Special Geometry, Hessian Structures and Applications

Gabriel Lopes Cardoso[a], Thomas Mohaupt[b,*]

[a] *Center for Mathematical Analysis, Geometry and Dynamical Systems, Department of Mathematics, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisboa, Portugal*
[b] *Department of Mathematical Sciences, University of Liverpool, Peach Street, Liverpool L69 7ZL, UK*

Abstract

The target space geometry of abelian vector multiplets in $\mathcal{N} = 2$ theories in four and five space-time dimensions is called special geometry. It can be elegantly formulated in terms of Hessian geometry. In this review, we introduce Hessian geometry, focussing on aspects that are relevant for the special geometries of four- and five-dimensional vector multiplets. We formulate $\mathcal{N} = 2$ theories in terms of Hessian structures and give various concrete applications of Hessian geometry, ranging from static BPS black holes in four and five space-time dimensions to topological string theory, emphasizing the role of the Hesse potential. We also discuss the r-map and c-map which relate the special geometries of vector multiplets to each other and to hypermultiplet geometries. By including time-like dimensional reductions, we obtain theories in Euclidean signature, where the scalar target spaces carry para-complex versions of special geometry.

Keywords: Supergravity, String Theory, Differential Geometry, Black Holes

*Corresponding Author
Email addresses: gcardoso@math.tecnico.ulisboa.pt (Gabriel Lopes Cardoso), Thomas.Mohaupt@liv.ac.uk (Thomas Mohaupt)

Contents

1. Introduction

Theories with 8 supercharges hold an interesting position between semi-realistic, but analytically un-tractable theories with 4 supercharges, and theories with more than 8 supercharges, which are analytically tractable, but have a two-derivative Lagrangian which is completely determined by their matter content. In contrast, the couplings of theories with 8 supercharges are functions of the scalar fields, and subject to interesting and complicated quantum corrections. We will refer to theories with 8 conserved real supercharges as $\mathcal{N} = 2$ theories, irrespective of the space-time dimension. This amounts to counting supersymmetries in multiples of the minimal number of supercharges of a four-dimensional theory.

Vector multiplets in $\mathcal{N} = 2$ theories contain gauge fields together with scalars and fermions. We will restrict ourselves to abelian vector fields, in which case one can take linear combinations of vector fields. By supersymmetry this imprints itself onto the scalars, leading to an affine structure and a scalar geometry which is 'special.' In four dimensions, where vector fields can couple to both electric and magnetic charges, linear transformations of vector fields and *electric-magnetic duality transformations* combine to a *symplectic group action* on the field strengths and their duals. By supersymmetry this imprints itself on the scalars, which in four dimensional are complex-valued, and leads to a Kähler geometry with 'special features.' While in rigid supersymmetry the number of scalar fields and vector fields is balanced, the coupling to Poincaré supergravity creates a mismatch, because the Poincaré supergravity multiplet contributes an additional vector field, the graviphoton. An elegant way to handle this is to employ the *gauge equivalence* between a theory of n vector multiplets coupled to Poincaré supergravity and a theory of $n+1$ superconformal vector multiplets coupled to conformal supergravity (the Weyl multiplet) and one additional auxiliary supermultiplet (which we will take to be a hypermultiplet). In the superconformal theory there now is a balance between $n + 1$ scalar fields and $n + 1$ vector fields. The superconformal symmetry gives the scalar geometry an additional *conical structure*. When recovering the Poincaré supergravity theory by imposing gauge fixing conditions, one scalar is eliminated, which corresponds to taking the *superconformal quotient* of the superconformal scalar manifold by a group action. In this way the scalar geometry of vector multiplets coupled to Poincaré supergravity can be understood as the projectivisation of the scalar geometry of the associated superconformal theory.

We will refer to the scalar geometries of five- and four-dimensional vector multiplets as *special geometries*. One characteristic feature of five- and four-dimensional vector multiplets is that all couplings of the two-derivative Lagrangians are encoded in a single function, the *Hesse potential*. In particular, the metric of the scalar manifolds of rigid vector multiplets are Hessian metrics, that is, the metric coefficients are the second derivatives of a real function, when written in affine coordinates with respect to a flat torsion-free connection. While the scalar metrics of local vector multiplets are not Hessian themselves, they can still be expressed in terms of the Hesse potential of the associated superconformal theory. In four dimensions, one can alternatively express the couplings in terms of a holomorphic function, the *prepotential*. This is in fact the pre-dominant point of view in the literature. In this review we will emphasize the role of the Hesse potential because (i) this makes manifest the similarities between five- and four-dimensional vector multiplets, (ii) the Hesse potential of four-dimensional vector multiplets transforms covariantly under symplectic transformations, while the prepotential does not. As a consequence, using the Hesse potential has advantages in many applications. We will review *Hessian geometry* and *special real geometry* in section 2, *electric-magnetic duality* in section 4, and *special Kähler geometry* in section 5. Based on this we discuss *five-dimensional vector multiplets* in section 3 and *four-dimensional vector multiplets* in section 6.

In Table 1 we list the acronyms and defining data of the types of special geometries relevant for five- and four-dimensional vector multiplets. One recurrent theme is that for each type of special geometry there is an affine, a conical and a projective version, which schematically are related like this:

$$\text{Affine} \xrightarrow{+\text{Homothety}} \text{Conical} \underset{\text{Cone}}{\overset{\text{Quotient}}{\rightleftarrows}} \text{Projective} \tag{1}$$

This is meant to indicate that the conical type is a special form of the affine type, which is characterized by the presence of a homothetic Killing vector field satisfying certain compatibility conditions. The projective version is obtained by taking the quotient of the conical version by a group action, which contains the action generated by the homothetic Killing vector field. Conversely, the conical type of the geometry is realized as a cone which has the projective geometry as its base. While the affine version corresponds to rigid vector multiplets, the conical version corresponds to superconformal vector multiplets, and the projective version corresponds to vector multiplets coupled to Poincaré supergravity. The relation between conical and projective geometry reflects the gauge equivalence between conformal supergravity and Poincaré supergravity. Five- and four-dimensional vector multiplets realize a real and a complex version of this scheme with group actions of $\mathbb{R}^{>0}$ and of $\mathbb{C}^*$ by real and by complex scale transformations, respectively. If we include hypermultiplets, there is as well a quaternionic version of this scheme. Hypermultiplets can be obtained by reduction of four-dimensional vector multiplets to three dimensions, followed by the dualization of the three-dimensional vector fields into scalars. Since hypermultiplets only contain scalars and fermions, their scalar geometry does not change under dimensional reduction, and is of the same type in any dimension where hypermultiplets can be defined. The upper limit is $d = 6$, which is the largest dimension where a supersymmetry algebra with 8 real supercharges can be constructed. The scalar geometries of hypermultiplets are quaternionic geometries, more precisely they are hyper-Kähler for rigid hypermultiplets, hyper-Kähler cones (or, conical hyper-Kähler) for superconformal hypermultiplets, and quaternionic Kähler (or, quaternion-Kähler) for hypermultiplets coupled to Poincaré supergravity. While we will focus on vector multiplets in this review, we will talk about hypermultiplets in the context of dimensional reduction, and regard their scalar geometries as the quaternionic versions of special geometry. The real, complex and quaternionic versions of special geometry are related by dimensional reduction, which induce maps called the r-map and the c-map between the scalar geometries. This is summarized in Table 2.

When discussing dimensional reduction in section 8, we also include dimensional reduction over time. This allows to construct theories with *Euclidean supersymmetry*. For four-dimensional vector multiples and for hypermultiplets the special geometry of the scalar manifold is modified, and now is of *para-complex* and of *para-quaternionic* type, respectively.

In addition to reviewing the construction of bosonic Lagrangians and discussing the resulting scalar geometries, we present a number of important applications: static BPS black holes in four and in five space-time dimensions in the presence of Weyl square interactions (sections 9 and 10); deformed special Kähler geometry and topological string theory (section 7); F-functions for point-particle Lagrangians (section 4), for the Born-Infeld-dilaton-axion system

ASR = affine special real	(M, g, ∇)
CASR = conical affine special real	(M, g, ∇, ξ)
PSR = projective special real	$(\bar{M}, \bar{g})$, $\bar{M} = M/\mathbb{R}^{>0} \cong \mathcal{H} \stackrel{\iota}{\to} M$ $\bar{g} = \iota^* g$
ASK = affine special Kähler	(M, J, g, ∇)
CASR = conical affine special Kähler	(M, J, g, ∇, ξ)
PSK = projective special Kähler	$(\bar{M}, \bar{J}, \bar{g})$, $\bar{M} = M/\mathbb{C}^* = M//U(1)$ $\pi^* \bar{g} = \iota^* g$, $\mathcal{H} \stackrel{\pi}{\to} \bar{M} = \mathcal{H}/U(1)$ $\mathcal{H} \stackrel{\iota}{\to} M$

Table 1: This table summarizes the acronyms we use for the various special geometries. The second column contains the essential geometrical data for each type of geometry. ∇ indicates a 'special' connection, which in particular is flat and torsion-free. ξ indicates a homothetic Killing vector field which gives the manifold locally the structure of a cone. A 'bar' indicates a 'projectivized' manifold which has been obtained by taking the orbits of a group action, which always includes the homothetic Killing vector field ξ. As usual π and ι indicate projections and immersions, respectively, and $*$ a pull-back. We refer to the corresponding sections of this review for precise definitions.

(section 11) and for a particular STU-model (section 12). In all these applications, the Hesse potential plays an important role: the semi-classical entropy of BPS black holes is obtained from the Hesse potential by Legendre transformation; the holomorphic anomaly equation of topological string theory is encoded in a Hessian structure; point-particle Lagrangians admit a reformulation in terms of a Hesse potential; the Hesse potential approach to the STU-model yields important information about the function F that encodes the Wilsonian Lagrangian of the model.

The topics and applications we chose to cover in this report are based on research papers and review articles which we will be referring to in the various sections comprising this report. The papers we chose to cite represent a small subset of the many papers that have been published over the past decades on the subject of special geometry at large. It would be impossible to refer to all these papers, and hence we have opted to cite only those which we used to write this report.

Finally, we have assembled extensive appendices on the mathematics and physics background of the report, for the benefit of the reader.

$$
\begin{array}{lccccc}
\text{Five-dimensional vector multiplets:} & \text{ASR} & \xrightarrow{\text{conical}} & \text{CASR} & \underset{\text{Cone}}{\overset{\text{Quotient}}{\rightleftarrows}} & \text{PSR} \\
 & \downarrow r & & & & \downarrow \bar{r} \\
\text{Four-dimensional vector multiplets:} & \text{ASK} & \xrightarrow{\text{conical}} & \text{CASK} & \underset{\text{Cone}}{\overset{\text{Quotient}}{\rightleftarrows}} & \text{PSK} \\
 & \downarrow c & & & & \downarrow \bar{c} \\
\text{Hypermultiplets:} & \text{HK} & \xrightarrow{\text{conical}} & \text{HKC} & \underset{\text{Cone}}{\overset{\text{Quotient}}{\rightleftarrows}} & \text{QK}
\end{array}
$$

Table 2: The real, complex and quaternionic versions of special geometry are related by the r-map and c-map, which are induced by dimensional reduction. A 'bar' indicates the supergravity version of either map. In the quaternionic case HKC stands for 'hyper-Kähler cone,' which is commonly used instead of 'conical hyper-Kähler', or CHK, which would be in line with the terminology we use for vector multiplets. QK stands for quaternionic Kähler. Precise definitions are given in the respective sections of this review.

2. Hessian geometry and special real geometry

In this section we introduce Hessian geometry, focussing on the aspects that are relevant for the special geometries of five- and four-dimensional vector multiplets. A comprehensive treatment of Hessian geometry can be found in [1]. Special emphasis is put on *conical Hessian manifolds*, that is Hessian manifolds admitting a homothetic Killing vector field. Such manifolds can be 'projectivized', that is the space of orbits of the homothetic Killing vector field carries a Riemannian metric, which, while not being Hessian, is determined by the Hesse potential of the conical Hessian manifold. Conical Hessian manifolds admit a Hesse potential which is a homogeneous function. The special real geometry of five-dimensional vector multiplets is obtained by restricting to Hesse potentials which are homogeneous cubic polynomials. The material on conical Hessian and special real geometry is partly based on [2, 3, 4, 5, 6, 7].

2.1. Hessian manifolds

In this subsection we provide the definition of a Hessian manifold, both in terms of local coordinates, and coordinate-free.

Definition 1. Hessian manifolds and Hessian metrics in terms of coordinates. *A pseudo-Riemannian*[1] *manifold* (M, g) *is called a* Hessian manifold *if it admits local coordinates* q^a*, such that the metric coefficients are the Hessian of a real function* H*, called the* Hesse potential*:*

$$g_{ab} = \partial_a \partial_b H := \partial^2_{a,b} H := H_{ab} \ . \tag{2}$$

[1]See A.4 for a review and for our conventions.

Such metrics are called Hessian metrics.

The relation (2) is not invariant under general coordinate transformations, but only under affine transformations. The definition implies that the manifold can be covered by special coordinate systems, related to each other by affine transformations, such that (2) holds in every coordinate patch. This is equivalent to the existence of a flat, torsion-free connection ∇, for which the special coordinates q^a are affine coordinates. Equivalently, the differentials dq^a define a parallel coordinate frame, $\nabla dq^a = 0$.[2] The flat torsion-free connection ∇ gives M the structure of an *affine manifold*. By the Poincaré lemma, the integrability condition

$$\partial_a g_{bc} = \partial_b g_{ac} = \partial_c g_{ba} \tag{3}$$

is necessary and locally sufficient for the existence of a Hesse potential. Passing to general coordinates, we see that the rank three tensor $S = \nabla g$ must be totally symmetric. We thus arrive at the following coordinate-free definition:

Definition 2. Hessian manifolds, Hessian metrics and Hessian structures. *A* Hessian manifold (M, g, ∇) *is a pseudo-Riemannian manifold* (M, g) *equipped with a flat, torsion-free connection* ∇*, such that the covariant rank three tensor* $S = \nabla g$ *is totally symmetric. The pair* (g, ∇) *defines a* Hessian structure *on* M*, and the metric* g *is called a* Hessian metric.

Locally, a Hessian metric takes the form $g = \nabla dH$, where the Hesse potential H is unique up to affine transformations. When using affine coordinates we can write $g = \partial^2 H$, since the connection ∇ acts by partial derivatives.

We note in passing that one example of a symmetric Hessian manifold which is prominent in physics is anti-de Sitter space [8]. Applications of Hessian manifolds to superconformal quantum mechanics have been discussed in [9] and [10, 11, 12]. Superconformal quantum mechanics on special Kähler manifolds, which as we will see later are in particular Hessian manifolds, has been discussed in [13, 14].

2.2. The dual Hessian structure

Hessian structures always come in pairs. This will play an important role later when we discuss electric-magnetic duality, special Kähler geometry, and black hole entropy functions.

[2] Frames and connections are reviewed in A.3 and A.5.

Definition 3. Dual affine coordinates. *If* q^a *are* ∇*-affine coordinates for a Hessian metric with Hesse potential* H*, then*

$$q_a := H_a := \partial_a H \tag{4}$$

are the associated dual affine coordinates*.*

Note that in general $q_a \neq H_{ab} q^b$. This reflects that q^a and q_a are functions on M, and not the components of a vector field or differential form. The matrix H^{ab} of metric coefficients with respect to the dual coordinates is determined by

$$g = H_{ab} dq^a dq^b = H^{ab} dq_a dq_b \ , \tag{5}$$

which implies that the matrix H^{ab} is the inverse of the matrix H_{ab}.

Definition 4. Dual connection on a Hessian manifold. *Let* (M, g, ∇) *be a Hessian manifold with Levi-Civita connection* D*. Then*

$$\nabla_{\text{dual}} = 2D - \nabla \ , \tag{6}$$

is called the dual connection *to* ∇*.*

Remark 1. Dual Hessian structures and the dual Hesse potential. The dual connection is flat and torsion-free, and defines a second Hessian structure on (M, g), called the *dual Hessian structure*. The ∇_{dual}-affine coordinates are the dual coordinates q_a introduced above, and the *dual Hesse potential* H_{dual} is related to H by a Legendre transformation,

$$H_{\text{dual}} = q^a H_a - H \ . \tag{7}$$

The matrix of metric coefficients with respect to the dual Hessian structure is the inverse matrix H^{ab} of H_{ab}:

$$H^{ab} = \frac{\partial^2 H_{\text{dual}}}{\partial q_a \partial q_b} \ . \tag{8}$$

We refer to section 2.3 of [1] for more details on the dual Hessian structure.

2.3. Conical Hessian manifolds

We now consider the case where the Hesse potential is a homogeneous function. This is relevant for both five- and four-dimensional vector multiplet theories.

Definition 5. Homogeneous functions. *A real function* H *is* homogeneous of degree n *in the variables* q^a *if*

$$H(\lambda q^a) = \lambda^n H(q^a) \ , \quad \lambda \in \mathbb{R}^* \ . \tag{9}$$

This is equivalent to the *Euler relation*

$$L_\xi H = q^a \partial_a H = nH \ , \tag{10}$$

where $\xi = q^a \partial_a$ is the so-called *Euler vector field* with respect to the coordinates q^a, and where L_ξ is the Lie derivative.[3] The k-th derivative of a homogeneous function of degree n is a homogeneous function of degree $n-k$. In local coordinates, we have the following hierarchy of relations:

$$q^a H_a = nH \ , \quad q^a H_{ab} = (n-1)H_b \ , \quad q^a H_{abc} = (n-2)H_{bc} \ , \ldots \tag{11}$$

Remark 2. Dual coordinates for homogeneous Hesse potentials. For a Hesse potential which is homogeneous of degree n, the dual coordinates $q_a = H_a$ have weight $n-1$, while the metric coefficients H_{ab} have weight $n-2$, and the dual metric coefficients H^{ab} have weight $2-n$. The Legendre transform defining the dual Hesse potential simplifies:

$$H_{\text{dual}} = q^a H_a - H = (n-1)H \ . \tag{12}$$

In particular $H_{\text{dual}} = -H$ for $n=0$ and $H_{\text{dual}} = H$ for $n=2$.

Definition 6. Homogeneous tensor fields. *A tensor field T is called* homogeneous of degree n *with respect to the action generated by a vector field ξ if*

$$L_\xi T = nT \ . \tag{13}$$

We will then also say that T has weight n. *The case $n=0$ corresponds to the special case of an invariant tensor.*

The Lie derivatives

$$L_\xi(\partial_a) = -\partial_a \ , \quad L_\xi(dq^a) = dq^a \ , \tag{14}$$

show that derivatives ∂_a have weight -1, while differentials dq^a have weight 1. Thus the components of a tensor T of type (p,q) and weight n have weight $n+p-q$.

Example: Consider the case where the metric g has weight n with respect to the Euler field ξ. Then

$$L_\xi g = ng \Leftrightarrow (L_\xi g)_{ab} = n g_{ab} \Leftrightarrow L_\xi(g_{ab}) = (n-2) g_{ab} \ . \tag{15}$$

Here $L_\xi(g_{ab}) = \xi^c \partial_c g_{ab}$ denotes the Lie derivative of the components of the metric *considered as functions.* This is to be distinguished from $(L_\xi g)_{ab} =$

[3] See A.3 for a review and our conventions.

$\xi^c \nabla_c g_{ab}$, which denotes the components of the tensor $L_\xi g$. The weight $n-2$ of the tensor components g_{ab} can be inferred from the following computation:

$$\begin{aligned} L_\xi g &= L_\xi(g_{ab}dq^a dq^b) = L_\xi(g_{ab})dq^a dq^b + g_{ab}L_\xi(dq^a)dq^b + g_{ab}dq^a L_\xi(dq^b) \\ &= (L_\xi(g_{ab}) + 2g_{ab})dq^a dq^b = (L_\xi g)_{ab}dq^a dq^b = n g_{ab}dq^a dq^b \ . \end{aligned} \tag{16}$$

Definition 7. Killing vector fields and homothetic Killing vector fields. *If the metric is a homogeneous tensor of weight $n \neq 0$ with respect to the action generated by a vector field ξ, then ξ is called a* homothetic Killing vector field *of weight n. If $n = 0$, then ξ is called a* Killing vector field.

Example: Let $g = \partial^2 H$ be a Hessian metric with a Hesse potential that is homogeneous of degree n. Then the Euler field ξ is a homothetic Killing vector field, and g has weight n. This follows immediately from $g = H_{ab}dq^a dq^b$.

Remark 3. Hypersurface orthogonality of the Euler field. If ξ is a homothetic Killing vector field for a Hessian metric g, then ξ is g-orthogonal to the level surfaces $H = \mathfrak{c}$ of the Hesse potential.

In ∇-affine coordinates this is manifest, since the dual coordinates are the components of a gradient:

$$(n-1)\partial_a H = (n-1)q_a = g_{ab}q^b \ . \tag{17}$$

Therefore the one-form $\xi^\flat = g_{ab}q^a dq^b = g(\xi, \cdot)$ dual to the Euler field ξ is exact, $\xi^\flat = (n-1)dH$. A vector T is tangent to the hypersurface $H = \mathfrak{c}$ if and only if it is annihilated by the one-form dH (equivalently, if it is orthogonal to the gradient of H). Therefore the vector field ξ is normal to the level surfaces of H:

$$0 = (n-1)dH(T) = \xi^\flat(T) = g(\xi, T) \ . \tag{18}$$

Note that the integrability condition $d\xi^\flat = 0$ is a special case of the Frobenius integrability condition for hypersurfaces, $\xi^\flat \wedge d\xi^\flat = 0$.[4]

Remark 4. The case $n = 1$ is to be discarded. Formula (17) shows that the case $n = 1$ is special. It corresponds to a linear Hesse potential for which the metric is totally degenerate, $H_{ab} = 0$. This case will be discarded in the following, since we are only interested in non-degenerate metrics.

[4]Hypersurface orthogonality and the Frobenius theorem are reviewed in A.7

Remark 5. The case $n = 0$ needs to be treated separately. The case $n = 0$, where ξ is a genuine Killing vector field, is interesting, but needs to be treated separately. In the following we will first consider the generic case $n \neq 0$ (with $n \neq 1$ understood), and then return to the case $n = 0$.

We would like to have a coordinate-free characterization of Hessian manifolds which admit homogeneous Hesse potentials. As a first step, we consider pseudo-Riemannian manifolds equipped with a homothetic Killing vector field which is the Euler field with respect to an affine structure. At this point it is not relevant whether the pseudo-Riemannian metric is Hessian or not. Since we admit indefinite metrics, the Euler field might become null, $g(\xi, \xi) = 0$. We will need to divide by the function $g(\xi, \xi)$ and therefore we require that ξ is *nowhere isotropic*, that is $g(\xi, \xi) \neq 0$ on the whole manifold M. Thus ξ is either globally time-like or globally space-like.

Definition 8. n-conical pseudo-Riemannian manifolds. An n-conical pseudo-Riemannian manifold (M, g, ∇, ξ) *is a pseudo-Riemannian manifold* (M, g) *equipped with a flat, torsion-free connection* ∇ *and a nowhere isotropic vector field* ξ*, such that*

$$D\xi = \frac{n}{2} Id_{TM} \;, \quad \nabla\xi = Id_{TM} \;. \tag{19}$$

Here D *is the Levi-Civita connection of the metric* g*, and* $D\xi$*,* $\nabla\xi$ *are endomorphism of the tangent bundle* TM *of* M*, that is, tensor fields of type* $(1,1)$*. Equivalently one can write*

$$D_X\xi = \frac{n}{2} X \;, \quad \nabla_X \xi = X \;, \quad \forall X \in \mathfrak{X}(M) \;, \tag{20}$$

where $\mathfrak{X}(M)$ *are the smooth vector fields on* M*.*

The condition $\nabla_X\xi = X$ implies that ξ is the Euler field with respect to ∇-affine coordinates q^a. Note that if this condition is dropped, we can change the value of n by rescaling ξ. One could in particular choose $n = 2$, which leads to the standard definition of a metric cone (or Riemannian cone). But since we are ultimately interested in Hessian manifolds, we insist on the existence of an affine structure, which prevents us from changing the value of n.

By decomposition of $D_X\xi = \frac{n}{2}X$ into its symmetric and anti-symmetric part we see that this condition is equivalent to ξ being a closed, hence hypersurface orthogonal, and homothetic Killing vector field:

$$D\xi = \frac{n}{2}\mathrm{Id} \Leftrightarrow \begin{cases} L_\xi g = ng \;, \\ d\xi^\flat = 0 \;. \end{cases} \tag{21}$$

In general local coordinates x^m this reads

$$D_m \xi_n = \frac{n}{2} g_{mn} \Leftrightarrow \begin{cases} (L_\xi g)_{mn} = D_m \xi_n + D_n \xi_m = n g_{mn} \ , \\ \partial_m \xi_n - \partial_n \xi_m = 0 \ . \end{cases} \tag{22}$$

Remark 6. Standard form of an n-conical metric. If (M, g) is an n-conical pseudo-Riemannian manifold, then g can locally be written in the from

$$g = \pm r^{n-2} dr^2 + r^n h \ , \tag{23}$$

where h is a pseudo-Riemannian metric on an immersed hypersurface $\iota \ : \mathcal{H} \to M$. For $n = 2$ this is the local form of a metric cone $(\mathbb{R}^{>0} \times \mathcal{H}, dr^2 + r^2 h)$ over a pseudo-Riemannian manifold $(\mathcal{H}, h)$.

We now give a proof following [7], which generalizes the treatment of Riemannian cones in [15]. The vector field ξ is hypersurface orthogonal and therefore locally the gradient of a function H, $\xi_m = \partial_m H$. The level surfaces of H are orthogonal to the integral lines of ξ. Combining this with the homothetic Killing equation shows that H is a potential for the metric:

$$(L_\xi g)_{mn} = D_m \xi_n + D_n \xi_m = n g_{mn} \Rightarrow D_m \partial_n H = \frac{n}{2} g_{mn} \ . \tag{24}$$

Differentiating the norm[5] $g(\xi, \xi)$ of ξ gives

$$\partial_p (g^{mn} \partial_m H \partial_n H) = 2 D_p (g^{mn} \partial_m H) \partial_n H = 2 g^{mn} D_p \partial_m H \partial_n H = n \partial_p H \ , \tag{25}$$

so that upon choosing a suitable integration constant,

$$g(\xi, \xi) = nH \ . \tag{26}$$

We use H as a coordinate along the integral lines of ξ, and extend this to a local coordinate system $\{H, x^i\}$ on M. For x^i we choose coordinates on the level surfaces of H, by picking any local coordinates on one level surfaces and extending them to M by the requirement that points on different level surfaces have the same coordinates x^i if they lie on the same integral line of ξ. Since the level surfaces are orthogonal to the integral lines of ξ, the metric has a block structure:

$$g = g_{HH} dH^2 + g_{ij} dx^i dx^j \ . \tag{27}$$

[5]Since we work with indefinite metrics, we use the term ‘norm’ for square-norm $g(\xi, \xi)$.

Using that $dH(\xi) = g(\xi, \xi)$ and $dx^i(\xi) = 0$ we find $g_{HH} = (nH)^{-1}$, and thus

$$g = \frac{dH^2}{nH} + g_{ij}dx^i dx^j \ , \quad \xi = nH\partial_H \ . \tag{28}$$

Introducing a new transverse coordinate $r > 0$ by $r^n = \pm nH$, this becomes

$$g = \pm r^{n-2}dr^2 + g_{ij}dx^i dx^j \ , \quad \xi = r\partial_r \ . \tag{29}$$

Note that we have to allow a relative sign between r and H, because H can be positive or negative, while r is positive. Using that $L_\xi dx^i = 0$, the homothetic Killing equation $L_\xi g = ng$ implies

$$(L_\xi g)_{ij} = r\partial_r g_{ij} = ng_{ij} \ . \tag{30}$$

Thus the functions $g_{ij}(r, x) = g_{ij}(r, x^1, \ldots, x^n)$ are homogeneous of degree n in ρ, and therefore

$$g_{ij}(r, x) = r^n h_{ij}(x) \ , \tag{31}$$

where $h_{ij} = h_{ij}(x)$ only depend on x^i, but not on r. Thus locally g takes the form

$$g = \pm r^{n-2}dr^2 + r^n h_{ij}(x)dx^i dx^j \ . \tag{32}$$

This is the local standard form of a n-conical metric. For $n = 2$ this is the metric on a pseudo-Riemannian cone, see A.9.

We observe that while our derivation is not valid for $n = 0$, the formula we have obtained still makes sense, since

$$g = \pm\frac{dr^2}{r^2} + h_{ij}dx^i dx^j \tag{33}$$

is a product metric on $\mathbb{R}^{>0} \times \mathcal{H}$, with isometric action of $\xi = r\partial_r$ by dilatation. Introducing a new radial coordinate ρ by $d\rho = \frac{dr}{r}$, this becomes the standard product metric

$$g = \pm d\rho^2 + h_{ij}dx^i dx^j \tag{34}$$

on $\mathbb{R} \times \mathcal{H}$, where the isometric action of $\xi = \partial_\rho$ is now by translation. The product form of the metric does not follow automatically from the n-conical conditions with $n = 0$, which imply $L_\xi g = 0$ and $d\xi^\flat = 0$. But if we impose in addition that ξ has constant norm, $g(\xi, \xi) = \mathfrak{c} \neq 0$, where we used that ξ is nowhere isotropic, we can show that g is a product metric, as follows. We choose a coordinate ρ by setting $\xi = \sqrt{|\mathfrak{c}|}\partial_\rho$ and extend this to a local coordinate system on M by choosing coordinates x^i on the hypersurfaces $\rho = \text{const.}$ orthogonal to the integral lines of ξ. In this coordinate system

$$g = g_{\rho\rho} d\rho^2 + g_{ij}(\rho, x) dx^i dx^j \ . \tag{35}$$

Since $g(\xi, \xi) = g_{\rho\rho}|\mathfrak{c}| = \mathfrak{c}$ it follows that $g_{\rho\rho} = \pm 1$, depending on whether ξ is time-like or space-like. Since by construction $L_\xi dx^i = 0$, the Killing equation $L_\xi g = 0$ implies that g_{ij} is independent of ρ:

$$(L_\xi g)_{ij} = \partial_\rho g_{ij} = 0 \ . \tag{36}$$

We can therefore interpret g_{ij} as a metric h_{ij} on any of the hypersurfaces $\rho =$ const. Thus we have shown that g locally takes the form (A.90)

$$g = \pm d\rho^2 + h_{ij} dx^i dx^j = \pm \frac{dr^2}{r^2} + h_{ij} dx^i dx^j \ , \tag{37}$$

of a product metric.

Relation to affine coordinates. The standard coordinates (r, x^i) on an n-conical Riemannian manifold can be related to a ∇-affine coordinate q^a by setting

$$(q^a) = (q^0, q^i) = (r, r x^i) \ . \tag{38}$$

The Jacobian of this transformation is

$$\frac{D(q^0, q^i)}{D(r, x^j)} = \begin{pmatrix} \frac{\partial r}{\partial r}|_{x^j} & \frac{\partial r}{\partial x^j}|_r \\ \frac{\partial r x^i}{\partial r}|_{x^j} & \frac{\partial r x^i}{\partial x^j}|_r \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x^i & r\delta^i_j \end{pmatrix} \ , \tag{39}$$

and therefore

$$\xi = r \frac{\partial}{\partial r} = r \left(\frac{\partial q^a}{\partial r} \frac{\partial}{\partial q^a} \right) = q^a \frac{\partial}{\partial q^a} \ . \tag{40}$$

The coordinates q^a have weight 1, the derivatives ∂_a have weight -1, and the metric coefficients g_{ab} are homogeneous functions of degree $(n-2)$ in q^a. We remark that the coordinates q^a can be viewed as *homogeneous coordinates* (also called *projective coordinates*) for the hypersurfaces $r = const.$, for which x^i are inhomogeneous coordinates.

So far we have not required that the pseudo-Riemannian metric g is Hessian. By adding this requirement we arrive at the following definition:

Definition 9. **n-conical Hessian manifolds.** *An* n-conical Hessian manifold (M, g, ∇, ξ) *is an n-conical pseudo-Riemannian* (M, g, ∇, ξ) *manifold which is Hessian, that is, ∇g is totally symmetric.*

Remark 7. n-conical Hessian manifolds admit a Hesse potential which is homogeneous of degree n. If (M, g, ∇, ξ) is an n-conical Hessian manifold with local affine coordinates q^a, then the function

$$H = \frac{1}{n(n-1)} q^a q^b g_{ab} \ , \tag{41}$$

which is homogeneous of degree n, is a Hesse potential for g.

The function H is manifestly homogeneous of degree n. By differentiating (41) twice and using the homogeneity relations (11) for g_{ab}, we obtain $H_{ab} = \partial_a \partial_b H = g_{ab}$, which shows that H is indeed a Hesse potential for g_{ab}. We remark that (41) does not apply to the degenerate case $n = 1$, which we discard, and to the interesting case $n = 0$, which we will consider separately below.

Definition 10. Conical affine coordinates. *Let (M, g, ∇, ξ) be an n-conical Hessian manifold. Then ∇-affine coordinates q^a are called* conical ∇-affine coordinates *if the Hesse potential is homogeneous of degree n in q^a.*

The homogeneity of H is only preserved under linear transformations, but not under translations. Therefore conical ∇-affine coordinates are unique up to linear coordinate changes. In the following it is understood that ∇-affine coordinates on a conical Hessian manifold are always chosen to be conical.

0-conical Hessian manifolds

We now turn to the special case $n = 0$, where the Euler field ξ acts isometrically on the Hessian metric g. Metrics of this type can be constructed by taking Hesse potentials of the form

$$\tilde{H} = \mathfrak{a} \log(\mathfrak{b} H) \ , \tag{42}$$

where $\mathfrak{a}, \mathfrak{b}$ are real constants, and where H is a homogeneous function of degree $n > 1$.[6] We will see later that certain constructions involving vector multiplets (superconformal quotients and dimensional reduction) naturally involve replacing a homogeneous Hesse potential by its logarithm. The constants $\mathfrak{a}, \mathfrak{b}$ have been introduced so that we can match our results with various conventions used in the physics literature.

Note that the Hesse potential $\tilde{H}$ is not a homogeneous function, since it transforms with a shift under $q^a \mapsto \lambda q^a$. However, its k-th derivatives are

[6]Except where the determination of signatures is concerned, we only use $n \neq 0$, $n \neq 1$ in the following. For physics applications we will need the cases $n = 2$ and $n = 3$.

homogeneous functions of degree $-k$ for any $k \geq 1$. The tensor $\tilde{g} = \tilde{H}_{ab} dq^a dq^b$ is homogeneous of degree zero, and defines a 0-conical Hessian metric. The first derivatives

$$\tilde{q}_a := \tilde{H}_a = \partial_a \tilde{H} = \mathfrak{a} \frac{H_a}{H} \tag{43}$$

of $\tilde{H}$ are homogeneous of degree -1. They define the coordinate system dual to the affine coordinates q^a with respect to the Hesse potential $\tilde{H}$. The overall sign of H does not have any effect on expressions which involve derivatives of $\tilde{H}$ only, since these expressions are invariant under $H \to -H$. In particular, the Hessian metrics $H_{ab} = \partial^2_{a,b} H$ and $-H_{ab} = \partial^2_{a,b}(-H)$ give rise to the same Hessian metric $\tilde{H}_{ab}$ if we 'take the log of the Hesse potential.'

Explicitly, the metric coefficients associated with the Hesse potential $\tilde{H}$ are:

$$\tilde{H}_{ab} = \partial^2_{a,b} \tilde{H} = \mathfrak{a} \, \frac{H H_{ab} - H_a H_b}{H^2} \ . \tag{44}$$

The following relations implied by the homogeneity of H are useful:

$$q^a q_a = q^a H_a = nH \ , \quad H_{ab} q^b = (n-1) q_a \ , \quad q^a q^b H_{ab} = n(n-1) H \ , \tag{45}$$

The dual affine coordinates $\tilde{q}_a$ with respect to $\tilde{H}$ satisfy

$$\tilde{H}_{ab} q^b = -\mathfrak{a} \frac{q_a}{H} = -\tilde{q}_a \ . \tag{46}$$

To compare the n-conical metric $\partial^2 H$ and the 0-conical metric $\partial^2 \tilde{H}$, we evaluate them on the Euler field ξ, which is orthogonal to the level surfaces of H and $\tilde{H}$, and on a vector field T, which is tangent to the level surfaces.

- Components transversal to the foliation $\mathcal{H}_{\mathfrak{c}}$.

$$g(\xi, \xi) = H_{ab} q^a q^b = n(n-1) H \ , \tag{47}$$

$$\tilde{g}(\xi, \xi) = \tilde{H}_{ab} q^a q^b = -\mathfrak{a} n \ . \tag{48}$$

  The $\tilde{g}$-norm $\tilde{g}(\xi, \xi)$ of ξ is constant on M, while the g-norm $g(\xi, \xi)$ depends on the leaf $\mathcal{H}_{\mathfrak{c}}$.

- Mixed components. If T is tangent to $\mathcal{H}_{\mathfrak{c}} = \{H = \mathfrak{c}\}$, then

$$dH(T) = T^a H_a = T^a q_a = 0 \ . \tag{49}$$

  Therefore

$$g(T, \xi) = H_{ab} T^a q^b = T^a q_a = 0 \ , \quad \tilde{g}(T, \xi) = \tilde{H}_{ab} T^a q^b = 0 \ . \tag{50}$$

- Tangential components:

$$\tilde{g}(T,T) = \mathfrak{a}\frac{HH_{ab}T^aT^b - T^aq_aq_bT^b}{H^2} = \frac{\mathfrak{a}}{H}H_{ab}T^aT^b = \frac{\mathfrak{a}}{H}g(T,T) \ . \quad (51)$$

These component are proportional for constant H.

Since the tangential components of both metrics are proportional for any fixed leaf $\mathcal{H}_\mathfrak{c}$, their pullbacks to the embedded hypersurfaces[7]

$$\iota_\mathfrak{c} \ : \mathcal{H}_\mathfrak{c} = \{q \in M | H(q) = \mathfrak{c}\} \to M \quad (52)$$

are proportional:

$$g_\mathfrak{c} = \iota_\mathfrak{c}^* \partial^2 H = \frac{\mathfrak{c}}{\mathfrak{a}} \iota_\mathfrak{c}^* \partial^2 \tilde{H} \ . \quad (53)$$

On the hypersurface $\mathcal{H} = \mathcal{H}_{\mathfrak{c}=1}$:

$$g_\mathcal{H} = \iota^* \partial^2 H = \frac{1}{\mathfrak{a}} \iota^* \partial^2 \log H \ . \quad (54)$$

By choosing $\mathfrak{a} = 1$ we can make the pullbacks equal. Note that the transversal components of both metrics are different. In particular both metrics have different signatures. On a leaf $\mathcal{H}_\mathfrak{c}$ we have

$$\mathfrak{a} g(T,T) = \mathfrak{c}\tilde{g}(T,T) \ , \quad (55)$$

$$g(\xi,\xi) = n(n-1)\mathfrak{c} \ , \quad \tilde{g}(\xi,\xi) = \tilde{H}_{ab}dq^adq^b = -n\mathfrak{a} \ . \quad (56)$$

Thus if g and $\tilde{g}$ have the same signature on tangent vectors, $\mathfrak{ac} > 0$, then they have different signature in the transverse direction.[8]

Remark 8. The dual Hessian structure and dual Hesse potential for a Hessian manifold with logarithmic Hesse potential. The Hesse potential $\tilde{H}_{\text{dual}}$ dual to $\tilde{H}$ is defined by

$$\tilde{H}^{ab} = \frac{\partial \tilde{H}_{\text{dual}}}{\partial \tilde{q}_a \partial \tilde{q}_b} \ , \quad (57)$$

where $\tilde{H}^{ab}$ is the inverse of $\tilde{H}_{ab}$. By a straightforward computation one finds $\tilde{H}_{\text{dual}} = -\tilde{H}$.[9] This is consistent with (12), which, however, cannot be applied directly, because $\tilde{H}$ is not a homogeneous function of degree zero.

---

[7]Immersions and embeddings are review in A.1. Since we are interested in comparing local expressions for various tensor fields, there is no loss of generality in assuming that the hypersurfaces $\mathcal{H}_{gc}$ are embedded.

[8]Here we use the assumption $n > 1$, which applies for the application to vector multiplets, where $n = 2$ or $n = 3$. Otherwise all expressions in this section are valid for $n \neq 0$, $n \neq 1$.

[9]We remark that in the physics literature, i.p. in [16], the dual Hesse potential was defined without minus sign. Here we use the definition given in [1].

2.4. Projectivization of conical Hessian manifolds

The relation between the manifolds (M, g_M) and $(\mathcal{H}, g_{\mathcal{H}})$ can be interpreted as a quotient, and $(\mathcal{H}, g_{\mathcal{H}})$ can be viewed as the projectivization of the conical manifold (M, g_M), with respect to the homothetic action of ξ. This construction is related to the so-called superconformal quotients in the physics literature. In particular the *real superconformal quotient* relating the scalar geometry of five-dimensional superconformal vector multiplets to the geometry of vector multiplets coupled to Poincaré supergravity is a special case of the quotient relating (M, g_M) and $(\mathcal{H}, g_{\mathcal{H}})$.

If (M, g_M, ∇, ξ) is a conical Hessian manifold we can consider the space of orbits $\bar{M} = M/\langle\xi\rangle \cong M/\mathbb{R}^{>0}$ of the action of ξ on M. We will assume that this quotient is well-behaved, so that $\bar{M}$ is a smooth manifold. To induce a metric $g_{\bar{M}}$ on the quotient, we need a symmetric, second rank co-tensor g^*_M on M which is *projectable*, that is, invariant under the action of ξ, $L_\xi g^*_M = 0$ and transversal to the action of ξ, $g^*_M(\xi, \cdot) = 0$. The second condition implies that g^* is not a metric on M, because it has a kernel which contains ξ. In order that it projects to a metric $g_{\bar{M}}$ on $\bar{M}$, the kernel of g^*_M must be one-dimensional, that is, it is spanned by ξ. Since the hypersurfaces $\mathcal{H}_\mathfrak{c}$ are transversal to ξ, any of them can be used as a a set of representatives for the orbit space $M/\langle\xi\rangle$, that is $\bar{M} \cong \mathcal{H}_\mathfrak{c}$. We can view M as a real line bundle, $\pi : M \to \bar{M}$ over $\bar{M} \cong \mathcal{H}$, and the invariant tensor g^*_M is equal to the pull-back of $g_{\bar{M}} = g_{\mathcal{H}}$ to M:

$$g^*_M = \pi^* g_{\bar{M}} = \pi^* g_{\mathcal{H}} \; . \tag{58}$$

The conical metric g_M is neither invariant nor transversal with respect to the action of ξ, but there is a natural way to construct a projectable tensor g^*_M out of g_M using the conical Hessian structure. Moreover, the induced metric $g_{\bar{M}}$ agrees, up to conventional normalization, with the pull-back $g_{\mathcal{H}}$ of the conical metric g_M to $\mathcal{H}$. Since g_M transforms with a different weight n under ξ, we can obtain an invariant tensor by multiplication with the appropriate power of H. In fact, we have seen that taking the logarithm of a homogeneous Hesse potential automatically associates a 0-conical Hessian metric to an n-conical one. To obtain a projectable tensor, it remains to add an ξ-invariant symmetric rank two co-tensor such that the resulting tensor becomes transversal to ξ. For this it is helpful to consider the one-form

$$d \log H = H^{-1} H_a dq^a \tag{59}$$

which vanishes on tangent vectors T to the surfaces $H = \mathfrak{c}$, while being constant along integral lines of ξ:

$$d \log H(T) = 0 \; , \quad d \log H(\xi) = H^{-1} H_a q^a = n \; . \tag{60}$$

By taking linear combinations between the 0-conical Hessian metric and the square of this one form, we obtain a family of ξ-invariant symmetric rank two co-tensors:

$$g_M^{(\alpha)} = \mathfrak{a} \frac{H H_{ab} - \alpha H_a H_b}{H^2} \, dq^a dq^b = \mathfrak{a} \left(H^{-1} H_{ab} dq^a dq^b - \alpha (d \log H)^2 \right) \; . \tag{61}$$

Note that only $\alpha = 1$ corresponds to a Hessian metric. Now we look for a critical value α^* of α where $g_M^{(\alpha)}$ becomes transversal to ξ:

$$\begin{aligned} 0 \quad &= \quad g_M^{(\alpha)}(\xi, \cdot) = \mathfrak{a} H^{-2} \left(H H_{ab} - \alpha H_a H_b \right) q^a dq^b = \mathfrak{a} H^{-2} H H_b \left((1-\alpha) n - 1 \right) dq^b \\ &\Rightarrow \quad \alpha = \frac{n-1}{n} =: \alpha_* \; . \end{aligned} \tag{62}$$

Note that as a function of α the norm $g(\xi, \xi)$ of ξ changes sign at $\alpha = \alpha^*$. Therefore $g_M^{(\alpha)}$ changes signature when crossing the critical value where it degenerates.

Thus we have identified the projectable tensor

$$g_M^* = g_M^{(\alpha_*)} = \mathfrak{a} \left(H^{-1} H_{ab} dq^a dq^b - \frac{n-1}{n} (d \log H)^2 \right) \; , \tag{63}$$

which defines a non-degenerate metric $g_{\bar{M}}$ on the quotient space $\bar{M} = M / \mathbb{R}^{>0}$. Since the hypersurfaces $\mathcal{H}_\mathfrak{c}$ are transversal to the integral lines of ξ, we can pick any such hypersurface to represent the quotient space. On tangent vectors T, S to $\mathcal{H}$, g_M^* agrees, up to a constant factor, with g_M, and therefore with the pull-back of g_M to $\mathcal{H}$:

$$g_M^*(T, S)_{\mathfrak{c}=1} = \mathfrak{a} H_{ab} T^a S^b = \mathfrak{a} g_M(T, X)_{\mathfrak{c}=1} = \mathfrak{a} g_{\mathcal{H}}(T, X) \; . \tag{64}$$

We remark that this construction can be viewed as a real analogue of the construction of the Fubini-Study metric on complex projective spaces, which itself is a special case of the complex version of the superconformal quotient (see for example [17]).

Finally we remark that the family $g_M^{(\alpha)}$ of ξ-invariant tensors can be generalized to families of symmetric tensors with given weight k under ξ. If H has weight n then metrics of the form

$$g^{(k, \alpha_1, \alpha_2)} = H^{k/n} \left(\alpha_1 g_M^* + \alpha_2 (d \log H)^2 \right) \; . \tag{65}$$

have weight k. This parametrization uses three building blocks: the projectable invariant tensor g^*_M, the quadratic differential $(d\log H)^2$ which vanishes on tangent vectors of the foliation $\mathcal{H}_c$, and the Hesse potential which determines the weight. By varying α_1 and α_2, the signature can be changed. All symmetric second rank co-tensors we need are included in this family.

2.5. Special real geometry

2.5.1. Affine special real manifolds as Hessian manifolds

We are now in position to define the scalar geometries five-dimensional vector multiplets. As we will see in section 3.1 the geometry of rigid five-dimensional vector multiplets is Hessian, and the scalar fields, which are the lowest components of vector multiplets, are 'special coordinates' on the scalar manifold.[10] Here special coordinates means affine coordinates with respect to the flat (or 'special') connection defining the Hessian structure. Supersymmetry imposes an additional condition because it implies the presence of a Chern-Simons term in the Lagrangian, whose gauge invariance (up to surface terms) restricts the Hesse potential to be a cubic polynomial. This leads to the following definition:

Definition 11. Affine special real manifolds (ASR manifolds). *An* affine special real manifold (M, g_M, ∇) *is a Hessian manifold with a Hesse potential that is a cubic polynomial in* ∇*-affine coordinates.*

We note that this definition is independent of the choice of special coordinates, since affine transformations preserve the degree of a polynomial. The ∇-affine coordinates of an ASR manifold are called *special real coordinates*, or special coordinates for short.

We can also define a conical version of affine special real geometry, which turns out to be the geometry of five-dimensional rigid *superconformal* vector multiplets, to be introduced in section 3.2.

Definition 12. Conical affine special real manifolds (CASR manifolds). *A* conical affine special real manifold (M, g_M, ∇, ξ) *is a 3-conical Hessian manifold whose Hesse potential is a homogeneous cubic polynomial in special coordinates.*

Finally, we can apply the quotient construction of section 2.4 to a CASR manifold. In this case we will refer to the quotient as the *real superconformal*

[10] More precisely the scalar fields are pullbacks from the scalar manifold to space-time of coordinate maps for the scalar manifold. See B.1.

quotient, because the resulting quotient manifolds occur as scalar target spaces for five-dimensional vector multiplet coupled to Poincaré supergravity, as we will see in section 3.3. This motivates the following definition:

Definition 13. Projective special real manifold (PSR manifold). *A projective special real manifold* $(\bar{M}, g_{\bar{M}})$ *is a pseudo-Riemannian manifold which can be obtained as the real superconformal quotient of a conical affine special real manifold* (M, g_M, ∇, ξ).

For later use we collect some formulae, which follow from those derived in the previous sections by specializing to the case $n = 3$. On a CASR manifold M we have the family

$$g_M^{(\alpha)} = \mathfrak{a}\frac{HH_{ab} - \alpha H_a H_b}{H^2} dq^a dq^b = \mathfrak{a}\left(H^{-1}H_{ab}dq^a dq^b - \alpha(d\log \mathfrak{b}H)^2\right) \tag{66}$$

of ξ-invariant symmetric rank 2 co-tensor fields. The following tensor fields are relevant for five-dimensional vector multiplet theories:

- The CASR metric
$$g_M = H_{ab}dq^a dq^b \; . \tag{67}$$
- The ξ-invariant metric
$$g_M^{(0)} = \mathfrak{a}H^{-1}H_{ab}dq^a dq^b \; , \tag{68}$$
which is a conformally rescaled version of the CASR metric $g_M = H_{ab}dq^a dq^b$.
- The 0-conical Hessian metric
$$g_M^{(1)} = \mathfrak{a}\partial^2 \log \mathfrak{b}H = \mathfrak{a}\frac{HH_{ab} - H_a H_b}{H^2} dq^a dq^b \; . \tag{69}$$
- The projectable tensor field
$$\begin{aligned} g_M^* &= \mathfrak{a}\left(H^{-1}H_{ab}dq^a dq^b - \frac{2}{3}(d\log H)^2\right) \\ &= \mathfrak{a}(H^{-1}H_{ab} - \frac{2}{3}H^{-2}H_a H_b)dq^a dq^b \; , \end{aligned} \tag{70}$$
where we used that $\alpha_* = \frac{2}{3}$ for $n = 3$. This tensor field projects to the PSR metric $g_{\bar{M}} = g_{\mathcal{H}}$.

We also note the norms of ξ with respect to these metrics:

$$g_M(\xi,\xi) = 6H \; , \quad g_M^{(0)}(\xi,\xi) = 6\mathfrak{a} \; , \quad g_M^*(\xi,\xi) = 0 \; , \quad g_M^{(1)}(\xi,\xi) = -3\mathfrak{a} \; . \tag{71}$$

As observed before, the signature of $g_M^{(\alpha)}$ changes at $\alpha = \alpha_* = \frac{2}{3}$.

2.5.2. Projective special real manifolds as centroaffine hypersurfaces

The original construction of five-dimensional vector multiplets coupled to supergravity [18] did not make use of the superconformal formalism. Instead the Poincaré supergravity Lagrangian and on-shell supertransformations were constructed directly. The resulting scalar manifold $\bar{M}$ was interpreted as a cubic hypersurface in $\mathbb{R}^{n+1}$, with a metric determined by the homogeneous cubic polynomial defining the embedding. We will not follow [18] in detail, but instead review the construction of [2], which realizes $\bar{M}$ as a so-called *centroaffine hypersurface* and allows to recover the local formulae of [18].

We start with $\mathbb{R}^{n+1}$ equipped with its standard flat connection ∂. Note that we do not introduce a metric on $\mathbb{R}^{n+1}$ so that the construction is done within the framework of affine differential geometry. The *position vector field* ξ is defined by $\xi(p) = p$ for all $p \in \mathbb{R}^{n+1}$. For linear coordinates h^I on $\mathbb{R}^{n+1}$ and ξ is the corresponding Euler field, $\xi = h^I \partial_I$.

Definition 14. PSR manifolds as centroaffine hypersurfaces. *A PSR manifold $\bar{M}$ is a connected immersed hypersurface*

$$\iota\ : \bar{M} \to \mathcal{H} := \{\mathcal{V} = 1\} \subset \mathbb{R}^{n+1} \tag{72}$$

where the homogeneous cubic polynomial

$$\mathcal{V} := C_{IJK} h^I h^J h^K \tag{73}$$

is assumed to be non-singular in a neighbourhood

$$U = U_\epsilon = \{\mathcal{V} = c | 1 - \epsilon < c < 1 + \epsilon\} \subset \mathbb{R}^{n+1} \tag{74}$$

of the hypersurface $\mathcal{H}$ for some $\epsilon > 0$.

We will assume that $\bar{M}$ is an embedded submanifold, so that we can identify $\bar{M}$ and $\mathcal{H}$. Let us verify that we can recover the alternative Definition 11. For a homogeneous cubic polynomial, the position vector field ξ is everywhere transversal to $\mathcal{H}$. This allows to define a metric $g_{\mathcal{H}}$ and a torsion-free connection ∇ on $\mathcal{H}$ by decomposing the connection ∂, acting on tangent vectors $X, Y \in T_p\mathcal{H}$, $p \in \mathcal{H}$, into a tangent and a transversal component:

$$\partial_X Y = \nabla_X Y + \frac{2}{3} g_{\mathcal{H}}(X, Y)\xi\ . \tag{75}$$

The factor $\frac{2}{3}$ is conventional. This construction is a special case of the construction of a *centroaffine hypersurface*, see A.10 for more details.

It is useful to introduce the totally symmetric trilinear from

$$C = C_{IJK} dh^I dh^J dh^K \ . \tag{76}$$

By contracting with the position vector ξ we obtain the following tensors:

1. The function
$$C(p,p,p) = C_{IJK} h^I h^J h^K = \mathcal{V} \ , \tag{77}$$
which defines the embedding.
2. The one-form
$$C(p,p,\cdot) = C_{IJK} h^I h^J dh^K = \frac{1}{3} d\mathcal{V} \ , \tag{78}$$
which is proportional to the differential of $\mathcal{V}$, and which therefore vanishes precisely on tangent vectors of $\mathcal{H}$.
3. The symmetric two-form
$$C(p,\cdot,\cdot) = C_{IJK} h^I dh^J dh^K = \frac{1}{6} \partial d\mathcal{V} \ , \tag{79}$$
which is proportional to the Hessian of the function $\mathcal{V}$. If this two-form is non-degenerate, it defines a Hessian metric on $U \subset \mathbb{R}^{n+1}$.

Since U is equipped with a 3-conical Hessian metric, we can identify it with the CASR manifold M of the previous section.

One defines the conjugate or dual coordinates

$$h_I := C_{IJK} h^J h^K \ , \tag{80}$$

so that $\mathcal{V} = h_I h^I$, $d\mathcal{V} = 3 h_I dh^I$. The dual coordinates h_I are, up to a numerical factor, the dual affine coordinates of the Hessian structure defined by $C(p,\cdot,\cdot)$.

We claim that $g_{\mathcal{H}}$ is proportional to the pullback of the Hessian metric $\partial d\mathcal{V}$ to $\mathcal{H}$:

$$g_{\mathcal{H}}(X,Y)_p = -3C(p,X,Y) = -\frac{1}{2} (\partial^2_{X,Y} \mathcal{V})_{|p} \ , \tag{81}$$

for all tangent vectors $X, Y = T_p \mathcal{H}$. To show this we extend the tangent vector fields X, Y to a neighbourhood $U = U_\epsilon$ of $\mathcal{H} \subset \mathbb{R}^{n+1}$, such that $X(\mathcal{V}) = Y(\mathcal{V}) = 0$. In other words the extended vector fields X, Y are tangent to the local foliation of $\mathbb{R}^{n+1}$ by hypersurfaces $\mathcal{H}_{\mathfrak{c}} = \{\mathcal{V} = \mathfrak{c}\}$. The Hessian of the function $\mathcal{V}$ is[11]

$$\partial^2_{X,Y} \mathcal{V} = X(Y(\mathcal{V})) - (\partial_X Y)(\mathcal{V}) = X^I Y^J \mathcal{V}_{IJ} \tag{82}$$

[11] We refer to A.5.4 for the definition of higher covariant derivatives with respect to vector fields, and the definition of the Hessian of a function with respect to a general linear connection.

so that on tangent vector fields of $\mathcal{H}$:

$$(\partial^2_{X,Y}\mathcal{V})_p = (-\partial_X Y)(\mathcal{V})_p = -3C(p,p,\partial_X Y)\ , \quad p \in \mathcal{H}\ . \tag{83}$$

In the second step we used the formula

$$Z(\mathcal{V})_p = Z^L\partial_L(C_{IJK}h^I h^J h^K) = 3Z^L C_{LJK}h^J h^K = 3C(p,p,Z)\ . \tag{84}$$

Using (75) we obtain

$$(\partial^2_{X,Y}\mathcal{V})_p = -3C(p,p,\nabla_X Y) - 2g_{\mathcal{H}}(X,Y)C(p,p,p) = -2g_{\mathcal{H}}(X,Y)\ , \tag{85}$$

where we used that $C(p,p,\cdot)$ vanishes on tangent vector of $\mathcal{H}$, and that $C(p,p,p) = 1$ for $p \in \mathcal{H}$. Thus $g_{\mathcal{H}}$ agrees with $-\frac{1}{2}\partial d\mathcal{V} = -3C(p,\cdot,\cdot)$ on tangent vectors, and we can therefore extend $g_{\mathcal{H}}$ to a Hessian metric $g = -\frac{1}{2}\partial d\mathcal{V}$ with Hesse potential $-\frac{1}{2}\mathcal{V}$ in a neighbourhood U_ϵ of $\mathcal{H}$. The metric $g = h_{IJ}dh^I dh^J$ is the 3-conical ASR metric denoted g_M, which occurred previously in the superconformal quotient construction. In local coordinates

$$h_{IJ} = -\frac{1}{2}\partial^2_{I,J}\mathcal{V} = -3C_{IJK}h^K\ . \tag{86}$$

The torsion-free connection ∇ is not the Levi-Civita connection D of the metric $g_{\mathcal{H}} = \iota^* g$. The connections ∇ and D can be related using a tensor S, which is defined in terms of the trilinear form C:

$$g(S_X Y, Z) = \frac{3}{2}C(X,Y,Z)\ , \tag{87}$$

where X, Y, Z are vector fields tangent to $\mathcal{H}$. Now we define a new connection D by[12]

$$D = \nabla - S\ . \tag{88}$$

To show that D is the Levi-Civita connection of $g_{\mathcal{H}}$ we must prove that D is metric and torsion-free. The total symmetry of the trilinear form implies $S_X Y = S_Y X$, and since ∇ is torsion-free, it follows that D is torsion-free. It remains to show D is metric, that is

$$(D_X g)(Y,Z) = Xg(Y,Z) - g(D_X Y, Z) - g(Y, D_X Z) = 0\ , \tag{89}$$

[12]Note that compared to [2] the symbols D and ∇ have been exchanged.

where X, Y, Z are tangent to $\mathcal{H}$. We extend X, Y, Z to $U = U_\epsilon$ such that $X(\mathcal{V}) = Y(\mathcal{V}) = Z(\mathcal{V}) = 0$. Substituting in $D = \nabla - S$ and using (75), together with the fact that ξ is g-orthogonal to tangent vectors, we find

$$\begin{aligned}(D_X g)(Y,Z) &= Xg(Y,Z) - g(\partial_X Y, Z) - g(Y, \partial_X Z) + g(S_X Y, Z) + g(Y, S_X Z) \\ &= (\partial_X g)(Y,Z) + 3C(X,Y,Z) \,, \end{aligned} \tag{90}$$

where we used the relation between the difference tensor S and the trilinear form C in the second step. Now we use that g is Hessian:

$$(\partial_X g)(Y,Z) = -\frac{1}{2}\partial^3_{X,Y,Z}\mathcal{V} = -3C(X,Y,Z) \,. \tag{91}$$

and therefore $(D_X g)(Y,Z) = 0$, as required to show that D is the Levi-Civita connection of $g_{\mathcal{H}}$. We remark that the metric $g_{\bar{M}} = g_{\mathcal{H}}$ defined on the hypersurface $\bar{M} = \mathcal{H}$ is not a Hessian metric. Moreover, the connections ∇ and D do not define flat connections on $\mathcal{H}$.

