

HYPERBOLIC ENTIRE FUNCTIONS WITH FULL HYPERBOLIC DIMENSION AND APPROXIMATION BY EREMENKO-LYUBICH FUNCTIONS

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ABSTRACT. We show that there exists a hyperbolic entire function f of finite order of growth such that the hyperbolic dimension—that is, the Hausdorff dimension of the set of points in the Julia set of f whose orbit is bounded—is equal to two. This is in contrast to the rational case, where the Julia set of a hyperbolic map must have Hausdorff dimension less than two, and to the case of all known explicit hyperbolic entire functions.

In order to obtain this example, we prove a general result on constructing entire functions in the *Eremenko-Lyubich class* \mathcal{B} with prescribed behavior near infinity, using Cauchy integrals. This result significantly increases the class of functions that were previously known to be approximable in this manner.

Furthermore, we show that the approximating functions are quasiconformally conjugate to their original models, which simplifies the construction of dynamical counterexamples. We also give some further applications of our results to transcendental dynamics.

1. INTRODUCTION

The Hausdorff dimension $\dim_{\mathbb{H}}(J(f))$ of the Julia set of a rational function f has been extensively studied. A related quantity, the *hyperbolic dimension* $\dim_{\text{hyp}}(f)$, was introduced by Shishikura [Sh] as the supremum over the Hausdorff dimensions of hyperbolic subsets of $J(f)$. (Here a *hyperbolic set* $K \subset J(f)$ is a compact, forward invariant subset of $J(f)$ such that sufficiently high iterates of f are expanding when restricted to K .)

Clearly the hyperbolic dimension is a lower bound for $\dim(J(f))$. If the rational function f is hyperbolic, then by definition $J(f)$ is a hyperbolic set itself, and hence

$$(1.1) \quad \dim J(f) = \dim_{\text{hyp}}(f).$$

It is natural to ask whether (1.1) holds more generally. In other words, how prevalent is expanding dynamics in the Julia set of a rational function?

1.1. Question.

Is there a rational function f such that $\dim J(f) \neq \dim_{\text{hyp}}(f)$?

The relation (1.1) is known to hold in a vast number of cases, including all non-recurrent rational functions. The tantalizing possibility of an example of a rational function where (1.1) fails was first suggested by results of Avila and Lyubich [AL] on

2010 *Mathematics Subject Classification.* 30D05, 30E10, 37F10 (Primary); 30D10, 30D15, 37F35 (Secondary).

This work was supported by EPSRC Fellowship EP/E052851/1.

Feigenbaum quadratic polynomials with periodic combinatorics. They show that, *if* such a map exists whose Julia set has positive area (and hence dimension 2), then its hyperbolic dimension would need to be strictly less than two. Since this article was submitted for publication, Avila and Lyubich have announced a proof that such Feigenbaum Julia sets of positive area do indeed exist, answering Question 1.1 in the positive. It remains open whether there is a rational function f such that $\dim_{\text{hyp}}(f) < \dim J(f) < 2$.

The results of Avila and Lyubich resonate strongly with the iteration theory of transcendental entire functions. Here, in stark contrast to the rational case, even *hyperbolic* functions (see Definition 1.2) frequently satisfy $\dim(J(f)) = 2$ and $\dim_{\text{hyp}}(f) < 2$. Stallard [S2] was the first to construct hyperbolic examples with $\dim_{\text{hyp}}(f) < \dim(J(f))$ (in slightly different terminology), while Urbański and Zdunik [UZ] proved that this situation occurs for hyperbolic exponential maps $f(z) = \exp(z) + a$, where $\dim(J(f)) = 2$ by a result of McMullen [McM]. This suggests that a systematic understanding of the measurable dynamics of transcendental entire functions is not only interesting in its own right, but can also help to shed further light on phenomena such as those discovered by Avila and Lyubich.

A class of hyperbolic entire functions that has received particular attention in recent years is given by those of finite order and disjoint type:

1.2. Definition.

A transcendental entire function $f : \mathbb{C} \rightarrow \mathbb{C}$ is *hyperbolic* if there exists a compact set $K \subset \mathbb{C}$ with $f(K) \subset \text{int}(K)$ such that the restriction

$$f : f^{-1}(\mathbb{C} \setminus K) \rightarrow \mathbb{C} \setminus K$$

is a covering map. An entire function is said to be *of disjoint type* if it is hyperbolic and the Fatou set $F(f)$ is connected, or equivalently if the set K can be chosen to be connected.

An entire function has *finite order* if, setting $\log_+ r = \max(0, \log r)$, we have

$$\limsup_{z \rightarrow \infty} \frac{\log_+ \log_+ |f(z)|}{\log_+ |z|} < \infty.$$

The topology of the Julia set of a hyperbolic entire function of finite order is completely understood [Ba1, BJR, R2, R3S]. Moreover, the Hausdorff dimension of $J(f)$ is equal to two in this case [Ba2], and the hyperbolic dimension is greater than one [BKZ]. More precisely, suppose that f is of disjoint type and finite order. Then:

- The Julia set is a disjoint uncountable union of curves to ∞ , each consisting of a finite *endpoint* and a *ray* connecting this endpoint to infinity. In fact, $J(f)$ is ambiently homeomorphic to a straight brush in the sense of [AO] (i.e., a certain universal plane topological object).
- The set of endpoints in $J(f)$ has Hausdorff dimension equal to two.
- The union of rays in $J(f)$ (without endpoints) has Hausdorff dimension equal to one.

Furthermore, for a large class of hyperbolic entire functions of finite order, including all hyperbolic maps in the exponential family $z \mapsto \exp(z) + a$, the trigonometric family

$z \mapsto a \exp(z) + b \exp(-z)$ and many others, the measurable dynamics is described in detail by the results of [MU1, MU2]. In particular, these maps satisfy $\dim_{\text{hyp}}(f) < 2 = \dim(J(f))$. This suggests the following problem.

1.3. Question.

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a hyperbolic transcendental entire function of finite order. Is it always the case that $\dim_{\text{hyp}}(f) < 2$?

In this article, we give a negative answer.

1.4. Theorem ((Hyperbolic functions with full hyperbolic dimension)). *There exists a transcendental entire function f of disjoint type and finite order such that $\dim_{\text{hyp}}(f) = 2$. Furthermore, f can be chosen such that $J(f)$ has positive measure.*

Approximation. In order to obtain the desired counterexample, we shall first construct a suitable “model function” with the desired behavior, and then approximate this map by an entire function. The idea of using approximation to construct interesting examples in complex dynamics was introduced by Eremenko and Lyubich [EL1], who used Arakelyan’s theorem [G, Satz IV.2.3], an important result of approximation theory. Given a closed set A , this theorem states that any function g , defined and continuous on A and holomorphic on its interior, can be uniformly approximated by entire functions if and only if A satisfies certain simple topological conditions.

Arakelyan’s theorem, while powerful, has the drawback that there is little we can say about the behavior of the approximating function f outside of the set A . In particular, we have no control over the set $\text{sing}(f^{-1})$ of critical and asymptotic values of f , which prevents us from being able to restrict the global function-theoretic or dynamical properties. Indeed, in order to obtain hyperbolic examples, we will at least need to be sure that the approximating function belongs to the *Eremenko-Lyubich class*

$$\mathcal{B} := \{f : \mathbb{C} \rightarrow \mathbb{C} \text{ transcendental entire: } \text{sing}(f^{-1}) \text{ is bounded}\},$$

which was introduced in [EL2]. A related problem, which we mention here for completeness although it is not treated in this article, is approximation by functions in the *Speiser class*

$$\mathcal{S} := \{f : \mathbb{C} \rightarrow \mathbb{C} \text{ transcendental entire: } \text{sing}(f^{-1}) \text{ is finite}\} \subset \mathcal{B}.$$

Leaving aside dynamics for a moment, let us discuss the question of the function-theoretic behavior that these maps can exhibit. As it turns out, this will be the main problem to deal with when constructing hyperbolic examples.

If $f \in \mathcal{B}$, then for sufficiently large $R > 0$, every component of $f^{-1}(\{|z| > R\})$ is simply connected and mapped by f as a universal covering. These components are called the *tracts* of f (over ∞). If $f \in \mathcal{B}$ and T is a tract of f as above, then we can define a branch of $\log f$ on T , and $\log f - \log R : T \rightarrow \mathbb{H}$ is a conformal isomorphism (where $\mathbb{H} := \{\text{Re } z > 0\}$ denotes the right half plane). Conversely, it is natural to ask which universal coverings can be approximated by entire functions in the class \mathcal{B} .

1.5. Question.

Suppose that $T \subset \mathbb{C}$ is a Jordan domain whose boundary passes through infinity and

that $\Psi : T \rightarrow \mathbb{H}$ is a conformal isomorphism with $f(\infty) = \infty$. Under which conditions on Ψ does there exist an entire function $f \in \mathcal{B}$ (resp. $f \in \mathcal{S}$) such that

$$f(z) = e^{\Psi(z)} + O(1), \quad z \in T ?$$

Arakelyan's theorem implies that such a function f always exists if we drop the requirement that $f \in \mathcal{B}$.

A well-known way of building functions with a given tract is to use Cauchy integrals; see e.g. [PS, Part III, problem 158]. Although this method is rather old, and has had many applications over the years, it does not seem to have been treated systematically in the classical literature. The only theorem of a general nature that we are aware of was recently stated in [R³S], following a construction from a paper by Eremenko and Gol'dberg [GE]. The result in [R³S, Proposition 7.1] states that approximation is always possible when Ψ is the restriction of a conformal isomorphism $\Psi : T' \rightarrow \Sigma$, where Σ is a sector, $\Sigma = \{z \in \mathbb{C} : |\arg z| < \pi/2 + \varepsilon\}$.

In [R³S], this general theorem is used to construct a counterexample to the so-called strong Eremenko conjecture. However, the requirement that Ψ extends to a conformal isomorphism onto a sector of opening angle greater than π is rather strong. It prevents, for instance, the construction of functions of lower order $1/2$, as well as of tracts such as the one depicted in [RRS, Figure 1].

In this note, we present a considerable strenghtening of [R³S, Proposition 7.1], which states that approximation is always possible if Ψ extends to a conformal isomorphism whose domain is only "slightly" larger than the half plane \mathbb{H} . It is convenient to first introduce the following definition.

1.6. Definition ((Model functions)).

A *model function* is a conformal isomorphism

$$\Psi : T \rightarrow H,$$

where

- $T \subset \mathbb{C}$ is an unbounded simply-connected domain;
- H is a simply-connected domain with $\mathbb{H} \subset H$ (recall that $\mathbb{H} := \{\operatorname{Re} z > 0\}$ denotes the right half plane);
- if $z_n \in T$ is a sequence with $f(z_n) \rightarrow \infty$ in H , then $z_n \rightarrow \infty$ in T .

1.7. Theorem ((Approximation of model functions)). *Let*

$$H := \{x + iy : x > -14 \log_+ |y|\},$$

where $\log_+(t) := \max(0, \log t)$, and let $\Psi : T \rightarrow H$ be a model function.

Set $g := \exp \circ \Psi$. Then there exists an entire function $f \in \mathcal{B}$ such that

$$f(z) = g(z) + O\left(\frac{1}{z}\right) \quad \text{when } z \in T, \quad \text{and}$$

$$f(z) = O\left(\frac{1}{z}\right) \quad \text{when } z \notin T$$

(as $z \rightarrow \infty$). If the domain T is symmetric with respect to the real axis and $\varphi(T \cap \mathbb{R}) \subset \mathbb{R}$, then f can be chosen such that $f(\mathbb{R}) \subset \mathbb{R}$.

Dynamical approximation. In order to use Theorem 1.7 to prove Theorem 1.4, we observe that the approximation automatically preserves dynamical features. The key fact is that, given our quality of approximation, the functions f and g are *quasiconformally equivalent near ∞* in the sense of [R2]:

1.8. Theorem ((Quasiconformal equivalence)). *Let f and g be as in Theorem 1.7, and let $R > 0$ be sufficiently large.*

Then there exists a quasiconformal homeomorphism $\varphi : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$g(z) = f(\varphi(z))$$

for all $z \in \mathbb{C}$ with $|g(z)| \geq R$.

This map is asymptotically conformal at ∞ ; more precisely,

$$\varphi(z) = z + O(1)$$

as $z \rightarrow \infty$.

By [R2], this implies that the functions are quasiconformally conjugate on the set of points whose orbits stay suitably large under iteration. In our setting we can even be sure (adapting ideas from [R2]) to obtain a *global* conjugacy on the Julia sets of the two functions, provided that the tract T is sufficiently well inside the domain $\{|z| > 1\}$.

1.9. Theorem ((Quasiconformal conjugacy)). *There is a universal constant $\rho_0 > 1$ with the following property.*

Let $\Psi : T \rightarrow H$ be as in Theorem 1.7, with the additional property that $T \subset \{|z| > \rho_0\}$.

Then, again setting $g := \exp \circ \Psi$, the function f in Theorem 1.7 can be chosen such that f and g are quasiconformally conjugate near their Julia sets.

More precisely, there is a quasiconformal homeomorphism $\vartheta : \mathbb{C} \rightarrow \mathbb{C}$ such that

$$\vartheta(g(z)) = f(\vartheta(z))$$

whenever $|g(z)| \geq \rho_0$, and ϑ restricts to a homeomorphism between the Julia set $J(g)$ (i.e., the set of points that remain in T under iteration of g) and the Julia set of f . Furthermore, the complex dilatation of ϑ equals zero almost everywhere on $J(g)$.

In order to prove Theorem 1.4, it will thus be sufficient to construct a model function $\Psi : T \rightarrow H$ for which the map $g := \exp \circ \Psi$ contains hyperbolic subsets of dimension arbitrarily close to 2 and which satisfies the hypotheses of Theorem 1.9. (Recall that quasiconformal mappings preserve sets of Hausdorff dimension 2.)

Further applications. Let us state two further new theorems that can be obtained from our approximation results via known constructions. (We refer to the articles in question for background on the questions answered by these examples.) The first is a strengthening of [R³S, Theorems 8.2 and 8.3].

1.10. Theorem ((Counterexamples of low growth to the strong Eremenko conjecture)). *There exists a disjoint-type transcendental entire function $f \in \mathcal{B}$ such that*

(a) $\log_+ \log_+ |f(z)| = (\log_+ |z|)^{1+o(1)}$ as $z \rightarrow \infty$,

(b) f has lower order $1/2$, and

(c) the Julia set $J(f)$ has no unbounded path-connected components.

Remark 1. We recall that any function $f \in \mathcal{B}$ must have lower order at least $1/2$ and that no function $f \in \mathcal{B}$ of finite (upper) order can satisfy (c) by [R³S].

Remark 2. The condition on the growth of f implies that $J(f)$ has Hausdorff dimension equal to two [BKS].

