

$$\begin{aligned}
&\tau_{12} = \tau_2 - \tau_1 \\
&\eta_1(k) = \left[x^T(k),\ \sum_{i=k-\tau_1}^{k-1} x^T(i),\ \sum_{i=k-\tau_2}^{k-\tau_1-1} x^T(i)\right]^T, \\
&\eta_2(k) = \left[x^T(k-1),\ f^T(y(k-1))\right]^T, \\
&\eta_3(i) = \left[x^T(i),\ \triangle x^T(i)\right]^T.
\end{aligned}$$

The second step is to find the conditions which can ensure the positive-definiteness of functional (4.3.10). Based on $R_1 > 0$ in (4.3.1), the following holds

$$\begin{aligned}
& \tau_1 \sum_{i=-\tau_1}^{-1} \sum_{j=k+i}^{k-1} \triangle x^T(j) R_1 \triangle x(j) \\
= \ & \tau_1 \sum_{j=k-\tau_1}^{k-1} \triangle x^T(j) R_1 \triangle x(j) \\
& + \tau_1 \sum_{i=-\tau_1+1}^{-1} \sum_{j=k+i}^{k-1} \triangle x^T(j) R_1 \triangle x(j) \\
> \ & \tau_1 \sum_{j=k-\tau_1}^{k-1} \triangle x^T(j) R_1 \triangle x(j) \\
& + \tau_1 \sum_{i=-\tau_1+1}^{-1} \sum_{j=k-1}^{k-1} \triangle x^T(j) R_1 \triangle x(j) \\
= \ & \tau_1 \sum_{j=k-\tau_1}^{k-1} \triangle x^T(j) R_1 \triangle x(j) \\
& + \tau_1 \sum_{i=-\tau_1+1}^{-1} \triangle x^T(k-1) R_1 \triangle x(k-1) \\
= \ & \tau_1(\tau_1 - 1)[x(k) - x(k-1)]^T R_1 [x(k) - x(k-1)] \\
& + \tau_1 \sum_{j=k-\tau_1}^{k-1} \triangle x^T(j) R_1 \triangle x(j). \qquad (4.3.11)
\end{aligned}$$

Based on $\hat{\Phi}_1 > 0$ in (4.3.2), the following is correct

$$
\begin{aligned}
& \tau_{12} \sum_{i=-\tau_2}^{-\tau_1-1} \sum_{j=k+i}^{k-1} \triangle x^T(j) R_2 \triangle x(j) + V_4(k) \\
= \; & \tau_{12} \sum_{i=-\tau_2}^{-\tau_1-1} \sum_{j=k+i}^{k-1} \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix}^T \hat{\Phi}_1 \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix} \\
= \; & \tau_{12} \sum_{j=k-\tau_2}^{k-1} \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix}^T \hat{\Phi}_1 \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix} \\
& + \tau_{12} \sum_{i=-\tau_2+1}^{-\tau_1-1} \sum_{j=k+i}^{k-1} \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix}^T \hat{\Phi}_1 \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix} \\
> \; & \tau_{12} \sum_{j=k-\tau_2}^{k-1} \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix}^T \hat{\Phi}_1 \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix} \\
& + \tau_{12} \sum_{i=-\tau_2+1}^{-\tau_1-1} \sum_{j=k-\tau_1}^{k-1} \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix}^T \hat{\Phi}_1 \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix} \\
= \; & \tau_{12} \sum_{j=k-\tau_1}^{k-1} \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix}^T \hat{\Phi}_1 \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix} \\
& + \tau_{12} \sum_{j=k-\tau_2}^{k-\tau_1-1} \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix}^T \hat{\Phi}_1 \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix} \\
& + \tau_{12}(\tau_{12}-1) \sum_{j=k-\tau_1}^{k-1} \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix}^T \hat{\Phi}_1 \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix} \\
= \; & \tau_{12}^2 \sum_{j=k-\tau_1}^{k-1} \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix}^T \hat{\Phi}_1 \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix} \\
& + \tau_{12} \sum_{j=k-\tau_2}^{k-\tau_1-1} \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix}^T \hat{\Phi}_1 \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix}.
\end{aligned}
\tag{4.3.12}
$$

Combining (4.3.10), (4.3.11), and (4.3.12) yields

$$
\begin{aligned}
& V_2(k)+V_3(k)+V_4(k) \\
> & \sum_{i=k-\tau_1}^{k-1} x^T(i)Q_1x(i)+\sum_{i=k-\tau_2}^{k-\tau_1-1} x^T(i)Q_2x(i) \\
& +\tau_1 \sum_{j=k-\tau_1}^{k-1} \triangle x^T(j)R_1\triangle x(j) \\
& +\tau_1(\tau_1-1)[x(k)-x(k-1)]^T R_1[x(k)-x(k-1)] \\
& +\tau_{12}^2 \sum_{j=k-\tau_1}^{k-1} \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix}^T \hat{\Phi}_1 \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix} \\
& +\tau_{12} \sum_{j=k-\tau_2}^{k-\tau_1-1} \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix}^T \hat{\Phi}_1 \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix} \\
= & \tau_1(\tau_1-1)[x(k)-x(k-1)]^T R_1[x(k)-x(k-1)] \\
& +\sum_{j=k-\tau_1}^{k-1} \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix}^T \hat{\Phi}_2 \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix} \\
& +\sum_{j=k-\tau_2}^{k-\tau_1-1} \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix}^T \hat{\Phi}_3 \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix},
\end{aligned}
\tag{4.3.13}
$$

where $\hat{\Phi}_2$ and $\hat{\Phi}_3$ are defined in (4.3.3) and (4.3.4), respectively. Furthermore, using $\hat{\Phi}_2 > 0, \hat{\Phi}_3 > 0$ and applying (4.2.5) to estimate the summation terms in (4.3.13) yield

$$
\begin{aligned}
& \sum_{j=k-\tau_1}^{k-1} \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix}^T \hat{\Phi}_2 \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix} \\
& \quad +\sum_{j=k-\tau_2}^{k-\tau_1-1} \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix}^T \hat{\Phi}_3 \begin{bmatrix} x(j) \\ \triangle x(j) \end{bmatrix} \\
> & \begin{bmatrix} \sum\limits_{i=k-\tau_1}^{k-1} x(i) \\ x(k)-x(t-\tau_1) \end{bmatrix}^T \frac{\hat{\Phi}_2}{\tau_1} \begin{bmatrix} \sum\limits_{i=k-\tau_1}^{k-1} x(i) \\ x(k)-x(t-\tau_1) \end{bmatrix} \\
& +\begin{bmatrix} \sum\limits_{i=k-\tau_2}^{k-\tau_1-1} x(i) \\ x(t-\tau_1)-x(t-\tau_2) \end{bmatrix}^T \frac{\hat{\Phi}_3}{\tau_{12}} \begin{bmatrix} \sum\limits_{i=k-\tau_2}^{k-\tau_1-1} x(i) \\ x(t-\tau_1)-x(t-\tau_2) \end{bmatrix}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& V_1(k)+V_2(k)+V_3(k)+V_4(k) \\
> & \begin{bmatrix} x(k) \\ \sum_{i=k-\tau_1}^{k-1} x(i) \\ \sum_{i=k-\tau_2}^{k-\tau_1-1} x(i) \end{bmatrix}^T P_1 \begin{bmatrix} x(k) \\ \sum_{i=k-\tau_1}^{k-1} x(i) \\ \sum_{i=k-\tau_2}^{k-\tau_1-1} x(i) \end{bmatrix} \\
& + \begin{bmatrix} x(k-1) \\ f(y(k-1)) \end{bmatrix}^T P_2 \begin{bmatrix} x(k-1) \\ f(y(k-1)) \end{bmatrix} \\
& +\tau_1(\tau_1-1)[x(k)-x(k-1)]^T R_1 [x(k)-x(k-1)] \\
& + \begin{bmatrix} \sum_{i=k-\tau_1}^{k-1} x(i) \\ x(k)-x(t-\tau_1) \end{bmatrix}^T \frac{\hat{\Phi}_2}{\tau_1} \begin{bmatrix} \sum_{i=k-\tau_1}^{k-1} x(i) \\ x(k)-x(t-\tau_1) \end{bmatrix} \\
& + \begin{bmatrix} \sum_{i=k-\tau_2}^{k-\tau_1-1} x(i) \\ x(t-\tau_1)-x(t-\tau_2) \end{bmatrix}^T \frac{\hat{\Phi}_3}{\tau_{12}} \begin{bmatrix} \sum_{i=k-\tau_2}^{k-\tau_1-1} x(i) \\ x(t-\tau_1)-x(t-\tau_2) \end{bmatrix} \\
= & \begin{bmatrix} \bar{e}_1\bar{\xi}(k) \\ \bar{e}_2\bar{\xi}(k) \\ \bar{e}_3\bar{\xi}(k) \end{bmatrix}^T P_1 \begin{bmatrix} \bar{e}_1\bar{\xi}(k) \\ \bar{e}_2\bar{\xi}(k) \\ \bar{e}_3\bar{\xi}(k) \end{bmatrix} + \begin{bmatrix} \bar{e}_6\bar{\xi}(k) \\ \bar{e}_1\bar{\xi}(k) \end{bmatrix}^T P_2 \begin{bmatrix} \bar{e}_6\bar{\xi}(k) \\ \bar{e}_1\bar{\xi}(k) \end{bmatrix} \\
& +\tau_1(\tau_1-1)[\bar{e}_1-\bar{e}_{11}]^T R_1 [\bar{e}_1-\bar{e}_{11}] \\
& + \begin{bmatrix} \bar{e}_2\bar{\xi}(k) \\ \bar{e}_1\bar{\xi}(k)-\bar{e}_4\bar{\xi}(k) \end{bmatrix}^T \frac{\hat{\Phi}_2}{\tau_1} \begin{bmatrix} \bar{e}_2\bar{\xi}(k) \\ \bar{e}_1\bar{\xi}(k)-\bar{e}_4\bar{\xi}(k) \end{bmatrix} \\
& + \begin{bmatrix} \bar{e}_3\bar{\xi}(k) \\ \bar{e}_4\bar{\xi}(k)-\bar{e}_5\bar{\xi}(k) \end{bmatrix}^T \frac{\hat{\Phi}_3}{\tau_{12}} \begin{bmatrix} \bar{e}_3\bar{\xi}(k) \\ \bar{e}_4\bar{\xi}(k)-\bar{e}_5\bar{\xi}(k) \end{bmatrix} \\
= & \bar{\xi}^T(k)\hat{\Phi}_4\bar{\xi}(k),
\end{aligned} \tag{4.3.14}
$$

where $\hat{\Phi}_4$ is defined in (4.3.5).

It follows from (4.3.5) and (4.3.14) that

$$
V(k) > \bar{\xi}^T(k)\hat{\Phi}_4\bar{\xi}(k) \geq \epsilon||x(k)||^2, \tag{4.3.15}
$$

where ϵ is a sufficient small positive scalar.

The third step is to find the conditions that can ensure the negative-definiteness of forward difference of functional (4.3.10). Defining the forward difference of Lyapunov functional as $\Delta V(k) = V(k+1) - V(k)$ and calculating it along the trajectories of digital filter (4.2.1)

$$\Delta V(k) = \sum_{i=1}^{4} \Delta V_i(k), \tag{4.3.16}$$

where $\Delta V_1(k)$ is given as

$$\begin{aligned}
& \Delta V_1(k) \\
= \;& V_1(k+1) - V_1(k) \\
= \;& \eta_1^T(k+1) P_1 \eta_1(k+1) - \eta_1^T(k) P_1 \eta_1(k) \\
& + \eta_2^T(k+1) P_2 \eta_2(k+1) - \eta_2^T(k) P_2 \eta_2(k) \\
= \;& \begin{bmatrix} x(k+1) \\ \sum\limits_{i=k-h_1+1}^{k} x(i) \\ \sum\limits_{i=k-h_2+1}^{k-h_1} x(i) \end{bmatrix}^T P_1 \begin{bmatrix} x(k+1) \\ \sum\limits_{i=k-h_1+1}^{k} x(i) \\ \sum\limits_{i=k-h_2+1}^{k-h_1} x(i) \end{bmatrix} \\
& - \begin{bmatrix} x(k) \\ \sum\limits_{i=k-h_1}^{k-1} x(i) \\ \sum\limits_{i=k-h_2}^{k-h_1-1} x(i) \end{bmatrix}^T P_1 \begin{bmatrix} x(k) \\ \sum\limits_{i=k-h_1}^{k-1} x(i) \\ \sum\limits_{i=k-h_2}^{k-h_1-1} x(i) \end{bmatrix} \\
& + \begin{bmatrix} x(k) \\ f(y(k)) \end{bmatrix}^T P_2 \begin{bmatrix} x(k) \\ f(y(k)) \end{bmatrix} \\
& - \begin{bmatrix} x(k-1) \\ f(y(k-1)) \end{bmatrix}^T P_2 \begin{bmatrix} x(k-1) \\ f(y(k-1)) \end{bmatrix} \\
= \;& \begin{bmatrix} e_1 \xi(k) \\ (e_6 + e_2 - e_3)\xi(k) \\ (e_7 + e_8 + e_3 - e_5)\xi(k) \end{bmatrix}^T P_1 \begin{bmatrix} e_1 \xi(k) \\ (e_6 + e_2 - e_3)\xi(k) \\ (e_7 + e_8 + e_3 - e_5)\xi(k) \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
& -\begin{bmatrix} e_2\xi(k) \\ e_6\xi(k) \\ (e_7+e_8)\xi(k) \end{bmatrix}^T P_1 \begin{bmatrix} e_2\xi(k) \\ e_6\xi(k) \\ (e_7+e_8)\xi(k) \end{bmatrix} \\
& +\begin{bmatrix} e_2\xi(k) \\ e_1\xi(k) \end{bmatrix}^T P_2 \begin{bmatrix} e_2\xi(k) \\ e_1\xi(k) \end{bmatrix} - \begin{bmatrix} e_{11}\xi(k) \\ e_2\xi(k) \end{bmatrix}^T P_2 \begin{bmatrix} e_{11}\xi(k) \\ e_2\xi(k) \end{bmatrix} \\
= \;& \xi^T(k)(\Pi_1^T P_1 \Pi_1 - \Pi_2^T P_1 \Pi_2 + \Pi_3^T P_2 \Pi_3 - \Pi_4^T P_2 \Pi_4)\xi(k) \\
= \;& \xi^T(k)\Phi_1\xi(k), \qquad (4.3.17)
\end{aligned}
$$

with Π_1, Π_2, and Φ_1 being defined in Theorem 4.

$\Delta V_2(k)$ is given as

$$
\begin{aligned}
& \Delta V_2(k) \\
= \;& V_2(k+1) - V_2(k) \\
= \;& \sum_{i=k-\tau_1+1}^{k} x^T(i)Q_1x(i) - \sum_{i=k-\tau_1}^{k-1} x^T(i)Q_1x(i) \\
& + \sum_{i=k-\tau_2+1}^{k-\tau_1} x^T(i)Q_2x(i) - \sum_{i=k-\tau_2}^{k-\tau_1-1} x^T(i)Q_2x(i) \\
= \;& x^T(k)Q_1x(k) - x^T(k-\tau_1)Q_1x(k-\tau_1) \\
& + x^T(k-\tau_1)Q_2x(k-\tau_1) - x^T(k-\tau_2)Q_2x(k-\tau_2) \\
= \;& \xi^T(k)(e_2^T Q_1 e_2 - e_3^T(Q_1 - Q_2)e_3 - e_5^T Q_2 e_5)\xi(k) \\
= \;& \xi^T(k)\Phi_2\xi(k), \qquad (4.3.18)
\end{aligned}
$$

with Φ_2 being defined in Theorem 4.

$\Delta V_3(k)$ is given as

$$\begin{aligned}\Delta V_3(k) &= V_3(k+1) - V_3(k)\\ &= \tau_1 \sum_{i=-\tau_1}^{-1} \sum_{j=k+i+1}^{k} \triangle x^T(j) R_1 \triangle x(j)\\ &\quad -\tau_1 \sum_{i=-\tau_1}^{-1} \sum_{j=k+i}^{k-1} \triangle x^T(j) R_1 \triangle x(j)\\ &\quad +\tau_{12} \sum_{i=-\tau_2}^{-\tau_1-1} \sum_{j=k+i+1}^{k} \triangle x^T(j) R_2 \triangle x(j)\\ &\quad -\tau_{12} \sum_{i=-\tau_2}^{-\tau_1-1} \sum_{j=k+i}^{k-1} \triangle x^T(j) R_2 \triangle x(j)\\ &= \tau_1^2 \triangle x^T(k) R_1 \triangle x(k) + \tau_{12}^2 \triangle x^T(k) R_2 \triangle x(k)\\ &\quad -J_1 - J_2 - J_3, \end{aligned} \tag{4.3.19}$$

with

$$\begin{aligned} J_1 &= \tau_1 \sum_{j=k-\tau_1}^{k-1} \triangle x^T(j) R_1 \triangle x(j),\\ J_2 &= \tau_{12} \sum_{j=k-\tau(t)}^{k-\tau_1-1} \triangle x^T(j) R_2 \triangle x(j),\\ J_3 &= \tau_{12} \sum_{j=k-\tau_2}^{k-\tau(t)-1} \triangle x^T(j) R_2 \triangle x(j). \end{aligned}$$

$\Delta V_4(k)$ is given as

$$\begin{aligned}\Delta V_4(k) &= V_4(k+1) - V_4(k)\\ &= \tau_{12} \sum_{i=-\tau_2}^{-\tau_1-1} \sum_{j=k+i+1}^{k} \eta_3^T(j) Z \eta_3(j)\\ &\quad -\tau_{12} \sum_{i=-\tau_2}^{-\tau_1-1} \sum_{j=k+i}^{k-1} \eta_3^T(j) Z \eta_3(j)\\ &= \tau_{12}^2 \eta_3^T(k) Z \eta_3(k) - J_4 - J_5, \end{aligned} \tag{4.3.20}$$

with

$$J_4 = \tau_{12} \sum_{j=k-\tau(t)}^{k-\tau_1-1} \eta_3^T(j) Z \eta_3(j),$$
$$J_5 = \tau_{12} \sum_{j=k-\tau_2}^{k-\tau(t)-1} \eta_3^T(j) Z \eta_3(j).$$

For symmetric matrix T_1 and T_2, the following two zero-value equations are satisfied:

$$\begin{aligned}\mathcal{Z}_1 &= \tau_{12} x(k-\tau_1)^T T_1 x(k-\tau_1) \\ &\quad - \tau_{12} x(k-\tau(k))^T T_1 x(k-\tau(k)) \\ &\quad - \tau_{12} \sum_{j=k-\tau(k)}^{k-\tau_1-1} \eta_3^T(i) \begin{bmatrix} 0 & T_1 \\ T_1 & T_1 \end{bmatrix} \eta_3(i) \\ &= 0,\end{aligned} \tag{4.3.21}$$

$$\begin{aligned}\mathcal{Z}_2 &= \tau_{12} x(k-\tau(k))^T T_2 x(k-\tau(k)) \\ &\quad - \tau_{12} x(k-\tau_2)^T T_2 x(k-\tau_2) \\ &\quad - \tau_{12} \sum_{j=k-\tau_2}^{k-\tau(k)-1} \eta_3^T(i) \begin{bmatrix} 0 & T_2 \\ T_2 & T_2 \end{bmatrix} \eta_3(i) \\ &= 0,\end{aligned} \tag{4.3.22}$$

which implies

$$\begin{aligned}&\mathcal{Z}_1 + \mathcal{Z}_2 \\ = \; &\tau_{12} \xi^T(k) \Big[e_3^T T_1 e_3 - e_4^T (T_1 - T_2) e_4 - e_5^T T_2 e_5 \Big] \xi(k) \\ &- J_6 - J_7,\end{aligned} \tag{4.3.23}$$

where

$$J_6 = \tau_{12} \sum_{j=k-\tau(k)}^{k-\tau_1-1} \eta_3^T(i) \begin{bmatrix} 0 & T_1 \\ T_1 & T_1 \end{bmatrix} \eta_3(i),$$
$$J_7 = \tau_{12} \sum_{j=k-\tau_2}^{k-\tau(k)-1} \eta_3^T(i) \begin{bmatrix} 0 & T_2 \\ T_2 & T_2 \end{bmatrix} \eta_3(i).$$

Based on Lemma 13-15, the following inequalities hold, if (4.2.8), (4.2.10) and (4.2.11) hold, for any positive diagonal matrices S, S_1, S_2 and D and any matrices N, M, M_1, and M_2:

$$\begin{aligned} \mathcal{Z}_3 &= 2\Big[y(k) - f(y(k))\Big]^T D\Big[f(y(k)) - Ly(k)\Big] \\ &\geq 0, \qquad (4.3.24) \\ \mathcal{Z}_4 &= 2\Big[y^T(k)S + x^T(k)M + f^T(y(k))N\Big]\Big[y(k) - f(y(k))\Big] \\ &\geq 0, \qquad (4.3.25) \\ \mathcal{Z}_5 &= 2\Big[y^T(k)S_1 + x^T(k-\tau(k))M_1\Big]\Big[y(k) - f(y(k))\Big] \\ &\geq 0, \qquad (4.3.26) \\ \mathcal{Z}_6 &= 2\Big[y^T(k)S_2 + x^T(k-1)M_2\Big]\Big[y(k) - f(y(k))\Big] \geq 0. \qquad (4.3.27) \end{aligned}$$

Moreover, based on the definition of $\xi(k)$, it can be found that several vectors therein satisfy the following conditions:

$$\upsilon_2(k) = (\tau(k) - \tau_1 + 1)\upsilon_4(k) - x(k - \tau_1),$$
$$\upsilon_3(k) = (\tau_2 - \tau(k) + 1)\upsilon_5(k) - x(k - \tau(k)),$$

which can lead to the following zero-value terms

$$\mathcal{Z}_7 = 2\xi^T(k)e_g^T U_1\Big[(\tau(k) - \tau_1 + 1)e_9 - e_7 - e_3\Big]\xi(k) = 0, \qquad (4.3.28)$$
$$\mathcal{Z}_8 = 2\xi^T(k)e_g^T U_2\Big[(\tau_2 - \tau(k) + 1)e_{10} - e_8 - e_4\Big]\xi(k) = 0, \qquad (4.3.29)$$

with U_1 and U_2 being any matrices.