2.6. Conical and projective special real geometry in local coordinates

In this section we derive explicit expressions for various quantities in terms of local coordinates on the CASR manifold M and on the PSR manifold $\bar{M} \cong \mathcal{H}$. Since we are interested in local expressions we assume that $\mathcal{H}$ is embedded into M, rather than only immersed, and take M to be foliated by hypersurfaces $\mathcal{H}_\mathfrak{c}$. We will relate the notation and convention used in the previous sections to those of [18], where the geometry of five-dimensional vector multiplets coupled to Poincaré supergravity was derived originally.

As in section 2.5.2 and in [18] affine coordinates on $M \cong U \subset \mathbb{R}^{n+1}$ are denoted h^I, $I = 0, \ldots, n$, local coordinates on $\mathcal{H}$ are denoted ϕ^x, $x = 1, \ldots, n$ and the Hesse potential is denoted $\mathcal{V}$. In section 2.3 these quantities were denoted q^a, x^i and H, respectively. On M we are using a second coordinate system, which consists of a coordinate along the integral lines of the Euler field ξ, together with coordinates on the level surfaces of the Hesse potential. Since the Euler field is transversal to $\mathcal{H}$, the CASR manifold M is foliated by hypersurfaces $\mathcal{H}_\mathfrak{c} = \{\mathcal{V} = \mathfrak{c}\}$. We can extend the coordinates ϕ^x to M by imposing that two points $p \in \mathcal{H}$ and $p' \in \mathcal{H}_\mathfrak{c}$ have the same coordinates ϕ^x is they lie on the same integral line of ξ. With regard to the transversal coordinate, the two natural choices are ρ and $r = e^\rho$, defined by

$$\xi = h^I \partial_I = \partial_\rho = r\partial_r \,. \tag{92}$$

The differential $\frac{\partial h^I}{\partial \phi^x}$ of the embedding

$$\iota_{\mathfrak{c}} \ : \mathcal{H}_{\mathfrak{c}} \ni \phi^x \mapsto h^I \in U \tag{93}$$

allows to pull-back tensor components to $\mathcal{H}_{\mathfrak{c}}$. Following [18] we define the rescaled quantities

$$h^I_x = -\sqrt{\frac{3}{2}}\partial_x h^I \ , \quad h_{Ix} = \sqrt{\frac{3}{2}}\partial_x h_I \tag{94}$$

for later convenience. Given the definitions

$$\mathcal{V} = C_{IJK} h^I h^J h^K \ , \quad h_I = C_{IJK} h^J h^K \tag{95}$$

for the Hesse potential and for the dual coordinates,[13] we note the following relations:

$$h^I h_I = \mathcal{V} \Rightarrow h^I_x h_I = 0 = h^I h_{Ix} \ . \tag{96}$$

The second relation follows because derivatives ∂_x are taken along hypersurfaces $\mathcal{H}_{\mathfrak{c}}$. Note that here and in the following some of our relations will differ from those found in [18] by factors of $\mathcal{V}$. The reason is that the relations given in [18] are valid on $\mathcal{H}$, that is for $\mathcal{V} = 1$, whereas we extend these relations to all of M. We now specify the relevant rank two symmetric tensor fields on M.

- The CASR metric on M is

$$g_M = -\frac{1}{2}\partial^2 \mathcal{V} = h_{IJ} dh^I dh^J \ , \quad h_{IJ} = -\frac{1}{2}\partial^2_{I,J}\mathcal{V} = -3 C_{IJK} h^K \ . \tag{97}$$

  Compared to section 2.3 this corresponds to the choice $H = -\frac{1}{2}\mathcal{V}$ while identifying the coordinates h^I with the coordinates q^a.

- The 0-conical metric on M is

$$g^{(1)}_M = -\frac{1}{3}\partial^2 \log \mathcal{V} = a_{IJ} dh^I dh^J \ , \tag{98}$$

$$a_{IJ} = -\frac{1}{3}\partial^2_{I,J} \log \mathcal{V} = \frac{-2C_{IJK}h^K\mathcal{V} + 3h_I h_J}{\mathcal{V}^2} \ . \tag{99}$$

  Compared to section 2.3 this corresponds to the choices $\mathfrak{a} = -\frac{1}{3}$ and $\mathfrak{b} = -\frac{1}{2}$. We note that with this convention ξ has unit norm, $g^{(1)}_M(\xi, \xi) = 1$, while on tangent vectors T, S we find $g^{(1)}_M(T, S) = \frac{3}{2\mathcal{V}} g_M(T, S)$.

[13] Remember that the h_I then differ from the standard dual coordinates of Hessian geometry by a factor.

- The projectable tensor on M is

$$g_M^* = \frac{-2C_{IJK}h^K\mathcal{V} + 2h_Ih_J}{\mathcal{V}^2}dh^I dh^J \ , \tag{100}$$

since

$$g_M^*(\xi, \cdot) = \frac{-2C_{IJK}h^I h^K\mathcal{V} + 2h^I h_I h_J}{\mathcal{V}^2} = 0 \ . \tag{101}$$

Note that

$$g_M^{(1)} = g_M^* + \frac{h_I h_J}{\mathcal{V}^2}dh^I dh^J \tag{102}$$

is the product decomposition of the 0-conical metric into the projectable tensor and the square of a one-form dual to the Euler field ξ.

- The PSR metric $g_{\mathcal{H}}$ is the pullback of the CASR metric g_M to $\mathcal{H}$, but differs by a factor $\frac{3}{2}$ from the pullback of the 0-conical metric $g_M^{(1)}$, which makes the definition (94) convenient:

$$g_{xy} = h_{IJ}\partial_x h^I \partial_y h^J = a_{IJ}h_x^I h_y^J \ . \tag{103}$$

We would also like to give expressions for the horizontal lifts of tensors from $\mathcal{H}$, or more generally from $\mathcal{H}_{\mathfrak{c}}$, to M. For this it is useful to note that

$$h_I = \mathcal{V}a_{IJ}h^J \ , \quad h_{Ix} = \mathcal{V}a_{IJ}h_x^J \ , \quad h_{Ix}h_y^I = \mathcal{V}a_{IJ}h_x^I h_y^J = \mathcal{V}g_{xy} \ . \tag{104}$$

We also define

$$h_I^x = g^{xy}h_{Iy} \ , \quad h^{Ix} = g^{xy}h_y^I \ . \tag{105}$$

Then the quantities h_I^x can be used to lift tensors from $\mathcal{H}$ to M, and to convert tensors from coordinates (ρ, ϕ^x) to coordinates h^I. For example, the components of the horizontal lift of g_{xy} to M are

$$\frac{3}{2}\frac{1}{\mathcal{V}^2}g_{xy}h_I^x h_J^y = \frac{3}{2}g_{IJ}^* = \frac{3}{2}\left(\frac{-2C_{IJK}h^K\mathcal{V} + 2h_Ih_J}{\mathcal{V}^2}\right) \ . \tag{106}$$

To verify this we evaluate the tensor on the left hand side on the coordinate frame

$$\xi = h^I\partial_I = \partial_\rho \ , \quad \partial_u = \partial_u h^I \partial_I \ . \tag{107}$$

Firstly, $g_{IJ}^* h^J = 0$, so that ξ is in the kernel. On tangent vectors we find

$$\left(\frac{3}{2\mathcal{V}^2}g_{xy}h_I^x h_J^y\right)\partial_u h^I \partial_v h^J = \frac{1}{\mathcal{V}^2}g_{xy}h_I^x h_y^J h_u^I h_v^J = g_{uv} \ , \tag{108}$$

as required.

Similarly, the 0-conical metric a_{IJ} can be decomposed into a term proportional to the horizontal lift of g_{xy} and an orthogonal complement:

$$a_{IJ} = \frac{g_{xy} h_I^x h_J^y + h_I h_J}{\mathcal{V}^2} \ . \tag{109}$$

This can again be verified by evaluation on the coordinate frame $\partial_\rho, \partial_x$. We find

$$a_{IJ} h^I h^J = \frac{h_I h^I h_J h^J}{\mathcal{V}^2} = 1 = g_M^{(1)}(\xi, \xi) \tag{110}$$

and

$$a_{IJ} \partial_u h^I \partial_v h^J = \frac{2}{3\mathcal{V}^2} g_{xy} h_I^x h_J^y h_u^I h_v^J = \frac{2}{3} g_{uv} = g_M^{(1)}(\partial_u, \partial_v) \ , \tag{111}$$

while $a_{IJ} h^I h_x^J = 0 \Rightarrow g_M^{(1)}(\xi, \partial_x) = 0$, thus verifying that (109) are the coefficients of the 0-conical metric $g_M^{(1)}$. To convert these coefficients from the linear coordinates h^I to the coordinates (ρ, ϕ^x), we compute

$$\begin{aligned} a_{IJ} dh^I dh^J &= \mathcal{V}^{-2} g_{xy} h_I^x h_J^y \frac{2}{3} h_u^I h_v^J d\phi^u d\phi^v + \mathcal{V}^{-2} h_I h_J h^I h^J d\rho^2 \\ &= \frac{2}{3} g_{xy} d\phi^x d\phi^y + d\rho^2 \ . \end{aligned} \tag{112}$$

Here we have substituted in a_{IJ} and used that $h_I^x h^I = 0$ to simplify the first term. In the second step we used

$$\xi = h^I \partial_I = \partial_\rho \Rightarrow \xi^\flat = d\rho = \mathcal{V}^{-1} h_I dh^I \ . \tag{113}$$

Next, we express the connections D and ∇ in local coordinates ϕ^x on $\mathcal{H}$, following [2]. Let X be a vector field tangent to $\mathcal{H}$. Then

$$X = X^I \partial_I = X^x \partial_x \Rightarrow X^I = X^x \partial_x h^I \ . \tag{114}$$

Equation (75) becomes

$$\partial_x (Y^y \partial_y h^I) = (\nabla_x Y^y)(\partial_y h^I) + \frac{2}{3} g_{xy} h^I Y^y \ . \tag{115}$$

Rewriting (87) in local coordinates we obtain the relation

$$\nabla_x Y^y = D_x Y^y + \frac{3}{2} C_{xz}^y Y^z \tag{116}$$

between the connections D and ∇ evaluated on tangent vectors X, Y, where

$$C_{xyz} := C_{IJK} \partial_x h^I \partial_y h^J \partial_z h^K \ . \tag{117}$$

is the pullback of the trilinear form to $\mathcal{H}$. Combining (115) and (116) we obtain

$$\begin{aligned}
&\partial_x(Y^y\partial_y h^I) = (D_xY^y)(\partial_y h^I) + Y^y D_x\partial_y h^I = (\nabla_x Y^y)(\partial_y h^I) + \frac{2}{3}g_{xy}h^I Y^y \\
\Rightarrow\quad &Y^y D_x\partial_y h^I = (\nabla_x Y^y - D_x Y^y)\partial_y h^I + h^I g_{xy}Y^y = \frac{3}{2}C^z_{xy}Y^y\partial_z h^I + \frac{2}{3}h^I g_{xy}Y^y \\
\Rightarrow\quad &D_x\partial_y h^I = \frac{3}{2}C^z_{xy}\partial_z h^I + \frac{2}{3}h^I g_{xy}\ . \qquad (118)
\end{aligned}$$

The corresponding formula (2.16) in [18] is

$$D_x h^I_y = -\sqrt{\frac{2}{3}}\left(g_{xy}h^I + T^z_{xy}h^I_z\right) \Leftrightarrow D_x\partial_y h^I = \frac{2}{3}g_{xy}h^I - \sqrt{\frac{2}{3}}T^z_{xy}\partial_z h^I\ . \qquad (119)$$

Matching with our formula requires

$$\frac{3}{2}C_{xyz} = -\sqrt{\frac{2}{3}}T_{xyz} \Rightarrow T_{xyz} = -\left(\frac{3}{2}\right)^{3/2} C_{IJK}\partial_x h^I\partial_y h^J\partial_z h^K\ . \qquad (120)$$

The constant tensor C_{IJK} on M can be decomposed as

$$C_{IJK} = \frac{5}{2\mathcal{V}^2}h_I h_J h_K + \frac{3}{2}a_{(IJ}h_{K)} + \frac{1}{\mathcal{V}^2}T_{xyz}h^x_I h^y_J h^z_K\ . \qquad (121)$$

To verify this decomposition we contract C_{IJK} with the vectors of the frame $\xi = h^I\partial_I = \partial_\rho$ and $\partial_x = \partial_x h^I\partial_I$.

- Contraction with three tangent vectors gives precisely the pullback of C_{IJK} to $\mathcal{H}_\mathfrak{c}$

$$C_{IJK}\partial_x h_I\partial_y h_J\partial_z h_K = C_{xyz} = C(\partial_x, \partial_y, \partial_z)\ . \qquad (122)$$

- Contracting once with the Euler field ξ we obtain the two-form $C(\xi, \cdot, \cdot)$ with components $C_{IJK}h^K$ on the left hand side. When applying the same contraction on the right hand side the third term does not contribute, and the contributions from the first and second term combine in $C_{IJK}h^K$.

We remark that the corresponding formula (2.12) of [18] is recovered for $\mathcal{V} = 1$. In [18] one can also find expressions for the curvature tensors of the CASR metric g_M and of the PSR metric $g_{\mathcal{H}}$, but we will not need these for our applications.

3. Five-dimensional vector multiplets

3.1. Rigid vector multiplets

In this section we present rigid five-dimensional vector multiplets, focussing on the bosonic part of the Lagrangian. We follow [19], where an off-shell realization has been worked out, based on the work of [20] on the superconformal case. The components of a five-dimensional rigid off-shell vector multiplet are

$$(A_\mu, \lambda^i, \sigma, Y^{ij}) \ , \tag{123}$$

where $\mu = 0, 1, 2, 3, 4$ is the Lorentz index, and $i, j = 1, 2$ is an internal index, transforming in the fundamental representation of the R-symmetry group $SU(2)_R$. R-symmetry indices i, j are raised and lowered using

$$(\varepsilon_{ij}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{124}$$

and $\varepsilon^{ij} := \varepsilon_{ij}$.[14] A_μ is a vector field, λ^i, $i = 1, 2$ is an $SU(2)_R$ doublet of symplectic Majorana spinors, σ is a real scalar, and $Y^{ij} = Y^{ji}$ are auxiliary fields, subject to the reality condition

$$(Y^{ij})^* = Y^{kl}\varepsilon_{ki}\varepsilon_{lj} = Y_{ij} \ . \tag{125}$$

Thus Y^{ij} has three independent real components. Taking into account the reality conditions, a vector multiplet has 8 bosonic and 8 fermionic off-shell degrees of freedom. These reduce to $4 + 4$ on-shell degrees of freedom upon imposing the equations of motion.

We consider an arbitrary number of vector multiplets, labelled by $I = 1, \ldots, n$. The bosonic part of the Lagrangian worked out in [19] is

$$\begin{aligned} L &= h_{IJ}\left(-\frac{1}{2}\partial_\mu\sigma^I\partial^\mu\sigma^J - \frac{1}{4}F^I_{\mu\nu}F^{J\,\mu\nu} + Y^I_{ij}Y^{J\,ij}\right) \\ &\quad -h_{IJK}\frac{1}{24}\epsilon^{\mu\nu\lambda\rho\sigma}A^I_\mu F^J_{\nu\lambda}F^K_{\rho\sigma} \ . \end{aligned} \tag{126}$$

Here h_I, h_{IJ}, h_{IJK} denote derivatives of a function h of the scalar fields σ^I,

$$h_I = \partial_I h \ , \quad h_{IJ} = \partial^2_{I,J} h \ , \quad h_{IJK} = \partial^2_{I,J,K} h \ . \tag{127}$$

Since the Chern-Simons term must be gauge invariant up to boundary terms, h_{IJK} must be constant, which implies that h must be a cubic polynomial. The special case where h is a quadratic polynomial corresponds to a free theory, while lower degrees of h lead to degenerate kinetic terms and can be discarded.

[14] Note that (ε^{ij}) is minus the inverse of (ε_{ij}). This choice is consistent with the NW-SE convention for the $SU(2)_R$ indices.

Thus the scalar manifold of a theory of five-dimensional rigid vector multiplets is an affine special real manifold, as defined in section 2.5, see Definition 11.

We remark that compared to [19] we have changed the definition of the ϵ-tensor by a sign, but we have kept the relation $\gamma_{\mu\nu\rho\sigma\tau} = +i\epsilon_{\mu\nu\rho\sigma\tau}\mathbb{1}$, which determines the sign of the Chern-Simons term, by simultaneously changing the representation of the Clifford algebra. We refer to [21] for a systematic discussion of the relative factors and signs between the terms in the supersymmetry variations and in the Lagrangians of five-dimensional vector multiplets. Note that in [21], the same convention $\epsilon_{01235} = 1$ for the ϵ-tensor was used as in this review, but in combination with a different sign in the relation between $\gamma_{\mu\nu\rho\sigma\tau}$ and the ϵ-tensor (that is, $\gamma_{\mu\nu\rho\sigma\tau} = -i\epsilon_{\mu\nu\rho\sigma\tau}\mathbb{1}$) this resulted in a Chern-Simon term with opposite sign compared to (126). The choices made in this review are more convenient for matching with the supergravity literature.

3.2. Rigid superconformal vector multiplets

We next specialize to the case where the vector multiplet theory is superconformal, following [20]. Superconformal invariance implies the Hesse potential must be a *homogeneous* cubic polynomial, which makes the scalar manifold a conical affine special real manifold in the sense of Definition 12. For later convenience we choose the Hesse potential

$$h = -\frac{1}{2} C_{IJK}\sigma^I\sigma^J\sigma^K \ , \tag{128}$$

where C_{IJK} are constants. Then

$$h_{IJ} = -3C_{IJK}\sigma^K \ , \quad h_{IJK} = -3C_{IJK} \ , \tag{129}$$

and the rigid superconformal vector multiplet Lagrangian is:

$$\begin{aligned} L \;\; = \;\; & 3C_{IJK}\sigma^K \left(\frac{1}{4} F^I_{\mu\nu} F^{J\,\mu\nu} + \frac{1}{2}\partial_\mu\sigma^I\partial^\mu\sigma^J - Y^I_{ij}Y^{Jij} \right) \\ & + \frac{1}{8}\epsilon^{\mu\nu\rho\sigma\lambda} C_{IJK} A^I_\mu F^J_{\nu\rho} F^K_{\sigma\lambda} \ , \end{aligned} \tag{130}$$

where we omitted all fermionic terms.

3.3. Superconformal matter multiplets coupled to superconformal gravity

We will follow the superconformal approach to construct a theory of n vector multiplets coupled to Poincaré supergravity. A comprehensive review of the superconformal approach can be found in the textbook [22], and the elements relevant for this review have been collected in B.5. The superconformal approach is based on the observation that a theory of n vector multiplets and n_H hypermultiplets coupled to Poincaré supergravity is *gauge equivalent* to a theory of $n+1$ superconformal vector multiplets and n_H+1 superconformal hypermultiplets coupled to conformal supergravity. Gauge equivalence means that the Poincaré supergravity theory is obtained from the superconformal theory by gauge fixing those superconformal symmetries that do not belong to the Poincaré supersymmetry algebra. Conversely, a Poincaré supergravity theory can be extended to a superconformal theory by adding one vector and one hypermultiplet which act as superconformal compensators. That is, the additional symmetries are introduced by adding new degrees of freedom.

3.3.1. Coupling of vector multiplets

The bosonic Lagrangian for a rigid superconformal vector multiplet theory was given in (130). Since we need to start with $n+1$ superconformal vector multiplets we change the range of the indices $I, J, \ldots$ to $I, J = 0, 1, \ldots, n$. The next step is to promote the superconformal symmetry to a local symmetry, and to add at least one hypermultiplet. Gauging the superconformal symmetry involves replacing partial derivatives by superconformal covariant derivatives, which contain the superconformal connections, or, in physics terminology, the superconformal gauge fields. The superconformal gauge fields belong to the so-called Weyl multiplet, together with certain auxiliary fields. We refer to B.5 for an overview. Our presentation will follow [23], but we will only retain the connections and auxiliary fields which are relevant for the bosonic vector multiplet Lagrangian. The bosonic part of the locally superconformally invariant vector multiplet Lagrangian can be brought to the form

$$\begin{aligned} L_V \quad = \quad & 3C_{IJK}\sigma^K \left[\frac{1}{2}\mathcal{D}_\mu \sigma^I \mathcal{D}^\mu \sigma^J + \frac{1}{4} F^I_{\mu\nu} F^{\mu\nu J} - Y^I_{ij} Y^{ijJ} - 3\sigma^I F^J_{\mu\nu} T^{\mu\nu} \right] \\ & + \frac{1}{8} C_{IJK} e^{-1} \epsilon^{\mu\nu\rho\sigma\tau} A^I_\mu F^J_{\nu\rho} F^K_{\sigma\tau} \\ & + C_{IJK}\sigma^I \sigma^J \sigma^K \left(\frac{1}{8} R + 4D + \frac{39}{2} T_{\mu\nu} T^{\mu\nu} \right) \ . \end{aligned} \tag{131}$$

Here

$$\mathcal{D}_\mu \sigma^I = (\partial_\mu - b_\mu)\sigma^I \ , \tag{132}$$

where b_μ is the gauge field for dilatations. $T_{\mu\nu}$ and D are auxiliary fields belonging to the Weyl multiplet. In the so-called K-gauge, to be introduced below, R becomes the Ricci scalar associated to the space-time metric $g_{\mu\nu}$ with vielbein e_μ^a and vielbein determinant e. We refer to B.5 for details regarding the vielbein ad Ricci scalar. In (131) we have adapted the Lagrangian of [23] to our conventions. This changes the sign in front of the Ricci tensor and removes a factor $-i$ from the Chern-Simons term.[15]

3.3.2. Coupling of hypermultiplets

The bosonic part of the locally superconformal hypermultiplet Lagrangian is

$$L_H = -\frac{1}{2}\varepsilon^{ij}\Omega_{\alpha\beta}\mathcal{D}_\mu A_i^\alpha A_j^\beta + \chi\left(-\frac{3}{16}R + 2D + \frac{3}{4}T_{\mu\nu}T^{\mu\nu}\right) . \quad (133)$$

Here A_i^α, where $\alpha = 1, \ldots, 2n_H + 2$ and $i, j = 1, 2$ encode the $4n_H + 4$ scalar degrees of freedom of the hypermultiplets. The quantity χ is the so-called *hyper-Kähler potential* and satisfies

$$\varepsilon_{ij}\chi = \Omega_{\alpha\beta}A_i^\alpha A_j^\beta . \quad (134)$$

We refer to B.5 for explicit expressions for the covariant derivative $\mathcal{D}_\mu A_i^\alpha$ and the quantity $\Omega_{\alpha\beta}$. The scalar geometry of rigid hypermultiplets is hyper-Kähler. If superconformal symmetry is imposed the scalar multiplet is a hyper-Kähler cone, that is, it admits a holomorphic and homothetic action of the group $\mathbb{H}^*$ of invertible quaternions. The relevant concepts of hyper-Kähler geometry are briefly reviewed in A.21.

[15] Note that [23] use an imaginary totally antisymmetric tensor defined by $\varepsilon_{01235} = i = i\epsilon_{01235}$. Taking this into account the relation which determines the sign of the Chern-Simons term is the same: $\gamma_{\mu\nu\rho\sigma\tau} = \varepsilon_{\mu\nu\rho\sigma\tau}\mathbb{1} = i\epsilon_{\mu\nu\rho\sigma\tau}\mathbb{1}$.

3.3.3. Poincaré supergravity

Combining the bosonic vector multiplet and hypermultiplet Lagrangians, we obtain:

$$\begin{aligned} L &= 3C_{IJK}\sigma^K \left[\frac{1}{2}\mathcal{D}_\mu \sigma^I \mathcal{D}^\mu \sigma^J + \frac{1}{4} F^I_{\mu\nu} F^{\mu\nu J} - Y^I_{ij} Y^{ijJ} - 3\sigma^I F^J_{\mu\nu} T^{\mu\nu} \right] \\ &+ \frac{1}{8} C_{IJK} e^{-1} \epsilon^{\mu\nu\rho\sigma\tau} A^I_\mu F^J_{\nu\rho} F^K_{\sigma\tau} \\ &+ \frac{1}{8} R \left(C_{IJK}\sigma^I\sigma^J\sigma^K - \frac{3}{2}\chi \right) + D\left(2\chi + 4C_{IJK}\sigma^I\sigma^J\sigma^K\right) \\ &+ T^{ab}T_{ab}\left(\frac{3}{4}\chi + \frac{39}{2} C_{IJK}\sigma^I\sigma^J\sigma^K \right) \\ &- \frac{1}{2}\varepsilon^{ij}\Omega_{\alpha\beta}\mathcal{D}_\mu \mathcal{A}^a_i \mathcal{D}^\mu \mathcal{A}^\beta_j \ . \end{aligned} \tag{135}$$

The auxiliary field Y^I_{ij} has the field equation $Y^I_{ij} = 0$ and can be eliminated trivially. The algebraic field equation for the auxiliary field D can be used to eliminate χ:

$$\chi = -2C_{IJK}\sigma^I\sigma^J\sigma^K \ . \tag{136}$$

Substituting this back into the Lagrangian, we obtain

$$\begin{aligned} L &= 3C_{IJK}\sigma^K \left[\frac{1}{2}\mathcal{D}_\mu \sigma^I \mathcal{D}^\mu \sigma^J + \frac{1}{4} F^I_{\mu\nu} F^{\mu\nu J} - 3\sigma^I F^J_{\mu\nu} T^{\mu\nu} \right] \\ &+ \frac{1}{8} C_{IJK} e^{-1} \epsilon^{\mu\nu\rho\sigma\tau} A^I_\mu F^J_{\nu\rho} F^K_{\sigma\tau} \\ &+ \frac{1}{2} R C_{IJK}\sigma^I\sigma^J\sigma^K \\ &+ 18 T^{ab}T_{ab} C_{IJK}\sigma^I\sigma^J\sigma^K \\ &- \frac{1}{2}\varepsilon^{ij}\Omega_{\alpha\beta}\mathcal{D}_\mu \mathcal{A}^a_i \mathcal{D}^\mu \mathcal{A}^\beta_j \ . \end{aligned} \tag{137}$$

In the next step we gauge-fix those superconformal transformations which are not super-Poincaré transformations. Local dilatations are gauge-fixed by the so-called D-gauge which imposes that the Einstein-Hilbert term acquires its canonical form:

$$C_{IJK}\sigma^I\sigma^J\sigma^K = \kappa^{-2} \ , \tag{138}$$

where $\kappa = \sqrt{8\pi G_N}$ is the gravitational coupling constant and G_N is Newton's gravitational constant. This implies that $\chi = -2\kappa^{-2}$, which because of (134) removes one real scalar degree of freedom from the hypermultiplet sector. The superconformal symmetries include an $SU(2)$ symmetry which acts in the adjoint representation on the hypermultiplet scalars. Gauge fixing this symmetry removes another three real scalar degrees of freedom. If we consider only one hypermultiplet at the superconformal level, i.e. $n_H = 0$, then all bosonic hypermultiplet degrees of freedom are removed and we can drop the last line in (137).[16] Since we are interested in the vector multiplet Lagrangian, we will assume this here. Note that since $\chi \neq 0$, consistency of the procedure requires that at least one superconformal hypermultiplet is present. This hypermultiplet is needed as a superconformal compensator.

For completeness we briefly mention what happens for $n_H > 0$. The gauge fixing removes one hypermultiplet, leaving a theory with n_H hypermultiplets. The resulting scalar manifold of dimension $4n_H$ is a quaternion-Kähler manifold. It was shown in [24] that the scalar geometry of hypermultiplets coupled to supergravity is quaternion-Kähler. In the superconformal approach the quaternion-Kähler manifold arises as the superconformal quotient of a hyper-Kähler cone [25, 26, 27, 28]. We remark that the hypermultiplet Lagrangian only couples gravitationally to the vector multiplet Lagrangian, and thus can always be truncated out consistently. We now return to the case $n_H = 0$.

The special superconformal transformations are gauge-fixed by the so-called K-gauge, which eliminates the dilatation gauge field: $b_\mu = 0$. This replaces the covariant derivatives $\mathcal{D}_\mu \sigma^I$ by partial derivatives $\partial_\mu \sigma^I$. Then the bosonic Lagrangian is

$$\begin{aligned} L &= 3C_{IJK}\sigma^K \left(\frac{1}{2}\partial_\mu \sigma^I \partial^\mu \sigma^J + \frac{1}{4}F^I_{\mu\nu}F^{J\mu\nu} - 3\sigma^I F^J_{\mu\nu}T^{\mu\nu} \right) \\ &\quad + \frac{1}{8}C_{IJK}e^{-1}\epsilon^{\mu\nu\rho\sigma\tau}A^I_\mu F^J_{\nu\rho}F^K_{\sigma\tau} \\ &\quad + \frac{1}{2\kappa^2}R + \frac{18}{\kappa^2}T_{\mu\nu}T^{\mu\nu} \ . \end{aligned} \tag{139}$$

Now we eliminate the auxiliary field $T_{\mu\nu}$ using its algebraic equation of motion

$$T_{\mu\nu} = \frac{\kappa^2}{4}C_{IJK}\sigma^I \sigma^J F^K_{\mu\nu} \ , \tag{140}$$

resulting in

$$\begin{aligned} L &= \frac{3}{2}C_{IJK}\sigma^K \partial_\mu \sigma^I \partial^\mu \sigma^J + \frac{1}{2\kappa^2}R + \frac{1}{8}C_{IJK}e^{-1}\epsilon^{\mu\nu\rho\sigma\tau}A^I_\mu F^J_{\nu\rho}F^K_{\sigma\tau} \\ &\quad - \frac{3}{8}(-2C_{IJK}\sigma^K + 3\kappa^2 C_{IAB}\sigma^A\sigma^B C_{JCD}\sigma^C\sigma^D)F^I_{\mu\nu}F^{J\mu\nu} \ . \end{aligned} \tag{141}$$

[16] As required for consistency, gauge fixing fermionic superconformal symmetries removes the fermionic partners of the four hypermultiplet scalars.

The scalar fields σ^I and couplings C_{IJK} are dimensionful. We define dimensionless scalars h^I and couplings $\mathcal{C}_{IJK}$ by

$$h^I := \kappa \sigma^I \ , \quad \mathcal{C}_{IJK} := \frac{1}{\kappa} C_{IJK} \ . \tag{142}$$

The scalars h^I satisfy

$$\mathcal{C}_{IJK} h^I h^J h^K = 1 \ . \tag{143}$$

It is convenient to define

$$h_I = \mathcal{C}_{IJK} h^J h^K \ . \tag{144}$$

In these new variables the Lagrangian becomes

$$\begin{aligned} L \quad = \quad & \frac{1}{2\kappa^2} R + \frac{3}{2\kappa^2} \mathcal{C}_{IJK} h^K \partial_\mu h^I \partial^\mu h^J \\ & - \frac{3}{8} \left(-2 \mathcal{C}_{IJK} h^K + 3 h_I h_J \right) F^I_{\mu\nu} F^{J\mu\nu} \\ & + \frac{\kappa}{8} \mathcal{C}_{IJK} e^{-1} \epsilon^{\mu\nu\rho\sigma\tau} A^I_\mu F^J_{\nu\rho} F^K_{\sigma\tau} \ . \end{aligned} \tag{145}$$

We would like to verify that this Lagrangian, which has been obtained using the superconformal approach, agrees with the bosonic part of the on-shell Poincaré supergravity Lagrangian constructed in [18]. The scalars h^I already have the same normalization. Following [18] we define

$$\overset{\circ}{a}_{IJ} := -2 \mathcal{C}_{IJK} h^K + 3 h_I h_J \ , \tag{146}$$

and note that

$$h_I = \overset{\circ}{a}_{IJ} \ h^J \ . \tag{147}$$

To obtain the same normalization of the vector fields as in [18] we define

$$\tilde{A}^I_\mu := \sqrt{\frac{3}{2}} A^I_\mu \ . \tag{148}$$

We also note that the scalar fields h^I are not independent, because they satisfy the constraint (143). This implies that $h_I \partial_\mu h^I = 0$. Using this, the bosonic Lagrangian takes the form

$$\begin{aligned} L \quad = \quad & \frac{1}{2\kappa^2} R - \frac{3}{4\kappa^2} \ \overset{\circ}{a}_{IJ} \ \partial_\mu h^I \partial^\mu h^J - \frac{1}{4} \ \overset{\circ}{a}_{IJ} \ \tilde{F}^I_{\mu\nu} \tilde{F}^{J\mu\nu} \\ & + \frac{\kappa}{6\sqrt{6}} e^{-1} \mathcal{C}_{IJK} \epsilon^{\mu\nu\rho\sigma\lambda} \tilde{A}^I_\mu \tilde{F}^J_{\nu\rho} \tilde{F}^K_{\sigma\lambda} \ . \end{aligned} \tag{149}$$

Finally, we introduce independent scalars ϕ^x, $x = 1, \ldots, n$ by solving the constraint (143). The metric g_{xy} for the target space of the scalars ϕ^x is obtained by re-writing the scalar term in the Lagrangian. The normalization chosen in [18] is such that

$$-\frac{1}{2}g_{xy}\partial_\mu\phi^x\partial^\mu\phi^y = -\frac{3}{4}\overset{\circ}{a}_{IJ}\,\partial_\mu h^I\partial^\mu h^J = -\frac{3}{4}\overset{\circ}{a}_{IJ}\,\frac{\partial h^I}{\partial\phi^x}\frac{\partial h^J}{\partial\phi^y}\partial_\mu\phi^x\partial^\mu\phi^y \ . \quad (150)$$

The resulting Lagrangian,

$$\begin{aligned} L &= \frac{1}{2\kappa^2}R - \frac{1}{2\kappa^2}g_{xy}\partial_\mu\phi^x\partial^\mu\phi^y - \frac{1}{4}\overset{\circ}{a}_{IJ}\,\tilde{F}^I_{\mu\nu}\tilde{F}^{J\mu\nu} \\ &\quad + \frac{\kappa}{6\sqrt{6}}e^{-1}\mathcal{C}_{IJK}\epsilon^{\mu\nu\rho\sigma\lambda}\tilde{A}^I_\mu\tilde{F}^J_{\nu\rho}\tilde{F}^K_{\sigma\lambda} \ , \end{aligned} \quad (151)$$

agrees with the corresponding terms in (2.7) of [18] upon setting $\kappa = 1$, and taking into account a relative sign in the definition of the Riemann tensor. When setting $\kappa = 1$ we see that g_{xy} is the PSR metric $g_{\mathcal{H}}$ associated with the Hesse potential $\mathcal{V} = \mathcal{C}_{IJK}h^Ih^Jh^K$ and $\overset{\circ}{a}_{IJ}$ the restriction of the corresponding 0-conical metric $g^{(1)}_M = a_{IJ}dh^Idh^J$ to $\mathcal{H} = \{\mathcal{V} = 1\}$, with the same normalization as in section 2.6. Note that for $\kappa = 1$ we have $h^I = \sigma^I$ and $C_{IJK} = \mathcal{C}_{IJK}$.

The decomposition (109) of a_{IJ} can be used to rewrite the Maxwell term[17]

$$\begin{aligned} -\frac{1}{4}\overset{\circ}{a}_{IJ}\,F^I_{\mu\nu}F^{J\mu\nu} &= -\frac{1}{4}g_{xy}h^x_Ih^y_JF^I_{\mu\nu}F^{J\mu\nu} - \frac{1}{4}h_Ih_JF^I_{\mu\nu}F^{J\mu\nu} \\ = \quad & -\frac{1}{4}g_{xy}\mathcal{F}^x_{\mu\nu}\mathcal{F}^{y\mu\nu} - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \ , \end{aligned} \quad (152)$$

where we have defined

$$\mathcal{F}_{\mu\nu} = h_IF^I_{\mu\nu} \ , \quad \mathcal{F}^x_{\mu\nu} = h^x_IF^I_{\mu\nu} \ . \quad (153)$$

The n field strengths $\mathcal{F}^x_{\mu\nu}$ belong to vector fields $\mathcal{A}^x_\mu$ which are the superpartners of the scalars ϕ^x under Poincaré supersymmetry. The additional field strength $\mathcal{F}_{\mu\nu}$ belongs to a vector field which is part of the Poincaré supergravity multiplet. In contrast, the $F^I_{\mu\nu}$ correspond to vector fields in the $n+1$ superconformal vector multiplets. Thus the decomposition into components tangential and orthogonal to $\mathcal{H}$ corresponds to mapping components of superconformal multiplets to the corresponding Poincaré vector multiplets. In the superconformal description there is a manifest linear action of the group $GL(n+1,\mathbb{R})$ on the field strength $F^I_{\mu\nu}$, and an associated action of the affine group $GL(n+1,\mathbb{R}) \ltimes \mathbb{R}^{n+1}$ on the scalars h^I. In the gauge-fixed description this is no longer manifest, because

[17] We set $\mathcal{V} = 1$.

there are only n independent scalars, but $n+1$ vector fields. For this reason it is often advantageous to work in the superconformal formulation of the theory.

We can now decide which signature we should choose for the CASR metric defining the superconformal theory. In the Poincaré theory, $\overset{\circ}{a}_{IJ}$ and g_{xy} must be positive definite, in order that the vector and scalar fields have positive kinetic energy. From the above decomposition of the Maxwell term it is clear that $\overset{\circ}{a}_{IJ}$ is positive definite if and only if g_{xy} is positive definite. Using the relations (66) – (71) between the metrics, we see that h_{IJ} must have Lorentz signature with the time-like direction along the integral lines of the Euler field ξ.[18] The direction normal to $\mathcal{H}$ corresponds to the extra 'compensating' vector multiplet, which shows that the kinetic term of the compensator has a flipped sign.

3.4. R^2-terms in five dimensions

We briefly describe the coupling of vector multiplets to R^2-interactions encoded in the square of the Weyl multiplet using the superconformal approach [29, 23].

The bosonic part of the Lagrangian containing the higher-derivative couplings reads, in the notation used in [23],[19]

$$\begin{aligned} L_{R^2} \;=\; & \tfrac{1}{64}\, c_I\, \sigma^I\, R_{ab}{}^{cd}(M) R_{cd}{}^{ab}(M) \\ & - \tfrac{3}{16} c_I \left(10\sigma^I T_{ab} - F^I_{ab}\right)\, R_{cd}{}^{ab}\, T^{cd} + \tfrac{3}{2} c_I \sigma^I\, T^{ab} [\mathcal{D}^c, \mathcal{D}_a] T_{bc} \\ & - c_I\, \sigma^I\, R_{ab} \left(T^{ac} T^b{}_c - \tfrac{1}{2} \eta^{ab} T^{cd} T_{cd}\right) + \dots\,, \end{aligned} \tag{154}$$

where R_{ab} denotes the Ricci tensor (B.32), and where we have only displayed the terms that are relevant for computing Wald's entropy of static BPS black holes, see section 9. We refer to [23] for the complete set of bosonic terms. The c_I denote arbitrary real constants.

Using (B.96) and (B.98), we obtain

$$R_{ab}{}^{cd}(M) = R_{ab}{}^{cd} - \tfrac{4}{3} \left(R_{[a}{}^{[c} - \tfrac{1}{8} R\, \delta_{[a}{}^{[c} \right) \delta_{b]}{}^{d]}\,, \tag{155}$$

which, in the K-gauge $b_\mu = 0$, denotes the Weyl tensor in five dimensions.

For future reference, we collect the bosonic terms in the R^2-corrected Lagrangian that are relevant for computing the entropy of static BPS black holes

[18] As we have seen, the overall sign of the CASR metric is not relevant.

[19] Note that our definition of the Riemann tensor differs from the one in [23] by an overall minus sign.

using Wald's definition of black hole entropy (B.134),

$$\begin{aligned} L &= 3\,C_{IJK}\,\sigma^K\left[\tfrac{1}{2}\,\mathcal{D}_\mu\sigma^I\mathcal{D}^\mu\sigma^J+\tfrac{1}{4}F^I_{\mu\nu}F^{\mu\nu J}-3\,\sigma^I F^J_{\mu\nu}\,T^{\mu\nu}\right]\\ &\quad+\tfrac{1}{8}\,R\left(-\tfrac{3}{2}\,\chi+C_{IJK}\,\sigma^I\sigma^J\sigma^K\right)\\ &\quad+T_{ab}T^{ab}\left(\tfrac{3}{4}\,\chi+\tfrac{39}{2}\,C_{IJK}\,\sigma^I\sigma^J\sigma^K\right)\\ &\quad+\tfrac{1}{64}\,c_I\,\sigma^I\,R_{ab}{}^{cd}(M)R_{cd}{}^{ab}(M)\\ &\quad-\tfrac{3}{16}c_I\left(10\sigma^I T_{ab}-F^I_{ab}\right)\,R_{cd}{}^{ab}\,T^{cd}+\tfrac{3}{2}c_I\sigma^I\,T^{ab}[\mathcal{D}^c,\mathcal{D}_a]T_{bc}\\ &\quad-c_I\,\sigma^I\,R_{ab}\left(T^{ac}T^b{}_c-\tfrac{1}{2}\eta^{ab}T^{cd}T_{cd}\right)\;. \end{aligned} \tag{156}$$

4. Electric-magnetic duality

Electric-magnetic duality in four dimensions is a characteristic feature of Maxwell's equations in vacuum. It describes the invariance of the combined system of equations of motion and Bianchi identities for the Maxwell gauge field A_μ under rotations of the electric field into the magnetic field and vice-versa. Electric-magnetic duality is also present in $\mathcal{N}=2$ supergravity theories coupled to abelian $\mathcal{N}=2$ vector multiplets in four dimensions [30, 31], and continues to hold when allowing for the coupling to a chiral background η [32]. Theories of this type are based on holomorphic functions $F(X,\eta)$, and electric-magnetic duality is defined in terms of a symplectic vector constructed from $F(X,\eta)$. This will be reviewed in the following subsections.

Non-holomorphic functions F are also of relevance and occur in various types of models [33, 34]. We will discuss three applications thereof, namely to point-particle Lagrangians that depend on coordinates and velocities, as well as on parameters η, in section 4.1 below, to topological string theory in section 7, and to the Born-Infeld-dilaton-axion system in section 11.

We begin by reviewing the formulation of point-particle Lagrangians in terms of a function F given in (157) below, following [33]. When passing over to the Hamiltonian description, one obtains a description based on a real Hesse potential associated to F. In this context, canonical transformations on phase space play a similar role to electric-magnetic duality transformations in Maxwell-type theories. Then we turn to electric-magnetic duality in Maxwell-type theories at the two-derivative level which arise in the $\mathcal{N}=2$ supergravity context, and subsequently we allow for the presence of a chiral background.

4.1. Point-particle models and F-functions

In the following, we will review [33] how general point-particle Lagrangians (that depend on coordinates and velocities, as well as on real parameters η) can be recast in terms of a function F of the form

$$F(x,\bar{x},\eta) = F^{(0)}(x) + 2i\,\Omega(x,\bar{x},\eta)\ , \tag{157}$$

where Ω is real. This is achieved with the help of a theorem that states that the dynamics of these models can be reformulated in terms of a symplectic vector $(X, \partial F/\partial X)$ constructed out of a complex function F of the form (157), whose real part comprises the canonical variables of the associated Hamiltonian.

Let us consider a point-particle model described by a Lagrangian L with n coordinates ϕ^i and n velocities $\dot{\phi}^i$. The associated canonical momenta $\partial L/\partial\dot{\phi}^i$ will be denoted by π_i. The Hamiltonian H of the system, which follows from L by Legendre transformation,

$$H(\phi,\pi) = \dot{\phi}^i\,\pi_i - L(\phi,\dot{\phi})\ , \tag{158}$$

depends on (ϕ^i, π_i), which are called canonical variables, since they satisfy the canonical Poisson bracket relations. The variables (ϕ^i,π_i) can be interpreted as local coordinates on a symplectic manifold called the classical phase space of the system. In these coordinates, the symplectic 2-form is $d\pi_i \wedge d\phi^i$. This 2-form is preserved under canonical transformations of (ϕ^i,π_i) given by

$$\begin{pmatrix} \phi^i \\ \\ \pi_i \end{pmatrix} \mapsto \begin{pmatrix} \tilde{\phi}^i \\ \\ \tilde{\pi}_i \end{pmatrix} = \begin{pmatrix} U^i{}_j & Z^{ij} \\ \\ W_{ij} & V_i{}^j \end{pmatrix} \begin{pmatrix} \phi^j \\ \\ \pi_j \end{pmatrix}\ , \tag{159}$$

where U, V, Z and W denote $n \times n$ matrices that satisfy the relations

$$\begin{aligned} U^T\,V - W^T\,Z = V^T\,U - Z^T\,W = \mathbb{1}\ , \\ U^T\,W = W^T\,U \quad , \quad Z^T\,V = V^T\,Z\ . \end{aligned} \tag{160}$$

Thus, the transformation (159) constitutes an element of $\mathrm{Sp}(2n,\mathbb{R})$. This transformation leaves the Poisson brackets invariant. The Hamiltonian transforms as a function under symplectic transformations, i.e. $\tilde{H}(\tilde{\phi},\tilde{\pi}) = H(\phi,\pi)$. When the Hamiltonian is invariant under a subset of $\mathrm{Sp}(2n,\mathbb{R})$ transformations, this subset describes a symmetry of the system. This invariance is often called duality invariance.

Now we give the theorem of [33] that states that the Lagrangian $L(\phi,\dot\phi)$ can be reformulated in terms of a complex function $F(x,\bar x)$ based on complex variables x^i, such that the canonical coordinates (ϕ^i,π_i) coincide with (twice) the real part of (x^i,F_i), where $F_i=\partial F(x,\bar x)/\partial x^i$.

Theorem 1. Point-particle Lagrangians and F-functions. *Given a Lagrangian $L(\phi,\dot\phi)$ depending on n coordinates ϕ^i and n velocities $\dot\phi^i$, with corresponding Hamiltonian $H(\phi,\pi)=\dot\phi^i\,\pi_i-L(\phi,\dot\phi)$, there exists a description in terms of complex coordinates $x^i=\frac{1}{2}(\phi^i+i\dot\phi^i)$ and a complex function $F(x,\bar x)$, such that,*

$$\begin{aligned} 2\,\mathrm{Re}\,x^i &= \phi^i\,,\\ 2\,\mathrm{Re}\,F_i(x,\bar x) &= \pi_i\,,\quad \textit{where}\;\; F_i=\frac{\partial F(x,\bar x)}{\partial x^i}\,. \end{aligned} \tag{161}$$

The function $F(x,\bar x)$ can be decomposed as

$$F(x,\bar x)=F^{(0)}(x)+2i\,\Omega(x,\bar x)\,, \tag{162}$$

where Ω is real. The decomposition (162) *may be subjected to the following equivalence transformation,*

$$F^{(0)}\mapsto F^{(0)}+g(x)\,,\qquad \Omega\mapsto\Omega-\mathrm{Im}\,g(x)\,, \tag{163}$$

which results in $F(x,\bar x)\mapsto F(x,\bar x)+\bar g(\bar x)$, and which leaves (x^i,F_i) invariant. The Lagrangian and Hamiltonian can then be expressed in terms of $F^{(0)}$ and Ω as,

$$\begin{aligned} L &= 4[\mathrm{Im}\,F-\Omega]\,, \qquad (164)\\ H &= -\,\mathrm{i}(x^i\,\bar F_{\bar\imath}-\bar x^{\bar\imath}\,F_i)-4\,\mathrm{Im}[F-\tfrac12 x^i\,F_i]+4\,\Omega\\ &= -\,\mathrm{i}(x^i\,\bar F_{\bar\imath}-\bar x^{\bar\imath}\,F_i)-4\,\mathrm{Im}[F^{(0)}-\tfrac12 x^i\,F^{(0)}_i]-2(2\,\Omega-x^i\Omega_i-\bar x^{\bar\imath}\Omega_{\bar\imath})\,, \end{aligned}$$

with $F_i=\partial F/\partial x^i, F^{(0)}_i=\partial F^{(0)}/\partial x^i, \Omega_i=\partial\Omega/\partial x^i$, and similarly for $\bar F_{\bar\imath},\bar F^{(0)}_{\bar\imath}$ and $\Omega_{\bar\imath}$.

Furthermore, the $2n$-vector (x^i,F_i) denotes a complexification of the phase space coordinates (ϕ^i,π_i) and transforms precisely as (ϕ^i,π_i) under symplectic transformations, i.e.

$$\begin{pmatrix} x^i\\ F_i(x,\bar x)\end{pmatrix}\mapsto\begin{pmatrix}\tilde x^i\\ \tilde F_i(\tilde x,\bar{\tilde x})\end{pmatrix}=\begin{pmatrix} U^i{}_j & Z^{ij}\\ W_{ij} & V_i{}^j\end{pmatrix}\begin{pmatrix} x^j\\ F_j(x,\bar x)\end{pmatrix}\,. \tag{165}$$

The equations (165) are integrable: the symplectic transformation yields a new function $\tilde F(\tilde x,\bar{\tilde x})=\tilde F^{(0)}(\tilde x)+2i\,\tilde\Omega(\tilde x,\bar{\tilde x})$, with $\tilde\Omega$ real.

Proof. We refer to [33] for the proof of the theorem. We note the following relations,

$$\begin{aligned} x^i &= \tfrac{1}{2}\Big(\phi^i + i\frac{\partial H}{\partial \pi_i}\Big) , \\ y_i &= \tfrac{1}{2}\Big(\pi_i - i\frac{\partial H}{\partial \phi^i}\Big) = \frac{\partial F(x,\bar{x})}{\partial x^i} . \end{aligned} \tag{166}$$

□

We close this subsection with the following comments. Firstly, we note that since both H and $F^{(0)} - \frac{1}{2}\, x^i\, F_i^{(0)}$ transform as functions under symplectic transformations, so does the following combination that appears in (164),

$$2\,\Omega - x^i\Omega_i - \bar{x}^{\bar{\imath}}\Omega_{\bar{\imath}} \ . \tag{167}$$

Secondly, the transformation law of $2\mathrm{i}\Omega_i = F_i - F_i^{(0)}$ under symplectic transformations is determined by the transformation behaviour of F_i and $F_i^{(0)}$, as described above. The transformation law of $2\mathrm{i}\Omega_{\bar{\imath}} = F_{\bar{\imath}}$, on the other hand, follows from the reality of $\tilde{\Omega}$,

$$\tilde{\Omega}_{\bar{\imath}} = (\overline{\tilde{\Omega}_i}) \ . \tag{168}$$

Thirdly, as indicated in (157), the function $F(x,\bar{x})$ may, in general, depend on a number of real parameters η that are inert under symplectic transformations. Without loss of generality, we may take η to be solely encoded in Ω, and, upon transformation, in $\tilde{\Omega}$ (we can use the equivalence relation (163) to achieve this). As discussed below in subsection 4.4.2, $\partial_\eta F = \partial F/\partial\eta$ transforms as a function under symplectic transformations [35].

4.2. Homogeneous $F(x,\bar{x},\eta)$

The theorem in subsection 4.1 did not assume any homogeneity properties for F. Here we will focus on the case when F is homogeneous of degree two and discuss some of the consequences of homogeneity [33]. This is the case that is relevant when coupling vector multiplets to supergravity. Moreover, it also covers other interesting systems, such as the Born-Infeld dilaton-axion system in an $AdS_2 \times S^2$ background, as we will explain in section 11.

Let us consider a function $F(x,\bar{x},\eta) = F^{(0)}(x) + 2i\Omega(x,\bar{x},\eta)$ that depends on a real parameter η, and let us discuss its behaviour under the scaling

$$x \mapsto \lambda\, x \ , \ \eta \mapsto \lambda^m\, \eta \tag{169}$$

with $\lambda \in \mathbb{R}\backslash\{0\}$. We take $F^{(0)}(x)$ to be quadratic in x, so that $F^{(0)}$ scales as $F^{(0)}(\lambda\, x) = \lambda^2\, F^{(0)}(x)$. This scaling behaviour can be extended to the full function F if we demand that the canonical pair (ϕ, π) given in (161) scales uniformly as $(\phi, \pi) \mapsto \lambda\,(\phi, \pi)$. Then we have

$$F(\lambda\, x, \lambda\, \bar{x}, \lambda^m\, \eta) = \lambda^2\, F(x, \bar{x}, \eta) \;, \tag{170}$$

which results in the homogeneity relation

$$2\, F = x^i\, F_i + \bar{x}^{\bar{\imath}}\, F_{\bar{\imath}} + m\, \eta\, F_\eta \;, \tag{171}$$

where $F_\eta = \partial F / \partial \eta$. Inspection of (158) shows that the associated Hamiltonian H scales with weight two as

$$H(\lambda\, \phi, \lambda\, \pi, \lambda^m\, \eta) = \lambda^2\, H(\phi, \pi, \eta) \;, \tag{172}$$

so that H satisfies the homogeneity relation,

$$2\, H = \phi\, \frac{\partial H}{\partial \phi} + \pi\, \frac{\partial H}{\partial \pi} + m\, \eta\, \frac{\partial H}{\partial \eta} \;. \tag{173}$$

Using (166), this can be written as

$$H = i \left(\bar{x}^{\bar{\imath}}\, F_i - x^i\, \bar{F}_{\bar{\imath}} \right) + \frac{m}{2}\, \eta\, \frac{\partial H}{\partial \eta} \;. \tag{174}$$

Next, using that the dependence on η is solely contained in Ω, we obtain

$$\frac{\partial H}{\partial \eta}|_{\phi, \pi} = - \frac{\partial L}{\partial \eta}|_{\phi, \dot{\phi}} = -4 \Omega_\eta \;, \tag{175}$$

where $\Omega_\eta = \partial \Omega / \partial \eta$. Thus, we can express (174) as

$$H = i \left(\bar{x}^{\bar{\imath}}\, F_i - x^i\, \bar{F}_{\bar{\imath}} \right) - 2\, m\, \eta\, \Omega_\eta \;. \tag{176}$$

This relation is in accordance with (164) upon substitution of the homogeneity relations $2 F^{(0)}(x) = x^i\, F^{(0)}_i$ and $2\, \Omega = x^i \Omega_i + \bar{x}^{\bar{\imath}} \Omega_{\bar{\imath}} + m\, \eta\, \Omega_\eta$ that follow from (171).

The Hamiltonian transforms as a function under symplectic transformations. Since the first term in (176) transforms as a function, it follows that Ω_η also transforms as a function. This is in accordance with the general result quoted at the end of subsection 4.1 which states that $\partial_\eta F$ transforms as a function.