Our second application strengthens a counterexample from [RRS].

1.11. Theorem ((Slowly escaping Devaney hairs)). *There exists a disjoint-type transcendental entire function $f \in \mathcal{B}$ such that*

- (a) $f(\mathbb{R}) \subset \mathbb{R}$ and $J(f) \cap \mathbb{R} = [a, \infty)$ for some $a > 0$;
- (b) every connected component of $J(f)$ is an arc connecting some finite endpoint to infinity, and $[a, \infty)$ is such a component;
- (c) every $x > a$ belongs to the escaping set $I(f) = \{z \in \mathbb{C} : f^n(z) \rightarrow \infty\}$;
- (d) the real axis does not intersect the fast escaping set $A(f)$ of points that escape to infinity “as fast as possible” in the sense of Bergweiler and Hinkkanen [BH].

We also note that the same construction as in the proof of Theorem 1.4, with different parameters, suggests a counterexample to the *area conjecture* of Epstein and Eremenko. However, while this construction yields a counterexample in the class of model functions, our approximation result does not allow us to construct such a counterexample in the class \mathcal{B} . This application and its background is discussed in [ER].

Some remarks about the proof. As already mentioned, the proof of Theorem 1.7 uses Cauchy integrals. More precisely, let $\gamma : (-\infty, \infty) \rightarrow T$ be defined by

$$\gamma(t) := \Psi^{-1}(it - 13 \log_+ |t| + 1)$$

and consider the function

$$h(z) := \frac{1}{2\pi i} \int_{\gamma} \frac{g(\zeta)}{\zeta - z} d\zeta.$$

We will show that the integral converges absolutely for $z \notin \gamma$ and that

$$h(z) = O\left(\frac{1}{z}\right)$$

as $z \rightarrow \infty$. It then follows that

$$f(z) := \begin{cases} h(z) + g(z) & z \in \tilde{T} \\ h(z) & z \notin \tilde{T} \end{cases}$$

is the desired entire function, where \tilde{T} is the component of $\mathbb{C} \setminus \gamma$ that is contained in T .

We should comment that the constant 14 appearing in the definition of H is not best possible: Our proof shows that it can be replaced by any constant that is larger than 13, and with some more careful estimates, it could be reduced further. However, our proof does not yield the analog of Theorem 1.7 for a domain of the form

$$H := \{x + iy : x > -\varepsilon \log_+ |y|\},$$

where $\varepsilon > 0$ is arbitrarily small.

We also note that, for the application in Theorem 1.4, it is important that the domain H is of the form as above (compare Remark 1 after the proof of Theorem 7.6): e.g. it would not be sufficient to be able to approximate functions $\Psi : T \rightarrow H$, where

$$H = \{x + iy : x > -|y|^\varepsilon\}.$$

Subsequent results. Motivated by our results, Chris Bishop [Bi2] has recently given a complete answer to Question 1.5, if one considers quasiconformal equivalence instead of uniform approximation: If $\Psi : T \rightarrow H$, then there is a function $f \in \mathcal{B}$, with a single tract, that is quasiconformally equivalent to $\exp \circ \Psi$ near ∞ . Furthermore, there is a function $f \in \mathcal{S}$ such that a suitable restriction of f is quasiconformally equivalent to $\exp \circ \Psi$. (In general, f cannot be constructed to have only a single tract.) These results even hold for arbitrary unions of tracts that accumulate only at infinity. In particular, Bishop's methods allow the construction of a counterexample to the area conjecture mentioned above, in the class \mathcal{S} , and indeed a counterexample [Bi1] to the stronger *order conjecture* of Adam Epstein, which asked whether the order of a transcendental entire function $f \in \mathcal{S}$ is invariant under quasiconformal equivalence.

Structure of the article. The first part of the paper deals with approximation and the proof of Theorem 1.7. In Section 2, we prove a technical result about the approximation of holomorphic functions using Cauchy integrals. (This covers a number of known constructions.) In Section 3, we collect some basic facts about hyperbolic geometry in plane domains; these are used in Section 4 to prove Theorem 1.7 in a slightly more general framework, using the results from Section 2. The short Section 5 is dedicated to verifying that our hypotheses in Theorem 1.7 indeed satisfy the assumptions used in Section 4.

The second part of the paper consists of Section 6, which establishes the results on quasiconformal equivalence and conjugacy.

Finally, Section 7 constructs the model function required for the proof of Theorem 1.4, while Section 8 briefly discusses Theorems 1.10 and 1.11.

We remark that the three parts of the paper can be read quite independently of each other (with the exception that the hyperbolic metric estimates of Section 3 will be used throughout).

Acknowledgments. I owe great thanks to Alexandre Eremenko, who introduced me to the method of approximation via Cauchy integrals by pointing me to the paper [GE], and who has shared many profound insights on this and related problems. I would also like to thank Adam Epstein, who led me to think about the area conjecture and to discover the basic structure of the example in Theorem 1.4, and Peter Hazard, stimulating conversations with whom resulted in the realization that this example could be adapted to yield functions with full hyperbolic dimension. Finally, I would like to thank Chris Bishop, Helena Mihaljević-Brandt, Phil Rippon, Gwyneth Stallard and Mariusz Urbański for interesting discussions about this work.

Basic notation. As usual, we denote by \mathbb{C} the complex plane. We also denote the right half plane by

$$\mathbb{H} := \{a + ib : a > 0, b \in \mathbb{R}\}$$

and the (Euclidean) disk of radius r around a point $z_0 \in \mathbb{C}$ by

$$\mathbb{D}_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}.$$

Euclidean distance is denoted dist ; e.g. $\text{dist}(A, z_0)$ is the Euclidean distance between a set $A \subset \mathbb{C}$ and the point z_0 .

As mentioned above, we set $\log_+(t) := \max(0, \log(t))$ for $t \geq 0$. We also define

$$|z|_+ := \max(|z|, 1) = \exp(\log_+ |z|)$$

for all $z \in \mathbb{C}$.

If $f : \mathbb{C} \rightarrow \mathbb{C}$ is a transcendental entire function, we denote by $\text{sing}(f^{-1})$ its set of critical and asymptotic values. (Here a is an asymptotic value if there is a curve $\gamma : [0, \infty) \rightarrow \mathbb{C}$ with $\gamma(t) \rightarrow \infty$ and $f(\gamma(t)) \rightarrow a$ for $t \rightarrow \infty$.) The closure of $\text{sing}(f^{-1})$ (in \mathbb{C}) is denoted $S(f) := \overline{\text{sing}(f^{-1})}$. An alternative definition of $S(f)$, which is the one we will be using, is as the smallest closed set that has the property that

$$f : \mathbb{C} \setminus f^{-1}(S(f)) \rightarrow \mathbb{C} \setminus S(f)$$

is a covering map.

2. APPROXIMATION USING CAUCHY INTEGRALS

In this section, we prove a general technical result about the approximation of holomorphic functions by Cauchy integrals. In Section 4, this will be used to deduce our main approximation theorem (Theorem 1.7).

2.1. Theorem ((Convergence of Cauchy integrals)). *Let $T \subset \mathbb{C}$ be a simply-connected domain and let $g : T \rightarrow \mathbb{C}$ be holomorphic. Let $\gamma : (-\infty, \infty) \rightarrow T$ be an injective and piecewise smooth curve such that $|\gamma(t)| \rightarrow \infty$ as $|t| \rightarrow \infty$, and let $\tilde{T} \subset T$ be the component of $\mathbb{C} \setminus \gamma$ that is contained in T . We assume that γ runs around \tilde{T} in clockwise direction.*

Suppose furthermore that there are constants $C_1, \dots, C_5 \geq 1$ and $\delta_1, \delta_2 \geq 0$ such that the following hold for all $\tau \in \mathbb{R}$ (recall that $|\tau|_+ = \max(|\tau|, 1)$):

- (a) $|\gamma(\tau)| \leq C_1 \cdot |\tau|_+$,
- (b) $|\gamma'(\tau)| \leq C_2 \cdot |\tau|_+^{\delta_1}$,
- (c) $|g(\gamma(\tau))| \leq C_3 \cdot |\tau|_+^{-(2+\delta_1+\delta_2)}$, and
- (d) if $|z - \gamma(\tau)| \leq |\tau|_+^{-\delta_2}/C_4$, then $z \in T$ and $|g(z)| \leq C_5 \cdot |\tau|_+^{-1}$.

Then

$$(2.1) \quad h(z) := \frac{1}{2\pi i} \int_{\gamma} \frac{g(\zeta)}{\zeta - z} d\zeta$$

defines a holomorphic function for $z \notin \gamma$, and

$$f(z) := \begin{cases} h(z) + g(z) & z \in \tilde{T} \\ h(z) & z \notin \tilde{T} \end{cases}$$

extends to an entire function $f : \mathbb{C} \rightarrow \mathbb{C}$.

Furthermore, there is a constant C_6 such that

$$|h(z)| \leq \frac{C_6}{|z|_+},$$

where C_6 depends only on C_1, \dots, C_5 ; more precisely, $C_6 = O(C_1 \cdot C_2 \cdot C_3 \cdot C_4 \cdot C_5)$.

Proof. We have

$$|g(\gamma(\tau))| \cdot |\gamma'(\tau)| \leq C_2 \cdot C_3 \cdot |\tau|_+^{-(2+\delta_2)},$$

hence

$$(2.2) \quad \begin{aligned} \int_{\gamma} |g(\zeta)| |d\zeta| &= \int_{-\infty}^{\infty} |g(\gamma(\tau))| \cdot |\gamma'(\tau)| d\tau \\ &\leq C_2 \cdot C_3 \cdot \int_{-\infty}^{\infty} |\tau|_+^{-(2+\delta_2)} d\tau = 2 \cdot C_2 \cdot C_3 \cdot \left(1 + \frac{1}{1 + \delta_2}\right) \leq 4 \cdot C_2 \cdot C_3. \end{aligned}$$

This implies that the integral in (2.1) is absolutely convergent and defines a holomorphic function h on $\mathbb{C} \setminus \gamma$. If $z_0 \in \gamma$, then we can modify the curve γ slightly to avoid the point z_0 , and thus see that the restriction of h to \tilde{T} has an analytic extension to a neighborhood of z_0 ; the same is true for the restriction $h|_{\mathbb{C} \setminus \tilde{T}}$. Using the residue theorem, we see that the two extensions differ exactly by the function $g(z)$ in a neighborhood of z_0 , which shows that the function f defined in the statement of the theorem does indeed extend to an entire function. (Compare also [R³S, Claim 2 in Section 7].)

Thus it remains to prove that $|h(z)| = O(1/|z|_+)$. The main problem is to estimate $h(z)$ when z is close to some point $\gamma(\tau_0)$. In this case, we will modify γ to a curve γ^z that avoids the disk D_{τ_0} of radius $\delta(\tau_0) := |\tau_0|_+^{-\delta_2}/C_4$ around $\gamma(\tau_0)$.

More precisely, let $z \in \mathbb{C} \setminus \gamma$. If $|\gamma(\tau) - z| > \delta(\tau)/2$ for all $\tau \in \mathbb{R}$, then we set $\gamma^z := \gamma$. Otherwise choose τ_0 with $|\gamma(\tau_0) - z| \leq \delta(\tau_0)/2$ such that $|\tau_0|$ is minimal. Let τ_1 and τ_2 be the smallest, respectively largest, values of τ for which $\gamma(\tau) \in \partial D_{\tau_0}$, and set

$$\gamma^z := \gamma((-\infty, \tau_1)) \cup \alpha \cup \gamma([\tau_2, \infty)),$$

where α is an arc of ∂D_{τ_0} chosen such that γ^z is homotopic to γ in $\mathbb{C} \setminus \{z\}$.

We then have

$$h(z) = \frac{1}{2\pi i} \int_{\gamma^z} \frac{g(\zeta)}{\zeta - z} d\zeta.$$

Hence

$$(2.3) \quad 2\pi|h(z)| \leq \int_{\partial D_{\tau_0}} \frac{|g(\zeta)|}{|\zeta - z|} |d\zeta| + \int_{\gamma \setminus \overline{D_{\tau_0}}} \frac{|g(\zeta)|}{|\zeta - z|} |d\zeta|.$$

To estimate the first integral, we bound $|\tau_0|_+$ from below in terms of $|z|_+$. We have

$$(2.4) \quad |\gamma(\tau_0) - z| \leq \frac{\delta(\tau_0)}{2} = \frac{|\tau_0|_+^{-\delta_2}}{2C_4} \leq 1.$$

Thus $|z|_+ \leq 2|\gamma(\tau_0)|_+$, and hence, by (a),

$$(2.5) \quad |\tau_0|_+ \geq \frac{|\gamma(\tau_0)|_+}{C_1} \geq \frac{|z|_+}{2C_1}.$$

So, by choice of D_{τ_0} and (d), we can bound the first integral from (2.3):

$$(2.6) \quad \int_{\partial D_{\tau_0}} \frac{|g(\zeta)| |d\zeta|}{|\zeta - z|} \leq \int_{\partial D_{\tau_0}} \frac{C_5 |d\zeta|}{|\tau_0| + \delta(\tau_0)} = \frac{2\pi C_5}{|\tau_0|} \leq \frac{4\pi C_1 C_5}{|z|_+}.$$

Now we turn to estimating the second integral in (2.3). If $\zeta = \gamma(\tau) \in \gamma^z \setminus \overline{D_{\tau_0}}$, then we have $|\zeta - z| > \delta(\tau_0)/2$. Using the definition of τ_0 and monotonicity of the function $|\tau| \mapsto \delta(\tau)$, we see that

$$(2.7) \quad |\gamma(\tau) - z| > \frac{\delta(\tau)}{2} = \frac{1}{2C_4 |\tau|_+^{\delta_2}}.$$

This estimate, together with (2.2), would be sufficient to prove that the integral in question, and hence $h(z)$, is bounded. In order to obtain the stronger fact that $h(z) = O(1/z)$, we subdivide the remaining part of the curve once more. (We note that this stronger bound is not required for the applications that we have in mind.)