Combining (4.3.16)-(4.3.29) yields

$$\begin{aligned} \Delta V(k) &= \sum_{i=1}^{4} \Delta V_i(k) \\ &\leq \sum_{i=1}^{4} \Delta V_i(k) + \sum_{i=1}^{8} \mathcal{Z}_i \\ &= \xi^T(k)\left(\sum_{i=1}^{6} \Phi_i\right)\xi(k) - \sum_{i=1}^{7} J_i. \qquad (4.3.30) \end{aligned}$$

Using $R_1 > 0$ and applying (4.2.4) of Lemma 10 to estimate J_1 yield

$$\begin{aligned} J_1 &= \tau_1 \sum_{j=k-\tau_1}^{k-1} \triangle x^T(j) R_1 \triangle x(j) \\ &\geq \begin{bmatrix} \kappa_1(k) \\ \kappa_2(k) \end{bmatrix}^T \begin{bmatrix} R_1 & 0 \\ 0 & 3R_1 \end{bmatrix} \begin{bmatrix} \kappa_1(k) \\ \kappa_2(k) \end{bmatrix} \\ &= \xi^T(k) \begin{bmatrix} e_2 - e_3 \\ e_2 + e_3 - \frac{2(e_2+e_6)}{\tau_1+1} \end{bmatrix}^T \begin{bmatrix} R_1 & 0 \\ 0 & 3R_1 \end{bmatrix} \\ &\quad \times \begin{bmatrix} e_2 - e_3 \\ e_2 + e_3 - \frac{2(e_2+e_6)}{\tau_1+1} \end{bmatrix} \xi(k) \\ &= \xi^T(k)\Phi_7\xi(k), \end{aligned} \tag{4.3.31}$$

where

$$\begin{aligned} \kappa_1(k) &= x(k) - x(k-\tau_1), \\ \kappa_2(k) &= x(k) + x(k-\tau_1) - \frac{2}{\tau_1+1}(v_1(k) + x(k)). \end{aligned}$$

Using $R_2 > 0$ and applying (4.2.4) of Lemma 10 to estimate J_2 yield

$$\begin{aligned} J_2 &= \tau_{12} \sum_{j=k-\tau(k)}^{k-\tau_1-1} \triangle x^T(i) R_2 \triangle x(i) \\ &\geq \frac{\tau_{12}}{\tau(k)-\tau_1} \begin{bmatrix} \kappa_3(k) \\ \kappa_4(k) \end{bmatrix}^T \begin{bmatrix} R_2 & 0 \\ 0 & 3R_2 \end{bmatrix} \begin{bmatrix} \kappa_3(k) \\ \kappa_4(k) \end{bmatrix} \\ &= \frac{\tau_{12}}{\tau(k)-\tau_1} \xi^T(k) \begin{bmatrix} e_3 - e_4 \\ e_3 + e_4 - 2e_9 \end{bmatrix}^T \begin{bmatrix} R_2 & 0 \\ 0 & 3R_2 \end{bmatrix} \\ &\quad \times \begin{bmatrix} e_3 - e_4 \\ e_3 + e_4 - 2e_9 \end{bmatrix} \xi(k) \\ &= \xi^T(k) \frac{\tau_{12}\Phi_{81}}{\tau(k)-\tau_1} \xi(k), \end{aligned} \tag{4.3.32}$$

where

$$
\begin{aligned}
\kappa_3(k) &= x(k-\tau_1) - x(k-\tau(k)), \\
\kappa_4(k) &= x(k-\tau_1) + x(k-\tau(k)) - 2\upsilon_4(k), \\
\Phi_{81} &= \begin{bmatrix} e_3 - e_4 \\ e_3 + e_4 - 2e_9 \end{bmatrix}^T \begin{bmatrix} R_2 & 0 \\ 0 & 3R_2 \end{bmatrix} \begin{bmatrix} e_3 - e_4 \\ e_3 + e_4 - 2e_9 \end{bmatrix}.
\end{aligned}
$$

Using $R_2 > 0$ and applying (4.2.4) of Lemma 10 to estimate J_3 yield

$$
\begin{aligned}
J_3 &= \tau_{12} \sum_{j=k-\tau_2}^{k-\tau(k)-1} \triangle x^T(i) R_2 \triangle x(i) \\
&\geq \frac{\tau_{12}}{\tau_2 - \tau(k)} \begin{bmatrix} \kappa_5(k) \\ \kappa_6(k) \end{bmatrix}^T \begin{bmatrix} R_2 & 0 \\ 0 & 3R_2 \end{bmatrix} \begin{bmatrix} \kappa_5(k) \\ \kappa_6(k) \end{bmatrix} \\
&= \frac{\tau_{12}}{\tau_2 - \tau(k)} \xi^T(k) \begin{bmatrix} e_4 - e_5 \\ e_4 + e_5 - 2e_{10} \end{bmatrix}^T \begin{bmatrix} R_2 & 0 \\ 0 & 3R_2 \end{bmatrix} \\
&\quad \times \begin{bmatrix} e_4 - e_5 \\ e_4 + e_5 - 2e_{10} \end{bmatrix} \xi(k) \\
&= \xi^T(k) \frac{\tau_{12}\Phi_{82}}{\tau_2 - \tau(k)} \xi(k), \qquad (4.3.33)
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa_5(k) &= x(k-\tau(k)) - x(k-\tau_2), \\
\kappa_6(k) &= x(k-\tau(k)) + x(k-\tau_2) - 2\upsilon_5(k), \\
\Phi_{82} &= \begin{bmatrix} e_4 - e_5 \\ e_4 + e_5 - 2e_{10} \end{bmatrix}^T \begin{bmatrix} R_2 & 0 \\ 0 & 3R_2 \end{bmatrix} \begin{bmatrix} e_4 - e_5 \\ e_4 + e_5 - 2e_{10} \end{bmatrix}.
\end{aligned}
$$

Using (4.3.8) and applying (4.2.5) Lemma 11 to estimate $J_4 + J_6$ yield

$$\begin{aligned}
J_4 \;&+\; J_6 = \tau_{12} \sum_{j=k-\tau(k)}^{k-\tau_1-1} \eta_3^T(i) \left(Z + \begin{bmatrix} 0 & T_1 \\ T_1 & T_1 \end{bmatrix} \right) \eta_3(i) \\
&\geq \frac{\tau_{12}}{\tau(k)-\tau_1} \begin{bmatrix} \upsilon_2(k) \\ \kappa_3(k) \end{bmatrix}^T \left(Z + \begin{bmatrix} 0 & T_1 \\ T_1 & T_1 \end{bmatrix} \right) \begin{bmatrix} \upsilon_2(k) \\ \kappa_3(k) \end{bmatrix} \\
&= \frac{\tau_{12}}{\tau(k)-\tau_1} \xi^T(k) \begin{bmatrix} e_7 \\ e_3 - e_4 \end{bmatrix}^T \left(Z + \begin{bmatrix} 0 & T_1 \\ T_1 & T_1 \end{bmatrix} \right) \\
&\quad \times \begin{bmatrix} e_7 \\ e_3 - e_4 \end{bmatrix} \xi(k) \\
&= \xi^T(k) \frac{\tau_{12}\Phi_{83}}{\tau(k)-\tau_1} \xi(k), \qquad (4.3.34)
\end{aligned}$$

where

$$\Phi_{83} = \begin{bmatrix} e_7 \\ e_3 - e_4 \end{bmatrix}^T \left(Z + \begin{bmatrix} 0 & T_1 \\ T_1 & T_1 \end{bmatrix} \right) \begin{bmatrix} e_7 \\ e_3 - e_4 \end{bmatrix}.$$

Using (4.3.9) and applying (4.2.5) Lemma 11 to estimate $J_5 + J_7$ yield

$$\begin{aligned}
J_5 \;&+\; J_7 = \tau_{12} \sum_{j=k-\tau_2}^{k-\tau(k)-1} \eta_3^T(i) \left(Z + \begin{bmatrix} 0 & T_2 \\ T_2 & T_2 \end{bmatrix} \right) \eta_3(i) \\
&\geq \frac{\tau_{12}}{\tau_2-\tau(k)} \begin{bmatrix} \upsilon_3(k) \\ \kappa_4(k) \end{bmatrix}^T \left(Z + \begin{bmatrix} 0 & T_2 \\ T_2 & T_2 \end{bmatrix} \right) \begin{bmatrix} \upsilon_3(k) \\ \kappa_4(k) \end{bmatrix} \\
&= \frac{\tau_{12}}{\tau_2-\tau(k)} \xi^T(k) \begin{bmatrix} e_8 \\ e_4 - e_5 \end{bmatrix}^T \left(Z + \begin{bmatrix} 0 & T_2 \\ T_2 & T_2 \end{bmatrix} \right) \\
&\quad \times \begin{bmatrix} e_8 \\ e_4 - e_5 \end{bmatrix} \xi(k) \\
&= \xi^T(k) \frac{\tau_{12}\Phi_{84}}{\tau_2-\tau(k)} \xi(k), \qquad (4.3.35)
\end{aligned}$$

where

$$\Phi_{84} = \begin{bmatrix} e_8 \\ e_4 - e_5 \end{bmatrix}^T \left(Z + \begin{bmatrix} 0 & T_2 \\ T_2 & T_2 \end{bmatrix} \right) \begin{bmatrix} e_8 \\ e_4 - e_5 \end{bmatrix}.$$

It follows from (4.3.32)-(4.3.35) that

$$\begin{aligned}
\sum_{i=2}^{7} J_i &\geq \xi^T(k)\left[\frac{\tau_{12}(\Phi_{81}+\Phi_{83})}{\tau(k)-\tau_1}+\frac{\tau_{12}(\Phi_{82}+\Phi_{84})}{\tau_2-\tau(k)}\right]\xi(k) \\
&= \xi^T(k)\left[\frac{\tau_{12}\Phi_{813}}{\tau(k)-\tau_1}+\frac{\tau_{12}\Phi_{824}}{\tau_2-\tau(k)}\right]\xi(k) \\
&= \xi^T(k)\left[\frac{\tau_{12}E_2^T\Xi_1 E_2}{\tau(k)-\tau_1}+\frac{\tau_{12}E_3^T\Xi_2 E_3}{\tau_2-\tau(k)}\right]\xi(k), \qquad (4.3.36)
\end{aligned}$$

where

$$\begin{aligned}
\Phi_{813} &= \begin{bmatrix} e_7 \\ e_3-e_4 \\ e_3+e_4-2e_9 \end{bmatrix}^T \left[\begin{array}{c|c} Z+\begin{bmatrix} 0 & T_1 \\ T_1 & R_2+T_1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 \end{bmatrix} & 3R_2 \end{array}\right] \\
&\quad \times \begin{bmatrix} e_7 \\ e_3-e_4 \\ e_3+e_4-2e_9 \end{bmatrix} \\
&= E_2^T\Xi_1 E_2, \\
\Phi_{824} &= \begin{bmatrix} e_8 \\ e_4-e_5 \\ e_4+e_5-2e_{10} \end{bmatrix}^T \left[\begin{array}{c|c} Z+\begin{bmatrix} 0 & T_2 \\ T_2 & R_2+T_2 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 \end{bmatrix} & 3R_2 \end{array}\right] \\
&\quad \times \begin{bmatrix} e_8 \\ e_4-e_5 \\ e_4+e_5-2e_{10} \end{bmatrix} \\
&= E_3^T\Xi_2 E_3.
\end{aligned}$$

For any matrix X, it follows from (4.2.6) of Lemma 12 that

$$\begin{aligned}
& \frac{\tau_{12}E_2^T\Xi_1E_2}{\tau(k)-\tau_1}+\frac{\tau_{12}E_3^T\Xi_2E_3}{\tau_2-\tau(k)} \\
\geq & \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}^T \begin{bmatrix} \frac{2\tau_2-\tau(k)-\tau_1}{\tau_{12}}\Xi_1 & X \\ * & \frac{\tau_2+\tau(k)-2\tau_1}{\tau_{12}}\Xi_2 \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix} \\
& -\frac{\tau_2-\tau(k)}{\tau_{12}}E_2^TX\Xi_1^{-1}X^TE_2 \\
& -\frac{\tau(k)-\tau_1}{\tau_{12}}E_3^TX^T\Xi_2^{-1}XE_3 \\
= & \ \Phi_8-\bar{\Phi}_8,
\end{aligned} \tag{4.3.37}$$

where

$$\begin{aligned}
\bar{\Phi}_8 = & \frac{\tau_2-\tau(k)}{\tau_{12}}E_2^TX\Xi_1^{-1}X^TE_2 \\
& +\frac{\tau(k)-\tau_1}{\tau_{12}}E_3^TX^T\Xi_2^{-1}XE_3.
\end{aligned}$$

Based on (4.3.30), (4.3.31), (4.3.36) and (4.3.37), it can be obtained that

$$\begin{aligned}
\Delta V(k) \leq & \ \xi^T(k)\left(\sum_{i=1}^{6}\Phi_i-\Phi_7-\Phi_8+\bar{\Phi}_8\right)\xi(k) \\
= & \ \xi^T(k)\left(\Psi+\bar{\Phi}_8\right)\xi(k).
\end{aligned} \tag{4.3.38}$$

It can be checked that $\Psi+\bar{\Phi}_8$ is convex with respect to $\tau(t)$, which from the convex combination technique shows the following holds

$$\Psi+\bar{\Phi}_8<0, \forall\tau(t)\in\{\tau_1,\tau_2\} \tag{4.3.39}$$

$$\Rightarrow \Psi+\bar{\Phi}_8<0, \forall\tau(t)\in[\tau_1,\tau_2]. \tag{4.3.40}$$

Based on the Schur complement, (4.3.39) is equivalent to LMIs (4.3.6) and (4.3.7). Thus, if LMIs (4.3.6) and (4.3.7) holds, then (4.3.40) holds, which combined with (4.3.38) leads to

$$\Delta V(k)\leq-\varepsilon||x(k)||^2, \tag{4.3.41}$$

for a sufficient small $\varepsilon>0$.

Therefore, under conditions (4.2.8), (4.2.10), (4.2.11) and (4.3.1)-(4.3.9), digital filter (4.2.1) is stable. This completes the proof. ■

Remark 11. *On the one hand, compared with the simple functionals used for discussing the digital filters in the previous literature [112, 140, 142, 143], extra free matrices are introduced by the augmented terms, $V_1(k)$ and $V_4(k)$, of the proposed augmented functional (4.3.10), especially, $f(y(k-1))$ included in $V_1(k)$ is used to construct functional at the first time, which can provide extra freedom during checking the feasibility of criterion. Thus, Theorem 4 in this chapter has less conservatism than the ones in the previous literature.*

Remark 12. *On the other hand, compared with the criteria of digital filters developed in [112, 140, 142, 143], Theorem 4 in this chapter has less conservatism due to the improvement during the estimation of $\Delta V(k)$. In order to accurately estimate the summation term arising in $\Delta V(k)$, i.e., $J_i, i = 1, 2, \cdots, 7$, Lemma 15 and several other techniques, which are proved to be helpful to reduce the conservatism during the investigation of discrete-time delay systems, are applied to reduce the estimation error, summarized as follows:*

(1) Due to adding of $\mathcal{Z}_5$ and $\mathcal{Z}_6$ respectively defined in (4.3.26) *and* (4.3.27) *into the $\Delta V(k)$, several cross terms are introduced to give the relationship between the delayed states, $x(k-\tau(k))$ and $x(k-1)$, and the nonlinear function, $f(y(k))$, which are not used in the literature [112, 140, 142, 143] and provide extra freedom for finding the solutions of conditions in Theorem 4 so as to reduce the conservatism.*

(2) The summation terms defined by J_1, J_2 and J_3 are bounded by using the Wirtinger-based summation inequality in this chapter, while the similar terms are estimated based on a more conservative inequality in the previous literature [112, 140, 142, 143], i.e. the Jensen-based summation inequality.

(3) Two zero-value equations $\mathcal{Z}_7$ and $\mathcal{Z}_8$, respectively defined in (4.3.28) *and* (4.3.29)*, are developed and introduced into the $\Delta V(k)$, which adds many cross-terms into Theorem 4. The presence of free matrices U_1 and U_2 in Theorem 4 can increase the feasibility of the conditions of Theorem 4 so as to reduce the conservatism.*

Remark 13. *Remark 11 and 12 show the improvements of Theorem 4 compared with the ones for digital filters reported in literature. In fact, compared with the techniques developed for the analysis of linear discrete-time system with a time-varying delay, novel treatments are also used to develop Theorem 4. More specifically, compared with the functionals used in the related works, in which each term of functionals is usually required to be positive, the functional* (4.3.10) *constructed in this chapter is relaxed by considering all terms together and requiring the sum of all terms to be positive. That is to say, the positive-definite condition of functional* (4.3.10) *is relaxed, (i.e.,* $P_1 > 0$*,* $P_2 > 0$*,* $Q_1 > 0$*,* $Q_2 > 0$*, and* $Z > 0$ *are removed), which helps to reduce the conservatism. Moreover,* $x(k-1)$ *included in* $V_1(k)$ *has not been used for the investigation of linear discrete-time system with a time-varying delay.*

In order to easily show the advantage of the proposed Theorem 4, the following corollary is developed by requiring each term of LKF is positive and setting $P_2 = 0, S_1 = 0, S_2 = 0, M_1 = 0, M_2 = 0$.