In certain situations, such as in the study of BPS black holes in $\mathcal{N} = 2$ supergravity theories [36], the discussion needs to be extended to a parameter

η that is complex, so that now we consider a function $F(x, \bar{x}, \eta, \bar{\eta}) = F^{(0)}(x) + 2i\,\Omega(x, \bar{x}, \eta, \bar{\eta})$ that scales as follows (with $\lambda \in \mathbb{R}\backslash\{0\}$),

$$F(\lambda\, x, \lambda\, \bar{x}, \lambda^m\, \eta, \lambda^m\, \bar{\eta}) = \lambda^2\, F(x, \bar{x}, \eta, \bar{\eta}) \; . \tag{177}$$

The extension to a complex η results in the presence of an additional term on the right hand side of (171) and (173),

$$\begin{aligned} 2\,F &= x^i\,F_i + \bar{x}^{\bar{\imath}}\,F_{\bar{\imath}} + m\,(\eta\,F_\eta + \bar{\eta}F_{\bar{\eta}}) \;, \\ 2\,H &= \phi\,\frac{\partial H}{\partial \phi} + \pi\,\frac{\partial H}{\partial \pi} + m\left(\eta\,\frac{\partial H}{\partial \eta} + \bar{\eta}\,\frac{\partial H}{\partial \bar{\eta}}\right) \;, \end{aligned} \tag{178}$$

and hence

$$H = i\left(\bar{x}^{\bar{\imath}}\,F_i - x^i\,\bar{F}_{\bar{\imath}}\right) + \frac{m}{2}\left(\eta\,\frac{\partial H}{\partial \eta} + \bar{\eta}\,\frac{\partial H}{\partial \bar{\eta}}\right) \; . \tag{179}$$

Then, since the dependence on η and $\bar{\eta}$ is solely contained in Ω, we obtain

$$H = i\left(\bar{x}^{\bar{\imath}}\,F_i - x^i\,\bar{F}_{\bar{\imath}}\right) - 2\,m\,(\eta\,\Omega_\eta + \bar{\eta}\,\Omega_{\bar{\eta}}) \; . \tag{180}$$

This is in accordance with (164) upon substitution of the homogeneity relations $2F^{(0)}(x) = x^i\,F_i^{(0)}$ and $2\,\Omega = x^i\Omega_i + \bar{x}^{\bar{\imath}}\Omega_{\bar{\imath}} + m\,(\eta\,\Omega_\eta + \bar{\eta}\,\Omega_{\bar{\eta}})$ that follow from (178).

The above extends straightforwardly to the case of multiple real or complex parameters.

4.3. Duality covariant complex variables

The Hamiltonian (164) is given in terms of complex fields x^i and $\bar{x}^{\bar{\imath}}$. It may also depend on parameters η, in which case the transformation law of x^i under symplectic transformations (165) will depend on η. It is therefore convenient to introduce duality covariant complex variables t^i, whose symplectic transformation law is independent of η. These variables ensure that when expanding the Hamiltonian in powers of η, the resulting expansion coefficients transform covariantly under symplectic transformations. This expansion can also be organized by employing a suitable covariant derivative. We review these aspects following [33].

We take the Hamiltonian (164) to depend on a single real parameter η that is inert under symplectic transformations. The discussion can be extended to the case of multiple real external parameters in a straightforward manner. We define complex variables t^i by [37],

$$\begin{aligned} 2\,\mathrm{Re}\,t^i &= \phi^i \ , \\ 2\,\mathrm{Re}\,F_i^{(0)}(t) &= \pi_i \ . \end{aligned} \tag{181}$$

Then, the vector $(t^i, F_i^{(0)}(t))$ describes a complexification of (ϕ^i, π_i) that transforms as in (159) under symplectic transformations. This yields the transformation law

$$\tilde{t}^i = U^i{}_j\, t^j + Z^{ij} F_j^{(0)}(t) \ , \tag{182}$$

which is independent of η. The new variables t^i are related to the x^i by (c.f. (161))

$$\begin{aligned} 2\mathrm{Re}\,t^i &= 2\mathrm{Re}\,x^i \ , \\ 2\mathrm{Re}\,F_i^{(0)}(t) &= 2\mathrm{Re}\,F_i(x,\bar{x},\eta) \ . \end{aligned} \tag{183}$$

We may now view H either as a function of t^i and $\bar{t}^{\bar{\imath}}$, or as a function of x^i and $\bar{x}^{\bar{\imath}}$. Differentiating $H(\phi, \pi(x,\bar{x},\eta),\eta)$ with respect to η yields

$$\left.\frac{\partial H}{\partial \eta}\right|_{x,\bar{x}} = \left.\frac{\partial H}{\partial \eta}\right|_{\phi,\pi} + \left.\frac{\partial H}{\partial \pi_k}\right|_{\mathrm{Re}\,x} \frac{\partial \pi_k}{\partial \eta} = \left.\frac{\partial H}{\partial \eta}\right|_{t,\bar{t}} + \left.\frac{\partial H}{\partial \pi_k}\right|_{\mathrm{Re}\,x} \left(F_{k\eta} + \bar{F}_{\bar{k}\eta}\right) \ , \tag{184}$$

where $F_{\eta k} = \partial^2 F / \partial\eta \partial x^k$, etc., and where on the right hand side we used $\pi_k = 2\mathrm{Re}\,F_k(x,\bar{x},\eta)$. Next, we use the conversion formula

$$\left.\frac{\partial H}{\partial \pi_k}\right|_{\mathrm{Re}\,x} = \left.\frac{\partial H}{\partial \mathrm{Im}\,x^i}\right|_{\mathrm{Re}\,x} \frac{\partial \mathrm{Im}\,x^i}{\partial \pi_k} = - \left.\frac{\partial H}{\partial \mathrm{Im}\,x^i}\right|_{\mathrm{Re}\,x} \hat{N}^{ik} \ , \tag{185}$$

where $\hat{N}^{ik}$ denotes the inverse of

$$-\frac{\partial \pi_k}{\partial \mathrm{Im}\,x^i} = -i\left(\frac{\partial}{\partial x^i} - \frac{\partial}{\partial \bar{x}^{\bar{\imath}}}\right)\left(F_k + \bar{F}_{\bar{k}}\right) = -i\left[F_{ik} - \bar{F}_{\bar{\imath}\bar{k}} - F_{k\bar{\imath}} + \bar{F}_{\bar{k}i}\right] = \hat{N}_{ik} \ . \tag{186}$$

Note that $\hat{N}_{ik}$ is a real symmetric matrix.,

$$\hat{N}_{ik} = -i\left[F_{ik}^{(0)} - \bar{F}_{\bar{\imath}\bar{k}}^{(0)}\right] + 2\left(\Omega_{ik} + \Omega_{\bar{\imath}\bar{k}} - \Omega_{k\bar{\imath}} - \Omega_{i\bar{k}}\right) \ . \tag{187}$$

We obtain

$$\partial_\eta H|_{t,\bar{t}} \quad = \quad \mathcal{D}_\eta H|_{x,\bar{x}} \ , \tag{188}$$

where $\mathcal{D}_\eta$ is given by

$$\mathcal{D}_\eta = \partial_\eta + i\,\hat{N}^{ij}\left(F_{\eta j} + \bar{F}_{\eta\bar{j}}\right)\left(\partial_i - \partial_{\bar{\imath}}\right)\ . \tag{189}$$

$\mathcal{D}_\eta$ acts as a covariant derivative for symplectic transformations. Applying multiple covariant derivatives $\mathcal{D}_\eta$ on any symplectic function depending on x^i and $\bar{x}^{\bar{\imath}}$, will again yield a symplectic function. For instance, consider applying $\mathcal{D}^2_\eta$ on $H(\phi, \pi(x,\bar{x},\eta), \eta)$ given in (164),

$$\mathcal{D}^2_\eta H(x,\bar{x},\eta) = -\,4\left[\partial^2_\eta\Omega - 2\,\hat{N}^{ij}\,\partial_\eta\left(\Omega_i - \Omega_{\bar{\imath}}\right)\,\partial_\eta\left(\Omega_j - \Omega_{\bar{j}}\right)\right]\ . \tag{190}$$

As discussed in section 4.4.2, while $\partial^2_\eta\Omega$ does not transform as a function under symplectic transformations, there exists a modification of it, given by (190), such that the modified expression transforms as a function.

4.4. Maxwell-type theories

Now we turn to Maxwell-type theories in four dimensions, namely, we consider the Maxwell sector of $\mathcal{N} = 2$ supergravity theories coupled to abelian $\mathcal{N} = 2$ vector multiplets. Below we will use some of the ingredients that go into the construction of these theories. We refer to section 6 for a detailed description of these theories. In the following, we review electric-magnetic duality in these theories, first at the two-derivative level, and then in the presence of an arbitrary chiral background field.

4.4.1. Electric-magnetic duality at the two-derivative level

The Wilsonian effective action is a local action that describes the effective dynamics at long distances [38]. The Wilsonian effective action describing the coupling of n abelian $\mathcal{N} = 2$ vector supermultiplets to four-dimensional $\mathcal{N} = 2$ supergravity at the two-derivative level is encoded in a holomorphic function $F(X)$, called the prepotential, which depends on $n+1$ complex scalar fields X^I $(I = 0, 1, \ldots n)$ and which is a homogeneous function of degree two under complex rescalings [31],

$$F(\lambda X) = \lambda^2\,F(X) \quad , \quad \lambda \in \mathbb{C}\backslash\{0\}\ , \tag{191}$$

from which one infers the relations

$$\begin{aligned} 2F &= F_I\,X^I\ , \\ F_I &= F_{IJ}\,X^J\ , \\ 0 &= F_{IJK}\,X^K\ , \end{aligned} \tag{192}$$

where $F_I = \partial F/\partial X^I$, $F_{IJ} = \partial^2 F/\partial X^I \partial X^J$, $F_{IJK} = \partial^3 F/\partial X^I \partial X^J \partial X^K$.

The resulting equations of motion for the abelian gauge fields A_μ^I only involve their field strengths $F_{\mu\nu}^I$. The combined system of equations of motion and Bianchi identities for the abelian gauge fields are invariant under so-called electric-magnetic duality transformations, which constitute symplectic $Sp(2n+2,\mathbb{R})$ transformations [31]. These transformations also induce $Sp(2n+2,\mathbb{R})$ transformations of the symplectic vector (X^I, F_I), as follows [32].

Consider the following Lagrangian for Maxwell fields A_μ^I,

$$L = -\frac{i}{4}\Big(\bar{F}_{IJ}\, F_{\mu\nu}^{+I} F^{+\mu\nu J} + 2\, \mathcal{O}^+_{\mu\nu I}\, F^{+\mu\nu I} - F_{IJ}\, F_{\mu\nu}^{-I} F^{-\mu\nu J} - 2\, \mathcal{O}^-_{\mu\nu I}\, F^{-\mu\nu I}\Big)\,, \tag{193}$$

where $F_{\mu\nu}^{\pm I}$ denote the (anti-)selfdual field strengths (c.f. (B.45)), and where we allow for a linear coupling of the field strengths $F_{\mu\nu}^{\pm I}$ to tensors $\mathcal{O}^\pm_{\mu\nu I}$. A Lagrangian of this form arises when considering the part of the $\mathcal{N}=2$ Wilsonian effective Lagrangian that describes the coupling of vector multiplets to $\mathcal{N}=2$ supergravity at the two-derivative level, c.f. (430).

We define the dual field strength by

$$G_{\mu\nu I} = \sqrt{-g}\, \varepsilon_{\mu\nu\rho\sigma}\, \frac{\partial L}{\partial F_{\rho\sigma}{}^I}\,. \tag{194}$$

Decomposing it into (anti-)selfdual parts $G^\pm_{\mu\nu I}$,

$$G^\pm_{\mu\nu I} = \pm 2i\, \frac{\partial L}{\partial F^{\pm}_{\rho\sigma}{}^I}\,, \tag{195}$$

we obtain

$$G^+_{\mu\nu I} = \bar{F}_{IJ} F^{+J}_{\mu\nu} + \mathcal{O}^+_{\mu\nu I}\,, \quad G^-_{\mu\nu I} = F_{IJ} F^{-J}_{\mu\nu} + \mathcal{O}^-_{\mu\nu I}\,. \tag{196}$$

The Bianchi identities and equations of motion for the abelian gauge fields take the form

$$\partial^\mu \big(F^{+I}_{\mu\nu} - F^{-I}_{\mu\nu}\big) = 0\,, \qquad \partial^\mu \big(G^+_{\mu\nu I} - G^-_{\mu\nu I}\big) = 0\,. \tag{197}$$

The combined system (197) is invariant under the transformation

$$\begin{pmatrix} F^{\pm I}_{\mu\nu} \\ G^\pm_{\mu\nu I} \end{pmatrix} \mapsto \begin{pmatrix} \tilde{F}^{\pm I}_{\mu\nu} \\ \tilde{G}^\pm_{\mu\nu I} \end{pmatrix} = \begin{pmatrix} U^I_J & Z^{IJ} \\ W_{IJ} & V_I{}^J \end{pmatrix} \begin{pmatrix} F^{\pm J}_{\mu\nu} \\ G^\pm_{\mu\nu J} \end{pmatrix}\,, \tag{198}$$

where U^I_J, $V_I{}^J$, W_{IJ} and Z^{IJ} are constant real $(n+1)\times(n+1)$ submatrices. We demand the transformation matrix in (198) to be invertible. Since we may rescale the field strengths $F^I_{\mu\nu}$ by a real constant, we impose the normalization $\det(U^T V - W^T Z) = 1$. Thus, the transformation matrix in (198) belongs to $SL(2n+2, \mathbb{R})$.

Next, decomposing the transformed field strenghts $\tilde{F}^{\pm I}_{\mu\nu}$, $\tilde{G}^{\pm}_{\mu\nu I}$ as in (196),

$$\tilde{G}^{+}_{\mu\nu I} = \bar{\tilde{F}}_{IJ} \tilde{F}^{+J}_{\mu\nu} + \tilde{\mathcal{O}}^{+}_{\mu\nu I} \, , \quad \tilde{G}^{-}_{\mu\nu I} = \tilde{F}_{IJ} \tilde{F}^{-J}_{\mu\nu} + \tilde{\mathcal{O}}^{-}_{\mu\nu I} \, , \tag{199}$$

we infer that under (198), F_{IJ} transforms as

$$\tilde{F}_{IJ} = (W_{IL} + V_I{}^K F_{KL}) \left[\mathcal{S}^{-1}\right]^L{}_J \quad , \quad \mathcal{S}^I{}_J = U^I{}_J + Z^{IK} F_{KJ} \, . \tag{200}$$

Then, demanding that $\tilde{F}_{IJ}$ is a symmetric matrix yields the condition

$$\begin{aligned} U^T W - W^T U + (U^T V - W^T Z)\, F - F\, (U^T V - W^T Z)^T & \\ + F\, (Z^T V - V^T Z)\, F = 0 \, , & \end{aligned} \tag{201}$$

where in this equation F denotes the matrix F_{IJ}. By comparing terms with the same power of F_{IJ}, we infer the conditions $U^T W = W^T U$ and $Z^T V = V^T Z$. In addition, the combination $U^T V - W^T Z$ needs to be proportional to the identity matrix, since the terms linear in F_{IJ} need to cancel for general F_{IJ} [39]. These conditions, when combined with the property that the transformation matrix belongs to $SL(2n+2, \mathbb{R})$, imply that the transformation matrix in (198) must be an element of $Sp(2n+2, \mathbb{R})$. Indeed, defining

$$\Delta = \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \, , \tag{202}$$

and demanding Δ to be a symplectic matrix, i.e.

$$\Delta^{-1} = \Omega \, \Delta^{\mathrm{T}} \, \Omega^{-1} \qquad \text{where} \quad \Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} , \tag{203}$$

gives

$$U^{\mathrm{T}} V - W^{\mathrm{T}} Z = V^{\mathrm{T}} U - Z^{\mathrm{T}} W = \mathbb{1} \, , \; U^{\mathrm{T}} W = W^{\mathrm{T}} U \, , \; Z^{\mathrm{T}} V = V^{\mathrm{T}} Z \tag{204}$$

as a consequence of $\Delta^{-1} \, \Delta = \mathbb{1}$, and

$$U V^{\mathrm{T}} - Z W^{\mathrm{T}} = V U^{\mathrm{T}} - W Z^{\mathrm{T}} = \mathbb{1} \, , \; U Z^{\mathrm{T}} = Z U^{\mathrm{T}} \, , \; W V^{\mathrm{T}} = V W^{\mathrm{T}} \tag{205}$$

as a consequence of $\Delta\,\Delta^{-1} = \mathbb{1}$.

Furthermore, we infer from (199) that under (198), the tensors $\mathcal{O}^{\pm}_{\mu\nu I}$ transform as

$$\tilde{\mathcal{O}}^{+}_{\mu\nu I} = \tilde{\mathcal{O}}^{+}_{\mu\nu J}\,\big[\bar{\mathcal{S}}^{-1}\big]^{J}{}_{I} \quad , \quad \tilde{\mathcal{O}}^{-}_{\mu\nu I} = \tilde{\mathcal{O}}^{-}_{\mu\nu J}\,\big[\mathcal{S}^{-1}\big]^{J}{}_{I}\ . \tag{206}$$

Next, we note that the transformation (200) of F_{IJ} is induced by the following transformation of the scalar fields X^I,

$$\begin{pmatrix} X^I \\ F_I \end{pmatrix} \mapsto \begin{pmatrix} \tilde{X}^I \\ \tilde{F}_I \end{pmatrix} = \begin{pmatrix} U^I_J & Z^{IJ} \\ W_{IJ} & V_I{}^J \end{pmatrix} \begin{pmatrix} X^J \\ F_J \end{pmatrix}\ , \tag{207}$$

which is the aforementioned $Sp(2n{+}2,\mathbb{R})$ transformation of the vector (X^I, F_I). Indeed, using (207), one derives

$$\frac{\partial \tilde{F}_I}{\partial X^J} = \tilde{F}_{IK}\left(U^K{}_J + Z^{KL}\,F_{LJ}\right) = W_{IJ} + V_I{}^L\,F_{LJ}\ . \tag{208}$$

For $N_{IJ} \equiv 2\mathrm{Im}\,F_{IJ}$, one obtains the transformation law

$$\begin{aligned} \tilde{N}_{IJ} &= N_{KL}\,\big[\bar{\mathcal{S}}^{-1}\big]^{K}{}_{I}\,\big[\mathcal{S}^{-1}\big]^{L}{}_{J}\ , \\ \tilde{N}^{IJ} &= \bar{\mathcal{S}}^{I}{}_{K}\,\mathcal{S}^{J}{}_{L}\,N^{KL} = \mathcal{S}^{I}{}_{K}\,\mathcal{S}^{J}{}_{L}\left(N^{KL} - i\mathcal{Z}^{KL}\right)\ , \end{aligned} \tag{209}$$

where

$$\mathcal{Z}^{IJ} = [\mathcal{S}^{-1}]^{I}{}_{K} Z^{KJ}\ . \tag{210}$$

Note that $\mathcal{Z}$ is a symmetric matrix by virtue of (205).

Owing to the symplectic condition (203), the quantities $\tilde{F}_I$ can be written as the derivative of a new function $\tilde{F}(\tilde{X})$ with respect to the new coordinates $\tilde{X}^I$,

$$\begin{aligned} \tilde{F}(\tilde{X}) &= \tfrac{1}{2}\big(U^{\mathrm{T}}W\big)_{IJ}X^I X^J + \tfrac{1}{2}\big(U^{\mathrm{T}}V + W^{\mathrm{T}}Z\big)_I{}^J X^I F_J + \tfrac{1}{2}\big(Z^{\mathrm{T}}V\big)^{IJ} F_I F_J \\ &= F(X) + \tfrac{1}{2}\big(U^{\mathrm{T}}W\big)_{IJ}X^I X^J + \big(W^{\mathrm{T}}Z\big)_I{}^J X^I F_J + \tfrac{1}{2}\big(Z^{\mathrm{T}}V\big)^{IJ} F_I F_J\ , \end{aligned} \tag{211}$$

where we made use of the homogeneity property (192). Note that $F(X)$ does not transform as a function under symplectic transformations (207), i.e. $\tilde{F}(\tilde{X}) \neq F(X)$. Its geometrical meaning will be discussed in subsection 5.4.2.

Two $\mathcal{N} = 2$ Wilsonian effective Lagrangians that are encoded in $F(X)$ and $\tilde{F}(\tilde{X})$, respectively, represent equivalent vector multiplet theories coupled to

$\mathcal{N} = 2$ supergravity. On the other hand, symplectic transformations that constitute a symmetry of the theory are transformations (207) for which

$$\tilde{F}(\tilde{X}) = F(\tilde{X}) \,, \tag{212}$$

since they leave the field equations invariant. Differentiating (212) with respect to $\tilde{X}^I$ gives $\tilde{F}_I(\tilde{X}) = \partial F(\tilde{X})/\partial \tilde{X}^I$, which means that the transformation law for $F_I(X)$ given in (207) is induced by substituting $\tilde{X}$ for X in $F_I(X)$. This yields a practical way for checking whether a symplectic transformation constitutes an invariance of the theory. Note that the property (212) does not imply that $F(X)$ is an invariant function; inspection of (211) shows that $\tilde{F}(\tilde{X}) \neq F(X)$, and hence $F(\tilde{X}) \neq F(X)$.

4.4.2. Electric-magnetic duality in a chiral background

Let us now briefly summarize various features of electric-magnetic duality in the presence of a chiral background field [32]. We refer to sections 6.4 and 7 for an extensive discussion of supergravity theories in the presence of a chiral background field, and for the relation with Hessian geometry.

We consider the Wilsonian effective action describing the coupling of $\mathcal{N} = 2$ supergravity to abelian vector multiplets in the presence of a chiral background field $\hat{A}$. The action is now encoded in a holomorphic function $F(X, \hat{A})$ which is homogeneous of degree two under complex rescalings, i.e.

$$F(\lambda X, \lambda^w \hat{A}) = \lambda^2 \, F(X, \hat{A}) \quad , \quad \lambda \in \mathbb{C}\backslash\{0\} \,, \tag{213}$$

where w denotes the scaling weight of $\hat{A}$, which we take to be non-vanishing. From (213) one infers the relation

$$2F(X, \hat{A}) = X^I \, F_I(X, \hat{A}) + w\hat{A} \, F_A(X, \hat{A}) \,, \tag{214}$$

where we introduced the notation $F_I(X, \hat{A}) = \partial F(X, \hat{A})/\partial X^I$, $F_A(X, \hat{A}) = \partial F(X, \hat{A})/\partial \hat{A}$. Symplectic transformations act on $(X^I, F_I(X, \hat{A}))$ as in (207),

$$\begin{aligned} \tilde{X}^I &= U^I{}_J \, X^J + Z^{IJ} \, F_J(X, \hat{A}), \\ \tilde{F}_I(\tilde{X}, \hat{A}) &= V_I{}^J \, F_J(X, \hat{A}) + W_{IJ} \, X^J \,, \end{aligned} \tag{215}$$

and they leave $\hat{A}$ inert. We will now show that $F_A(X, \hat{A})$ transforms as a function under symplectic transformations. It follows that the combination

$F(X,\hat{A}) - \frac{1}{2}X^I F_I(X,\hat{A})$ also transforms as a function due to the relation (214),

$$\tilde{F}(\tilde{X},\hat{A}) - \tfrac{1}{2}\tilde{X}^I \tilde{F}_I(\tilde{X},\hat{A}) = F(X,\hat{A}) - \tfrac{1}{2}X^I F_I(X,\hat{A})\,. \tag{216}$$

We start from the second relation in (215) and differentiate with respect to X^J keeping $\hat{A}$ fixed. This gives

$$\tilde{F}_{IK} = \big(W_{IP} + V_I{}^L\, F_{LP}\big)\,\big[\mathcal{S}^{-1}\big]^P{}_K\;, \tag{217}$$

where

$$\frac{\partial \tilde{X}^I}{\partial X^J} \equiv \mathcal{S}^I{}_J = U^I{}_J + Z^{IK}\, F_{KJ}(X,\hat{A})\;. \tag{218}$$

Taking the transposed of this equation, one verifies that $\tilde{F}_{IK}$ is symmetric in I and K, i.e. $\tilde{F}_{IK} = \tilde{F}_{KI}$.

Next, we differentiate the second relation in (215) with respect to $\hat{A}$, keeping X^I fixed. This yields

$$\tilde{F}_{IA}(\tilde{X},\hat{A}) = \Big(V_I{}^K - \tilde{F}_{IL}\, Z^{LK}\Big)\, F_{KA}(X,\hat{A})\;. \tag{219}$$

Using (217), we obtain for the transposed of the matrix on the right hand side of (219),

$$V^T - Z^T\tilde{F} = \mathcal{S}^{-1}\;, \tag{220}$$

where here $\tilde{F}$ denotes the symmetric matrix $\tilde{F}_{IJ}$. Hence,

$$\tilde{F}_{IA}(\tilde{X},\hat{A}) = F_{KA}(X,\hat{A})\big[\mathcal{S}^{-1}\big]^K{}_I\;, \tag{221}$$

With this result, and using $F_{KA}(X,\hat{A})\big[\mathcal{S}^{-1}\big]^K{}_I = \partial(F_A(X,\hat{A}))/\partial\tilde{X}^I$, we obtain

$$\tilde{F}_A(\tilde{X},\hat{A}) = F_A(X,\hat{A})\;, \tag{222}$$

up to terms that are independent of X^I, and which we drop, since they are not relevant for the vector multiplet Lagrangian. Thus, $F_A(X,\hat{A})$ transforms as a function under symplectic transformations.

Defining $N_{IJ} \equiv 2\mathrm{Im}\, F_{IJ}$ and $N^{IJ} \equiv \big[N^{-1}\big]^{IJ}$, and using (217), one obtains the transformation laws

$$\begin{aligned}
\tilde{N}_{IJ} &= N_{KL}\,\big[\bar{\mathcal{S}}^{-1}\big]^K{}_I\,\big[\mathcal{S}^{-1}\big]^L{}_J\,,\\
\tilde{N}^{IJ} &= N^{KL}\,\bar{\mathcal{S}}^I{}_K\,\mathcal{S}^J{}_L\,,\\
\tilde{F}_{IJK} &= F_{MNP}\,\big[\mathcal{S}^{-1}\big]^M{}_I\,\big[\mathcal{S}^{-1}\big]^N{}_J\,\big[\mathcal{S}^{-1}\big]^P{}_K\,.
\end{aligned} \tag{223}$$

Using (222), one finds

$$\tilde{F}_{AA}(\tilde{X},\hat{A}) = F_{AA}(X,\hat{A}) - F_{AI}(X,\hat{A})\,F_{AJ}(X,\hat{A})\,\mathcal{Z}^{IJ}\,, \tag{224}$$

where

$$\mathcal{Z}^{IJ} \equiv [\mathcal{S}^{-1}]^I{}_K\, Z^{KJ}\,, \tag{225}$$

which is symmetric in I and J, see below (205). This shows that, while F_A transforms as a function under symplectic transformations, higher derivatives of F with respect to $\hat{A}$, such as F_{AA}, do not transform as functions under symplectic transformations. Combinations that do transform as symplectic functions can be generated systematically, as follows [32]. Assume that $G(X,\hat{A})$ transforms as a function under symplectic transformations. Then, also $\mathcal{D}G(X,\hat{A})$ transforms as a symplectic function (c.f. (189)), where

$$\mathcal{D} \equiv \frac{\partial}{\partial\hat{A}} + iF_{AI}N^{IJ}\frac{\partial}{\partial X^J}\,, \tag{226}$$

as one readily verifies using (223). Consequently one can introduce a hierarchy of symplectic functions $F^{(n)}(X,\hat{A})$, which are modifications of $F_{A\cdots A}$,

$$F^{(n)}(X,\hat{A}) \equiv \frac{1}{n!}\mathcal{D}^{n-1}F_A(X,\hat{A}) \quad , \quad n\geq 1\,. \tag{227}$$

While $F^{(1)}$ is holomorphic, all the higher $F^{(n)}$ (with $n \geq 2$) are non-holomorphic. This lack of holomorphy is governed by the following equation (with $n \geq 2$),

$$\frac{\partial F^{(n)}}{\partial \bar{X}^I} = \tfrac{1}{2}\bar{F}_I{}^{JK}\sum_{r=1}^{n-1}\frac{\partial F^{(r)}}{\partial X^J}\,\frac{\partial F^{(n-r)}}{\partial X^K}\,, \tag{228}$$

where $\bar{F}_I{}^{JK} = \bar{F}_{ILM}\,N^{LJ}N^{MK}$.

In section 7 we will relate the covariant derivative (226) and the holomorphic anomaly equation (228) to properties of Hessian structures in the presence of a chiral background field, (c.f. (479) and (485)).

5. Special Kähler geometry

In this section we discuss special Kähler geometry from the mathematical point of view. The definition is ultimately motivated by physics: special Kähler geometry is the geometry of $\mathcal{N}=2$ vector multiplets. As we have seen in the previous section, the field equations of theories of abelian vector fields are invariant under symplectic transformations, which generalize the electric-magnetic rotations of Maxwell theory. In $\mathcal{N} = 2$ vector multiplets, which contain scalars and fermions together with vector fields, this extends to an action of the symplectic group on all fields, which imposes strong constraints on the scalar geometry. In short, special Kähler manifolds are Kähler manifolds equipped with a flat connection ∇ which is compatible with the symplectic structure, in the sense that symplectic transformations act linearly on ∇-affine coordinates. Moreover, the Kähler metric is Hessian with ∇ as the associated flat connection.

Special Kähler geometry has undergone various re-formulations over the past 30 years. Our approach blends the original definition [40] in terms of special coordinates and using the superconformal calculus with the intrinsic construction of [41] and the universal construction of [42], which allows to relate the former two approaches. Other formulations of special Kähler geometry will be discussed in section 5.4.

5.1. Affine special Kähler geometry

We will first present an intrinsic definition, and introduce special real and special holomorphic coordinates, the Hesse potential and the holomorphic prepotential. Then we give two extrinsic constructions, firstly as a Kählerian Lagrangian immersion into a complex symplectic vector space, secondly as a parabolic affine hypersphere immersed into a real space. The holomorphic prepotential and the Hesse potential are the generating functions for these two immersions.

5.1.1. The intrinsic definition

We start with the relatively recent definition given in [41], which is intrinsic in the sense of only using data involving the tangent bundle and associated bundles. Our presentation is based on [41] and [42].

Definition 15. Affine special Kähler manifolds (ASK manifolds). *An affine special Kähler manifold (M, J, g, ∇) is a Kähler manifold (M, J, g) endowed with a flat, torsion-free connection ∇, such that*

1. ∇ *is symplectic, that is, the Kähler form $\omega = g(\cdot, J\cdot)$ is parallel: $\nabla\omega = 0$.*
2. ∇J *is covariantly closed, $d_\nabla J = 0$.*

In the second condition, $J \in \Gamma(\mathrm{End}(TM)) \cong \Gamma(TM \otimes T^*M)$ is regarded as a vector valued one-form, $J \in \Omega^1(M, TM)$. This condition can be rephrased as

$\nabla J \in \mathcal{T}^1_2(M) = \Gamma(TM \otimes T^*M \otimes T^*M)$ being symmetric:

$$(\nabla_X J)(Y) = (\nabla_Y J)(X) \;, \quad \forall X, Y \in \mathfrak{X}(M) \;. \tag{229}$$

The definition implies that $\nabla g \in \mathcal{T}^0_3(M)$ is totally symmetric, and therefore ASK manifolds are Hessian. On a Hermitian manifold any two of the three tensor fields g, J and ω determine the third,[20] and this allows to replace condition 2 by the alternative condition

2'. $\nabla g \in \mathcal{T}^0_3(M)$ is completely symmetric.

Thus we may say that an ASK manifold is a Kähler manifold with a compatible Hessian structure. The associated flat connection ∇ is called the *special connection*. If we impose that ∇-affine coordinates on M are ω-Darboux coordinates, this restricts our freedom of making affine transformations to those where the linear part is symplectic. We will call the corresponding group $\mathrm{Aff}_{Sp(\mathbb{R}^{2n})}(\mathbb{C}^{2n}) = Sp(\mathbb{R}^{2n}) \ltimes \mathbb{C}^{2n} \subset \mathrm{Aff}(\mathbb{C}^{2n})$ the *affine symplectic group*.

We will now verify the statements made in the preceding paragraphs using special real coordinates. Since the connection ∇ is flat and torsion-free, we can choose local ∇-affine coordinates q^a which define a parallel coframe $e^a = dq^a$, $\nabla e^a = 0$ and a parallel frame $e_a = \partial_a = \frac{\partial}{\partial q^a}$, $\nabla e_a = 0$, see A.5.3. Such coordinates are unique up to affine transformations. The connection ∇ is symplectic, and therefore

$$\nabla\omega = \nabla\left(\frac{1}{2}\omega_{ab} e^a \wedge e^b\right) = \frac{1}{2}\partial_c \omega_{ab} e^c \otimes e^a \wedge e^b + \frac{1}{2}\omega_{ab}\nabla(e^a \wedge e^b) = 0 \;. \tag{230}$$

In ∇-affine coordinate the second term vanishes, and the symplectic form ω has constant coefficients:

$$\nabla\omega = 0 \Rightarrow \partial_c \omega_{ab} = 0 \;. \tag{231}$$

We can fix a standard form for the constant antisymmetric matrix ω_{ab}. The conventional choice we make is

$$\omega = \frac{1}{2}\omega_{ab} dq^a \wedge dq^b = \Omega_{ab} dq^a \wedge dq^b = 2dx^I \wedge dy_I \;, \tag{232}$$

where

$$\Omega_{ab} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \;. \tag{233}$$

[20] See A.13

The coordinates $q^a = (x^I, y_I)$ are called *special real coordinates*, and the splitting of the q^a into x^I and y_I corresponds to the choice of a *polarization*, that is, a splitting of the symplectic vector space T_pM, $p \in M$ into two maximally isotropic subspaces. The special real coordinates q^a are ω-Darboux coordinates, but differ from standard Darboux coordinates by a factor $\sqrt{2}$.[21] Choosing special real coordinates restricts our freedom to perform coordinate transformations to affine symplectic transformations,

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto M \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} , \quad M \in Sp(2n, \mathbb{R}) , \ \ a, b \in \mathbb{R}^n . \tag{234}$$

Next, we evaluate the condition $d_\nabla J = 0$ in special real coordinates, using the rules for the covariant exterior derivative from A.5.4:

$$0 = d_\nabla J = d(J^a_b e^b) \otimes e_a + J^a_b e^b \otimes \nabla e_a . \tag{235}$$

In the co-frame $e^a = dq^a$ this condition reduces to

$$d_\nabla J = (\partial_c J^a_{\ b})(dq^c \wedge dq^b) \otimes \partial_a = 0 \Rightarrow \partial_{[c} J^a_{\ b]} = 0 . \tag{236}$$

To relate this to ∇J being symmetric, note that

$$\nabla_X J = X^a (\partial_a J^c_{\ b}) e^b \otimes e_c + J^c_b \nabla_X (e^b \otimes e_c) \tag{237}$$

reduces in special real coordinates to

$$\nabla_X J = (X^a \partial_a J^c_{\ b}) dq^b \otimes \partial_c \tag{238}$$

so that

$$(\nabla_X J)(Y) = X^a Y^b (\partial_a J^c_{\ b}) \partial_c . \tag{239}$$

Using (236) we see that

$$d_\nabla J = 0 \Leftrightarrow (\nabla_X J)(Y) = (\nabla_Y J)(X) , \quad \forall X, Y \in \mathfrak{X}(M) , \tag{240}$$

that is, ∇J is symmetric, $\nabla J \in \Gamma(\mathrm{Sym}^2(T^*M) \otimes TM)$. Metric and Kähler form are related by

$$\omega(X, Y) = g(X, JY) \Leftrightarrow g(X, Y) = -\omega(X, JY) . \tag{241}$$

[21] Darboux coordinates are usually normalized such that $\omega = \frac{1}{2}\Omega_{ab} d\tilde{q}^a \wedge d\tilde{q}^b = d\tilde{x}^I \wedge d\tilde{y}_I$.

In local coordinates this implies

$$\omega_{ab} = g_{ac} J^c_{\ b} \Leftrightarrow g_{ab} = -\omega_{ac} J^c_{\ b} \Leftrightarrow J^a_{\ b} = g^{ac}\omega_{cb} \ . \tag{242}$$

In special real coordinates,

$$(\nabla_X g)(Y, Z) = \partial_c g_{ab} X^c Y^a Z^b \ . \tag{243}$$

Expressing g_{ab} in terms of ω_{ab} and $J^a_{\ b}$, and using that ω_{ab} is constant in special real coordinates, we find

$$\partial_c g_{ab} = -\omega_{ad}\partial_c J^d_{\ b} \ . \tag{244}$$

Using (236) we obtain that $\partial_c g_{ab}$ is totally symmetric, which shows that g is Hessian. It is also clear that for a flat, torsion-free and symplectic connection ∇, g being Hessian implies that ∇J is symmetric, so that condition 2 in the definition of an ASK manifold can be replaced by condition 2'.

For later use we collect further local formulae in special real coordinates. The metric is Hessian,

$$g = H_{ab} dq^a dq^b \ , \quad H_{ab} = \frac{\partial^2 H}{\partial q^a \partial q^b} \ . \tag{245}$$

We denote the inverse metric coefficients by H^{ab}. The inverse of $\Omega_{ab} = \frac{1}{2}\omega_{ab}$ is

$$\Omega^{ab} = 2\omega^{ab} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \ . \tag{246}$$

Using that $J^a{}_b = H^{ac}\omega_{cb}$ and $J^a{}_c J^c{}_b = -\delta^a_b$ we obtain,

$$\frac{1}{2}\Omega^{ab} = -2H^{ac}H^{bd}\Omega_{cd} \Leftrightarrow H_{ab}\Omega^{bc}H_{cd} = -4\Omega_{ad} \ , \tag{247}$$

where the numerical factors are due to the normalization of Ω_{ab}. The components of the complex structure in terms of H_{ab} and Ω_{ab} are:

$$J^a{}_b = 2H^{ac}\Omega_{cb} = -\frac{1}{2}\Omega^{ac}H_{cb} \ . \tag{248}$$

As on any Hessian manifold, there is a dual special connection $\nabla_{\text{dual}} = 2D - \nabla$, whose affine coordinates are the dual special real coordinates,

$$q_a := H_a := \frac{\partial H}{\partial q^a} \ . \tag{249}$$

As discussed in section 2.2, the metric coefficients with respect to q_a are given by the inverse matrix H^{ab}:

$$g = H^{ab} dq_a dq_b \ , \quad H^{ac} H_{cb} = \delta^a_c \ , \tag{250}$$

and the dual Hesse potential is obtained by a Legendre transformation:

$$H^{ab} = \frac{\partial^2 H_{\text{dual}}}{\partial q_a \partial q_b} \ , \quad H_{\text{dual}} = q^a q_a - H \ . \tag{251}$$

The dual special coordinates q_a are ω-Darboux coordinates:

$$\omega = \Omega_{ab} dq^a \wedge dq^b = 2dx^I \wedge dy_I = -\frac{1}{4}\Omega^{ab} dq_a \wedge dq_b = 2du^I \wedge dv_I \ , \tag{252}$$

corresponding to the dual polarization

$$q_a =: (2v_I, -2u^I) \ . \tag{253}$$

Special real coordinates are adapted to the symplectic and Hessian structure of an ASK manifold. We now turn to the complex aspects of ASK geometry, following [41]. The complexified tangent bundle $T_{\mathbb{C}}M$ of M decomposes into the holomorphic tangent bundle $T^{(1,0)}M$ and the anti-holomorphic tangent bundle $T^{(0,1)}M$,[22]

$$T_{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M \ , \tag{254}$$

which can be characterized as the eigendistributions of the complex structure J,

$$TM^{(1,0)} = \ker(J - i\mathbb{1}) \ , \quad TM^{(0,1)} = \ker(J + i\mathbb{1}) \ . \tag{255}$$

Similarly, the complexified cotangent bundle decomposes as $T^*_{\mathbb{C}}M = T^{*(1,0)}M \oplus T^{*(0,1)}M$. Since $d_\nabla J = 0$, the projection operator

$$\Pi^{(1,0)} = \frac{1}{2}(\mathbb{1} + iJ) \in \Gamma(T^*_{\mathbb{C}}M \otimes T^{(1,0)}M) \ : \quad T_{\mathbb{C}}M \to T^{(1,0)}M \tag{256}$$

satisfies $d_\nabla \Pi^{(1,0)} = 0$. Hence locally $\Pi^{(1,0)} = d_\nabla \zeta = \nabla\zeta$, where ζ is a complex, not necessarily holomorphic vector field, which is unique up to a flat complex vector field.[23] In special real coordinates ζ has an expansion[24]

$$\zeta = X^I \frac{\partial}{\partial x^I} + W_I \frac{\partial}{\partial y_I} \ , \tag{257}$$

[22] See A.11 for some background on complex manifolds.

[23] The relevant properties of the exterior covariant derivative d_∇ are reviewed in A.5.4.

[24] Compared to [41] we have changed the relative sign between the two terms of ζ to be consistent with our conventions.

where X^I, W_I are complex functions on M. Then

$$\Pi^{(1,0)} = dX^I \otimes \frac{\partial}{\partial x^I} + dW_I \otimes \frac{\partial}{\partial y_I} \ , \qquad (258)$$

where $dX^I, dW_I \in T^{*(1,0)}M$, which implies that the functions X^I, W_I are holomorphic. Since $\mathrm{Re}(\Pi^{(1,0)}) = \mathrm{Id}_{TM}$ it follows that

$$\mathrm{Re}(dX^I) = dx^I \ , \quad \mathrm{Re}(dW_I) = dy_I \ . \qquad (259)$$

Using that the real differentials dx^I are linearly independent, it can be shown that the differentials dX^I are linearly independent over $\mathbb{C}$, and therefore the holomorphic functions X^I define a local holomorphic coordinate system on M [41, 42]. These are the so-called *special holomorphic coordinates*, often simply called special coordinates. Similarly, the functions W_I define another holomorphic coordinate system on M, which is called the dual (holomorphic) special coordinate system.

Since $\frac{\partial}{\partial X^I}$ is of type $(1,0)$, that is $\Pi^{(1,0)}\frac{\partial}{\partial X^I} = \frac{\partial}{\partial X^I}$, it follows that

$$\frac{\partial}{\partial X^I} = \frac{\partial}{\partial x^I} + \frac{\partial W_K}{\partial X^I}\frac{\partial}{\partial y_K} \ . \qquad (260)$$

The Kähler form $\omega = 2dx^I \wedge dy_I = \frac{1}{2}(dX^I + d\bar{X}^I) \wedge (dW_I + d\bar{W}_I)$ must be a $(1,1)$-form, therefore

$$0 = dX^I \wedge dW_I = dX^I \wedge \frac{\partial W_I}{\partial X^J} dX^J \Rightarrow \frac{\partial W_I}{\partial X^J} = \frac{\partial W_J}{\partial X^I} \ . \qquad (261)$$

This implies that locally W_I is the holomorphic gradient of a function $F(X^I)$, called the *prepotential*, which is determined up to a constant:

$$W_I = \frac{\partial F}{\partial X^I} =: F_I \ , \quad \frac{\partial W_I}{\partial X^J} = \frac{\partial^2 F}{\partial X^I \partial X^J} =: F_{IJ} \ . \qquad (262)$$

The Kähler form can be expressed in terms of the prepotential as

$$\omega = -\frac{i}{2} N_{IJ} dX^I \wedge d\bar{X}^J \ , \qquad (263)$$

where

$$N_{IJ} = 2\mathrm{Im}F_{IJ} = -i(F_{IJ} - \bar{F}_{IJ}) \ , \qquad (264)$$

and where $\bar{F}_{IJ}$ is the complex conjugate of F_{IJ}. The corresponding Kähler metric and Hermitian form are

$$g = N_{IJ} dX^I d\bar{X}^J \ , \quad \gamma = g + i\omega = N_{IJ} dX^I \otimes d\bar{X}^J \ . \qquad (265)$$

Since

$$N_{IJ} = \frac{\partial^2 K}{\partial X^I \partial \bar{X}^J} \; , \quad K = i(X^I \bar{F}_I - F_I \bar{X}^I) \; , \tag{266}$$

the function K is a Kähler potential. The choice of the sign in the definition of K is conventional. Sometimes N_{IJ} and K are defined with an additional minus sign. Note that to obtain a model where N_{IJ} is positive definite, or more generally is non-degenerate and carries a specific signature, one may need to restrict the coordinates X^I to a suitable domain. This has to be analysed model by model.

We have now recovered the original definition of ASK manifolds in terms of local formulae in special coordinates [40]: *an ASK manifold is a Kähler manifold where the Kähler potential admits a holomorphic prepotential.*[25]

5.1.2. Extrinsic construction as a Kählerian Lagrangian immersion

The intrinsic definition of [41] has an extrinsic counterpart: every simply connected ASK manifold can be realized as a Kählerian Lagrangian immersion into the standard complex symplectic vector space $V = T^*\mathbb{C}^n \cong \mathbb{C}^{2n}$ [42]. Lagrangian immersions have a potential, which for ASK manifolds is the holomorphic prepotential.

We start with the standard complex symplectic vector space $V = T^*\mathbb{C}^n$ equipped with complex Darboux coordinates (X^I, W_I), the standard complex symplectic form $\Omega = dX^I \wedge dW_I$, and the standard real structure defined by complex conjugation $\tau : V \to V,\ v \mapsto \tau v = \bar{v}$.[26] The set of fixed points of the real structure τ are the real points $V^\tau = T^*\mathbb{R}^n \cong \mathbb{R}^{2n} \subset \mathbb{C}^{2n}$. Given these data we can define the Hermitian form

$$\gamma_V = i\Omega(\cdot, \tau\cdot) = i\left(dX^I \otimes d\overline{W}_I - dW_I \otimes d\overline{X}^I\right) = g_V + i\omega_V \; , \tag{267}$$

which has complex signature (n, n). Its real part defines a flat Kähler metric of real signature $(2n, 2n)$, with associate Kähler form ω_V, and complex structure I_V.

Let M be a connected complex manifold of complex dimension n. A holomorphic immersion $\phi : M \to V$ is called *non-degenerate* if $g_M := \phi^* g_V$ is

[25] Note that Kähler potentials are only determined up to Kähler transformations, and the formula expressing K in terms of F provides only a subclass of the Kähler potentials for a given ASK metric.

[26] See A.18 for a few additional remarks regarding complex symplectic manifolds.

non-degenerate, where $\phi^* g_V$ denotes the pull-back of the metric g_V by ϕ to M, see A.6. In this case g_M is a Kähler metric on M, which in general has indefinite signature. Therefore non-degenerate holomorphic immersions are also called *Kählerian* immersions. One can show that $\phi^* g_V$ being non-degenerate is equivalent to $\omega_M := \phi^*\omega_V$ being non-degenerate, and also to $\gamma_M := \phi^*\gamma_V$ being non-degenerate.

A holomorphic immersion $\phi : M \to V$ is called *Lagrangian* if $\phi^*\Omega = 0$. It has been shown in [42] that a Kählerian Lagrangian immersion $M \to V$ induces on M the structure of an affine special Kähler manifold. Conversely every simply connected affine special Kähler manifold admits a Kählerian Lagrangian immersion which induces its ASK structure. The immersion is unique up to transformations of V which leave the data (I_V, Ω, τ) invariant. These transformations act on complex Darboux coordinates as

$$\begin{pmatrix} X^I \\ W_I \end{pmatrix} \mapsto M \begin{pmatrix} X^I \\ W_I \end{pmatrix} + \begin{pmatrix} A^I \\ B_I \end{pmatrix} \;, \quad M \in \mathrm{Sp}(2n,\mathbb{R}) \;, \quad A^I, B_I \in \mathbb{C} \;, \tag{268}$$

and belong to the subgroup $\mathrm{Aff}_{Sp(\mathbb{R}^{2n})}(\mathbb{C}^{2n}) = Sp(2n,\mathbb{R}) \ltimes \mathbb{C}^{2n}$ of the complex affine group $GL(2n,\mathbb{C}) \ltimes \mathbb{C}^{2n}$.

The *Liouville form* $\lambda = W_I dX^I$ of V is a potential for the symplectic form: $d\lambda = -\Omega$. Therefore its pullback $\phi^*\lambda$ under the Lagrangian immersion ϕ is locally exact and admits a holomorphic potential F, defined on some domain $U \subset M$:

$$dF = \phi^*\lambda \;. \tag{269}$$

The pullbacks $\tilde{X}^I = \phi^* X^I$, $\tilde{W}_I = \phi^* W_I$ are holomorphic functions on M. Since ϕ is non-degenerate one can pick n independent functions and use them as local holomorphic coordinates on M. By applying a symplectic transformation if necessary one can always arrange that $\tilde{X}^I$ are local holomorphic coordinates on M. In this case the functions $\tilde{W}_I$ form a second 'dual' holomorphic coordinate system, which we will discuss in more detail in section 5.1.4. We can always choose $U \subset M$ small enough so that ϕ becomes an embedding. In this case we do not need to distinguish by notation between (X^I, W_I) and $(\tilde{X}^I, \tilde{W}_I)$. If we use special coordinates X^I on M then $dF = W_I dX^I$, implying $W_I = F_I = \frac{\partial F}{\partial X^I}$. Note that the integrability condition $F_{IJ} = \partial_I W_J = \partial_J W_I = F_{JI}$ is satisfied since ϕ is Lagrangian. The immersion ϕ locally takes the form

$$\mathbb{C}^n \supset U \ni (X^I) \mapsto (X^I, W_I) \in T^*U \subset \mathbb{C}^{2n} , \tag{270}$$

where we identify U with a domain in $\mathbb{C}^n$ using the coordinates X^I. We can also identify ϕ with dF and $\phi(U)$ with the graph

$$\left\{ (X^I, W_I) \in \mathbb{C}^{2n} | (X^I) \in U, W_I = \frac{\partial F}{\partial X^I} \right\} \tag{271}$$

of dF over U. With these properties and identifications $U \subset \mathbb{C}^n$ is called an *affine special Kähler domain*.

We proceed by deriving local expressions for the metric g_M, Kähler form ω_M and special connection ∇ on M. We decompose the complex Darboux coordinates on V into their real and imaginary parts:

$$X^I = x^I + iu^I , \quad W_I = y_I + iv_I . \tag{272}$$

Then

$$\gamma_V = g_V + i\omega_V = \frac{1}{2} \left(dx^I dv_I - du^I dy_I \right) + i \left(dx^I \wedge dy_I + du^I \wedge dv_I \right) \tag{273}$$

and

$$\Omega = dx^I \wedge dy_I - du^I \wedge dv_I + i \left(du^I \wedge dy_I + dx^I \wedge dv_I \right) . \tag{274}$$

By pullback we define the functions $\tilde{x}^I = \mathrm{Re}(\phi^* x^I)$, $\tilde{y}_I = \mathrm{Re}(\phi^* y_I)$ on M. Since the immersion is Lagrangian,

$$\mathrm{Re}(\phi^* \Omega) = 0 \Rightarrow d\tilde{x}^I \wedge d\tilde{y}_I = d\tilde{u}^I \wedge d\tilde{v}_I , \tag{275}$$

and therefore

$$\omega_M = d\tilde{x}^I \wedge d\tilde{y}_I + d\tilde{u}^I \wedge d\tilde{v}_I = 2(d\tilde{x}^I \wedge d\tilde{y}_I) . \tag{276}$$

For a simply connected ASK manifold M, $(\tilde{x}^I, \tilde{y}_I)$ are globally defined functions, but they are only global coordinates if the immersion ϕ is an embedding. By restricting to a domain $U \subset M$ where ϕ becomes an embedding, we can use $(\tilde{x}^I, \tilde{y}_I)$ as coordinates and do not need to distinguish them from (x^I, y_I) by notation. They are Darboux coordinates for the Kähler form ω_M, and define a flat, torsion-free, symplectic connection ∇ by $\nabla dx^I = 0$, $\nabla dy_I = 0$. One can show that ∇ is the special connection occurring in the intrinsic definition, and that (x^I, y_I) are the corresponding special real coordinates.

Next, we work out some expressions in terms of special holomorphic coordinates. The pull-back of the Hermitian form γ_V is

$$\gamma_M = \phi^* \gamma_V = i \left(dX^I \otimes d\bar{F}_I - dF_I \otimes d\bar{X}^I \right) = N_{IJ} dX^I \otimes d\bar{X}^J \ , \tag{277}$$

where

$$N_{IJ} = 2\mathrm{Im} F_{IJ} = \frac{\partial^2 K}{\partial X^I \partial \bar{X}^J} \ , \ \ K = i(X^I \bar{F}_I - F_I \bar{X}^I) \ . \tag{278}$$

By decomposing $\gamma_M = g_M + i\omega_M$ we obtain a non-degenerate, in general indefinite Kähler metric

$$g_M = N_{IJ} dX^I d\bar{X}^J \ , \tag{279}$$

with associated Kähler form

$$\omega_M = -\frac{i}{2} N_{IJ} dX^I \wedge d\bar{X}^J \ . \tag{280}$$

Thus we have recovered all the local expressions of section 5.1.1.

We note that the characteristic property of a Kählerian immersion, the non-degeneracy of $g_M = \phi^* g_V$ corresponds in special coordinates to $F_{IJ} = \partial^2_{IJ} F$ having an invertible imaginary part. A holomorphic one-form $\phi = dF$ is called regular if $\det(\mathrm{Im} F_{IJ}) \neq 0$. It follows that locally every regular closed holomorphic one-form defines a Lagrangian Kählerian immersion.

We conclude this section by expanding on some details. Firstly, the image $\phi(U)$ of $U \subset M$ is not automatically a graph, although this is the generic situation. For special choices of ϕ the functions X^I on U are not independent and do not define a holomorphic coordinate system on U. This can be detected by W_I not satisfying the integrability condition for the existence of of a prepotential F with gradient $F_I = W_I$. In this situation one can choose local holomorphic coordinates z^I on U and work with the functions $(X^I(z), F_I(z))$. As we will discuss in section 5.4.1, the map

$$z \mapsto (X^I(z), F_I(z)) \tag{281}$$

can be interpreted as a holomorphic section of a line bundle over M. We will discuss definitions of ASK geometry based on line bundles in section 5.4.3. Finally, only simply connected ASK manifolds admit a global immersion into $V \cong T^*\mathbb{C}^n$. As far as the local description is concerned this is not an issue, as we can restrict to simply connected submanifolds $U \subset M$. In order to obtain a global construction of general, not necessarily simply connected, ASK manifolds, the vector space V must be replaced by an affine bundle with fibre V. This will be discussed in section 5.4.2.

5.1.3. Extrinsic construction as a parabolic affine hypersphere

Affine special Kähler manifolds admit a second extrinsic construction, which is real rather than complex, with the Hesse potential as generating function. Our presentation follows [43, 44].

In this construction the ASK manifold M is immersed into $\mathbb{R}^{2n+1}$ as a hypersurface

$$\varphi : M \to \mathbb{R}^{2n+1} \,. \tag{282}$$

Using the standard connection ∂ (defined by the partial derivative with respect to linear coordinates) on $\mathbb{R}^{2n+1}$ and a vector field ξ which is transversal to M, one can give M the structure of an *affine hypersphere*, see A.10. The decomposition

$$\partial_X Y = \nabla_X Y + g(X,Y)\xi \tag{283}$$

of derivatives of vector fields X, Y tangent to M defines a torsion-free connection ∇ and a so-called Blaschke metric g on M. The connection ∇ is flat if the vector field ξ is chosen such that its integral lines are parallel on $\mathbb{R}^{2n+1}$, $\partial\xi = 0$, and thus do not intersect at finite points. This makes M a *parabolic* (or improper) *affine hypersphere*. A parabolic affine hypersphere is called *special* if there exists an almost complex structure J on M such that J is skew with respect to the Blaschke metric g, and such that the fundamental form $\omega = g(\cdot, J\cdot)$ is ∇-parallel. It has been shown in [43] that if $\varphi : M \to \mathbb{R}^{2n+1}$ is a special parabolic affine hypersphere with data (J, ω, ∇), then (M, J, g, ∇) is an affine special Kähler manifold. Conversely, any simply connected ASK manifold admits an immersion as a special parabolic affine hypersphere. The immersion is unique up to unimodular affine transformations of $\mathbb{R}^{2n+1}$. In terms of ∇-affine coordinates (x^I, y_I) on M, the immersion takes the form

$$\varphi \;: M \to \mathbb{R}^{2n+1} \,, (x^I, y_I) \mapsto \varphi_F = (x^I, y_I, H(x,y)) \,, \tag{284}$$

where H is the Hesse potential of the ASK manifold.

Since any ASK manifold can also be characterized locally by a holomorphic prepotential F, the Hesse potential H and the prepotential F determine each other. It has been shown in [43] that their relation is

$$H(x,y) = 2\mathrm{Im}(F(X(x,y))) - 2\mathrm{Re}(F_I(x,y))\mathrm{Im}X^I(x,y) \,, \tag{285}$$

where $x^I = \mathrm{Re}(X^I)$ and $y^I = \mathrm{Re}(F_I)$. That is, the Hesse potential is twice the Legendre transform of the imaginary part of the prepotential. Note that compared to the 'full' Legendre transformation which replaces all affine coordinates by their duals, $q^a \mapsto q_a$, this is a 'partial' Legendre transformation, where $u^I = \mathrm{Im}(X^I)$ is replaced by $y_I = \mathrm{Re}(F_I)$ as an independent variable, $(x^I, u^I) \mapsto (x^I, y_I)$.

5.1.4. Dual coordinate systems

In section 2.2 we have seen that Hessian structures always come in pairs, with associated dual affine coordinate systems q^a and q_a. This extends to ASK manifolds through the existence of a dual (or conjugate) special connection, $\nabla^{(J)}$, which coincides with the dual connection in the Hessian sense. Consequently, apart from the special holomorphic coordinates $X^I = x^I + iu^I$ and the special real coordinates (x^I, y_I) an ASK manifold has dual special holomorphic coordinates $W_I = F_I = y_I + iv_I$ and dual special real coordinates $(2v_I, -2u^I)$, c.f. (253). For the discussion of dual special connections we follow [42].

Given a connection ∇ and an invertible endomorphism field $A \in \Gamma(\mathrm{End}(TM))$ one can define a new connection by

$$\nabla^{(A)} X = A \nabla (A^{-1} X) \ . \tag{286}$$

For a flat connection on a complex manifold (M, J) one can in particular define the one-parameter family of flat connections $\nabla^\theta := \nabla^{\exp(\theta J)}$. By Taylor expanding $\exp(\theta J)$ and using that $J^2 = -\mathrm{Id}$, we find that the connections ∇ and ∇^θ are related by

$$\nabla^\theta = \nabla + A^\theta \ , \quad \text{where} \quad A^\theta = e^{\theta J} \nabla (e^{-\theta J}) = -\sin\theta e^{\theta J} \nabla J \ . \tag{287}$$

Note that this family of connections is periodic in θ and thus is parametrized by S^1. If (M, J, ω, ∇) is an ASK manifold with special connection ∇, then $(M, J, \omega, \nabla^\theta)$ is an ASK manifold with special connection ∇^θ, for any value of θ. As Kähler manifolds such manifolds are identical. In the physics literature ASK manifolds are usually identified if their special connections differ by $A = e^{\theta J}$, see section 5.4.3.