Define $\Theta := \max\left(1, \frac{|z|_+}{2C_1}\right)$. For $\tau \leq \Theta$, we then have

$$|\gamma(\tau)| \leq C_1 \cdot |\tau|_+ \leq C_1 \cdot \Theta = \max\left(\frac{|z|_+}{2}, C_1\right) =: R.$$

We use this to estimate the integral over the curve

$$\gamma_1^z := \gamma^z \cap \gamma([- \Theta, \Theta]).$$

The idea is that $|\zeta - z|$ is (at least) comparable to $|z|_+$ for all points ζ on this curve. Indeed, suppose that $|z|_+ < 2C_1$. Then $\Theta = 1$, and hence, by (2.7),

$$(2.8) \quad |\zeta - z| \geq \frac{1}{2C_4 \Theta^{\delta_2}} = \frac{1}{2C_4} > \frac{|z|_+}{4C_1 C_4}.$$

If $|z|_+ \geq 2C_1$, then $|\zeta| \leq R = |z|_+/2$, hence again $|\zeta - z| \geq |z|_+/2 > |z|_+/(4C_1 C_4)$. Thus, using (2.2):

$$(2.9) \quad \begin{aligned} \int_{\gamma_1^z} \frac{|g(\zeta)|}{|\zeta - z|} |d\zeta| &\leq \frac{4C_1 C_4}{|z|_+} \int_{\gamma_1^z} |g(\zeta)| |d\zeta| \\ &\leq \frac{4C_1 C_4}{|z|_+} \int_{\gamma} |g(\zeta)| |d\zeta| \leq \frac{16C_1 C_2 C_3 C_4}{|z|_+}. \end{aligned}$$

It remains to deal with the part of the curve given by

$$\gamma_2^z := \gamma \setminus (\overline{D_{\tau_0}} \cup \gamma([- \Theta, \Theta])).$$

We use (2.7), as well as (b) and (c) to obtain

$$(2.10) \quad \begin{aligned} \int_{\gamma_2^z} \frac{|g(\zeta)|}{|\zeta - z|} |d\zeta| &\leq \int_{|\tau| > \Theta} 2C_4 |\tau|^{\delta_2} \cdot |g(\gamma(\tau))| \cdot |\gamma'(\tau)| |d\tau| \\ &\leq 4 \cdot C_2 \cdot C_3 \cdot C_4 \cdot \int_{\Theta}^{\infty} t^{\delta_2} \cdot t^{-(2+\delta_1+\delta_2)} \cdot t^{\delta_1} dt \\ &= 4 \cdot C_2 \cdot C_3 \cdot C_4 \cdot \int_{\Theta}^{\infty} \tau^{-2} d\tau = 4C_2 C_3 C_4 \Theta^{-1} \leq \frac{8C_1 C_2 C_3 C_4}{|z|_+}. \end{aligned}$$

Combining the estimates (2.6), (2.9) and (2.10), the proof is complete. ■

The following proposition shows that any function approximating a universal covering must itself have a logarithmic singularity over infinity.

2.2. Proposition. *Let $\Psi : T \rightarrow \mathbb{H}$ be a model function, and set $g := \exp \circ \Psi$. Suppose that $f : T \rightarrow \mathbb{C} \setminus \{0\}$ is a holomorphic function with $|f(z) - g(z)| \leq M$ for some $M > 0$ and all $z \in T$. Define*

$$T' := \{z \in T : |f(z)| > 2M\}.$$

Then T' is a simply-connected domain and $f : T' \rightarrow \{|z| > 2M\}$ is a universal covering map.

Proof. Let us set $T'' := \{z \in T : |g(z)| > M\}$. Then T'' is simply-connected, $|f(z)| > 0$ for all $z \in T''$ and $T' \subset T''$. It follows from the minimum principle that T' is simply-connected. We can define a branch $F : T' \rightarrow \mathbb{H}$ of $\log f$. By continuity, we have $\operatorname{Re} F(z) \rightarrow \log(2M)$ as z tends to a point in the boundary of T' (in \mathbb{C}). We claim that $|F(z)| \rightarrow \infty$ as $z \rightarrow \infty$. Indeed, by assumption we have, for all $z \in T'$,

$$|\operatorname{Re} F(z) - \operatorname{Re} \Psi(z)| = \left| \log \frac{|f(z)|}{|g(z)|} \right| \leq \log 2.$$

Furthermore, the argument of $f(z)$ and $g(z)$ differs by less than π , and hence $|\operatorname{Im} F(z) - \operatorname{Im} \Psi(z)|$ is contained in the union

$$\bigcup_{k \in \mathbb{Z}} ((2k - 1)\pi, (2k + 1)\pi).$$

Since T' is connected, it follows that

$$|\operatorname{Im} F(z) - \operatorname{Im} \Psi(z)| \leq (2k + 1)\pi$$

for some $k \in \mathbb{Z}$. Hence

$$|F(z) - \Psi(z)| \leq K$$

for a suitable constant $K > 0$, which proves our claim that $|F(z)| \rightarrow \infty$ as $z \rightarrow \infty$.

So F is a proper map, and hence has some well-defined degree d . F extends to a degree d map from the boundary of T' (in the Riemann sphere) to $\{\operatorname{Re} z = \log(2M)\} \cup \{\infty\}$. Since ∞ only has one preimage, it follows that $d = 1$. Thus F is a conformal isomorphism, and $f = \exp \circ F$ is a universal covering map, as claimed. \blacksquare

Before proving our main approximation result in Section 4, let us note that Theorem 2.1 includes the examples from [PS] and [S1].

2.3. Corollary. *Let $p > 0$ and set*

$$\begin{aligned} S_1 &:= \{x + iy : x > 0 \text{ and } |y| < \pi\}, \quad \text{and} \\ S_2 &:= \{x + iy : |y| \leq \pi x / [(1 + p)(\log(x))^p], x \geq 3\}. \end{aligned}$$

Also set

$$g_1(z) := e^{e^z} \quad \text{and} \quad g_2(z) := \exp(e^{(\log z)^{1+p}})$$

(where g_1 is defined on \mathbb{C} , and g_2 on $\mathbb{C} \setminus (-\infty, 1]$). Let γ_j be the boundary of S_j , described in clockwise direction. Then

$$f_j(z) := \frac{1}{2\pi i} \int_{\gamma_j} \frac{g_j(\zeta)}{\zeta - z} d\zeta, \quad \notin \bar{S},$$

extends to an entire function $f_j : \mathbb{C} \rightarrow \mathbb{C}$. Furthermore, $f_j \in \mathcal{B}$ and

$$f_j(z) = \begin{cases} g_j(z) + O(1/z) & z \in S_j \\ O(1/z) & \text{otherwise} \end{cases} \quad \text{as } z \rightarrow \infty.$$

Proof. It is easy to see that the parametrizations of γ_1 and γ_2 by arc-length satisfy the assumptions of Theorem 2.1, say with $\delta_1 = \delta_2 = 0$, and T being the domain of definition of g_j . Hence $f_j(z)$ is indeed defined and extends to an entire function with the stated asymptotics. The fact that f_j belongs to the class \mathcal{B} follows from Proposition 2.2. ■

3. THE HYPERBOLIC METRIC OF SIMPLY-CONNECTED DOMAINS

We frequently use the *hyperbolic metric* in a domain $U \subset \mathbb{C}$ that omits more than two points. (For an introduction to the hyperbolic metric, see e.g. [BM].) We denote distance with respect to this metric by dist_U , and the density of the metric by ρ_U . That is,

$$\text{dist}_U(z, w) = \inf_{\gamma} \int_0^1 |\gamma'(t)| \rho_U(\gamma(t)) dt,$$

where the infimum is taken over all curves $\gamma : [0, 1] \rightarrow U$ with $\gamma(0) = z$ and $\gamma(1) = w$.

We shall routinely use a number of standard facts about the hyperbolic metric.

3.1. Proposition ((Properties of the hyperbolic metric)).

- (a) The hyperbolic metric in the right half plane \mathbb{H} is given by $\rho_{\mathbb{H}} = \frac{1}{\text{Re } z}$. In particular, $\text{dist}_{\mathbb{H}}(1, x) = \log x$ for $x \geq 1$ and $\text{dist}_{\mathbb{H}}(x, x + ix) \leq 1$ for every $x > 0$.
- (b) In the strip $S = \{|\text{Im } z| < \pi\}$, we have $\text{dist}_S(z, w) \geq |\text{Re } z - \text{Re } w|/2$ for all $z, w \in S$.
- (c) If $V \subset W$, then $\rho_V(z) \geq \rho_W(z)$ for all $z \in V$.
- (d) If $V, W \subset \mathbb{C}$ are hyperbolic and $f : V \rightarrow W$ is a conformal isomorphism, then f is a hyperbolic isometry; i.e. $\rho_V(z) = |f'(z)| \cdot \rho_W(f(z))$.
- (e) If $V \subset \mathbb{C}$ is simply connected, then $1/(2 \text{dist}(z, \partial V)) \leq \rho_V(z) \leq 2/\text{dist}(z, \partial V)$ for all $z \in V$.

Let us make two more simple observations about the hyperbolic metric in simply-connected domains.

3.2. Lemma ((Hyperbolic distance and Euclidean distance)). *Let $V \subset \mathbb{C}$ be a simply-connected domain, and let $z, w \in V$. Then*

$$\text{dist}_V(z, w) \geq \frac{1}{2} \log \left(1 + \frac{|z - w|}{\text{dist}(z, \partial V)} \right).$$

Proof. Set $\delta := \text{dist}(z, \partial V)$. Let $\gamma : [0, T] \rightarrow V$ be a curve connecting z and w , parametrized by Euclidean arc-length. So $T \geq \text{dist}(z, w)$. Then we have $\text{dist}(\gamma(t), \partial V) \leq \delta + t$. Thus, by Proposition 3.1 (e),

$$\begin{aligned} \int_0^T |\gamma'(t)| \rho_V(t) dt &\geq \int_0^T \frac{dt}{2 \text{dist}(\gamma(t), \partial V)} \geq \frac{1}{2} \int_0^T \frac{dt}{\delta + t} \\ &= \frac{1}{2} (\log(\delta + T) - \log(\delta)) = \frac{1}{2} \log \left(1 + \frac{T}{\delta} \right). \quad \blacksquare \end{aligned}$$

3.3. Lemma ((Bounded hyperbolic diameter of Euclidean disks)). *Let $V \subset \mathbb{C}$ be a simply-connected domain, let $z_0 \in V$ and let $\Delta \in (0, 2]$. Define $\delta := \Delta \cdot \text{dist}(z_0, \partial V)/4$.*

If $z \in V$ with $|z - z_0| \leq \delta$, then $\text{dist}_V(z, z_0) \leq \Delta$.

Proof. Set $d := |z - z_0|$ and let $\gamma : [0, d] \rightarrow V$ be the straight line segment connecting z_0 and z , parametrized by arc-length. Then

$$\text{dist}(\gamma(t), \partial V) \geq \frac{\text{dist}(z_0, \partial V)}{2},$$

and thus, again using Proposition 3.1 (e),

$$\text{dist}_V(z, z_0) \leq \int_0^d \rho_V(\gamma(t)) dt \leq \int_0^d \frac{4dt}{\text{dist}(z_0, \partial V)} = \frac{4 \cdot d}{\text{dist}(z_0, \partial V)} \leq \Delta. \quad \blacksquare$$

Finally, we will on occasion use the following version of the Ahlfors distortion theorem [A, Corollary to Theorem 4.8].

3.4. Theorem ((Ahlfors distortion theorem)). *Let $V \subset \mathbb{C}$ be a simply connected domain, and let $z, w \in V$ with $a := \text{Re } z < \text{Re } w =: b$. Let $\sigma_z, \sigma_w \subset V$ be the maximal vertical line segments passing through z resp. w .*

Set $S = \{a + ib : |b| \leq \pi\}$, and let $\varphi : V \rightarrow S$ be a conformal isomorphism such that $\varphi(\sigma_z)$ and $\varphi(\sigma_w)$ both separate $-\infty$ from $+\infty$ in S (i.e., they connect the upper and lower boundaries of the strip S), and such that $\varphi(\sigma_z)$ separates $\varphi(\sigma_w)$ from $-\infty$ (i.e., $\varphi(\sigma_z)$ is to the left of $\varphi(\sigma_w)$ in S).

For $a \leq x \leq b$, let $\vartheta(x)$ denote the shortest length of a vertical line segment at real part x that separates z from w in V .

If $\int_a^b dx/\vartheta(x) \geq 1/2$, then

$$\varphi(b) - \varphi(a) \geq 2\pi \int_a^b \frac{dx}{\vartheta(x)} - 2 \log 32.$$

We also note the following fact, which is closely related to the distortion theorem:

3.5. Lemma ((Geodesics in quadrilaterals)). *Let V be a simply-connected domain that is symmetric with respect to the real axis. Let γ_1 and γ_2 be two cross-cuts of V that are symmetric with respect to the real axis, with $\gamma_1 \cap \gamma_2 = \emptyset$, and suppose that the quadrilateral Q bounded by γ_1 and γ_2 in V has modulus at least $1/2$. (I.e., the extremal length of the family of curves connecting γ_1 and γ_2 in V is at least $1/2$.)*

Then \bar{Q} contains a geodesic of V that is symmetric with respect to the real axis.

Proof. Let S denote the strip $\{a + ib : |b| \leq \pi\}$ and let $\varphi : V \rightarrow S$ be a conformal isomorphism that takes $\mathbb{R} \cap V$ to the real axis. Set $\tilde{Q} := \varphi(Q)$; then \tilde{Q} is a quadrilateral in S , symmetric with respect to the real axis, of modulus at least $1/2$. We must show that \tilde{Q} contains a vertical segment connecting the two boundary components of S .

The exponential map takes \tilde{Q} to an annulus of modulus at least $1/2$, slit along an interval of the positive real axis, which separates 0 from ∞ . By Teichmüller's modulus theorem [A, Theorem 4-7], the closure of this annulus contains a round circle centered at the origin, which completes the proof. \blacksquare

4. APPROXIMATION OF MODEL FUNCTIONS

We now turn to proving Theorem 1.7. As already mentioned, this result is “best” possible with our method, in the sense that the domain H is chosen as close to a right half plane as possible while still guaranteeing convergence of the Cauchy integral. However, sometimes it is convenient to use other image domains, e.g. because it might be possible to write down an explicit mapping function for these. We will therefore work in a somewhat more general setting. In particular, we recover the results of [R³S] as a special case.

4.1. Standing Assumption ((Assumption on H and γ)).

$H \subset \mathbb{C}$ is a simply connected domain containing the right half plane \mathbb{H} . Furthermore,

$$\alpha : (-\infty, \infty) \rightarrow H$$

is a piecewise smooth injective curve for which there exist positive constants A_1, A_2, A_3, A_4 and Δ with $A_1, A_4 > 1$ such that, for all $t \in \mathbb{R}$:

- (a) $\operatorname{Re} \alpha(t) \leq -13 \log_+(t) + \log A_1$.
- (b) $|\alpha'(t)| \leq A_2$. (If t belongs to the discrete set where α is not differentiable, this means that both the left and right derivatives are bounded by A_2 .)
- (c) $\operatorname{dist}_H(\alpha(t), 1 + it) \leq A_3$. (Recall that dist_H denotes the hyperbolic distance in H .)
- (d) If $\zeta \in H$ with $\operatorname{dist}_H(\alpha(t), \zeta) \leq \Delta$, then $\operatorname{Re} \zeta \leq -4 \log_+ |t| + \log A_4$.