Corollary 1. *For given scalars* l_i*,* τ_1 *and* τ_2*, digital filter (4.2.1) with time-varying satisfying (4.2.2) is asymptotically stable if there exists positive definite symmetric matrices* P_1*,* Z*,* Q_1, Q_2, R_1, R_2*, symmetric matrices* T_1, T_2*, positive diagonal matrices* S, D*, and any matrices* X*,* U_1, U_2*,* M, N*, such that condition (4.2.8) and LMIs (4.3.6)-(4.3.9) are feasible.*

Remark 14. *Although Theorem 4 has less conservatism, compared with the ones in [112, 140, 143], with the help of the techniques summarized in Remarks 3-5, it is still a sufficient criterion and provides conservative results. Many new techniques developed for linear time-delay systems, such as improved functionals [157, 158] and improved bounding inequalities [146], can be used to further reduce the conservatism. In addition, the method proposed in this chapter can be extended to deal with other types of systems with time delays [143] and the practical systems affected by communication delays and/or saturation constraint [56, 159–163].*

4.4 Numerical examples

In this section, two numerical examples are given to demonstrate the effectiveness and advantages of the proposed method.

Example 1. *Consider digital filter (4.2.1) with the following parameters*

$$l_1 = -1,\ l_2 = 0, \tag{4.4.1}$$

$$l_{11} = l_{12} = l_{21} = l_{22} = 1, \tag{4.4.2}$$

$$A = \begin{bmatrix} 0.3 & -0.4 \\ 0.5 & 0.7 \end{bmatrix}, \tag{4.4.3}$$

$$A_d = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}. \tag{4.4.4}$$

For different given lower bounds of $\tau(k)$, τ_1,

the allowably maximal values of τ_2 calculated by Theorem 4, Corollary 1, and the ones provided by the criteria of [112,140,143] are listed in Table 4.1. It is clearly seen that Theorem 4 provides less conservative results than the ones reported in [112,140,143], which shows that the improvements, summarized in Remarks 3 and 4, indeed reduce the conservatism. Moreover, the results provided by Theorem 4 are less conservative than those provided by Corollary 1, which means that the relaxed positive-definite condition in Theorem 4 and the cross terms introduced by Lemma 15 indeed reduce the conservatism as analyzed in Remarks 4 and 5. Therefore, the advantages of the proposed method is verified.

When the lower bound of time-varying delay is 1, i.e., $\tau(k) \geq \tau_1 = 1$, it is calculated from Theorem 4 that the allowably maximal value of τ_2, $\tau_{\max}$, is 43 and

Table 4.1: Allowably maximal value of τ_2, $\tau_{\max}$, for various τ_1 (Example 2)

	τ_1			
Criteria	1	3	5	10
Theorem 1 [140]	8	10	12	17
Corollary 1 [143]	11	13	15	20
Theorem 2 [112]	11	13	15	20
Corollary 1	15	17	19	24
Theorem 4	43	45	47	52

the related feasible solutions of the conditions of Theorem 4 are given as follows

$$P_1 = \begin{bmatrix} 1.8501 & 0.7162 & 0.0050 & 0.0085 & -0.0020 & 0.0017 \\ 0.7162 & 1.1126 & -0.0043 & 0.0031 & -0.0011 & -0.0013 \\ 0.0050 & -0.0043 & 0.0006 & 0.0003 & -0.0007 & 0.0006 \\ 0.0085 & 0.0031 & 0.0003 & 0.0050 & -0.0007 & -0.0003 \\ -0.0020 & -0.0011 & -0.0007 & -0.0007 & -0.0002 & 0.0001 \\ 0.0017 & -0.0013 & 0.0006 & -0.0003 & 0.0001 & 0.0003 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0.0082 & -0.0036 & -0.0104 & -0.0022 \\ -0.0036 & 0.0089 & 0.0109 & -0.0024 \\ -0.0104 & 0.0109 & 0.9051 & 0.1608 \\ -0.0022 & -0.0024 & 0.1608 & 0.6397 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 0.0702 & 0.0129 \\ 0.0129 & 0.0411 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0.0333 & 0.0053 \\ 0.0053 & 0.0177 \end{bmatrix},$$

$$R_1 = \begin{bmatrix} 0.0116 & -0.0006 \\ -0.0006 & 0.0074 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 0.3433 & -0.0088 \\ -0.0088 & 0.1474 \end{bmatrix},$$

$$Z = \begin{bmatrix} 0.0013 & 0.0007 & 0.0010 & -0.0004 \\ 0.0007 & 0.0010 & 0.0013 & 0.0001 \\ 0.0010 & 0.0013 & 0.0023 & 0.0001 \\ -0.0004 & 0.0001 & 0.0001 & 0.0006 \end{bmatrix}.$$

That is, digital filter (4.2.1) with parameters given in (4.4.1)-(4.4.4) and time-varying delay satisfying $1 \leq \tau(k) \leq 43$ is asymptotically stable since one can find the Lyapunov functional (4.3.10) with matrices $P_1, P_2, Q_1, Q_2, R_1, R_2, Z$ being given above. Simulation results are also obtained to verify this case.

Let the initial condition $x(k) = [-0.4,\ 0.1]^T$, $k \in [-43, 0]$ and the delay is a random value within $[1, 43]$. The responses of the corresponding time delay and two state variables of digital filter are shown in Fig. 4.1. It is clearly observed that the digital filter is stable. Therefore, the effectiveness of the proposed criterion is verified.

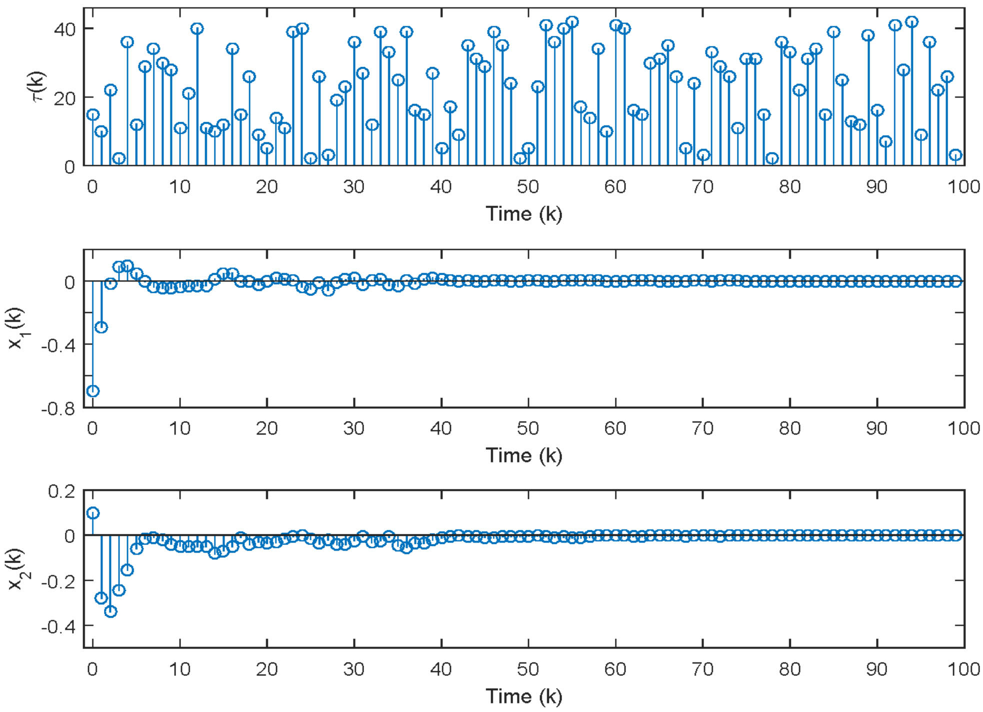

Figure 4.1: The time delay and state trajectories (Example 2).

Example 2. *Consider digital filter (4.2.1) with the following parameters*

$$l_1 = -1,\ l_2 = 0,\ l_3 = 1, \tag{4.4.5}$$

$$l_{11} = l_{21} = 0,\ l_{12} = l_{22} = l_{13} = l_{23} = 1, \tag{4.4.6}$$

$$A = \begin{bmatrix} 0.8 & -1.75 & -2.5 \\ -0.1 & -0.5 & -0.6 \\ 0.1 & 0.1 & 0.5 \end{bmatrix}, \tag{4.4.7}$$

$$A_d = \begin{bmatrix} 0.001 & 0.01 & -0.01 \\ 0 & 0.01 & 0 \\ -0.01 & 0.01 & 0.01 \end{bmatrix}. \tag{4.4.8}$$

For different given lower bounds of $\tau(k)$, τ_1, the allowably maximal values of τ_2 calculated by Theorem 4, Corollary 1, and the ones provided by the criteria of [112,140,143] are listed in Table 4.2. It is clearly seen that Theorem 4 provides less conservative results than the others, which shows the advantages of the proposed Theorem 4.

Table 4.2: Allowably maximal value of τ_2, $\tau_{\max}$, for various τ_1 (Example 3)

	τ_1				
Criteria	5	10	15	20	25
Theorem 1 [140]	33	38	43	48	53
Corollary 1 [143]	34	39	43	48	53
Theorem 2 [112]	34	39	43	48	53
Corollary 1	47	52	57	62	67
Theorem 4	49	54	59	64	69

From Table 4.2, digital filter (4.2.1) with parameters given in (4.4.5)-(4.4.8) and time-varying delay satisfying $5 \le \tau(k) \le 49$ is asymptotically stable. Simulation results are also obtained to verify this observation.

Let the initial condition $x(k) = [-0.4, 0.02, 0.3]^T$, $k \in [-49, 0]$ and the delay is a random value within $[5, 49]$. The responses of the time delay together with two state variables of digital filter are shown in Fig. 4.2. It is clearly observed that digital filter is stable. Therefore, the effectiveness of the proposed criterion is verified.

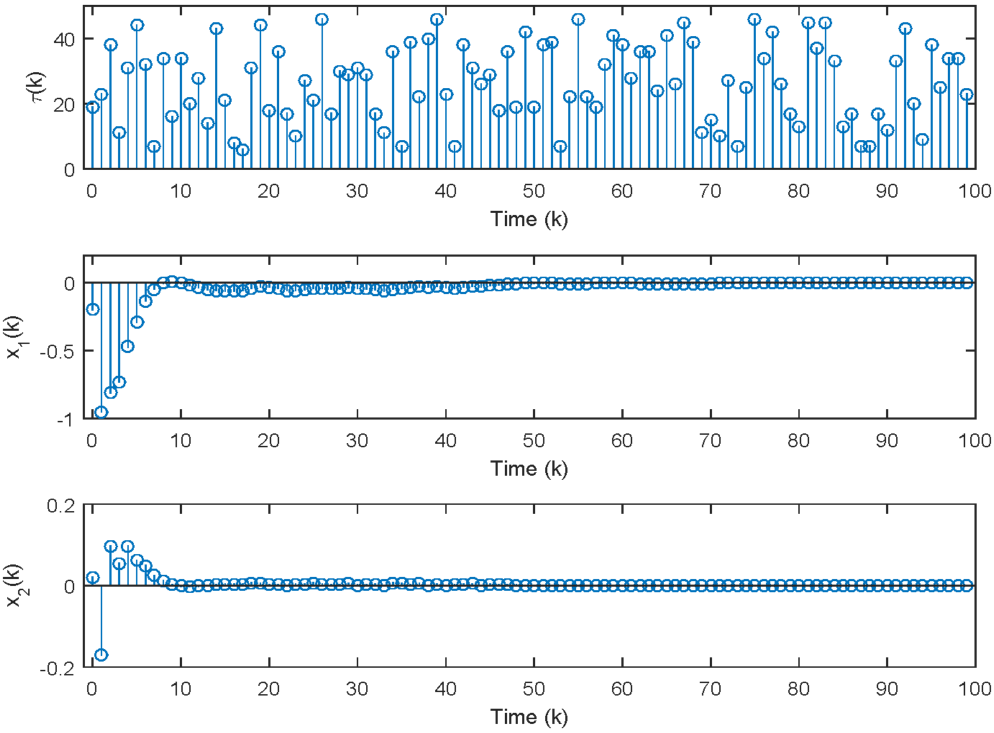

Figure 4.2: The time delay and state trajectories (Example 3).

4.5 Conclusion

This chapter has investigated the stability analysis of fixed-point state-space digital filters with generalized overflow arithmetic and a time-varying delay. A new stability criterion has been developed to assess the influence of the time delay on the stability of digital filter. The criterion has less conservatism in comparison to the ones reported in the previous literature due to two aspects of improvements. A new Lyapunov functional with several augmented terms and relaxed positive-definite condition has been constructed, and free matrices therein provides extra freedom of checking the conditions of stability criterion. And, Lemma 15, together with several new techniques (such as Wirtinger-based inequality, zero-value equations, the extended reciprocally convex matrix inequality), has been used to estimate the summation terms arising in the forward difference of functional, which leads to a smaller estimate gap than the methods used in the literature. Finally, two numerical examples have been given to show the effectiveness and advantages of the proposed stability criterion.

Chapter 5

Delay dependent stability analysis via relaxed three integral inequality method and its application to optimal economic dispatch

5.1 Introduction

As mentioned by [164], the delay decomposition approach is the best one to handle the stability of system with constant delay and obviously the less conservative stability results can be obtained when the delay decomposition approach is extended to the case of time-varying [165]. Generally, the main aim is to derive a maximum allowable delay bound of the time-delay such that the time delays'system is asymptotically stable for any delay size less than this maximum delay [166]. However, the purpose of reducing conservatism is still limited due to the estimated derivative of LKF.

In the construction of LKF to obtain delay-dependent criterion, a double integral term which showed below is frequently applied:

$$V_r(t) = \int_{-h}^{0} \int_{t+s}^{t} \dot{x}^T(u) R \dot{x}(u) \, du \, ds \tag{5.1.1}$$

where h is respectively the upper bound of a time-varying delay and x(t) is the system state. Then the following single integral terms with time-varying delay information will appear in the forward difference of $V_r(t)$:

$$\bar{S}(t) = -\int_{t-d(t)}^{t} \dot{x}^T(s)R\dot{x}(s)\,ds - \int_{t-h}^{t-d(t)} \dot{x}^T(s)R\dot{x}(s)\,ds \tag{5.1.2}$$

As mentioned by Zhang [167], the key problem during the criterion-deriving is how to estimate the lower bound of the above single integral term. There are plenty of works to find the optimal solution which is estimated the derivative of LKF, but relaxed integral inequalities is one of the best methods. This chapter provides further study on combined the delay decomposition approach and relaxed integral inequalities. Like most of researches, the research of Zhang just studies the derivative which includes the two single integral terms with time-varying delay information. However, in the delay decomposition approach, there are three single integral terms showing below in the derivative. This motivates to prove a new inequality which is suitable for this situation.

$$\begin{aligned} S(t) \;\; &= \;\; -\int_{t-\alpha d(t)}^{t} \dot{x}^T(s)R\dot{x}(s)\,ds - \int_{t-d(t)}^{t-\alpha d(t)} \dot{x}^T(s)R\dot{x}(s)\,ds \\ &\quad\; -\int_{t-h}^{t-d(t)} \dot{x}^T(s)R\dot{x}(s)\,ds \end{aligned} \tag{5.1.3}$$

Finally, stability analysis of linear systems with time-varying delays is further studied and a new criterion to understand the effect of delays on the system stability is developed. Three relaxed integral inequalities are developed to estimate the derivative of LKF by considering three integral terms together. Then combined with the delay decomposition approach.

The main contribution of the chapter is summarized as follows:

- A dynamic market model of ED problem is proposed. Stability of the resulting dynamical model is investigated and the region of attraction around the equilibrium of interest is established.

- By developing a delay decomposition approach, the integral interval $[t-h, t]$ is decomposed into three integral intervals $[t-h, t-d(t)]$, $[t-d(t), t-\alpha d(t)]$

and $[t - \alpha d(t), t]$. Since a tuning parameter α is introduced, the information about $x(t - \alpha d(t))$ can be taken into full consideration, and thus the upper bound of S(t) can be estimated more exactly no matter the delay derivative exists or not.

- A new stability criterion with less conservatism in comparison with the existing ones is established. A new LKF with delay-product-type terms is constructed, and its derivative is estimated via a novel relaxed integral inequality and combined a delay decomposition approach which mentioned above. A more general type of Lyapunov functionals is defined in this chapter, and new delay-dependent sufficient stability criteria are obtained in terms of LMIs. It is shown that the presented stability conditions are much less conservative than the existing ones.

- A dynamic market model that incorporates the interaction between real-time pricing, physical constraints, and demand response based loads is proposed. In contrast to the papers above, this market model is directly linked with the standard market clearing structure so that the relation between the state-variables of the dynamic model and the primal variables of the power dispatch model is transparent. And secondary, stability of the resulting dynamical model which consists of three main participants, generating company(GenCo), consumers company(Conco), and independent system operator (ISO) is investigated and the region of attraction around the equilibrium of interest is established.

Notations: Throughout this chapter, the superscripts T and -1 mean the transpose and the inverse of a matrix, respectively; $\mathcal{R}^n$ denotes the n-dimensional Euclidean space; $\|\cdot\|$ refers to the Euclidean vector norm; $P > 0$ (≥ 0) means P is a real symmetric and positive-definite (semi-positive-definite) matrix; I and 0 stand for the identity matrix and the zero-matrix, respectively; $diag\{\cdot\}$ denotes the block-diagonal matrix; the symmetric term in the symmetric matrix is denoted by $*$; and $He\{X\} = X + X^T$.

5.2 Problem formulation and preliminaries

Consider the following linear system with a time-varying delay:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t - d(t)), & t \geq 0 \\ x(t) = \phi(t), & t \in [-h, 0] \end{cases} \tag{5.2.1}$$

where $x(t) \in R^n$ is the system state, A and A_d are the system matrices, the initial condition $\phi(t)$ is a continuously differentiable function, and d(t) is the time-varying delay satisfying

$$0 \leq d(t) \leq h \tag{5.2.2}$$

$$|\dot{d}(t)| \leq \mu \tag{5.2.3}$$

where h and μ are constant.

This chapter aims to derive a new delay-dependent stability criteria for analyzing the stability of system(5.2.1) by using delay decomposition approach which combined relaxed integral inequalities. Inspired by Gouaisbaut's work [168], a constant $\alpha \leq 1$ will be added on the delay h as a new delay $\alpha d(t)$, and satisfy

$$0 \leq \alpha d(t) \leq \alpha h \tag{5.2.4}$$

$$|\alpha \dot{d}(t)| \leq \alpha \mu \tag{5.2.5}$$

As mentioned previously, there are three single integral terms when we derive the LKF which satisfied delay decomposition approach and relaxed integral inequalities. Consequently, we firstly aim to develop a new inequalities which have three relaxed integral inequalities. Then we prepare to estimate the new stability criteria.

For the sake of notational simplicity, some common settings are showed below:

$$\begin{aligned} &e_i = \begin{bmatrix} 0_{n\times(i-1)n}, & I, & 0_{n\times(7-i)n} \end{bmatrix}, \; i = 1, 2, \ldots, 5 \\ &E_i = \begin{bmatrix} 0_{n\times(i-1)n}, & I, & 0_{n\times(3-i)n} \end{bmatrix}, \; i = 1, 2, 3 \\ &e_s = Ae_1 + A_d e_3 \\ &t_d = t - d(t), \;\; t_h = t - h, \;\; t_a = t - \alpha d(t), \;\; t_1 = t - d_1(t) \\ &d_1(t) = \alpha d(t), \;\; d_2(t) = d(t) - \alpha d(t). \end{aligned}$$

The following lemma was used for developing the main results.