The connection

$$\nabla^{\pi/2} = \nabla^{(J)} = \nabla - J \nabla J \tag{288}$$

is called the connection conjugate to ∇. The convex combination

$$D := \frac{1}{2}(\nabla + \nabla^{(J)}) \tag{289}$$

of special connections satisfies $DJ = 0$. For ASK manifolds the connection D is metric compatible, and since it is by construction also torsion-free, D is the Levi-Civita connection.

This implies that for ASK manifolds where the complex structure is ∇-parallel, $\nabla J = 0$ (rather than just d_∇-closed), the Kähler metric is flat: $\nabla J = 0$ implies $\nabla = \nabla^{(J)} = D$, so that the Levi-Civita connection D is flat. In local special coordinates, this corresponds to the case where the Hesse potential and the prepotential are quadratic polynomials. Physics-wise, these are free theories.

Comparing (289) to (6) shows that the conjugate connection $\nabla^{(J)}$ coincides with the dual connection $\nabla_{\rm dual}$ in the Hessian hence. This implies that the special real coordinates with respect to $\nabla^{(J)}$ are the dual special real coordinates $q_a = H_a = (2v_I, -2u^I)$. The corresponding dual holomorphic coordinates are $W_I = y_I + iv_I$.

5.1.5. Symplectic transformations for special complex and special real coordinates

In this section we derive explicit formulae which relate the local expressions for the metric and other tensors in complex and real special coordinates. We also study how various quantities transform under symplectic transformations.

We start with comparing the coefficients of the metric in special holomorphic coordinates X^I and in special real coordinates $q^a = (x^I, y_I)$:

$$g_M = N_{IJ} dX^I d\bar{X}^J = H_{ab} dq^a dq^b \ . \tag{290}$$

We need to express the Hessian (H_{ab}) of H in terms of the matrices $R = (R_{IJ}) = (2\text{Re}(F_{IJ}))$ and $N = (N_{IJ}) = (2\text{Im}(F_{IJ}))$, which are twice the real and imaginary part, respectively, of the holomorphic Hessian (F_{IJ}) of the prepotential F. This amounts to performing a coordinate transformation from the real coordinates (x^I, u^I) underlying the complex coordinates $X^I = x^I + iu^I$ to the special real coordinates $q^a = (x^I, y_I)$. By taking derivatives of the relations

$$\begin{aligned} X^I &= x^I + iu^I(x,y) \ , \\ F_I &= y_I + iv_I(x,y) \ , \end{aligned} \tag{291}$$

we obtain the components of the Jacobians of the coordinate transformations

$$(x,u) \mapsto (x,y) \ , \quad (x,y) \mapsto (x,u) \ . \tag{292}$$

When taking derivatives of a function of the form $\tilde{f}(x,u) = f(x, y(x,u))$ we need to employ the chain rule

$$\tilde{f}_{x^I} = f_{x^I} + f_{y_K} \frac{\partial y_K}{\partial x^I} \; , \quad \tilde{f}_{u^I} = f_{y_K} \frac{\partial y_K}{\partial u^I} \; , \tag{293}$$

where we use the short-hand notation $f_{x^I} = \frac{\partial f}{\partial x^I}$, etc.

Using this we obtain the Jacobians

$$\frac{D(x,u)}{D(x,y)} = \begin{pmatrix} \mathbb{1} & 0 \\ \frac{\partial u}{\partial x}\Big|_y & \frac{\partial u}{\partial y}\Big|_x \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ N^{-1}R & -2N^{-1} \end{pmatrix} \tag{294}$$

and

$$\frac{D(x,y)}{D(x,u)} = \begin{pmatrix} \mathbb{1} & 0 \\ \frac{\partial y}{\partial x}\Big|_u & \frac{\partial y}{\partial u}\Big|_x \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ \frac{1}{2}R & -\frac{1}{2}N \end{pmatrix} . \tag{295}$$

Together with further relations given in B.3 one obtains

$$(H_{ab}) = \begin{pmatrix} N + RN^{-1}R & -2RN^{-1} \\ -2N^{-1}R & 4N^{-1} \end{pmatrix} , \tag{296}$$

where $N^{-1} = (N^{IJ})$ is the inverse of $N = (N_{IJ})$. As discussed in section 5.1.3 the Hesse potential H is related to the imaginary part of the prepotential by a Legendre transformation:

$$H(q) = H(x,y) = 2\mathrm{Im}F(x + iu(x,y)) - 2y_I u^I(x,y) . \tag{297}$$

We can also express the metric in dual special real coordinates q_a:

$$g = H_{ab} dq^a dq^b = H^{ab} dq_a dq_b , \tag{298}$$

where, as for any Hessian metric, the metric coefficients H^{ab} with respect to the dual coordinates are the inverse of H_{ab}, hence

$$(H^{ab}) = \begin{pmatrix} N^{-1} & \frac{1}{2}N^{-1}R \\ \frac{1}{2}RN^{-1} & \frac{1}{4}(N + RN^{-1}R) \end{pmatrix} . \tag{299}$$

The dual Hesse potential H_{dual},

$$H^{ab} = \frac{\partial^2 H_{\mathrm{dual}}}{\partial q_a \partial q_b} \tag{300}$$

is related to the Hesse potential by a full Legendre transformation[27]

$$H_{\text{dual}} = q^a q_a - H \ , \tag{301}$$

as discussed in section 2.2.

Special real coordinates are unique up to affine transformations with linear part in $\mathrm{Sp}(\mathbb{R}^{2n}) = \mathrm{Sp}(2n, \mathbb{R})$. In the following we discard translations and focus on linear symplectic transformations, under which the coordinates q^a transform as

$$q^a \mapsto \mathcal{O}^a{}_b q^b \ , \tag{302}$$

where $\mathcal{O} = (\mathcal{O}^a{}_b)$ is a symplectic matrix:

$$\mathcal{O}^b{}_a \Omega_{bc} \mathcal{O}^c{}_d = \Omega_{ad} \Leftrightarrow \mathcal{O}^T \Omega \mathcal{O} = \Omega \ . \tag{303}$$

We will call any object transforming in the fundamental representation of $\mathrm{Sp}(2n, \mathbb{R})$ a *symplectic vector*. Objects p_a which transform in the contragradient representation,

$$p_a \mapsto \mathcal{O}_a{}^b p_b \ , \tag{304}$$

where

$$\mathcal{O}_a{}^b \mathcal{O}^c{}_b = \delta^a_c \Leftrightarrow (\mathcal{O}_a{}^b) = \mathcal{O}^{T,-1} \ , \tag{305}$$

will be called *symplectic co-vectors*. The matrix Ω intertwines the two representations: if q^a is a symplectic vector, then $\Omega_{ab} q^b$ is a symplectic co-vector. Similarly, we define *symplectic tensors* as objects which have components with several upper and lower indices, such that each upper index transforms in the fundamental and each lower index transforms in the contragradient representation.

As an example, the metric $g = H_{ab} dq^a dq^b$ is an invariant symmetric rank two co-tensor, and since dq^a transform in the fundamental representation, the components H_{ab} of g transform as follows:

$$H_{ab} \mapsto \mathcal{O}_a{}^b \mathcal{O}_c{}^d H_{cd} \ . \tag{306}$$

Therefore

$$H^{ab} \mapsto \mathcal{O}^a{}_b \mathcal{O}^c{}_d H^{cd} \ , \tag{307}$$

[27] We call this a 'full' Legendre transformation because it involves all of the variables. In contrast the Legendre transformation relating the Hesse potential and the prepotential only involves half of the coordinates, $(x^I, y_I) \mapsto (x^I, u^I)$, and therefore we will call it a 'partial' Legendre transformation.

which implies that the dual coordinates $q_a = H_a$ transform contragradiently,

$$q_a \mapsto \mathcal{O}_a{}^b q_b \ . \tag{308}$$

Consistency requires that the Hesse potential H must be a symplectic function since $H_{ab} = \partial^2_{a,b} H$. The tensor Ω_{ab} is by definition an invariant tensor, and the complex structure J^a_b is a symplectic tensor of type $(1,1)$. Therefore all quantities we have defined using special real coordinates and dual special real coordinates are tensor components which transform as indicated by their indices.

In contrast, quantities expressed in terms of special holomorphic coordinates do not transform as tensor components in general. Since $X^I = x^I + iu^I$, $W_I = y_I + iv_I$, where $q^a = (x^I, y_I)$ and $q_a = (2v_I, -2u^I)$, it is clear that $(X^I, F_I)^T$ is a complex linear combination of symplectic vectors and therefore a complex symplectic vector. As in section 4.4.1 we set

$$\mathcal{O} = \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \ , \tag{309}$$

so that

$$U^T V - W^T Z = V^T U - Z^T W = \mathbb{1} \ , \quad U^T W = W^T U \ , \quad Z^T V = V^T Z \tag{310}$$

and

$$\begin{aligned} X^I &\mapsto U^I{}_J X^J + Z^{IJ} F_J \ , \\ F_I &\mapsto V_I{}^J F_J + W_{IJ} X^J \ . \end{aligned} \tag{311}$$

The special holomorphic coordinates X^I comprise half of the components of a symplectic vector and therefore do not define a symplectic tensor by themselves. We have already seen in section 4.4.1 that the holomorphic prepotential $F(X^I)$ is not a symplectic function. There we worked out the explicit transformation formula for the special case of prepotentials which are homogeneous of degree two. We will provide a general formula for the transformation of the prepotential together with a geometrical interpretation in section 5.4.2

Similarly, N_{IJ}, N^{IJ} and other expressions involving holomorphic indices do not transform as symplectic tensors, as we have already seen in section 4.4.1. By contracting the symplectic vector $(X^I, F_I)^T$ with its complex conjugate, we obtain a symplectic function, namely the Kähler potential:

$$K = i(X^I \bar{F}_I - F_I \bar{X}^I) \ . \tag{312}$$

Comparing to section 4 we see that the holomorphic and real formalism of special geometry are related in a way similar to the relation between the Lagrangian and Hamiltonian formalism of mechanics. In particular, the real (Hamiltonian) formalism is covariant with respect to symplectic transformations, whereas the holomorphic (Lagrangian) formalism is not. We remark that the functions $(q^a) = (x^I, y_I)$, always define local coordinates on M, irrespective of whether the 'symplectic frame' (X^I, F_I) allows a prepotential or not. For simply connected ASK manifolds q^a are in fact globally defined functions, since the immersion ϕ is global. Note, however that they only define a global coordinate system on M if ϕ is a global embedding, which need not be the case even if ϕ is a global immersion. In contrast, X^I, which are half of a set of complex coordinates (X^I, F_I) on V, only define local complex coordinates on $U \subset M$ if $\phi(U) \subset V$ is the graph of a map $V \to V\ : X^I \mapsto W_I = F_I(X)$.

5.2. Conical affine special Kähler geometry

When extending $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 2$ superconformal symmetry, two additional bosonic symmetries become relevant for vector multiplets: dilatations $\mathbb{R}^{>0}$ and phase transformations $U(1)$. On the scalar fields these are realized as a holomorphic homothetic action of $\mathbb{C}^* \cong \mathbb{R}^{>0} \times U(1)$. To obtain a superconformal Lagrangian, the prepotential must be homogeneous of degree two under complex scale transformations $X^I \mapsto \lambda X^I$, $\lambda \in \mathbb{C}^*$, while the Hesse potential must be homogeneous of degree two under real scale transformations $q^a \to \lambda q^a$, $\lambda \in \mathbb{R}^{>0}$, and invariant under $U(1)$ transformations. We will follow [42, 45].

Definition 16. Conical affine special Kähler manifolds (CASK manifolds). *A conical affine special Kähler manifold $(M, g, \omega, \nabla, \xi)$ is an affine special Kähler manifold (M, g, ω, ∇) equipped with a nowhere null vector field ξ, such that*

$$D\xi = \nabla\xi = \mathrm{Id}_{TM} \ , \tag{313}$$

where D is the Levi-Civita connection of g.

From section 2.3 we know that (313) implies that (M, g, ∇, ξ) is a 2-conical Riemannian manifold,[28] hence a Riemannian cone in the standard sense. Since (M, g, ∇) is in addition Hessian, it is a 2-conical Hessian manifold in the sense

[28] As usual we admit indefinite signature.

of Definition 9 given in section 2.3, and admits a Hesse potential which is homogeneous of degree 2 under the $\mathbb{R}^{>0}$-transformations generated by ξ:

$$L_\xi g = 2g \; , \quad L_\xi H = 2H \; . \tag{314}$$

In addition M is Kähler, and the ASK conditions imply that the vector field $J\xi$ is isometric, $L_{J\xi} g = 0$, and preserves the homogeneous Hesse potential, $L_{J\xi} H = 0$. The two vector fields $\{\xi, J\xi\}$ commute and generate a holomorphic, homothetic $\mathbb{C}^*$-action on M. On a CASK manifold one may choose, at least locally, *conical special real coordinates* $q^a = (x^I, y_I)$ such that the homothetic Killing vector field takes the form

$$\xi = q^a \frac{\partial}{\partial q^a} = x^I \frac{\partial}{\partial x^I} + y_I \frac{\partial}{\partial y_I} \; . \tag{315}$$

Such coordinates are unique up to symplectic transformations, since the compatibility with the conical structure prevents us from admitting translations. A holomorphic immersion $\phi : M \to V \cong T^*\mathbb{C}^n$ is called *conical* if the position vector field ξ^V on V is tangent along ϕ. If $\phi : M \to V$ is a conical Kählerian Lagrangian immersion of a complex connected manifold (M, J) with induced data (g, ∇, ξ), then (M, J, g, ∇, ξ) is a conical affine special Kähler manifold. Conversely, any simply connected CASK manifold can be realized as a conical Kählerian Lagrangian immersion [45].

By considering an open subset $U \subset M$ if necessary, we can assume that ϕ is an embedding. Using this we can easily verify those local formulae that do not follow from previous results on Hessian manifolds using conical special real coordinates. For reference we first collect some useful relations following from homogeneity:

$$q^a H_a = 2H \; , \quad q^a H_{ab} = H_b = q_b \; , \quad q^a q^b H_{ab} = 2H \; , \quad q^a H_{abc} = 0 \; . \tag{316}$$

For CASK manifolds the special real coordinates q^a and dual special real coordinates $q_a = H_a$ are related by $q_a = H_{ab} q^b$, $q^a = H^{ab} q_b$. This is a special feature of Hesse potentials which are homogeneous of degree two, compare (45).

Using that $J^a_{\ b} = -\frac{1}{2}\Omega^{ac} H_{cb}$, and the above homogeneity properties, the components of $J\xi$ are

$$J\xi = J^a_{\ b} q^b \partial_a = \frac{1}{2} H_b \Omega^{ba} \partial_a \; . \tag{317}$$

From this we see immediately that ξ and $J\xi$ commute, and therefore generate an abelian transformation group

$$[\xi, J\xi] = L_\xi J\xi = -L_{J\xi}\xi = 0 \,. \tag{318}$$

The Lie derivatives of the differentials are

$$L_\xi dq^a = \frac{\partial q^a}{\partial q^b} dq^b = dq^a \,, \quad L_{J\xi} dq^a = \frac{\partial (J^a_c q^c)}{\partial q^b} dq^b = -\frac{1}{2}\Omega^{ab} H_{bc} dq^c \,. \tag{319}$$

The Lie derivatives of the Hesse potential

$$L_\xi H = 2H \,, \quad L_{J\xi} H = \frac{1}{2} H_a \Omega^{ab} H_b = 0 \tag{320}$$

show that H is $J\xi$-invariant. We also list the Lie derivatives of $q_a = H_a$

$$L_\xi H_a = H_a \,, \quad L_{J\xi} H_a = \frac{1}{2} H_c \Omega^{cb} H_{ba} = 2\Omega_{ab} q^b \,, \tag{321}$$

and the Lie derivatives of the second derivatives of H

$$L_\xi H_{ab} = 0 \,, \quad L_{J\xi} H_{ab} = \frac{1}{2} H_c \Omega^{cd} H_{dab} = 0 \,. \tag{322}$$

The last equality follows from differentiating $H_{ab}\Omega^{bc}H_{cd} = -4\Omega_{ad}$ upon contraction with q^a and using homogeneity. Combining results, we find that $J\xi$ is a Killing vector field, $L_{J\xi} g = L_{J\xi}(H_{ab} dq^a dq^b) = 0$.

In summary we have the following infinitesimal $\mathbb{C}^*$-action:

$$[\xi, J\xi] = 0 \,, \quad L_\xi g = 2g \,, \quad L_{J\xi} g = 0 \,. \tag{323}$$

Moreover, the action of $J\xi$ is ω-Hamiltonian:

$$\begin{aligned} \omega(J\xi, X) &= g(J\xi, JX) = g(\xi, X) = q^a H_{ab} X^b = H_a X^a \\ &= X^a \partial_a H = X(H) = dH(X) \,, \quad \forall X \in \mathfrak{X}(M) \,, \end{aligned} \tag{324}$$

hence

$$\omega(J\xi, \cdot) = dH(\cdot) \,, \tag{325}$$

with moment map[29]

$$H = \frac{1}{2} H_{ab} q^a q^b = \frac{1}{2} g(\xi, \xi) \,. \tag{326}$$

At each point, the vector fields ξ and $J\xi$ define two distinguished directions, which correspond to the radial and angular direction of a complex cone whose

[29] See A.14 for a brief review of Hamiltonian vector fields and moment maps.

base is spanned by the vector fields in $\langle \xi, J\xi \rangle^{\perp}$. It is useful for the following discussion of project special Kähler manifolds to decompose the metric and other tensors into tangential and transversal parts with respect to the $\mathbb{C}^*$-action. For this purpose we introduce the one-forms

$$\alpha = dH = H_a dq^a \ , \quad \beta = q^a \Omega_{ab} dq^b \ , \tag{327}$$

which, up to normalization, are dual to the vector fields ξ and $J\xi$:

$$\begin{aligned} \alpha(\xi) = 2H \ , \qquad & \alpha(J\xi) = 0 \ , \quad \alpha(X) = 0 \ , \\ \beta(\xi) = 0 \ , \qquad & \beta(J\xi) = H \ , \quad \beta(X) = 0 \ , \end{aligned} \tag{328}$$

for all $X \in \langle \xi, J\xi \rangle^{\perp}$.[30] The forms α, β carry weight 2 under the dilatations generated by ξ and are invariant under the $U(1)$ transformations generated by $J\xi$:

$$\begin{aligned} L_\xi \alpha = 2\alpha \ , \qquad & L_{J\xi}\alpha = 0 \ , \\ L_\xi \beta = 2\beta \ , \qquad & L_{J\xi}\beta = 0 \ . \end{aligned} \tag{329}$$

Note that the scaling weight of any tensor which transforms homothetically under $\mathbb{C}^*$ can be changed by multiplying it with the appropriate power of the Hesse potential. In particular any tensor transforming with a definite scaling weight can be made invariant, and

$$\begin{aligned} \tilde{g}_M^{(A,B,C)} &= AH^{-1} g_M + BH^{-2}\alpha^2 + CH^{-2}\beta^2 \\ &= \left(A\frac{H_{ab}}{H} + B\frac{H_a H_b}{H^2} + C\frac{\Omega_{ac} q^c \Omega_{bd} q^d}{H^2} \right) dq^a dq^b \end{aligned} \tag{330}$$

is a family of $\mathbb{C}^*$-invariant symmetric rank two co-tensor fields which includes the conformally rescaled metric $H^{-1} g_M$ as the special case $A = 1, B = C = 0$. We can obtain a tensor field which is transversal to the $\mathbb{C}^*$-action by imposing

$$\tilde{g}_M^{(A,B,C)}(\xi, \cdot) = \tilde{g}_M^{(A,B,C)}(J\xi, \cdot) = 0 \Rightarrow B = -\frac{A}{2} \ , \quad C = -2A \ . \tag{331}$$

Thus the transversal part, which has a two-dimension kernel spanned by $\{\xi, J\xi\}$ is

$$\tilde{g}_M^{(0),A'} = A' H_{ab}^{(0)} dq^a dq^b \ , \tag{332}$$

[30] Here $\perp$ denotes orthogonality with respect to $g_M = H_{ab} dq^a dq^b$.

where

$$H^{(0)}_{ab} = -\frac{1}{2H}H_{ab} + \frac{1}{4H^2}H_aH_b + \frac{1}{H^2}\Omega_{ac}q^c\Omega_{bd}q^d \ , \quad A' = -2A \ . \tag{333}$$

Solving (333) for the CASK metric g_M we obtain

$$H_{ab} = (-2H)H^{(0)}_{ab} + \frac{1}{2H}H_aH_b + \frac{2}{H}\Omega_{ac}q^c\Omega_{bd}q^d \ . \tag{334}$$

This is an orthogonal decomposition of H_{ab} into projections onto the distributions $\langle \xi, J\xi\rangle^{\perp}$, $\langle \xi \rangle$ and $\langle J\xi \rangle$. The signatures of H_{ab} and $H^{(0)}_{ab}$ are related: if $H^{(0)}_{ab}$ is positive definite on $\langle \xi, J\xi\rangle^{\perp}$, then the CASK metric g_M has complex Lorentz signature $(\mp\mp\pm\cdots\pm)$. The overall sign of H_{ab} depends on the sign of the Hesse potential H.

For later use we define the tensor field

$$\hat{g}_M = \hat{H}_{ab}dq^adq^b \ , \quad \hat{H}_{ab} = -\frac{1}{2}H_{ab} + \frac{2}{H}\left(\frac{1}{4}H_aH_b + \Omega_{ac}q^c\Omega_{bd}q^d\right) \ , \tag{335}$$

which differs from the CASK metric by an overall factor $-\frac{1}{2}$ and a sign flip along the distribution spanned by ξ and $J\xi$. Thus $\hat{g}_M$ is positive or negative definite if the CASK metric g_M has complex Lorentz signature. The tensor $\hat{H}_{ab}$ and its inverse $\hat{H}^{ab}$ are related to the complex symmetric matrix

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + i\frac{N_{IK}X^KN_{JL}X^L}{X^MN_{MN}X^N} \tag{336}$$

by

$$\hat{H}_{ab} = \begin{pmatrix} \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \end{pmatrix} \ , \quad \hat{H}^{ab} = \begin{pmatrix} \mathcal{I}^{-1} & \mathcal{I}^{-1}\mathcal{R} \\ \mathcal{R}\mathcal{I}^{-1} & \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} \end{pmatrix} \ , \tag{337}$$

where $\mathcal{N}_{IJ} = \mathcal{R}_{IJ} + i\mathcal{I}_{IJ}$. We will see in section 6 that $\mathcal{N}_{IJ}$ is the coefficient matrix of the terms quadratic in the abelian field strengths in the Lagrangian for four-dimensional vector multiplets coupled to Poincaré supergravity. While its real version $\hat{H}_{ab}$ is a symplectic tensor, the complex matrix $\mathcal{N}_{IJ}$ transforms fractionally linearly under symplectic transformations,

$$\mathcal{N} \mapsto (W + V\mathcal{N})(U + Z\mathcal{N})^{-1} \ . \tag{338}$$

Observe that the relation between $\hat{H}_{ab}$ and $\mathcal{N}_{IJ}$ is analogous to the one between H_{ab} and F_{IJ}, in particular both F_{IJ} and $\mathcal{N}_{IJ}$ transform fractionally linearly.

Another natural symmetric tensor field on M is the 0-conical Hessian metric that we obtain by taking the logarithm of the Hesse potential H as a Hesse potential. Choosing the normalization

$$\tilde{H} = -\frac{1}{2} \log |H| \,, \tag{339}$$

we obtain

$$\begin{aligned} \tilde{g} &= \tilde{H}_{ab} dq^a dq^b = \tilde{g}_M^{(-1/2,1/2,0)} \,, &(340)\\ \tilde{H}_{ab} &= \partial^2_{a,b} \tilde{H} = -\frac{1}{2H} H_{ab} + \frac{1}{2H^2} H_a H_b &(341)\\ &= H^{(0)}_{ab} + \frac{1}{4H^2} H_a H_b - \frac{1}{H^2} \Omega_{ac} q^c \Omega_{bd} q^d \\ &= \frac{1}{H} \hat{H}_{ab} - \frac{2}{H^2} \Omega_{ac} q^c \Omega_{bd} q^d \,. \end{aligned}$$

This tensor differs from the CASK metric by an overall factor $-2H$, which makes it $\mathbb{C}^*$-invariant. Its signature differs from the one of H_{ab} by a sign flip along ξ. Thus if the CASK metric has complex Lorentz signature $(\mp, \mp, \pm, \ldots, \pm)$, then $\tilde{H}_{ab}$ has real Lorentz signature $(\pm, \mp, \pm, \ldots, \pm)$, with the time-like direction generated by $J\xi$.

5.3. Projective special Kähler geometry

In sections 2.4 and 2.5 we have discussed the real superconformal quotient, which relates affine and projective special real geometry. Similarly, given a CASK manifold M we can obtain a *projective special Kähler manifold* $\bar{M}$ by a complex quotient construction. To do this we construct a Kähler metric on the orbit space $\bar{M} = M/\mathbb{C}^*$, of the $\mathbb{C}^*$-action on a CASK manifold M. Since the CASK metric $g_M = H_{ab} dq^a dq^b$ transforms homothetically, we can make it $\mathbb{C}^*$-invariant through multiplication by a multiple H^{-1}. To obtain a projectable tensor $\tilde{g}^{(0)}_M$, we then take the transversal part:

$$\tilde{g}^{(0)}_M = H^{(0)}_{ab} dq^a dq^b \,, \quad H^{(0)}_{ab} = -\frac{1}{2H} H_{ab} + \frac{1}{4H^2} H_a H_b + \frac{1}{H^2} \Omega_{ac} q^c \Omega_{bd} q^d \,. \tag{342}$$

As in (332) we have chosen $A' = 1 \Leftrightarrow A = -\frac{1}{2}$, to be consistent with supergravity conventions. By projection onto orbits $\tilde{g}^{(0)}_M$ defines a non-degenerate metric $\bar{g}_{\bar{M}}$ on $\bar{M}$, which conversely lifts to $\tilde{g}^{(0)}_M$ under the pullback of the projection $\pi : M \to \bar{M}$, that is $\tilde{g}^{(0)}_M = \pi^* \bar{g}_{\bar{M}}$.

The quotient by the holomorphic homothetic $\mathbb{C}^*$-action will be referred to as the (complex) superconformal quotient. In order for $\bar{g}_{\bar{M}}$ to be well defined, we need that $g(\xi, \xi) = -2H \neq 0$. Moreover, we need to assume that the quotient by the $\mathbb{C}^*$-action is well behaved. This gives rise to the following definitions [42, 45]:

Definition 17. Regular conical affine special Kähler manifold. *A conical affine special Kähler manifolds (M, J, g, ∇, ξ) is called* regular *if the function $g(\xi, \xi) = -2H$ is nowhere vanishing on M, and if the canonical quotient $\pi : M \to \bar{M}$ onto the space of orbits of $\mathbb{C}^*$ on M is a holomorphic submersion onto a Hausdorff manifold.*

Definition 18. Projective special Kähler manifold (PSK manifold). *A projective special Kähler manifold $(\bar{M}, g_{\bar{M}})$ is a (possibly indefinite) Kähler manifold which can be obtained as the superconformal quotient of a regular CASK manifold (M, g_M, ∇, ξ).*

In supergravity applications, $\bar{g}_{\bar{M}}$ is the metric on the manifold parametrized by the physical scalar fields, and therefore must be positive definite. The results of the preceding section imply that the underlying CASK metric must then have complex Lorentz signature $(\mp, \mp, \pm, \ldots, \pm)$, where the time-like directions are along the orbits of the $\mathbb{C}^*$-action generated by $\langle \xi, J\xi \rangle$. In physics these directions correspond to an additional vector multiplet acting as a conformal compensator. Note that an overall sign flip of g_M does not change $\bar{g}_{\bar{M}}$. The tensor field $\hat{g}_M$ defined in (335) also plays a role in physics. It is proportional to the vector field metric and therefore must have definite signature. This is automatic if $\bar{g}_{\bar{M}}$ is positive definite.

The superconformal quotient can be interpreted as a Kähler quotient, that is as a symplectic quotient consistent with a Kähler structure, see also A.15. To see how this work we use the holomorphic parametrization of the CASK manifold and follow the original construction of [40]. When using special coordinates X^I, the homothetic Killing vector fields take the form

$$\xi = X^I \frac{\partial}{\partial X^I} + cc \;, \quad J\xi = iX^I \frac{\partial}{\partial X^I} + cc \;. \tag{343}$$

The superconformal quotient proceeds in two steps. First the coordinates X^I are restricted to the hypersurface

$$S = \{X^I \in M | i(X^I \bar{F}_I - F_I \bar{X}^I) = -1\} \;. \tag{344}$$

In physics the condition $i(X^I \bar{F}_I - F_I \bar{X}^I) = -1$ is called the D-gauge, because it fixes the local dilatation symmetry which is part of the superconformal group.

We will discuss the physics aspects in section 6, while a review of the superconformal formalism can be found in B.4.

Since $J\xi$ acts isometrically on S, one can take the quotient with respect to the $U(1)$ group action and obtains $\bar{M} = S/U(1)$. To recognize this construction as a Kähler quotient, we note that

$$i(X^I \bar{F}_I - F_I \bar{X}^I) = N_{IJ} X^I \bar{X}^J = K_{CASK} = H_{ab} q^a q^b = 2H = g(\xi, \xi) \tag{345}$$

is the norm of the homothetic Killing vector field ξ, which is proportional to the Hesse potential, which is the moment map for the Hamiltonian isometric $U(1)$ action on the CASK manifold M, see (324). This shows that $S/U(1) = M//U(1)$ is a symplectic quotient with respect to the Hamiltonian isometric action of $J\xi$ on M. Moreover (M, g_M) is a Riemannian cone over (S, g_S) with g_M and g_S related by $g_M = dr^2 + r^2 g_S$. Since (M, g_M) is Kähler, it follows that (S, g_S) is Sasakian.[31] The metric induced by g_S on the quotient $\bar{M} = S/U(1) = M//U(1)$ is Kähler, as we will show below, and therefore $M//U(1)$ is a Kähler quotient.

To show that the metric is Kähler, we express the projectable tensor $H^{(0)}_{ab} dq^a dq^b$ in holomorphic coordinates. Rather than performing the coordinate transformation from special real to special holomorphic coordinates, we start with $g_M = N_{IJ} dX^I d\bar{X}^J$ and construct a tensor which is projectable onto the orbits of the $\mathbb{C}^*$-action. The resulting tensor $\tilde{g}^{(0)}_M = N^{(0)}_{IJ} dX^I d\bar{X}^J$ has the components

$$N^{(0)}_{IJ} = -\frac{N_{IJ}}{N_{MN} X^M \bar{X}^N} + \frac{N_{IK} \bar{X}^K N_{JL} X^L}{(N_{MN} X^M \bar{X}^N)^2} \ . \tag{346}$$

To see that this is correct we note that the components $N^{(0)}_{IJ}$ are homogeneous of degree -2, so that $\tilde{g}^{(0)}_M$ is invariant under ξ. Moreover

$$N^{(0)}_{IJ} X^I = 0 = N^{(0)}_{IJ} \bar{X}^J \ , \tag{347}$$

which shows that $\tilde{g}^{(0)}_M$ is transversal to the actions generated by ξ and $J\xi$. Therefore this tensor field is projectable. The CASK Kähler potential $K = g(\xi, \xi)$ is a global function, and we can use it provide a global expression for $\tilde{g}^{(0)}_M$:

$$\tilde{g}^{(0)}_M = -\frac{\partial\bar{\partial} K}{K} + \frac{\partial K \bar{\partial} K}{K^2} \ , \tag{348}$$

[31] Sasakian geometry is reviewed in A.17.

where $\partial\bar{\partial}K = g_M$ is the CASK metric. By inspection

$$\tilde{g}_M^{(0)} = \partial\bar{\partial}\left(-\log(-K)\right) \,, \tag{349}$$

so that the degenerate symmetric rank two co-tensor field $\tilde{g}_M^{(0)}$ has a 'Kähler like' potential

$$\mathcal{K}(X,\bar{X}) = -\log(-K) = -\log(-N_{IJ}X^I\bar{X}^J) = -\log(-i(X^I\bar{F}_I - F_I\bar{X}^I)) \,. \tag{350}$$

Upon projection onto $\bar{M}$ the 'Kähler like' potential $K(X,\bar{X})$ becomes a genuine Kähler potential for $\bar{g}_{\bar{M}}$. We will obtain an expression in terms of local coordinates on $\bar{M}$ in section 5.4.1, see (372).

Rather than viewing $\bar{M}$ as an abstract quotient, one usually prefers to realize it concretely as a submanifold $\bar{M} \subset S \subset M$. In physics describing $\bar{M}$ as a submanifold corresponds to imposing a $U(1)$ gauge on top of the D-gauge. There is no canonical choice for a $U(1)$ gauge. The only canonical choice would be to take a hypersurface in S which is orthogonal at every point to orbits of the $U(1)$-action. However, S is a contact manifold and the distribution defined by this condition is a contact distribution, and therefore not integrable.[32] This situation is different from the first step, where we defined S as the level set of the symplectic function $g(\xi,\xi)$, which is a moment map for the Hamiltonian $U(1)$ action generated by $J\xi$, see (324), (345). There are two ways to proceed. One can choose a gauge, for example by imposing that one of the special holomorphic coordinates is real, such as $\bar{X}^0 = X^0$. This will always break the full symplectic covariance that we have preserved so far, because the orthogonality to the $U(1)$-orbits was the only remaining symplectically invariant equation involving ξ and $J\xi$. Alternatively, one can work 'upstairs,' on S or M using $U(1)$-invariant quantities or $\mathbb{C}^*$-invariant quantities, respectively. This has the advantage of preserving symplectic covariance, and we will see how it is done in the following.

5.4. Other formulations of special Kähler geometry

5.4.1. Formulation in terms of line bundles

In the physics literature, special Kähler geometry is often presented in a slightly different language where the quantities (X^I, F_I) are interpreted as sections of a line bundle $\mathcal{U}^{\bar{M}} \to \bar{M}$. In this section we explain how this formulation

[32]Contact structures and their relation to integrability are reviewed in A.16.

can be recovered from the immersion $M \to T^*\mathbb{C}^{n+1}$ discussed in section 5.1.2, following [45].

The universal line bundle

We start by recalling that on a holomorphic Hermitian vector bundle there is a unique connection, called the Chern connection, which is simultaneously holomorphic and Hermitian, see A.12. Consider the open set of non-isotropic vectors $V' = \{v \in V | \gamma(v, v) \neq 0\} \subset V = T^*\mathbb{C}^{n+1}$ of the vector space V. The space of complex lines $P(V') = \{[v] = \mathbb{C}v | v \in V'\}$ is the projectivization of V. Then the trivial vector bundle $\mathcal{V} := P(V') \times V \to P(V')$ equipped with the standard Hermitian metric $\gamma = \Omega(\cdot, \bar{\cdot})$ is a holomorphic Hermitian vector bundle, with Chern connection $d = \partial + \bar{\partial}$. The *universal bundle* $\mathcal{U} \to P(V')$ is defined as the holomorphic line sub-bundle of $\mathcal{V}$ whose fibre $\mathcal{U}_p$ over $p = [v] \in P(V')$ is the corresponding line $\mathbb{C}v$. The Chern connection on $\mathcal{U}$ is given by the γ-orthogonal projection of the flat Chern connection d of $\mathcal{V}$:

$$\mathcal{D}_X v := \pi_{\mathcal{U}} d_X v = \frac{\gamma(d_X v, v)}{\gamma(v, v)} v \,, \tag{351}$$

where X is a complex vector field on $P(V')$ and v a section of $\mathcal{U} \subset \mathcal{V}$.

Pull-back of universal line bundle to the CASK manifold M

If (M, J, g, ∇, ξ) is a regular CASK manifold, then we have the following commutative diagram:

$$\begin{CD} M @>{\phi}>> V' \\ @V{\pi}VV @VV{\pi_V}V \\ \bar{M} @>{\bar{\phi}}>> P(V') \end{CD} \tag{352}$$

Remark 9. The projectivization $P(V')$ of the symplectic manifold V' is a *contact manifold*, see A.16. The holomorphic map $\bar{\phi}$ is a *Legendrian immersion* induced by the holomorphic Lagrangian immersion ϕ.

The map $f = \bar{\phi} \circ \pi = \pi_V \circ \phi : M \to P(V')$ defines the pull-back $(\mathcal{U}^M, \mathcal{D})$ of the universal bundle $(\mathcal{U}, \mathcal{D})$, where we use the same symbol $\mathcal{D}$ for the Chern connection on $\mathcal{U}$ and its pull-back to $\mathcal{U}^M$:

$$\begin{CD} \mathcal{U}^M @>>> \mathcal{U} \\ @A{\phi}AA\!\!@VVV @VV{\pi_{\mathcal{U}}}V \\ M @>{\bar{\phi}\circ\pi = \pi_V \circ \phi}>> P(V') \end{CD} \tag{353}$$

The holomorphic Lagrangian immersion $\phi : M \to V$ can be regarded as a holomorphic section of $\mathcal{U}^M$, as follows: according to A.6 the pull-back bundle $\mathcal{U}^M$ is defined as

$$\mathcal{U}^M = \{(m,u) \in M \times \mathcal{U} | (\pi_V \circ \phi)(m) = \pi_{\mathcal{U}}(u)\} . \tag{354}$$

A section of $s : P(V') \to \mathcal{U}$ of $\mathcal{U} \to P(V')$ has the form $s(p) = v_p$, where $v_p \in V$ is a vector such that $[v_p] = p \in P(V')$. For v_p we can choose any vector on the line p. The pull-back section $\mathcal{U}^M \to M$ has the form $((\pi_V \circ \phi)^* s)(m) = v_{(\pi_V \circ \phi)(m)}$, where $v_{(\pi_V \circ \phi)(m)}$ is a vector on the same line in V as $\phi(m)$. If we choose a section s of $\mathcal{U}$ such that $s(p) = \phi(m)$ for $p = \pi_V \phi(m)$, then the corresponding pull-back section $m \mapsto \phi(m)$ can be identified with ϕ. The local form of ϕ, regarded as a section of $\mathcal{U}^M \to M$ is

$$\phi : M \to \mathcal{U}^M : (X^I) \mapsto (X^I, F_I(X)) , \tag{355}$$

where X^I are special holomorphic coordinates on M, $F_I = \partial F / \partial X^I$, and where F is the prepotential of ϕ regarded as a local holomorphic Lagrangian immersion $\phi : M \to V$.

Since the Chern connection $\mathcal{D}$ on the universal line bundle $\mathcal{U}$ is defined by orthogonal projection, the pull-back connection satisfies

$$\mathcal{D}_I \phi = i A^h_I \phi , \quad \mathcal{D}_{\bar{I}} \phi = 0 , \tag{356}$$

where $\mathcal{D}_I := \mathcal{D}_{\partial_I}$ and $\mathcal{D}_{\bar{I}} = \mathcal{D}_{\partial_{\bar{I}}}$, and where the components A^h_I of the connection one-form $i A^h_I dX^I$ of the pull-back connection are:

$$iA^h_I = \frac{\gamma(\partial_I \phi, \phi)}{\gamma(\phi,\phi)} = \frac{i(\partial_I X^J \bar{F}_J - \partial_I F_J \bar{X}^J)}{i(X^K \bar{F}_K - F_K \bar{X}^K)} , \tag{357}$$

$$iA^h_{\bar{I}} = \frac{\gamma(\partial_{\bar{I}} \phi, \phi)}{\gamma(\phi,\phi)} = \frac{i(\partial_{\bar{I}} X^J \bar{F}_J - \partial_{\bar{I}} F_J \bar{X}^J)}{i(X^K \bar{F}_K - F_K \bar{X}^K)} = 0 . \tag{358}$$

We can also express the pull-back connection with respect to a unitary (unit norm) section $\phi_1 = \phi / ||\phi||$, $||\phi|| := \sqrt{|\gamma(\phi,\phi)|}$, where $\gamma(\phi,\phi) = i(X^I \bar{F}_I - F_I \bar{X}^I)$. The unitary section ϕ_1 can be interpreted as a section of a principal $U(1)$ bundle $\mathcal{P}^M \to M$, to which the holomorphic line bundle $\mathcal{U}^M \to M$ is associated. Let $\mathcal{D}$ be a principal connection on $\mathcal{P}^M$ with connection one-form $iA_I dX^I + iA_{\bar{I}} d\bar{X}^I$, so that covariant derivatives of sections of $\mathcal{P}^M$ take the form

$$\mathcal{D}_I \phi_1 = i A_I \phi_1 , \quad \mathcal{D}_{\bar{I}} \phi_1 = i A_{\bar{I}} \phi_1 . \tag{359}$$

We require that the pull-back connection on $\mathcal{U}^M$ is induced by this principal connection. Then we can read off the components $(A_I, A_{\bar I})$ of the connection one-form of $\mathcal{P}^M$ by comparing (359) to the covariant derivatives of unit sections of $\mathcal{U}^M$. Using that

$$D_I\phi_1 = \frac{D_I\phi}{||\phi||} - \frac{\phi}{||\phi||^2}\partial_I||\phi|| \;, \quad D_{\bar I}\phi_1 = -\frac{\phi}{||\phi||^2}\partial_{\bar I}||\phi|| \;, \tag{360}$$

we find:

$$A_I = \frac{1}{2}A_I^h \;, \quad A_{\bar I} = \frac{1}{2}\overline{A_I^h} \;. \tag{361}$$

Pull back of the universal line bundle to the PSK manifold $\bar M$

By choosing a section $s : \bar M \to M$ of the $\mathbb{C}^*$-bundle $\pi : M \to \bar M$ we can regard $\bar M$ as an embedded submanifold, at least locally. We can also use $\bar\phi : \bar M \to P(V')$ to obtain the pull-back bundle $(\mathcal{U}^{\bar M}, \mathcal{D})$ of the universal bundle $(\mathcal{U}, \mathcal{D})$:

$$\begin{array}{ccccc} \mathcal{U}^{\bar M} & \longrightarrow & \mathcal{U}^M & \longrightarrow & \mathcal{U} \\ {\scriptstyle s^*\phi}\uparrow\downarrow & & {\scriptstyle \phi}\uparrow\downarrow & & \downarrow \\ \bar M & \xrightarrow{\;s\;} & M & \xrightarrow{\pi_V\circ\phi = \bar\phi\circ\pi} & P(V') \end{array} \qquad \bar M \xrightarrow{\;\bar\phi\;} P(V') \tag{362}$$

The local form of the pull-back of $\phi : M \to \mathcal{U}^M$ by $s : \bar M \to M$ to a section $s^*\phi : \bar M \to \mathcal{U}^{\bar M}$ is

$$s^*\phi : \bar M \to \mathcal{U}^{\bar M} \;\; (\zeta^a) \mapsto (X^I(\zeta), F_I(\zeta)) \;, \tag{363}$$

where ζ^a are local holomorphic coordinates on $\bar M$, and where X^I and F_I depend holomorphically on ζ^a. Evaluating the pull-back connection on a holomorphic section $s : \bar M \to M$ we obtain

$$\mathcal{D}_a s = iA_a^h s = i\partial_a X^I A_I^h s = \frac{\gamma(\partial_a\phi, \phi)}{\gamma(\phi,\phi)} s = \frac{\partial_a X^I \bar F_I - \partial_a F_I \bar X^I}{X^I \bar F_I - F_I \bar X^I} s \;, \quad \mathcal{D}_{\bar a} s = 0 \;. \tag{364}$$

On a unitary section $s_1 = s/||s||$ we pull back the principal connection $(A_I, A_{\bar I})$ of $\mathcal{P}^M$ to obtain a principal connection with components

$$A_a = \partial_a X^I A_I + \partial_a \overline{X^I} A_{\bar I} \;, \;\; A_{\bar a} = \partial_{\bar a} X^I A_I + \partial_{\bar a}\overline{X^I} A_{\bar I} \;, \tag{365}$$

The local components $(X^I(\zeta,\bar\zeta), F_I(\zeta,\bar\zeta))$ of the pull-back of ϕ by a unit section s_1 satisfy

$$\gamma(\phi,\phi) = \gamma(\phi(s_1),\phi(s_1)) = i(X^I(\zeta,\bar\zeta)\bar F_I(\zeta,\bar\zeta) - F_I(\zeta,\bar\zeta)\bar X^I(\zeta,\bar\zeta)) = \pm 1 \tag{366}$$

and depend non-holomorphically on the local holomorphic coordinates ζ^a. Evaluating the connection on a unit section s_1 we find

$$\begin{aligned} \mathcal{D}_a s_1 &= iA_a s_1 = \frac{\gamma(\partial_a \phi, \phi) - \gamma(\phi, \partial_{\bar a}\phi)}{2\gamma(\phi,\phi)} s_1 \ , && (367) \\ \mathcal{D}_{\bar a} s_1 &= iA_{\bar a} s_1 = \frac{\gamma(\partial_{\bar a} \phi, \phi) - \gamma(\phi, \partial_{a}\phi)}{2\gamma(\phi,\phi)} s_1 \ . && (368) \end{aligned}$$

Note that for a unitary section $\gamma(\phi,\phi) = \pm 1$ and $\gamma(\partial_a\phi, \phi) = -\gamma(\phi, \partial_{\bar a}\phi)$. Also note that $A_{\bar a} = \overline{A_a}$. In terms of components $(X^I(\zeta, \bar\zeta), F_I(\zeta,\bar\zeta))$ the components of the connection one form are

$$iA_a = -iA_{\bar a} = -\frac{i}{2} \frac{X^I \overleftrightarrow{\partial}_a \bar F_I - F_I \overleftrightarrow{\partial}_a \bar X^I}{i(X^I \bar F_I - F_I \bar X^I)} \ , \qquad (369)$$

where $i(X^I \bar F_I - F_I \bar X^I) = \pm 1$, and where we use the notation $a \overleftrightarrow{\partial} b = (a\partial b - (\partial a) b)$.

Pull-back of the universal line bundle to space-time N

Our last step is to consider the situation where a physical theory defined on space-time N contains massless scalar fields with values in a PSK manifold $\bar M$. The Lagrangian description of such scalar fields is given by a non-linear sigma model, see B.1 for details. The scalar fields are the components of a map $\mathcal{Z} : N \to \bar M$ from space-time N into a PSK manifold $\bar M$. This defines a further pull-back $(\mathcal{U}^N, \mathcal{D})$ of the universal bundle to a line bundle over space-time.

$$\begin{array}{ccccccc} \mathcal{U}^N & \longrightarrow & \mathcal{U}^{\bar M} & \longrightarrow & \mathcal{U}^M & \longrightarrow & \mathcal{U} \\ \mathcal{X}^*\phi \uparrow\downarrow & & s^*\phi \uparrow\downarrow & & \phi \uparrow\downarrow & & \downarrow \\ N & \xrightarrow{\mathcal{Z}} & \bar M & \xrightarrow{s} & M & \xrightarrow{\pi_V \circ \phi = \bar\phi \circ \pi} & P(V') \\ & & \mathcal{X} & & & \bar\phi & \end{array} \qquad (370)$$

Introducing local coordinates x^μ on space-time, sections of the pull-back of the universal bundle by a holomorphic section take the following form in terms of components:

$$\mathcal{X}^*\phi : N \to \mathcal{U}^N \qquad (x^\mu) \mapsto (X^I(\zeta(x)), F_I(\zeta(x))) \qquad (371)$$

Given a set of local holomorphic coordinates z^a on $\bar M$, we can choose a local holomorphic non-vanishing function h on $\bar M$ and set $X^0 = h(z)$. Then $X^a(z) = h(z) z^a$, and we can interpret the conical holomorphic special coordinates X^I as local functions on $\bar M$. Since $z^a = X^a / X^0$, the local holomorphic coordinates z^a

are the 'inhomogeneous' special holomorphic coordinates on $\bar{M}$ associated to the special holomorphic coordinates X^I on M, which can be viewed as projective coordinates (or homogeneous coordinates) on $\bar{M}$.[33]

This construction provides us with a section $s : \bar{M} \to M : z^a \mapsto X^I(z)$. By making a holomorphic coordinate transformation $z^a \mapsto \zeta^a$ on $\bar{M}$ we can then go from special holomorphic coordinates z^a to general holomorphic coordinates ζ^a. Given a holomorphic section $(X^I(\zeta), F_I(\zeta))$ of $\mathcal{U}^{\bar{M}} \to \bar{M}$, we can obtain an expression for the Kähler metric $g_{\bar{M}}$. Firstly, we locally identify $\bar{M}$ with an embedded complex submanifold of M using the section $s : \bar{M} \to M \;\; : \zeta^a \mapsto X^I(\zeta)$. The metric $g_{\bar{M}}$ is obtained by pulling back the projectable tensor $\tilde{g}^{(0)}_M$, see (348), that we have built out of the Kähler metric g_M. According to (350) the tensor $\tilde{g}^{(0)}_M$ has a 'Kähler like' potential $\mathcal{K}(X, \bar{X})$. Since $(X^I(z), F_I(z))$ are local holomorphic functions on $\bar{M}$, it follows that $g_M = \iota^* \tilde{g}^{(0)}_M$ is a Kähler metric $g_{\bar{M}} = \partial\bar{\partial}\mathcal{K}$ with Kähler potential

$$\mathcal{K} = -\log(-i(X^I(\zeta)\bar{F}_I(\bar{\zeta}) - F_I(\zeta)\bar{X}^I(\bar{\zeta}))) \; . \tag{372}$$

We also note that the pullback of the Chern connection to the pull back bundle $\mathcal{U}_N \to N$ over space-time by a unitary section is

$$\begin{aligned} A_\mu(x) &= \partial_\mu \zeta^a(x) A_a(\zeta(x), \bar{\zeta}(x)) + \partial_\mu \bar{\zeta}^{\bar{a}}(x) A_{\bar{a}}(\zeta(x), \bar{\zeta}(x)) \qquad (373) \\ &= -\frac{1}{2} \frac{X^I \overleftrightarrow{\partial}_\mu \bar{F}_I - F_I \overleftrightarrow{\partial}_\mu \bar{X}^I}{i(X^I \bar{F}_I - F_I \bar{X}^I)} = -\frac{i}{2} \frac{N_{IJ}((\partial_\mu X^I)\bar{X}^J - X^I \partial_\mu \bar{X}^J)}{N_{KL} X^K \bar{X}^L} \; , \end{aligned}$$

where $i(X^I\bar{F}_I - F_I\bar{X}^I) = N_{IJ}X^I\bar{X}^J = \pm 1$. We will see in section 6 that this pull back connection is equal, *up to an overall minus sign*, to the $U(1)$ connection used in the superconformal calculus (see also B.4).

5.4.2. Formulation in terms of an affine bundle, and why the prepotential transforms as it does

In this section we elaborate on the following two points:

1. The extrinsic realization of ASK manifolds [42] which we have described in section 5.1.2 only provides a global construction for simply connected ASK manifolds. It is desirable to have a generalization which allows the global construction of general ASK manifolds.

[33] The terms 'inhomogeneous coordinate' and 'homogeneous/projective coordinate' are used here as in projective geometry, for example for coordinates on the complex projective space $P_n(\mathbb{C})$.

2. The transformation properties of the holomorphic prepotential under symplectic transformations are complicated, and their geometric origin remains obscure. The prepotential is not a symplectic function, and when deriving its transformation formula by integrating the transformation formula (311) for the symplectic vector (X^I, F_I), this leaves the integration constant undetermined. For a homogeneous prepotential this constant is absent, and for degree two we found the complicated looking expression (211).

We will now report how these issues have been resolved in [46, 47]. The prepotential F can be defined as the potential of the Liouville form $\lambda = W_I dX^I$, restricted to a special Lagrangian submanifold $L \subset V = T^*\mathbb{C}^n$, where $d\lambda_{|L} = -\Omega_{|L} = 0$ and $\lambda_{|L} = F_I dX^I = dF$. Here we assume that the complex symplectic coordinates (X^I, W_I) on V have been chosen such that L is a graph. Then $W_I = F_I = \partial F/\partial X^I$ on L. From these expressions it is clear that neither λ nor F is invariant under symplectic transformations. However the one-form

$$\eta = X^I dW_I - W_I dX^I \tag{374}$$

is symplectically invariant, and, like the Liouville form, a potential for the complex symplectic form Ω, hence closed when restricted to a Lagrangian submanifold $L \subset V$:

$$d\eta = 2\Omega_{|L} = 0 \; . \tag{375}$$

Consequently η is locally exact on L and admits a potential f, which is a symplectic function, and which is unique up to an additive constant,

$$\eta_{|L} = -df \; . \tag{376}$$

We will call the potential f a *Lagrange potential*, and note that Lagrange potentials and prepotentials are related by

$$2F = f + X^I F_I \Leftrightarrow f = 2F - X^I F_I \; . \tag{377}$$

From the physics literature it is well known that the combination $F - \frac{1}{2}X^I F_I$ is a symplectic function [32]. We now see that this function is, up to normalization, the Lagrange potential associated to F.

Let now M be a connected, but not necessarily simply connected ASK manifold. Then the above applies locally, if we choose a domain $U \subset M$ which is small enough to admit a Lagrangian Kählerian embedding $\phi : U \to V \cong T^*\mathbb{C}^n \cong \mathbb{C}^{2n}$. Such an embedding identifies U with a Lagrangian submanifold $L \subset \mathbb{C}^{2n}$. On each such L we have a symplectically invariant one-form $\eta = X^I dW_I - W_I dX^I$ and can choose a Lagrange potential f. Then (L, f) is called a *Lagrangian pair*, and a Lagrangian pair (L, f) is called *Kählerian* if the restriction of the Hermitian form $\gamma = i\Omega(\cdot, \bar{\cdot})$ is non-degenerate. Lagrangian pairs are related to each other by a group action. The relevant group is $G_{\mathbb{C}} := Sp(\mathbb{C}^{2n}) \ltimes \mathrm{Heis}_{2n+1}(\mathbb{C})$, where $\mathrm{Heis}_{2n+1}(\mathbb{C})$ is the $(2n+1)$-dimensional complex Heisenberg group. The group $G_{\mathbb{C}}$ is a central extension of the complex symplectic affine $Sp(\mathbb{C}^{2n}) \ltimes \mathbb{C}^{2n} \subset \mathrm{Aff}(\mathbb{C}^{2n})$. We will see that the central extension is needed to include the freedom of shifting Lagrange potentials and prepotentials by a constant, and refer to A.19 for further details about the group $G_{\mathbb{C}}$, its subgroups and its representations. The group $G_{\mathbb{C}}$ maps a given Lagrangian pair (L, f) to the new pair

$$g \cdot (L, f) = (\bar{\rho}(g)L, g \cdot f) \ , \tag{378}$$

where $g = (M, s, v) \in G_{\mathbb{C}}$, with $M \in Sp(\mathbb{C}^{2n})$, $s \in \mathbb{C}$ central, $v \in \mathbb{C}^{2n}$ a translation, where $\bar{\rho}$ is the affine representation of $G_{\mathbb{C}}$ obtained by 'forgetting the centre', that is by the natural action of $(M, v) \in Sp(\mathbb{C}^{2n}) \ltimes \mathbb{C}^{2n}$, and where

$$g \cdot f = f \circ g^{-1} + \Omega(\cdot, v) - 2s \tag{379}$$

is the new Lagrange potential. While the first term is the natural action of the affine group on functions, the second and third term correspond to translations and to central transformations, respectively. In particular, the third term, which represents the action of the centre of the group $G_{\mathbb{C}}$, corresponds to shifting the Lagrange potential, and the associated prepotential, by a constant.

To describe the local embedding of an ASK manifold, we can only admit Lagrangian pairs which are Kählerian. The subgroup of $G_{\mathbb{C}}$ acting on Kählerian Lagrangian pairs is $G_{SK} = Sp(\mathbb{R}^{2n}) \ltimes \mathrm{Heis}_{2n+1}(\mathbb{C}) \subset G_{\mathbb{C}}$, which is a central extension of the affine symplectic group $\mathrm{Aff}_{Sp(\mathbb{R}^{2n})}(\mathbb{C}^{2n}) = Sp(\mathbb{R}^{2n}) \ltimes \mathbb{C}^{2n}$ which we have encountered before.

We need a further definition. A *special Kähler pair* (ϕ, F) is a Kählerian Lagrangian embedding $\phi : U \to \phi(U) \subset V$, which induces on U the restriction of the ASK structure of M, together with the choice of a prepotential F. For each U, one denotes by $\mathcal{F}(U)$ the set of all special Kähler pairs, where only domains U are admitted where $\mathcal{F}(U) \neq \emptyset$. A Kählerian Lagrangian pair (ϕ, F)

determines a Lagrangian pair (L, f) with Lagrangian submanifold $L := \phi(U)$ and Lagrange potential f given by

$$\phi^* f = 2F - X^I W_I \ , \tag{380}$$

where ϕ has components $\phi = (X^I, W_I)$. Formula (380) relates Lagrange potentials and prepotentials. By assumption the functions X^I define special coordinates on U and if we identify U with $\phi(U)$ we can omit ϕ^* in (380) and relate F and f as functions of holomorphic special coordinates. Then we are back to (377).

The group G_{SK} acts on the special Kähler pairs $\mathcal{F}(U)$ by

$$g \cdot (\phi, F) := (g\phi \ , g \cdot F) \ , \tag{381}$$

where

$$g\phi = \bar{\rho}(g) \circ \phi \ , \tag{382}$$

with $\bar{\rho}$ the same representation of G_{SK} as above and

$$g \cdot F := F - \frac{1}{2} X^I W_I + \frac{1}{2} X'^I W'_I + \frac{1}{2} (g\phi)^* \Omega(\cdot, v) - s \ , \tag{383}$$

where $\phi = (X^I, W_I)$ and $g\phi = (X'^I, W'_I)$ are the local expressions for ϕ and $g\phi$. The somewhat complicated transformation formula (383) for prepotentials follows from the formula (379) for Lagrange potentials together with (380).