We refer to the pair of H and α as the *initial configuration*. The bounds below will depend on this initial choice.

Remark 1. The final two conditions may seem somewhat technical. Roughly, they mean that the curve α stays within a comparable distance from both ∂H and the line $\{\operatorname{Re} \zeta = 1\}$; compare Section 5.

Remark 2. It is not difficult to see that the choice

$$H := \{x + iy : x > -14 \log_+ |y|\}$$

used in the statement of Theorem 1.7 and the curve

$$\alpha(t) := it - 13 \log_+ |t| + 1$$

satisfy our standing assumption. For completeness, we provide the argument in Section 5.

Remark 3. In applications, the domain H and the curve α will be fixed, so dependence on the initial configuration will not usually be important. However, we note that our bounds will depend only on the constants A_1 to A_4 and Δ , but not otherwise on α and H .

4.2. Standing Assumption ((Model function)).

Furthermore,

$$\Psi : T \rightarrow H$$

is a model function in the sense of Definition 1.6 (where H is the domain from Standing Assumption 4.1). We additionally assume, by way of normalization, that that $1 \in T$, $0 \in \partial T$, $\operatorname{dist}(1, \partial T) = 0$ and $\Psi(1) = 1$.

Let V be a component of $\exp^{-1}(T)$ and let $G : V \rightarrow H$ be the conformal isomorphism $G := \Psi \circ \exp$. We also set $g := \exp \circ \Psi$. Note that we have $g \circ \exp = \exp \circ G$.

Finally, we set $\beta := G^{-1} \circ \alpha$ and $\gamma := \exp \circ \beta = \Psi^{-1} \circ \alpha$. Let \tilde{T} be the component of $\mathbb{C} \setminus \gamma$ that is contained in T .

We will now show that (under these assumptions), we can apply Theorem 2.1 to T and a reparametrization of γ .

4.3. Lemma ((Growth and distance to boundary)). *There are constants M_1 and M_2 , depending only on the initial configuration, such that*

$$|\gamma(t)| \leq M_1 \cdot |t|_+^4 \quad \text{and} \quad \text{dist}(\gamma(t), \partial T) \geq M_2 \cdot |t|_+^{-4}$$

for all $t \in \mathbb{R}$.

Proof. We set $C := A_3 + 1$. Using the fact that hyperbolic distances in H are smaller than those in the half plane \mathbb{H} (recall Proposition 3.1), we see that

$$\begin{aligned} \text{dist}_H(1, \alpha(t)) &\leq \text{dist}_{\mathbb{H}}(1, |t|_+) + \text{dist}_{\mathbb{H}}(|t|_+, |t|_+ + ti) \\ &\quad + \text{dist}_{\mathbb{H}}(|t|_+ + ti, 1 + ti) + \text{dist}_H(1 + ti, \alpha(t)) \\ &\leq \log_+ |t| + 1 + \log_+ |t| + A_3 = C + 2 \log_+ |t|. \end{aligned}$$

We now use the Ahlfors distortion theorem, Theorem 3.4, to deduce the desired estimate. Let σ_1 be the maximal vertical line segment in V containing $0 = G^{-1}(1)$, and let σ_2 be the maximal vertical line segment containing $\beta(t)$. Set $S = \{a + ib : |b| < \pi\}$ and let $\varphi : V \rightarrow S$ be a conformal isomorphism such that $\text{Re } \varphi(0) = 0$, such that $\varphi(\sigma_1)$ and $\varphi(\sigma_2)$ both connect the upper and lower boundaries of S , and such that $\varphi(\sigma_2)$ is to the right of $\varphi(\sigma_1)$. This is always possible: pick two prime ends ζ_1 and ζ_2 (if V is a Jordan domain, this simply means picking two points on ∂V) such that σ_1 separates ζ_1 from σ_2 and σ_2 separates σ_1 from ζ_2 . We then choose φ such that $\varphi(\sigma_1) = -\infty$ and $\varphi(\sigma_2) = +\infty$.

Recall that V does not intersect its own translates by integer multiples of $2\pi i$, and hence does not contain any vertical segments of height greater than 2π . If $\text{Re } \beta(t) \geq \pi$, then Theorem 3.4 and Proposition 3.1 (b) imply that

$$2 \text{dist}_S(\varphi(0), \varphi(\beta(t))) \geq |\varphi(\beta(t)) - \varphi(0)| \geq \text{Re } \beta(t) - D,$$

where $D = 2 \log 32$ is a universal constant. Thus

$$\text{Re } \beta(t) \leq \max(\pi, D + 2C) + 4 \log_+ |t|.$$

Recalling that $\gamma(t) = \exp(\beta(t))$, the first claim is proved.

Similarly, we can estimate $\text{dist}(\gamma(t), \partial T)$, using Lemma 3.2. Indeed, set $z := \gamma(t)$ and $\delta := \text{dist}(z, \partial T)$. Recall that $\text{dist}(1, \partial T) = 1$, so $|z - 1| \geq 1 - \delta$, and hence

$$C + 2 \log_+ |t| \geq \text{dist}_H(1, \alpha(t)) = \text{dist}_T(1, z) \geq \frac{1}{2} \log \left(1 + \frac{|z - 1|}{\delta} \right) \geq \frac{1}{2} \log \frac{1}{\delta}$$

by Lemma 3.2. Exponentiating this inequality and rearranging, we see that

$$\delta \geq (|t|_+)^{-4} \cdot \exp(-2C),$$

as desired. ■

4.4. Corollary. *There is a constant M_3 , depending only on the initial configuration, such that*

$$|\gamma'(t)| \leq M_3 \cdot |t|_+^4$$

for all $t \in \mathbb{R}$.

Furthermore, there is a constant M_4 , depending only on the initial configuration, with the following property. If $t \in \mathbb{R}$ and $z \in \mathbb{C}$ with

$$|z - \gamma(t)| \leq M_4 \cdot |t|_+^{-4},$$

then $z \in T$ and $|g(z)| \leq A_4 \cdot |t|_+^{-4}$.

Proof. We have

$$|\gamma'(t)| = |\alpha'(t)| \cdot |(G^{-1})'(\alpha(t))| \cdot \exp(\operatorname{Re} \beta(t)).$$

The first term is bounded by our standing assumption that $|\alpha'(t)| \leq A_2$. To estimate the second term, we use hyperbolic geometry: G is a conformal isomorphism and V and H are both simply connected, so

$$(G^{-1})'(\alpha(t)) = \frac{\rho_H(\alpha(t))}{\rho_V(\beta(t))} \leq 4 \frac{\operatorname{dist}(\beta(t), \partial V)}{\operatorname{dist}(\alpha(t), \partial H)}.$$

Since \exp is injective on V , we have $\operatorname{dist}(z, \partial V) \leq \pi$ for all $z \in V$. We also note that $\operatorname{dist}(\alpha(t), \partial H) \geq 1/D$ for some constant D that depends only on A_3 . Indeed, if $\operatorname{Re} \alpha(t) \geq 1/2$, there is nothing to prove (since $\mathbb{H} \subset H$). Otherwise, we have $\operatorname{dist}(\alpha(t), 1+it) \geq 1/2$ and $\operatorname{dist}_H(\alpha(t), 1+it) \leq A_3$, and the claim follows from Lemma 3.2.

Finally,

$$\exp(\operatorname{Re} \beta(t)) = |\gamma(t)| \leq M_1 \cdot |t|_+^4$$

by Lemma 4.3. Combining these estimates, we see that

$$|\gamma'(t)| \leq 4\pi \cdot A_2 \cdot D \cdot M_1 \cdot |t|_+^4.$$

To prove the second claim, let us assume without loss of generality that $\Delta \leq 2$ and set

$$M_4 := \frac{M_2 \cdot \Delta}{4}.$$

Suppose that $t \in \mathbb{R}$ and $z \in \mathbb{C}$ are as in the claim; then

$$|z - \gamma(t)| \leq M_4 \cdot |t|_+^{-4} = \frac{\Delta}{4} \cdot M_2 \cdot |t|_+^{-4} \leq \frac{\Delta}{4} \cdot \operatorname{dist}(\gamma(t), \partial T)$$

by Lemma 4.3. Hence we can apply Lemma 3.3 to see that $\operatorname{dist}_T(z, \gamma(t)) \leq \Delta$, and thus $\operatorname{dist}_H(\Psi(z), \alpha(t)) \leq \Delta$. By the standing assumption, it follows that

$$|g(z)| = \exp(\operatorname{Re} \Psi(z)) \leq A_4 \cdot |t|_+^{-4}. \quad \blacksquare$$

We now ready to apply Theorem 2.1 to conclude:

4.5. Corollary ((Approximation by entire functions)). *In the setting of Standing Assumption 4.1, let $\Psi : T \rightarrow H$ be any model function. Define $g(z) := \exp(\Psi(z))$, $\gamma := \Psi^{-1} \circ \alpha$ and $z_0 := \Psi^{-1}(1)$. Then*

$$h(z) := \frac{1}{2\pi i} \int_{\gamma} \frac{g(\zeta)}{\zeta - z} d\zeta$$

defines a holomorphic function for $z \notin \gamma$. This function satisfies

$$|h(z)| \leq M_5 \quad \text{and} \quad |h(z)| \leq \max(|z_0|_+, \text{dist}(z_0, \partial T)) \cdot \frac{M_6}{|z|_+}$$

for all z . Here the constants M_5 and M_6 depend only on the initial configuration.

Furthermore,

$$f(z) := \begin{cases} h(z) + g(z) & z \in \tilde{T} \\ h(z) & z \notin \tilde{T} \end{cases}$$

extends to an entire function $f : \mathbb{C} \rightarrow \mathbb{C}$ with $S(f) \subset \overline{\mathbb{D}_{2M_5}(0)}$.

Proof. Let us first assume that T and Ψ are normalized as in Standing Assumption 4.2; i.e., $0 \in \partial T$, $\text{dist}(1, \partial T) = 1$ and $\Psi(1) = 1$.

We reparametrize the curve γ by the substitution

$$\tau := \begin{cases} t^4 & t \geq 1 \\ t & |t| < 1 \\ -t^4 & t \leq -1. \end{cases}$$

By Lemma 4.3 and Corollary 4.4, we have

- (a) $|\gamma(\tau)| \leq M_1 \cdot |\tau|_+$;
- (b) $|d\gamma(\tau)/d\tau| = \frac{1}{4} \cdot |\gamma'(t)| \cdot |\tau|_+^{-3/4} \leq \frac{M_4}{4} \cdot |\tau|_+^{1/4}$;
- (c) $|g(\gamma(\tau))| = \exp(\text{Re } \alpha(t)) \leq A_1 \cdot |t|_+^{-13} = A_1 \cdot |\tau|_+^{-(2+1/4+1)}$;
- (d) if $|z - \gamma(\tau)| \leq M_4 \cdot |\tau|_+^{-1}$, then $z \in T$ and $|g(z)| \leq A_4 \cdot |\tau|_+^{-1}$.

Thus the claims on the convergence and asymptotics of $h(z)$ and the analytic continuation of $f(z)$ follow from Theorem 2.1. The claim regarding the singular values of f follows from Proposition 2.2.

For general T and Ψ , we normalize and apply the case that was just established. More precisely, set $z_0 := \Psi^{-1}(1)$ and let $a \in \partial T$ be a point whose distance to z_0 is minimal; define $\alpha := z_0 - a$. Then the function

$$\tilde{\Psi}(z) := \Psi(\alpha z + a)$$

satisfies Standing Assumption 4.2. Let \tilde{h} be the corresponding function

$$\tilde{h}(z) = \frac{1}{2\pi i} \int_{\tilde{\gamma}} \frac{\tilde{g}(\zeta)}{\zeta - z} d\zeta = h(\alpha z + a),$$

where $\tilde{\gamma}(t) = (\gamma(t) - a)/\alpha$. As we have just seen,

$$|h(z)| = \left| \tilde{h} \left(\frac{z - a}{\alpha} \right) \right| \leq \frac{M_5}{|(z - a)/\alpha|_+} \leq M_5$$

for a constant M_5 depending only on the initial configuration. It is elementary to verify that

$$\left| \frac{z - a}{\alpha} \right|_+ \geq \frac{|z|_+}{2 \max(|a|_+, |\alpha|_+)}.$$

(E.g., distinguish between the cases $|z| < 2|a|_+$ and $|z| \geq 2|a|_+$.) Hence

$$|h(z)| \leq 2 \max(|a|_+, |\alpha|_+) \cdot \frac{M_5}{|z|_+} \leq 4 \max(|z_0|_+, |\alpha|) \cdot \frac{M_5}{|z|_+},$$

as desired. ■

4.6. Remark ((Dependence of the bounds in Theorem 1.7)).

In the next section, we shall carry out the simple verification that the choice of H and γ from Theorem 1.7 satisfies Standing Assumption 4.1. Hence Corollary 4.5 implies the theorem.

In particular, we note that the approximating function f from Theorem 1.7 satisfies $|f(z)| \leq M$ outside T and $|f(z) - g(z)| \leq M$ in T , where $M = M_5$ is a universal constant. This fact will be used in the second part of the paper.

5. VALID INITIAL CONFIGURATIONS

5.1. Proposition. *Let $\rho : \mathbb{R} \rightarrow [0, \infty)$ be a continuous function that is increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$. Set*

$$H := \{z \in \mathbb{C} : \operatorname{Re} z > -\rho(\operatorname{Im} z)\}.$$

Suppose that

$$\alpha : (-\infty, \infty) \rightarrow H$$

is a piecewise smooth injective curve such that (a) and (b) of Standing Assumption 4.1 are satisfied. If furthermore $\operatorname{Im} \alpha(t) = t$ for all $t \in \mathbb{R}$ and

$$c \cdot |\operatorname{Re} \alpha(t)| \leq \operatorname{dist}(\alpha(t), \partial H) \leq C |\operatorname{Re} \alpha(t)|$$

when $|t|$ is sufficiently large, where c and C are positive constants, then Standing Assumption 4.1 is satisfied.

Proof. Observe first that, if $a_1, a_2, b \in \mathbb{R}$ are such that $-\rho(b) < a_1 < a_2$, then

$$\operatorname{dist}(a_1 + bi, \partial H) < \operatorname{dist}(a_2 + bi, \partial H).$$

Now let $t \in \mathbb{R}$; by (a) of Standing Assumption 4.1, we may assume that $|t|$ is sufficiently large that $\operatorname{Re} \alpha(t) < -1$. Then

$$\begin{aligned} \operatorname{dist}_H(\alpha(t), 1 + it) &\leq \int_0^{1 - \operatorname{Re} \alpha(t)} \rho_H(\alpha(t) + x) dx \leq \int_0^{1 - \operatorname{Re} \alpha(t)} \frac{2 dx}{\operatorname{dist}(\alpha(t), \partial H)} \\ &\leq 2 \frac{1 - \operatorname{Re} \alpha(t)}{c \cdot |\operatorname{Re} \alpha(t)|} \leq \frac{4}{c}. \end{aligned}$$

So requirement (c) holds when $|t|$ is large enough.