Lemma 16. *([167]) For a $n \times n$ symmetric matrix $R > 0$ and any $2n \times 2n$ matrix S_1 satisfying $\begin{bmatrix} R_1 & S_1 \\ * & R_1 \end{bmatrix} \geq 0$ with $R_1 = diag\{R, 3R\}, the S(t)$ defined in (5.1.2) can be estimated as:*

$$S(t) \leq -\frac{1}{h}\zeta_1(t) g_0^T \chi_0 g_0 \zeta_1(t) \tag{5.2.6}$$

where

$$\begin{aligned}
\zeta_1(t) &= \begin{bmatrix} x^T(t), & x^T(t-d(t)), & x^T(t-h), & V_1^T(t), & V_2^T(t) \end{bmatrix}^T \\
V_1(t) &= \int_{t-d(t)}^{t} \frac{x(s)}{d(t)} ds, \quad V_2(t) = \int_{t-h}^{t-d(t)} \frac{x(s)}{h-d(t)} ds \\
g_0 &= \begin{bmatrix} G_1^T & G_2^T \end{bmatrix}^T, \quad G_1 = \begin{bmatrix} e_1 - e_2 \\ e_1 + e_2 - 2e_4 \end{bmatrix}, \quad G_2 = \begin{bmatrix} e_2 - e_3 \\ e_2 + e_3 - 2e_5 \end{bmatrix} \\
\chi_0 &= \begin{bmatrix} Z_1 & S_1 \\ * & Z_1 \end{bmatrix},
\end{aligned}$$

5.3 Improvement stability criterion based on relaxed three integral inequality

This section discusses the methods of estimating S(t). The commonly used method based on

Lemma 17. *For a block $2n \times 2n$ symmetric matrix $Z_1 = diag(R, 3R)$, with $R > 0$ and any $2n \times 2n$ matrix S_1, S_2, S_3, let $\chi_i > 0 (i = 1, 2, 3)$, the S(t) can be estimated as:*

$$S(t) \leq -\frac{1}{h}\zeta_2(t) g^T (\psi_2 - \phi) g \zeta_2(t) \tag{5.3.1}$$

where

$$
\begin{aligned}
S(t) &= -\int_{t-d_1(t)}^{t} \dot{x}^T(s)R\dot{x}(s)ds - \int_{t-d(t)}^{t-d_1(t)} \dot{x}^T(s)R\dot{x}(s)ds - \int_{t-h}^{t-d(t)} \dot{x}^T(s)R\dot{x}(s)ds \\
\zeta_2(t) &= \begin{bmatrix} x^T(t), & x^T(t-d_1(t)), & x^T(t-d(t)), & x^T(t-h), & V_3^T(t), & V_4^T(t), & V_5^T(t) \end{bmatrix}^T \\
V_3(t) &= \int_{t-d_1(t)}^{t} \frac{x(s)}{d_1(t)}ds, \quad V_4(t) = \int_{t-d(t)}^{t-d_1(t)} \frac{x(s)}{d_2(t)}ds, \quad V_5(t) = \int_{t-h}^{t-d(t)} \frac{x(s)}{h-d(t)}ds \\
g &= \begin{bmatrix} G_a^T & G_b^T & G_c^T \end{bmatrix}^T \\
G_a &= \begin{bmatrix} e_1 - e_2 \\ e_1 + e_2 - 2e_5 \end{bmatrix}, \quad G_b = \begin{bmatrix} e_2 - e_3 \\ e_2 + e_3 - 2e_6 \end{bmatrix}, \quad G_c = \begin{bmatrix} e_3 - e_4 \\ e_3 + e_4 - 2e_7 \end{bmatrix} \\
\psi_1 &= \begin{bmatrix} Z_1 & S_1 & S_2 \\ * & Z_2 & S_3 \\ * & * & Z_3 \end{bmatrix}, \\
\chi_{10} &= \begin{bmatrix} Z_1 & S_1 \\ * & Z_1 \end{bmatrix}, \quad \chi_{20} = \begin{bmatrix} Z_2 & S_2 \\ * & Z_2 \end{bmatrix}, \quad \chi_{30} = \begin{bmatrix} Z_3 & S_3 \\ * & Z_3 \end{bmatrix}, \\
\psi_2 &= \begin{bmatrix} \frac{2h-d_1(t)}{h}Z_1 & \frac{2h-d(t)}{h}S_1 & \frac{h+d_2(t)}{h}S_2 \\ * & \frac{2h-d_2(t)}{h}Z_2 & \frac{h+d_1(t)}{h}S_3 \\ * & * & \frac{h+d_1(t)}{h}Z_3 \end{bmatrix} \\
\phi &= \frac{h-d(t)}{h}\left(\begin{bmatrix} S_2 \\ S_3 \\ 0 \end{bmatrix} Z_1^{-1} \begin{bmatrix} S_2^T & S_3^T & 0 \end{bmatrix} \right) \\
&\quad + \frac{d_2(t)}{h}\left(\begin{bmatrix} S_1 \\ 0 \\ S_3^T \end{bmatrix} Z_2^{-1} \begin{bmatrix} S_1^T & 0 & S_3 \end{bmatrix} \right) + \frac{d_1(t)}{h}\left(\begin{bmatrix} 0 \\ S_1^T \\ S_2^T \end{bmatrix} Z_3^{-1} \begin{bmatrix} 0 & S_1 & S_2 \end{bmatrix} \right)
\end{aligned}
$$

Proof: By setting $\lambda(s,a,b) = \frac{2s-b-a}{b-a}$, The following integral can be calculated by the Matlab:

$$\int_a^b \dot{x}(s)ds = x(b) - x(a) \tag{5.3.2}$$

$$\int_a^b \lambda(s,a,b)\dot{x}(s)ds = x(b) + x(a) - \frac{2}{b-a}\int_a^b x(s)ds \tag{5.3.3}$$

$$\int_a^b \lambda(s,a,b)ds = 0 \tag{5.3.4}$$

$$\int_a^b \lambda^2(s,a,b)ds = \frac{b-a}{3} \tag{5.3.5}$$

$$\begin{bmatrix} M_{2i-1}R_i^{-1}M_{2i-1}^T & M_{2i-1}R_i^{-1}M_{2i}^T & M_{2i-1} \\ * & M_{2i}R_i^{-1}M_{2i}^T & M_{2i} \\ * & * & R_i \end{bmatrix} \geq 0 \tag{5.3.6}$$

$$\Pi_i = -\int_{t_{i+1}}^{t_i} \begin{bmatrix} g_1 \\ f_i g_1 \\ \dot{x}(s) \end{bmatrix}^T \begin{bmatrix} M_{2i-1}R_i^{-1}M_{2i-1}^T & M_{2i-1}R_i^{-1}M_{2i}^T & M_{2i-1} \\ * & M_{2i}R_i^{-1}M_{2i}^T & M_{2i} \\ * & * & R_i \end{bmatrix} \begin{bmatrix} g_1 \\ f_i g_1 \\ \dot{x}(s) \end{bmatrix} ds \leq 0$$

$$\begin{aligned}
g_1 &= [G_1^T, G_2^T, G_3^T]^T \xi(t), \quad f_1 = \lambda(s, t-d_1(t), t), \\
f_2 &= \lambda(s, t-d(t), t-d_1(t)), \quad f_3 = \lambda(s, t-d_1(t), t-h) \\
t_1 &= t, \quad t_2 = t-d_1(t), \quad t_3 = t-d(t), \quad t_4 = t-h \\
M_1 &= -\frac{1}{h}[R_1,\ 0,\ X_1^T,\ X_2^T]^T, \quad M_2 = -\frac{1}{h}[0,\ 3R_1,\ X_3^T,\ X_4^T]^T \\
M_3 &= -\frac{1}{h}[Y_1^T,\ R_2,\ 0,\ X_5^T]^T, \quad M_4 = -\frac{1}{h}[Y_2^T,\ 0,\ 3R_2,\ X_6^T]^T \\
M_5 &= -\frac{1}{h}[Y_3^T,\ Y_5^T,\ R_3,\ 0]^T, \quad M_6 = -\frac{1}{h}[Y_4^T,\ Y_6^T,\ 0,\ 3R_3]^T \\
Z_i &= diag\{R_i,\ 3R_i\}\,(i = 1,2,3) \\
S_1 &= [X_1,\ X_3]^T = [Y_1,\ Y_2],\ S_2 = [X_2,\ X_4]^T = [Y_3,\ Y_4], \\
S_3 &= [X_5,\ X_6]^T = [Y_5,\ Y_6]
\end{aligned}$$

$$
\begin{aligned}
&- \int_{t-d_1(t)}^{t} \begin{bmatrix} g_1 \\ f_1 g_1 \end{bmatrix}^T \begin{bmatrix} M_1 R_1^{-1} M_1^T & M_1 R_1^{-1} M_2^T \\ * & M_2 R_1^{-1} M_2^T \end{bmatrix} \begin{bmatrix} g_1 \\ f_1 g_1 \end{bmatrix} ds \\
&= -\int_{t-d_1(t)}^{t} g_1^T M_1 R_1^{-1} M_1^T g_1 ds - 2\int_{t-d_1(t)}^{t} f_1 g_1^T M_1 R_1^{-1} M_2^T g_1 ds \\
&\quad -\int_{t-d_1(t)}^{t} f_1 g_1^T M_2 R_1^{-1} M_2^T f_1 g_1 ds \\
&= -g_1^T M_1 R_1^{-1} M_1^T g_1 (t-(t-d_1(t))) - 2\int_{t-d_1(t)}^{t} f_1 ds g_1^T M_1 R_1^{-1} M_2^T g_1 \\
&\quad -\int_{t-d_1(t)}^{t} f_1^2 ds g_1^T M_2 R_1^{-1} M_2^T g_1 \\
&= -d_1(t) g_1^T M_1 R_1^{-1} M_1^T g_1 - 0 * g_1^T M_1 R_1^{-1} M_2^T g_1 - \frac{d_1(t)}{3} g_1^T M_2 R_1^{-1} M_2^T g_1 \\
&= -\frac{d_1(t)}{h^2} \xi(t) \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}^T \begin{bmatrix} Z_1 & S_1 & S_2 \\ S_1^T & S_1^T Z_1^{-1} S_1 & S_1^T Z_1^{-1} S_2 \\ S_2^T & S_2^T Z_1^{-1} S_1 & S_2^T Z_1^{-1} S_2 \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} \xi(t) \qquad (5.3.7)
\end{aligned}
$$

$$
-2\int_{t-d_1(t)}^{t} \begin{bmatrix} g_1 \\ f_1 g_1 \end{bmatrix}^T \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \dot{x}(s) ds = \frac{1}{h} \xi(t) \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}^T \begin{bmatrix} 2Z_1 & S_1 & S_2 \\ S_1^T & 0 & 0 \\ S_2^T & 0 & 0 \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} \xi(t)
$$

$$
\begin{aligned}
&- \int_{t-d(t)}^{t-d_1(t)} \begin{bmatrix} g_1 \\ f_2 g_1 \end{bmatrix}^T \begin{bmatrix} M_3 R_2^{-1} M_3^T & M_3 R_2^{-1} M_4^T \\ * & M_4 R_2^{-1} M_4^T \end{bmatrix} \begin{bmatrix} g_1 \\ f_2 g_1 \end{bmatrix} ds \\
&= -\frac{d(t)-d_1(t)}{h^2} \xi(t) \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}^T \begin{bmatrix} S_1 Z_2^{-1} S_1^T & S_1 & S_1 Z_2^{-1} S_3 \\ S_1^T & Z_2 & S_3 \\ S_3^T Z_2^{-1} S_1^T & S_3^T & S_3^T Z_2^{-1} S_3 \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} \xi(t)
\end{aligned}
$$

$$
-\ 2\int_{t-d(t)}^{t-d_1(t)} \begin{bmatrix} g_1 \\ f_2 g_1 \end{bmatrix}^T \begin{bmatrix} M_3 \\ M_4 \end{bmatrix} \dot{x}(s) ds = \frac{1}{h} \xi(t) \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}^T \begin{bmatrix} 0 & S_1 & 0 \\ S_1^T & 2Z_2 & S_3 \\ 0 & S_3^T & 0 \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} \xi(t)
$$

$$
\begin{aligned}
- & \int_{t-h}^{t-d(t)} \begin{bmatrix} g_1 \\ f_3 g_1 \end{bmatrix}^T \begin{bmatrix} M_5 R_3^{-1} M_5^T & M_5 R_3^{-1} M_6^T \\ * & M_6 R_3^{-1} M_6^T \end{bmatrix} \begin{bmatrix} g_1 \\ f_3 g_1 \end{bmatrix} ds \\
= & -\frac{h-d(t)}{h^2}\xi(t) \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}^T \begin{bmatrix} S_2 Z_3^{-1} S_2^T & S_2 Z_3^{-1} S_3^T & S_2 \\ S_3 Z_3^{-1} S_2^T & S_3 Z_3^{-1} S_3^T & S_3 \\ S_2^T & S_3^T & Z_3 \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} \xi(t)
\end{aligned}
$$

$$
-2\int_{t-h}^{t-d(t)} \begin{bmatrix} g_1 \\ f_3 g_1 \end{bmatrix}^T \begin{bmatrix} M_5 \\ M_6 \end{bmatrix} \dot{x}(s) ds = \frac{1}{h}\xi(t) \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & S_2 \\ 0 & 0 & S_3 \\ S_2^T & S_3^T & 2Z_3 \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} \xi(t)
$$

$$
\sum_{i=1}^{3} \Pi_i = S(t) + \frac{1}{h}\xi(t) \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}^T \Theta \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} \xi(t)
$$

$$
\begin{aligned}
\Theta = & \begin{bmatrix} Z_1 & S_1 & S_2 \\ * & Z_2 & S_3 \\ * & * & Z_3 \end{bmatrix} \\
& + \begin{bmatrix} \frac{d(t)-d_1(t)}{h} T_1 + \frac{h-d(t)}{h} T_2 & \frac{h-d(t)}{h} N_1 & \frac{d(t)-d_1(t)}{h} N_2 \\ * & \frac{d_1(t)}{h} T_3 + \frac{h-d(t)}{h} T_4 & \frac{d_1(t)}{h} N_3 \\ * & * & \frac{d_1(t)}{h} T_5 + \frac{d(t)-d_1(t)}{h} T_6 \end{bmatrix} \\
= & \begin{bmatrix} \frac{2h-d_1(t)}{h} Z_1 & \frac{2h-d(t)}{h} S_1 & \frac{h+d_2(t)}{h} S_2 \\ * & \frac{2h-d_2(t)}{h} Z_2 & \frac{h+d_1(t)}{h} S_3 \\ * & * & \frac{h+d_1(t)}{h} Z_3 \end{bmatrix} - \frac{h-d(t)}{h} \begin{bmatrix} S_2 \\ S_3 \\ 0 \end{bmatrix} Z_1^{-1} \begin{bmatrix} S_2^T & S_3^T & 0 \end{bmatrix} \\
& -\frac{d_2(t)}{h} \begin{bmatrix} S_1 \\ 0 \\ S_3^T \end{bmatrix} Z_2^{-1} \begin{bmatrix} S_1^T & 0 & S_3 \end{bmatrix} - \frac{d_1(t)}{h} \begin{bmatrix} 0 \\ S_1^T \\ S_2^T \end{bmatrix} Z_3^{-1} \begin{bmatrix} 0 & S_1 & S_2 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
T_1 &= Z_1 - S_1 Z_2^{-1} S_1^T \quad T_2 = Z_1 - S_2 Z_3^{-1} S_2^T \quad T_3 = Z_2 - S_1^T Z_1^{-1} S_1 \\
T_4 &= Z_2 - S_3 Z_3^{-1} S_3^T \quad T_5 = Z_3 - S_2^T Z_1^{-1} S_2 \quad T_6 = Z_3 - S_3^T Z_2^{-1} S_3 \\
N_1 &= S_1 - S_2 Z_3^{-1} S_3^T \quad N_2 = S_2 - S_1 Z_2^{-1} S_3 \quad N_3 = S_3 - S_1^T Z_1^{-1} S_2
\end{aligned}
$$

$$\psi_2 = \begin{bmatrix} \frac{2h-d_1(t)}{h}Z_1 & \frac{2h-d(t)}{h}S_1 & \frac{h+d_2(t)}{h}S_2 \\ * & \frac{2h-d_2(t)}{h}Z_2 & \frac{h+d_1(t)}{h}S_3 \\ * & * & \frac{h+d_1(t)}{h}Z_3 \end{bmatrix} \tag{5.3.8}$$

$$\begin{aligned} \phi &= \frac{h-d(t)}{h}\left(\begin{bmatrix} S_2 \\ S_3 \\ 0 \end{bmatrix} Z_1^{-1}\begin{bmatrix} S_2^T & S_3^T & 0 \end{bmatrix}\right) \\ &+\frac{d_2(t)}{h}\left(\begin{bmatrix} S_1 \\ 0 \\ S_3^T \end{bmatrix} Z_2^{-1}\begin{bmatrix} S_1^T & 0 & S_3 \end{bmatrix}\right) + \frac{d_1(t)}{h}\left(\begin{bmatrix} 0 \\ S_1^T \\ S_2^T \end{bmatrix} Z_3^{-1}\begin{bmatrix} 0 & S_1 & S_2 \end{bmatrix}\right) \end{aligned} \tag{5.3.9}$$

let

$$\begin{aligned} \dot{V}(t) &\leq P(t) + S(t) \\ &\leq \zeta_2^T(t)(\Xi - \frac{1}{h}g^T(\psi_2 - \phi)g)\zeta_2(t) \\ &= \zeta_2^T(t)(\Xi - \frac{1}{h}g^T\psi_2 g + \frac{1}{h}g^T\phi g)\zeta_2(t) \end{aligned}$$

therefore, $\Xi - \frac{1}{h}g^T\psi g + \frac{1}{h}g^T\phi g < 0$, leads to $\dot{V}(t) \leq \epsilon \parallel x(t) \parallel^2$ for a sufficient small scalar $\epsilon > 0$. using Schur complement we can got when $d(t) = 0$,

$$\Xi - \frac{1}{h}g^T\psi_2 g + \frac{1}{h}g^T\phi g < 0$$

$$\Xi - \frac{1}{h}g^T\psi_2 g + \frac{1}{h}g^T\left(\begin{bmatrix} S_2 \\ S_3 \\ 0 \end{bmatrix} Z_1^{-1}\begin{bmatrix} S_2^T & S_3^T & 0 \end{bmatrix}\right) g < 0$$

$$\Sigma_1 = \begin{bmatrix} \Xi - \frac{1}{h}g^T\psi_2 g & g^T\begin{bmatrix} S_2 \\ S_3 \\ 0 \end{bmatrix} \\ * & -hZ_1 \end{bmatrix} < 0$$

when $d(t) = h$,

$$\Xi - \frac{1}{h} g^T \psi_2 g + \frac{1}{h} g^T \phi g < 0$$

$$\Xi - \frac{1}{h} g^T \psi_2 g + g^T \frac{d_2(t)}{h^2} \left(\begin{bmatrix} S_1 \\ 0 \\ S_3^T \end{bmatrix} Z_2^{-1} \begin{bmatrix} S_1^T & 0 & S_3 \end{bmatrix} \right) g$$

$$+ g^T \frac{d_1(t)}{h^2} \left(\begin{bmatrix} 0 \\ S_1^T \\ S_2^T \end{bmatrix} Z_3^{-1} \begin{bmatrix} 0 & S_1 & S_2 \end{bmatrix} \right) g < 0$$

$$\Sigma_2 = \begin{bmatrix} \Xi - \frac{1}{h} g^T \psi_2 g & g^T \begin{bmatrix} S_1 \\ S_3^T \end{bmatrix} & g^T \begin{bmatrix} S_1^T \\ S_2^T \end{bmatrix} \\ * & -\frac{h^2}{d_2(t)} Z_2 & 0 \\ * & * & -\frac{h^2}{d_1(t)} Z_3 \end{bmatrix} < 0$$