By specialization to the subgroup $Sp(\mathbb{R}^{2n}) \subset G_{SK}$ we see that under symplectic transformations $g = (M, 0, 0)$:

$$F \to F' - \frac{1}{2} X^I W_I + \frac{1}{2} X'^I W'_I \Leftrightarrow F' - \frac{1}{2} X'^I W'_I = F - \frac{1}{2} X^I W_I \ . \tag{384}$$

This is the standard formula for the transformation of the prepotential, now derived without the ambiguity of adding a constant. The observation that $F - \frac{1}{2} X^I F_I$ is a symplectic function is now explained by this function being proportional to the associated Lagrange potential. For CASK manifolds, symplectic transformations act on the set of homogeneous prepotentials of degree two. Note that two is the only degree of homogeneity for the prepotential, where F_I has the same degree of homogeneity as X^I, so that a linear combination of X^I and F_I transforms homogeneously.

Let us now turn our attention to how a global construction of ASK manifolds can be achieved by glueing together special Kähler pairs. We will only give a summary and refer the interested reader to [46, 47] for details. The group G_{SK} acts simply transitively on the set $\mathcal{F}(U)$ of special Kähler pairs for fixed $U \subset M$. One can show that by letting U vary over M one obtains a G_{SK} principal bundle $P \to M$, called the *bundle of special Kähler pairs*, which comes equipped with a flat connection. The group G_{SK} admits a linear representation $\rho : G_{SK} \to Sp(\mathbb{R}^{2n})$ which defines a flat real symplectic vector bundle $(V_{\mathbb{R}}, \Omega, \nabla)$ of rank $2n$, such that $\nabla\Omega = 0$. By complex linear extension $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C}$ we obtain a flat symplectic holomorphic vector bundle $(V_{\mathbb{C}}, \Omega, \nabla)$, with $\nabla\Omega = 0$, where we use the same symbol for ∇, Ω and their extensions. The complex symplectic form Ω on $V_{\mathbb{C}}$ defines a Hermitian metric $\gamma = i\Omega(\cdot, \tau\cdot)$, where τ is complex conjugation. Since the group $Sp(\mathbb{R}^{2n})$ acts on $\mathbb{C}^{2n}$, the complex vector bundle $V_{\mathbb{C}}$ is associated to the G_{SK} principal bundle of special Kähler pairs through the (extension of the) linear representation ρ.

One can further show that M being an ASK manifold implies that $V_{\mathbb{C}}$ admits a global holomorphic section $\Phi : M \to V_{\mathbb{C}}$, such that

$$(\nabla\Phi)^*\Omega \quad = \quad 0\,, \tag{385}$$

$$(\nabla\Phi)^*\gamma \quad \text{is} \quad \text{non-degenerate.} \tag{386}$$

The map $\nabla\Phi : TM \to V_{\mathbb{C}}$ is a morphism of holomorphic vector bundles. The global section Φ generalizes the global immersion $M \to V$ of simply connected ASK manifolds, with conditions (385) and (386) corresponding to the requirements that ϕ must be symplectic ($\phi^*\Omega = 0$) and Kählerian ($\phi^*\gamma$ non-degenerate). This construction does not yet encode the freedom of making translations. To include these we need to introduce a flat complex affine bundle $A \to M$ modelled on $V_{\mathbb{C}}$,[34] which can also be defined as the affine bundle associated to the principal bundle $P \to M$ of special Kähler pairs by the affine representation $\bar{\rho} : G_{SK} \to \mathrm{Aff}_{\mathrm{Sp}(\mathbb{R}^{2n})}(\mathbb{C}^{2n})$ on $\mathbb{C}^{2n}$.

One then obtains the following theorem, which generalizes the construction of [42]:

Theorem 2. Extrinsic construction of general affine special Kähler manifolds (Theorem 3.5.4 of [47]). *Let M be a complex manifold, and $A \to M$ be a flat complex affine bundle modelled on the complex vector bundle $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C}$, where (V, Ω, ∇) is a flat real symplectic vector bundle such that $\nabla\Omega = 0$. If there is a global holomorphic section $\Phi : M \to A$ such that the*

[34]See A.1 for the definition of an affine bundle.

conditions (385) *and* (386) *are satisfied, then* M *carries the structure of an affine special Kähler manifold, and* A *is associated to the principal* G_{SK} *bundle of special Kähler pairs by the affine representation* $\bar{\rho} : G_{SK} \to \mathit{Aff}_{\mathrm{Sp}(\mathbb{R}^{2n})}(\mathbb{C}^{2n})$ *acting on* $\mathbb{C}^{2n}$.

Conversely, if M *is an affine special Kähler manifold, then the associated complex affine bundle* $A \to M$ *corresponding to the affine representation* $\bar{\rho}$: $G_{SK} \to \mathit{Aff}_{\mathrm{Sp}(\mathbb{R}^{2n})}(\mathbb{C}^{2n})$ *acting on* $\mathbb{C}^{2n}$ *has a global section* $\Phi : M \to A$*, which satisfies the conditions* (385) *and* (386).

5.4.3. Comparison to the literature

In this section we will compare the definitions we have given for affine and projective special Kähler geometry with other definitions in the literature. So far we have covered the original definition [40] of PSK geometry, which was expressed in terms of special holomorphic coordinates and based on the superconformal tensor calculus; the intrinsic definition of [41], and the extrinsic construction of [42], which has extended the earlier work [48, 49] into the framework of *special complex geometry,* which contains special Kähler geometry as a subset. An alternative 'bilagrangian' extrinsic construction of ASK manifolds has been given in [50].

In between [40] and [41] various other formulations of special Kähler geometry have been presented in the physics literature. Common themes in these approaches are: (i) to have manifest holomorphic coordinate invariance of the formalism, that is, to use general holomorphic coordinates instead of special holomorphic coordinates, and (ii) to avoid using the prepotential explicitly, because the prepotential is not a symplectic function, and because there are (non-generic) symplectic frames where no prepotential exists. This leads one to work with a collection $\Phi(z) = (X^I(z), F_I(z))$ of holomorphic functions defined on local coordinate charts, which are glued together by transition functions, and which are are interpreted as defining a global section of a vector bundle. Equivalently, one can use a unit section $\Phi_1(z, \bar{z}) = (X^I(z, \bar{z}), F_I(z, \bar{z}))$, which then is not holomorphic. In this setting special Kähler geometry is defined by imposing suitable conditions on this section which allow to define a non-degenerate special Kähler metric, and, more generally, to obtain all the local expressions needed to have a well defined vector multiplet Lagrangian. Since these approaches are covered by excellent reviews, articles and books including [51, 52, 53, 54, 22], which contain comprehensive bibliographies, we only mention a few selected papers in the following. The work of [55] gave a geometric definition of PSK manifolds in terms of holomorphic vector bundles, which was motivated by the insight that special Kähler geometry plays an important role in the geometry of moduli spaces of Calabi-Yau compactifications of string theory, see also section 5.5. The so-called rheonomic approach to supergravity, see [56] for a review, was applied to $\mathcal{N} = 2$ vector multiplets in [57, 58] to obtain a formulation based on general holomorphic coordinates. Issues relating to the (non-)existence of a prepotential were discussed in [59]. This is particularly relevant for gauged supergravity, that is for supergravity theories with non-abelian gauge symmetries or charged matter multiplets, because the gauging breaks the continuous symplectic symmetry and distinguishes a discrete subset of frames. Gauged supergravity is outside the scope of this review. The formulation of special Kähler geometry in terms of real symplectic coordinates was discussed in [60, 61].

For a more detailed comparison between the approach presented in this review and alternative formulations, we use [53], where various definitions of special Kähler geometry have been collected and compared to each other, and [54], which has extended these definitions to arbitrary target space signature.

For ASK manifolds, the transition functions given in [53] take the form

$$\left(\begin{array}{c} X^I(z_{(i)}) \\ F_I(z_{(i)}) \end{array} \right) = e^{ic_{(ij)}} M_{(ij)} \left(\begin{array}{c} X^I(z_{(j)}) \\ F_I(z_{(j)}) \end{array} \right) + b_{(ij)} \; . \tag{387}$$

Here the indices i, j refer to two overlapping patches $U_i, U_j \subset M$, $(M_{(ij)}, b_{(ij)}) \in \mathrm{Sp}(\mathbb{R}^{2n}) \times \mathbb{C}^{2n}$, are transition functions corresponding to affine symplectic transformations, and $e^{ic_{(ij)}} \in U(1)$ are constant $U(1)$ phases. While $(M_{(ij)}, b_{(ij)})$ realize the affine representation $\bar{\rho}$ of the group G_{SK}, and therefore can be interpreted as transition functions of the complex affine bundle $A \to M$, the phases $e^{ic_{(ij)}}$ reflect an additional freedom which is not present in [41], [42], where the special connection ∇ is part of the data defining an ASK manifold. As discussed in section 5.1.4, special connections always come in S^1-families. While the underlying Kähler manifold is the same, ASK manifolds with different special connections from the same S^1-family are considered distinct according to the definitions in [41], [42]. However, this choice does not influence the Kähler metric and other data needed to build a vector multiplet Lagrangian, and therefore definitions in the physics literature do not require to fix the special connection. The phases $e^{ic_{(ij)}}$ in the transition functions (387) reflect the freedom of choosing different special connections $\nabla_{(i)}$ and $\nabla_{(j)}$ from the same S^1-family on U_i and U_j. Thus compared to the complex affine bundle $A \to M$ transition functions of the form (387) define a bundle which is modified by a 'twist.' It would be interesting to describe this twist within the framework of [42, 47]. Moreover, since we are not aware of explicit examples where the additional freedom of rotating the special connection is actually used, it would be interesting to find explicit examples.

Let us also have a look at the definition of PSK manifolds given in [53]. In this case the transition functions between patches $U_i, U_j \subset \bar{M}$ are of the form

$$\begin{pmatrix} X^I(z_{(i)}) \\ F_I(z_{(i)}) \end{pmatrix} = e^{f_{(ij)}(z)} M_{(ij)} \begin{pmatrix} X^I(z_{(j)}) \\ F_I(z_{(j)}) \end{pmatrix} , \tag{388}$$

where $f_{(ij)}(z)$ are holomorphic functions and $M_{(ij)} \in Sp(2n+2, \mathbb{R})$. Such transition functions correspond to a product bundle $\mathcal{L} \otimes \mathcal{H} \to \bar{M}$, where $\mathcal{L}$ is a holomorphic line bundle and $\mathcal{H}$ is a flat symplectic vector bundle. If $\mathcal{H} \to \bar{M}$ is trivial we can identify $\mathcal{L}$ with the pull-back line bundle $\mathcal{U}^{\bar{M}} \to \bar{M}$. If $\mathcal{H}$ is non-trivial, we expect that this bundle will arise when applying the construction of section 5.4.1 to the complex affine bundle $A \to M$. We remark that the special connection ∇ on M does not induce a flat connection on $\bar{M}$, since the superconformal quotient includes dividing out the isometric $U(1)$-action which acts by rotation on the S^1-family of special connections. It would be interesting to have an intrinsic characterization of PSK manifolds, which then could be related to the constructions in terms of line bundles and vector bundles.

Finally, another global condition which is included explicitly in the definition [53] of PSK manifolds is that $\bar{M}$ should be a *Kähler-Hodge manifold*. In the mathematical literature a Kähler manifold $\bar{M}$ is called a Kähler-Hodge manifold or Kähler manifold of restricted type if its Kähler form ω defines an integral cohomology class, $[\omega] \in H^2(\bar{M}, \mathbb{Z})$. For compact $\bar{M}$ this implies that $\bar{M}$ is a projective variety, that is, embeddable into complex projective space. In supergravity a normalization condition for the Kähler form arises since the fields transform under the local action of the group $U(1)$, which in the superconformal approach is part of the superconformal group. One must therefore impose that these transformations are globally well defined on the scalar manifold. This also applies to $\mathcal{N} = 1$ supergravity, which like $\mathcal{N} = 2$ has a local $U(1)$ group action on its scalar manifold $\bar{M}$. For compact $\bar{M}$ it was shown in [62] that this implies that the Kähler form must define an even integer class in $H^2(\bar{M}, \mathbb{Z})$. That the condition is even-ness rather than integrality results from the normalization of the $U(1)$ charges, which are half-integer valued for fermions. In the physics literature the term Kähler-Hodge is used for Kähler manifolds which are target spaces of supermultiplets that can be coupled consistently to supergravity. Most standard examples for PSK are open domains which have trivial topology, so that $[\omega] = 0$, and the Kähler-Hodge condition is automatically satisfied. For non-compact scalar manifolds with non-trivial topology the global well-definedness of $U(1)$-transformations can impose non-trivial conditions. A recent comprehensive analysis has shown that a scalar manifold $\bar{M}$ is an admissible target space for chiral supermultiplets coupled to $\mathcal{N} = 1$ supergravity if it admits a so-called chiral triple [63]. If space-time is a spin manifold, then every Kähler-Hodge manifold admits a chiral triple, irrespective of whether it is compact or non-compact [63]. In [54] it was shown that 'projective Kähler manifolds', that is scalar manifolds constructed as Kähler quotients using the superconformal calculus are automatically Kähler-Hodge. Since we have defined PSK manifolds as Kähler quotients of CASK manifolds, there is no need to require the Kähler-Hodge property explicity.

5.5. Special geometry and Calabi-Yau three-folds

The geometry of moduli spaces of Calabi-Yau three-folds provides natural realizations of special real and special Kähler geometry. These moduli spaces appear in compactifications of supergravity and of string and M-theory on Calabi-Yau three-folds. The scalar manifolds in physical applications usually combine moduli which correspond to deformations of the Calabi-Yau metric with moduli associated with the deformations of antisymmetric tensor fields. We start with the discussion of the moduli of the Calabi-Yau metric, and then turn to the moduli spaces of string compactifications. In this section we assume knowledge of some mathematical concepts, including holonomy, de Rham and Dolbeault cohomology,[35] Hodge numbers, homology, Poincaré duality, the cup and intersection product. Since this material is not needed in other parts of this review, we will not explain these concepts in detail, but refer the readers to [64] Vol 2 and [65, 66], on which this section is partly based.

A *Calabi-Yau n-fold X* is a $2n$-dimensional compact Riemannian manifold with holonomy group contained in $SU(n) \subset U(n) \subset SO(2n)$. This implies that X is Kähler, but it is more restrictive than that, by excluding a subgroup $U(1) \subset$

[35]Some aspects of Dolbeault cohomology are presented in A.11.

$U(n)$ from the holonomy, which implies that the metric is Ricci-flat. Therefore a Calabi-Yau manifold can alternatively be defined as a Kähler manifold admitting a Ricci-flat metric.[36]

We now specialize to Calabi-Yau three-folds. In the following it is understood that the holonomy group is not contained in $SU(2) \subset SU(3)$, thus excluding the cases where $X = K3 \times T^2$ (which is hyper-Kähler with holonomy $SU(2)$), and where $X = T^6$ (which is flat). The moduli space arising when dimensionally reducing the Einstein-Hilbert term on X is the space $\mathcal{M}_{\text{Ricci}}$ of Ricci-flat metrics on X. If the field equations of a higher-dimensional theory of gravity and matter admit a solution where space-time takes the form $\mathbb{R}^{1,3} \times X$ with a metric $\eta_{1,3} \times g$, which is the product of the four-dimensional Minkowski metric $\eta_{1,3}$ with a Ricci-flat metric g on X, then the four-dimensional massless fields corresponding to zero modes of the higher-dimensional metric are: (i) the four-dimensional graviton, equivalently, linearized fluctuations of the Minkowski metric $\eta_{1,3}$, (ii) four-dimensional vector fields in the adjoint representation of the isometry group of g, (iii) four-dimensional scalars in one-to-one correspondence with linearly independent solutions of the linearized Einstein equation on X. If X is a Calabi-Yau three-fold, then there are no continuous isometries and hence no massless vector fields descending from the higher-dimensional metric. A Ricci-flat metric on X is consistent with the field equations if the energy-momentum tensor has no non-zero components along X. In this case the Einstein equations reduce to the condition that X is Ricci-flat, and scalar zero modes of the metric parametrize the moduli space of Ricci-flat metrics on X.[37] The linearized form of the Ricci-flatness condition is a Laplace-type equation for the so-called Lichnerowicz Laplacian, whose zero modes are the moduli scalars. They enter into the low-energy effective four-dimensional action through a non-linear sigma model with target space $\mathcal{M}_{\text{Ricci}}$, equipped with the metric

$$G(\delta g_{(1)}, \delta g_{(2)}) = \frac{1}{V} \int_X \delta g_{(1)mn} \delta g_{(2)pq} g^{mp} g^{nq} \sqrt{g} d^6 x \ , \tag{389}$$

where $x^m, m = 1, \ldots, 6$ are coordinates on X, where $g = (g_{mn})$ is the metric

[36]For string theory compactifications the metric is only Ricci-flat to leading order in α' for $n > 2$, but this does not affect the following discussion.

[37]In string theory the Einstein equations receive α'-corrections. This leads to corrections to the metric on the moduli space, which can be computed using two topologically twisted versions of string theory [66].

on X, where $\delta g_{(i)mn}$ $i = 1, 2$ are infinitesimal deformations of the metric, and where V is the volume of X.

For Calabi-Yau three-folds $\mathcal{M}_{\text{Ricci}}$ is locally isometric to the product of the moduli space $\mathcal{M}_{\text{cplx}}$ of complex structures on X and the moduli space $\mathcal{N}_{\text{Kahler}}$ of Kähler structures on X,

$$\mathcal{M}_{\text{Ricci}} \cong \mathcal{M}_{\text{cplx}} \times \mathcal{N}_{\text{Kahler}} . \tag{390}$$

This factorization is a special feature of Calabi-Yau three-folds. The definition of a Kähler form requires the choice of a complex structure, and in general the space $\mathcal{N}_{\text{Kahler}}$ of Kähler forms of a complex manifold is fibred over its space $\mathcal{M}_{\text{cplx}}$ of complex structures. However, for Calabi-Yau three-folds the Kähler structure and complex structure can locally be varied independently. From the physics perspective this is predicted by supersymmetry, since in $\mathcal{N} = 2$ theories both types of moduli belong to different types of multiplets. In terms of complex coordinates u^a, $a = 1, 2, 3$ on the Calabi-Yau three-fold X with metric g, complex structure J and Kähler form ω, deformations of the complex structure J correspond to deformations of the Kähler metric $g_{a\bar{b}}$ which have the form $\delta g_{ab}, \delta g_{\bar{a}\bar{b}}$ and therefore are not Hermitian with respect to the undeformed complex structure J. In contrast deformations of the Kähler structure correspond to deformations $\delta g_{a\bar{b}}$ of the metric, which are Hermitian with respect to the complex structure J but change the Kähler form ω of X.

Infinitesimal deformations of a complex structure $J \in \Gamma(\text{End}(TX))$ are generated by holomorphic vector-valued one forms $\tau = \tau^a{}_{\bar{b}} \partial_a \otimes d\bar{u}^{\bar{b}}$, $\bar{\partial}\tau = 0$. Two such forms generate equivalent deformations if they differ by an $\bar{\partial}$-exact form, therefore complex structure deformations are classified by $H^1(X, T_{\mathbb{C}}X)$, the first Dolbeault cohomology group of X with values in the complexified tangent bundle. On a Calabi-Yau three-fold there exists a holomorphic, covariantly constant $(3, 0)$-form Ω, called the holomorphic top-form, which is unique up to complex rescalings $\Omega \to \lambda\Omega$, where $\lambda \in \mathbb{C}^*$. This provides an isomorphism between $T_{\mathbb{C}}X$ and $\Lambda^2 T^*_{\mathbb{C}}X$ by

$$\phi^a \mapsto \psi_{bc} = \Omega_{abc}\phi^a , \tag{391}$$

which implies the relation

$$H^1(X, T_{\mathbb{C}}X) \cong H^1(X, \Lambda^2 T^*_{\mathbb{C}}X) \cong H^{2,1}_{\bar{\partial}}(X) , \tag{392}$$

so that complex structure deformations of Calabi-Yau three-folds are parametrized by the Dolbeault cohomology group $H^{2,1}_{\bar{\partial}}(X)$, that is, by equivalence classes of

$\bar{\partial}$-closed $(2,1)$-forms modulo $\bar{\partial}$-exact forms, which are related to vector-valued one-forms by the isomorphism,

$$\phi_{ab\bar{c}} = \Omega_{abd}\tau^{d}_{\ \bar{c}} \ . \tag{393}$$

The dimension of $H^{2,1}(X)$ (considered as a vector space) is given by the Hodge number $h^{2,1} \geq 0$, which is a topological invariant of X. Since there is a one-to-one correspondence between linearly independent harmonic (p,q)-form on X and elements of $H^{p,q}(X)$, one can choose harmonic $(2,1)$-forms to generate the complex structure deformations. The expansion of a general harmonic $(2,1)$-form ϕ in a basis ϕ_A, $A = 1, \ldots, h^{2,1}$,

$$\phi = z^A \phi_A \ , \quad z^A \in \mathbb{C} \tag{394}$$

provides local coordinates z^A on $\mathcal{M}_{\text{cplx}}$. The metric on $\mathcal{M}_{\text{cplx}}$ is induced by the standard scalar product $(\alpha, \beta) = \int_X \alpha \wedge *\beta$ between $(2,1)$-forms. To see that this metric is Kähler, and more specifically projective special Kähler, one uses the relation between complex structures on X and the periods of the holomorphic top-form Ω. Choosing a complex structure on X is equivalent to specifying a decomposition of the third de-Rham cohomology group into Dolbeault cohomology groups,

$$H^3(X) = H^{3,0}_{\bar{\partial}}(X) \oplus H^{2,1}_{\bar{\partial}}(X) \oplus H^{1,2}_{\bar{\partial}}(X) \oplus H^{0,3}_{\bar{\partial}}(X) \ . \tag{395}$$

Such a decomposition is obtained by picking one of the $b_3 = 1 + h^{2,1} + h^{2,1} + 1$ harmonic forms, where b_3 is the third Betti number, and declaring it to be the holomorphic top form. More precisely, the complex structure does not depend on the explicit choice of Ω, but only on the corresponding 'complex direction', since we can rescale $\Omega \mapsto \lambda\Omega$, $\lambda \in \mathbb{C}^*$.

We now choose a basis (A^I, B_I), $I = 0, \ldots, h^{2,1}$ of the third homology group $H_3(X, \mathbb{Z})$ of X, with normalization

$$A^I \cdot B_J = \delta^I_J = -B_J \cdot A^I \ , \tag{396}$$

where $\cdot$ denotes the intersection product (which is defined by counting intersection points between submanifolds, weighted with orientation). The periods

$$X^I(z) := \int_{A^I} \Omega \ , \quad F_I(z) := \int_{B_I} \Omega \ , \tag{397}$$

of the holomorphic top-form depend holomorphically on the complex coordinates z^A on $\mathcal{M}_{\text{cplx}}$. Poincaré duality provides an isomorphism between the homology groups $H_p(X,\mathbb{Z})$ and the cohomology groups $H^{6-p}(X,\mathbb{Z})$,[38]

$$C \mapsto [C] \; , \;\; \text{such that} \quad \int_C \beta = \int_X [C] \wedge \beta \; , \tag{398}$$

for all $\beta \in \Omega^p(X)$. Poincaré duality maps the intersection product of cycles (defined by counting intersection points weighted by orientation) to the cup product of cohomology cycles (induced by the wedge product of forms). This allows to define a basis (α_I, β^I) of $H^3(X,\mathbb{Z})$ dual to the basis (A^I, B_I) of $H_3(X,\mathbb{Z})$:

$$\int_{A^J} \alpha_I = \int \alpha_I \wedge \beta^J = \delta^I_J \; , \quad \int_{B_J} \beta^I = \int \beta^I \wedge \alpha_J = -\delta^I_J \; . \tag{399}$$

In terms of this basis the top-form has the expansion

$$\Omega = X^I \alpha_I - F_I \beta^I \; . \tag{400}$$

Since only half of the periods are independent, X^I can be chosen to parametrize the possible choices of a top-form out of the harmonic three-forms. It follows that the X^I can be used as projective coordinates for $\mathcal{M}_{\text{cplx}}$. At this point the relation to CASK and PSK manifolds becomes obvious. It turns out that the metric on $\mathcal{M}_{\text{cplx}}$, which is defined by the scalar product between $(2,1)$ forms, is a Kähler metric with Kähler potential

$$K = -\log\left[i \int_X \Omega \wedge \bar{\Omega} \right] = -\log\left[-i \left(X^I(z) \bar{F}_I(z) - F_I(z) \bar{X}^I(z) \right) \right] \; . \tag{401}$$

This is a PSK metric, given in terms of a holomorphic section $(X^I(z), F_I(z))$ of the complex line bundle $\mathcal{L} \to \bar{M} = \mathcal{M}_{\text{cplx}}$. The associated CASK metric also has a natural interpretation. If we do not only choose a complex structure, but in addition a specific top-form compatible with this structure, the resulting space, which is parametrized by the independent periods X^I, is a complex cone over $\mathcal{M}_{\text{cplx}}$ which carries the structure of a CASK manifold.

We now turn to infinitesimal deformations $\delta g_{a\bar{b}}$ of the Ricci flat metric which preserve the complex structure. In local complex coordinates u^a on X the Kähler forms is given by $\omega = i g_{a\bar{b}} du^a \wedge d\bar{u}^{\bar{b}}$, and therefore such deformations change

[38] The ring $\mathbb{Z}$ can be replaced by $\mathbb{R}$ or $\mathbb{C}$.

the Kähler form. Since the Kähler form is a closed real $(1,1)$-form, it defines a class in

$$H^{1,1}(X,\mathbb{R}) := H^2(X,\mathbb{R}) \cap H^{1,1}(X,\mathbb{C}) \ , \tag{402}$$

which labels Kähler structures on X. Changes of the Kähler structure are changes of the Kähler form by a real $(1,1)$-form which is closed but not exact. As representatives one can choose $h^{1,1}$ linearly independent harmonic $(1,1)$-forms ω_x, $x = 1, \ldots h^{1,1}$. Then the expansion of the Kähler form in terms of this basis,

$$\omega = t^x \omega_x \ , \quad t^x \in \mathbb{R} \ , \tag{403}$$

provides real coordinates on the space of $\mathcal{N}_{\text{Kahler}}$ of Kähler structures. We remark that $\mathcal{N}_{\text{Kahler}} \subsetneq H^{1,1}(X,\mathbb{R}) \cong \mathbb{R}^{h^{1,1}}$, since when deforming the Kähler form we need to preserve the positivity of the metric g on X. This can be expressed by the requirement that the volumes of X and of all its complex submanifolds must be positive. The top exterior power of the Kähler form is proportional to the volume form of (X,g). The volume V of X is given by

$$V = \frac{1}{3!} \int_X \omega \wedge \omega \wedge \omega \ . \tag{404}$$

Moreover the Kähler form is a so-called *calibrating form* for holomorphic curves C and holomorphic surfaces S in X, that is

$$\text{Vol}(C) = \int_C \omega \ , \quad \text{Vol}(S) = \frac{1}{2} \int_S \omega \wedge \omega \ . \tag{405}$$

The conditions

$$\int_C \omega > 0 \ , \quad \int_S \omega \wedge \omega > 0 \ , \quad \int_X \omega \wedge \omega \wedge \omega > 0 \tag{406}$$

define the *Kähler cone* of X, the space of positive Kähler classes, which is $\mathcal{N}_{\text{Kahler}}$.

Using the basis ω_x, the volume takes the form

$$V = \frac{1}{3!} C_{xyz} t^x t^y t^z \ , \tag{407}$$

where the quantities

$$C_{xyz} := \int_X \omega_x \wedge \omega_y \wedge \omega_z \tag{408}$$

are topological invariants, called *triple intersection numbers*. To explain this name, we use the isomorphism $H^2(X,\mathbb{Z}) \cong H_4(X,\mathbb{Z})$ provided by Poincaré duality, which maps the cup product of cohomology classes of closed differential forms to the intersection product of homology classes of closed submanifolds. This implies that

$$C_{xyz} = D_x \cdot D_y \cdot D_z \ , \tag{409}$$

where $D_x,\ x = 1, \ldots h^{1,1} = b_2$ is the basis of $H_4(X, \mathbb{Z})$ dual to the basis ω_x of $H^2(X, \mathbb{Z})$, and where $\cdot$ is the intersection product between homological four-cycles.[39]

Since the moduli dependence of the volume V is given by a homogeneous symmetric polynomial in the Kähler moduli t^x, we can use it to define a 3-conical metric, in fact an ASR metric, on $\mathcal{N}_{\text{Kahler}}$. Its logarithm $\log V$ defines the associated 0-conical Hessian metric. The metric on $\mathcal{N}_{\text{Kahler}}$ obtained by dimensional reduction of the Einstein-Hilbert action is the metric induced by the scalar product of $(1,1)$ forms. Its metric coefficients with respect to the basis ω_x can be shown to be of the form

$$G_{xy} = G(\omega_x, \omega_y) = \frac{\partial^2 \log V}{\partial t^x \partial t^y} \ . \tag{410}$$

Thus the metric on the Kähler cone is 0-conical with a Hesse potential given by the logarithm of the volume. The associated PSR metric on hypersurfaces of constant volume also has a natural interpretation. It is the metric on the moduli space $\mathcal{N}^V_{\text{Kahler}}$ of Kähler structures at fixed volume. As we have seen in section 2.5 this metric is obtained by pulling back either the 3-conical metric $\partial^2 V$ or the 0-conical metric $\partial^2 \log V$ to the hypersurface $\mathcal{N}^V_{\text{Kahler}} \subset \mathcal{N}_{\text{Kahler}}$.

In physics applications it is $\mathcal{N}^V_{\text{Kahler}}$ rather than $\mathcal{N}_{\text{Kahler}}$ which appears as the target space of a sigma model, and therefore it must carry a positive definite metric. From section 2.5 we know that the PSR metric is positive definite if the 0-conical metric $\partial^2 \log V$ is positive definite, and equivalently if the 3-conical metric $\partial^2 V$ has real Lorentz signature $(1, h^{1,1} - 1)$. These conditions are indeed satisfied in Calabi-Yau compactifications. The distinction between time-like and space-like directions with respect to $\partial^2 V$ in the space of $(1,1)$-forms corresponds to the so-called Lefschetz decomposition of $H^2(X, \mathbb{R})$ into 'primitive forms', which are orthogonal to the Kähler form ω, and the direction parallel to the Kähler form.

[39] b_2 is the second Betti number of X. Note that for Calabi-Yau three-folds $h^{2,0} = h^{0,2} = 0$, hence $b_2 = h^{1,1}$.

A related but different question is to determine the maximal domain in $H^{1,1}(X,\mathbb{R}) \cong \mathbb{R}^{h^{1,1}}$ where the PSR metric is positive definite. The boundary of this region can be characterized using the simpler 3-conical metric $\partial^2 V$ by $\det(\partial^2 V) = 0$. Note that the region where the PSR metric is positive definite is in general larger than the Kähler cone. Therefore it is important to keep track of Kähler cone of the underlying Calabi-Yau manifold when working with an effective supergravity theory. For example, in [67, 68] it has been shown that naked singularities which are naively present in some solutions of five-dimensional supergravity are unphysical if the theory is obtained as a Calabi-Yau compactification of eleven-dimensional supergravity, because at the singularity the scalar fields take values which are inside the domain where the PSR metric is positive definite, but outside the Kähler cone of the underlying Calabi-Yau manifold. If the theory is considered as embedded into M-theory one needs to modify the effective Lagrangian when the boundary of the Kähler cone is reached, even though all data in the Lagrangian, and solutions with space or time dependent moduli, remain smooth at this point. The modification of the Lagrangian corresponds to continuing into the Kähler cone of another Calabi-Yau manifold, which differs from the original one by a transition which changes the topology. At the boundary of the Kähler cone additional massless vector or hypermultiplets are present. Integrating out these multiplets induces threshold corrections to the couplings in the effective Lagrangian for the remaining modes, which for five-dimensional vector multiplets induce finite shifts of the coefficients of the Hesse potential [69, 67, 68]. The proper treatment of this subtlety removes naked singularities which are naively present in domain and black hole solutions with non-constant scalars. In this sense, the Kähler cone acts as a cosmic censor.

So far we have been discussing the moduli space of Ricci-flat metrics on X. In supergravity and string compactifications, there are additional moduli resulting from the dimensional reduction of various p-form fields. Massless scalar fields arise whenever the components of such a p-form along X are harmonic forms on X. The number of massless scalars is given by the corresponding Hodge number. Such massless scalars are moduli, unless the effective theory contains a potential for them, which is not the case for Calabi-Yau compactifications in the absence of fluxes. A particular role is played by the Kalb-Ramond two-form field B of string theory. When reducing a type-II string theory on a Calabi-Yau three-fold, the B-field gives rise to $h^{1,1}$ real moduli, which naturally combine with the moduli of the Kähler structure. Defining a complexified Kähler form and expanding in the basis ω_A of $H^{1,1}(X,\mathbb{R})$,

$$\omega_{\mathbb{C}} = B + i\omega = z^A \omega_A \ , \quad z^A \in \mathbb{C} \ , \quad A = 1, \ldots h^{1,1} \ , \tag{411}$$

we obtain complex coordinates z^A on the moduli space $\mathcal{M}_{\text{Kahler}}$ of *complexified Kähler structures*. This space turns out to be a Kähler manifold with a Kähler potential that is obtained from the Hesse potential $\log V$, as follows.

Generally, given a Hessian manifold N of dimension n with local coordinates t^A and Hesse potential H, we can extend this to a complex manifold $M \cong \mathbb{R}^n \times N$, with coordinates $z^A = s^A + it^A$. The Hessian metric on N can be extended to a Kähler metric on M with Kähler potential

$$K(z,\bar{z}) = H(\text{Im}(z)) \ . \tag{412}$$

This defines a Kähler metric with metric coefficients[40]

$$g_{A\bar{B}} = \frac{\partial^2 K}{\partial z^A \partial \bar{z}^{\bar{B}}} = 4 \frac{\partial^2 H}{\partial t^A \partial t^B} \ , \tag{413}$$

which has an isometry group which contains the n commuting shifts $z^A \mapsto z^A + r^A$, $r^A \in \mathbb{R}$.

In the case at hand, we can use the Hesse potential $-\log V$ of a 0-conical Hessian metric on $\mathcal{N}_{\text{Kahler}}$ as a Kähler potential for a Kähler metric $\mathcal{M}_{\text{Kahler}}$.[41] To see that this metric is actually a PSK metric, we introduce projective coordinates X^I, $I = 0, \ldots h^{1,1}$ on $\mathcal{M}_{\text{Kahler}}$ by choosing local holomorphic functions $X^I(z)$ such that $X^A/X^0 = z^A$. Then we define the holomorphic function, homogeneous of degree two,

$$F = \frac{1}{3!} \frac{C_{ABC} X^A X^B X^C}{X^0} \ . \tag{414}$$

It is straightforward to see that

$$\begin{aligned} & -i\left(X^I \bar{F}_I - F_I \bar{X}^I\right) \\ = \quad & -\frac{i}{3!}|X^0|^2 C_{ABC}(z^A - \bar{z}^{\bar{A}})(z^B - \bar{z}^{\bar{B}})(z^C - \bar{z}^{\bar{C}}) = 8|X^0|^2 V \ . \end{aligned} \tag{415}$$

[40] We do not correct for the factor 4, which comes from the Jacobian, so that the two metrics differ by a constant factor. This is does not matter here, since we only want to illustrate the principle. In applications the normalization is fixed by the Lagrangian of the explicit model one considers.

[41] The minus sign is introduced for consistency with the literature.

Theory	Number of vector multiplets	Moduli space	Geometry
M-theory	$h^{1,1}-1$	$\mathcal{N}^V_{\text{Kahler}}$	PSR
II A	$h^{1,1}$	$\mathcal{M}_{\text{Kahler}}$	PSK
II B	$h^{2,1}$	$\mathcal{M}_{\text{cplx}}$	PSK

Table 3: This table shows which moduli of a Calabi-Yau compactification sit in vector multiplets. PSR=projective special real, PSK = projective special Kähler

Therefore the Kähler potentials $-\log V$ and $-\log\left(-i(X^I\bar{F}_I - F_I\bar{X}^I)\right)$ differ by a Kähler transformation and define the same Kähler metric on $\mathcal{M}_{\text{Kahler}}$.[42] Therefore the metric on $\mathcal{M}_{\text{Kahler}}$ is a PSK metric with prepotential (414) and Kähler potential

$$K = -\log\left(-i(X^I\bar{F}_I - F_I\bar{X}^I)\right) \ . \tag{416}$$

We remark that in string theory the 'very special' cubic form of the prepotential only holds to leading order in perturbation theory and is subject to complicated corrections.

We conclude by indicating how the moduli of Calabi-Yau compactifications of eleven-dimensional supergravity and of type-II string theories are distributed among five- and four-dimensional $\mathcal{N} = 2$ supermultiplets. Moduli are either allocated to vector multiplets, where the geometry of the target space is PSR or PSK, or to hypermultiplets, where the geometry is quaternionic Kähler, denoted QK in the tables. The dimension of a quaternionic Kähler manifold is divisible by four. The maximal dimension of a Kähler submanifold of a QK manifold is half of the total dimension [70]. Hypermultiplets contain a mixture of moduli of the metric, moduli resulting from reducing p-form gauge fields, and, for type II string theory, the dilaton and the axion obtained from dualizing the Kalb-Ramond two-form. The PSK spaces $\mathcal{M}_{\text{cplx}}$ and $\mathcal{M}_{\text{Kahler}}$ are Kähler submanifolds of hypermultiplet target manifolds, at least to lowest order in α'.

Table 3 lists vector multiplet moduli, Table 4 lists hypermultiplet moduli. In compactifications from eleven to five dimensions, the moduli of the real Kähler form split: the volume modulus sits in a hypermultiplet, the remaining Kähler moduli parametrizing the fixed volume hypersurface in the Kähler cone sit in

[42] Note that the domains where the argument of the logarithm is positive agree. Therefore both Kähler potentials are defined over the same domain.

Theory	Number of hypermultiplets	Moduli space	Geometry
M-theory/IIA	$h^{2,1}+1$	$\mathcal{M}_{\rm cplx} \subset \mathcal{M}_{HM}$	PSK $\subset$ QK
II B	$h^{1,1}+1$	$\mathcal{M}_{\rm Kahler} \subset \mathcal{M}_{HM}$	PSK $\subset$ QK

Table 4: This table shows which moduli of a Calabi-Yau compactification sit in hypermultiplets. PSK = projective special Kähler, QK = quaternionic Kähler.

vector multiplets. Note that this split is required in order to obtain a PSR manifold. The volume modulus and the complex structure modulus sit in hypermultiplets together with moduli coming from reducing p-form gauge fields. In compactifications from ten to four dimensions the moduli of the Kähler form and those of the Kalb-Ramond B-field combine into complex moduli. Depending on whether one considers the IIA or IIB theory, either the moduli of the complexified Kähler form or the moduli of the complex structure moduli sit in vector multiplets. The remaining moduli of the metric sit in hypermultiplets together with the dilaton, the axion obtained by dualizing the Kalb-Ramond field, and moduli associated to p-form gauge fields in the Ramond-Ramond sector.

6. Four-dimensional vector multiplets

6.1. Rigid vector multiplets

The field content of a four-dimensional rigid abelian vector multiplet is $(X, \Omega_i, A_\mu, Y_{ij})$ [71]. X denotes a complex scalar field; A_μ denotes an abelian gauge field with field strength $F = dA$; Ω_i denotes an $SU(2)_R$ doublet of chiral fermions; Y_{ij} denotes an $SU(2)_R$ triplet of scalar fields, i.e. Y_{ij} is a symmetric matrix satisfying the reality condition

$$Y_{ij} = \varepsilon_{ik}\,\varepsilon_{jl}\,Y^{kl} \quad , \quad Y^{ij} = (Y_{ij})^* \; . \tag{417}$$

Thus, off-shell, an abelian vector multiplet has eight bosonic and eight fermionic real degrees of freedom.

We are interested in the Lagrangian describing the dynamics of n abelian vector multiplets. These vector multiplets will be labelled by an index $I = 1, \ldots, n$. The Lagrangian is encoded [31] in a holomorphic function $F(X)$, called the prepotential. We denote holomorphic derivatives of $F(X)$ with respect to X^I by $F_I = \partial F/\partial X^I$, $F_{IJ} = \partial^2 F/\partial X^I \partial X^J$, etc. We denote the complex conjugate of X^I by $\bar{X}^I$, and anti-holomorphic derivatives of $\bar{F}(\bar{X})$ by $\bar{F}_I = \partial\bar{F}/\partial\bar{X}^I$, etc.

The bosonic part of the Lagrangian reads

$$L = -N_{IJ}\,\partial_\mu X^I\,\partial^\mu \bar{X}^J + \left(\tfrac{1}{4} i\, F_{IJ}\, F^{-I}_{\mu\nu} F^{\mu\nu - J} - \tfrac{1}{8} i\, F_{IJ}\, Y^I_{ij} Y^{J\,ij} + \text{h.c.}\right) \ , \quad (418)$$

where

$$F^I_{\mu\nu} = 2\partial_{[\mu} A^I_{\nu]} \ , \quad (419)$$

and where N_{IJ} is given by (264). Note that the kinetic terms for the scalar fields and for the abelian gauge fields are determined in terms of N_{IJ},

$$L_{\text{kin}} = -N_{IJ}\,\partial_\mu X^I\,\partial^\mu \bar{X}^J - \tfrac{1}{8}\,N_{IJ}\,F^I_{\mu\nu} F^{\mu\nu J} \ . \quad (420)$$

The kinetic term for the scalar fields describes a sigma-model, whose target space is an affine special Kähler (ASK) manifold. This is a Riemannian manifold with Kähler metric $N_{IJ} = \partial^2 K(X,\bar{X})/\partial X^I \partial \bar{X}^J$ and Kähler potential (266).

As discussed in subsection 5.1.1, the metric g of an ASK manifold, when expressed in terms of special real coordinates $q^a = (x^I, y_I) = (\text{Re}\, X^I, \text{Re}\, F_I)$, is Hessian,

$$g = N_{IJ}\, dX^I d\bar{X}^J = H_{ab} dq^a dq^b \quad , \quad a,b = 1,\ldots,2n \ , \quad (421)$$

where $H_{ab} = \partial^2 H/\partial q^a \partial q^b$ is determined in terms of the real Hesse potential H. The Hesse potential H is related to the prepotential F by Legendre transformation, c.f. (285). As in subsection 5.1.4, we decompose (X^I, F_I) into real and imaginary parts,

$$\begin{aligned} X^I &= x^I + iu^I \ , \\ F_I &= y_I + iv_I \ . \end{aligned} \quad (422)$$

Next, we perform the Legendre transform of the imaginary part of F with respect to u^I, thereby replacing u^I by y_I as independent variables,

$$H(x,y) = 2\,\text{Im}\, F(x+iu) - 2\, y_I\, u^I \ , \quad (423)$$

where

$$\frac{\partial \text{Im}\, F}{\partial u^I} = y_I \ . \quad (424)$$

The latter expresses u as a function of (x,y), locally, and inserting this expression on the right hand side of (423) yields $H(x,y)$.

6.2. Rigid superconformal vector multiplets

Next, we specialize to the case where the vector multiplet theory is superconformal. This implies that $F(X)$ must be homogeneous of degree 2 under complex scalings,

$$F(\lambda X) = \lambda^2 \, F(X) \quad , \quad \lambda \in \mathbb{C}^* \; , \tag{425}$$

from which one infers the relations (192). The associated Hesse potential is homogeneous of degree 2, and the scalar manifold is a conical affine special Kähler manifold.

6.3. Superconformal matter multiplets coupled to conformal supergravity

As in the five-dimensional case, we will follow the superconformal approach to construct a theory of n abelian vector multiplets coupled to Poincaré supergravity. This is based on the fact that a theory of n vector multiplets and n_H hypermultiplets coupled to Poincaré supergravity is *gauge equivalent* to a theory of $n+1$ superconformal vector multiplets and n_H+1 superconformal hypermultiplets coupled to conformal supergravity.

6.3.1. Coupling of vector multiplets

First, we consider the coupling of $n+1$ abelian vector multiplets to conformal supergravity at the two-derivative level. The index I labelling these abelian vector multiplets now runs over $I = 0, 1, \ldots, n$. The component fields of the abelian vector multiplets carry the Weyl and chiral weights given in Table B.14. Then, using (B.82), we have

$$\mathcal{D}_\mu X^I = (\partial_\mu - \, b_\mu + i \, A_\mu) \, X^I \; . \tag{426}$$

The bosonic part of the Lagrangian describing the coupling of abelian vector multiplets to conformal supergravity reads,

$$\begin{aligned} L \;\; = \;\; & \Big[i\mathcal{D}^\mu F_I \, \mathcal{D}_\mu \bar{X}^I - iF_I \, \bar{X}^I (-\tfrac{1}{6} R - D) - \tfrac{1}{8} i F_{IJ} \, Y^I_{ij} Y^{Jij} \\ & + \tfrac{1}{4} i F_{IJ} (F^{-I}_{\mu\nu} - \tfrac{1}{2} \bar{X}^I T^-_{\mu\nu})(F^{\mu\nu\,-J} - \tfrac{1}{2} \bar{X}^J T^{\mu\nu -}) \\ & - \tfrac{1}{4} i F_I (F^{+I}_{\mu\nu} - \tfrac{1}{2} X^I T^+_{\mu\nu}) T^{\mu\nu +} - \tfrac{1}{8} i F \, T^+_{\mu\nu} T^{\mu\nu +} + \text{h.c.} \Big] \; . \end{aligned} \tag{427}$$

This equals

$$\begin{aligned} L = & -N_{IJ} \mathcal{D}^\mu X^I \, \mathcal{D}_\mu \bar{X}^J - i \left(F_I \, \bar{X}^I - X^I \bar{F}_I \right) (-\tfrac{1}{6} R - D) + \tfrac{1}{8} \, N_{IJ} \, Y^I_{ij} Y^{Jij} \\ & + \left(-\tfrac{1}{4} i \bar{F}_{IJ} F^{+I}_{\mu\nu} F^{\mu\nu\,+J} - \tfrac{1}{16} N_{IJ} X^I X^J T^+_{\mu\nu} T^{\mu\nu +} + \tfrac{1}{4} \, N_{IJ} X^I \, F^{+J}_{\mu\nu} T^{\mu\nu\,+} + \text{h.c.} \right) \; . \end{aligned} \tag{428}$$

6.3.2. Coupling of hypermultiplets

We consider the coupling of $r = n_H + 1$ hypermultiplets that are neutral with respect to the gauge symmetries of the vector multiplets. We follow the presentation given in [28], which is based on sections $A_i{}^\alpha(\phi)$ of an $\mathrm{Sp}(r) \times \mathrm{Sp}(1)$ bundle ($\alpha = 1, \ldots, 2r; i = 1, 2$) which depend on scalar fields ϕ^A, defined in the context of a so-called hyper-Kähler cone of dimension $4r$.

The bosonic part of the Lagrangian describing the coupling of hypermultiplets to conformal supergravity is given by

$$-\tfrac{1}{2}\varepsilon^{ij}\,\bar{\Omega}_{\alpha\beta}\,\mathcal{D}_\mu A_i{}^\alpha\,\mathcal{D}^\mu A_j{}^\beta + \chi(-\tfrac{1}{6}R + \tfrac{1}{2}D)\;, \tag{429}$$

where the hyper-Kähler potential χ and the covariant derivative $\mathcal{D}_\mu A_i{}^\alpha(\phi)$ are given in (B.94).

6.3.3. Poincaré supergravity

Combining the bosonic Lagrangians (428) and (429), we obtain

$$\begin{aligned} L \;=\; & \left[i\left(\bar{X}^I F_I - X^I \bar{F}_I\right) - \chi\right]\tfrac{1}{6}\,R + \left[i\left(\bar{X}^I F_I - X^I \bar{F}_I\right) + \tfrac{1}{2}\chi\right] D \\ & - N_{IJ}\mathcal{D}^\mu X^I\,\mathcal{D}_\mu \bar{X}^J + \tfrac{1}{8}\,N_{IJ}\,Y_{ij}^I Y^{Jij} \\ & + \left(-\tfrac{1}{4}i\bar{F}_{IJ}F_{\mu\nu}^{+I}F^{\mu\nu\,+J} - \tfrac{1}{16}N_{IJ}X^I X^J T_{\mu\nu}^{+}T^{\mu\nu+}\right. \\ & \left.\quad + \tfrac{1}{4}\,N_{IJ}X^I\,F_{\mu\nu}^{+J}T^{\mu\nu\,+} + \text{h.c.}\right) - \tfrac{1}{2}\varepsilon^{ij}\,\bar{\Omega}_{\alpha\beta}\,\mathcal{D}_\mu A_i{}^\alpha\,\mathcal{D}^\mu A_j{}^\beta\;. \end{aligned} \tag{430}$$

Note that the field D does not have a kinetic term: it appears as a multiplier. Its field equation yields the condition

$$\chi = -2i\left(\bar{X}^I F_I - X^I \bar{F}_I\right)\;. \tag{431}$$

Similarly, the field equation for Y_{ij} is simply

$$Y_{ij} = 0\;. \tag{432}$$

Inserting (431) and (432) into (430) yields

$$\begin{aligned} L \;=\; & i\left(\bar{X}^I F_I - X^I \bar{F}_I\right)\tfrac{1}{2}\,R - N_{IJ}\mathcal{D}^\mu X^I\,\mathcal{D}_\mu \bar{X}^J \\ & + \left(-\tfrac{1}{4}i\bar{F}_{IJ}F_{\mu\nu}^{+I}F^{\mu\nu\,+J} - \tfrac{1}{16}N_{IJ}X^I X^J T_{\mu\nu}^{+}T^{\mu\nu+}\right. \\ & \left.\quad + \tfrac{1}{4}\,N_{IJ}X^I\,F_{\mu\nu}^{+J}T^{\mu\nu\,+} + \text{h.c.}\right) - \tfrac{1}{2}\varepsilon^{ij}\,\bar{\Omega}_{\alpha\beta}\,\mathcal{D}_\mu A_i{}^\alpha\,\mathcal{D}^\mu A_j{}^\beta\;. \end{aligned} \tag{433}$$

Next, we use the symmetries of conformal supergravity to impose gauge conditions. We begin by fixing the freedom to perform dilations (whose generator is

D, see Table B.7), by picking

$$i\left(\bar{X}^I F_I - X^I \bar{F}_I\right) = \kappa^{-2} \quad , \quad \mathrm{D-gauge} \ . \tag{434}$$

This is the so-called D-gauge. Here $\kappa^2 = 8\pi G_N$, where G_N denotes the Newton's constant. With this choice, we obtain the Einstein-Hilbert term (B.62). In the following, we set $\kappa^2 = 1$. Note that with the choice (434), we obtain

$$\chi = -2 \ , \tag{435}$$

which shows that at least one hypermultiplet is needed in order to obtain the Einstein-Hilbert term (B.62). The condition (435) removes one real bosonic degree of freedom in the hypermultiplet sector. Fixing the freedom under $SU(2)_R$ transformations (c.f. (B.94)) removes three additional real degrees of freedom in the hypermultiplet sector, so that in total, we have removed four real degrees of freedom. This amounts to removing the bosonic degrees of freedom of one hypermultiplet. There are then $r - 1 = n_H$ physical hypermultiplets left. We will not consider them any further, and hence we drop them in what follows.

Now we pick the K-gauge (B.72), which removes the dilational connection b_μ from the covariant derivatives (426) and (B.94). Next, varying with respect to the $U(1)$ connection A_μ gives (c.f. (373))

$$A_\mu = \tfrac{1}{2} i \frac{N_{IJ}\left((\partial_\mu X^I)\, \bar{X}^J - X^I\, \partial_\mu \bar{X}^J\right)}{N_{KL} X^K \bar{X}^L} \ . \tag{436}$$

In the D-gauge (434), where $N_{KL} X^K \bar{X}^L = -1$, this becomes

$$\begin{aligned} A_\mu &= \tfrac{1}{2} i\, N_{IJ}\left(X^I\, \partial_\mu \bar{X}^J - (\partial_\mu X^I)\, \bar{X}^J\right)_{|_{N_{KL}X^K\bar{X}^L=-1}} \ , \\ &= -\tfrac{1}{2} i \left(\partial_a K\, \partial_\mu z^a - \partial_{\bar{a}} K \partial_\mu \bar{z}^a\right) \\ &= A_a\, \partial_\mu z^a - A_{\bar{a}} \partial_\mu \bar{z}^a \ , \end{aligned} \tag{437}$$

where $z^a = X^a/X^0$ denote complex physical scalar fields ($a = 1, \ldots, n$), and where $K(z, \bar{z})$ denotes the Kähler potential given in (B.104). This is in agreement with (B.122). The connection A_μ is the pull-back to space-time of the connection A_a given in (369). Finally, varying with respect to $T^+_{\mu\nu}$ gives

$$T^+_{\mu\nu} = 2 \frac{N_{IJ} X^I}{N_{KL} X^K X^L}\, F^{+J}_{\mu\nu} \ . \tag{438}$$

Thus, at the two-derivative level, the fields A_μ and $T^{\pm}_{\mu\nu}$ are auxiliary fields.

Inserting these various expressions into (433), using the relation (B.126) and dropping terms that involve physical hypermultiplets, we obtain the following gauge fixed Lagrangian,

$$\begin{aligned} L &= \tfrac{1}{2}R - g_{a\bar{b}}\,\partial_\mu z^a\,\partial^\mu \bar{z}^b + \left(-\tfrac{1}{4}i\mathcal{N}_{IJ}F^{+I}_{\mu\nu}F^{\mu\nu\,+J} + \text{h.c.}\right) \\ &= \tfrac{1}{2}R - g_{a\bar{b}}\,\partial_\mu z^a\,\partial^\mu \bar{z}^b + \tfrac{1}{4}\text{Im}\mathcal{N}_{IJ}\,F^I_{\mu\nu}\,F^{\mu\nu J} - \tfrac{i}{4}\text{Re}\mathcal{N}_{IJ}\,F^I_{\mu\nu}\,\tilde{F}^{\mu\nu J}, \end{aligned} \tag{439}$$

where

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + i\frac{N_{IP}X^P\,N_{JQ}X^Q}{N_{KL}X^KX^L} \tag{440}$$

with $N_{KL}X^K\bar{X}^L = -1$. The resulting Lagrangian describes the bosonic part of the action for vector multiplets coupled to Poincaré supergravity. It is obtained from the action for matter multiplets coupled to conformal supergravity by using two compensating multiplets: one vector multiplet, and one hypermultiplet.

6.4. Coupling to a chiral background

The construction of the action (430) describing the coupling of abelian vector multiplets and neutral hypermultiplets to conformal supergravity at the two-derivative level can be extended, within the superconformal approach, to allow for the presence of a chiral background field [32]. This is achieved by allowing the function $F(X)$ that enters in the construction of (430), to depend on an additional holomorphic field $\hat{A}$, so that now $F(X,\hat{A})$. The background field $\hat{A}$ is introduced as the lowest component of a chiral supermultiplet. Compatibility with superconformal symmetry determines the scaling behaviour of the chiral multiplet, while insisting on a local supersymmetric action implies that the dependence on the chiral multiplet is holomorphic. Therefore, the function F has to be (graded) homogeneous of degree two, that is

$$F(\lambda X, \lambda^w \hat{A}) = \lambda^2 F(X,\hat{A})\;, \quad \lambda \in \mathbb{C}^*\;, \tag{441}$$

where w is the weight of $\hat{A}$ under scale transformations. It follows that F satisfies the relation,

$$X^I F_I + w\hat{A}\,F_A = 2F\,. \tag{442}$$

Here F_I and F_A denote the derivatives of $F(X,\hat{A})$ with respect to X^I and $\hat{A}$, respectively.

We denote the component fields of the chiral background superfield with a caret. We focus on the bosonic component fields, which we denote by $\hat{A}$,

$\hat{B}_{ij}$, $\hat{F}^-_{ab}$ and by $\hat{C}$. Here $\hat{A}$ and $\hat{C}$ denote complex scalar fields, appearing at the θ^0- and θ^4-level of the chiral background superfield, respectively, while the symmetric complex SU(2) tensor $\hat{B}_{ij}$ and the anti-selfdual Lorentz tensor $\hat{F}^-_{ab}$ reside at the θ^2-level.

In the presence of the chiral background, the action (430) becomes encoded in $F(X,\hat{A})$, and reads as follows,

$$\begin{aligned} L \;=\; & \Big[i\mathcal{D}^\mu F_I\,\mathcal{D}_\mu \bar{X}^I - iF_I\,\bar{X}^I(-\tfrac{1}{6}R - D) - \tfrac{1}{8}iF_{IJ}\,Y^I_{ij}Y^{Jij} - \tfrac{1}{4}i\hat{B}_{ij}\,F_{AI}Y^{Iij} \\ & + \tfrac{1}{4}iF_{IJ}(F^{-I}_{ab} - \tfrac{1}{2}\bar{X}^I T^-_{ab})(F^{ab-J} - \tfrac{1}{2}\bar{X}^J T^{ab-}) \\ & - \tfrac{1}{8}iF_I(F^{+I}_{ab} - \tfrac{1}{2}X^I T^+_{ab})T^{ab+} + \tfrac{1}{2}i\hat{F}^{-ab}\,F_{AI}(F^{-I}_{ab} - \tfrac{1}{2}\bar{X}^I T^-_{ab}) \\ & + \tfrac{1}{2}iF_A\hat{C} - \tfrac{1}{8}iF_{AA}(\varepsilon^{ik}\varepsilon^{jl}\hat{B}_{ij}\hat{B}_{kl} - 2\hat{F}^-_{ab}\hat{F}^{-ab}) - \tfrac{1}{8}iF\,T^+_{ab}T^{ab+} + \text{h.c.}\Big] \\ & - \tfrac{1}{2}\varepsilon^{ij}\,\bar{\Omega}_{\alpha\beta}\,\mathcal{D}_\mu A_i{}^\alpha\,\mathcal{D}^\mu A_j{}^\beta + \chi(-\tfrac{1}{6}R + \tfrac{1}{2}D)\,. \end{aligned} \tag{443}$$

The last line pertains to the hypermultiplets, as discussed in subsection 6.3.2.