Finally, let $\Delta > 0$. Then by Lemma 3.2, the hyperbolic disk of radius Δ around $\alpha(t)$ is contained in the Euclidean disk around $\alpha(t)$ of radius

$$(e^{2\Delta} - 1) \cdot \operatorname{dist}(\alpha(t), \partial H).$$

On the other hand, we have

$$-\operatorname{Re} \alpha(t) > 13 \log_+ |t| - \log A_1 > 8 \log_+ |t|,$$

provided $|t|$ is sufficiently large, and hence

$$-\operatorname{Re} \alpha(t) - 4 \log_+ |t| > -\operatorname{Re} \alpha(t)/2 > \frac{\operatorname{dist}(\alpha(t), \partial H)}{2C}.$$

So if we choose $\Delta := \log(1 + 1/(4C))/2$, we have

$$\operatorname{Re} \zeta < -4 \log_+ |t|$$

whenever $\operatorname{dist}_H(\zeta, \gamma(t)) \leq \Delta$ (still under the assumption that t is sufficiently large). Hence requirement (d) also holds when $|t|$ is sufficiently large.

We obtain the requirements for all $t \in \mathbb{R}$ by choosing A_3 and A_4 sufficiently large. \blacksquare

5.2. Corollary ((Two valid initial configurations)). *Set either*

$$\begin{aligned} H &:= \{x + iy : x > -14 \log_+ |y|\} \quad \text{and} \\ \alpha(t) &:= it - 13 \log_+ |t| + 1 \end{aligned}$$

or

$$\begin{aligned} H &:= \{x + iy : x > -2M \cdot |y|\} \quad \text{and} \\ \alpha(t) &:= it - M|t| + 1 \end{aligned}$$

(where $M > 0$ is a constant).

Then H and α satisfy the conditions of Standing Assumption 4.1.

Proof. By the previous proposition, we only need to check (a) and (b). The first of these is self-evident. The second is also immediate; in the first case we have

$$\alpha'(t) = \begin{cases} i - \frac{13}{t} & |t| > 1 \\ i & -1 < t < 1. \end{cases}$$

and in the second

$$\alpha'(t) = i - M \cdot \operatorname{sign}(t). \quad \blacksquare$$

6. QUASICONFORMAL EQUIVALENCE AND CONJUGACY

We begin with a simple lemma on obtaining a quasiconformal map on a vertical strip that extrapolates between the identity on one boundary and a map that is not too far from the identity on the other.

6.1. Lemma. *Let $R > 0$, and let $\tau : \mathbb{R} \rightarrow \mathbb{C}$ be a differentiable function with the property that there is $\eta < 1/2$ with $|\tau'(t)| \leq \eta$ and $|\tau(t)| \leq \eta R$ for all $t \in \mathbb{R}$.*

Let S denote the strip $\{a + ib : 0 < a < R\}$. Then there is a quasiconformal map ϑ , defined on S , such that $\vartheta(it) = it$ and

$$\vartheta(R + it) = R + it + \tau(t)$$

for all $t \in \mathbb{R}$.

Furthermore, the complex dilatation $\mu_\vartheta = (d\vartheta/d\bar{z})/(d\vartheta/dz)$ is bounded by $\eta/(1 - \eta)$ almost everywhere.

Proof. The map ϑ is defined simply by linear interpolation along horizontal line segments:

$$(6.1) \quad \vartheta(z) := z + \frac{\operatorname{Re} z}{R} \cdot \tau(\operatorname{Im}(z)).$$

This map satisfies the required boundary conditions. Clearly ϑ is injective when restricted to a fixed horizontal line segment $\{z \in S : \operatorname{Im} z = t\}$, and this line segment is

mapped to the straight segment L_t that connects it and $R + it + \tau(t)$. In order to prove that ϑ is a homeomorphism onto its image, we must show that no two of these image line segments intersect.

This follows from the assumptions on τ . Indeed, let $t_0 \in \mathbb{R}$. From $|\tau(t_0)|/R \leq \eta < 1/2$, we see that the line segment L_{t_0} is sloped at an angle strictly between $-\pi/4$ and $\pi/4$. On the other hand, from $|\tau'| \leq \eta < 1/2$ we see that the argument of the derivative

$$\frac{d\vartheta(R + it)}{dt} = i + \tau'(t)$$

lies strictly between $\pi/4$ and $3\pi/4$. This implies that, for $t > t_0$, the point $\vartheta(R + it)$ lies above the line through it and $R + it$, and for $t < t_0$ it lies below this line. Hence the line segments L_t and L_{t_0} do not intersect for $t \neq t_0$.

So it remains to estimate the complex dilatation of ϑ . If we set $h(z) := \frac{\operatorname{Re} z}{R} \cdot \tau(\operatorname{Im}(z))$, then we have

$$\frac{\partial \vartheta}{\partial z} = 1 + \frac{\partial h}{\partial z} \quad \text{and} \quad \frac{\partial \vartheta}{\partial \bar{z}} = \frac{\partial h}{\partial \bar{z}}.$$

Furthermore, writing $z = x + iy$, we have $h(z) = \tau(y) \cdot x/R$, and hence

$$\frac{\partial h(z)}{\partial x} = \frac{\tau(y)}{R} \quad \text{and} \quad \frac{\partial h(z)}{\partial y} = \frac{x}{R} \cdot \tau'(y).$$

Thus

$$\left| \frac{\partial h(z)}{\partial z} \right|, \left| \frac{\partial h(z)}{\partial \bar{z}} \right| \leq \frac{1}{2} \left(\frac{|\tau(y)|}{R} + |\tau'(y)| \right) \leq \eta.$$

Hence we have seen that

$$\left| \frac{\partial \vartheta}{\partial z} \right| \geq 1 - \eta \quad \text{and} \quad \left| \frac{\partial \vartheta}{\partial \bar{z}} \right| \leq \eta.$$

It follows that ϑ is quasiconformal and satisfies the stated bound on its dilatation. \blacksquare

As in [R2], it will be useful to work in logarithmic coordinates when proving our equivalence and conjugacy statements. Hence we shall initially state our results for the following class of functions introduced in [R2, R³S].

6.2. Definition ((The class \mathcal{B}_{\log}^p)).

A holomorphic function

$$F : \mathcal{V} \rightarrow H$$

is said to belong to the class \mathcal{B}_{\log}^p if

- (A) H is a $2\pi i$ -periodic unbounded Jordan domain that contains a right half-plane.
- (B) $\mathcal{V} \neq \emptyset$ is $2\pi i$ -periodic and $\operatorname{Re} z$ is bounded from below in \mathcal{V} .
- (C) F is $2\pi i$ -periodic.
- (D) Each component T of \mathcal{V} is an unbounded Jordan domain that is disjoint from all its $2\pi i\mathbb{Z}$ -translates. For each such T , the restriction $F : T \rightarrow H$ is a conformal isomorphism with $F(\infty) = \infty$. (T is called a *tract* of F ; we denote the inverse of $F|_T$ by F_T^{-1} .)
- (E) The components of \mathcal{V} accumulate only at ∞ ; i.e., if $z_n \in \mathcal{V}$ is a sequence of points no two of which belong to the same component of \mathcal{V} , then $z_n \rightarrow \infty$.

Remark 1. In [R2], the class of functions described in Definition 6.2 is simply called \mathcal{B}_{\log} , while in [R³S], that notation is used for the larger set obtained by omitting the periodicity requirement (C). Subsequent papers such as [RRS] followed the latter convention, hence we use \mathcal{B}_{\log}^p for the class above.

Remark. Let $f \in \mathcal{B}$ and let $R > 0$ be sufficiently large to ensure that $\mathbb{D}_R(0)$ contains the set $S(f) \cup \{0\} \cup \{f(0)\}$. Set $W := \mathbb{C} \setminus \overline{\mathbb{D}_R(0)}$, $H := \exp^{-1}(W) = \{z \in \mathbb{C} : \operatorname{Re} z > \log R\}$ and $\mathcal{V} := \exp^{-1}(f^{-1}(W))$. Then every component V of \mathcal{V} is a Jordan domain whose boundary passes through infinity, and $f \circ \exp : V \rightarrow W$ is a universal covering. Hence we can define a function $F : \mathcal{V} \rightarrow H$ that belongs to \mathcal{B}_{\log}^p and satisfies $\exp \circ F = f \circ \exp$. Such a function is called a *logarithmic transform of f* (or “ f in logarithmic coordinates”); this is the motivation for the definition of \mathcal{B}_{\log}^p .

In the introduction, we stated our results only for functions with a single tract. However, the equivalence and conjugacy results in this section actually hold for functions in the class \mathcal{B}_{\log}^p that are sufficiently close to each other, even if there are infinitely many tracts. The key statement is about *quasiconformal equivalence*:

6.3. Theorem ((QC equivalence in the class \mathcal{B}_{\log}^p)). *Suppose that $G : \mathcal{V}_G \rightarrow \mathbb{H}$ and $F : \mathcal{V}_F \rightarrow H$ belong to the class \mathcal{B}_{\log}^p , and that there is a constant $M > 0$ with*

$$\mathcal{V}_G \supset \mathcal{V}_F \supset \{G^{-1}(z) : \operatorname{Re} z > M\}$$

and $|F(z) - G(z)| \leq M$ for all $z \in \mathcal{V}_F$.

Then for every $R \geq 4(M + 2\pi)$, there exists a quasiconformal map $\Phi : \mathbb{C} \rightarrow \mathbb{C}$, commuting with translation by $2\pi i$, such that

$$F(\Phi(z)) = G(z) \quad \text{whenever } \operatorname{Re} G(z) \geq R.$$

Moreover, the complex dilatation μ_Ψ of Ψ satisfies

$$|\mu_\Psi(z)| \leq 4 \cdot \frac{M}{R}$$

for almost all $z \in \mathbb{C}$.

Furthermore, $\Phi(z) = z$ when $z \notin \mathcal{V}_G$ and when $z \in \mathcal{V}_G$ with $\operatorname{Re} G(z) \leq R/2$, and

$$|\Phi(z) - z| \leq \sup\{|F(\zeta) - G(\zeta)| : |\zeta - z| \leq K\} \leq M$$

otherwise, where $K = 2\pi \cdot (1 + \log 2)$ is a universal constant.

Proof. We begin with a simple observation regarding the structure of H and \mathcal{V}_F .

Claim. The range H of F contains the half plane $\{z \in \mathbb{H} : \operatorname{Re} z > 2M\}$. Furthermore, if V is a component of \mathcal{V}_G , there is a unique component \tilde{V} of \mathcal{V}_F contained in V .

Proof. Let V be a component of \mathcal{V}_G . Then, by assumption, there is a component \tilde{V} of \mathcal{V}_F that contains the connected set $G_V^{-1}(\{\operatorname{Re} z > M\})$, and this component is contained in V . Now $\operatorname{Re} G$ is bounded on $V \setminus \tilde{V}$, and hence $\operatorname{Re} F$ is bounded on $(V \cap \mathcal{V}_F) \setminus \tilde{V}$. Since $\operatorname{Re} F$ is unbounded on every connected component of \mathcal{V}_F , we have seen that indeed $\tilde{V} = V \cap \mathcal{V}_F$.

Furthermore, let $z \in \mathbb{H}$ with $\operatorname{Re} z > 2M$ and consider the circle C of radius M around z . Then $G|_V^{-1}(C)$ is a simple closed curve in \tilde{V} , and it follows from the assumption that

its image under F winds once around z . Since H is simply-connected, it follows that $z \in H$ as claimed. \triangle

Claim. Let V and \tilde{V} be as above. Then there is a quasiconformal map $\varphi_V : \mathbb{H} \rightarrow \mathbb{H}$ with $\varphi_V(z) = z$ when $\operatorname{Re} z \leq R/2$ and $\varphi_V(z) = G(F|_{\tilde{V}}^{-1}(z))$ when $\operatorname{Re} z \geq R$. Furthermore, the complex dilatation of φ_V is bounded by $4M/R < 1$.

Proof. We construct this map using Lemma 6.1. Indeed, set $h(z) := G(F_{\tilde{V}}^{-1}(z)) - z$. Then $|h(z)| \leq M$ for all $z \in \mathcal{V}_F$. Furthermore, if $z \in \mathbb{H}$ with $\operatorname{Re} z \geq R$, then the domain of h contains the disk of radius $R - 2M \geq R/2$. Hence by the Cauchy inequality, we see that

$$|h'(z)| \leq \frac{2M}{R}.$$

Thus, if we set $\tau(t) := h(it + R)$, we have

$$|\tau'(t)| \leq \frac{2M}{R} < \frac{1}{2} \quad \text{and} \quad \frac{2|\tau(t)|}{R} \leq \frac{2M}{R} < \frac{1}{2}$$

for all t . Hence we can apply Lemma 6.1 to obtain a quasiconformal map ϑ on the strip between real parts $R/2$ and R , such that ϑ is the identity on the left boundary of the strip and agrees with $G(F_{\tilde{V}}^{-1}(z)) = z + \tau(z)$ on the right boundary. Hence

$$\varphi_V(z) := \begin{cases} z & \text{if } \operatorname{Re} z \leq R/2 \\ \vartheta(z) & \text{if } R/2 < \operatorname{Re} z < R \\ G(F_{\tilde{V}}^{-1}(z)) & \text{if } \operatorname{Re} z \geq R \end{cases}$$

is the desired quasiconformal homeomorphism. (Note that the map is quasiconformal near points with real parts equal to R or $R/2$, since a straight line is quasiconformally removable.) The bound on the dilatation also follows from Lemma 6.1. \triangle

Now let $z \in \mathbb{C}$. If $z \notin \mathcal{V}_G$, then we define $\Phi(z) := z$. Otherwise, let V be the component of \mathcal{V}_G containing z and define

$$\Phi(z) := G_V^{-1}(\varphi_V(G(z))).$$

Then $\Phi : \mathbb{C} \rightarrow \mathbb{C}$ is a homeomorphism. Furthermore, Φ is quasiconformal on \mathcal{V}_G and agrees with the identity outside this set. Hence, the map is quasiconformal everywhere by Royden's glueing lemma ([Be, Lemma 2], [DH, Lemma 2]), and satisfies the stated dilatation bound. If $\operatorname{Re} G(z) \geq R$ and V is the component of \mathcal{V}_G containing z , then

$$F(\Phi(z)) = F(G_V^{-1}(\varphi_V(G(z)))) = G(z)$$

by construction.