Remark 15. *In the proof of Lemma 17, the interval*$[t-h, t]$ *is divided into three subintervals* $[t-h, t-d(t)]$, $[t-d(t), t-\alpha d(t)]$ *and*$[t-\alpha d(t), t]$, *the information of delayed state* $x(t-\alpha d(t))$ *and* $x(t-d(t))$ *can be taken into account. It is clear that the Lyapunov function defined in Lemma 17 are more general than the ones in some existing results. Since the delay decomposition approach is introduced in time delay, it is clear that the stability results are based on the delay decomposition approach. When the positions of delay decomposition are varied,the stability results of proposed criteria are also different.In order to obtain the optimal delay decomposition sequence. The proposed stability conditions are much less conservative and are more general than some existing results.*

5.3.1 New theorem which using lemma 17

Theorem 5. *For given scalars* α, *and* μ, *system (5.2.1) with the time-varying delay satisfying (5.2.2) and (5.2.3) is asymptotically stable if one of the following conditions holds*

A1:there exist a $5n \times 5n$*-matrix* $P = P^T \geqslant 0$, $2n \times 2n$*-matrices* $Q_i = Q_i^T (i = 1, 2, 3)$, $R = R^T > 0$,$Q_4 = Q_4^T > 0$, *and any* $2n \times 2n$*-matrices* $S_i (i = 1, 2, 3)$, *such* *that the following conditions hold*

$$\chi_1 = \begin{bmatrix} R_1 & S_1 \\ * & R_1 \end{bmatrix} \geq 0, \ \chi_2 = \begin{bmatrix} R_1 & S_2 \\ * & R_1 \end{bmatrix} \geq 0, \tag{5.3.10}$$

$$\chi_3 = \begin{bmatrix} R_1 & S_3 \\ * & R_1 \end{bmatrix} \geq 0 \tag{5.3.11}$$

$$\Psi_1 < 0 \tag{5.3.12}$$

A2:there exist a $5n \times 5n$*-matrix* $P = P^T \geqslant 0$*,* $2n \times 2n$*-matrices* $Q_i = Q_i^T (i = 1, 2, 3)$*,*$Q_4 = Q_4^T > 0$*,* $R = R^T > 0$*, and any* $2n \times 2n$*-matrices* $S_i (i = 1, 2, 3)$*, such that the following conditions hold*

$$\Phi_1 = \begin{bmatrix} \Psi_2|_d(t) = 0 & X_1 \\ * & -R_1 \end{bmatrix} < 0 \tag{5.3.13}$$

$$\Phi_2 = \begin{bmatrix} \Psi_2|_d(t) = h & X_2 & X_3 \\ * & -R & 0 \\ * & * & -R_1 \end{bmatrix} < 0 \tag{5.3.14}$$

where

$$\Psi_1 = \Xi_1 + \Xi_2 + \Xi_3 + \Xi_4 - \Xi_5 \tag{5.3.15}$$

$$\Psi_2 = \Xi_1 + \Xi_2 + \Xi_3 + \Xi_4 - \Xi_5 - \Xi_6 \tag{5.3.16}$$

$$\Xi_1 = He(F_1^T P F_a) \tag{5.3.17}$$

$$\begin{aligned} \Xi_2 = & \begin{bmatrix} e_1 \\ 0 \end{bmatrix}^T Q_1 \begin{bmatrix} e_1 \\ 0 \end{bmatrix} - \begin{bmatrix} e_4 \\ e_3 - e_4 \end{bmatrix}^T Q_3 \begin{bmatrix} e_4 \\ e_3 - e_4 \end{bmatrix} \\ & -(1 - \alpha \dot{d}(t))(\begin{bmatrix} e_2 \\ e_1 - e_2 \end{bmatrix}^T Q_1 \begin{bmatrix} e_2 \\ e_1 - e_2 \end{bmatrix} - \begin{bmatrix} e_2 \\ e_2 - e_3 \end{bmatrix}^T Q_2 \begin{bmatrix} e_2 \\ e_2 - e_3 \end{bmatrix}) \quad (5.3.18) \\ & -(1 - \mu)(\begin{bmatrix} e_3 \\ e_2 - e_3 \end{bmatrix}^T Q_2 \begin{bmatrix} e_3 \\ e_2 - e_3 \end{bmatrix} - \begin{bmatrix} e_3 \\ 0 \end{bmatrix}^T Q_3 \begin{bmatrix} e_3 \\ 0 \end{bmatrix}) \\ & +2\alpha d(t) \begin{bmatrix} 0 \\ e_s \end{bmatrix}^T Q_1 \begin{bmatrix} e_6 \\ e_1 - e_6 \end{bmatrix} - 2(1 - \alpha) d(t)(1 - \alpha \dot{d}(t)) \begin{bmatrix} 0 \\ e_5 \end{bmatrix}^T Q_2 \begin{bmatrix} e_7 \\ e_7 - e_3 \end{bmatrix} \\ & +2(h - d(t))(1 - \dot{d}(t)) \begin{bmatrix} 0 \\ e_5 \end{bmatrix}^T Q_3 \begin{bmatrix} e_8 \\ e_3 - e_8 \end{bmatrix} \quad (5.3.19) \end{aligned}$$

$$\Xi_3 = he_s^T R e_s \tag{5.3.20}$$

$$\Xi_4 = \begin{bmatrix} e_s^T Q_4 e_s - (1-\dot{d}(t)) e_5^T Q_4 e_5 \end{bmatrix} \tag{5.3.21}$$

$$\Xi_5 = \frac{1}{h} g^T \chi g \tag{5.3.22}$$

$$\Xi_6 = \frac{h-d(t)}{h} g_1^T \chi_1 g_1 + \frac{(1-\alpha)d(t)}{h} g_2^T \chi_2 g_2 + \frac{\alpha d(t)}{h} g_2^T \chi_3 g_2 \tag{5.3.23}$$

$$\chi = \begin{bmatrix} R_1 & S_1 & S_2 \\ * & R_1 & S_3 \\ * & * & R_1 \end{bmatrix}, \ R_1 = \begin{bmatrix} R & 0 \\ * & 3R \end{bmatrix} \tag{5.3.24}$$

$$F_1 = \begin{bmatrix} e_1 \\ e_3 \\ \alpha d(t) e_6 \\ (1-\alpha) d(t) e_7 \\ (h-d(t)) e_8 \end{bmatrix}, \ F_a = \begin{bmatrix} e_s \\ e_5 \\ e_1 - [1-\alpha\dot{d}(t)] e_2) \\ [1-\alpha\dot{d}_1(t)] e_2 - [1-\dot{d}(t)] e_3 \\ [1-\dot{d}(t)] e_3 - e_4 \end{bmatrix} \tag{5.3.25}$$

$$g = \begin{bmatrix} G_a^T & G_b^T & G_c^T \end{bmatrix}^T, \ g_1 = \begin{bmatrix} G_a^T & G_b^T \end{bmatrix}^T, \tag{5.3.26}$$

$$g_2 = \begin{bmatrix} G_a^T & G_c^T \end{bmatrix}^T, \ g_3 = \begin{bmatrix} G_b^T & G_c^T \end{bmatrix}^T \tag{5.3.27}$$

$$G_a = \begin{bmatrix} e_1 - e_2 \\ e_1 + e_2 - 2e_6 \end{bmatrix}, \ G_b = \begin{bmatrix} e_2 - e_3 \\ e_2 + e_3 - 2e_7 \end{bmatrix}, \tag{5.3.28}$$

$$G_c = \begin{bmatrix} e_3 - e_4 \\ e_3 + e_4 - 2e_8 \end{bmatrix} \tag{5.3.29}$$

$$e_j = \begin{bmatrix} 0_{n\times(i-1)n}, & I_n, & 0_{n\times(8-i)n} \end{bmatrix} \tag{5.3.30}$$

$$e_s = Ae_1 + A_d e_3 \tag{5.3.31}$$

$$X_1 = \sqrt{\frac{1}{h}} g_1^T \begin{bmatrix} S_2^T & S_3^T \end{bmatrix}^T, \ X_2 = \sqrt{\frac{1-\alpha}{h}} g_2^T \begin{bmatrix} S_1^T & S_3 \end{bmatrix}^T, \tag{5.3.32}$$

$$X_3 = \sqrt{\frac{\alpha}{h}} g_3^T \begin{bmatrix} S_1 & S_2 \end{bmatrix}^T \tag{5.3.33}$$

Proof: The following LKF candidate with two delay-product-type terms is constructed

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) \tag{5.3.34}$$

$$
\begin{aligned}
V_1(t) &= \bar{x}^T(t)P\bar{x}(t)\\
V_2(t) &= \int_{t-\alpha d(t)}^{t}\begin{bmatrix} x(s)\\ \int_s^t \dot{x}(u)du\end{bmatrix}^T Q_1 \begin{bmatrix} x(s)\\ \int_s^t \dot{x}(u)du\end{bmatrix} ds\\
&\quad + \int_{t-d(t)}^{t-\alpha d(t)}\begin{bmatrix} x(s)\\ \int_{t-d(t)}^{s} \dot{x}(u)du\end{bmatrix}^T Q_2 \begin{bmatrix} x(s)\\ \int_{t-d(t)}^{s} \dot{x}(u)du\end{bmatrix} ds\\
&\quad + \int_{t-h}^{t-d(t)}\begin{bmatrix} x(s)\\ \int_{s}^{t-d(t)} \dot{x}(u)du\end{bmatrix}^T Q_3 \begin{bmatrix} x(s)\\ \int_{s}^{t-d(t)} \dot{x}(u)du\end{bmatrix} ds\\
V_3(t) &= h\int_{-h}^{0}\int_{t+s}^{t}\dot{x}^T(u)R\dot{x}(u)\,du\,ds\\
V_4(t) &= \int_{t-d(t)}^{t}\dot{x}^T(s)Q_4x(s)\,ds
\end{aligned}
$$

If the matrices in $V(x_t)$ satisfying $P > 0$ $Q_i = Q_i^T$ $(i = 1,2,3)$, and $R > 0$, then $V(x_t) \geq \varepsilon_1||x(t)||$ for a scalar $\varepsilon_1 > 0$. On the other side, the conditions guaranteeing the negative definite of the derivative of $V(x_t)$ are discussed. let

$$
\begin{aligned}
\xi_1(t) &= \Big[x^T(t),\ x^T(t-\alpha d(t)),\ x^T(t-d(t)),\ x^T(t-h),\ \dot{x}^T(t-d(t))\\
&\qquad \int_{t-\alpha d(t)}^{t}\frac{x^T(s)}{\alpha d(t)}ds,\ \int_{t-d(t)}^{t-\alpha d(t)}\frac{x^T(s)}{(1-\alpha)d(t)}ds,\ \int_{t-h}^{t-d(t)}\frac{x^T(s)}{h-d(t)}ds,\ \Big]
\end{aligned}
$$

Calculating the derivative of $V_1(x_t)$ yields

$$
\begin{aligned}
\dot{V}_1(t) &= 2\bar{x}^T(t)P\dot{\bar{x}}(t)\\
&= 2\begin{bmatrix} x(t)\\ x(t-d(t))\\ \int_{t-\alpha d(t)}^{t} x(s)ds\\ \int_{t-d(t)}^{t-\alpha d(t)} x(s)ds\\ \int_{t-h}^{t-d(t)} x(s)ds\end{bmatrix} P \begin{bmatrix} \dot{x}(t)\\ (1-\dot{d}(t))\dot{x}^T(t-d(t))\\ x(t)-[1-\alpha\dot{d}(t)]x(t-\alpha d(t))\\ [1-\alpha\dot{d}(t)]x(t-\alpha d(t))-[1-\dot{d}(t)]x(t-d(t))\\ [1-\dot{d}(t)]x(t-d(t))-x(t-h)\end{bmatrix}\\
&= \xi_1^T(t)He(F_1^TPF_a)\xi_1(t)\\
&= \xi_1^T(t)\Xi_1\xi_1(t)
\end{aligned} \tag{5.3.35}
$$

where F_1 and F_a are defined in (5.3.25) and Ξ_1 are defined in (5.3.32).

Calculating the derivative of the $V_2(x_t)$ along the solutions of system leads to

$$
\begin{aligned}
\dot{V}_2(t) &= \begin{bmatrix} x(t) \\ 0 \end{bmatrix}^T Q_1 \begin{bmatrix} x(t) \\ 0 \end{bmatrix} - (1-\alpha\dot{d}(t)) \begin{bmatrix} x(t-\alpha d(t)) \\ \int_{t-\alpha d(t)}^{t} \dot{x}(u)\,du \end{bmatrix}^T Q_1 \begin{bmatrix} x(t-\alpha d(t)) \\ \int_{t-\alpha d(t)}^{t} \dot{x}(u)\,du \end{bmatrix} \\
&+2\alpha d(t) \begin{bmatrix} 0 \\ \dot{x}(t) \end{bmatrix}^T Q_1 \begin{bmatrix} \frac{1}{\alpha d(t)} \int_{t-\alpha d(t)}^{t} x(s)\,ds \\ x(t) - \frac{1}{\alpha d(t)} \int_{t-\alpha d(t)}^{t} x(s)\,ds \end{bmatrix} \\
&+(1-\alpha\dot{d}(t)) \begin{bmatrix} x(t-\alpha d(t)) \\ \int_{t-d(t)}^{t-\alpha d(t)} \dot{x}(u)du \end{bmatrix}^T Q_2 \begin{bmatrix} x(t-\alpha d(t)) \\ \int_{t-d(t)}^{t-\alpha d(t)} \dot{x}(u)du \end{bmatrix} \\
&-(1-\dot{d}(t)) \begin{bmatrix} x(t-d(t)) \\ 0 \end{bmatrix}^T Q_2 \begin{bmatrix} x(t-d(t)) \\ 0 \end{bmatrix} \\
&-2(1-\alpha)d(t)(1-\dot{d}(t)) \begin{bmatrix} 0 \\ \dot{x}(t-d(t)) \end{bmatrix}^T Q_2 \begin{bmatrix} \frac{1}{(1-\alpha)d(t)} \int_{t-d(t)}^{t-\alpha d(t)} x(s)\,ds \\ \frac{1}{(1-\alpha)d(t)} \int_{t-d(t)}^{t-\alpha d(t)} x(s)\,ds - x(t-d(t)) \end{bmatrix} \\
&+(1-\dot{d}(t)) \begin{bmatrix} x(t-d(t)) \\ 0 \end{bmatrix}^T Q_3 \begin{bmatrix} x(t-d(t)) \\ 0 \end{bmatrix} \\
&- \begin{bmatrix} x(t-h) \\ \int_{t-h}^{t-d(t)} \dot{x}(u)\,du \end{bmatrix}^T Q_3 \begin{bmatrix} x(t-h) \\ \int_{t-h}^{t-d(t)} \dot{x}(u)\,du \end{bmatrix} \\
&+2(h-d(t))(1-\dot{d}(t)) \begin{bmatrix} 0 \\ \dot{x}(t-d(t)) \end{bmatrix}^T Q_3 \begin{bmatrix} \frac{1}{h-d(t)} \int_{t-h}^{t-d(t)} x(s)\,ds \\ x(t-d(t)) - \frac{1}{h-d(t)} \int_{t-h}^{t-d(t)} x(s)\,ds \end{bmatrix} \\
&\le \xi_1^T(t)[\begin{bmatrix} e_1 \\ 0 \end{bmatrix}^T Q_1 \begin{bmatrix} e_1 \\ 0 \end{bmatrix} - \begin{bmatrix} e_4 \\ e_3 - e_4 \end{bmatrix}^T Q_3 \begin{bmatrix} e_4 \\ e_3 - e_4 \end{bmatrix} \\
&-(1-\alpha\dot{d}(t))(\begin{bmatrix} e_2 \\ e_1 - e_2 \end{bmatrix}^T Q_1 \begin{bmatrix} e_2 \\ e_1 - e_2 \end{bmatrix} - \begin{bmatrix} e_2 \\ e_2 - e_3 \end{bmatrix}^T Q_2 \begin{bmatrix} e_2 \\ e_2 - e_3 \end{bmatrix}) \\
&-(1-\dot{d}(t))(\begin{bmatrix} e_3 \\ 0 \end{bmatrix}^T (Q_2 - Q_3) \begin{bmatrix} e_3 \\ 0 \end{bmatrix}) + 2\alpha d(t) \begin{bmatrix} 0 \\ e_s \end{bmatrix}^T Q_1 \begin{bmatrix} e_6 \\ e_1 - e_6 \end{bmatrix} \\
&-2(1-\alpha)d(t)(1-\alpha\dot{d}(t)) \begin{bmatrix} 0 \\ e_5 \end{bmatrix}^T Q_2 \begin{bmatrix} e_7 \\ e_7 - e_2 \end{bmatrix} \\
&+2(h-d(t))(1-\dot{d}(t)) \begin{bmatrix} 0 \\ e_5 \end{bmatrix}^T Q_3 \begin{bmatrix} e_8 \\ e_3 - e_8 \end{bmatrix}]\xi_1(t) \\
&= \xi_1^T(t)\Xi_2\xi_1(t)
\end{aligned}
\tag{5.3.36}
$$

where Ξ_2 is defined in (5.3.32). Calculating the derivative of the $V_3(x_t)$ yields

$$\begin{aligned}\dot{V}_3(t) &= h^2\dot{x}^T(t)R\dot{x}(t) - h\int_{t-h}^{t}\dot{x}^T(s)R\dot{x}(s)ds \\ &= \xi_1^T(t)\Xi_3\xi_1(t) - hS(t) \end{aligned} \tag{5.3.37}$$

Calculating the derivative of $V_4(x_t)$ yields

$$\begin{aligned}\dot{V}_4(t) &= \dot{x}^T(t)Q_4\dot{x}(t) - (1-\dot{d}(t))\dot{x}^T(t-d(t))Q_4\dot{x}(t-d(t)) \\ &= \xi_1^T\left[e_s^TQ_4e_s - (1-\dot{d}(t))e_5^TQ_4e_5\right]\xi_1 \\ &= \xi_1^T\Xi_4\xi_1\end{aligned}$$

On the one hand, if applying past inequality to estimate $S(t)$ appearing in (5.3.36), the $V(t)$ can be estimated as

$$\begin{aligned}\dot{V}(t) &\leq \xi_1^T(t)\left(\Xi_1+\Xi_2+\Xi_3+\Xi_4 - \begin{bmatrix}G_a\\G_b\\G_c\end{bmatrix}^T\begin{bmatrix}R_1 & S_1 & S_2\\ * & R_1 & S_3\\ * & * & R_1\end{bmatrix}\begin{bmatrix}G_a\\G_b\\G_c\end{bmatrix}\right)\xi_1(t) \\ &= \xi_1^T(t)\Psi_1\xi_1\end{aligned} \tag{5.3.38}$$

where Ψ_1 is defined in (5.3.15). Therefore, $\Psi_1 < 0$ leads to $\dot{V}(t) \leq -\varepsilon_2||x(t)||^2$ for a sufficient small scalar $\varepsilon_2 > 0$.