6.4.1. Coupling to R^2 terms

When identifying the chiral background superfield with the square of the Weyl superfield, the action (443) will contain higher-derivative curvature terms proportional to the square of the Weyl tensor. In this case the chiral weight w in (441) equals $w = 2$, and the bosonic fields of the chiral background superfield becomes identified with

$$\begin{aligned} \hat{A} &= 4\,(T^-_{ab})^2\,, \\ \hat{B}_{ij} &= -32\,\varepsilon_{k(i}R(\mathcal{V})^k{}_{j)ab}\,T^{ab-}\,, \\ \hat{F}^{-ab} &= 32\,\mathcal{R}(M)_{cd}{}^{ab}\,T^{cd-}\,, \\ \hat{C} &= 64\,\mathcal{R}(M)^{-cd}{}_{ab}\,\mathcal{R}(M)^{-ab}_{cd} + 32\,R(\mathcal{V})^{-ab\,k}{}_l\,R(\mathcal{V})^{-l}_{ab\;k} - 64\,T^{ab-}\,D_a\,D^c T^+_{cb}\,. \end{aligned} \tag{444}$$

In these expressions, we have suppressed all terms that involve fermionic fields. The curvatures appearing in (444) are given by

$$\begin{aligned} R(\mathcal{V})_{\mu\nu}{}^i{}_j &= 2\partial_{[\mu}\mathcal{V}^i_{\nu]j} + \mathcal{V}^i_{[\mu k}\mathcal{V}^k_{\nu]j} \\ \mathcal{R}(M)_{ab}{}^{cd} &= R_{ab}{}^{cd} + 8f_{[a}{}^{[c}\delta_{b]}{}^{d]} - \tfrac{1}{8}\Big(T^{cd+}\,T^-_{ab} + T^+_{ab}\,T^{cd-}\Big)\,, \end{aligned} \tag{445}$$

where we recall that $R_{ab}{}^{cd}$ is computed using the spin connection (B.67). Note that the T^2-modification in (445) exactly cancels the T^2-terms contained in $f_\mu{}^a$, as can be verified by using the relation (B.80),

$$\mathcal{R}(M)_{ab}{}^{cd} = C_{ab}{}^{cd} - D\,\delta_{[a}{}^{[c}\delta_{b]}{}^{d]} - 2i\,\tilde{R}_{[a}{}^{[c}(T)\,\delta_{b]}{}^{d]}\,, \tag{446}$$

where

$$C_{ab}{}^{cd} = R_{ab}{}^{cd} - 2\left(R_{[a}{}^{[c} - \tfrac{1}{6} R\, \delta_{[a}{}^{[c}\right) \delta_{b]}{}^{d]} \;. \tag{447}$$

In the K-gauge (B.72), $C_{ab}{}^{cd}$ denotes the Weyl tensor, and $\hat{C}$ includes a term proportional to the square of the anti-selfdual part of the Weyl tensor,

$$\hat{C} = 64\, C^{-cd}{}_{ab}\, C^{-ab}_{cd} + \ldots \tag{448}$$

The term $T^{ab-}\, D_a\, D^c T^+_{cb}$ in (444) is written out in (B.92).

Observe that the $U(1)$ connection A_μ and the field T^-_{ab} cannot any longer be eliminated in closed form, as in (436) as in (438) at the two-derivative level, but only iteratively. In particular, T^-_{ab} can be eliminated iteratively by means of an expansion of $F(X, \hat{A})$ in powers of $\hat{A}$,

$$F(X, \hat{A}) = \sum_{n=0}^{\infty} F^{(n)}(X)\, \hat{A}^n \;, \tag{449}$$

which generates an expansion with infinitely many higher-derivative terms that are all proportional to $\hat{C}$. This results in an action that contains infinitely many higher-derivative terms that are proportional to the square of the anti-selfdual part of the Weyl tensor. Such an action is naturally interpreted as a Wilsonian effective action.

7. Hessian geometry in the presence of a chiral background

In this section, we discuss the geometric meaning of deformations of the prepotential function $F^{(0)}(X)$ by chiral background fields, such as in (441). We begin by considering holomorphic deformations of $F^{(0)}(X)$. We use the description of affine special Kähler manifolds as immersions, to introduce the notion of deformed affine special Kähler manifolds [72]. We then discuss the existence of a Hessian structure on these deformed manifolds, and relate the Hessian structure to the holomorphic anomaly equation for a hierarchy of symplectic functions.

Subsequently, we turn to non-holomorphic deformations of $F^{(0)}(X)$. We follow [72].

7.1. Holomorphic deformation of the immersion

We deform the prepotential $F^{(0)}(X)$ by allowing for the presence of a complex deformation parameter Υ. The prepotential $F^{(0)}(X)$ gets replaced by the generalized prepotential $F(X, \Upsilon)$, which is holomorphic in X^I and Υ.

7.1.1. Holomorphic family of immersions

The geometric model for the deformation parametrized by Υ is a map [72]

$$\phi : \hat{M} := M \times \mathbb{C} \to V \ , \quad (X^I, \Upsilon) \mapsto (X^I, F_I(X, \Upsilon)) \ , \tag{450}$$

which can be interpreted as a holomorphic family of immersions $\phi_\Upsilon : M \to V \ , \quad (X^I) \mapsto (X^I, F_I(X, \Upsilon))$, that define a family of affine special Kähler structures on M.

Next, we define a metric and a two-form on $\hat{M} = M \times \mathbb{C}$ by pulling back the canonical Hermitian form γ_V given in (267),

$$\gamma = \phi^* \gamma_V = g + i\omega = N_{IJ} dX^I \otimes d\bar{X}^J + i\bar{F}_{I\Upsilon} dX^I \otimes d\bar{\Upsilon} - iF_{I\Upsilon} d\Upsilon \otimes d\bar{X}^I \ , \tag{451}$$

where $N_{IJ} = -i(F_{IJ} - \bar{F}_{IJ})$, and where $F_{I\Upsilon} = \partial_I \partial_\Upsilon F$. We assume that γ is non-degenerate. Denoting the holomorphic coordinates on $\hat{M}$ by $(v^A) = (X^I, \Upsilon)$, we obtain for the metric on $\hat{M}$,

$$g = g_{A\bar{B}} \, dv^A d\bar{v}^B = N_{IJ} dX^I d\bar{X}^J + i\bar{F}_{I\Upsilon} dX^I d\bar{\Upsilon} - iF_{J\Upsilon} d\Upsilon d\bar{X}^J \ , \tag{452}$$

which is a Kähler metric $g_{A\bar{B}} = \partial_A \partial_{\bar{B}} K$ with Kähler potential

$$K = -i \left(\bar{X}^I F_I(X, \Upsilon) - X^I \, \bar{F}_I(\bar{X}, \bar{\Upsilon}) \right) \ . \tag{453}$$

The associated Kähler form is

$$\omega = -\frac{i}{2} N_{IJ} dX^I \wedge d\bar{X}^J + \frac{1}{2} \bar{F}_{I\Upsilon} dX^I \wedge d\bar{\Upsilon} - \frac{1}{2} F_{I\Upsilon} d\Upsilon \wedge d\bar{X}^I \ . \tag{454}$$

The Kähler metric $g_{A\bar{B}}$ has occured in the deformed sigma model [73], which provides a field theoretic realization of the set-up just described.

For latter use, we introduce the decomposition

$$2F_{IJ} = R_{IJ} + iN_{IJ} \ , \tag{455}$$

where $R_{IJ} = 2 \, \mathrm{Re} \, F_{IJ}$, $N_{IJ} = 2 \, \mathrm{Im} \, F_{IJ}$. We denote the inverse of N_{IJ} by $N^{-1} = (N^{IJ})$.

7.1.2. The Hesse potential

We now define special real coordinates and a Hesse potential in presence of the deformation. We then show that the Kähler metric g on $\hat{M}$ given in (452) is no longer Hessian. There is, however, another metric on $\hat{M}$ that is Hessian.

We denote this metric by g^H. We show that $\hat{M} = M \times \mathbb{C}$ can be equipped with a Hessian structure (∇, g^H), where $g^H \neq g$.

We introduce real coordinates $(q^a) = (x^I, y_I)$ as in (291)

$$X^I = x^I + iu^I(x, y, \Upsilon, \bar{\Upsilon}) \;, \quad F_I = y_I + iv_I(x, y, \Upsilon, \bar{\Upsilon}) \;. \tag{456}$$

Then, the (generalized) Hesse potential is defined by a Legendre transform of the generalized prepotential $F(X, \Upsilon)$,

$$H(x, y, \Upsilon, \bar{\Upsilon}) = 2\,\mathrm{Im}\, F(x + iu(x, y, \Upsilon, \bar{\Upsilon}), \Upsilon) - 2y_I u^I(x, y, \Upsilon, \bar{\Upsilon}) \;. \tag{457}$$

Note that H is homogeneous of degree two.

We will be interested in the coordinate transformations

$$\begin{aligned} (x, u, \Upsilon, \bar{\Upsilon}) &\mapsto (x, y, \Upsilon, \bar{\Upsilon}) \;, \\ (x, y, \Upsilon, \bar{\Upsilon}) &\mapsto (x, u, \Upsilon, \bar{\Upsilon}) \;. \end{aligned} \tag{458}$$

To convert from one coordinate system to the other one, we use the following formulae when differentiating a function $\tilde{f}(x, u, \Upsilon, \bar{\Upsilon}) = f(x, y(x, u, \Upsilon, \bar{\Upsilon}), \Upsilon, \bar{\Upsilon})$,

$$\begin{aligned} \left.\frac{\partial \tilde{f}}{\partial x^I}\right|_u &= \left.\frac{\partial f}{\partial x^I}\right|_y + \left.\frac{\partial f}{\partial y_K}\right|_x \frac{\partial y_K}{\partial x^I} \;, \\ \left.\frac{\partial \tilde{f}}{\partial u^I}\right|_x &= \left.\frac{\partial f}{\partial y_K}\right|_x \frac{\partial y_K}{\partial u^I} \;, \\ \left.\frac{\partial \tilde{f}}{\partial \Upsilon}\right|_{x,u} &= \left.\frac{\partial f}{\partial \Upsilon}\right|_{x,y} + \left.\frac{\partial f}{\partial y_K}\right|_x \frac{\partial y_K}{\partial \Upsilon} \;. \end{aligned} \tag{459}$$

We refer to B.3, where we have collected various formulae with details on the conversion (458).

The Kähler metric g in (452), when expressed in coordinates $(q^a, \Upsilon, \bar{\Upsilon})$, takes the form

$$g = \frac{\partial^2 H}{\partial q^a \partial q^b} dq^a dq^b + \frac{\partial^2 H}{\partial q^a \partial \Upsilon} dq^a d\Upsilon + \frac{\partial^2 H}{\partial q^a \partial \bar{\Upsilon}} dq^a d\bar{\Upsilon} \;, \tag{460}$$

where

$$\left(\frac{\partial^2 H}{\partial q^a \partial q^b}\right) = \begin{pmatrix} N + RN^{-1}R & -2RN^{-1} \\ -2N^{-1}R & 4N^{-1} \end{pmatrix} \;, \tag{461}$$

and

$$\begin{aligned}
\frac{\partial^2 H}{\partial x^I \partial \Upsilon} &= 2\bar{F}_{IM} N^{MN} F_{N\Upsilon} \ , \quad \frac{\partial^2 H}{\partial x^I \partial \bar{\Upsilon}} = 2F_{IM} N^{MN} \bar{F}_{N\Upsilon} \ , \\
\frac{\partial^2 H}{\partial y_I \partial \Upsilon} &= -2N^{IJ} F_{J\Upsilon} \ , \quad \frac{\partial^2 H}{\partial y_I \partial \bar{\Upsilon}} = -2N^{IJ} \bar{F}_{J\Upsilon} \ .
\end{aligned} \tag{462}$$

In the undeformed case ($\Upsilon = 0$), the Kähler metric is also Hessian. In the deformed case ($\Upsilon \neq 0$), this is not any longer the case. This can be seen as follows.

First, we note that $\hat{M}$ can be equipped with a Hessian structure (∇, g^H). This requires the existence of a flat, torsion-free connection ∇, which can be constructed as follows. For fixed Υ, the map $\phi_\Upsilon : M \to V$ induces an affine special Kähler structure, with special connection ∇ and ∇-affine coordinates $(q^a) = (x^I, y_I)$. We can extend ∇ to a flat, torsion-free connection on $\hat{M} = M \times \mathbb{C}$ by imposing

$$\nabla dx^I = 0 \ , \quad \nabla dy_I = 0 \ , \quad \nabla d\Upsilon = 0 \ , \quad \nabla d\bar{\Upsilon} = 0 \ . \tag{463}$$

Since M can be covered by special real coordinate systems, we may extend these relations to $\hat{M}$, providing it with the affine structure required to define a flat, torsion-free connection on $\hat{M}$.

Now we define the metric g^H to be the Hessian metric of the (generalized) Hesse potential (457). Upon computing its components explicitly, we find that g^H differs from the Kähler metric g by

$$g^H - g = \partial^2 H_{|x,y} = \frac{\partial^2 H}{\partial \Upsilon \partial \Upsilon} d\Upsilon d\Upsilon + 2\frac{\partial^2 H}{\partial \Upsilon \partial \bar{\Upsilon}} d\Upsilon d\bar{\Upsilon} + \frac{\partial^2 H}{\partial \bar{\Upsilon} \partial \bar{\Upsilon}} d\bar{\Upsilon} d\bar{\Upsilon} \ , \tag{464}$$

where

$$\begin{aligned}
\frac{\partial^2 H}{\partial \Upsilon \partial \bar{\Upsilon}} &= N^{IJ} F_{I\Upsilon} \bar{F}_{J\Upsilon} \ , \quad \frac{\partial^2 H}{\partial \Upsilon \partial \Upsilon} = -iF_{\Upsilon\Upsilon} + N^{IJ} F_{I\Upsilon} F_{J\Upsilon} \ , \\
\frac{\partial^2 H}{\partial \bar{\Upsilon} \partial \bar{\Upsilon}} &= i\bar{F}_{\Upsilon\Upsilon} + N^{IJ} \bar{F}_{I\Upsilon} \bar{F}_{J\Upsilon} \ .
\end{aligned} \tag{465}$$

We remark that these metric coefficients are symplectic functions [32], which is necessary in order that $g^H - g$ is a well defined tensor field (which we know to be the case, because g^H and g are both metric tensors). We further remark that

$$2H = K - 2i\Upsilon F_\Upsilon + 2i\bar{\Upsilon}\bar{F}_\Upsilon \tag{466}$$

differs from the Kähler potential (453) by a Kähler transformation. Therefore $2H$, taken as a Kähler potential, defines the same Kähler metric $g = g^K$ as

K. However, when taking K as a Hesse potential one does not get the Hessian metric g^H. Note that the Hesse potential (466) is the sum of two symplectic functions, namely K and $\mathrm{Im}\,(\Upsilon F_\Upsilon)$, c.f. subsection 4.1.

Thus, the Kähler metric g on $\hat{M}$ is not Hessian with respect to the affine structure that we have defined on $\hat{M}$, i.e. $g \neq g^H$.

7.1.3. Deformed affine special Kähler geometry

Next we show that $\hat{M}$ carries a deformed version of affine special Kähler geometry. Namely, we show that $(\hat{M} = M \times \mathbb{C}, J, g)$ is a Kähler manifold with Kähler form ω, equipped with a flat, torsion-free connection ∇ for which $\nabla\omega \neq 0$ and $d_\nabla J \neq 0$. The non-vanishing of $\nabla\omega$ and $d_\nabla J$ is controlled by the symplectic function F_Υ.

We will call such manifolds deformed affine special Kähler manifolds. Since our definition involves the map ϕ defined in (450), this is not an intrinsic definition, but the name for a specific construction.

We have already established that g is a Kähler metric with Kähler form ω, c.f. (452) and (454). To compare the latter with the two-form $2dx^I \wedge dy_I$, which is the Kähler form on M, we compute

$$\begin{aligned} 2dx^I \wedge dy_I \quad &= \quad -\frac{i}{2}N_{IJ}dX^I \wedge d\bar{X}^J - \frac{1}{2}F_{I\Upsilon}d\Upsilon \wedge d\bar{X}^I + \frac{1}{2}\bar{F}_{I\Upsilon}dX^I \wedge d\bar{\Upsilon} \\ &\quad + \frac{1}{2}F_{I\Upsilon}dX^I \wedge d\Upsilon + \frac{1}{2}\bar{F}_{I\Upsilon}d\bar{X}^I \wedge d\bar{\Upsilon} \ , \end{aligned} \tag{467}$$

and therefore the Kähler form can be written as

$$\omega = 2dx^I \wedge dy_I - \frac{1}{2}F_{I\Upsilon}dX^I \wedge d\Upsilon - \frac{1}{2}\bar{F}_{I\Upsilon}d\bar{X}^I \wedge d\bar{\Upsilon} \ . \tag{468}$$

This shows that $2dx^I \wedge dy_I$, when considered as a two-form on $\hat{M}$, is not of type $(1,1)$ (since ω is, and both differ by pure forms). Using the rewriting

$$F_{I\Upsilon}dY^I \wedge d\Upsilon = dF_\Upsilon \wedge d\Upsilon = -d(\Upsilon dF_\Upsilon) \ , \tag{469}$$

we find

$$\omega = 2dx^I \wedge dy_I + \frac{1}{2}d(\Upsilon dF_\Upsilon) + \frac{1}{2}d(\bar{\Upsilon}d\bar{F}_\Upsilon) \ . \tag{470}$$

Thus the difference between the Kähler forms ω of $\hat{M}$ and $2dx^I \wedge dy_I$ of M is exact. The deformation involves the function $F_\Upsilon = \partial_\Upsilon F$. The latter is a symplectic function, c.f. (222). It contains all the information about the deformation.

Next we compute

$$\nabla\omega = -\frac{1}{2}d(F_{I\Upsilon}) \otimes (dX^I \wedge d\Upsilon) + c.c. \;, \tag{471}$$

and hence, ω is not parallel. Thus, the connection ∇ is not a symplectic connection on $\hat{M}$. This shows that while $(\hat{M}, g, \omega, \nabla)$ is Kähler, it is not special Kähler. The deformation is controlled by an exact form, which is determined by the symplectic function F_Υ.

Next, we show that the complex structure J is not covariantly closed, i.e. $d_\nabla J \neq 0$. To compute the exterior covariant derivative of the complex structure J, we note that the vector fields $\partial_{x^I}, \partial_{y_I}, \partial_\Upsilon, \partial_{\bar{\Upsilon}}$ define a ∇-parallel frame which is dual to the ∇-parallel co-frame $dx^I, dy_I, d\Upsilon, d\bar{\Upsilon}$. Using this one obtains

$$\nabla\frac{\partial}{\partial X^I} = \nabla\left(\frac{1}{2}\frac{\partial}{\partial x^I} + \frac{1}{2}F_{IJ}\frac{\partial}{\partial y_J}\right) = \frac{1}{2}dF_{IJ} \otimes \frac{\partial}{\partial y_J} \;. \tag{472}$$

Using that $d_\nabla J = dJ^a e_a - J^a \wedge d_\nabla e_a$ where e_a is any basis of sections of $T\hat{M}$, so that $d_\nabla e_a = \nabla e_a$, we find

$$d_\nabla J = \left(-idX^I \wedge \frac{1}{2}dF_{IJ} + c.c.\right) \otimes \frac{\partial}{\partial y_J} \;. \tag{473}$$

Note the rewriting

$$dX^I \wedge dF_{IJ} = dX^I \wedge F_{IJ\Upsilon}d\Upsilon = -d(F_{IJ}dX^I) = d(F_{I\Upsilon}d\Upsilon) \;,$$

where we used symmetry of F_{IJ} and the chain rule. Therefore

$$d_\nabla J = (-id(F_{I\Upsilon}d\Upsilon) + c.c.) \otimes \frac{\partial}{\partial y_I} = \left(-iF_{IJ\Upsilon}dX^J \wedge d\Upsilon + c.c.\right) \otimes \frac{\partial}{\partial y_I} \;, \tag{474}$$

which is non-vanishing. As a consistency check, observe that $d_\nabla^2 = 0$, which must hold because ∇ is flat. Note that the non-vanishing of $d_\nabla J$ is expressed in terms of an exact form constructed out of the function F_Υ.

In summary, $(\hat{M} = M \times \mathbb{C}, J, g)$ is a Kähler manifold with Kähler form ω, equipped with a flat, torsion-free connection ∇, with non-vanishing $\nabla\omega$ and $d_\nabla J$ given by (471) and (474).

For completeness we remark that the pullback of the complex symplectic form Ω of V is non-vanishing,[43]

$$\phi^*\Omega = F_{I\Upsilon}dX^I \wedge d\Upsilon = -d(\Upsilon dF_\Upsilon) \;, \tag{475}$$

where the right hand side is exact and controlled by F_Υ.

[43] Obviously, $\hat{M}$ cannot be a (locally immersed) Lagrangian submanifold of V on dimensional grounds.

7.1.4. Holomorphic anomaly equation from the Hessian structure

Next, we turn to the study of the integrability condition for the existence of a Hesse potential H on $\hat{M}$, and we reinterpret it as a holomorphic anomaly equation for a hierarchy of symplectic functions constructed from F_Υ.

In (463) we showed that $\hat{M}$ can be equipped with a Hessian structure (∇, g^H). Then, in ∇-affine coordinates $Q^a = (x^I, y_I, \Upsilon, \bar{\Upsilon})$, the totally symmetric covariant rank three tensor $S = \nabla g^H$ has components $S_{abc} = \partial_a g_{bc}$ which satisfy the integrability condition $\partial_a g_{bc} = \partial_b g_{ca} = \partial_c g_{ab}$. One particular integrability relation is

$$S_{x^I \Upsilon\Upsilon} = S_{\Upsilon x^I \Upsilon} \ , \tag{476}$$

i.e.

$$\partial_{x^I} g^H_{\Upsilon\Upsilon}\big|_y = \partial_\Upsilon g^H_{x^I\Upsilon}\big|_{x,y} \ , \tag{477}$$

with metric components given by (c.f. (462) and (465))

$$\begin{aligned} g^H_{x^I\Upsilon} &= 2\bar{F}_{IJ} N^{JK} F_{K\Upsilon} \ , \\ g^H_{\Upsilon\Upsilon} &= -iD_\Upsilon F_\Upsilon \ , \end{aligned} \tag{478}$$

where the derivative D_Υ,

$$D_\Upsilon = \frac{\partial}{\partial\Upsilon}\bigg|_X + iN^{IJ} F_{J\Upsilon} \frac{\partial}{\partial X^I} \ , \tag{479}$$

is the symplectic covariant derivative that was introduced in (189), and which takes the form (479) when acting on a holomorphic $F(X, \Upsilon)$.

We now evaluate equation (477) in coordinates $(X^I, \bar{X}^I, \Upsilon, \bar{\Upsilon})$, using the Jacobian (B.54), to obtain

$$\begin{aligned} S_{x^I\Upsilon\Upsilon} &= \frac{\partial g^H_{\Upsilon\Upsilon}}{\partial x^I}\bigg|_y = \frac{\partial g^H_{\Upsilon\Upsilon}}{\partial x^I}\bigg|_u + \frac{\partial g^H_{\Upsilon\Upsilon}}{\partial u^K}\frac{\partial u^K}{\partial x^I} \ , \quad \text{where} \quad \frac{\partial}{\partial x^I}\bigg|_u = \frac{\partial}{\partial X^I} + \frac{\partial}{\partial \bar{X}^I} \ , \\ S_{\Upsilon x^I\Upsilon} &= \frac{\partial g^H_{x^I\Upsilon}}{\partial\Upsilon}\bigg|_{x,y} = \frac{\partial g^H_{x^I\Upsilon}}{\partial\Upsilon}\bigg|_{x,u} + \frac{\partial g^H_{x^I\Upsilon}}{\partial u^K}\frac{\partial u^K}{\partial\Upsilon} \ . \end{aligned} \tag{480}$$

We compute

$$
\begin{aligned}
\left.\frac{\partial g^H_{\Upsilon\Upsilon}}{\partial x^I}\right|_u &= -i\frac{\partial}{\partial \bar{X}^I} D_\Upsilon F_\Upsilon - i\bar{F}_{\bar{I}}{}^{KL} F_{K\Upsilon} F_{L\Upsilon} \;, \\
\left.\frac{\partial g^H_{\Upsilon\Upsilon}}{\partial u^K}\right|_x &= \left(\frac{\partial}{\partial X^K} - \frac{\partial}{\partial \bar{X}^K}\right)\left(F_{\Upsilon\Upsilon} + iN^{KL}F_{K\Upsilon}F_{L\Upsilon}\right) \\
&= F_{K\Upsilon\Upsilon} - F_K{}^{PQ}F_{P\Upsilon}F_{Q\Upsilon} - 2N^{PQ}F_{KP\Upsilon}F_{Q\Upsilon} - \bar{F}_K{}^{PQ}F_{P\Upsilon}F_{Q\Upsilon} \;, \\
\left.\frac{\partial g^H_{x^I\Upsilon}}{\partial \Upsilon}\right|_{x,u} &= -iF_{I\Upsilon\Upsilon} + F_{\Upsilon I}{}^J F_{J\Upsilon} + \left(F_{IJ} + \bar{F}_{IJ}\right)\left(iF_\Upsilon{}^{JK}F_{K\Upsilon} + F_{\Upsilon\Upsilon}{}^J\right) \;, \\
\left.\frac{\partial g^H_{x^I\Upsilon}}{\partial u^K}\right|_x &= F_{IK\Upsilon} + iF_{IK}{}^L F_{L\Upsilon} + i\left(F_{IL} + \bar{F}_{IL}\right)\left(iF_K{}^{LP}F_{P\Upsilon} + F_{\Upsilon K}{}^L\right) \\
&\quad -i\bar{F}_{IK}{}^L F_{L\Upsilon} - \left(F_{IL} + \bar{F}_{IL}\right)\bar{F}_K{}^{LP}F_{P\Upsilon} \;, \qquad (481)
\end{aligned}
$$

where indices are raised using N^{IJ}. Then, the integrability condition (477) results in

$$\frac{\partial}{\partial \bar{X}^I} D_\Upsilon F_\Upsilon = \bar{F}_{IJK} N^{JP} N^{KQ} F_{P\Upsilon} F_{Q\Upsilon} \;. \qquad (482)$$

We now explore the consequences of (482). To this end, we first define [32] a hierarchy of symplectic functions through covariant derivatives of the holomorphic symplectic function $F_\Upsilon(X, \Upsilon)$,

$$\Phi^{(n)}(X, \bar{X}, \Upsilon, \bar{\Upsilon}) = \frac{1}{n!} D_\Upsilon^{n-1} F_\Upsilon \quad , \quad n \in \mathbb{N} \;, \qquad (483)$$

and $\Phi^{(0)} = 0$. Note that $\Phi^{(1)}$ is the only holomorphic function in this hierarchy. Then, (482) can be expressed as

$$\frac{\partial \Phi^{(2)}}{\partial \bar{X}^I} = \frac{i}{2}\frac{\partial N^{JK}}{\partial \bar{X}^I} F_{J\Upsilon} F_{K\Upsilon} = \frac{1}{2}\bar{F}_I^{JK} \partial_J \Phi^{(1)} \partial_K \Phi^{(1)} \;, \qquad (484)$$

where $\bar{F}_I^{JK} = \bar{F}_{IPQ} N^{PJ} N^{QK}$. Thus, the integrability condition (477) results in (484) which captures the non-holomorphicity of $\Phi^{(2)}$.

Using (484) as a starting point, one derives, by complete induction, the following holomorphic anomaly equation,

$$\frac{\partial \Phi^{(n)}}{\partial \bar{X}^I} = \frac{1}{2}\bar{F}_I^{JK} \sum_{r=1}^{n-1} \partial_J \Phi^{(r)} \partial_K \Phi^{(n-r)} \;, \;\; n \geq 2 \;, \qquad (485)$$

which captures the departure from holomorphicity of the $\Phi^{(n)}$, with $n \geq 2$. In doing so, one uses [33]

$$\partial_{\bar{I}} F_\Upsilon = 0 \;, \quad D_\Upsilon \bar{F}_{IJK} = 0 \quad , \quad [D_\Upsilon, N^{IJ}\partial_J] = 0 \;. \qquad (486)$$

For example, to derive the anomaly equation for $\Phi^{(3)}$, we need to evaluate

$$\begin{aligned} \partial_{\bar{I}} D^2_\Upsilon F_\Upsilon &= D_\Upsilon \partial_{\bar{I}} D_\Upsilon F_\Upsilon + i(\partial_{\bar{I}} N^{JK}) F_{J\Upsilon} \partial_K D_\Upsilon F_\Upsilon \\ &= 3\bar{F}_I^{JK} \partial_J F_\Upsilon \partial_K D_\Upsilon F_\Upsilon \, . \end{aligned} \tag{487}$$

Using that $D^{n-1}_\Upsilon F_\Upsilon = n! \Phi^{(n)}$ this becomes

$$\partial_{\bar{I}} \Phi^{(3)} = \bar{F}_I^{JK} \partial_J \Phi^{(1)} \partial_K \Phi^{(2)} = \frac{1}{2} \bar{F}_I^{JK} \sum_{r=1}^{2} \partial_J \Phi^{(r)} \partial_K \Phi^{(3-r)} \, . \tag{488}$$

Next, we define

$$F^{(n)}(X, \bar{X}) = \Phi^{(n)}(X, \bar{X}, \Upsilon = \bar{\Upsilon} = 0) \, . \tag{489}$$

The $F^{(n)}(X, \bar{X})$ satisfy the holomorphic anomaly equation

$$\frac{\partial F^{(n)}}{\partial \bar{X}^I} = \frac{1}{2} \bar{F}_I^{(0)JK} \sum_{r=1}^{n-1} \partial_J F^{(r)} \partial_K F^{(n-r)} \; , \quad n \geq 2 \, . \tag{490}$$

Here, $\bar{F}_I^{(0)JK}$ is computed from the undeformed function $F^{(0)}(X) = F(X, \Upsilon)_{|\Upsilon=0}$, i.e. $\bar{F}_I^{(0)JK} = \bar{F}_I^{JK}{}_{|\Upsilon=0}$.

The hierarchy of equations (485) can be re-organized into a master anomaly equation, by introducing

$$G(X, \bar{X}, \Upsilon, \bar{\Upsilon}, \mu) = \sum_{n=0}^{\infty} \mu^{n+1} \Phi^{(n+1)}(X, \bar{X}, \Upsilon, \bar{\Upsilon}) \, , \tag{491}$$

where μ denotes an expansion parameter. Then, the function G satisfies the master anomaly equation

$$\frac{\partial}{\partial \bar{X}^I} G = \frac{1}{2} \bar{F}_I^{JK} \partial_J G \partial_K G \, . \tag{492}$$

Finally, one may ask whether other components of $S = \nabla g^H$ will give rise to additional non-trivial differential equations. To investigate this, we now consider the component $S_{x^I \Upsilon \bar{\Upsilon}} = \partial_{x^I} g^H_{\Upsilon \bar{\Upsilon}} \big|_y$, which is constructed out of the metric component $g^H_{\Upsilon \bar{\Upsilon}} = N^{IJ} F_{I\Upsilon} \bar{F}_{J\Upsilon}$. Evaluating the relation $S_{x^I \Upsilon \bar{\Upsilon}} = S_{\bar{\Upsilon} x^I \Upsilon} = \partial_{\bar{\Upsilon}} \, g^H_{x^I \Upsilon} \big|_{x,y}$ in supergravity variables we find that it is identically satisfied. Thus, the only non-trivial differential equation resulting from $g^H_{\Upsilon\Upsilon}$ and $g^H_{\Upsilon\bar{\Upsilon}}$ is encoded in the relation $S_{x^I \Upsilon \Upsilon} = S_{\Upsilon x^I \Upsilon}$.

7.2. Non-holomorphic deformation

Next, we extend the discussion to a non-holomorphic generalized prepotential $F = F(X, \bar{X}, \Upsilon, \bar{\Upsilon})$ by considering a non-holomorphic map $\phi : \hat{M} \to V$.

Since F and F_Υ are no longer holomorphic, they will have non-vanishing derivatives with respect to $\bar{X}^I$ and $\bar{\Upsilon}$. To distinguish between these various derivatives, we will, in the following, use a notation that involves 'unbarred' indices $I, J, \ldots$ and 'barred' indices $\bar{I}, \bar{J}, \ldots$.

7.2.1. Non-holomorphic deformation of the prepotential

We generalize the map (450) to

$$\phi\ : \hat{M} = M \times \mathbb{C} \to V\ , \quad (X^I, \Upsilon) \mapsto (X^I, F_I(X, \bar{X}, \Upsilon, \bar{\Upsilon}))\ , \tag{493}$$

where $F_I = \partial F / \partial X^I$, can be obtained from a generalized prepotential F. We assume that F has the form [74]

$$F(X, \bar{X}, \Upsilon, \bar{\Upsilon}) = F^{(0)}(X) + 2i\Omega(X, \bar{X}, \Upsilon, \bar{\Upsilon})\ , \tag{494}$$

where $F^{(0)}$ is the undeformed prepotential, and where Ω is a real-valued function that describes the deformation.[44]

The holomorphic deformation is recovered when Ω is harmonic. This makes use of the observation that the complex symplectic vector (X^I, F_I) does not uniquely determine the prepotential F [34]. If we make a transformation

$$\begin{aligned} F^{(0)}(X) &\mapsto F^{(0)}(X) + g(X, \Upsilon)\ , \\ \Omega(X, \bar{X}, \Upsilon, \bar{\Upsilon}) &\mapsto \Omega(X, \bar{X}, \Upsilon, \bar{\Upsilon}) - \frac{1}{2i}(g(X, \Upsilon) - \bar{g}(\bar{X}, \bar{\Upsilon}))\ , \end{aligned} \tag{495}$$

where $g(X, \Upsilon)$ is holomorphic, then F changes by an antiholomorphic function, $F \mapsto F + \bar{g}$, and the symplectic vector (X^I, F_I) and the map ϕ are invariant. If Ω is harmonic,

$$\Omega(X, \bar{X}, \Upsilon, \bar{\Upsilon}) = f(X, \Upsilon) + \bar{f}(\bar{X}, \bar{\Upsilon})\ , \tag{496}$$

we can make a transformation with $g = 2if$ and obtain

$$F \mapsto F^{(0)}(X) + 2if(X, \Upsilon) =: F(X, \Upsilon)\ , \tag{497}$$

which is a holomorphically deformed prepotential, as considered in subsection 7.1. If, however, Ω is not harmonic, then we have a genuine generalization which requires us to consider non-holomorphic generalized prepotentials.

[44]This function is not to be confused with the complex symplectic form on the vector space V introduced in subsection 5.1.2.

7.2.2. Non-holomorphic deformation and geometry

We proceed by analysing the geometry induced by pulling back the standard Hermitian form γ_V of V given by (267) to $\hat{M}$ using (493),

$$\begin{aligned}\gamma &= -i(F^{(0)}_{IJ} - \bar{F}^{(0)}_{\bar{I}\bar{J}})dX^I \otimes d\bar{X}^J + 2(\Omega_{IJ} + \Omega_{\bar{I}\bar{J}})dX^I \otimes d\bar{X}^J + 2\Omega_{\bar{I}J}dX^I \otimes dX^J \\ &\quad +2\Omega_{I\bar{J}}d\bar{X}^I \otimes d\bar{X}^J + 2\Omega_{\bar{I}\bar{\Upsilon}}dX^I \otimes d\bar{\Upsilon} + 2\Omega_{I\Upsilon}d\Upsilon \otimes d\bar{X}^I + 2\Omega_{\bar{I}\Upsilon}dX^I \otimes d\Upsilon \\ &\quad +2\Omega_{I\bar{\Upsilon}}d\bar{\Upsilon} \otimes d\bar{X}^I \ . \end{aligned} \tag{498}$$

By decomposing $\gamma = g + i\omega$, we obtain the following metric on $\hat{M}$,

$$\begin{aligned} g &= -i(F^{(0)}_{IJ} - \bar{F}^{(0)}_{\bar{I}\bar{J}})dX^I d\bar{X}^J + 2(\Omega_{IJ} + \Omega_{\bar{I}\bar{J}})dX^I d\bar{X}^J \\ &\quad +2\Omega_{\bar{I}J}dX^I dX^J + 2\Omega_{I\bar{J}}d\bar{X}^I d\bar{X}^J + 2\Omega_{\bar{I}\bar{\Upsilon}}dX^I d\bar{\Upsilon} + 2\Omega_{I\Upsilon}d\Upsilon d\bar{X}^I \\ &\quad +2\Omega_{\bar{I}\Upsilon}dX^I d\Upsilon + 2\Omega_{I\bar{\Upsilon}}d\bar{\Upsilon} d\bar{X}^I \ . \end{aligned} \tag{499}$$

This expression shows that g is not Hermitian, and hence not Kähler with respect to the natural complex structure J. The non-Hermiticity is encoded in the mixed derivatives $\Omega_{I\bar{J}}$, which makes it manifest that it is related to the non-harmonicity of Ω. This metric occurs in the sigma model discussed in [73].

The imaginary part of γ defines a two-form on $\hat{M}$,

$$\begin{aligned} \omega &= \frac{1}{2i}(-i(F^{(0)}_{IJ} - \bar{F}^{(0)}_{\bar{I}\bar{J}}))dX^I \wedge d\bar{X}^J - i(\Omega_{\bar{I}\bar{J}} + \Omega_{IJ})dX^I \wedge d\bar{X}^J \\ &\quad -i\Omega_{\bar{I}J}dX^I \wedge dX^J + i\Omega_{I\bar{J}}d\bar{X}^I \wedge d\bar{X}^J - i\Omega_{\bar{I}\bar{\Upsilon}}dX^I \wedge d\bar{\Upsilon} - i\Omega_{I\Upsilon}d\Upsilon \wedge d\bar{X}^I \\ &\quad -i\Omega_{\bar{I}\Upsilon}dX^I \wedge d\Upsilon + i\Omega_{I\bar{\Upsilon}}d\bar{X}^I \wedge d\bar{\Upsilon} \ . \end{aligned} \tag{500}$$

This two-form is no longer of type $(1,1)$ with respect to the standard complex structure, which is consistent with the non-Hermiticity of g. However, ω is still closed

$$d\omega = 0 \ , \tag{501}$$

and hence $(\hat{M}, \omega)$ is a symplectic manifold.

The difference between the symplectic forms ω of $\hat{M}$ and $2dx^I \wedge dy_I$ of M is exact,

$$\omega = 2dx^I \wedge dy_I + \frac{1}{2}d(\Upsilon dF_\Upsilon) + \frac{1}{2}d(\bar{\Upsilon} d\bar{F}_{\bar{\Upsilon}}) + \partial\bar{\partial}F \ , \tag{502}$$

where $\partial = dX^I \otimes \partial_{X^I} + d\Upsilon \otimes \partial_\Upsilon$. Compared to (470) there is an additional term which measures the non-holomorphicity of the generalized prepotential.

7.2.3. The Hesse potential

We introduce real coordinates $(q^a) = (x^I, y_I)$ by

$$X^I = x^I + iu^I(x,y,\Upsilon,\bar{\Upsilon}) \;, \quad F_I(X,\bar{X},\Upsilon,\bar{\Upsilon}) = y_I + iv_I(x,y,\Upsilon,\bar{\Upsilon}) \;. \tag{503}$$

We introduce the combinations [73]

$$N_{\pm IJ} = N_{IJ} \pm 2\mathrm{Im} F_{I\bar{J}} = -i(F_{IJ} - \bar{F}_{\bar{I}\bar{J}} \pm F_{I\bar{J}} \mp \bar{F}_{\bar{I}J}) \tag{504}$$

and

$$R_{\pm IJ} = R_{IJ} \pm 2\mathrm{Re} F_{I\bar{J}} = F_{IJ} + \bar{F}_{\bar{I}\bar{J}} \pm F_{I\bar{J}} \pm \bar{F}_{\bar{I}J} \;. \tag{505}$$

Note that $N_-^T = N_-$, while $R_\pm^T = R_\mp$.

In the presence of a non-holomorphic deformation, the Hesse potential is defined as the Legendre transform of

$$L = 2\mathrm{Im}F - 2\Omega = 2\mathrm{Im}F^{(0)} + 2\Omega \;, \tag{506}$$

c.f. (164) (the normalization used here differs from the one in (164) by a factor 2). As explained in section 4.1, the function L can be interpreted as a Lagrange function, and the Hesse potential as the corresponding Hamilton function. Thus, the Hesse potential associated to $F(X,\bar{X},\Upsilon,\bar{\Upsilon})$ is

$$H(x,y,\Upsilon,\bar{\Upsilon}) = -i(F - \bar{F}) - 2\Omega - 2u^I y_I \;. \tag{507}$$

We now compute the associated Hessian metric g^H by taking derivatives of H with respect to the coordinates $(Q^A) = (q^a, \Upsilon, \bar{\Upsilon})$, where $(q^a) = (x^I, y_I)$. To convert from coordinates $(x^I, u^I, \Upsilon, \bar{\Upsilon})$ to coordinates (Q^A) and back, we use the Jacobians (B.57) and (B.58). We obtain for the components of the Hessian metric g^H,

$$\begin{aligned}
\frac{\partial H}{\partial q^a \partial q^b} &= \begin{pmatrix} N_+ + R_- N_-^{-1} R_+ & -2R_- N_-^{-1} \\ -2N_-^{-1} R_+ & 4N_-^{-1} \end{pmatrix} \;, \\
\frac{\partial^2 H}{\partial x^I \partial \Upsilon} &= -i(F_{I\Upsilon} - \bar{F}_{\bar{I}\Upsilon}) + R_{-IK} N_-^{KJ} (F_{J\Upsilon} + \bar{F}_{\bar{J}\Upsilon}) \;, \\
\frac{\partial^2 H}{\partial y_I \partial \Upsilon} &= -2N_-^{IK} (F_{K\Upsilon} + \bar{F}_{\bar{K}\Upsilon}) \;,
\end{aligned} \tag{508}$$

together with their complex conjugates, and

$$\begin{aligned}
\frac{\partial^2 H}{\partial \Upsilon \partial \bar{\Upsilon}} &= -iF_{\Upsilon\bar{\Upsilon}} + N_-^{IJ} (\bar{F}_{\bar{I}\bar{\Upsilon}} - \bar{F}_{I\bar{\Upsilon}})(F_{\Upsilon J} - F_{\Upsilon \bar{J}}) = -iD_\Upsilon F_{\bar{\Upsilon}} \;, \\
\frac{\partial^2 H}{\partial \Upsilon \partial \Upsilon} &= -iD_\Upsilon F_\Upsilon \;, \quad \frac{\partial^2 H}{\partial \bar{\Upsilon} \partial \bar{\Upsilon}} = i\overline{D_\Upsilon F_\Upsilon} \;,
\end{aligned} \tag{509}$$

where

$$D_\Upsilon = \partial_\Upsilon + iN_-^{IJ}(F_{\Upsilon J} - F_{\Upsilon \bar{J}}) \left(\frac{\partial}{\partial X^I} - \frac{\partial}{\partial \bar{X}^I} \right) \tag{510}$$

is the symplectically covariant derivative introduced in (189).

As before (c.f. subsection 7.1.2), the Hessian metric g^H differs from the metric g in (499) (induced by pulling back g_V using ϕ) by differentials involving derivatives of H with respect to $\Upsilon, \bar{\Upsilon}$,

$$g^H = g + \partial^2 H\big|_{x,y} \;, \tag{511}$$

where

$$\partial^2 H\big|_{x,y} = \frac{\partial^2 H}{\partial\Upsilon\partial\Upsilon} d\Upsilon d\Upsilon + 2\frac{\partial^2 H}{\partial\Upsilon\partial\bar{\Upsilon}} d\Upsilon d\bar{\Upsilon} + \frac{\partial^2 H}{\partial\bar{\Upsilon}\partial\bar{\Upsilon}} d\bar{\Upsilon} d\bar{\Upsilon} \;. \tag{512}$$

7.2.4. Hierarchy of non-holomorphic symplectic functions

The function $F_\Upsilon = \partial_\Upsilon F$ is a non-holomorphic symplectic function, c.f. (222). Using the symplectically covariant derivative D_Υ given in (510), we construct a hierarchy of symplectic functions by

$$\Phi^{(n+1)}(X, \bar{X}, \Upsilon, \bar{\Upsilon}) = \frac{1}{(n+1)!} D_\Upsilon^n F_\Upsilon(X, \bar{X}, \Upsilon, \bar{\Upsilon}) \quad , \quad n \in \mathbb{N}_0 \;. \tag{513}$$

Then, we define symplectic functions $F^{(n)}(X, \bar{X})$ by

$$F^{(n)}(X, \bar{X}) = \Phi^{(n)}(X, \bar{X}, \Upsilon, \bar{\Upsilon})\Big|_{\Upsilon = \bar{\Upsilon} = 0} \;, \quad n \geq 1 \;. \tag{514}$$

The functions $F^{(n)}(X, \bar{X})$ with $n \geq 2$ will satisfy a holomorphic anomaly equation, whose precise form depends on the details of the non-holomorphic deformation.

7.2.5. The holomorphic anomaly equation of perturbative topological string theory

For a specific deformation, the resulting holomorphic anomaly equation is the one of perturbative topological string theory [75]. Namely, let us first rescale $F^{(n)} \mapsto 2iF^{(n)}$, for convenience. Now we take Υ to be real, and $F^{(1)}$ to be

$$F^{(1)} = f^{(1)} + \bar{f}^{(1)} + \alpha \ln \det N_{IJ}^{(0)} \;. \tag{515}$$

Here, $\alpha \in \mathbb{R}$ is the deformation parameter, and $N_{IJ}^{(0)}$ equals $N_{IJ}^{(0)} = -i(F_{IJ}^{(0)} - \bar{F}_{IJ}^{(0)})$. When $\alpha = 0$, $F^{(1)}$ is the real part of a holomorphic function $f^{(1)}(X)$.

For the α-deformation, the holomorphic anomaly equation satisfied by the $F^{(n)}$ with $n \geq 2$ is given by (see [34])

$$\frac{\partial}{\partial \bar{X}^K} F^{(n)} = i \bar{F}_K^{(0)IJ} \left(\sum_{r=1}^{n-1} \partial_I F^{(r)} \partial_J F^{(n-r)} - 2\alpha D_I \partial_J F^{(n-1)} \right) \quad , \quad n \geq 2 \; , \tag{516}$$

where $\bar{F}_K^{(0)IJ} = \bar{F}^{(0)}_{KQP} N^{(0)QI} N^{(0)PJ}$. The covariant derivative D_I, when acting on a vector V_J, takes the form

$$D_I V_J = \partial_I V_J - \Gamma_{IJ}{}^K V_K \; , \tag{517}$$

where $\Gamma_{IJ}{}^K$ is the Levi-Civita connection associated with the Kähler metric of the undeformed theory (i.e. the Kähler metric computed from $F^{(0)}(X)$). When $\alpha = 0$, this anomaly equation reduces to the one given in (490), upon undoing the rescaling $F^{(n)} \mapsto 2iF^{(n)}$ performed above. When $\alpha = -1/2$, (516) is the holomorphic anomaly equation of perturbative topological string theory [75, 76]. Let us display the expression for $F^{(2)}$ obtained by solving the anomaly equation [75, 77, 76],

$$\begin{aligned} F^{(2)}(X, \bar{X}) = {} & f^{(2)} - N_{(0)}^{IJ} \big(f_I^{(1)} - \mathrm{i}\alpha \, F^{(0)}_{IKL} N_{(0)}^{KL} \big) \big(f_J^{(1)} - \mathrm{i}\alpha \, F^{(0)}_{JPQ} N_{(0)}^{PQ} \big) \\ & + 2\alpha \, N_{(0)}^{IJ} f_{IJ}^{(1)} - \alpha^2 \big[\mathrm{i} N_{(0)}^{IJ} N_{(0)}^{KL} F^{(0)}_{IJKL} - \tfrac{2}{3} N_{(0)}^{IJ} F^{(0)}_{IKL} N_{(0)}^{KP} N_{(0)}^{LQ} F^{(0)}_{JPQ} \big], \end{aligned} \tag{518}$$

with holomorphic input data $f^{(1)}(X)$ and $f^{(2)}(X)$.

The expressions for the higher $F^{(n)}(X, \bar{X})$ become very lengthy quickly, see the expression for $F^{(3)}(X \, \bar{X})$ given in Appendix D of [34]. The non-holomorphicity of $F^{(n)}(X, \bar{X})$ is entirely contained in the quantities $N^{(0)}_{IJ}, N^{IJ}_{(0)}$. Observe that $F^{(1)}$ is real, while the higher $F^{(n)}$ $(n \geq 2)$ are not.

7.2.6. Holomorphic anomaly equation from the Hessian structure

The holomorphic anomaly equation (516) is encoded in the underlying Hesse structure, namely in the relation

$$S_{x^I \Upsilon \Upsilon} = S_{\Upsilon x^I \Upsilon} \; , \tag{519}$$

which the totally symmetric rank three tensor $S = \nabla g^H$ has to satisfy, where g^H denotes the Hessian metric computed in subsection 7.2.3. We refer to [72] for the somewhat technical verification of this assertion, where this was shown for the case of the anomaly equation for $F^{(2)}$. Thus, the holomorphic anomaly equation (516) is intimately related to the existence of a Hessian structure on $\hat{M}$.

8. Dimensional reduction over space and time. Euclidean special geometry

In this section we will review how the special geometries of five- and four-dimensional vector multiplets are related to each other, and to the special geometry of hypermultiplets, by dimensional reduction. We take this opportunity to also discuss how special geometry gets modified for theories defined on a Euclidean space'time,' by including time-like dimensional reductions. We will focus on presenting and discussing key facts and results while referring to the literature for details.

8.1. Space-like and time-like dimensional reductions

Space-like and time-like dimensional reductions of Lagrangians differ by specific relative signs between terms. We illustrate this with a simple example, a theory involving a free massless scalar σ and an abelian vector field A_μ in $n+1$ dimensions,

$$S = \int d^{n+1}x \left(-\frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) . \tag{520}$$

Upon dimensional reduction, the vector field A_μ decomposes into a vector field A_m and a scalar $b = A_*$, where $*$ is the index of the direction we reduce over. The reduced Lagrangian, where we only keep the massless modes, is

$$S = \int d^n x \left(-\frac{1}{2}\partial_m \sigma \partial^m \sigma + \frac{1}{2}\varepsilon \partial_m b \partial^m b - \frac{1}{4} F_{mn} F^{mn} \right) , \tag{521}$$

where $\varepsilon = -1$ for a space-like reduction and $\varepsilon = 1$ for a time-like reduction.[45] Thus in a Euclidean theory obtained by time-like dimensional reduction, the sign of the kinetic term of the scalar b is inverted and the Euclidean action is indefinite. This distinguishes such Euclidean theories from Euclideanized theories obtained by Wick rotation, see section 8.2 for discussion.

For space-like reductions we can combine the real scalars σ and b into a complex scalar $X = \sigma + ib$. For time-like reductions there are two ways to proceed. Either we can use *adapted real coordinates* which are lightcone coordinates with

[45] Note that part of the literature on dimensional reduction defines ε with the opposite sign.

respect to the scalar target space, $X_\pm = \sigma \pm b$, or we can introduce *para-complex coordinates* by employing para-complex numbers $z = x + ey$, $x, y \in \mathbb{R}$, where the para-complex unit e satisfies

$$e^2 = 1 \; , \quad \bar{e} = -e \; . \tag{522}$$

The anti-linear involution $\bar{\cdot}$ is called *para-complex conjugation.* The para-complex numbers $C := \mathbb{R} \oplus e\mathbb{R}$ form a real algebra, but not a number field, and not even a division algebra. Zero divisors correspond to 'lightcone directions', since $(1+e)(1-e) = 0$. Nevertheless one can use para-complex numbers to define various types of structures on differentiable manifolds, which are analogous to those based on complex numbers, such as complex, Hermitian and Kähler structures. Para-complex geometries are useful to formulate special geometry in Euclidean signature [19, 78, 45, 17], and more recently have taken on a role in generalized and doubled geometry as well [79, 80, 81, 82, 83]. We provide some background information in A.20, and refer to [84] for a historical review.

One advantage of working with para-complex scalar fields is that it makes the similarities between space-like and time-like reductions manifest. In particular one can introduce an *ε-complex notation* by

$$i_\varepsilon = \left\{ \begin{array}{cc} i & \text{for } \; \varepsilon = -1 \; , \\ e & \text{for } \; \varepsilon = 1 \; , \end{array} \right\} \Rightarrow i_\varepsilon^2 = \varepsilon \; , \quad \overline{i_\varepsilon} = -i_\varepsilon \; . \tag{523}$$

In ε-complex notation, the reduced Lagrangian (521) becomes

$$S = \int d^n x \left(-\frac{1}{2} \partial_m X \partial^m \bar{X} - \frac{1}{4} F_{mn} F^{mn} \right) \; , \quad \text{where} \quad X = \sigma + i_\varepsilon b \; . \tag{524}$$

8.2. Euclidean and Euclideanized theories

Before proceeding we need to clarify the distinction between Euclidean and Euclideanized theories. In this review a 'Euclidean supersymmetric theory' or 'Euclidean supergravity theory' is a theory with a Lagrangian which is invariant under the Euclidean supersymmetry algebra. This is true in particular for theories which are obtained by a time-like dimensional reduction, but one can also construct Euclidean theories ab initio, starting from the Euclidean supersymmetry algebra, see for example [21], or by analytical continuation of Killing spinor equations, see for example [85]. In contrast by a 'Euclideanized theory' we refer to a theory which has been obtained from a theory in Lorentz signature by applying a Wick rotation. From the previous section it is clear that for four-dimensional theories which can be obtained by dimensional reduction from five dimensions, the Euclidean and Euclideanized theory will in general have bosonic Lagrangians which differ by relative signs for some of the scalars. For theories containing fermions the additional complication arises that reality condition are signature dependent, which can lead to a doubling of the fermionic degrees of freedom upon Euclideanization. In four dimensions Majorana spinor exist in Lorentzian, but not in Euclidean signature, which in particular implies that there is no Euclidean '$\mathcal{N} = 1$' supersymmetry algebra with four real supercharges. One can still define a meaningful Euclideanization of four-dimensional $\mathcal{N} = 1$ theories within the Osterwalder-Schrader formalism [86]. In this approach one uses a modified Hermiticity condition in the Euclidean theory, and supersymmetry is encoded in Euclidean Ward identities which become the standard supersymmetric Ward identities upon continuation to Lorentz signature. An alternative proposal for the Euclideanization of supersymmetric theories with extended supersymmetry, where there is no issue with the doubling of fermionic degrees of freedom, is to modify the Wick rotation such that the resulting theory has an action which is invariant under Euclidean supersymmetry [87, 88, 89]. For a certain class of theories, which include the bosonic parts of four-dimensional vector multiplet theories, Euclidean and Euclideanized actions can be mapped to each other using that the Hodge dualization of axion-like scalars does not commute with a Wick rotation [45].

Since Euclidean actions obtained by a time-like reduction can be indefinite, while a well-behaved Euclidean functional integral requires an Euclidean action which is bounded from below, one might think that only Euclideanized theories can provide the proper starting point for defining supersymmetric theories. However, the situation is more complicated for various reasons. Firstly, in Euclidean signature the Hodge-dualization of p-form fields changes the sign of their 'kinetic term' and thus relates definite and indefinite actions.[46] Secondly real integrals can be dominated by complex saddle points and the functional integral of a supersymmetric theory can be dominated by real BPS solutions to an indefinite Euclidean action. In particular, this is the case for the D-instanton so-

[46]See section 8.4.1.

lutions of type-IIB string theory [90]. At least in simple examples one can show explicitly that Euclidean and Euclideanized actions can be used alternatively to perform a saddle point evaluation of the same functional integral, using different 'integration contours' in complexified field space [91]. This suggest to construct theories on space-times of different signatures as different real forms of master theories on with a complexified field space on a complexified space-time. In this context it is natural to also consider space-time signature other than Euclidean and Lorentzian, see below.

Euclidean actions also serve a practical role as part of generating techniques for stationary solutions of theories in Lorentzian signature [92, 93]. Upon time-like dimensional reduction one obtains an auxiliary Euclidean theory, whose field equations are often easier to solve. Solutions of the reduced Euclidean theory can then be lifted to stationary solutions of the Lorentzian theory. This can be viewed as generating 'solitons' (stationary finite energy solutions of a Lorentzian theory) from 'instantons' (finite action solutions of Euclidean theory). With proper attention to boundary terms one can show that the instanton action of certain Euclidean solutions agrees exactly with the ADM mass of the black hole solutions obtained by lifting [45]. This 'reduction/oxidation' method is not limited to BPS solutions and can be used to generate non-extremal solutions.

Some remarks on general space-time signatures

Once time-like T-dualities are admitted, the web of string dualities relates theories in different space-time signatures [94, 95, 96]. The maximally supersymmetric supergravity theories in ten and eleven dimensions can all be related to real forms of a single complex ortho-symplectic Lie superalgebra [97, 98]. Five- and four-dimensional vector multiplets for all possible space-time signatures have been obtained in [99, 21, 100].