To prove the final statement, we first observe that

$$|G'(z)| \geq 1$$

whenever $\operatorname{Re} G(z) \geq 4\pi$ by [EL2, Lemma 1]. If $z \in \tilde{V}$ with $\Phi(z) \neq z$, then by construction we have $\operatorname{Re} G(z) \geq R/2 \geq 4\pi$ and, likewise, $\operatorname{Re} G(\Phi(z)) \geq 4\pi$. Thus

$$|z - \Phi(z)| \leq |G(z) - G(\Phi(z))| = |G(z) - \varphi_V(G(z))|,$$

where V is the component of \mathcal{V}_G containing z .

If $\operatorname{Re} G(z) \geq R$, let us set $\omega := G(z)$; otherwise we set $\omega := R + i \cdot \operatorname{Im}(G(z))$. Then we have

$$|G(z) - \varphi_V(G(z))| \leq |h(\omega)|$$

by the definition of φ_V , provided we use the formula (6.1) for ϑ from Lemma 6.1.

Writing $z_1 := F_{\tilde{V}}^{-1}(\omega)$, we have seen that

$$|z - \Phi(z)| \leq |h(\omega)| = |G(z_1) - F(z_1)|.$$

It remains to estimate $|z - z_1|$. The hyperbolic distance in \mathbb{H} between $G(z)$ and ω is at most $\log 2$. Furthermore, the Euclidean distance in \mathbb{H} between $\omega = F(z_1)$ and $G(z_1)$ is bounded by M . Since $\operatorname{Re} \omega \geq R \geq 2M$, the hyperbolic distance in \mathbb{H} between these two points is at most 1.

So the hyperbolic distance between $G(z)$ and $G(z_1)$ is bounded by $1 + \log 2$, and thus the hyperbolic distance in V between z and z_1 is also bounded by this constant. Using the standard estimate on the hyperbolic metric, and the fact that V does not intersect its translates by multiples of $2\pi i$, we see that indeed

$$|z - z_1| \leq 2\pi \cdot (1 + \log 2). \quad \blacksquare$$

Proof of Theorem 1.8. We can let $F, G \in \mathcal{B}_{\log}^{\text{p}}$ be logarithmic transforms of f and g , respectively. More precisely, we assume without loss of generality that $0 \notin T$ and set $G := \Psi \circ \exp$, which (suitably restricted) is an element of $\mathcal{B}_{\log}^{\text{p}}$ with domain \mathcal{V}_G and range \mathbb{H} .¹ If we choose $\mu > 0$ sufficiently large, then we can likewise define a map $F : \mathcal{V}_F \rightarrow \{z \in \mathbb{H} : \operatorname{Re} z > \mu\}$ that belongs to the class $\mathcal{B}_{\log}^{\text{p}}$ and satisfies $\exp \circ F = f \circ \exp$ and $\mathcal{V}_F \subset \mathcal{V}_G$. Then F and G satisfy the hypotheses of the previous theorem.

In fact, recall that $|f(z) - g(z)| \leq C/|z|$ for a suitable constant C (provided z is sufficiently large). Since the exponential map is expanding on a right half plane, it follows that we can choose the logarithmic transform F in such a way that

$$(6.2) \quad |F(z) - G(z)| \leq C \cdot e^{-\operatorname{Re} z}$$

for all $z \in \mathcal{V}_F$.

Now choose R sufficiently large that we can apply Theorem 6.3, and let Φ be the quasiconformal map obtained from the theorem. Then $\varphi(e^z) := e^{\Phi(z)}$ defines a quasiconformal map that satisfies $g(z) = f(\varphi(z))$ whenever $|g(z)| \geq e^R$. To estimate the asymptotics of φ at ∞ , observe that

$$|\Phi(z) - z| \leq C \cdot e^K \cdot e^{-\operatorname{Re} z},$$

where K is the universal constant from the previous theorem. Let us set $w := e^z$. If $\operatorname{Re} z$ is sufficiently large, we have

$$\begin{aligned} |w - \varphi(w)| &= |w| \cdot \left| 1 - \frac{e^{\Phi(z)}}{e^z} \right| \leq |w| \cdot (1 + e^{\operatorname{Re}(\Phi(z)-z)}) \\ &\leq 2|w| \cdot |\Phi(z) - z| \leq 2|w| \cdot C \cdot e^K \cdot e^{-\operatorname{Re} z} = 2Ce^K. \quad \blacksquare \end{aligned}$$

¹This need not quite be true if the boundary of the domain of Ψ is not a Jordan curve, since the boundary of the range of Ψ has the interval from $-i$ to i in common with \mathbb{H} . This problem is easily dealt with by first conjugating g by $z \mapsto e^\varepsilon \cdot z$ for some small $\varepsilon > 0$.

6.4. Theorem ((QC conjugacy in the class \mathcal{B}_{\log}^p)). *Suppose that F, G, M and R are as in Theorem 6.3. Suppose furthermore that*

$$\mathcal{V}_G \subset \{z \in \mathbb{H} : \operatorname{Re} z > R\}.$$

Then there exists a quasiconformal homeomorphism $\Theta : \mathbb{C} \rightarrow \mathbb{C}$, commuting with translation by $2\pi i$, such that

$$\Theta(G(z)) = F(\Theta(z))$$

whenever $z \in \mathcal{V}_G$ with $\operatorname{Re} G(z) \geq R$.

The complex dilatation μ_Θ satisfies $|\mu_\Theta| \leq 4M/R$ almost everywhere, and $\mu_\Theta = 0$ almost everywhere on the Julia set $J(G) = \{z \in \mathcal{V}_G : G^n(z) \in \mathcal{V}_G \text{ for all } n\}$.

Furthermore, suppose that $|F(z) - G(z)| \rightarrow 0$ uniformly as $\operatorname{Re} z \rightarrow \infty$. Then

$$\sup_{z \in J_Q(G)} |\Theta(z) - z| \rightarrow 0$$

as $Q \rightarrow \infty$, where

$$J_Q(G) = \{z \in J(G) : \operatorname{Re} G^j(z) \geq Q \text{ for all } j \geq 0\}.$$

Proof. This theorem essentially follows from the corresponding results in [R2]. However, for completeness we shall sketch the proof, which is not difficult in our case.

Let Φ be the map from Theorem 6.3. We define a sequence of quasiconformal maps Φ_j by $\Phi_0 := \Phi$ and

$$\Phi_{j+1}(z) := \begin{cases} F_{\tilde{\nu}}^{-1}(\Phi_j(G(z))) & \text{if } z \in \mathcal{V}_G \text{ and } \operatorname{Re} G(z) > R \\ \Phi(z) & \text{otherwise.} \end{cases}$$

If $z \in \mathcal{V}_G$ and $\operatorname{Re} G(z) = R$, then $G(z) \notin \mathcal{V}_G$, and hence

$$F_{\tilde{\nu}}^{-1}(\Phi_j(G(z))) = F_{\tilde{\nu}}^{-1}(G(z)) = \Phi(z).$$

Hence the maps match up on the boundary, and each Φ_j is a homeomorphism. It follows from Royden's glueing lemma that the Φ_j are all quasiconformal, with the same bound on the dilatation as Φ . Furthermore, because all Φ_j agree on $\mathbb{C} \setminus \mathcal{V}_G$, it follows by induction that Φ_j and Φ_{j+1} agree on $\mathbb{C} \setminus G^{-j}(\mathcal{V}_G)$. Hence the sequence of maps stabilize on the open set $\mathbb{C} \setminus J(G)$. This set is dense in \mathbb{C} because G is expanding with respect to the hyperbolic metric of \mathbb{H} (compare [R2, Lemma 2.3]). Together with the compactness property of quasiconformal maps, this implies that $\Phi_j \rightarrow \Theta$ for a quasiconformal map Θ . By construction, $\Theta \circ G = F \circ \Theta$ whenever $z \in \mathcal{V}_G$ and $\operatorname{Re} z \geq R$.

Each map Φ_j is conformal on a neighborhood of $J(G)$, implying the statement about the dilatation on $J(G)$.

The last claim follows easily from the fact that G is expanding on $J_Q(G)$ and the final statement in Theorem 6.3. ■

Proof of Theorem 1.9. Assuming ρ_0 was chosen sufficiently large, we apply Theorem 1.7 to obtain a function f approximating g . Recall that $|f(z) - g(z)| \leq \mu$ on \mathcal{T} , where $\mu > 1$ is a universal constant (Remark 4.6).

As in the proof of Theorem 1.8 and of Proposition 2.2, we can let $G : \mathcal{V}_G \rightarrow \mathbb{H}$ and $F : \mathcal{V}_F \rightarrow \{a + ib : a > \log(2\mu)\}$ be logarithmic transforms of f and g , respectively.

Furthermore, F can be chosen such that

$$|F(z) - G(z)| \leq M$$

for a universal constant M , and we can assume that M is chosen so large that $\mathcal{V}_F \supset \{G^{-1}(z) : \operatorname{Re} z > M\}$. In other words, the hypotheses of Theorem 6.3 are satisfied.

If ρ_0 was chosen sufficiently large, then $R := \log \rho_0$ satisfies $R \geq 4(M + 2\pi)$, and hence we can apply Theorem 6.4 to obtain a conjugacy Θ between G and F . Defining $\vartheta(\exp(z)) := \exp(\Theta(z))$ yields the desired conjugacy between g and f . ■

6.5. Remark ((Additional properties of the conjugacy)).

It follows from the proof that, in the setting of Theorem 1.9, the following additional statements hold:

- The quasiconformal dilatation of ϑ tends to zero as $\inf_{z \in T} |z| \rightarrow \infty$.
- $\sup_{z \in J_Q(g)} d_{\log}(z, \vartheta(z)) \rightarrow 0$ as $Q \rightarrow \infty$, where

$$J_Q(g) := \{z \in J(g) : |g^n(z)| \geq Q \text{ for all } n \geq 1\}$$

and d_{\log} denotes the distance with respect to the metric $\frac{|dz|}{|z|}$.

7. FUNCTIONS WITH FULL HYPERBOLIC DIMENSION

We now turn to proving Theorem 1.4. That is, we construct an entire function $f \in \mathcal{B}$ that is hyperbolic and whose Julia set contains hyperbolic sets of Hausdorff dimension arbitrarily close to two. To do so, we will construct a suitable model function $\Psi : T \rightarrow H$, where H is as in Theorem 1.7, and apply Theorem 1.9.

Idea of the construction. Before we give the details, let us broadly outline the idea. For simplicity, let us consider models $\Psi : T \rightarrow \mathbb{H}$ (where \mathbb{H} is the right half plane). It turns out that changing to the domain H from Theorem 1.7 does not add significant new issues.

We discuss the construction in logarithmic coordinates. Suppose that $\Psi : T \rightarrow \mathbb{H}$ is a model function, with $\bar{T} \subset \mathbb{H}$, and let V be a component of $\mathcal{V} := \exp^{-1}(T)$; we define $G := \Psi \circ \exp$. Then $G : \mathcal{V} \rightarrow \mathbb{H}$ is $2\pi i$ -periodic and $G|_V$ is a conformal isomorphism between V and \mathbb{H} .

The basic set-up of the construction is somewhat reminiscent of the proof [BKZ] that the hyperbolic dimension of a function with a logarithmic tract over infinity is always strictly greater than one. Let $K > 0$ be sufficiently large, and let Q be the square $Q = \{a + ib : K < a < 3K; |b| < K\}$, centered at the point $2K$. We shall build a finite iterated function system (compare [MaU] or also [R1, Definition 2.10]) on the square Q , each of whose branches is of the form $z \mapsto G^{-1}(z + 2\pi ik)$, for some $k \in \mathbb{Z}$ and some branch of G^{-1} . Then the union of all $2\pi i\mathbb{Z}$ -translates of the limit set of this function system is invariant under G . Projecting by the exponential map, we hence obtain a hyperbolic set for the map $g := \exp \circ \Psi$; the question is how to construct the tract T and the function system in such a way that the Hausdorff dimension of this limit set is close to two.

Suppose that we are given points $\omega_1, \dots, \omega_m \in V$ with $G(\omega_j) = 2K + 2\pi ik_j$, for some $k_j \in \mathbb{Z}$, such that the ω_j have real parts between K and $3K$. Suppose furthermore that

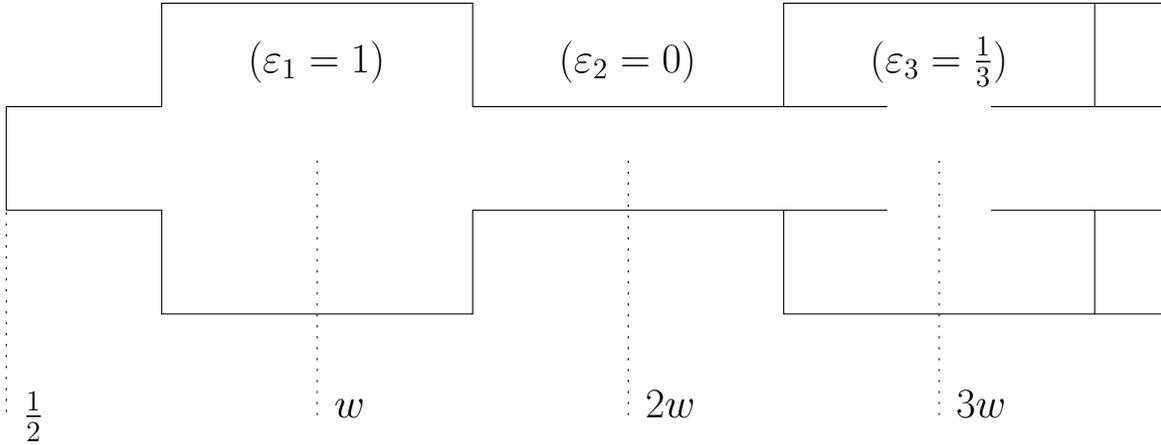


FIGURE 1. The domain $V = V((\varepsilon_k)_{k \in \mathbb{N}})$

$|k_j - k_i| > K/\pi$ for $i \neq j$. Then, for each j , there are approximately K/π points ω_j^ℓ in $\omega_j + 2\pi i\mathbb{Z}$ that are themselves contained in the square Q . For each such point, we can define a conformal map φ_j^ℓ on Q by $\varphi_j^\ell(z) = G^{-1}(z + 2\pi i k_j)$, where the branch of G^{-1} is chosen such that $\varphi_j^\ell(2K) = \omega_j^\ell$.

These maps form a conformal iterated function system on Q (assuming that each $\varphi_j^\ell(Q)$ does not intersect the boundary of Q , which will not be difficult to ensure). Note that each φ_j^ℓ is a contraction, and the contraction factor is on the order of

$$\frac{\rho_{\mathbb{H}}(2K)}{\rho_V(\omega_j)} \approx \frac{\text{dist}(\omega_j, \partial V)}{K}.$$

Hence the size of the contraction factor depends on the distance of ω_j to the boundary of V . So we should try to construct the tract V in such a way that the curve

$$\Gamma_P := \{z \in V : \text{Re } G(z) = P\},$$

where $P = 2K$, isn't always too close to the boundary of V .