On the other hand, if applying equality (5.1.3) to estimate $S(t)$ appearing in (5.3.37), the $V(t)$ can be estimated as

$$\begin{aligned}\dot{V}(t) &\leq \xi_1^T(t)\left\{\Xi_1+\Xi_2+\Xi_3+\Xi_4 - \left(\begin{bmatrix}G_a\\G_b\\G_c\end{bmatrix}^T\begin{bmatrix}R_1 & S_1 & S_2\\ * & R_1 & S_3\\ * & * & R_1\end{bmatrix}\begin{bmatrix}G_a\\G_b\\G_c\end{bmatrix}\right.\right. \\ &\quad + (h-d(t))\begin{bmatrix}G_1\\G_2\end{bmatrix}^T\left(\begin{bmatrix}Z_1 & S_1\\ S_1^T & Z_1\end{bmatrix} - \begin{bmatrix}S_2\\S_3\end{bmatrix}Z_1^{-1}\begin{bmatrix}S_2^T & S_3^T\end{bmatrix}\right)\begin{bmatrix}G_1\\G_2\end{bmatrix} \\ &\quad + ((1-\alpha)d(t))\begin{bmatrix}G_1\\G_3\end{bmatrix}^T\left(\begin{bmatrix}Z_2 & S_2\\ S_2^T & Z_2\end{bmatrix} - \begin{bmatrix}S_1\\S_3^T\end{bmatrix}Z_2^{-1}\begin{bmatrix}S_1^T & S_3\end{bmatrix}\right)\begin{bmatrix}G_1\\G_3\end{bmatrix} \\ &\quad \left.\left. + \alpha d(t)\begin{bmatrix}G_2\\G_3\end{bmatrix}^T\left(\begin{bmatrix}Z_3 & S_3\\ S_3^T & Z_3\end{bmatrix} - \begin{bmatrix}S_1^T\\S_2^T\end{bmatrix}Z_3^{-1}\begin{bmatrix}S_1 & S_2\end{bmatrix}\right)\begin{bmatrix}G_2\\G_3\end{bmatrix}\right)\right\}\xi_1(t) \\ &= \xi_1^T(t)(\Psi_2+\Xi_{b1})\xi_1\end{aligned} \tag{5.3.39}$$

where Ψ_2 is defined in (5.3.16), and

$$\begin{aligned}\Xi_{b1} &= \frac{(1-\alpha)d(t)}{h} g_1^T \begin{bmatrix} S_2 \\ S_3 \end{bmatrix} Z_1^{-1} \begin{bmatrix} S_2^T & S_3^T \end{bmatrix} g_1 \\ &+ \frac{(1-\alpha)d(t)}{h} g_2^T \begin{bmatrix} S_1 \\ S_3^T \end{bmatrix} Z_2^{-1} \begin{bmatrix} S_1^T & S_3 \end{bmatrix} g_2 + \frac{\alpha d(t)}{h} g_3^T \begin{bmatrix} S_1^T \\ S_2^T \end{bmatrix} Z_3^{-1} \begin{bmatrix} S_1 & S_2 \end{bmatrix} g_3\end{aligned} \tag{5.3.40}$$

Therefore, $\Phi_1 < 0$ and $\Phi_2 < 0$, which is equivalent to $\Psi_2 < 0$ and $\Psi_2 + \Xi_{b1} < 0$ based on the Schur complement and convex combination method, leads to $\dot{V}(t) \leq -\varepsilon_2 ||x(t)||^2$ for a sufficient small scalar $\varepsilon_2 > 0$.

Remark 16. *It is clear that the Lyapunov function defined in A2 Theorems 1 are more general than the A1 in [167]. The proposed stability conditions are much less conservative and are more general than some existing results.*

5.4 Numerical example

Two numerical examples listed in Table 5.1 are used to check the conservatism of the stability criteria. The conservatism of the criteria is checked based on the calculated maximal admissible delay upper bounds. Consider system (5.2.1) with the parameters show in Table 5.1 Applied these two numerical examples separately

Table 5.1: systems used as numerical examples.

Criteria	A	A_d
1	$\begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$
2	$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$

in control system 5.2.1 and use Theorem 1 to calculate. The results show in table 5.2 and 5.3.

Based on the results listed in table 5.2 and 5.3, the new resulted is better than previous work. It can be found that the proposed Theorem 1 can provide less conservative results.

Table 5.2: MADUPs for various $\mu = -\mu_1 = \mu_2$(Example 1).

Criteria	$\mu = -\mu_1 = \mu_2$			
	0	0.1	0.5	0.8
Corollary 3 [169]	4.472	3.669	2.337	1.934
Theorem 1 [170]	4.975	3.869	2.337	1.934
Theorem 2 [171]	5.120	4.081	2.528	2.152
Theorem 3 [172]	6.117	4.794	2.682	1.957
Theorem 1.C1 [167]	6.059	4.703	2.420	2.137
Theorem 1.A2				
$\alpha = 1$	6.0593	4.8172	2.9990	2.5854
$\alpha = 0.9$	6.0811	4.8322	2.9832	2.5578
$\alpha = 0.7$	6.0176	4.7714	2.9612	2.5497
$\alpha = 0.5$	5.9878	4.7404	2.9615	2.5621
$\alpha = 0.3$	6.0200	4.7698	2.9834	2.5819
$\alpha = 0.1$	6.0818	4.8269	3.0044	2.5961
$\alpha = 0$	6.0593	4.8162	2.9978	2.5858

- The regular method which divided the interval into two subintervals can be showed in inequality(5.3.37) by recognise the $\alpha = 0$ or $\alpha = 1$. The difference between the delay decomposition approach which divided the interval into two subintervals and three subintervals can be find in above tables. The results show that Theorem 1.A2 provides better result when $0 < \alpha < 1$ than $\alpha = 0$ or $\alpha = 1$.

- The results show that Theorem 1.A2 provides better result than Theorem 1 A1, which verifies the less conservatism of inequality (5.3.37).

- The results show that Theorem 1.A2 provides better result than previous results show in [169] [167].

Table 5.3: MADUPs for various $\mu = -\mu_1 = \mu_2$(Example 2).

Criteria	$\mu = -\mu_1 = \mu_2$		
	0	0.1	0.5
Corollary 3 [169]	2.52	1.75	1.61
Theorem 1 [170]	2.523	2.028	1.622
Theorem 2.C1 [167]	3.136	2.386	1.775
Theorem 2.C2 [167]	3.136	2.397	1.787
Theorem 1.A2			
$\alpha = 1$	3.0347	2.0744	1.9137
$\alpha = 0.9$	3.0716	2.0836	1.9117
$\alpha = 0.7$	3.0886	2.0721	1.8999
$\alpha = 0.5$	3.0895	2.0663	1.8945
$\alpha = 0.3$	3.0907	2.4210	1.9242
$\alpha = 0.1$	3.0727	2.4051	1.9054

5.4.1 Application on load frequency control system

In power systems, LFC has been used effectively for many years. As mentioned in Chapter 2, time delays arising in feedback measurement channel and the forward control channel and calculated its upper bounds are helpful for controller design.

By defining virtual state and measurement output vectors as $x(t) = [\Delta f,\ \Delta P_m,\ \Delta P_v,\ \int ACE]^T$ and $y(t) = [ACE,\ \int ACE]^T$, respectively, and the closed-loop LFC system can be expressed as follows which defined in [173]:

$$\dot{x}(t) = Ax(t) + A_d x(t - d_1(t) - d_2(t)) \tag{5.4.1}$$

where

$$x(t) = \begin{bmatrix} \Delta f \\ \Delta P_m \\ \Delta P_v \\ \int ACE \end{bmatrix}, \quad A = \begin{bmatrix} -\frac{D}{M} & \frac{1}{M} & 0 & 0 \\ 0 & -\frac{1}{T_{ch}} & \frac{1}{T_{ch}} & 0 \\ -\frac{1}{RT_g} & 0 & -\frac{1}{T_g} & 0 \\ \beta & 0 & 0 & 0 \end{bmatrix},$$

$$A_d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{K_p\beta}{T_g} & 0 & 0 & -\frac{K_I}{T_g} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with the parameters given in [55]: $M = 10$, $D = 1$, $T_{ch} = 0.3$, $T_g = 0.1$, $R = 0.05$, and $\beta = 21$.

Table 5.4: MADUPs for various $\mu = -\mu_1 = \mu_2 = 0.9$(LFC).

Criteria	α				NoVs
	0	0.1	0.9	1	
[55]	4.672				
[173]	6.882				$28n^2 + 10n$
Theorem 1.A1	6.8599	6.8847	6.7854	6.8482	
Theorem 1.A2	6.8612	6.9086	6.8523	6.8659	$33n^2 + 7n$

The results listed in table 5.4. It can be clearly find the Theorem 1.A2 offers much better upper bound than previous results. It can be found the results also better than the sum of two time delays which shows in Chapter 2.

5.5 Economic dispatch

Economic dispatch (ED) ensures the operation of power system in the most economical condition minimizing the cost of generating electricity and satisfying the balance of supply and demand as well as the output constraints of each generator.

In recent years, with the fast development of renewable energy, more and more distributed generators integrated into the distributed system of power system. As a result, power systems have become much more decentralized than before. Considering those DGs, the traditional centralized solution will not be the best way to solve the ED. Moreover, the distributed solution can provide more robust, economic and efficient way to solve the ED in distributed power systems. For example, in [174], a distributed algorithm for electricity market is proposed to protect the privacy of load aggregators and generators, and to reduce the computational complexity of the centralized approach. This is because that the connectivity of the distributed structure is usually higher than the centralized structure (i.e., star network), and the calculation can be done by a bunch of cheap devices in parallel in a distributed system instead of the expensive centralized control center. Besides these advantages, the development of distributed control system also benefits the conventional centralized power system, since the distributed control system can be implemented as an auxiliary or backup system for the centralized control system to improve the robustness and efficiency of the power system.

Distributed economic dispatch methods have been proposed recently. One of the major methods is the consensus-based distributed ED approaches [175]. These methods are mainly based on the consensus protocol from graph theory method. Since the necessary condition for the solution to the ED problem requires the Lagrange multipliers of all generators to be identical, the consensus protocol was applied to achieve all equal Lagrange multipliers in these researches. The solution of the distributed ED problem should also meet the power balance constraint (the power supply is equal to the power consumption), this value is estimated locally by each generator through an estimation algorithm and the result of the estimation heavily depends on the initial value [176].

In distributed optimization problems, communication among agents is a key issue. In practice, communication delays are inevitable and may destabilize the system dynamics. In multi agent systems, there are many literature [177–180] dealing with the impact of communication delays. While, only few of them investigate the impact of delay on distributed ED problem. Yang et al. [24] introduced the distributed algorithm for ED problem with uniform communication delays, while communication delays are different in the practical communication network. Most of above mentioned researches have assumed that communication delays are uniform for each component. In practical power system, it is very hard to guarantee this hypothesis, especially as the network grows larger and larger. However, the impacts of communication delays analyse via the Jensen's Inequality and the results still suffer from large conservatism.

5.5.1 Modeling economic dispatch problem

In the microgrid, the model of economic dispatching problem introduces renewable energy in addition to the model of electricity market. Based on this, a simple state are assumed. There are conventional generators and new energy generators in microgrid and the cost of renewable generator is to be zero. The cost function of each conventional generator is defined as given below[27]

$$C_i(P_{Gi}) = \frac{1}{2}c_{Gi}(P_{Gi})^2 + b_{Gi}P_{Gi} + a_{Gi} \tag{5.5.1}$$

$$P_{Gi}^{\min} \leq P_{Gi} \leq P_{Gi}^{\max} \tag{5.5.2}$$

where P_{Gi} is the active power generated by each generating unit $i \in G_f = \{1, 2..., N_G\}$; a_{Gi}, b_{Gi} and c_{Gi} are generators cost coefficients; and P_{Gi}^{min} and P_{Gi}^{max} are the lower and upper bound of power generation of the generator i.

The goal of ED problem is to minimize total cost while meeting total demands and satisfying the output limits of generators. The objective function can be stated as

$$\begin{array}{ll} \min & \left(\sum_{i=1}^{n_C} C_i\left(P_{Gi}\right)\right) \\ \text{subject to} & \sum_{i=1}^{n_G} P_{Gi} + \sum_{l=1}^{n_R} P_{Rl} = P_D \\ & P_{Gi}^{\min} \leq P_{Gi} \leq P_{Gi}^{\max} \end{array} \tag{5.5.3}$$

where n_G is the number of conventional generators; n_R is the number of renewable generators; P_{Rl} is the active power generated by the renewable generator l; and P_D is the total active power demand.

5.5.2 Centralized approach for economic dispatch problem

For ease of calculation, it is assumed that the conventional generator does not have constraints at fisrt. The Lagrange multiplier method can be used to solve the ED problme and the Lagrange function is established for(6)

$$\Gamma = \sum_{i=1}^{n_G} C_i\left(P_{Gi}\right) + \lambda\left(P_D - \sum_{i=1}^{n_C} P_{Gi} - \sum_{l=1}^{n_R} P_{Rl}\right) \tag{5.5.4}$$

where λ is the Lagrange multiplier.

From the first-order optimality conditions, the optimal solution for (5.5.4) is

$$\frac{\partial C_i}{\partial P_{Gi}} - \lambda = 0 \tag{5.5.5}$$

The partial derivatives of the (5.5.1)

$$\frac{\partial C_i}{\partial P_{Gi}} = \alpha_i P_{Gi} + \beta_i \tag{5.5.6}$$

From (5.5.5) and (5.5.6), we get

$$P_{Gi} = \frac{\lambda - \beta_i}{\alpha_i} \tag{5.5.7}$$

Substituting P_{Gi} into the equality constraint, the centralized optimal solution without the constraints can be obtained as

$$\lambda^* = \frac{P_D - \sum_{l=1}^{n_R} P_{Rl} + \sum_{i=1}^{n_G} \frac{\beta_i}{\alpha_i}}{\sum_{i=1}^{n_G} \frac{1}{\alpha_i}} \tag{5.5.8}$$

Next, the constraint of each conventional generator are considered below. If the conventional generator does not exceed the bound, P_{Gi} can be calculated as (5.5.7). If the conventional generator exceeds its upper or lower bound, then $P_{Gi} = P_{Gi}^{\max}$ or $P_{Gi} = P_{Gi}^{\min}$. Thus, the generation of the conventional generator can be calculated as

$$P_{Gi} = \begin{cases} p_{Gi}^{\max}, & \frac{\lambda - \beta_i}{\alpha_i} > p_{Gi}^{\max} \\ p_{Gi}^{\min}, & \frac{\lambda - \beta_i}{\alpha_i} < p_{Gi}^{\min} \\ \frac{\lambda - \beta_i}{\alpha_i} & \text{otherwise} \end{cases} \tag{5.5.9}$$

Let Ω denote the set of conventional generators that exceed their bounds. Then the centralized optimal solution with the constraints $\tilde{\lambda}$ is

$$\widetilde{\lambda} = \frac{P_D - \sum_{l=1}^{n_R} P_{Rl} - \sum_{i \in \Omega} P_{Gi} + \sum_{i \notin \Omega} \frac{\beta_i}{\alpha_i}}{\sum_{i \notin \Omega} \frac{1}{\alpha_i}} \tag{5.5.10}$$

5.5.3 Distributed algorithm with the communication delays

In this section, a multiagent system consisting of n_G agents is considered. Each agent only has access to its local cost function and only needs to receive information from its neighbors. The goal for all agents is to solve Problem (5.5.4) via cooperation. In practical ED problem, due to the distance among the generations, it is necessary to consider communication delays among the generations. Considering the complicated working environment, time-varying delays have more practical significance in engineering applications.

5.5.4 Distributed algorithm without generation constraint

Based on the optimality analysis in the aforementioned section, all generators operate at the optimal configuration. A fully distributed algorithm with time-varying communication delays to solve ED problem for microgrids, as shown[22]

$$\begin{cases} \dot{\widetilde{P}}_{Gi}(t) = -\sum_{j=1}^{n_G} a_{ij}\left(\lambda_j(t-\tau(t)) - \lambda_i(t-\tau(t))\right) \\ \dot{\lambda}_i(t) = \kappa\left(-P_{Gi}(t) - \widetilde{P}_{Gi}(t)\right) \\ \qquad + \sum_{j=1}^{n_G} a_{ij}\left(\lambda_j(t-\tau(t)) - \lambda_i(t-\tau(t))\right) \end{cases} \tag{5.5.11}$$

where $\widetilde{P}_{Gi}$ is the estimated active power generated by the conventional generator i; $\tau(t) \in [0, \overline{\tau})$ denotes the communication delays, $\overline{\tau} > 0$; and κ is the model parameter. For calculation purpose, I rewrite (5.5.11) in the dynamic model:

$$\dot{y}(t) = Ay(t) + A_d y(t-\tau) + b \tag{5.5.12}$$

$$y(t) = \begin{bmatrix} P_{g_i} \\ \lambda_i(t) \end{bmatrix}, \quad A = \begin{bmatrix} -\alpha_{g_i} & 0 \\ -0 & 0 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & L \\ 0 & L \end{bmatrix}, \quad b = \begin{bmatrix} \beta_{g_i} \\ 0 \end{bmatrix} \tag{5.5.13}$$

where $\mathbf{P}(t) = (P_{G1}, \ldots, P_{Gn_G})^T$, and $\lambda(t) = (\lambda_1, \ldots, \lambda_{n_G})^T$; Firstly, a undirected and strongly connected communication topoloigy G is condiered Which means $\mathbf{1}_n^T \mathbf{L} = \mathbf{0}_n^T$

The state $x_2 = y_2 - y_2^*$ can be used to replace the gap between the solution of the model $y = [P_{g_i}^T, \lambda^T]^T$ and the optimal solution $y^* = [P^{*T}_{g_i}, \lambda^{*T}_n]^T$, and coincides with it at infinity if the market is stable. Then electricity market model with two generator and one consumer system can be rewrite as following system

$$\dot{x}(t) = Ax(t) + A_d x\left(t - d_k(t) - \tau\right) \tag{5.5.14}$$

Then the above stability analysis method can be used in system 5.5.14

5.6 Conclusion

This chapter has investigated the stability of linear systems with time-varying delay. A novel inequalities with three integral terms for the stability analysis of this system is proposed. The delay decomposition approach which compared the above inequalities has been used in this chapter. Those techniques have led to a stability criterion with less conservatism in comparison with the existing criteria. Then the effect of the delays on system stability can be assessed accurately by using the proposed stability criterion. A numerical example has been used to demonstrate the advantages of proposed method. The application on LFC system and ED problem has been used to show advantage.

Chapter 6

Dynamic stability of electricity market with time delay

6.1 Introduction

For more than one decade, the electricity market, as a part of electricity industry restructuring, have experienced huge transformations all around the world. This restructuring has a series of reforms, which mainly include the building of spot market and the separation of electricity market into generation, transmission, distribution and load. The reason for restructuring is because the conventional electricity supply market, where generation, transmission and distribution were regarded as a whole electricity supply, was vertically integrated in the electricity markets as a monopoly utility prior to electricity industry restructuring. However, with the occurrence of the electricity supply industry competition in restructuring, electricity markets will no longer be charged by a single electricity supply industry [23]. The restructuring for generation market and the load market is deregulation. As for the transmission and distribution networks, they will remain regulated, but they must be open to all customers. Therefore, in order to study electricity markets, modelling electricity markets is a key step.

Different electricity market models have been used in the literature to capture various aspects of power market dynamics from bilateral contracts, power exchanges and Pool markets to data based time-series models, game theoretical formulations and the dynamical modeling of supply, (elastic) demand, and real-time pricing. Among all these models, the price signal is setting to control the electricity market operation. And in most of these research, the price signal is directly considered as a market clearing price. However, the price signals can be sent through any communication networks or devices [181]. The above communication will lead to the price signal's arrival delay. As the price signal is the key information to control the electricity market, the research on the impact of price signal with time delay needs to be carefully studied. The time delay of price signal is considered by [29]. In related researches, there are two kinds of time delay such as market clearing time, which ranges from a few minutes to as much as an hour [182] and the delay between the market clearing time and signal process.