8.3. Reduction from five to four dimensions: the r-map

8.3.1. Reduction without gravity: the rigid r-map

We now turn to the dimensional reduction of the five-dimensional bosonic vector multiplet Lagrangian (126), following [19], and treating space-like and time-like reduction in parallel. Upon reduction, the five-dimensional vector fields A^I_μ decompose into four-dimensional vector fields A^I_m and scalars b^I, which we combine with five-dimensional scalars σ^I to ε-complex scalars $X^I = \sigma^I +$

$i_\varepsilon b^I$. The four-dimensional field strengths are decomposed into selfdual and antiselfdual parts according to

$$F^{\pm}_{mn} = \frac{1}{2}(F_{mn} \pm \tilde{F}_{mn}) = \frac{1}{2}(F_{mn} \pm \frac{1}{2i_\varepsilon}\varepsilon_{mnpq}F^{pq}) \ . \tag{525}$$

The couplings of the four-dimensional theory are encoded in an ε-holomorphic prepotential $F(X)$, which up to a constant factor is obtained by extending the Hesse potential $h(\sigma)$ of the five-dimensional theory from real to ε-complex values:

$$F(X) = -\frac{1}{2i_\varepsilon}h(\sigma + i_\varepsilon b) \ . \tag{526}$$

We extend our previous definitions according to[47]

$$\begin{aligned} &R_{IJ} = F_{IJ} + \bar{F}_{IJ} \ , \quad N_{IJ} = -i_\varepsilon(F_{IJ} - \bar{F}_{IJ}) = \frac{\partial^2 K}{\partial X^I \partial \bar{X}^J} \ , \\ &K = i_\varepsilon(X^I \bar{F}_I - F_I \bar{X}^I) \ . \end{aligned} \tag{527}$$

The resulting four-dimensional bosonic Lagrangian takes the form

$$\begin{aligned} L &= -N_{IJ}\partial_m X \partial^m \bar{X} + \left(\frac{i_\varepsilon}{4}F_{IJ}F^{I-}_{mn}F^{J-mn} + h.c.\right) + \cdots \qquad (528) \\ &= -N_{IJ}\partial_m X^I \partial^m \bar{X}^J - \frac{1}{8}N_{IJ}F^I_{mn}F^{Jmn} - \frac{1}{16}R_{IJ}\varepsilon^{mnpq}F^I_{mn}F^J_{pq} + \cdots \ , \end{aligned}$$

where we have omitted the auxiliary fields Y^I_{ij}.

We now turn to the relation between the scalar manifolds M of the five-dimensional theory and N of the four-dimensional theory. The ASR metric $g_M = h_{IJ}d\sigma^I d\sigma^J$ is mapped to the *affine special ε-Kähler metric*

$$g_N = N_{IJ}(\sigma)d\sigma^I d\sigma^J - \varepsilon N_{IJ}(\sigma)db^I db^J = N_{IJ}dX^I d\bar{X}^J \ , \tag{529}$$

with ε-holomorphic prepotential (526). Since the fields b^I take values in $\mathbb{R}^n$, where n is the number of five-dimensional vector multiplets, we can identify N with the tangent bundle of M, that is $N \cong TM$. The metric (529) only depends on the scalars σ^I and therefore has an isometry group which contains the constant shifts $b^I \mapsto b^I + \beta^I$, where $(\beta^I) \in \mathbb{R}^n$. These isometries are relicts of the five-dimensional abelian gauge symmetry. Moreover, the metric (529) is block-diagonal with respect to σ^I and b^I. This decomposition has an invariant

[47] In [19] N_{IJ} and K were defined with the opposite sign. This has been compensated for by changing the overall sign of F. Apart from this, some fields have to be rescaled by constant factors.

meaning, because the special connection ∇ of the ASR manifold M can be used to decompose

$$T_X N = T_X N^{\mathrm{vert}} \oplus T_X N^{\mathrm{hor}}_{\nabla} \cong T_\sigma M \oplus T_\sigma M \ , \quad X \in N = TM \ , \quad \sigma = \pi(X) \in M \ , \tag{530}$$

where $\pi : N = TM \to M$ is the canonical projection. The vertical space can be identified with $T_{\pi(X)}M = T_\sigma M$ using the projection

$$T_X N^{\mathrm{vert}} := \ker(d\pi_X) \cong T_\sigma M \ . \tag{531}$$

While in general there is no canonical complement of $T_X N^{\mathrm{vert}} \subset T_X N$, the connection ∇ defines a horizontal subbundle $TN^{\mathrm{hor}}_{\nabla}$, which is spanned by vectors tangent to the horizontal lifts of curves on N. This can be used to identify the horizontal subspace $T_X N^{\mathrm{hor}}_{\nabla}$ with the tangent space $T_\sigma M$ using the projection:

$$d\pi_X|_{T_X N^{\mathrm{hor}}_{\nabla}} \ : T_X N^{\mathrm{hor}}_{\nabla} \stackrel{\cong}{\longrightarrow} T_\sigma M \ . \tag{532}$$

A similar construction based on the Levi-Civita connection D is used to define the so-called Sasaki metric on the tangent bundle $N = TM$ of a Riemannian manifold M, which has a block-diagonal structure like in (529). The 'Sasaki-like' metric g_N on the tangent bundle $N = TM$ of an ASR manifold with metric g_M is defined by the special connection ∇ instead of D, and it comes in two versions, labelled by ε, which differ by a relative sign of the metric along the horizontal and vertical distribution. It has been shown in [19] that if (M, g_M, ∇) is an ASR manifold, then $N = TM$ carries the structure of an affine special ε-Kähler manifold (N, J_N, g_N, ∇_N), where the metric g_N, the ε-complex structure J_N and the special connection ∇_N can be constructed out of the ASR data. The map

$$r_\varepsilon : \{\text{ASR manifolds}\} \to \{\text{AS}\varepsilon\text{K manifolds}\} \ : M \mapsto N = TM \tag{533}$$

is called the *rigid r-map*.

While we have omitted the supersymmetry transformations and fermionic terms of the Lagrangian, these can be found in [19]. We remark that by dimensional reduction one can only obtain a subset of the four-dimensional vector multiplet theories, namely those where the prepotential is a cubic polynomial in the ε-complex special coordinates X^I. Such prepotentials are called *very special.* However, the only terms not obtained by dimensional reduction are four-fermion terms which are proportional to the fourth derivatives F_{IJKL}, $\bar{F}_{IJKL}$ of the prepotential. To obtain the general four-dimensional Lagrangian one takes F to be a general ε-holomorphic function. Then the Lagrangian is only invariant up to terms generated by variation of terms involving the third derivatives of the prepotential. The four-fermion terms are determined by imposing that their variation restores the supersymmetry invariance of the Lagrangian [19].

We remark that Euclidean supersymmetric theories, and in fact supersymmetric theories on space-times of arbitrary signature can also be constructed ab initio, rather than by dimensional reduction. In particular, five-dimensional rigid off shell vector multiplets and their Lagrangians have been obtained for all signatures (t, s), $t + s = 5$ in [21].

8.3.2. Reduction with gravity: the supergravity r-map

We now turn to the more interesting case of performing the reduction in supergravity. When starting in five dimensions with $n_{(5)}$ vector multiples coupled to Poincaré supergravity, we end up in four dimensions with $n_{(4)} = n_{(5)} + 1$ vector multiplets coupled to Poincaré supergravity, because the five-dimensional supergravity multiplet decomposes into the four-dimensional supergravity multiplet and an additional Kaluza-Klein vector multiplet. The five-dimensional metric decomposes as

$$g_{\mu\nu} dx^{\mu} dx^{\nu} = -\varepsilon e^{2\sigma} \left(dx^{*} + \mathcal{A}_m dx^m\right)^2 + g_{mn} dx^m dx^n \; , \tag{534}$$

where g_{mn} is the four-dimensional metric with signature $(\varepsilon, +, +, +)$, $\mathcal{A}_m$ is the Kaluza-Klein vector and σ is the Kaluza-Klein scalar.

We start from (149), (150), (151) with $\kappa = 1$ and relabel $I = 0, \ldots, n_{(5)}$ into $a = 1, \ldots, n_{(5)} + 1 = n_{(4)}$, so that we can use $I, J = 0, \ldots n_{(4)}$ to label four-dimensional vector multiplets. It is convenient to work with the constrained scalars h^a, subject to $\mathcal{V}(h) = C_{abc} h^a h^b h^c = 1$, instead of the physical scalars ϕ^x. Upon reduction, one can then define new scalars

$$y^a := 6^{1/3} e^{\sigma} h^a \; . \tag{535}$$

These are $n_{(4)}$ unconstrained real scalars which encode the Kaluza-Klein scalar through

$$\mathcal{V}(y) = C_{abc} y^a y^b y^c = 6 e^{3\sigma} \; , \tag{536}$$

while the physical five-dimensional scalars ϕ^x can be parametrized by the independent ratios $h^x / h^{n_{(4)}}$, $x = 1, \ldots, n_{(5)}$. The real scalars y^a are combined with the scalar components $x^a \propto A^a_*$ of the five-dimensional gauge fields into ε-complex scalars $z^a := x^a + i_\varepsilon y^a$. With this convention the five-dimensional gauge symmetry induces an invariance of the four-dimensional theory under real shifts of the scalars z^a, that is under $z^a \mapsto z^a + r^a$, with $(r^a) \in \mathbb{R}^{n_{(4)}}$. Thus the scalar manifold looks locally like a higher-dimensional version of the upper half plane.[48] For the vector fields it is necessary to take field dependent (x^a-dependent) linear combinations in order to make the four-dimensional gauge symmetry manifest. Moreover, to arrive at standard four-dimensional conventions, fields need to be rescaled by constant factors, see [45] for details. The resulting bosonic Lagrangian takes the form[49]

$$\begin{aligned} L &= \frac{1}{2}R - \bar{g}_{ab}\partial_m z^a \partial^m \bar{z}^b + \frac{1}{4}\mathrm{Im}\mathcal{N}_{IJ}F^I_{mn}F^{Jmn} + \frac{\varepsilon}{4}\mathrm{Re}\mathcal{N}_{IJ}F^I_{mn}\frac{1}{2}\varepsilon^{mnpq}F^J_{pq} \\ &= \frac{1}{2}R - \bar{g}_{ab}\partial_m z^a \partial^m \bar{z}^b + \frac{1}{4}\mathrm{Im}\mathcal{N}_{IJ}F^I_{mn}F^{Jmn} + \frac{\varepsilon i_\varepsilon}{4}\mathrm{Re}\mathcal{N}_{IJ}F^I_{mn}\tilde{F}^{J|pq} \\ &= \frac{1}{2}R - \bar{g}_{ab}\partial_m z^a \partial^m \bar{z}^b + \left(\frac{1}{4i_\varepsilon}\mathcal{N}_{IJ}F^{+I}_{mn}F^{+Jmn} + h.c.\right) , \end{aligned} \tag{537}$$

which generalizes (439) to the ε-complex case. As in the rigid case only a subclass of four-dimensional theories can be obtained by reduction. Given a five-dimensional Hesse potential of the form $h = C_{abc}h^a h^b h^c$ the ε-holomorphic prepotentials resulting from reduction have the very special form

$$F = -\frac{1}{6}C_{abc}\frac{X^a X^b X^c}{X^0} , \tag{538}$$

where X^I, $I = 0, \ldots, n_V^{(4)}$ are related to the physical scalars z^a by $z^a = X^a/X^0$. It was shown in [45] that the superconformal quotient admits an ε-complex generalization, which for $\varepsilon = 1$ connects conical affine special para-Kähler manifolds N to projective special para-Kähler manifolds $\bar{N}$. In the para-complex version of the quotient, $\mathbb{C}^* = \mathbb{R}^{>0} \times U(1)$ is replaced by $C^* = \mathbb{R}^{>0} \times SO(1,1)$. The group $SO(1,1)$ replacing $U(1)$ is the abelian factor of the R-symmetry group $SO(1,1) \times SU(2)$ of the four-dimensional Euclidean supersymmetry algebra [19, 45]. With suitable conventions, all local formulae of special Kähler

[48] There are other conventions in the supergravity literature where the axion-like scalars are taken to be the imaginary rather than real parts, in particular the fields S, T, U of the much studied STU-model are defined that way.

[49] Compared to [45], there is an explicit factor ε in the last term to account for the different definition of the ϵ-tensor in Lorentzian signature.

geometry have ε-complex extensions. In particular

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} - i\varepsilon \frac{N_{IK}X^K N_{JL}X^L}{N_{MN}X^M X^N} \tag{539}$$

generalizes (440) while

$$\bar{K} = -\log\left(-i_\varepsilon(X^I \bar{F}_I - F_I \bar{X}^I)\right) = -\log(-K) \tag{540}$$

generalizes (350). The expressions (346) for the projectable tensor, and $K = N_{IJ}X^I\bar{X}^J = -1$ for the D-gauge are valid for both values of ε.

Dimensional reduction relates the scalar manifolds of the two theories by assigning to every PSR manifold $\bar{M}$ of real dimension $n = n_{(5)}$ a PSεK manifold $\bar{N}$ of real dimension $2n + 2 = 2n_{(4)}$. The additional two scalars come from the reduction of the five-dimensional supergravity multiplet, one from the metric, one from the graviphoton. The resulting map

$$\bar{r}_\varepsilon \ : \{\text{PSR manifolds}\} \to \{\text{PS}\varepsilon\text{K manifolds}\}\ , \quad \bar{M}_n \mapsto \bar{N}_{2n+2} \tag{541}$$

is called the *supergravity r-map.*

For dimensional reasons $\bar{N} \ncong T\bar{M}$, which raises the question how to understand the geometry of this map. So far we have considered Poincaré supergravity in an on-shell formulation. For many purposes, including having full manifest symplectic covariance, working off-shell, and including higher derivatives, one needs to have the superconformal off-shell version of the dimensional reduction, and for the $\bar{r}$-map. Here we focus on the scalar geometry. The full off-shell reductions of five-dimensional superconformal vector and hypermultiplets coupled to the Weyl multiplet can be found in [101, 102]. In the superconformal setting we have an $(n+1)$-dimensional real cone M_{n+1} over $\bar{M}_n$ and a $(2n+4)$-dimensional ε-complex cone N_{2n+4} over $\bar{N}_{2n+2}$. Since the superconformal theories are gauged versions of rigid superconformal theories, it is natural to apply the rigid r-map to M_{n+1}. This yields an ASεK manifold $\hat{N}_{2n+2}$, which is not conical. Note that the dimensional reduction of a rigid superconformal symmetry breaks conformal symmetry, as follows immediately from our results on the r-map. The cubic Hesse potential of $\bar{M}_{n+1}$ maps to a cubic prepotential for $\hat{N}_{2n+2}$, but rigid superconformal symmetry requires a prepotential which is homogeneous of degree two. To lift the supergravity r-map $\bar{M}_n \mapsto \bar{N}_{2n+2}$ to a map $M_{n+1} \mapsto N_{2n+4}$ between the associated conical manifolds, one needs to combine the rigid r-map $M_{n+1} \mapsto \hat{N}_{2n+2}$ with another map called the 'conification map' con : $\hat{N}_{2n+2} \mapsto N_{4n+2}$, which canonically, that is without arbitrary choices and only using given data, assigns a cone N_{4n+2} to the non-conical manifold $\hat{N}_{2n+2}$.

Such a conification map has been constructed, for the case of space-like reduction ($\varepsilon = 1$) in [46],[47].[50] This conification map induces a map $\hat{N}_{2n+2} \mapsto \bar{N}_{2n+2}$ between ASK manifolds and PSK-manifolds of the same dimension, called the ASK/PSK correspondence. The situation is summarized in the following diagram

$$\begin{array}{ccccc} M & \xmapsto{r} & \hat{N} & \xmapsto{\text{con}} & N \\ \Big\downarrow{\scriptstyle \mathrm{SC}} & & & \searrow^{\text{ASK/PSK}} & \Big\downarrow{\scriptstyle \mathrm{SC}} \\ \bar{M} & & \xmapsto[\bar{r}]{} & & \bar{N} \end{array} \qquad (542)$$

where 'SC' indicates a superconformal quotient. Since the rigid r-map relates a cubic Hesse potential $h(\sigma^a)$ to a cubic prepotential $F_{\hat{N}}(X^a)$, one expects that the conification map yields a prepotential of the form $F_N(X^I) = F_{\hat{N}}(X^a)/X^0$ for the CASεK manifold N. While this turns out to be correct, we stress that it is not clear a priori how to formulate the relation between $\hat{N}$ and N in a way that is independent of a choice of coordinates. Note that the special coordinates X^a on $\hat{N}$ are unique up to transformations in $Sp(2n+2, \mathbb{R}) \ltimes \mathbb{C}^{2n+2}$, while the conical special coordinates X^I on N are unique up to transformations in $Sp(2n+4, \mathbb{R})$. Understanding the geometric meaning of the conification of $\hat{N}$ into N requires in particular to relate these two group action to one another.

The conification map

The concepts of Lagrangian pairs and of special Kähler pairs, which were introduced in section 5.4.2, are needed for defining the conification of ASK manifolds. It turns out that the conification map can be formulated such that it applies to any ASK-manifold, not only to those which can be obtained using the rigid r-map:

$$\text{con} : \{\text{ASK manifolds}\} \to \{\text{CASK manifolds}\} , \;\; \hat{N}_{2n} \mapsto N_{2n+2} . \qquad (543)$$

Compared to the previous paragraphs we have shifted $n \mapsto n-1$ in order to stress that this construction is valid for any ASK manifold.[51]

[50] While it should be straightforward to extend this to the para-complex setting, we restrict ourselves to reviewing published work.

[51] The case $n = 0$ can be interpreted as mapping the zero-dimensional ASK manifold $\{pt\}$ consisting of a single point to the CASK manifold $\mathbb{C}$ with its standard flat metric, correspond-

Consider the complex symplectic vector space $\mathbb{C}^{2n+2}$ with Darboux coordinates (X^I, W_I), where $I = 0, \ldots, n$. The vector field ∂_{W_0} is Hamiltonian with moment map X^0 and the symplectic reduction[52] with respect to ∂_{W_0} can be identified with the symplectic vector space $\mathbb{C}^{2n}$ with Darboux coordinates (X^a, W_a), $a = 1, \ldots, n$:

$$\{X^0 = 1\}/\langle \partial_{W_0} \rangle \cong \mathbb{C}^{2n} \; . \tag{544}$$

In section 5.4.2 we introduced the group $G_{\mathbb{C}} = Sp(\mathbb{C}^{2n}) \ltimes \mathrm{Heis}_{2n+1}(\mathbb{C})$ which acts on Lagrangian pairs by the affine representation $\bar{\rho} : G_{\mathbb{C}} \to \mathrm{Aff}_{Sp(\mathbb{C}^{2n})}(\mathbb{C}^{2n})$. As shown in [46] and reviewed in A.19 this affine representation can be extended to a linear symplectic representation of $G_{\mathbb{C}}$ on $\mathbb{C}^{2n+2}$. Based on this observation, the conification of ASK manifolds can be formulated locally using Lagrangian pairs and special Kähler pairs, and then globalized using a principal bundle based on the subgroup $G_{SK} \subset G_{\mathbb{C}}$. Recall from section 5.4.2 that any ASK manifold can be described locally by a special Kähler pair (ϕ, F), that is an embedding $\phi : \hat{N} \supset U \to \mathbb{C}^{2n}$ defined by a prepotential F, where $\phi = dF$. The special Kähler pair (ϕ, F) determines a Lagrangian pair (L, f), consisting of a Lagrangian submanifold $L \subset \mathbb{C}^{2n}$ together with a Lagrange potential f. To describe CASK manifolds in this approach one needs to add the condition that the embedding ϕ is conical, as defined in section 5.2. The corresponding Lagrangian submanifolds are called *regular Lagrangian cones*. Proposition 3.4 of [46] establishes a one-to-one correspondence between Lagrangian pairs in $\mathbb{C}^{2n}$ and regular Lagrangian cones in $\mathbb{C}^{2n+2}$, provided by two maps called conification con and reduction red $=$ con^{-1}. The action of the group $G_{SK} \subset G_{\mathbb{C}}$ is equivariant with respect to these maps, which allows to define the conification of special Kähler pairs. Up to the action of G_{SK} the conification works by 'homogenization' of the prepotential,

$$F_{\hat{N}}(X^1, \ldots, X^n) \mapsto F_N(X^0, X^1, \ldots X^n) = (X^0)^2 F_{\hat{N}}(X^1/X^0, \ldots, X^n/X^0) \; . \tag{545}$$

Interestingly, only the action of the subgroup $G = Sp(\mathbb{R}^{2n}) \ltimes \mathrm{Heis}_{2n+1}(\mathbb{R})$ preserves the induced Kähler metric on the Lagrangian cone. This means that the

ing to a quadratic prepotential.

[52]See A.14 for a review of Hamiltonian vector fields, moment maps and symplectic reductions.

supergravity r-map admits non-trivial deformations, which at the level of the prepotential correspond to adding terms of the form

$$\delta F = i(a_{0a} X^0 X^a + c(X^0)^2) \ , \quad a_{0a}, c \in \mathbb{R} \ . \tag{546}$$

We will discuss the physical interpretation of these deformations below. Having defined the conification of special Kähler pairs, the extension to the conification of general ASK manifolds uses the flat G_{SK}-principal bundle of special Kähler pairs introduced in section 5.4.2. Roughly speaking, starting with a local conification of $\hat{N}$ using a special Kähler pair (ϕ, F) one obtains the global conification N of $\hat{N}$ by maximal analytical extension of (ϕ, F). We refer to [46] for details.

8.3.3. The deformed supergravity r-map

Using the conification $\hat{N}_{2n+2} \mapsto N_{2n+4}$ we obtain the ASK/PSK correspondence $\hat{N}_{2n+2} \mapsto \bar{N}_{2n+2}$, while composing the rigid r-map with the conification map we can lift the supergravity r-map to the superconformal level, $M_{n+1} \mapsto N_{2n+4}$. More precisely, while the homogenized prepotential (545) matches with the result of the reduction of five-dimensional vector multiplets, the conification map allows to include the non-trivial deformations (546). Such terms are allowed for four-dimensional vector multiplets, but disappear when a decompactification limit to five-dimensions is performed [103, 104]. Terms of the form (546) with $a_{0a} = 0$ but $c \neq 0$ do actually occur in string theory. In type-II compactifications on Calabi-Yau three-folds they arise as worldsheet instantons with a coefficient proportional to the Euler number χ of the three-fold [105, 106], while in heterotic compactifications on $K3 \times T^2$ they are part of the one-loop corrections and proportional to an expansion coefficient of a (model dependent) modular form [107, 108, 109, 110]. We remark that deformations where $a_{0a} \neq 0$ and $c = 0$ do not have a known realization in string theory. Note that δF in (546) has purely imaginary coefficients, and is therefore distinct from terms of the form $\hat{\delta} F = \frac{1}{24} c_{2I} X^0 X^I$, which arise in IIA-compactifications, where c_{2I} are the components of the second Chern class. Terms of the form $\hat{\delta} F$ have real coefficients, and can be absorbed by a symplectic transformation. Thus they do not provide a non-trivial deformation, while (546) does.

8.4. Reduction from four to three dimensions: the c-map

8.4.1. Reductions to three dimensions

Compared to the generic situation considered in section 8.1, reductions to three dimensions have an enhanced number of scalar fields, because abelian vector fields can be dualized into scalars. Consider the generalized Maxwell Lagrangian

$$L(A) = -\frac{1}{2} F \wedge *F \tag{547}$$

for a p-form field strength $F = dA$ in $n = t+s$ dimensions, where t is the number of time-like dimensions, more precisely, the number of negative eigenvalues of the metric. By promoting the Bianchi identity $dF = 0$ to a field equation using a Lagrange multiplier $(n-p-1)$-form B, and subsequently eliminating F by its algebraic equation of motion, one arrives, after dropping any boundary terms resulting from integration by parts, at the dual Lagrangian

$$\tilde{L}(B) = (-)^t \frac{1}{2} G \wedge *G \,, \tag{548}$$

where $G = dB$ is the Hodge dual of F. Note that the sign of the generalized Maxwell term flips whenever the number of time-like dimensions (negative eigenvalues of the metric) is even, in particular in Euclidean signature, while it remains the same for an odd number of time-like dimensions, in particular in Lorentzian signature.[53]

Consider now starting with a four-dimensional action with one ε_1-complex scalar and one abelian gauge field.

$$S = \int d^4x \left(-\frac{1}{2} \partial_\mu X \partial^\mu \bar{X} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) . \tag{549}$$

Upon reduction to three dimensions we end up with four real scalars: the real and imaginary parts[54] of $X = \sigma + i_{\varepsilon_1} b$, the component $p = A_*$ of the four-dimensional vector field along the direction we reduce over, and the scalar s we gain by dualizing the three-dimensional abelian vector field A_m. For $\varepsilon_1 = -1$ we take the four-dimensional theory to have signature $(-+++)$ and consider both a space-like reduction, $\varepsilon_2 = -1$, and a time-like reduction, $\varepsilon_2 = 1$. For $\varepsilon_1 = 1$, we take the four dimensional theory to have signature $(++++)$, and only a space-like reduction, $\varepsilon_2 = -1$ is possible.

The corresponding three-dimensional actions are

$$S = \int d^3x \left(-\frac{1}{2} \partial_m \sigma \partial^m \sigma + \frac{\varepsilon_1}{2} \partial_m b \partial^m b + \frac{\varepsilon_2}{2} \partial_m p \partial^m p - \frac{\varepsilon_1 \varepsilon_2}{2} \partial_m s \partial^m s \right) . \tag{550}$$

[53] Irrespective of whether we choose a mostly plus or mostly minus convention for the metric, the sign of the kinetic energy is preserved in Lorentzian and reversed in Euclidean signature.

[54] Here and in the following 'real part' and 'imaginary part' is short for 'ε-real part' and 'ε-imaginary part,' respectively.

For $\varepsilon_1 = \varepsilon_2 = -1$ we can combine the four real scalars into one scalar valued in the quaternions $\mathbb{H}_{-1} := \mathbb{H}$,

$$q = \sigma + ib + jp + kq \ , \quad \bar{q} = \sigma - ib - jp - kq \ , \qquad (551)$$

where i, j, k anticommute pairwise, and where $i^2 = j^2 = k^2 = -1$. In the other cases we can combine them into a scalar valued in the algebra $\mathbb{H}_1$ of para-quaternions, where two of the complex units are replaced by para-complex units. For example, for $\varepsilon_1 = -1$, $\varepsilon_2 = 1$ we can use (551) with $j^2 = k^2 = 1$.[55] To treat both cases in parallel we use an ε-quaternionic notation where $\mathbb{H}_\varepsilon$ denotes the quaternions for $\varepsilon = -1$ and the para-quaternions for $\varepsilon = 1$.[56] The resulting action takes the form

$$S = \int d^3x \left(-\frac{1}{2} \partial_m q \partial^m \bar{q} \right) \ . \qquad (552)$$

For theories with several interacting scalars this type of rewriting is not practical, but it illustrates that the target space geometries that one obtains by dimensionally reducing four-dimensional vector multiplets are ε-quaternionic geometries. More specifically, when dimensionally reducing vector multiplets and dualizing all the three-dimensional vector fields, the resulting supersymmetry representations are hypermultiplets, and the target space geometry is ε-hyper-Kähler (ε-HK) in the rigid and ε-quaternionic Kähler (ε-QK), a.k.a. ε-quaternion-Kähler in the supergravity case.

8.4.2. The rigid c-map

We now turn to the reduction of a four-dimensional bosonic on-shell vector multiplet Lagrangian of the form (528). This section is based on [78], to which we refer for details.[57] The parameter $\varepsilon_1 = \pm 1$ labels the four-dimensional scalar target geometry, which is affine special Kähler for Lorentzian and affine special para-Kähler for Euclidean space-time signature, with a general ε_1-holomorphic prepotential. The second parameter $\varepsilon_2 = \pm 1$ distinguishes between space-like reduction and time-like reduction, where the latter is only possible if we start in

[55] See A.21 for a brief review of quaternions, para-quaternions, and the related 'ε-quaternionic' geometric structures.

[56] We write $\varepsilon_1, \varepsilon_2$ for signs related to the four-dimensional Lagrangian and its reduction to three dimensions, respectively, while using ε when talking about ε-complex structures in general.

[57] The conventions used in [78] are slightly different from those used in this review, which leads to various constant rescalings of fields.

Lorentzian signature. After dualization of the three-dimensional vector fields, the Lagrangian takes the form

$$\begin{aligned} L &= -N_{IJ}\partial_m X^I \partial^m \bar{X}^J + \varepsilon_2 (N_{IJ} - \varepsilon_1 R_{IK} N^{KL} R_{LJ}) \partial_m p^I \partial^m p^J \\ &\quad + 4\varepsilon_1\varepsilon_2 R_{IK} N^{KJ} \partial_m p^I \partial^m s_J - 4\varepsilon_2\varepsilon_2 N^{IJ} \partial_m s_I \partial^m s_J \ . \end{aligned} \tag{553}$$

Here $p^I \propto A^I_*$ are the scalar components of the four-dimensional vector fields and s_I the scalars obtained from dualizing the three-dimensional vector fields.

The target space geometry of the three-dimensional theory is hyper-Kähler for $\varepsilon_1 = \varepsilon_2 = -1$ [111] and para-hyper-Kähler for $\varepsilon_1\varepsilon_2 = -1$ [78]. It is possible to combine the real fields (p^I, s_I) into ε-complex coordinates W_I (where $\varepsilon = -\varepsilon_1\varepsilon_2$) and to make the ε-hyper-Kähler geometry of the target space manifest by finding explicit expressions for the three ε-complex structures and for an ε-Kähler potential in terms of the special geometry data of the four-dimensional theory [78]. Alternatively, one can work in real coordinates. The ε_1-complex version of the expression (296) for the Hessian metric on the four-dimensional scalar target space is

$$(H_{ab}) = \begin{pmatrix} N_{IJ} - \varepsilon_1 R_{IK} N^{KJ} R_{JL} & 2\varepsilon_1 R_{IK} N^{KJ} \\ 2\varepsilon_1 N^{IK} R_{KJ} & -4\varepsilon_1 N^{IJ} \end{pmatrix} . \tag{554}$$

Replacing the ε_1-complex scalars X^I by special real coordinates q^a, and combining the remaining real scalars into the symplectic vector $\hat{q}^a = (p^I, s_I)$, we obtain

$$L = -H_{ab}(q)\partial_m q^a \partial^m q^b + \varepsilon_2 H_{ab}(q) \partial_m \hat{q}^a \partial^m \hat{q}^b \ . \tag{555}$$

From this expression it is manifest that we can interpret the target space N of the three-dimensional theory as the tangent bundle $N = TM$ of the ASε_1K target manifold M of the four-dimensional theory, equipped with the Sasaki-like metric

$$ds^2_{N=TM} = H_{ab}(dq^a dq^b - \varepsilon_2 d\hat{q}^a d\hat{q}^b) \ . \tag{556}$$

Similar to the case of the rigid r-map, the special connection ∇ of the ASε_1K manifold M can be used to perform a canonical splitting of TN into a horizontal and a vertical distribution. Moreover, the special geometry data of M can be used to show that $N = TM$ globally carries the structure of an ε-HK manifold. The map induced by dimensional reduction of four-dimensional vector multiplets is called the rigid c-map:

$$c_{\varepsilon_1,\varepsilon_2} : \{\text{AS}\varepsilon_1\text{K manifolds}\} \to \{\varepsilon-\text{HK manifolds}\} , \quad M_{2n} \mapsto N_{4n} \cong TM . \tag{557}$$

Depending on $\varepsilon_1, \varepsilon_2$ there are three subcases:

1. The spatial c-map, or simply, the (rigid) c-map: $\varepsilon_1 = \varepsilon_2 = -1$, and $\varepsilon = -\varepsilon_1\varepsilon_2 = 1$. This corresponds to the standard, space-like reduction of vector multiplets in Lorentzian signature, and was first described in [111]. All involved scalar target space geometries are positive definite.[58]
2. The temporal c-map, $\varepsilon = -1, \varepsilon_2 = 1$ and $\varepsilon = -\varepsilon_1\varepsilon_2 = 1$. This corresponds to the time-like reduction of a Lorentzian vector multiplet theory and relates a positive definite scalar geometry to one with neutral signature.
3. The Euclidean c-map, $\varepsilon_1 = 1, \varepsilon_2 = -1$ and $\varepsilon = -\varepsilon_1\varepsilon_2 = 1$. This corresponds to the space-like reduction of a Euclidean vector multiplet theory and relates two target space geometry with neutral signature.

We remark that instead of setting $N = TM$, we can alternatively take $N = T^*M$, since the metric allows to identify tangent spaces with cotangent spaces. Then

$$ds^2_{N=T^*M} = H_{ab}dq^a dq^b - \varepsilon_2 H^{ab} d\hat{q}_a d\hat{q}_b , \tag{558}$$

where H^{ab} is the inverse of H_{ab} and $d\hat{q}_a = H_{ab}dq^b$.[59] Thus the cotangent bundle of an ASε_1K manifold is an ε-HK manifold [111, 78]. This is a stronger result than for generic Kähler manifolds, where it is known that the cotangent bundle admits the structure of an HK manifold locally, in a neighbourhood of its zero section [112, 113].

8.4.3. The supergravity c-map and its deformation

We finally turn to the reduction of four-dimensional vector multiplets coupled to supergravity to three dimensions. Our starting point is the bosonic on-shell Lagrangian (537) in Lorentzian or Euclidean space-time signature, with a general ε_1-holomorphic prepotential. The four-dimensional metric is decomposed according to

$$ds^2_4 = g_{\mu\nu}dx^\mu dx^\nu = -\varepsilon_2 e^{\phi}(dx^* + V_m dx^m)^2 + e^{-\phi} g_{mn} dx^m dx^n , \tag{559}$$

[58] That is, if we impose positive kinetic energy for all fields. Mathematically we can also consider scalar target spaces with indefinite metrics.

[59] The integrability condition for the local existence of the functions $\hat{q}_a$, which are fibre coordinates on T^*M, follows from H_{ab} being the components of a Hessian metric on M.

where V_m is the Kaluza-Klein vector and ϕ the Kaluza-Klein scalar. After reduction to three dimensions, all abelian vector fields are dualized into scalars. The bosonic field content of the resulting three-dimensional theory is:

- The three-dimensional metric g_{mn}.
- The $n = n_{(4)}$ ε_1-complex four-dimensional scalars z^A, where $n_{(4)}$ is the number of four-dimensional vector multiplets.
- The $n+1$ real scalars $\zeta^I \propto A^I_*$ obtained by reducing the $n+1$ four-dimensional vector fields A^I_μ.
- The $n+1$ real scalars $\tilde{\zeta}_I$ obtained by dualizing the $n+1$ three-dimensional vector fields A^I_m.
- The Kaluza-Klein scalar ϕ and the scalar $\tilde{\phi}$ obtained by dualizing the Kaluza-Klein vector V_m.

The three-dimensional metric does not carry local degrees of freedom while the $4n+4$ real scalars $\mathrm{Re}(z^A), \mathrm{Im}(z^A), \zeta^I, \tilde{\zeta}_I, \phi, \tilde{\phi}$ are the bosonic components of $4n+4$ hypermultiplets, coupled to three-dimensional Poincaré supergravity. The three-dimensional Lagrangian is [114, 17]

$$\begin{aligned} L_3^{(\varepsilon_1,\varepsilon_2)} &= \frac{1}{2}R_3 - \bar{g}_{A\bar{B}}\partial_m z^A \partial^m \bar{z}^{\bar{B}} - \frac{1}{4}\partial_m\phi\partial^m\phi \\ &+\varepsilon_1 e^{-2\phi}\left[\partial_m\tilde{\phi} + \frac{1}{2}\left(\zeta^I\partial_m\tilde{\zeta}_I - \tilde{\zeta}_I\partial_m\zeta^I\right)\right]^2 \\ &-\frac{\varepsilon_2}{2}e^{-\phi}\left[\mathcal{I}_{IJ}\partial_m\zeta^I\partial^m\zeta^J - \varepsilon_1\mathcal{I}^{IJ}\left(\partial_m\tilde{\zeta}_I - \mathcal{R}_{IK}\partial_m\zeta^K\right)^2\right] . \end{aligned} \tag{560}$$

The target space geometry of hypermultiplets coupled to supergravity is quaternionic-Kähler [24]. For Euclidean hypermultiplets obtained by dimensional reduction the target space geometry is para-quaternionic Kähler [19, 17]. The map between scalar geometries induced by dimensional reduction of four-dimensional vector multiplets coupled to supergravity is called the supergravity c-map:

$$\bar{c}_{(\varepsilon_1,\varepsilon_2)} : \{\mathrm{PS}\varepsilon_1\mathrm{K\ \ manifolds}\} \to \{\varepsilon - \mathrm{QK\ \ manifolds}\} , \ \ \bar{M}_{2n} \mapsto \bar{N}_{4n+4} . \tag{561}$$

The properties of the three types of supergravity c-maps are summarized in Table 5.

c-map	Space-time signature	scalar geometry	scalar manifold signature
spatial	$(1,3) \mapsto (1,2)$	PSK $\mapsto$ QK	$(2p,2q) \mapsto (4p+4,4q)$
temporal	$(1,3) \mapsto (0,3)$	PSK $\mapsto$ PQK	$(2p,2q) \mapsto (2d,2d)$
Euclidean	$(0,4) \mapsto (0,3)$	PSPK $\mapsto$ PQK	$(r,r) \mapsto (2d,2d)$

Table 5: This table summarizes the relations between the space-time signatures, target space geometries and target space-signatures for the 3 types of supergravity c-maps. We include the case where the PSK manifold has indefinite signature, which is mathematically well defined, but corresponds to a vector multiplet theory where some of the fields have negative kinetic energy. In this case the QK manifold obtained by the spatial supergravity c-map is also indefinite. Para-Kähler and para-QK manifolds always have neutral signature. Manifolds of dimension $2n$ map to manifolds of dimension $4n+4$, therefore $d = p+q+2$ in row 2 and $d = r+1$ in row 3.

Showing that the scalar target manifold $\bar{N}_{4n+4}$ of the Lagrangian (560) is ε-quaternionic Kähler is somewhat involved, in particular if one wants to have a global description of $\bar{N}_{4n+4}$. There are various ways to describe the geometry of $\bar{N}$, which we discuss in turn.

Supergravity c-map spaces as group bundles

The first description of the geometry of $\bar{N}$ is based on an observation of [114] for the case $\varepsilon_1 = \varepsilon_2 = -1$: when restricting to constant values of z^i, the metric on the corresponding subspace is Kähler, only depends on the number of vector multiplets, and is in fact the metric of a Riemannian symmetric space. It was shown in [4] that if the underlying PSK manifold $\bar{M}$ is a PSK domain, then the image under the supergravity c-map is a QK domain of the form $\bar{N} = \bar{M} \times G$, where G is a solvable Lie group, and where the QK metric $g_{\bar{N}}$ is a 'bundle metric' $g_{\bar{N}} = g_{\bar{M}} + g_G(p)$, where $g_G(p)$ is a family of left-invariant metric on G parametrized by $p \in \bar{M}$. It was also shown in [4] that this construction can be 'globalized,' that is one can apply the supergravity c-map domain-wise and then glue together the resulting QK domains consistently and uniquely to obtain a QK manifold. Moreover, it was shown that the supergravity c-map preserves geodesic (and hence metric) completeness, that is, if $\bar{M}$ is complete so is its image $\bar{N}$ under the supergravity c-map. Except for the completeness result (which heavily relies on the involved metrics being definite), the description of $\bar{N}$ by gluing domains should also apply to the case where $\varepsilon_1\varepsilon_2 = -1$. In [17] it was shown that the image of a PSε_1K domain $\bar{M}$ under $c_{(\varepsilon_1,\varepsilon_2)}$ takes the form

$\bar{N} = \bar{M} \times G$, with a bundle metric $g_{\bar{N}} = g_{\bar{M}} + g_G(p)$, where G is a solvable Lie group. The solvable Lie groups G and left-invariant metrics on G were found to be the following:

1. $\varepsilon_1 = \varepsilon_2 = -1$. This is the standard (spatial) supergravity c-map which was already considered in [114]. The solvable Lie group is the Iwasawa subgroup of $U(n+2,1)$ and can be identified globally with the complex hyperbolic space

$$\mathbb{C}H^{n+2} = \mathrm{U}(n+2,1)/(U(n+2) \times U(1)) \quad (562)$$

equipped with a positive definite Kähler metric of constant holomorphic sectional curvature -1.[60] The metric on the resulting QK manifold $\bar{N} = \bar{M} \times G$ is positive definite.[61]

2. $\varepsilon_1 = -1$, $\varepsilon_2 = 1$. This is the temporal supergravity c-map. The group G is again the Iwasawa subgroup of $U(n+2,1)$, but with a different, indefinite left-invariant metric. It can be identified locally with the indefinite complex hyperbolic space

$$\mathbb{C}H^{1,n+1} \cong U(1,n+2)/(U(1,n+1) \times U(1)) \quad (563)$$

equipped with a pseudo-Kähler metric of complex signature $(1, n+1)$ and constant holomorphic sectional curvature -1. Note that for non-compact symmetric spaces of indefinite signature the Iwasawa subgroup does not act transitively, so that we cannot identify G globally with the above symmetric space. However, it can be shown that G acts with an open orbit, thus allowing the identification of G with an open subset of the symmetric space. The signature of the resulting space $\bar{M} \times G$ is neutral $(2n+2, 2n+2)$, as required for a para-quaternionic Kähler manifold.

3. $\varepsilon_1 = 1$, $\varepsilon_2 = -1$. This is the Euclidean supergravity c-map. The solvable Lie group G is the Iwasawa subgroup of $SL(n+3,\mathbb{R})$ and can be identified locally with para-complex hyperbolic space

$$CH^{n+2} \cong SL(n+3,\mathbb{R})/S(GL(1) \times GL(n+2)) \quad (564)$$

[60] The ε-holomorphic sectional curvature of an ε-Kähler manifold (M, J, g) is $\langle (R(X,JX)JX, X \rangle / \langle X \wedge JX, X \wedge X \rangle$, where X is a vector field, where $\langle \cdot, \cdot \rangle$ is the scalar product between tensors induced by the metric, and where R is the curvature tensor. It can be interpreted as the sectional curvature of the ε-complex line $X \wedge JX$ [115, 17].

[61] We assume, here and in the following case, that the target space metric of the four-dimensional vector multiplet theory is positive definite.

equipped with a para-Kähler metric of real signature $(n+2, n+2)$ and of constant para-holomorphic sectional curvature -1. The signature of $\bar{M} \times G$ is $(2n+2, 2n+2)$, as required for a para-quaternionic Kähler manifold.

The simplest examples for $\bar{N}$ are the 'universal hypermultiplets' obtained by reducing pure four-dimensional $\mathcal{N}=2$ supergravity. In this case $\bar{M} = \{pt\}$ and $\bar{N}$ is locally isometric to one of the following manifolds:

1. For the spatial supergravity c-map, $\varepsilon_1 = \varepsilon_2 = -1$, the target space is globally isometric to

$$\mathbb{C}H^2 \cong SU(2,1)/(U(2) \times U(1)) \ . \tag{565}$$

This is the 'universal hypermultiplet' which is obtained by the reduction of pure $\mathcal{N}=2$ supergravity. In general c-map spaces the universal hypermultiplet spans a distinguished subspace. Note however, that once string corrections to the hypermultiplet metric are taken into account the universal hypermultiplet ceases to be an identifiable, 'universal' part of the scalar manifold [116].

2. For the temporal supergravity c-map, $\varepsilon_1 = -1, \varepsilon_2 = 1$, the target space is locally isometric to

$$\mathbb{C}H^{1,1} \cong U(2,1)/(U(1,1) \times U(1)) \ . \tag{566}$$

This is target space for a time-like reduction of pure $\mathcal{N}=2$ supergravity.

3. For the Euclidean supergravity c-map, the target space is

$$CH^2 \cong SL(3,\mathbb{R})/S(GL(1) \times GL(2)) \ . \tag{567}$$

This target space does not only arise in the dimensional reduction of pure $\mathcal{N}=2$ Euclidean supergravity [89], but also when dualizing the so-called double tensor multiplet in Euclidean signature [117]. This reflects that Euclidean actions which differ by sign flips can be related by using that dimensional reduction/lifting, Wick rotation and Hodge dualization do not commute which each other, see also [45], as discussed in section 8.2.

Conification of ε-HK manifolds

We now turn to another way of describing the scalar manifold $\bar{N}$. As in the case of the supergravity r-map, one can lift the supergravity c-map to the superconformal level. Within the superconformal formalism, it is not possible to formulate hypermultiplets off-shell with a finite number of auxiliary fields. However, as long as the hypermultiplet manifold has sufficiently many isometries, hypermultiplets can be dualized into tensor multiplets, which admit a superconformal off-shell representation [118]. Alternatively, the projective superspace formalism can be used to describe hypermultiplets off-shell, see also section 8.4.6. Off-shell formulations of the supergravity c-map were obtained in [119] using projective superspace, and in [120, 121] using the superconformal formalism.

We will review the global geometric construction of the supergravity c-map given in [17], which is inspired by the superconformal approach and which provides all the data necessary for describing the theory at the superconformal level. This description also allows a complete and relatively short proof that spaces in the image of the supergravity c-map are global ε-quaternionic Kähler manifolds. Moreover, this proof also applies to a one-parameter family of non-trivial deformations of the metric obtained from the supergravity c-map.

When working with hypermultiplets the situation regarding the scalar target spaces is the ε-quaternionic analogon of the real and complex settings for five- and four-dimensional vector multiplets. To each ε-QK manifold $\bar{N}_{4n+4}$ describing $n+1$ hypermultiplets coupled to Poincaré supergravity, one can associate an ε-HK cone N_{4n+8}, that is an ε-HK manifold with a homothetic action of the group $\mathbb{H}_\varepsilon^*$ of invertible ε-quaternions, such that $\bar{N} \cong N/\mathbb{H}_\varepsilon^*$. Conversely N is an $\mathbb{H}_\varepsilon^*$-bundle over $\bar{N}$. We remark that while it would be more in line with our terminology for vector multiplets to use the term 'conical ε-HK manifold', we follow the literature in using 'ε-HK cone' instead.

One can obtain the superconformal lift $\tilde{c} : M_{2n+2} \mapsto N_{4n+8}$ of the supergravity c-map $c : \bar{M}_{2n} \mapsto \bar{N}_{4n+4}$ by composing the rigid c-map $c : M_{2n+2} \mapsto \hat{M}_{4n+4} \cong TM$ with a conification map $\text{con} : \hat{M}_{4n+4} \mapsto M_{4n+8}$. The situation is summarized by the following diagram:

$$\begin{array}{ccccc} & & \overset{\tilde{c}}{\frown} & & \\ M & \overset{c}{\longmapsto} & \hat{N} & \overset{\text{con}}{\longmapsto} & N \\ \Big\downarrow {\scriptstyle \text{SC}} & & & \searrow^{\varepsilon\text{HK/QK}} & \Big\downarrow {\scriptstyle \text{SC}} \\ \bar{M} & & \underset{\bar{c}}{\longmapsto} & & \bar{N} \end{array} \qquad (568)$$

This diagram induces a correspondence between ε-HK and ε'-QK manifolds of the same dimension.[62]

The ε-HK/QK correspondence

The correspondence can be formulated independently of the supergravity c-map, and then also applies to ε-HK manifolds $\hat{N}$ which are not in the image of the rigid c-map, but specify the conditions stated below. The resulting ε-HK/QK correspondence generalizes the HK/QK correspondence of [122], which was applied to the space-like supergravity c-map in [123] in the context of the twistor approach, see also section 8.4.6 below. We follow [124, 125, 126], who have extended the HK/QK correspondence to arbitrary signature and to the para-complex setting.

For an ε-HK manifold $\hat{N}$ to admit a conification N the following conditions must hold:

1. $\hat{N}$ admits a time-like or space-like Killing vector field Z, which is ε-holomorphic with respect to an ε-complex structure J_1, which is part of ε-HK structure. The Killing vector field Z is Hamiltonian with respect to the corresponding ε-Kähler form ω_1, that is, there exists a function f such that $df = -\omega_1(Z\cdot, \cdot)$.
2. The functions f and $f_1 = f - \frac{1}{2}g(Z, Z)$ are nowhere zero.
3. The Killing vector field Z rotates the other two ε-complex structures, J_2 and J_3, of the ε-HK structure, that is, $L_Z J_2 = 2\varepsilon J_3$.

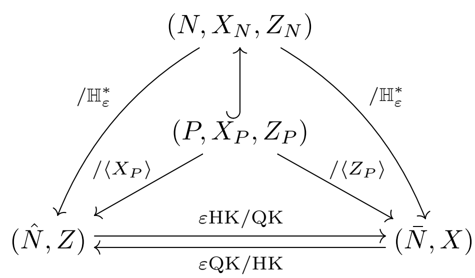
Having constructed an ε-HK cone N_{4n+8}, one obtains a corresponding ε-QK manifold N_{4n+4} by a superconformal quotient.[63] Conversely, given any ε-QK manifold $\bar{N}_{4n+4}$, there always exist the associated ε-HK cone (which for $\varepsilon = -1$ is also known as the Swann bundle) N_{4n+8}. One can then obtain an ε-HK manifold N_{4n+4} by taking an ε-HK quotient, provided that N_{4n+8} admits a tri-holomorphic Killing vector field X_N which commutes with the Euler field ξ of the cone N_{4n+8}, acts freely and satisfies a technical condition regarding the level sets of its moment map. The existence of such a vector field follows from the existence of a space-like or time-like Killing vector field X on $\bar{N}$, again subject to a technical condition.[64]

[62] Note that $\varepsilon \neq \varepsilon'$ can occur, see Table 5.

[63] Since the construction involves the moment map of Z explicitly, the correspondence allows a one-parameter deformation to be discussed in 8.4.3.

[64] We refer to [126] for the details.

It turns out that the ε-HK/QK correspondence can be formulated without using the ε-HK cone N_{4n+8} explicitly. Roughly speaking, three of the four extra dimensions of the cone do not play an essential role, so that one can take a shortcut and relate $\hat{N}_{4n+4}$ and $\bar{N}_{4n+4}$ via a manifold P_{4n+5} of real dimension $4n+5$. The manifold P_{4n+5} is a rank one principal bundle over N_{4n+4} with principal action generated by a vector field X_P, and simultaneously a rank one principal bundle over $\bar{N}_{4n+4}$ with principal action generated by a vector field Z_P. The vector fields X_P and Z_P are lifts of the Killing vector fields X on $\bar{N}_{4n+4}$ and Z on $\hat{N}_{4n+4}$ that we mentioned before. The manifold P_{4n+5} is a submanifold of the ε-HK cone N_{4n+8}, and taking quotients of P_{4n+5} with respect to the principal actions of X_P and Z_P is consistent with taking an ε-HK quotient and an ε-QK quotient of N_{4n+8}, respectively. The situation is summarized in the following diagram:



Explicit expressions for the all relevant geometric data on $\hat{N}, \bar{N}, N$ and P can be found in [124, 125, 126].

A symplectic parametrization of the supergravity c-map

The space P_{4n+5} appears naturally in the dimensional reduction of four-dimensional vector multiplets, if we use special real coordinate for the CASK manifold M and insist on maintaining manifest symplectic invariance after dimensional reduction. This leads to a reformulation of (560) in terms of a gauged sigma model with target space P_{4n+5}, which is equivalent to a sigma model with target space $\bar{N}_{4n+4}$ [127].

This reformulation requires a couple of steps. First we replace the four-dimensional scalars z^A by the projective scalars X^I which take values in M_{2n+2}:

$$\bar{g}_{A\bar{B}}\partial_m z^A \partial^m \bar{z}^{\bar{B}} = \tilde{g}^{(0)}_{IJ}\partial_m X^I \partial^m \bar{X}^J \ . \tag{569}$$

Here $\tilde{g}_M^{(0)} = \pi^* \bar{g}_{\bar{M}}$ is the lift of the PSK metric to the CASK manifold. Since $\tilde{g}_M^{(0)}$ has a two-dimensional kernel, this rewriting does not increase the number of propagating degrees of freedom. The right hand side can be viewed as a gauged sigma model, where the connection gauging the action of $\mathbb{C}^*_{\varepsilon_1} \cong \mathbb{R}^{>0} \times U(1)_{\varepsilon_1}$ on M has been integrated out.[65] Here

$$U(1)_{\varepsilon_1} = \begin{cases} U(1) , & \text{for } \varepsilon_1 = -1 , \\ GL(1, \mathbb{R}) , & \text{for } \varepsilon_1 = 1 , \end{cases} \tag{570}$$

is the ε_1-unitary group which is part of the R-symmetry group of the supersymmetry algebra. Rewriting the vector field couplings in terms of X^I is trivial since the matrix $\mathcal{N}_{IJ} = \mathcal{R}_{IJ} + i_{\varepsilon_1} \mathcal{I}_{IJ}$ is homogeneous of degree zero.

The second step is to make the field redefinition $Y^I := e^{\phi/2} X^I$, which absorbs the Kaluza-Klein scalar ϕ into the superconformal scalars X^I. If we impose the D-gauge on X^I, then ϕ can be expressed as a function of the new scalars Y^I:

$$-i_{\varepsilon_1}(X^I \bar{F}_I - F_I \bar{X}^I) = 1 \Rightarrow -i_{\varepsilon_1}(Y^I \bar{F}_I - F_I \bar{Y}^I) = e^{\phi} . \tag{571}$$

From now on we do not regard ϕ as an independent field, but as a function of the fields Y^I. Since the fields Y^I are subject to $U(1)_{\varepsilon_1}$-gauge transformations, the $(n+1)$ ε_1-complex scalars Y^I represent $2n+1$ propagating degrees of freedom. Geometrically, $2n$ scalars correspond to excitations transverse to the $\mathbb{C}^*_{\varepsilon_1}$-action on M and thus to the independent four-dimensional scalars z^i, while the additional scalar corresponds to the radial direction of the real cone $M = \mathbb{R}^{>0} \times S$, where S is the ε_1-Sasakian submanifold of M defined by the D-gauge.

The third step is to use special real coordinates on M. Since Y^I can be interpreted as special ε-holomorphic coordinates on M, we can define associated special real coordinates $q^a = (x^I, y_I)$,

$$Y^I = x^I + i_{\varepsilon_1} u^I(x, y) , \quad F_I(Y) = y_I + i_{\varepsilon_1} v_I(x, y) , \tag{572}$$

which compared to the usual special real coordinates have been rescaled by a factor $e^{\phi/2}$ involving the Kaluza-Klein scalar ϕ.

[65] This proceeds by imposing the K-gauge $b_\mu = 0$ on (426) and then eliminating the $U(1)$ gauge field A_μ by its equation of motion, see section 6.

The fourth step is to express the vector field coupling matrix $\mathcal{N}_{IJ}$ in terms of the tensor field $\hat{H}_{ab}$ using (337). Finally, instead of using the tensors H_{ab} and $\hat{H}_{ab}$ it is convenient to express all couplings in terms of the Hessian metric

$$\tilde{H}_{ab} = \partial^2_{a,b}\left[-\frac{1}{2}\log(-2H)\right] = -\frac{1}{2H}H_{ab} + \frac{1}{2H^2}H_a H_b \tag{573}$$

where

$$H(q^a) = -\frac{1}{2}e^{\phi} = \frac{i_\varepsilon}{2}(Y^I\bar{F}_I - F_I\bar{Y}^I) \tag{574}$$

is the Hesse potential for the CASε_1K-metric on M. Defining $\hat{q}^a := \frac{1}{2}(\zeta^I, \tilde{\zeta}_I)$ and

$$(\Omega_{ab}) = \begin{pmatrix} 0 & \mathbb{1}_{n+1} \\ -\mathbb{1}_{n+1} & 0 \end{pmatrix} \tag{575}$$

we can rewrite (560) in the form [127, 17]

$$\begin{aligned} L_3^{(\varepsilon_1,\varepsilon_2)} &= \frac{1}{2}R_3 - \tilde{H}_{ab}\left(\partial_m q^a\partial^m q^b - \varepsilon_2\partial_m\hat{q}^a\partial^m\hat{q}^b\right) \\ &\quad + \frac{\varepsilon_1}{H^2}\left(q^a\Omega_{ab}\partial_m q^b\right)\left(q^a\Omega_{ab}\partial^m q^b\right) \\ &\quad - \frac{2\varepsilon_1\varepsilon_2}{H^2}\left(q^a\Omega_{ab}\partial_m\hat{q}^b\right)\left(q^a\Omega_{ab}\partial^m\hat{q}^b\right) \\ &\quad + \frac{\varepsilon_1}{4H^2}\left(\partial_m\tilde{\phi} + 2\hat{q}^a\Omega_{ab}\partial_m\hat{q}^b\right)\left(\partial^m\tilde{\phi} + 2\hat{q}^a\Omega_{ab}\partial^m\hat{q}^b\right) . \end{aligned} \tag{576}$$

This is a non-linear sigma model for $4n+5$ real scalars $q^a, \hat{q}^a, \tilde{\phi}$ coupled to gravity. Its target space P_{4n+5} is the total space of the rank one principal bundle $\pi : P_{4n+5} \to \bar{N}_{4n+4}$ which occurs when constructing the supergravity c-map using the ε-HK/QK correspondence. Since the scalar fields q^a are subject to $U(1)_{\varepsilon_1}$ gauge transformations, there are only $4n+4$ propagating degrees of freedom. The symmetric tensor

$$\begin{aligned} g_P &= \tilde{H}_{ab}(dq^a dq^b - \varepsilon_2 d\hat{q}^a d\hat{q}^b) - \frac{\varepsilon_1}{H^2}(q^a\Omega_{ab}dq^b)^2 + \frac{2\varepsilon_1\varepsilon_2}{H^2}(q^a\Omega_{ab}d\hat{q}^b)^2 \\ &\quad - \frac{\varepsilon_1}{4H^2}(d\tilde{\phi}^2 + 2\hat{q}^a\Omega_{ab}d\hat{q}^b)^2 \end{aligned} \tag{577}$$

defined by the Lagrangian (576) has a one-dimensional kernel and is projectable with respect to the $U(1)_{\varepsilon_1}$-action. Thus (576) is a gauged non-linear sigma model (with the $U(1)_{\varepsilon_1}$-connection integrated out), and defines, by projection onto orbits, a non-linear sigma model with target space $\bar{N} = P/U(1)_{\varepsilon_1}$ and ε-QK metric $g_{\bar{N}}$, where $g_P = \pi^* g_{\bar{N}}$.