We will show that it is possible to ensure that Γ_P stays a fixed distance away from ∂V at regular intervals. The construction depends on a sequence $\Xi = (\varepsilon_k)_{k \in \mathbb{N}}$ of numbers $\varepsilon_k \in [0, 1]$. The domain V consists of a central strip of fixed height $2h < 2\pi$, joined to a sequence of equally spaced, equally sized chambers (on both sides) of width w . The connection between these chambers is opened by a fraction of ε_k . That is, if $\varepsilon_k = 0$, then the chamber is completely closed off, whereas if $\varepsilon_k = 1$, the chamber is completely open; see Figure 1.

When a chamber is completely open, i.e. $\varepsilon_k = 1$, then—provided k is large enough—the curve Γ_P will run close to the boundary of the chamber. On the other hand, if $\varepsilon_k = 0$, then the chamber is completely closed off, and hence Γ_P cannot enter it. By continuity, it is possible to ensure that Γ_P runs through the central point of the chamber. This suggests that, for suitable choice of the sequence Ξ , all the chambers between real parts K and $3K$ will contain a point ω_j with $\text{dist}(\omega_j, \partial V) > \delta$, for some fixed δ , and with $G(\omega_j) = 2K + 2\pi i k_j$, as above. The number of points ω_j^ℓ then is roughly $2K^2/w \cdot \pi$; i.e. grows quadratically with K , while the contraction factor is of order $1/K$. Thus the

Hausdorff dimension of the corresponding limit set tends to 2 as $K \rightarrow \infty$. By making sure that the above properties hold for a sequence K_j tending to infinity, the hyperbolic dimension of the resulting function is equal to two.

The hyperbolic metric in H . Before we provide the details of the construction that was just outlined, let us make some observations about the hyperbolic metric of

$$H := \{x + iy : x > -14 \log_+ |y|\}.$$

7.1. Lemma ((Hyperbolic geometry of H)). *The segment $[0, \infty)$ is a hyperbolic geodesic in H . Furthermore, there is a constant $C_1 > 1$ such that, for every $z_0 \in H$, the hyperbolic geodesic of H that contains z_0 and is perpendicular to (and symmetric with respect to) the real axis is contained in $\{z \in H : |z_0|/C_1 < |z| < C_1|z_0|\}$.*

Furthermore, there is a constant C_2 with the following property. If $x \geq 2$ and $z \in H$ with $1 \leq \operatorname{Re} z \leq x/2$, then

$$\operatorname{dist}_H(x, z) \geq \frac{\log\left(\frac{x}{\operatorname{Re} z}\right)}{C_2}.$$

Proof. The first claim is clear because the domain is symmetric with respect to the real axis.

To prove the second claim, let us use the term ‘‘vertical geodesic’’ to refer to geodesics that are perpendicular to the real axis. Consider the quadrilateral in H bounded by the arcs $\sigma_{|z_0|/C_1}$ and $\sigma_{|z_0|}$, where $\sigma_t = \{z \in H : |z| = t\}$ and $C_1 > 1$. By the comparison principle for extremal length, the modulus of this quadrilateral is greater than that of the slit annulus

$$\{z \in \mathbb{C} : 1/C_1 < |z|/|z_0| < 1, z \notin (-\infty, 0)\}.$$

The latter modulus is equal to $(\log C_1)/2\pi$. Hence, by Lemma 3.5, if $C_1 > e^\pi$, then this quadrilateral contains a vertical geodesic of H . For the same reason, there is a vertical geodesic between $\sigma_{|z_0|}$ and $\sigma_{C_1|z_0|}$. The vertical geodesic passing through z must lie between these two geodesics, and hence lies between $\sigma_{|z_0|/C_1}$ and $\sigma_{C_1|z_0|}$, as claimed.

To verify the final claim, let $z = P + iy$, and assume without loss of generality that $y \geq 0$. Applying Lemma 3.2, we see that

$$\operatorname{dist}_H(x, z) \geq \frac{1}{2} \log \left(1 + \frac{|x - z|}{\operatorname{dist}(z, \partial H)} \right).$$

We have $\operatorname{dist}(z, \partial H) \leq P + 14 \log_+ y$. If $\log_+ y \leq P$, then

$$\frac{|x - z|}{\operatorname{dist}(z, \partial H)} \geq \frac{x - P}{15P} \geq \frac{x}{30P} \geq \frac{1}{30} \sqrt{\frac{x}{P}}.$$

On the other hand, if $\log_+ y > P$, then

$$\begin{aligned} \frac{|x - z|}{\operatorname{dist}(z, \partial H)} &\geq \frac{\max(x - P, y)}{15 \log y} \geq \frac{\max(x - P, y)}{15 \log(\max(x - P, y))} \\ &\geq \frac{\sqrt{\max(x - P, y)}}{15} \geq \frac{\sqrt{x - P}}{15} \geq \frac{\sqrt{x}}{30} \geq \frac{1}{30} \sqrt{\frac{x}{P}}. \end{aligned}$$

(Here we used that $t/\log t \geq \sqrt{t}$ for $t > 1$.) So in either case we have

$$\text{dist}_H(x, z) \geq \frac{1}{2} \log \left(1 + \frac{1}{30} \sqrt{\frac{x}{P}} \right),$$

which implies that we can set

$$C_2 := \sup_{t \geq 2} \frac{2 \log t}{\log(1 + \sqrt{t}/30)} < \infty. \quad \blacksquare$$

Description of V and parameter selection. In a slight modification of the construction described above, we will allow the parameter sequence $\Xi = (\varepsilon_k)_{k \in \mathbb{N}}$ to take values $\varepsilon_k \in [0, 1] \cup \{1^*\}$. Here $\varepsilon_k = 1^*$ will mean that the chamber is not only completely open, but if furthermore also $\varepsilon_{k+1} = 1^*$, then the wall between the two chambers is removed. The reason for this is that we wish to show that our example can be chosen in such a way that the Julia set has positive measure, and this requires us to introduce long parts of the tract that have height 2π . (Readers interested only in an example with hyperbolic dimension equal to two can ignore this possibility in the following.)

We fix the width $w := 2\pi$ of the chambers and also set $h := \pi/3$. Then, using the convention that $|1^*| = 1$, V is given by

$$\begin{aligned} V := V(\Xi) := & \left\{ a + ib : a > \frac{1}{2}, |b| < h \right\} \\ & \cup \bigcup_{k \in \mathbb{N}: \varepsilon_k \neq 0} \left\{ a + ib : |a - kw| < \frac{w}{2}, h < |b| < \pi \right\} \\ & \cup \bigcup_{k \in \mathbb{N}} \left\{ a + ib : |a - kw| < \frac{|\varepsilon_k| \cdot w}{2}, |b| = h \right\} \\ & \cup \bigcup_{k \in \mathbb{N}: \varepsilon_k = \varepsilon_{k+1} = 1^*} \left\{ a + ib : a = \frac{(2k+1)w}{2}, |b| < \pi \right\}. \end{aligned}$$

Let $G : V \rightarrow H$ be a conformal isomorphism with $G(1) = 1$ and $G'(1) > 0$. Because the tract is symmetric with respect to the real axis, we have $G([1, \infty)) = [1, \infty)$, and $G(\bar{z}) = \overline{G(z)}$. The dynamical model function Ψ we later approximate will be given by $\Psi(\rho_0 \cdot \exp(z)) := G(z)$, where ρ_0 is the constant from Theorem 1.9 and Ξ is a suitably chosen sequence.

We now proceed to investigate the behavior of the function G , for a given sequence $\Xi = (\varepsilon_k)_{k \in \mathbb{N}}$. All definitions in the following depend on Ξ , but for simplicity of notation, we often suppress this dependence. We also emphasize that any constants appearing in the results will be *independent* of Ξ , unless explicitly stated otherwise.

A key fact is that the tract $V(\Xi)$ depends continuously on Ξ , using the product topology on $([0, 1] \cup \{1^*\})^{\mathbb{N}}$ and the Carathéodory kernel topology for the domains. In particular, the inverse $G^{-1} : H \rightarrow V$, and thus also the function G itself, depends continuously on Ξ in the topology of locally uniform convergence.

We begin by estimating $|G(z)|$ independently of Ξ :

7.2. Lemma. *There are constants $C_3 > 1$ and $C_4 > 0$ such that*

$$\frac{\operatorname{Re} z}{C_3} - C_4 \leq \log |G(z)| \leq C_3 \operatorname{Re} z + C_4$$

for all $z \in V$ with $\operatorname{Re} z \geq w$.

Proof. By the standard estimate on the hyperbolic metric (or, alternatively, by the Ahlfors distortion theorem), it follows that there is a constant $C_3 > 1$ such that

$$x/C_3 \leq \log G(x) \leq C_3 x$$

for all $x \in V \cap \mathbb{R}$ with $x \geq w/2 > 1$.

Now let $z \in V$ with $\operatorname{Re} z \geq w$. We consider the vertical geodesic γ of V passing through z (i.e., the unique geodesic through z that intersects the real axis perpendicularly.) Let x be the point of intersection of γ with the real axis. It follows from Lemma 7.1 that $G(x)/C_1 \leq |G(z)| \leq C_1 \cdot G(x)$; hence

$$(7.1) \quad x/C_3 - \log C_1 \leq \log |G(z)| \leq C_3 x + \log C_1.$$

Furthermore, let $k \geq 2$ be maximal with $kw/2 \leq \operatorname{Re} z$. Then the quadrilateral in V bounded by the vertical cross-cuts $\{\zeta \in V : \operatorname{Re} \zeta = (k-1)w/2\}$ and $\{\zeta \in V : \operatorname{Re} \zeta = kw/2\}$ has modulus at least $1/2$ (by choice of w and the comparison principle for extremal length). Hence, by Lemma 3.5 this quadrilateral contains a vertical geodesic of V , and in particular we must have $x \geq (k-1)w/2$. Similarly we have $x \leq (k+2)w/2$, and hence

$$|x - \operatorname{Re} z| \leq w.$$

Combining this with (7.1), the proof is complete. ■

For $P > 0$, define as above

$$\Gamma_P := \{z \in V : \operatorname{Re} G(z) = P\}.$$

We shall define *signed hyperbolic distance* from Γ_P in V by setting

$$\delta(z, P) := \begin{cases} -\operatorname{dist}_V(z, \Gamma_P) & \text{if } \operatorname{Re} G(z) > P; \\ 0 & \text{if } \operatorname{Re} G(z) = P \\ \operatorname{dist}_V(z, \Gamma_P) & \text{if } \operatorname{Re} G(z) < P; \\ \infty & \text{if } z \notin V. \end{cases}$$

We observe that $\delta(z, P)$ depends continuously on Ξ for fixed z and P . Hence, setting $\zeta_k := kw + i \cdot 2\pi/3$, we see that

$$\delta_{P,k}(\Xi) := \delta(\zeta_k, P)$$

is a continuous function of Ξ .

7.3. Lemma. (a) *For all $k \in \mathbb{N}$ and all $\Xi = (\varepsilon_k)_{k \in \mathbb{N}}$: if $\varepsilon_k = 0$, then $\delta_{P,k}(\Xi) = \infty$.*
 (b) *For all $P > 0$, there is $k_0 = k_0(P) \in \mathbb{N}$ (independent of the sequence $\Xi = (\varepsilon_k)_{k \in \mathbb{N}}$) such that, for all $k \geq k_0$: if $\varepsilon_k = 1$ (or $\varepsilon_k = 1^*$), then $\delta_{P,k} \leq -1$.*

Furthermore, there is a constant κ_2 such that $k_0(P) \leq \kappa_2 \cdot \log P$ for sufficiently large P .

Proof. The first part is trivial by definition.

To prove the second part, we can assume without loss of generality that $P \geq 1$. Suppose that $\varepsilon_k = 1$ or $\varepsilon_k = 1^*$ with $k \geq C_3 \cdot \log(2P)/w$, where C_3 is as in the proof of Lemma 7.2. Set $x := \operatorname{Re} \zeta_k = k \cdot w$.

By the standard estimate on the hyperbolic metric, the hyperbolic distance between ζ_k and $x := \operatorname{Re} \zeta_k = k \cdot w$ is bounded by some uniform constant C . (In fact, given our choice of w and h , we can take $C = 4$.) Hence, if $\delta_{P,k}(\Xi) > -1$, then the hyperbolic distance between x and Γ_P is bounded by $C + 1$. This is only possible if k is sufficiently small.

Indeed, we have $G(x) \geq \exp(x/C_3) \geq 2P$ by choice of k . Applying Lemma 7.1, we see that

$$C + 1 \geq \operatorname{dist}_V(x, \Gamma_P) \geq \frac{\log\left(\frac{G(x)}{P}\right)}{C_2} \geq \frac{kw}{C_2 \cdot C_3} - \frac{\log P}{C_2}.$$

The claim follows by rearranging. ■

Remark. We could have replaced the sequence (ξ_k) by any sequence whose hyperbolic distance from ξ_k is bounded (or does not grow too quickly). This would yield a stronger version of Theorem 7.4 below, but we will not require this extra generality.

We can now prove our parameter selection result.

7.4. Theorem. *Let $A \subset \mathbb{N}$, and suppose we are given values $(\tilde{\varepsilon}_k)_{k \in \mathbb{N} \setminus A}$ with $\tilde{\varepsilon}_k \in [0, 1] \cup \{1^*\}$, and a sequence $(P_k)_{k \in A}$ with $k \geq k_0(P_k)$ for all k .*

Then there is a sequence $\Xi = (\varepsilon_k)_{k \in \mathbb{N}}$ such that

- $\varepsilon_k = \tilde{\varepsilon}_k$ for $k \notin A$ and
- $\delta_{k, P_k}(\Xi) = 0$ (i.e., $\zeta_k \in \Gamma_{P_k}$) for all $k \in A$.

Proof. Let us first prove the result for *finite* subsets $A \subset \mathbb{N}$. If $m := \#A = 1$, then the claim simply corresponds to the intermediate value theorem.

For $m > 1$, the claim similarly follows by basic topology. More precisely, let us define a map from the m -cube $[0, 1]^A$ to itself. For $x \in [0, 1]^A$, let $\Xi(x)$ be the sequence defined by setting $\varepsilon_k = x_k$ for $k \in A$ and $\varepsilon_k = \tilde{\varepsilon}_k$ for $k \notin A$.