The communication infrastructure as a necessary intermediate for conveying price information to a real-time market, introduces certain challenges on the stable operation of the grid. The main challenge is the introduction of discrete time delay and constant or time-varying delays due to the related information processing and communication lags. The stability of the electricity grid and market operation can be endangered by these delays. Many researches focused on the communication in the electricity market. Various approaches have been developed were based on the Lyapunov-Krasovskii stability theory and the resulting stability criteria have a certain degree of conservativeness, which leads that the tolerance of the time delay where it usually indicates by the acceptable maximal delay bound or acceptable maximal sampling period. Many researches have obtained many significant results on the impacts of communication delays. Thus, the approach which considers the increasing of acceptable maximal delay bound is a significant issue and challenge within the context of electricity markets in microgrids and this is the motivation of this chapter to reduce the conservativeness for market optimization purpose.

In this chapter, an electricity market model and its dynamic characteristics are investigated. The main idea of this chapter is to introduce the time delay in the dynamic model of electricity market. A novel stability analysis method which proposed in Chapter 2 for control system with two additive communication delays combined with the input delay approach are applied in the electricity market model. Compared with the existing literature, the contributions of this chapter are summarized as follows:

1. A dynamic electricity market model that incorporates the interaction between real-time pricing, physical constraints and demand response based loads is developed. The stability of the resulting dynamical model which consists of three main participants, generating company, consumer company and system operator is investigated and the conservativeness is reduced by applying the new stability criterion.

2. A less conservative stability criterion which using Wirtinger-based inequality together with the reciprocally convex combination technique is applied. Compared with the existing criteria, several new techniques, including the novel method of the LKF constructing and the improved techniques for estimating the LKF and its derivative, are applied to reduce the conservativeness.

The remainder of this chapter is organized as follows: In section 2, the electricity market structure is introduced. Section 3 presents the dynamic model of electricity market and its stability properties are derived by using a new stability criterion. The calculation and simulation results are provided in section 4. Finally, section 5 gives summary.

List of Symbols:

$\mathbf{P_g}$ the amount of the generated power

$\mathbf{P_d}$ the amount of the consumed power

λ power price

$\mathbf{E}$ time integral of the difference in supply and demand

$\theta(\vartheta)$ the set of indices of generating units (demand) at node n

$\mathbf{\Omega}$ the set of indices of nodes connected to node n

$\delta_\mathbf{n}$ the voltage angle of bus n

$\mathbf{B_{nm}}$ the susceptance of line $n-m$

$\mathbf{P_{nm}^{max}}$ the transmission capacity limit of line $n-m$

6.2 Electricity market structure

The electricity market which considers in this thesis is wholesale and is assumed to function as follows: each GenCo submits the bidding stacks of each of its units to the ISO and each ConCo submits the bidding stack of each of its demands to the ISO too. Then, the market is cleared by the ISO. Prices and production and consumption schedules are given by an appropriate market-clearing procedure. Before discussing the details, some assumptions are given here. There is no monopoly existing in a deregulated market, which means the GenCo cannot manipulate the price which is the same for ConCo.

6.2.1 Generating company model

There are multiple generators in any grids. If it is assumed that N_G generators and the associated operating cost is denoted as $C_{g_i}(P_{g_i})$, the marginal cost function of each conventional generator can be defined as given below [183]

$$C_{g_i}(P_{g_i}) = \frac{1}{2}c_{g_i}(P_{g_i})^2 + b_{g_i}P_{g_i} + a_{g_i} \tag{6.2.1}$$

where P_{g_i} is the active power generated by each generator unit $i \in G_f = \{1, 2..., N_G\}$; a_{g_i}, b_{g_i} and c_{g_i} are generators cost coefficients; $P_{g_i}^{min}$ and $P_{g_i}^{max}$ are the lower and upper bound of power generated by the ith generator.

The goal of a GenCo is to maximize its overall profit π_{g_i} which is stated as

$$\max_{P_{g_i}} \pi_{g_i} = \max_{P_{g_i}}[\lambda_{g_i}P_{g_i} - C_{g_i}(P_{g_i})] \tag{6.2.2}$$

$$s.t. \quad \sum_{i=1}^{N_G} P_{g_i} = P_D \tag{6.2.3}$$

$$P_{g_i}^{min} \leq P_{g_i} \leq P_{g_i}^{max} \tag{6.2.4}$$

6.2.2 Consumer company model

Similar to the generators, there are multiple consumers in the most grids. If it is assumed that there are N_D consumers and the demand of each consumer $j \in D_q = \{1, 2..., N_D\}$, is denoted as P_{d_j}. The associated utility function is denoted as $U_{d_j}(P_{d_j})$, which represents the value of using electricity for the consumer and is defined as [183]

$$U_{d_j}(P_{d_j}) = \frac{1}{2}c_{d_j}(P_{d_j})^2 + b_{d_j}P_{d_j} + a_{d_j} \tag{6.2.5}$$

where a_{d_j}, b_{d_j} and c_{d_j} are consumer utility coefficients. The goal of the ConCo is to maximize the total profit, π_{d_j}, while consuming electricity. This profit, for a unit j connected to node n, is determined as the difference between the utility $U_{d_j}(P_{d_j})$ and the market clearing price. Assuming that the corresponding power consumed is denoted as P_{d_j}, the maximization problem can be posed as

$$\max_{P_{d_j}} \pi_{d_j} = \max_{P_{d_j}}[U_{d_j}(P_{d_j}) - \lambda_{d_j}P_{d_j}] \tag{6.2.6}$$

$$s.t. \quad \sum_{i=1}^{N_G} P_{g_i} = \sum_{j=1}^{N_D} P_{d_j} \tag{6.2.7}$$

$$P_{d_j}^{min} \leq P_{d_j} \leq P_{d_j}^{max} \tag{6.2.8}$$

6.2.3 Market clearing model

The market-clearing procedure consists of optimizing a cost function, subject to various network constraints. The most dominant network constraints are due to line capacity limits and network losses [184]. The power flow through any line is often limited due to technical constraints and is said to be congested when it approaches its maximum limit. This constraint is explicitly included in the model shown below. The second constraint is due to the network losses, most of which come from the heat losses in the power lines. For ease of exposition, such ohmic losses are not considered in this chapter.

The cost function that is typically used is referred to as social welfare. In economic definition, social welfare encapsulates the benefits of a market to the society. It is reasonable to increase this benefit as much as possible. Social welfare describes the aggregate well-being of GenCo and ConCo in a given market. It is consisted by two main parts which are ConCo surplus and GenCo profits. Social welfare is defined as S_W, which

$$S_W = \sum_{i=1}^{N_G} \pi_{g_i} + \sum_{j=1}^{N_D} \pi_{d_j} = \sum_{i=1}^{N_G}[\lambda_{g_i}P_{g_i} - C_{g_i}(P_{g_i})] + \sum_{j=1}^{N_D} \max_{P_{d_j}}[U_{d_j}(P_{d_j}) - \lambda_{d_j}P_{d_j}]$$

If it is assumed that there is no benefit for market operator, this means the payments of consumers will equal to the benefits of generators. Then, the social welfare can be defined as

$$S_W = \sum_{j \in D_q} U_{d_j}(P_{d_j}) - \sum_{i \in G_f} C_{g_i}(P_{g_i}) \tag{6.2.9}$$

where the first and second terms denote the revenue due to surpluses stemming from bids from GenCo and ConCo, respectively. The market-clearing procedure is given by

$$\min \quad -S_W \tag{6.2.10}$$

$$s.t. \quad \sum_{j \in \vartheta} P_{d_j} - \sum_{i \in \theta} P_{g_i} + \sum_{m \in \Omega} B_{nm}[\delta_n - \delta_m] = 0 \;\; \forall n \in N, \tag{6.2.11}$$

$$B_{nm}[\delta_n - \delta_m] \leq P_{nm}^{max} \;\; \forall n \in N; \;\; \forall m \in \Omega, \tag{6.2.12}$$

where $\theta(\vartheta)$ is denoted the set of indices of generating units (demand) at node n, Ω is the set of indices of nodes connected to node n, δ_n is the voltage angle of bus n, B_{nm} is the susceptance of line $n - m$ and P_{nm}^{max} is the transmission capacity limit of line $n - m$. The constraints (6.2.11) and (6.2.12) are due to power balance and capacity limits, respectively. The associated Lagrange multipliers λ_n and γ_{nm} can be indicated by each constraint. The underlying optimization problem of the ISO can therefore be defined as the optimization of (6.2.10) subject to constraints (6.2.11) and (6.2.12). The corresponding Lagrangian of the market clearing optimization problem is given by

$$L(x, \lambda_n, \gamma_{nm}) = \sum_{j \in D_q} U_{d_j}(P_{d_j}) - \sum_{i \in G_f} C_{g_i}(P_{g_i}) \tag{6.2.13}$$

$$+ \sum_{n+1}^{N} \lambda_n [\sum_{j \in \vartheta} P_{d_j} - \sum_{i \in \theta} P_{g_i} + \sum_{m \in \Omega} B_{nm}[\delta_n - \delta_m]] \tag{6.2.14}$$

$$+ \sum_{n+1}^{N} \sum_{m \in \Omega} \gamma_{nm}[B_{nm}[\delta_n - \delta_m] - P_{nm}^{max}] \tag{6.2.15}$$

where $x = \begin{bmatrix} P_G & P_D & \delta \end{bmatrix}^T$ is the primal optimization variable. The resulting solution can be determined, using Karush Kuhn Tucker(KKT) conditions [185], it will satisfy the following conditions:

$$\left(\frac{\mathrm{d}C_{g_i}(P_{g_i})}{\mathrm{d}P_{g_i}}\,|_{P^*_{g_i}} - \lambda^*_{n(i)}\right) = 0 \quad \forall i \in G_f \tag{6.2.16}$$

$$\left(\lambda^*_{n(i)} - \frac{\mathrm{d}U_{d_j}(P_{d_j})}{\mathrm{d}P_{d_j}}\,|_{P^*_{d_j}}\right) = 0 \quad \forall j \in D_q \tag{6.2.17}$$

$$\left(\sum_{m\in\Omega} B_{nm[\lambda^*_n - \lambda^*_m + \gamma^*_{nm} - \gamma^*_{mn}]}\right) = 0 \;\; \forall n \in N \tag{6.2.18}$$

$$\left(-\sum_{i\in\theta}\sum_{b=1}^{N_{g_i}P^*_{g_i}} + \sum_{j\in\vartheta}\sum_{k=1}^{N_{d_j}P^*_{d_j}} + \sum_{m\in\Omega} B_{nm}\left[\delta^*_n - \delta^*_m\right]\right) = 0 \;\; \forall n \in N \tag{6.2.19}$$

$$\gamma^*_{nm}\left(b_{nm}\left[\delta^*_n - \delta^*_m\right] - P^{max}_{nm}\right) = 0 \quad \forall n \in N,\;\; \forall m \in \Omega \tag{6.2.20}$$

Now, these decision variables will be connected to a Nash equilibrium approach. The equilibrium of the market model in (6.2.17)-(6.2.20) is the Nash equilibrium for GenCo and ConCo, which collectively optimizes the overall benefit of GenCo and ConCo denoted in (6.2.2) and (6.2.6) while satisfying constraints of system operator in (6.2.11) and (6.2.12).

It follows that the payoff functions denoted in (6.2.2), (6.2.6) and (6.2.10) are diagonally strictly concave and the corresponding constraints are concave functions. This implies that the market has a unique Pure Strategy Nash Equilibrium that is identical to the solution of the KKT denoted in (6.2.17)-(6.2.20). This implies that at the extremum,$P^*_{g_i}$, $P^*_{d_j}$, δ^*_n, λ^*_n, γ^*_{nm}, the profit of GenCo in (6.2.2) is maximized. That is,

$$\pi_{g_i}(P^*_{g_i}, P^*_{d_j}, \delta^*_n, \lambda^*_n, \gamma^*_{g_i},) \geq \pi_{g_i}(P_{g_i}, P^*_{d_j}, \delta^*_n, \lambda^*_n, \gamma^*_{g_i},)$$

Similarly, it follows that the the total profit of ConCo is maximized. That is,

$$\pi_{g_i}(P^*_{g_i}, P^*_{d_j}, \delta^*_n, \lambda^*_n, \gamma^*_{g_i},) \geq \pi_{g_i}(P^*_{g_i}, P_{d_j}, \delta^*_n, \lambda^*_n, \gamma^*_{g_i},)$$

This implies that at the extremum, the best response of the players is given by $P^*_{g_i}$, $P^*_{d_j}$, δ^*_n, λ^*_n and γ^*_{nm}, Therefore, this extremum coincides with the Nash equilibrium.

6.2.4 Optimization method

In the optimization area, [32] mentioned a general optimization method to process the core problem. It is introduced in this part for legibility.

Definition of optimization problem

A convex optimization problem can be formed as :

$$
\begin{aligned}
& Minimize f(x) \\
& s.t. g_i(x) = 0, \;\; \forall i = 1, ..., N \\
& \sum_{i=1}^{N} R_{ji} h_i(x) \leq c_j, \;\; \forall j = 1, ...L
\end{aligned}
$$

where R is a matrix of constants, f, g_i, h_i are differentiable functions and c_j are constants.

Dual decomposition

In order to derive the dual optimization problem, the Lagrangian function is defined as

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{N} \lambda_i g_i(x) + \sum_{j=1}^{L} \mu_j (R_{ji} h_i(x) - c_j) \tag{6.2.21}$$

where λ_i and $\mu_j \geq 0$ are (dual) Lagrangian multipliers for the equality and inequality constraints, and x is the primal variable. Denoting

$$D(\lambda, \mu) = inf_x L(x, \lambda, \mu) \tag{6.2.22}$$

the dual optimization problem is formulated as

$$
\begin{aligned}
& Maximize D(\lambda, \mu) \\
& s.t. \mu \leq 0, \;\; \forall j = 1, ..., L
\end{aligned}
$$

Under the condition that the original problem is strictly feasible, then there is no duality gap (i.e. the original and the dual problem have the same optimum). In this case, the dual problem can be solved instead of the original problem. In addition, the constraint set for the optimization problem is convex which allows to use the method of Lagrange multipliers and the KKT theorem.

Subgradient Algorithm

In generally, an iterative manner can more simply solve the above optimization problem. For this purpose, a gradient approach is often employed and is briefly described below. Since the ultimate goal of constraint optimization problem is the minimization of a Lagrangian function denoted as $L(x, \lambda, \mu)$ in (6.2.21), x, λ and μ can be progressively changed so that the minima-Lagrange multiplier pairs λ and μ satisfy the KKT conditions. In order to achieve these results, Primal-Dual interior point method is used, which is given by

$$x(t+h) = x(t) - k_x \bigtriangledown_x L(x, \lambda, \mu)h \qquad (6.2.23)$$

$$\lambda(t+h) = \lambda(t) - k_\lambda \bigtriangledown_\lambda L(x, \lambda, \mu)h \qquad (6.2.24)$$

$$\mu(t+h) = \mu(t) - k_\mu \left[\bigtriangledown_\mu L(x, \lambda, \mu)\right]_\mu^+ h \qquad (6.2.25)$$

where k_x, k_λ and k_μ are positive scaling parameters which control the amount of change in the direction of the gradient. Letting $h \to 0$,

$$\tau_x \dot{x}(t) = - \bigtriangledown_x L(x, \lambda, \mu) \qquad (6.2.26)$$

$$\tau_\lambda \dot{\lambda}(t) = \bigtriangledown_\lambda L(x, \lambda, \mu) \qquad (6.2.27)$$

$$\tau_\mu \dot{\mu}(t) = \left[\bigtriangledown_\mu L(x, \lambda, \mu)\right]_\mu^+ \qquad (6.2.28)$$

where $\tau_y = 1/k_y$ for $y = x$, λ and μ.

6.3 Stability analysis of electricity market with time delay

The stability analysis of electricity market with time delay has few research as the stability of the system can be ensured by reducing the economic income of the market participants. However, with the introduction of renewable energy and the rise of smart grid technologies such as distributed power grid, the economic cost of maintaining the stability of the electricity market becomes higher and higher, which makes its stability problem paid high attention.

6.3.1 Dynamic model of the electricity market

In the actual electricity market, the market participants are not able to know the competitors' decision and profit functions. They are unable to reach the equilibrium condition at once. In fact, each participant is rational and can only decide the production strategy according to the participant's expected profit at each iteration. For each market participant, the evaluation of the participant's own profit is more accurate than the predication of the competitors output. Therefore, market participants play a game with a dynamic adjustment to reach their Nash equilibrium. Now, a reasonable dynamic model of the market participant's interactions can be considered. It is assumed that if the game is not at equilibrium, each participant will attempt to change the participant's own strategy so as to obtain the maximum rate of change of the participant's own payoff function with respect to a change in the participant's own strategy.

By using the Subgradient Algorithm method which mentioned in preliminary part, the optimization probelm in (6.2.10)-(6.2.12) can be viewed alternately as a game between the GenCo, ConCo and ISO, where each of these three players attempts to maximize their own benefit [186]. Instead of solving (6.2.17)-(6.2.20) as a static optimization problem, it can be taken as a dynamic approach, inspired by (6.2.27)-(6.2.28). Supposing that the underlying primal and dual variables are perturbed from their corresponding equilibrium to P_{g_i}, P_{d_j}, λ_n and γ_{nm}. Using (6.2.17) and (6.2.2), a differential equation for the ith equation for the i GenCo $\forall i \in G_f$ can be derived as

$$\tau_{g_i} \dot{P}_{g_i} = \lambda(i) - c_{g_i} P_{g_i} + b_{g_i} \tag{6.3.1}$$

with the goal of driving its solution P_{g_i} to the equilibrium $P^*_{g_i}$ which solves (6.2.17). Similarly, using (6.2.18) and (6.2.6), a differential equation can be derived for the jth ConCo $\forall i \in D_q$ as

$$\tau_{d_j} \dot{P}_{d_j} = c_{d_j} P_{d_j} - b_{d_j} - \lambda(j) \tag{6.3.2}$$

where τ_{g_i} and τ_{d_j} are time-constants that can be adjusted so as to result in an optimal convergence of these solutions to the equilibrium in (6.2.17)-(6.2.20).

In addition, differential equations for the market clearing prices, congestion price and phase angles can be determined as

$$\tau_{\lambda_n}\dot{\lambda}_n = -\sum_{i\in\theta_n} P_{g_i} + \sum_{j\in\vartheta_n} P_{d_j} + \sum_{m\in\Omega_n} B_{nm}[\delta_n - \delta_m] \qquad (6.3.3)$$

$$\tau_{nm}\dot{\gamma}_{nm} = [B_{nm}(\gamma_n - \gamma_m - P_{nm}^{max})]^+_{\gamma_{nm}} \qquad (6.3.4)$$

$$\tau_{\delta_n}\dot{\delta}_n = -\sum_{m\in\Omega_n} B_{nm}[\lambda_n - \lambda_m + \gamma_{nm} - \gamma_{mn}] \qquad (6.3.5)$$

Equation (6.3.1)-(6.3.5) represent a dynamic model of the overall electricity market which consider congestion.