For $k \in A$ and $x \in [0, 1]^A$, we define

$$\tilde{\delta}_k(x) := \begin{cases} 0 & \delta_{k, P_k}(\Xi(x)) > 1, \\ 1 & \delta_{k, P_k}(\Xi(x)) < -1, \\ \frac{1 - \delta_{k, P_k}(\Xi(x))}{2} & \text{otherwise.} \end{cases}$$

Then $\varphi : (\varepsilon_k)_{k \in A} \mapsto (\tilde{\delta}_k)_{k \in A}$ is a continuous map of the m -cube to itself, and by Lemma 7.3 it maps any face (of any dimension) to itself. Hence the map between the homotopy (or homology) groups of the boundary of the cube induced by φ is the identity. This implies that φ must be surjective; and thus there exists $x \in [0, 1]^A$ such that $\tilde{\delta}_k(x) = 1/2$ for all $k \in A$.

This proves the theorem for finite A . If A is infinite, we take an increasing sequence of finite subsets A_k that exhaust A , and let Ξ be a limit of the corresponding sequences. The claim follows by continuity of the functions $\Xi \mapsto \delta_{k, P_k}(\Xi)$. ■

Proof of Theorem 1.4. We are now in a position to complete the construction, in line with the sketch we gave at the beginning of the section. Recall that we will need to move the tract sufficiently far to the right in logarithmic coordinates in order to apply Theorem 1.9; i.e., we are really interested in the dynamical behavior of the function $G(z - R_0)$, where $R_0 := \log \rho_0 > 0$.

7.5. Corollary. *There is a constant $K_0 > R_0$ with the following property. Suppose that $K_0 \leq K_1 < K_2 < \dots$ is a sequence of integers with $K_{i+1} > 3K_i$ for all $i \geq 1$. Then there is a sequence Ξ so that the corresponding function $G : V(\Xi) \rightarrow H$ satisfies*

$$\operatorname{Re} G(\zeta_k) = 2K_i$$

for all k with $K_i < w \cdot k + R_0 < 3K_i$ and such that $\varepsilon_k = 1^*$ for all other k .

Proof. This follows from the previous theorem. It only needs to be checked that $(K_i - R_0)/w > k_0(2K_i)$, provided K_0 was chosen sufficiently large, and this follows from the statement on the size of k_0 in Lemma 7.3. ■

For the remainder of the section, let us fix $1 \leq K_0 < K_1 < K_2 < \dots$ and let $V = V(\Xi)$ be as in the preceding corollary. We define

$$G_0(z) := G(z - R_0), \quad \Psi(\exp(z)) := G_0(z) \quad \text{and} \quad g(z) := \exp(\Psi(z)).$$

So $G_0 : V' \rightarrow H$ is a conformal isomorphism, where $V' = V + R_0$, and $\Psi : T \rightarrow H$ is a model function, where $T := \exp(V')$.

By definition, $T \subset \{|z| > \rho_0\}$, and hence we can apply Theorem 1.9 to Ψ . We obtain a disjoint-type entire function $f : \mathbb{C} \rightarrow \mathbb{C}$ with $|g(z) - f(z)| = O(1/|z|)$ and such that f and g are quasiconformally conjugate on their Julia sets.

7.6. Theorem. *The hyperbolic dimension of g is two. Hence also $\dim_{\text{hyp}}(f) = 2$. Both f and g have finite order.*

Proof. Let us fix i (sufficiently large) and $K := K_i$; we shall construct a hyperbolic set whose dimension tends to two as i tends to ∞ .

Let k^- and k^+ be the minimal resp. maximal values of k with $K + w/2 \leq w \cdot k + R_0 \leq 3K - w/2$. That is,

$$k^- := \left\lceil \frac{2K + w - 2R_0}{2w} \right\rceil \quad \text{and} \quad k^+ := \left\lfloor \frac{6K - w - 2R_0}{2w} \right\rfloor$$

Set $\zeta'_k := \zeta_k + R_0$. Then $\operatorname{Re} G_0(\zeta'_k) = 2K$ by choice of V ; observe also that $\operatorname{Im} G_0(\zeta'_k) > 0$ (since G_0 is real on the real axis, and the points ζ'_k lie above the real axis in V). Let us define

$$m_k := \left\lfloor \frac{\operatorname{Im} G_0(\zeta'_k)}{2\pi} \right\rfloor \quad \text{and} \quad \omega_k := G_0^{-1}(2K + 2\pi i m_k).$$

In other words, $G_0(\omega_k)$ is the $2\pi i\mathbb{Z}$ -translate between $2K$ and $G_0(\zeta'_k)$ that is closest to $G_0(\zeta'_k)$.

Let Q be the square of sidelength K centered at $2K$. For $k^- \leq k \leq k^+$, define

$$\varphi_k : Q \rightarrow V; z \mapsto G_0^{-1}(z + 2\pi i m_k).$$

We begin by showing that the φ_k do not contract too strongly.

Claim 1. There exists a universal constant λ_0 such that $|\varphi'_k(z)| \geq \lambda_0/K$ for all $k \in \{k^-, \dots, k^+\}$ and all $z \in Q$.

Proof. Each φ_k extends conformally to a square of sidelength $2K$ centered at $2K$. Hence it suffices to estimate the derivative of φ_k at the center of the square; the claim then follows from Koebe's distortion theorem. We have

$$(7.2) \quad |\varphi'_k(2K)| = \frac{\rho_H(G_0(\omega_k))}{\rho_{V'}(\omega_k)} \geq \frac{\text{dist}(\omega_k, \partial V')}{4 \text{dist}(G_0(\omega_k), \partial H)}.$$

We first note that the hyperbolic distance between ω_k and ζ'_k is uniformly bounded (and in fact tends to zero as i tends to infinity). By choice of Ξ , $\text{dist}(\zeta'_k, V')$ is uniformly bounded from below; hence $\text{dist}(\omega_k, V') > \delta_1$ for some universal $\delta_1 > 0$. Furthermore,

$$\begin{aligned} \text{dist}(G_0(\omega_k), \partial H) &\leq 2K + 14 \log_+(2\pi m_k) \leq 2K + 14 \log_+ |G(\zeta_k)| \\ &\leq 2K + C_3 \cdot \text{Re } \zeta_k + C_4 \leq 2K + 3C_3 K + C_4 \leq (2 + 3C_3 + C_4)K. \end{aligned}$$

Substituting these two estimates into (7.2) completes the proof of the claim. \triangle

Claim 2. For $k^- \leq k \leq k^+$, we have $(2k - 1)w/2 < \text{Re } \varphi_k(z) < (2k + 1)w/2$, provided i was chosen sufficiently large.

Proof. The hyperbolic distance in V' between ω_k and ζ'_k is uniformly bounded. The hyperbolic diameter of $Q + 2\pi i m_k$ in H , and hence the hyperbolic diameter of $\varphi_k(Q)$ in V' , is likewise uniformly bounded. Thus the hyperbolic distance between ζ'_k and $\varphi_k(z)$ is uniformly bounded, independently of k and $z \in Q$. On the other hand, as k tends to infinity (under the assumption that $K_i + w/2 < w \cdot k + \rho_0 < 3K_i - w/2$ for some i), we must have $\varepsilon_k \rightarrow 0$ by the same reasoning as in the proof of Lemma 7.3. This implies that, for sufficiently large k , the set $\varphi_k(Q)$ is contained in the k -th ‘‘chamber’’ of the tract V' , proving the claim. \triangle

Consider the conformal iterated function system on Q formed by the maps

$$\varphi_k^\ell(z) := \varphi_k(z) + 2\pi i \ell$$

for $k^- \leq k \leq k^+$ and $|\ell| \leq (K - \pi)/2\pi$. By the preceding lemma, and choice of ℓ , we have $\varphi_k^\ell(Q) \subset Q$ for all k and ℓ , and the images of Q under these maps have pairwise disjoint closures. The number N of functions in our IFS is

$$N = \left(2 \left\lfloor \frac{K - \pi}{2\pi} \right\rfloor + 1 \right) \cdot (k^+ - k^- + 1) \geq \lambda_1 \cdot K^2,$$

where λ_1 is a suitable constant (provided i , and hence K , is sufficiently large).

Let $X = X_i$ be the limit set of this iterated function system; i.e. X is the unique compact set with $X = \bigcup_{k,\ell} \varphi_k^\ell(X)$. We have

$$\dim_{\text{H}}(X_i) \geq \frac{\log N}{\inf_{k,\ell,z} \log |(\varphi_k^\ell)^{-1}'(z)|} \geq \frac{2 \log K + \log \lambda_1}{\log K - \log \lambda_0}.$$

(Compare e.g. [R1, Lemma 2.10].) So $\dim_{\text{H}}(X_i) \rightarrow 2$ as i , and hence K , tends to infinity.

To conclude the proof, first note that $\exp(X_i)$ is invariant under g by definition. Indeed, let $z \in X_i$; say $z \in \varphi_k^\ell(Q)$. Then

$$\begin{aligned} g(\exp(z)) &= \exp(G_0(z - 2\pi i\ell)) \\ &= \exp((\varphi_k)^\ell(z - 2\pi i\ell) - 2\pi i m_k) = \exp((\varphi_k^\ell)^{-1}(z)) \in \exp(X_i). \end{aligned}$$

So $\exp(X_i)$ is an invariant compact set for g . Every such set is a hyperbolic set for g (since g strictly expands the hyperbolic metric of T). This proves the claim for g , and the corresponding claim for the approximating function f follows because the two functions are quasiconformally conjugate, and quasiconformal maps preserve sets of Hausdorff dimension two. Furthermore g has finite order of growth by Lemma 7.2, and the same holds for f . ■

Remark 1. The key point in the proof at which the choice of the range H comes into play is Claim 1, which allows us to estimate the size of the pieces in the iterated function system from below. If our domain H was, say, instead given by a “parabola shape”

$$H = \{x + iy : x > -|y|^\rho\},$$

then the size of these pieces would shrink exponentially with k , and the proof breaks down completely.

Remark 2. The Ahlfors distortion theorem gives a precise value for the order of g (and f) in terms of the shape of the tract V , and hence it is not difficult to see that, by varying the height h of the tract, we can construct functions of any given order. Letting the height tend to zero (slowly) along the real axis, we can also construct functions of infinite order.

To complete the proof of Theorem 1.4, it remains to show that—provided the K_i were chosen appropriately—the Julia set of f has positive area. This follows by letting the sequence K_i grow extremely quickly, so as to leave intermediate pieces where the tract has height 2π , with the length of these pieces growing at least in an iterated exponential manner.

We can then apply [AB, Remark 3.1] to our function to see that the Julia set has positive area. That the hypotheses of this result are satisfied follows from [AB, Theorem 1.3]. We omit the details.

8. FURTHER APPLICATIONS

We now briefly comment on the construction of the other two counterexamples mentioned in the introduction.

First we comment on Theorem 1.10. In [R³S, Theorem 8.3], a model function $\Psi : T \rightarrow \mathbb{H}$ is constructed such that $z \mapsto e^{\Psi(z)}$ has the desired properties. Here the tract T is constructed as $T = \exp(V)$, where V is contained in a horizontal strip of height 2π . Furthermore, the domain V can be chosen to lie in any half plane $\{\operatorname{Re} \zeta > R\}$; in particular for the choice $R = R_0 = \log \rho_0$, where ρ_0 is as in Theorem 1.9.

By our theorems, we only need to show that, in this construction, we can replace the right half plane \mathbb{H} by the domain H from Theorem 1.7. This can be seen in two ways. Either we note that we can carry out the same construction in this setting also. Alternatively, let Ψ be the model actually constructed in [R³S], and consider $\tilde{\Psi} := \Psi \circ \Theta$,

where $\Theta : \mathbb{H} \rightarrow H$ is a conformal isomorphism. That this does not change the order of growth of the resulting function $e^{\tilde{\Psi}}$ is easy to see by estimating the asymptotics of the map Θ using standard methods. Thus it remains to show that the new map also has the property that the Julia set contains no unbounded path-connected components, and this follows exactly as in [R³S, pp. 109–110: Proof of Theorem 1.1].

Now we turn to Theorem 1.11. We indicate how to modify the proof in [RRS], where a model function $F : T \rightarrow \mathbb{H}$ is constructed with the desired properties, to instead yield a model function $F : T \rightarrow H$, where H is as in Theorem 1.7. Here T is a tract in logarithmic coordinates, so at the end of the construction, we will apply Theorem 1.7 to the map $\Psi(e^z) := F(z)$.

The form of the tract T , as described in [RRS, Section 6] is exactly the same; instead of choosing a conformal isomorphism $F : T \rightarrow \mathbb{H}$ with $F(1)$ and $F(\infty) = \infty$, one takes instead a conformal isomorphism $F : T \rightarrow H$ with the same properties. We need to show that [RRS, Proposition 6.3] also holds for this construction.

Most of the proof again goes through verbatim, except for the following points, which previously used explicit formulas for the hyperbolic metric in \mathbb{H} :

- Instead of the first displayed equation, we note that

$$\text{dist}_H(1, F(u)) = \int_1^{F(u)} \rho_H(t) dt,$$

and $1/t \geq \rho_H(t) \geq \frac{1}{2t}$ for all $t \in \mathbb{R}$, where the first inequality follows because $\mathbb{H} \subset H$ and the second from the standard estimate on the hyperbolic metric in a simply-connected domain. Thus we see that

$$\text{dist}_H(1, F(u)) \leq \log(F(u)) \leq 2 \text{dist}_H(1, F(u)).$$

We again have $\text{dist}_H(1, F(u)) = \text{dist}_T(1, u)$, and the proof proceeds as before.

- The estimate on $F(w_k)$ also uses the explicit formula for the hyperbolic metric; this estimate is used in the choice of ε_k in the inductive construction.

We could simply replace this estimate by that from Lemma 7.1, but instead we make a more qualitative argument. The key point is that, for fixed $T > 0$, the hyperbolic distance in H between R and the line $\{\text{Re } z = T\}$ tends to infinity as $R \rightarrow \infty$ by Lemma 7.1.

It follows that there is a function $\eta(R_0, r)$ with the following properties such that $R \geq R_0$ and $z \in H$ with $\text{dist}_H(R, z) \leq r$, then $\text{Re } z > \eta(R_0, r)$. Furthermore, for fixed r , we have $\lim_{R_0 \rightarrow \infty} \eta(R_0, r) = \infty$.

Now, in the inductive definition, we again choose ε_k after u_{k+1} has been chosen, but before r_{k+1} , in such a way that

$$\eta(b(r_k + 1), 12r_k) > u_{k+1} + \vartheta_k.$$

(This is possible because $b(r_k + 1) \rightarrow \infty$ as $\varepsilon_k \rightarrow 0$.)

Since $\text{dist}_H(F(w_k), F(r_k + 1)) \leq 12r_k$ and $F(r_k + 1) \geq b(r_k + 1)$, we then again have $\text{Re } F(w_k) > u_{k+1} + \vartheta_k$, as desired.

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