6.3.2 Impact of time delay on the stability of electricity market

The characteristic of above dynamic model is decentralized. The market clearing price is given at each node i. For research purpose, the next step focus on the dynamic model of electricity market without congestion. When the congestion are out of consideration, the market clearing price in whole market at time t can be uniform as λ. Then, the above differential equations for the dynamic model of overall electricity market without congestion are given by simplification above equations:

$$\tau_{g_i}\dot{P}_{g_i} = \lambda - c_{g_i}P_{g_i} - b_{g_i} + kE(t) \qquad (6.3.6)$$

$$\tau_{d_j}\dot{P}_{d_j} = c_{d_j}P_{d_j} - b_{d_j} - \lambda \qquad (6.3.7)$$

$$\tau_{\lambda}\dot{\lambda} = -\sum_{i\in\theta_n} P_{g_i} + \sum_{j\in\vartheta_n} P_{d_j} \qquad (6.3.8)$$

$$\dot{E} = \sum_{i\in\theta_n} P_{g_i} - \sum_{j\in\vartheta_n} P_{d_j} \qquad (6.3.9)$$

In a deregulated electricity market, discrete price signals are received and intervals equal to the market clearing time [29], i.e., the updating period of electricity price. In a real market, the price signals are sent through various communication networks and devices. The following new linear model with the sampled and delayed price is obtained:

$$\begin{cases} \tau_{g_i}\dot{P}_{g_i}(t) = \lambda\left(t_k - \tau\right) - b_{g_i} - c_{g_i}P_{g_i}(t) - kE(t) \\ \tau_{D_j}\dot{P}_{D_j}(t) = b_{D_j} + c_{D_j}P_{D_j}(t) - \lambda\left(t_k - \tau\right) \\ \dot{E}(t) = \sum P_{g_i} - \sum P_{D_j} \\ \tau_\lambda\dot{\lambda}(t) = -E(t) \end{cases} \tag{6.3.10}$$

where t_k is the updating instants of the price satisfying.

$$0 < t_{k+1} - t_k = T_{mct_k} \leq T_{mct}$$

with T_{mct_k} and T_{mct} being respectively the market clearing time for k and its maximal value; and τ is the communication delay.

6.3.3 Stability analysis of the electrical market with time delay

The market with one generator and one consumer

The electricity market model with one generator and one consumer leads i and j equal to 1. A continuous-time state equation can replace the system (6.3.10):

$$\dot{y}(t) = Ay(t) + A_d y(t-h) + b \tag{6.3.11}$$

$$y(t) = \begin{bmatrix} P_g \\ P_d \\ E(t)) \\ \lambda(t) \end{bmatrix}, \; A = \begin{bmatrix} -\frac{c_g}{\tau_g} & 0 & -\frac{k}{\tau_g} & 0 \\ 0 & \frac{c_d}{\tau_d} & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\tau_\lambda} & 0 \end{bmatrix}, \; A_d = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{\tau_g} \\ 0 & 0 & 0 & \frac{1}{\tau_d} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; b = \begin{bmatrix} \frac{b_g}{\tau_g} \\ \frac{b_d}{\tau_d} \\ 0 \\ 0 \end{bmatrix} \tag{6.3.12}$$

The solution of this model $y = [P_g^T, P_d^T, E^T, \lambda^T]^T$, converges to the equilibrium in (6.3.12), as $t \to \infty$ if the overall system is stable. At all other transient times, the trajectories represent the specific path that these variables take, when perturbed, as they converge towards the optimal solution. In other words, y is distinct from the optimal solution $y^* = [P{*}_g^T, P{*}_d^T, E^{*T}, \lambda^{*T}]^T$, and coincides with it at infinity if the market is stable. For defining a new state $x = y - y^*$, the stability of overall system at equilibrium point y^* is equivalent to the stability of the following system at zero-point.

$$\dot{x}(t) = Ax(t) + A_d x\left(t - d_k(t) - \tau\right) \tag{6.3.13}$$

It is effortless to discover that the above system is similar to the control system with two delays which is investigated in Chapter 2. Therefore, the Theorem can directly employ to discuss the stability of the electricity market with market clearing time $d_k(t)$ and time delays τ of price signal.

The market with two generators and one consumer

The electricity market model with two generators and one consumer leads $i = 1, 2$ and $j = 1$. A continuous-time state equation can replace the system (6.3.10):

$$\dot{y_2}(t) = A_2 y(t) + A_{d_2} y_2(t-h) + b_2 \tag{6.3.14}$$

$$y_2(t) = \begin{bmatrix} P_{g_i} \\ P_d \\ E(t)) \\ \lambda(t) \end{bmatrix}, \; A_2 = \begin{bmatrix} -\frac{c_{g_i}}{\tau_{g_i}} & 0 & -\frac{k}{\tau_{g_i}} & 0 \\ 0 & \frac{c_d}{\tau_d} & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\tau_\lambda} & 0 \end{bmatrix}, \; A_{d_2} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{\tau_{g_i}} \\ 0 & 0 & 0 & \frac{1}{\tau_d} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; b_2 = \begin{bmatrix} \frac{b_{g_i}}{\tau_{g_i}} \\ \frac{b_d}{\tau_d} \\ 0 \\ 0 \end{bmatrix} \tag{6.3.15}$$

The state $x_2 = y_2 - y_2^*$ is the same to replace the gap between the solution of the model $y = [P_{g_i}^T, P_d^T, E^T, \lambda^T]^T$ and the optimal solution $y^* = [P^{*T}_{g_i}, P^{*T}_{d}, E^{*T}, \lambda^{*T}]^T$, and coincides with it at infinity if the market is stable. Then, the electricity market model with two generator and one consumer system can be rewritten as following system

$$\dot{x_2}(t) = A_2 x_2(t) + A_{d_2} x_2\left(t - d_k(t) - \tau\right) \tag{6.3.16}$$

There is no difference except the order of matrix is larger than one supply system, which means the stability analysis method can also be used to solve this problem with the sharply increasing on the calculation.

6.4 Calculation and simulation results

6.4.1 One generator and one consumer

The case study which has one generator and one consumer in a electricity market are studied in this part to compare the advantage of the proposed Theorem. The parameters listed in Table 6.1 are used in this case study and the equilibrium point of this system is $P_g = 26.68MW$, $P_d = 26.68MW$, $\lambda = 4.66$ and $E = 0$. Assume that a sudden shortage of supply is occurred since a generator is out off grid, then P_g is reduced to 80% of steady state production.

Table 6.1: Parameters used in one generator, one consumer model

τ_g	c_g	b_g	τ_d	c_d	b_d	τ_λ	k
0.2	0.1	2	0.1	-0.2	10	100	0.1

Based on Theorem 1 and the criterion in [2.3.1 2.3.2 2.3.3], the values of τ with given market clearing time period $d_k(t)$ are calculated and partial results are show in Table 6.2. Based on 6.2, it can be found that the stability upper bound obtained by the Theorem 1 is larger than the previously results obtained by [187, 188]. In addition, the simulation results are shown in Fig. 6.1. It can also be observed that the electricity market is stable, which verifies the effectiveness of the proposed method.

Table 6.2: Upper bounds of τ for given $d_k(t)$

Criteria	$d_k(t)$		
	1.0	2.0	5.0
[187]	6.975	5.605	-
[188]	7.712	7.034	3.865
Theorem 1	8.161	7.823	5.142

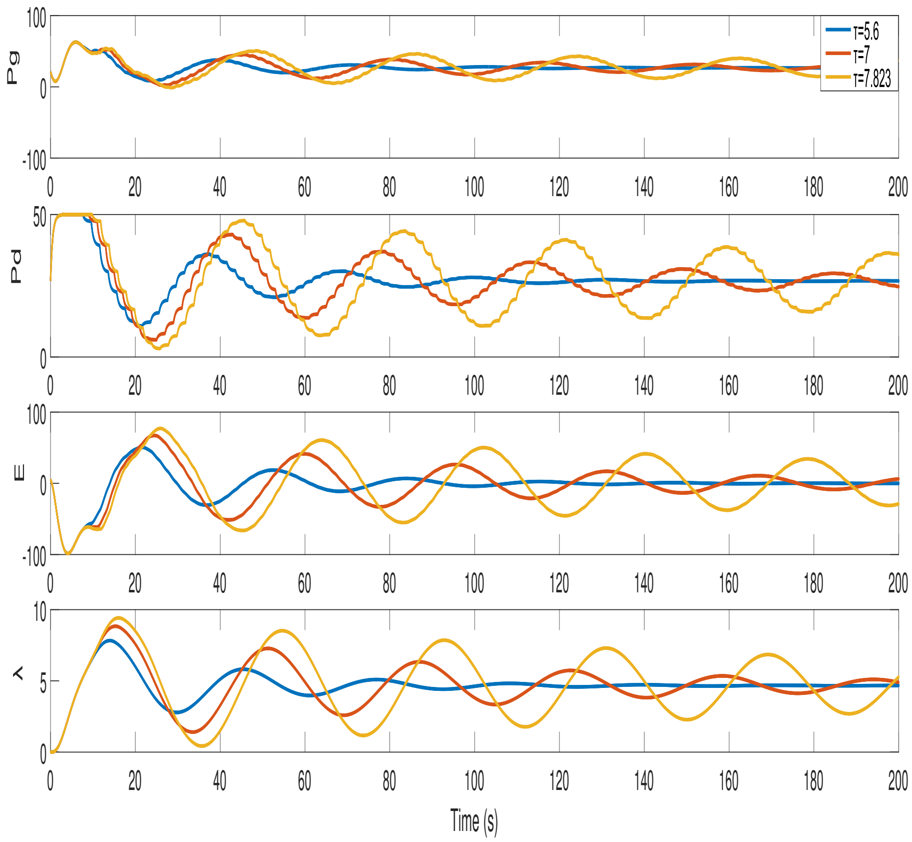

Figure 6.1: The responses of the system during $T_{mct} = 2$.

6.4.2 Two generators and one consumer

The case study which has two generators and one consumer in a electricity market are studied in this part to compare the advantage of the proposed Theorem. The parameters list in Table 6.3 are used in this case and the equilibrium point of this system is $P_{g_1} = 26.68MW$, $P_{g_2} = 0MW$, $P_d = 26.68MW$, $\lambda = 4.66$, and $E = 0$. The states keep with previous case except one spare generator are introduce in this case. Also assume that a sudden shortage of supply is occurred since a generator is out off grid, then P_g is reduced to 80% of steady state production. The values of τ with given market clearing time period $d_k(t)$ are calculated by Theorem 1 and the criterion in [2.3.1 2.3.2 2.3.3]. Partial results can be found in Table 6.4. The results show the stability upper bound and the simulation results of electricity market with

Table 6.3: Parameters used in two generators, one consumer model

Number	τ_g	c_g	b_g	τ_d	c_d	b_d	τ_λ	k
1	0.2	0.1	2	0.1	-0.2	10	100	0.1
2	0.3	0.15	1.5	-	-	-	-	0.2

clearing time equals to 2 are shown in Fig. 6.2. The simulation results show that the electricity market is stable, which verify the effectiveness of the proposed method. Compared the results achieved in two case studies, it can be found that the standby unit can enhance the stability of the whole system.

Table 6.4: Upper bounds of τ for given $d_k(t)$

Criteria	$d_k(t)$		
	2	5	10
[188]	12.71	8.34	5.15
Theorem 1	13.64	9.07	6.47

The numerical examples are simple with one or two generators and one consumer to demonstrate the proposed method can be applied in a electricity market. As the actual electricity market is much more complex, there is no related research give an optimized results with more complex electricity market equilibrium data. The proposed method will text in complex scenarios in the future work.

6.5 Conclusion

In this chapter, the dynamic model of electricity market has been introduced. Then, the stability of this model with communication delay has been analyzed. A stability criterion, which using Wirtinger-based inequality with less conservatism in comparison with the existing criteria is applied in the electricity market. Then, the effect of the delays on system stability can be assessed accurately by using the proposed stability criterion. Two case studies are provided. The calculation and simulation results have been obtained and shown to demonstrate the effectiveness

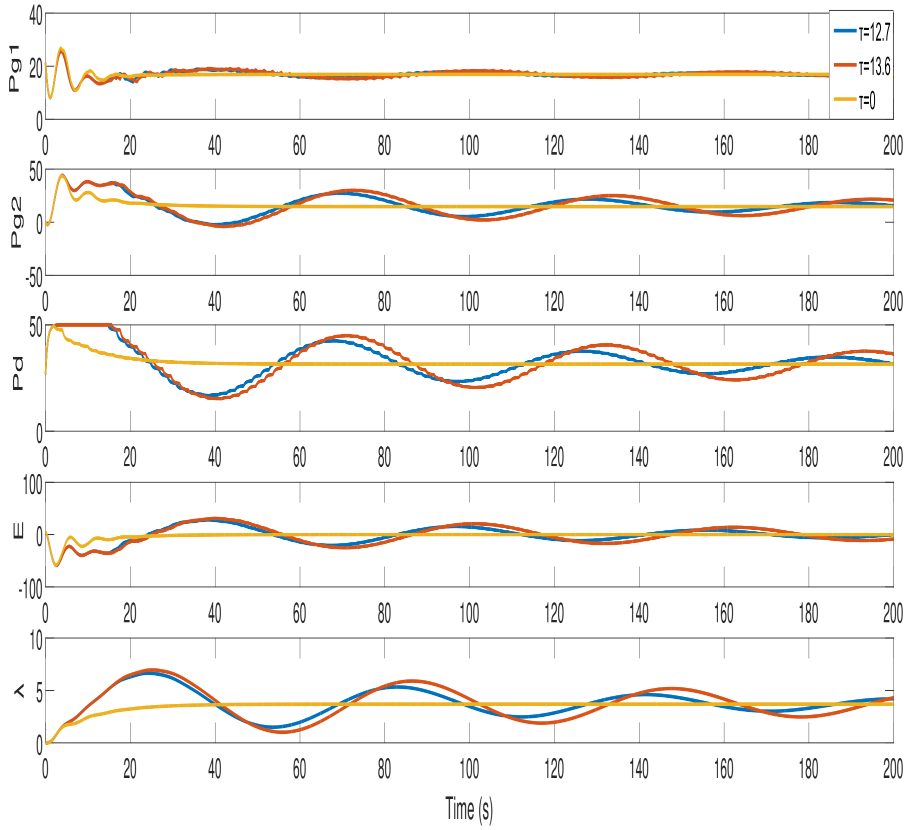

Figure 6.2: The responses of the system during $T_{mct} = 2$, $d_k(t) = 12.7$ and $d_k(t) = 13.6$

of the proposed method.

Chapter 7

Conclusions and Future Work

This chapter summarize the obtained results and contributions of this thesis. Suggestions for future investigations are also presented.

7.1 Conclusions

In this thesis, the stability analysis of time delay system and its application on the power system are investigated. Time-varying time delays resulting from the communication channel or a channel like communication network such as message RNA are considered in those industrial time delay systems. Delay dependent stability criteria with less conservatism have been proposed at first. Then less conservative estimation of delay margin is used to guide the synthesis of the controller with compromising between the dynamic performance and the delay margins.

First, four new delay dependent stability criteria have been proposed. Their effectiveness has been verified by numerical examples with the comparison with the approaches provided by previous researches. In Chapter 2, the stability of linear systems with two additive time-varying delays are analysed. The less conservative delay-dependent stability criterion is given by constructing a new delay-product-type LKF and the Wirtinger-based integral inequality, combing the analysis technique of inequalities with the reciprocally convex.

The stability analysis of the GRN with time-varying delay is investigated in

Chapter 3. A novel delay-dependent stability criterion has been derived. The novel Wirtinger-type double integral inequality has been developed for the double integral terms estimation. Chapter 4 has investigated the stability analysis of fixed-point state-space digital filters with generalized overflow arithmetic and a time-varying delay. The criterion has less conservatism in comparison to the ones reported in the previous literature due to two aspects of improvements. A new Lyapunov functional with several augmented terms and the relaxed positive-definite condition has been constructed, and free matrices therein provide extra freedom of checking the conditions of stability criterion, combining the analysis technique of inequalities such as Wirtinger-based inequality, zero-value equations and the extended reciprocally convex matrix inequality. In Chapter 5, the less conservative delay-dependent stability criterion is given by constructing the Lyapunov functional which was proposed in Chapter 2 based on the ideal of delay decomposition. The Wirtinger-type relaxed three integral inequalities combining reciprocally convex matrix inequality are used to analysis inequalities.

In the second part of this thesis, the proposed stability criteria are applied in different real industry systems with time delays, which include power systems (LFC, electricity market), GRN and digital filtering.

The first stability criterion is applied in load frequency control scheme of power systems in Chapter 2, where load frequency control is designed for maintaining the frequency at its required value. Since two time delays are introduced by the communication between feedback measurement channel and the forward control channel of the load frequency control, the proposed stability criterion is used to calculate the delay margins of those two additive time-varying delays. The obtained delay margins of the two-additive time-varying delays can be used to design a controller of the closed-loop system. According to the results provided, it can be found that the delay margins of time delays are larger than existing work in literature, which means obtained results is less conservative than previous work. By using these delay margins, system stability can be assessed accurately. In addition, when one of the delay change rate increases, the calculated sum of two time delay margins changes as the same time. Hence, it proves that the two delays cannot be considered as one delay when two delays are different. Finally, the simulation results of the power system with the calculated delay margins added are provided and it can be found that the power system with LFC is stable, which verifies the effectiveness of the proposed method.

For the deregulated electricity market, the first stability criterion is also applied to analyse the system considering the price signal with time delay. The price of electricity is used to calculate the optimal power flow and control the power system operation through generators, loads and market operators. Time delays are generally ignored in the related researches. Therefore, the optimal power flow which was calculated by previous researches is not the real optimal solutions, which can lead to energy and economic losses. In order to save the cost of participants, the stability analysis of the electricity market with time delay is studied. The first stability criterion is applied to the electricity market in Chapter 6 to investigate the effect of time delay. Two case studies which include generator and consumer are researched separately. It can be found that the stability upper bound obtained by the proposed criterion is larger than the previous results in the literature, which indicates that the proposed criterion can achieve less conservatism than existing results. Furthermore, the simulation results of the electricity market show that the electricity market is still stable under the calculated maximum time delay. It verifies the effectiveness of the proposed stability criterion. However, from the simulation results, the minimum time delay which makes the system unstable is greater than the calculated maximum time delay. In other words, there's still plenty of room for improvement.

In Chapter 3, the stability analysis of control systems is extended to a biological system for expanding the scope of application. The second stability criterion which is proposed in this thesis is applied in the genetic regulatory networks with nonlinear dynamical behaviour and time delay. It can be easily found from the results that the proposed stability criterion can provide the larger maximal admissible delay bounds for two cases than those given in the existing literature. It shows that the proposed criterion is indeed less conservative than the ones reported in the literature. In addition, the number of variables in the proposed criteria is smaller than previous research, which means the proposed method not only is less conservative, but also saves computing space.

The digital filter is a necessary element of everyday electronics. It is an effective device that produces the desired discrete-time output signal from the original input signal, which will unavoidably involve undesired information. Therefore, the analysis of digital filters is helpful for their implementation. Improved delay-dependent stability analysis of digital filters with time-varying delay and generalized overflow arithmetic are proposed in Chapter 4. The third stability criterion is used to analyse the digital filters which consider both nonlinearities and time delays. Two numerical examples have been given to show the effectiveness and advantages of the proposed stability criterion. In addition, the results with and without relaxed positive-definite condition and the cross-terms introduced by the Wirtinger-type inequality demonstrate that these techniques are effective in reducing conservatism.

7.2 Future work

The possible future work are listed based on the following ideas.

- In recent years, other integral inequalities, tighter than Wirtinger-based integral inequality, have been developed, for example, free-matrix-based inequality, auxiliary function-based integral inequalities and Bessel-Legendre inequality. The further improved stability criteria can be obtained by combining those inequalities and the proposed delay-product-type LKF, which will be further studied in future research.

- The dynamic model of electricity market is researched just with output constraint of each generator. However, as the real electricity market, the congestion and line loss can not be ignored. The difference between the optimized scheme and the actual system makes it difficult to apply the result directly in the actual power market. However, if the model of block management and line loss is too complicated, the calculation amount of stability analysis will be greatly increased. In the future work, stability analysis methods combining conservative reduction and computational load can be studied to make the electricity market model more realistic.

- The stability analysis of the optimal economic dispatch of microgrids considering communication delays are investigated by ignored the cost of renewable generators. Further more, there are plenty researches focus on demand response. The both goals of economic dispatch and demand response are reducing their cost and get more benefit from market. The dynamic of this gaming has few attention and this motivates the next research. In the future work, the model of ED problem considering will be more rigorous.

- The case studies in electricity market and ED problem are only based a very simple ideal-type model. The further improved case studies based on 39bus-IEEE model will be investigated.

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