As explained in section 5.3, there is no natural choice of an $U(1)_{\varepsilon_1}$-gauge which realizes the PSK manifold $\bar{M}$ canonically as an embedded submanifold of the CASK manifold M, because the distribution orthogonal to the $U(1)_{\varepsilon_1}$-action is not integrable. Similarly, there is no preferred way to identify $\bar{N}$ with a submanifold of P. Instead of making a conventional choice, it is possible and advantageous to work with the P-valued gauged sigma model. The coordinates we have constructed on P are either symplectic vectors, $q^a, \hat{q}^a$, or symplectic scalars, $\tilde{\phi}$. Fixing a $U(1)_{\varepsilon_1}$ gauge requires to impose a condition on q^a and symplectic covariance is lost. However for many purposes, including to prove that $(\bar{N}_{4n+4}, g_{\bar{N}})$ is ε-QK, one can work on P_{4n+5} and maintain symplectic covariance.

Describing the supergravity c-map using a gauged sigma model with target P_{4n+5} amounts to replacing the diagram (568) by

$$
\begin{array}{ccccc}
M & \overset{c}{\longmapsto} & \hat{N} \cong TM & \longmapsto & P \cong TM \times \mathbb{R} \\
\downarrow \text{SC} & & & \searrow^{\varepsilon\text{HK/QK}} & \downarrow /U(1)_{\varepsilon_1} \\
\bar{M} & & \underset{\bar{c}}{\longmapsto} & & \bar{N}
\end{array}
\tag{578}
$$

Defining the one-forms

$$\rho = H^{-1} q^a \Omega_{ab} dq^b \ , \quad \sigma = H^{-1} q^a \Omega_{ab} d\hat{q}^b \ , \quad \tau = H^{-1} \hat{q}^a \Omega_{ab} d\hat{q}^b \tag{579}$$

the projectable tensor (577) takes the form

$$g_P = \tilde{g}_{TM} - \varepsilon_1 \rho^2 + 2\varepsilon_1 \varepsilon_2 \sigma^2 - \varepsilon_1 (d\tilde{\phi} + \tau)^2 \tag{580}$$

where

$$\tilde{g}_{TM} = \tilde{H}_{ab}(dq^a dq^b - \varepsilon_2 d\hat{q}^a d\hat{q}^b) \tag{581}$$

is the image of $\tilde{g}_M = \tilde{H}_{ab} dq^a dq^b$ under the rigid c-map. The manifold P_{4n+5} is defined as $TM \times \mathbb{R}$, where $\mathbb{R}$ is parametrized by $\tilde{\phi}$. The tensor g_P is obtained by twisting the product metric $\tilde{g}_{TM} - \varepsilon_1 d\tilde{\phi}^2$ using the one forms ρ, σ, τ. The vector field $\partial/\partial\tilde{\phi}$ leaves g_P invariant and generates a principal action on P which allows to recover TM as a quotient $TM \cong P/\mathbb{R}$. We remark that one can replace $\mathbb{R}$ by S^1, which is indeed the choice usually made in the ε-HK/QK correspondence. The choice $\mathbb{R}$ is suitable for the supergravity c-map, where $\tilde{\phi}$ is the dualized Kaluza-Klein vector. The principal $\mathbb{C}^*_{\varepsilon_1}$-action on M can be lifted to TM and to P, which then allows to take a quotient of P by the principal action of

$U(1)_{\varepsilon_1} \subset \mathbb{C}^*_{\varepsilon_1}$. The tensor g_P is invariant under and transversal with respect to this group action and defines a non-degenerate metric $g_{\bar{N}}$ on $\bar{N} = P/U(1)_{\varepsilon_1}$.

Alternatively, we interpret the diagram 578 such that the rigid c-map is applied to the CASε_1K metric $g_M = H_{ab}dq^a dq^b$ to obtain the ε-HK metric $g_{TM} = H_{ab}(dq^a dq^b - d\hat{q}^a d\hat{q}^b)$. The tensor g_P is then obtained by a conformal rescaling and twisting by the one-forms $d\tilde{H} = \tilde{H}_a dq^a$ and $\tilde{H}_a d\hat{q}^a$ in addition to the modifications which relate $\tilde{g}_{TM}$ to g_P. Note that for both g_M and $\tilde{g}_M$ their relation to g_P is determined by the ε-HK/QK correspondence (equivalently, by conification), and therefore is canonical.

Proving that a metric is an ε-QK metric is usually difficult, because an ε-QK manifold need not admit any globally defined and integrable ε-complex structures. One advantage of constructing $\bar{N}$ as a quotient of $P \cong TM \times \mathbb{R}$ is that TM is ε-HK. This can be used to construct data on P which by projection define an ε-QK structure on $\bar{N}$, thus providing a concise proof that $\bar{N}$ is ε-QK. We refer to [17] for details. As shown there, calculations on P can be translated into calculations on $\bar{N}$ using local sections. The original proof of [114] that spaces in the image of spatial supergravity c-map are QK uses an adapted co-frame on $\bar{N}$. The approach of [17] also allows to show that ε-QK manifolds obtained from the supergravity c-map admit integrable ε-complex structures. In particular, the ε_1-complex structure of M induces an integrable ε_1-complex structure on $\bar{N}$ which is part of the ε-QK structure.[66] There also always exists a second integrable ε_1-complex structure, which is not part of the ε-QK structure, and which differs from the first integrable structure by a sign flip on a two-dimensional distribution. A third integrable structure only exists if the Hessian metric g_M on M has a quadratic Hesse potential, $\nabla g_M = 0$.

The parametrization (576) of the c-map has turned out to be useful for obtaining explicit non-extremal black hole and black brane in solutions, as well as cosmological solutions, for four-dimensional $\mathcal{N} = 2$ vector multiplets coupled to Poincaré supergravity, without and with gauging [16, 129, 130, 131].[67]

[66] For the spatial supergravity c-map this was first shown in [128].

[67] Non-extremal solutions for five-dimensional vector multiplets can be obtained in a similar way using the r-map [132].

From Griffiths to Weil flags

It was observed in [133] and [134] that the spatial supergravity c-map involves the so-called Weil intermediate Jacobian, which parametrizes Hodge structures on Calabi-Yau three-folds. Similarly, the rigid c-map involves the so-called Griffiths intermediate Jacobian [49]. While this can be interpreted in the context of Calabi-Yau compactifications, where the scalar manifold are related to the moduli spaces of complex and Kähler structures, the supergravity c-map is well defined for any theory of $\mathcal{N} = 2$ vector multiplets coupled to supergravity. Therefore, one should be able to understand the appearance of the Griffiths and Weil Jacobians without reference to Calabi-Yau manifolds. In [4] a geometrical interpretation was given based on the realization of CASK manifolds as Lagrangian cones in $V = \mathbb{C}^{2n+2} = T^*\mathbb{C}^{n+1}$ We have already noted that besides the CASK metric $g_M = H_{ab}dq^a dq^b$ the CASK manifold M admits another metric $\hat{g}_M = \hat{H}_{ab}dq^a dq^b$, which, up to an overall factor, differs by a sign flip along the distribution spanned by the vector fields ξ and $J\xi$ which generate the $\mathbb{C}^*$-action. This operation can be viewed as a reflection on V which induces an $Sp(\mathbb{R}^{2n+2})$ equivariant diffeomorphism between certain flag manifolds defined over V. These flag manifolds are of the same type as the Griffiths and and Weil intermediate Jacobians, and have therefore been dubbed Griffiths and Weil flags, respectively.

From our description of the supergravity c-map it is clear why it involves a map from Griffiths to Weil flags. In a rigid vector multiplet theory the matrix encoding the vector field couplings is H_{ab}, while in a local vector multiplet theory it is $\hat{H}_{ab}$. The rigid c-map generates a term of the form $H_{ab}d\hat{q}^a d\hat{q}^b$ in the metric on TM, which in the three-dimensional Lagrangian corresponds to the dimensional reduction of the vector field of a rigid vector multiplet theory. The twisting relating g_{TM} to g_P involves (among other things) the replacement $H_{ab}d\hat{q}^a d\hat{q}^b \mapsto \hat{H}_{ab}d\hat{q}^a d\hat{q}^b$, where the latter term corresponds in the three-dimensional Lagrangian to the dimensional reduction of the vector fields of local vector multiplets. Thus the HK/QK part of the supergravity c-map acts as a reflection on V which replaces Griffiths flags by Weil flags.

The deformed supergravity c-map

Similar to the ASK/PSK correspondence, the ε-QK-metrics obtained from the ε-HK/QK correspondence depend explicitly on the choice of a moment map for the ε-holomorphic vector field Z on $\hat{N}$. This results in a one-parameter family of metrics $g_{\bar{N}}^{(c)}$, with $c = 0$ corresponding to the supergravity c-map [123]. It has been shown directly, that is without invoking supersymmetry, that while the deformation is non-trivial, the metrics $g_{\bar{N}}^{(c)}$ with $c \neq 0$ are still ε-QK [124, 125, 126]. In the QK case the deformation corresponds, for a specific value of c, to the one-loop correction to the hypermultiplet metric in type-II Calabi-Yau compactifications [135, 136]. Explicit expressions for the generalization of (577), (580) can be found in [125, 126].

8.4.4. Results on completeness, classification and symmetries of PSR, PSK and QK manifolds

In this section we collect results on the geodesic completeness, classification and isometries of PSR, PSK and QK manifolds. Recall that a pseudo-Riemannian manifold is called *homogeneous* if its group of isometries acts transitively, and *globally symmetric* if every point is the fixed point of an involutive isometry. A pseudo-Riemannian manifold is called *geodesically complete*, if any geodesic can be extended to infinite affine parameter. If the metric is positive definite, geodesic completeness is equivalent to metric completeness. Pseudo-Riemannian symmetric spaces are in particular homogeneous, and homogeneous spaces are geodesically complete. Locally, symmetric spaces are characterized by their Riemann tensor being parallel. A manifold of *co-homogeneity* k is a manifold where the minimal co-dimension of orbits of the isometry group is k.

It was proved in [4] that for metrics of positive signature the supergravity r-map and c-map preserve geodesic completeness. This is useful for obtaining new results in Riemannian geometry, because it allows to generate complete PSK manifolds from complete PSR manifolds, and complete QK manifolds from complete PSK manifolds.

The r-map and c-map do not only preserve completeness, but also preserve isometries and in fact create new ones. The obvious induced isometries are those descending from higher-dimensional gauge symmetries whenever a vector field is reduced dimensionally. But there are additional 'hidden' symmetries as well. Without aim for completeness, some relevant references are [114, 137, 39, 138]. There is also a relation between symmetric PSR manifolds and Jordan algebras, as was already observed in [18]. This has been studied extensively in the literature, but lies outside the topic of this review. We refer the interested reader to [139, 140, 141] and references therein.

All homogeneous (and thus in particular all symmetric) PSR manifolds have been classified in [142]. A simple criterion for the completeness of PSR manifolds was proved in [143]. Complete PSR manifolds of dimension one and two have been classified in [4] and [144], respectively. Already in dimension two there are continuous families of non-isomorphic PSR spaces. Complete PSR manifolds based on reducible cubic polynomials have been classified in [145] and belong to four infinite series, two of which consist of homogenous spaces while the other two consist of spaces of co-homogeneity one.

Homogeneous (pseudo-)PSK manifolds of the form G/K, where G is a semi-simple Lie group and K a compact subgroup are automatically symmetric spaces [146]. Examples of PSK manifolds with co-homogeneity one have been constructed by applying the r-map to non-homogeneous PSR manifolds [145]. A general criterion of the geodesic completeness of PSK manifolds has been proved in [147].

The spatial c-map is a powerful tool for the construction and classification of quaternionic Kähler manifolds, which are the most complicated non-exceptional types of Riemannian manifolds with special holonomy. The hypermultiplet manifolds occurring in supergravity always have negative scalar curvature [24]. Alekseevskian spaces, that is homogeneous QK spaces of negative scalar curvature which admit a completely solvable[68] and simply transitive group of isometries have been classified in [148]. The classification of homogeneous QK manifolds generated by the supergravity c-map [142] contains a class of spaces not contained in the original list of [148]. It was then shown in [149] that these were the only cases missing, thus completing the classification. With the exception of the quaternionic hyperbolic spaces $\mathbb{H}H^{n+1}$, all Alekseevsky spaces can be obtained from the supergravity c-map.

Mathematically the supergravity c-map is extremely useful because it allows the explicit construction of non-homogeneous quaternionic-Kähler spaces. Moreover, since it preserves completeness, one can use complete, non-homogeneous PSK manifolds to obtain complete, non-homogeneous QK manifolds. Two infinite families of complete QK manifold of co-homogeneity 1 have been constructed in [145].

While the supergravity c-map preserves completeness, this is no longer true

[68] A solvable Lie group action is called completely solvable if the generators in the adjoint representation have real eigenvalues.

for the deformed supergravity c-map, that is if one includes a non-trivial constant $c \neq 0$ in the choice of the moment map for the vector field Z on the partner manifold under the HK/QK correspondence. However, one can show that every PSK manifold which exhibits so-called regular boundary behaviour is complete, and that its image under the deformed supergravity c-map is a complete QK manifold for $c \geq 0$ [147]. The same is true for complete PSK manifolds with a cubic prepotential, irrespective of their boundary behaviour [147]. This allows to construct a huge class of complete non-homogeneous QK manifolds. The results of [147] have a curious implication for physics, where in type-II Calabi-Yau compactifications the parameter c corresponds to the one-loop correction to the hypermultiplet metric and is proportional to the Euler number of the Calabi-Yau three-fold. Given that mirror symmetry is a symmetry of string theory and maps the Euler number to its negative, is it surprising that whether the one-loop correction preserves completeness depends on the sign of the Euler number. Understanding this observation will likely involve to also consider the effect of further (instanton) corrections to the hypermultiplet metric.

The supergravity c-map also allows to construct homogeneous and non-homogeneous pseudo-QK and para-QK spaces. Due to the lack of a completeness result comparable to [4] much less is known. The classification of symmetric pseudo-QK and para-QK spaces can be obtained from the classification of pseudo-Riemannian symmetric spaces by analysing their isotropy representations [150]. The Hesse potentials of symmetric PSK take the form $H = \sqrt{Q}$, where Q is a homogeneous polynomial of degree four. These polynomials have been determined in [151]. They have an immediate geometric interpretation for the associated QK manifold, because they determine the so-called quartic Weyl tensor, which is the traceless part of the curvature tensor of a QK manifold. Explicit descriptions for the homogeneous PSK manifolds in the image of the supergravity c-map, including their prepotentials, Kähler potentials and realizations as bounded open domains can be found in [152].

8.4.5. The c-map in string theory

In this review we have focussed on the c-map as a construction in supergravity. Originally the c-map was formulated in the context of string theory, more precisely type-II compactifications on Calabi-Yau three-folds [111]. By T-duality type-IIA string theory on $X \times S^1_R$, where X is Calabi-Yau three-fold and S^1_R is a circle of radius R (in string units), is equivalent to type-IIB string theory on $X \times S^1_{R^{-1}}$. By taking the limits $R \to 0$ and $R \to \infty$ one obtains a relation between type-IIA and type-IIB string theory compactified on the same Calabi-Yau manifold, thus somewhat complementary to mirror symmetry. This form of T-duality is often referred to as the c-map, though using the terminology of this review it actually combines the supergravity c-map and its inverse, as follows. Given a type-IIA compactification on X, we have an $\mathcal{N} = 2$ supergravity theory with $n_V^{(A)}$ vector multiplets and $n_H^{(A)}$ hypermultiplets and a scalar manifold $\bar{M}_{(A)} \times \bar{N}_{(A)}$ which is the product of a PSK and a QK manifold. Upon reduction to three dimensions, this becomes supergravity coupled to $n_V^{(A)} + 1 + n_H^{(A)}$ hypermultiplets, with scalar manifold $\bar{N}'_{(A)} \times \bar{N}_{(A)}$, where $\bar{N}'_{(A)}$ is the QK manifold obtained by applying the c-map to $\bar{M}_{(A)}$.

Applying T-duality and lifting back to four dimensions results in an effective IIB-theory with $n_V^{(B)} = n_H^{(A)} - 1$ vector multiplets and $n_H^{(B)} = n_V^{(A)} + 1$ hypermultiplets. The scalar manifold is $\bar{M}_{(B)} \times \bar{N}_{(B)}$, where $\bar{M}_{(B)}$ is the image of $\bar{N}_{(B)} = \bar{N}'_{(A)}$ under the inverse of the supergravity c-map. Note the shifts ± 1 in the number of multiplets which accounts for the degrees of freedom residing in the Poincaré supergravity multiplet.

This construction allows the obtain the tree-level hypermultiplet metrics for the type-IIA/B theory from the vector multiplet metrics of type-IIB/A. However type-II hypermultiplet metrics are subject to perturbative and non-perturbative corrections. The perturbative corrections arise at the one-loop level and have been discussed above. Non-perturbative corrections have been studied extensively by combining string dualties with the twistor approach, see below. For Calabi-Yau three-folds X which are K3-fibrations, the type-II compactification on X is believed to be dual to a heterotic compactification on $K3 \times T^2$ with a suitable choice of an $E_8 \times E_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$ vector bundle $V \to K3$. While heterotic hypermultiplet metrics are believed to be exact at string tree level, they are hard to compute because they are related to the moduli spaces of vector bundles (instantons) on a K3 surfaces.

8.4.6. The twistor approach and instanton corrections to hypermultiplet metrics

Every quaternonic Kähler space $\bar{N}_{4n}$ admits an associated twistor space $\mathcal{Z}_{4n+2}$, which, roughly speaking, is the $S^2 \cong P^1_{\mathbb{C}}$ bundle obtained by attaching to each point of $\bar{N}_{4n+2}$ the sphere $\{aJ_1 + bJ_2 + cJ_3 | a^2 + b^2 + c^2 = 1\}$ of complex structures generated by the endomorphisms J_1, J_2, J_3 which locally spann the quaternionic structure. The twistor space can be embedded into the HK cone (or Swann bundle) N_{4n+4} and thus 'sits half-ways' between $\bar{N}_{4n}$ and N_{4n+4}.

Twistor spaces have been used extensively to study the supergravity c-map and to obtain perturbative and non-perturbative corrections to hypermultiplet metrics. One advantage of this approach is that it allows to describe quaternionic Kähler spaces in terms of holomorphic data on the twistor space. The twistor approach is closely related to the projective superspace formulation of supersymmetry. This is complementary to the approach underlying this review, which focusses on Hessian structures and the superconformal formalism. We refer the interested reader to the literature, in particular to [139, 153] and references therein.

9. Static BPS black holes and entropy functions in five dimensions

The equations of motion of $\mathcal{N} = 2$ supergravity coupled to abelian vector multiplets in four and five space-time dimensions admit static, single-centre, extremal black hole solutions. These are black hole solutions whose near-horizon geometry is $AdS_2 \times S^p$, with $p = 2$ ($p = 3$) in four (five) space-time dimensions. These solutions are supported by the Maxwell charges as well as by the scalar fields of the theory. Asymptotically, these scalar fields take arbitrary values. When approaching the event horizon of the black hole (which at the two-derivative level is a Killing horizon [154]), the scalar fields flow to specific values that are entirely determined by the charges of the black hole [155, 156, 157]. This is the so-called attractor mechanism for extremal black holes, which can be explained by rewriting the equations of motion as gradient flow equations: regardless of their asymptotic values, the scalar fields are driven to specific values at the horizon. When these extremal black hole solutions are also supersymmetric, they are called BPS black holes.

In this section we study static BPS black holes in five dimensions. They are electrically charged. The gradient flow equations for these BPS black holes were originally obtained by studying the supersymmetry preserved by these solutions [158, 159]. Here, we derive them by performing a suitable rewriting of the underlying action [160, 161] .

The near-horizon geometry of these five-dimensional black hole solutions is described by an $AdS_2 \times S^3$ space-time. In this geometry, the attractor values for the scalar fields at the horizon can be obtained from a variational principle based on the so-called entropy function for extremal black holes [162]. Evaluating this entropy function at the extremum then yields the entropy of the black hole. For BPS black holes, the horizon values of the scalar fields can also be derived from a variational principle based on a different entropy function, called BPS entropy function. The BPS entropy function is constructed from the Hesse potential $\mathcal{V}$ of the CASR manifold discussed in section 2.6.

The above considerations based on the entropy function can be extended to the case where one considers BPS black hole solutions in $\mathcal{N} = 2$ supergravity theories in the presence of higher-derivative terms proportional to the square of the Weyl tensor [163, 164, 165, 23]. We discuss the effect of Weyl square interactions on the entropy of static BPS black holes.

9.1. Single-centre BPS black hole solutions through gradient flow equations

9.1.1. Action and line element ansatz

The action for $\mathcal{N} = 2$ Poincaré supergravity at the two-derivative level is given in (145). Here we set $\kappa^2 = 1$, i.e. $G^{-1} = 8\pi$, and we will denote the scalar fields h^I $(I = 0, \ldots, n)$ by X^A $(A = 0, \ldots, n)$, so that now

$$C_{ABC} X^A X^B X^C = 1 \, . \tag{582}$$

Correspondingly, we will denote the quantities h_I and $\overset{\circ}{a}_{IJ}$ introduced in (146) by X_A and G_{AB}, respectively,[69]

$$\begin{aligned} X_A &= C_{ABC} X^B X^C = G_{AB} \, X^B \, , \\ G_{AB} &= -2 \, C_{ABC} X^C + 3 X_A X_B \, . \end{aligned} \tag{583}$$

We note the following useful relations, which will be used in the following,

$$\begin{aligned} X_A \, X^A &= 1 \, , \\ X_A \, \partial_\mu X^A &= X^A \partial_\mu X_A = 0 \, , \\ G^{AB} \, \partial_\mu X_B &= -\partial_\mu X^A \, , \\ G_{AB} \partial_\mu X^A \partial^\mu X^B &= G^{AB} \partial_\mu X_A \partial^\mu X_B \, . \end{aligned} \tag{584}$$

Next, we display the part of the $\mathcal{N} = 2$ Poincaré supergravity action that is relevant for the purpose of obtaining gradient flow equations for static extremal

[69] We note that the normalizations used in this section differ slightly from those used in [160].

black hole solutions,

$$S = \int d^5x \sqrt{-g} \left(\frac{1}{2} R - \frac{3}{4} G_{AB}\, \partial_\mu X^A\, \partial^\mu X^B - \frac{3}{8}\, G_{AB} F^A_{\mu\nu}\, F^{B\mu\nu} \right) . \qquad (585)$$

We are interested in static solutions to the equations of motion, and hence we take the five-dimensional line element, the one-form gauge fields A^A and the scalar fields X^A to have the following form in adapted coordinates,

$$\begin{aligned} ds_5^2 &= g_{\mu\nu}\, dx^\mu\, dx^\nu = -f^2(r)\, dt^2 + f^{-1}(r)\, ds^2_{\rm GH} \ , \\ A^A &= \chi^A(r)\, dt \ , \\ X^A &= X^A(r) \ , \end{aligned} \qquad (586)$$

where $ds^2_{\rm GH}$ describes four-dimensional Euclidean flat space, which we write in the form

$$ds^2_{\rm GH} = r^{-1} \left(dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) \right) + r\, (d\psi + \cos\theta\, d\varphi)^2 \ . \qquad (587)$$

Here $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$, $\psi \in [0, 4\pi)$. Indeed, by changing the radial coordinate to

$$\rho^2 = 4\, r \ , \qquad (588)$$

one obtains

$$ds^2_{\rm GH} = d\rho^2 + \frac{\rho^2}{4} \left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 \right) = dx^m dx^m \quad , \quad m = 1, \dots, 4 \ , \qquad (589)$$

where

$$\begin{aligned} \sigma_1 &= -\sin\psi\, d\theta + \cos\psi\, \sin\theta\, d\varphi \ , \\ \sigma_2 &= \cos\psi\, d\theta + \sin\psi\, \sin\theta\, d\varphi \ , \\ \sigma_3 &= d\psi + \cos\theta\, d\varphi \ . \end{aligned} \qquad (590)$$

We take the electric field $\partial_\rho \chi^A$ to be sourced by electric charges which we denote by q_A, up to a normalization constant, so that $\partial_\rho \chi^A \sim f^2 G^{AB} q_B / \rho^3$, and hence

$$\partial_r \chi^A = -\frac{2}{3} \frac{f^2}{r^2}\, G^{AB} q_B \ . \qquad (591)$$

9.1.2. Gradient flow equations

Here we derive first-order flow equations for solutions of the form (586). These solutions describe single-centre static extremal black holes in a five-dimensional asymptotically flat space-time. We follow [160].

Inserting the ansatz (586) into the action (585) yields,

$$
\begin{aligned}
S \;=\; & \frac{1}{4}\int dt\, dr\, d\theta\, d\varphi\, d\psi\, \sin\theta \qquad (592)\\
& \Big[-3r^2 f^{-2}(f')^2 - 3r^2 G_{AB}(X^A)'(X^B)' + 3r^2 f^{-2} G_{AB}\chi'^A\,\chi'^B \\
& \quad +2\partial_r\left(r^2 f^{-1} f'\right)\Big] \,,
\end{aligned}
$$

where $\,' = \partial_r$. Introducing the radial coordinate

$$
\tau = \frac{1}{r}\;, \qquad (593)
$$

and using (584) as well as (591), this can can be rewritten into

$$
\begin{aligned}
S \;=\; & \frac{1}{4}\int dt\, dr\, d\theta\, d\varphi\, d\psi\, \sin\theta \\
& \Big[-3\,\tau^2\, G^{AB}\left(\partial_\tau X_A + f\,\partial_\tau f^{-1}\, X_A - \frac{2}{3}\,s\,f\,q_A\right) \\
& \qquad\qquad \left(\partial_\tau X_B + f\,\partial_\tau f^{-1}\, X_B - \frac{2}{3}\,s\,f\,q_B\right) \\
& +3\tau^2 f^{-2} G_{AB}\left(\partial_\tau\chi^A - \frac{2}{3} f^2 G^{AC} q_C\right)\left(\partial_\tau\chi^B - \frac{2}{3} f^2 G^{BD} q_D\right) \\
& +2\partial_r\left(r^2 f^{-1} f' - 2q_A\,\chi^A - 2\,s\,f\,q_A X^A\right)\Big] \,, \qquad (594)
\end{aligned}
$$

where $s = \pm 1$.

The last line in (594) denotes a total derivative. Thus, up to a total derivative term, S is expressed in terms of squares of first-order terms which, when requiring stationarity of S with respect to variations of the fields, results in

$$
\begin{aligned}
\partial_\tau X_A + f\,\partial_\tau f^{-1}\, X_A &= \frac{2}{3}\,s\,f\,q_A \,,\\
\partial_\tau \chi^A &= \frac{2}{3} f^2 G^{AB} q_B \,. \qquad (595)
\end{aligned}
$$

Contracting the first equation with X^A yields the flow equation for the warp factor f,

$$
\partial_\tau f^{-1} = \frac{2}{3}\,s\,q_A X^A \,. \qquad (596)
$$

The gradient flow equations (595) then take the equivalent form

$$
\begin{aligned}
\partial_\tau\left(f^{-1}\, X_A\right) &= \frac{2}{3}\,s\;q_A \,,\\
\partial_\tau f^{-1} &= \frac{2}{3}\,s\,q_A X^A \,,\\
\partial_\tau \chi^A &= \frac{2}{3} f^2 G^{AB} q_B \,. \qquad (597)
\end{aligned}
$$

It can be checked that the five-dimensional Einstein-, Maxwell- and scalar field equations of motion derived from (585) are satisfied by the solutions to the flow equations (597).

The first flow equation in (597) is solved by

$$f^{-1}\, X_A = \frac{2}{3}\, s\, H_A \quad , \quad H_A = h_A + q_A\, \tau \; , \tag{598}$$

where h_A denote integration constants. Contracting this with X_A results in

$$f^{-1} = \frac{2}{3}\, s\, H_A\, X^A \; . \tag{599}$$

One then verifies that this solves the flow equation for f^{-1} by virtue of the relation $X_A \partial_\tau X^A = 0$, c.f. (584). Thus, the flow equations (597) are solved by

$$\begin{aligned} f^{-1}\, X_A &= \frac{2}{3}\, s\, H_A \; , \\ f^{-1} &= \frac{2}{3}\, s\, H_A\, X^A \; , \\ \chi^A &= -s\, f\, X^A \; . \end{aligned} \tag{600}$$

In the following, we take $s = 1$ and we assume that the C_{ABC} in (582) are all positive, so that $X^A > 0$. Demanding $f^{-1} > 0$ along the flow, we infer that $H_A > 0$ along the flow, and hence also $h_A, q_A > 0$. The solution describes a static, electrically charged extremal black hole solution in five dimension, which is BPS [159]. The latter can be deduced as follows. The Lagrangian (594) contains the term $q_A G^{AB} q_B$, also called black hole potential $V_{\rm BH}$. It can be expressed in terms of the five-dimensional central charge,

$$Z_5 = q_A\, X^A \; , \tag{601}$$

as

$$V_{\rm BH} = q_A G^{AB} q_B = Z_5^2 + G^{AB}\, (D_A Z_5)\, (D_B Z_5) \; , \tag{602}$$

where

$$D_A Z_5 = q_A - X_A Z_5 \; . \tag{603}$$

Likewise, the gradient flow equations for f^{-1} and X^A can be expressed in terms of Z_5 and $D_A Z_5$ [166, 161],

$$\begin{aligned} \partial_\tau X^A &= -\frac{2}{3}\, f\, G^{AB}\, D_B Z_5 \; , \\ \partial_\tau f^{-1} &= \frac{2}{3}\, Z_5 \; . \end{aligned} \tag{604}$$

The scalar fields X^A stop flowing when $D_A Z_5 = 0 \, \forall A = 1, \ldots, n$. The latter corresponds to a critical point of the black hole potential. If at this critical point $Z_{5,\mathrm{crit}} \neq 0$, then the scalar fields X^A attain the constant values $X_A = q_A / Z_{5,\mathrm{crit}}$, and the warp factor f^{-1} becomes $f^{-1} = \frac{2}{3} \, Z_{5,\mathrm{crit}} \, \tau$. The associated line element describes the geometry of $AdS_2 \times S^3$, which is the near-horizon geometry of a static extremal black hole in five dimensions. Thus, when approaching the horizon of the black hole, the scalar fields X^A flow to a critical point of the black hole potential (602) satisfying $D_A Z_5 = 0$ with $Z_5 \neq 0$. Such a a critical point is a BPS critical point [156].

The black hole potential may have other critical points that are not BPS. Suppose that the black hole potential admits a second decomposition, in terms of a real quantity $W_5 = Q_A X^A$,

$$V_{\mathrm{BH}} = q_A G^{AB} q_B = W_5^2 + G^{AB} D_A W_5 D_B W_5 \; , \tag{605}$$

with $W_5 \neq Z_5$, and that it possesses a critical point $D_A W_5 = 0$ with $W_5 \neq 0$. Then this critical point is non-BPS, and it is associated to a non-BPS static extremal black hole solution that can be obtained by solving first-order flow equations of the form (604), but now with Z_5 replaced by W_5. This is so, because the rewriting of the action (592) using (605) proceeds in exactly the same manner as the one discussed above. Thus, in certain cases, non-BPS static extremal black holes solutions may be obtained by solving first-order flow equations [160].

9.2. Entropy functions for static BPS black holes

9.2.1. Entropy function at the two-derivative level

We consider the solution (600) with $s = 1$. In the coordinates (587), the near-horizon geometry of the BPS black hole is obtained by sending $r \to 0$. Inspection of (600) shows that in this limit the X^A become constant, while $f^{-1}(r) \propto 1/r$. Setting

$$f^{-1}(r) = \frac{v_2}{4\, r} \; , \tag{606}$$

with v_2 a positive constant, and inserting this into (586) shows that the near-horizon geometry of a static BPS black hole is $AdS_2 \times S^3$,

$$ds_5^2 = v_1 (-r^2 dt^2 + \frac{dr^2}{r^2}) + \frac{v_2}{4} \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right) + \frac{v_2}{4} \left(d\psi + \cos\theta \, d\varphi \right)^2 \; , \tag{607}$$

with $v_1 = v_2/4$. In this near-horizon geometry, the gauge potentials χ^A behave as $\chi^A(r) \propto r$, and hence we set

$$\chi^A(r) = e^A\, r \ , \tag{608}$$

with constant e^A. The near-horizon solution is thus specified by (607) and (608), and supported by constant X^A. The values of the X^A at the horizon are, according to the attractor mechanism for extremal black holes, specified by the charges carried by the black hole. These values can be determined by extremizing the so-called entropy function, to which we now turn.

We consider the reduced Lagrangian $\mathcal{F}_5$ which is obtained by evaluating the Lagrangian (585) in the near-horizon BPS black hole background (607), (608) and integrating over the horizon [167],

$$\begin{aligned} \mathcal{F}_5 &= \frac{1}{8\pi} \int d\psi\, d\theta\, d\phi \sqrt{-g}\, L_5 \ , \\ &= \pi\, \frac{v_1\, (v_2^3)^{1/2}}{4} \left[-\frac{1}{v_1} + \frac{3}{v_2} + \frac{3}{4} \frac{G_{AB}\, e^A\, e^B}{v_1^2} \right] \ . \end{aligned} \tag{609}$$

The entropy function is then given by the Legendre transform [162]

$$\mathcal{E}_5 = 2\pi \left(2\pi\, q_A\, e^A - \mathcal{F}_5 \right) \ . \tag{610}$$

The entropy function is a function of the constant parameters e^A, v_1, v_2, X^A. Extremizing the entropy function with respect to these parameters and evaluating the entropy function at the extremum, yields the entropy of the static BPS black hole expressed in terms of the charges q_A.

Varying the entropy function $\mathcal{E}_5$ with respect to the electric fields e^A and setting $\partial_e \mathcal{E}_5 = 0$ yields

$$\frac{3\pi}{8} \frac{(v_2^3)^{1/2}}{v_1}\, G_{AB}\, e^B = 2\pi\, q_A \ . \tag{611}$$

Varying $\mathcal{E}_5$ with respect to v_1, v_2 and setting the variations to zero yields

$$v_1 = \frac{v_2}{4} = G_{AB}\, e^A\, e^B \ . \tag{612}$$

Inserting (612) into $\mathcal{E}_5$ yields

$$\mathcal{E}_5 = \frac{\pi^2}{2} \left(v_2^3 \right)^{1/2} \ , \tag{613}$$

which equals the macroscopic entropy $\mathcal{S}_{\rm macro} = A_5/4$ of the static black hole, where A_5 denotes the horizon area. Using (611), we infer

$$\frac{9}{4}\, v_2\, G_{AB}\, e^A\, e^B = \frac{q_A\, G^{AB}\, q_B}{4\pi^2} \ , \tag{614}$$

and hence

$$v_2^2 = \frac{4}{9\pi^2}\, q_A\, G^{AB}\, q_B \ . \tag{615}$$

The horizon values of the X^A are determined in terms of the charges q_A by varying $\mathcal{E}_5$ with respect to the X^A and setting $\delta_X \mathcal{E}_5 = 0$. In doing so, one has to take into account the constraint (582), which implies

$$C_{ABC} X^A X^B \delta X^C = 0 \tag{616}$$

for arbitrary variations δX^C. Using the relation for G_{AB} given in (583), one obtains for $\delta_X \mathcal{E}_5 = 0$,

$$e^A e^B \delta X^C \left(-C_{ABC} + 3\, C_{ACE} X^E X_B + 3\, C_{BCE} X^E X_A\right) = 0 \ . \tag{617}$$

Setting $e^A = \gamma\, X^A$, as required for a BPS solution, solves (617) by virtue of (616). The scale factor γ is determined by inserting this expression into (615), which results in

$$\gamma = \frac{1}{2}\, \sqrt{v_2} \ . \tag{618}$$

Then, using (611), we infer

$$v_2\, X_A = \frac{8}{3}\, q_A \ . \tag{619}$$

This is the so-called attractor equation, whose solution determines the values of the scalar fields X^A at the horizon in terms of the charges carried by the BPS black hole. Contracting (619) with X^A yields $v_2 = \frac{8}{3} q_A X^A$, and using (582) one infers $v_2 \sim q$, and hence $\mathcal{S}_{\rm macro} \sim q^{3/2}$ [156].

9.2.2. BPS entropy function at the two-derivative level, the Hesse potential and its dual

The attractor equation (619) can also be derived from a variational principle based on a different entropy function, which we call the BPS entropy function. The BPS entropy function is constructed from the Hesse potential $\mathcal{V}$ of the CASR manifold discussed in section 2.6,

$$H(\mathcal{Y}) = \frac{1}{2}\, \mathcal{V}(\mathcal{Y}) = \frac{1}{2}\, C_{ABC}\, \mathcal{Y}^A\, \mathcal{Y}^B\, \mathcal{Y}^C \ , \tag{620}$$

where we have introduced

$$\mathcal{Y}^A = v_2^{1/2}\, X^A \ . \tag{621}$$

The BPS entropy function reads

$$\Sigma(\mathcal{Y}, q) = 4\, q_A\, \mathcal{Y}^A - H(\mathcal{Y}) \ . \tag{622}$$

Extremizing with respect to $\mathcal{Y}^A$ yields

$$C_{ABC}\, \mathcal{Y}^B\, \mathcal{Y}^C = \frac{8}{3}\, q_A \ , \tag{623}$$

which expresses the $\mathcal{Y}^A$ in terms of the charges q_A. The value of Σ at this extremum is

$$\Sigma(q) = C_{ABC}\, \mathcal{Y}^A\, \mathcal{Y}^B\, \mathcal{Y}^C = v_2^{3/2} \ , \tag{624}$$

and hence

$$\mathcal{S}_{\rm macro} = \frac{\pi^2}{2}\, \Sigma(q) \ . \tag{625}$$

Thus, upon extremization, the electric charges q_A become proportional to the dual special real coordinates, while the BPS entropy is proportional to the dual Hesse potential, evaluated on the background, c.f. (7).

9.2.3. R^2-corrected BPS entropy function, the Hesse potential and its dual

Now we allow for the presence of a specific class of R^2 terms in the $\mathcal{N} = 2$ supergravity Lagrangian, namely those arising from the coupling of vector multiplets to the square of the Weyl multiplet. The effect of these higher derivative terms on the near-horizon region of static BPS black hole solutions and on the associated BPS entropy has been thoroughly discussed in [163, 164, 165, 23]. We follow [23].

The coupling of vector multiplets to the square of the Weyl multiplet can be conveniently described using the superconformal approach to supergravity. This is reviewed in B.5. One salient feature is that the Lagrangian describing the couplings of vector multiplets to the square of the Weyl multiplet contains a term proportional to the square of the Weyl tensor, with coupling function $c_A\, X^A$, where c_A are constant coefficients [29].

We focus on solutions to the associated equations of motions that have full supersymmetry. These field configurations satisfy [23],

$$\partial_\mu X^A = 0 \ \ , \ \ F^A_{ab} = 4\, X^A\, T_{ab} \ \ , \ \ Y^{ij} = 0 \ \ , \ \ D = 0 \ \ , \ \ T_{ab} T^{ab} = \text{constant} \ . \tag{626}$$

The associated line element describes a circle fibred over an $AdS_2 \times S^2$ base,

$$\begin{aligned} ds^2 &= \frac{1}{16v^2}\left(-r^2dt^2 + \frac{dr^2}{r^2} + d\theta^2 + \sin^2\theta d\varphi^2\right) + e^{2g}\left(d\psi + B\right)^2 \ , \\ B &= -\frac{1}{4v^2}e^{-g}\left(T_{23}\, r\, dt - T_{01}\, \cos\theta\, d\varphi\right) \ , \\ v &= \sqrt{(T_{01})^2 + (T_{23})^2} \ . \end{aligned} \tag{627}$$

Here, T_{01} and T_{23} denote the non-vanishing components of T_{ab}, and they are associated with (t, r, θ, φ). v and e^g are constants. In the following, we focus on static configurations, and hence set $T_{23} = 0$. Introducing the notation ($T_{01} \neq 0$)

$$p^0 = \frac{e^{-g}}{4v^2}\, T_{01} \tag{628}$$

and using $v^2 = (T_{01})^2$, the line element (627) may be brought into the form

$$ds^2 = \frac{1}{16v^2}\left(-r^2dt^2 + \frac{dr^2}{r^2} + d\theta^2 + d\varphi^2 + \frac{1}{(p^0)^2}d\psi^2 + \frac{2}{p^0}\cos\theta d\varphi d\psi\right) \ . \tag{629}$$

Then, demanding $p^0 = 1$, in which case $e^{-g} = 4T_{01} = 4v > 0$, and fixing the periodicity of ψ to $\psi \in [0, 4\pi)$, the line element becomes the line element for $AdS_2 \times S^3$ given in (607), with $v_2 = 1/(4v^2)$. This is the near-horizon geometry of a static BPS black hole supported by electric charges q_A and constant scalar fields X^A. The latter are expressed in terms of the charges through the attractor equation

$$q_A = \frac{3e^g}{8T_{01}}\left(C_{ABC}X^BX^C - c_A(T_{01})^2\right) = \frac{3}{32v^2}\left(C_{ABC}X^BX^C - c_A\, v^2\right) \ , \tag{630}$$

where we normalized the charges as in (619).

In this background, the equation of motion for the auxiliary D-field takes the form

$$\chi = -2C_{ABC}X^AX^BX^C + 4c_AX^A(T_{01})^2 \ . \tag{631}$$

Then, imposing the normalization of the Einstein-Hilbert term (c.f. (156)),

$$C_{ABC}X^AX^BX^C - \frac{3}{2}\chi = 4 \ , \tag{632}$$

yields the constraint

$$C_{ABC}X^AX^BX^C = 1 + \frac{3}{2}c_AX^A(T_{01})^2 \ . \tag{633}$$

Introducing $\mathcal{Y}^A$ as in (621),

$$\mathcal{Y}^A = \frac{1}{2v}X^A \ , \tag{634}$$

we obtain

$$C_{ABC}\mathcal{Y}^A\mathcal{Y}^B\mathcal{Y}^C = \frac{1}{8v^3} + \frac{3}{16}c_A\mathcal{Y}^A \ . \tag{635}$$

The attractor equation (630) becomes

$$\hat{q}_A \equiv q_A + \frac{3}{32}c_A = \frac{3}{8}C_{ABC}\mathcal{Y}^B\mathcal{Y}^C \ . \tag{636}$$

The entropy of these static BPS black holes is computed using Wald's entropy formula (B.134). Using the R^2-corrected Lagrangian (156) in the background (626), (629), we obtain

$$\frac{\partial L}{\partial R_{\mu\nu\rho\sigma}}\,\varepsilon_{\mu\nu}\varepsilon_{\rho\sigma} = -C_{ABC}X^AX^BX^C \ . \tag{637}$$

Note that the contributions proportional to c_A have cancelled out. Then, using the line element (629)

$$\int_{S^3}\sqrt{h}\,d\Omega = \frac{\pi^2}{4v^3} \ , \tag{638}$$

we obtain for the macroscopic entropy of a static BPS black hole,

$$\mathcal{S}_{\text{macro}} = \frac{\pi^2}{2}C_{ABC}\mathcal{Y}^A\mathcal{Y}^B\mathcal{Y}^C \ , \tag{639}$$

with the $\mathcal{Y}^A$ expressed in terms of the charges through (636). The constant v in the line element (629) is determined through (635) in terms of the charges. This fully determines the near-horizon geometry of the static BPS black hole.

The attractor equation (636) can be obtained by extremizing the following BPS entropy function,

$$\Sigma(\mathcal{Y}, q) = 4\,\hat{q}_A\,\mathcal{Y}^A - H(\mathcal{Y}) \ , \tag{640}$$

with $H(\mathcal{Y})$ given as in (620). The BPS entropy function is thus given in terms of the dual Hesse potential, c.f. (7). The value of Σ at the extremum yields the entropy (639),

$$\mathcal{S}_{\text{macro}} = \frac{\pi^2}{2}\,\Sigma(q) \ . \tag{641}$$

10. Static BPS black holes and entropy functions in four dimensions

In four dimensions, the equations of motion of $\mathcal{N}=2$ supergravity coupled to abelian vector multiplets (without or with higher-derivative terms proportional to the square of the Weyl tensor) admit single-centre, dyonic, extremal black hole solutions. These are spherically symmetric solutions. When they are supersymmetric, they are called BPS solutions.

In so-called isotropic coordinates (t, r, θ, ϕ), the line element of a spherically symmetric space-time takes the form

$$ds^2 = -e^{2g(r)}\, dt^2 + e^{2f(r)} \left(dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right) \ . \tag{642}$$

At the two-derivative level, BPS black hole solutions satisfy $f = -g$ [168, 169]. In the following, we will restrict the discussion to the class of solutions with $f = -g$, and we will write their line element as

$$ds^2 = -e^{2U(r)}\, dt^2 + e^{-2U(r)} \left(dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right) \ . \tag{643}$$

Extremal black hole solutions carry electric and magnetic charges (q_I, p^I) associated with the abelian gauge fields A^I_μ of the theory,

$$\int_{S^2_\infty} d\theta d\phi\, F_{\theta\phi}{}^I = p^I \quad , \quad \int_{S^2_\infty} d\theta d\phi\, G_{\theta\phi\, I} = q_I \ , \tag{644}$$

where we integrate over an asymptotic two-sphere S^2_∞. Here, $G_{\theta\phi\, I}$ denotes the dual field strength defined in (194).

These black holes are furthermore supported by complex scalar fields X^I that reside in the vector multiplets. These scalar fields will, generically, have a non-trivial profile, i.e. $X^I = X^I(r)$. Asymptotically, the scalar fields take arbitrary values. When approaching the event horizon, the scalar fields flow to fixed values that are entirely determined by the charges of the black hole. This is the attractor mechanism for extremal black holes [155, 156, 157, 170]: the values of the scalar fields at the event horizon are attracted to specific values given in terms of the charges of the black hole, irrespective of their asymptotic values at spatial infinity. For BPS black holes at the two-derivative level, the flow to the event horizon is described by gradient flow equations for the scalar fields and for the metric factor e^U. These first-order flow equations can be obtained from a reduced action in one dimension, see subsection 10.1.

In the near-horizon region $r \approx 0$, the metric (643) takes the form

$$ds^2 = v_1 \left(-r^2\, dt^2 + \frac{dr^2}{r^2}\right) + v_2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) , \tag{645}$$

with $v_1 = v_2$, and describes the line element of an $AdS_2 \times S^2$ space-time. Here, v_1 denotes a constant whose value is entirely specified by the charges carried by the black hole, through the attractor mechanism. The attractor values for the scalar fields at the horizon can be obtained from a variational principle based on the so-called entropy function. Evaluating this entropy function at the extremum then yields the entropy of the black hole. This entropy function can be derived from the reduced action evaluated in the near-horizon geometry (645), as we will discuss in subsection 10.2.

The above considerations based on the entropy function can be extended to the case where one considers extremal black hole solutions in $\mathcal{N} = 2$ supergravity theories in the presence of higher-derivative terms proportional to the square of the Weyl tensor. Then, for BPS black holes, one still finds $v_1 = v_2$ [36], while for non-BPS black holes, one generically has $v_1 \neq v_2$ [171, 172]. For further reading on these topics, we refer to [173, 174, 175, 176, 177].

There are many other interesting aspects about black hole attractors which we do not describe in this review. These include: relations between topics in number theory and BPS black holes [178, 179, 180]; multicenter bound states of BPS black holes [181]; the OSV conjecture [182, 74]; the $4D/5D$ connection between black objects [183, 184, 185]; attractors and cosmic censorship [186]; rotating attractors [187]; the quantum entropy function [188, 189]; attractor flows and CFT [190]; a Riemann-Hilbert approach to rotating attractors [191]; hot attractors [192].

10.1. Single-centre BPS black hole solutions through gradient flow equations

In the following, we will derive gradient flow equations for BPS black hole solutions in $\mathcal{N} = 2$ supergravity theories at the two-derivative level in four dimensions [193, 194]. These are first-order flow equations that will be obtained from a reduced action based on the Lagrangian (439). The latter is evaluated in the background (643) and subsequently rewritten in terms of squares of first-order terms. We use the relation (B.126) to perform the rewriting in terms of (rescaled) scalar fields X^I, rather than in terms of scalar fields z^a, as follows [195].

We introduce rescaled scalar fields Y^I defined by

$$Y^I = e^{-U}\,\tilde{X}^I = e^{-U}\,\bar{\varphi}\,X^I \,, \tag{646}$$

where U denotes the metric factor in (643). Here, $\bar{\varphi}$ denotes a phase, with a $U(1)$-weight that is opposite to the one of X^I. Thus, $\tilde{X}^I = \bar{\varphi}\,X^I$ denotes a $U(1)$ invariant variable.

Using (B.126), we evaluate the Lagrangian (439) in the background (643), taking X^I and Y^I to be functions of r, only. We evaluate the covariant derivative (B.122),

$$\bar\varphi\,\mathcal{D}_r X^I = \partial_r \tilde X^I + i\,\hat A_r\,\tilde X^I \;, \tag{647}$$

where

$$\hat A_r = A_r + i\varphi\,\partial_r\bar\varphi \;, \tag{648}$$

where φ is the complex conjugate of $\bar\varphi$, and where A_r is given by (437). Then,

$$N_{IJ}\,\mathcal{D}_r X^I\,\mathcal{D}_r\bar X^J = N_{IJ}\,\tilde X'^I\,\bar{\tilde X}'^J + \hat A_r^2 \;, \tag{649}$$

where $\tilde X'^I = \partial_r\tilde X^I$. Observe that in view of $N_{IJ}X^I\,\bar X^J = -1$, we have

$$e^{-2U} = -N_{IJ}\,Y^I\,\bar Y^J \;, \tag{650}$$

as well as

$$e^{-2U}\,U' = \tfrac{1}{2}\,N_{IJ}\,\left(Y'^I\,\bar Y^J + Y^I\,\bar Y'^J\right) \;, \tag{651}$$

where we used the homogeneity property $F_{IJK}X^K = 0$. Similarly, using the homogeneity property of F_I, the connection $\hat A_r$ can be expressed in terms of the Y^I as

$$\hat A_r = -\tfrac{1}{2}\,e^{2U}\left[(F_I - \bar F_I)\,\partial_r(Y^I - \bar Y^I) - (Y^I - \bar Y^I)\,\partial_r(F_I - \bar F_I)\right] \;. \tag{652}$$

Extremal black holes carry electric and magnetic charges (q_I, p^I). Electric fields $E^I(r)$ and magnetic charges p_I are introduced as (c.f. (644))

$$F_{rt}{}^I = E^I \;, \qquad F_{\theta\phi}{}^I = \frac{p^I}{4\pi}\,\sin\theta\,. \tag{653}$$

The θ-dependence of $F_{\theta\phi}{}^I$ is fixed by rotational invariance.

Rather than using a description based on (p^I, E^I), we seek a description in terms of magnetic/electric charges (p^I, q_I). To introduce electric charges q_I, we consider the dual field strengths $G_{\mu\nu I}$ defined in (194). Adopting the conventions where $x^\mu = (t, r, \theta, \phi)$ and $\varepsilon_{tr\theta\phi} = 1$, it follows that, in the background (643),

$$G_{\theta\phi\,I} = -e^{-2U(r)}\,r^2\;\sin\theta\,\frac{\partial L}{\partial F_{rt}{}^I} = -e^{-2U(r)}\,r^2\;\sin\theta\,\frac{\partial L}{\partial E^I}\,. \tag{654}$$

Writing (c.f. (644))

$$G_{\theta\phi\,I} = \frac{q_I}{4\pi}\,\sin\theta\,, \tag{655}$$

where the θ-dependence is again fixed by rotational invariance, we infer

$$q_I = -4\pi\, e^{-2U(r)}\, r^2\, \frac{\partial L}{\partial E^I} \ . \tag{656}$$

We pass from a description based on (p^I, E^I) to a description based on (p^I, q_I) by means of the Legendre transform

$$L_{1d} = \left(\int d\theta\, d\phi\, \sqrt{-g}\, L\right) + q_I\, E^I \ , \tag{657}$$

with the Lagrangian L, given in (439), evaluated in the background (643).

To keep the discussion as simple as possible, let us first consider the case of electrically charged black holes. Subsequently we extend the discussion to the case of dyonic black holes.

Using (656), we obtain

$$E^I = e^{2U}\, \frac{\left[(\text{Im}\mathcal{N})^{-1}\right]^{IJ}\, q_J}{4\pi\, r^2} \ . \tag{658}$$

The resulting one-dimensional action $4\pi\, S_{1d} \equiv \int dr\, L_{1d}$ reads,

$$\begin{aligned} -S_{1d} \quad &= \quad \int dr\, r^2 \left[U'^2 + N_{IJ}\, \tilde{X}'^I\, \bar{\tilde{X}}'^J + \hat{A}_r^2 - \tfrac{1}{2}\, e^{2U}\, \frac{q_I\, \left[(\text{Im}\mathcal{N})^{-1}\right]^{IJ}\, q_J}{(4\pi\, r^2)^2}\right] \\ &\quad - \int dr\, \frac{d}{dr}\left(r^2\, U'\right) . \end{aligned} \tag{659}$$

In the following, we will, for notational simplicity, absorb a factor 4π into q_I, i.e. $q_I/(4\pi) \to q_I$.

Next, we rewrite (659) in terms of the rescaled variables Y^I. Using (651), we obtain the intermediate result

$$\begin{aligned} -S_{1d} \quad = \quad & \int dr\, r^2 \Big\{ 2\left[U' + e^{2U}\text{Re}\Big(\frac{Y^I\, q_I}{r^2}\Big)\right]^2 \\ & \qquad + e^{2U}\, N_{IJ}\, \left(Y'^I + N^{IK}\, \frac{q_K}{r^2}\right)\, \left(\bar{Y}'^J + N^{JL}\, \frac{q_L}{r^2}\right) + \hat{A}_r^2 \\ & \qquad - \tfrac{1}{2}\, \frac{e^{2U}}{r^4}\, q_I\, \left[(\text{Im}\mathcal{N})^{-1}\right]^{IJ}\, q_J - \frac{e^{2U}}{r^4}\, q_I\, N^{IJ}\, q_J \\ & \qquad - 2e^{4U}\left[\text{Re}\left(\frac{Y^I\, q_I}{r^2}\right)\right]^2 \Big\} \\ & - \int dr\, \frac{d}{dr}\left[r^2\, U' + 2e^{2U}\, \text{Re}\left(Y^I\, q_I\right)\right] \ . \end{aligned} \tag{660}$$

Then, using the second identity in (B.128) we obtain

$$-S_{1d} = S_{\rm square} + S_{\rm TD}\ , \tag{661}$$

where

$$\begin{aligned} S_{\rm square} &= \int dr\, r^2 \Big\{ 2\Big[U' + e^{2U} {\rm Re}\left(\frac{Y^I\, q_I}{r^2}\right)\Big]^2 \\ &\qquad + e^{2U}\, N_{IJ}\, \left(Y'^I + N^{IK}\, \frac{q_K}{r^2}\right)\, \left(\bar{Y}'^J + N^{JL}\, \frac{q_L}{r^2}\right) \\ &\qquad + \hat{A}_r^2 + 2e^{4U}\, \left[{\rm Im}\left(\frac{Y^I\, q_I}{r^2}\right)\right]^2 \Big\}\ , \end{aligned} \tag{662}$$

and

$$S_{\rm TD} = -\int dr\, \frac{d}{dr}\left[r^2\, U' + 2\, e^{2U}\, {\rm Re}\left(Y^I\, q_I\right)\right]\ . \tag{663}$$

The above results can be easily extended to the case of dyonic black holes, as follows. First, we view the term $q({\rm Im}\mathcal{N})^{-1}q$ in the action (659) as part of the black hole potential (B.132),

$$V_{\rm BH} = -\tfrac{1}{2}\, q_I\, \left[({\rm Im}\mathcal{N})^{-1}\right]^{IJ}\, q_J = g^{i\bar{\jmath}}\, \mathcal{D}_i Z\, \bar{\mathcal{D}}_{\bar{\jmath}} \bar{Z} + |Z|^2\ , \tag{664}$$

where $Z(X) = -q_I\, X^I$. Turning on magnetic charges p^I amounts to extending $Z(X)$ as in (B.130),

$$Z(X) = p^I\, F_I(X) - q_I\, X^I = \left(p^I\, F_{IJ} - q_J\right) X^J = -\hat{q}_I\, X^I\ , \tag{665}$$

where[70]

$$\hat{q}_I = q_I - F_{IJ}\, p^J\ . \tag{666}$$

Then, the extension to the dyonic case proceeds by replacing q_I with $\hat{q}_I$ in (661), which results in

$$\begin{aligned} S_{\rm square} &= \int dr\, r^2 \Big\{ 2\Big[U' + e^{2U} {\rm Re}\left(\frac{Y^I\, \hat{q}_I}{r^2}\right)\Big]^2 \\ &\qquad + e^{2U}\, N_{IJ}\, \left(Y'^I + N^{IK}\, \frac{\bar{\hat{q}}_K}{r^2}\right)\, \left(\bar{Y}'^J + N^{JL}\, \frac{\hat{q}_L}{r^2}\right) \\ &\qquad + \hat{A}_r^2 + 2e^{4U}\, \left[{\rm Im}\left(\frac{Y^I\, \hat{q}_I}{r^2}\right)\right]^2 \Big\}\ , \end{aligned} \tag{667}$$

[70]Here we subject p^I to the same rescaling as the q_I, i.e. $p^I/(4\pi) \to p^I$.

and

$$S_{\rm TD} = - \int dr\, \frac{d}{dr} \left[r^2\, U' + 2\, e^{2U}\, {\rm Re}\left(Y^I\, \hat{q}_I \right) \right] \ . \tag{668}$$

Now we vary $S_{\rm square}$ with respect to U and to Y^I, respectively. The vanishing of these variations can be achieved by setting the variation of the individual squares in $S_{\rm square}$ to zero,

$$\begin{aligned} U' &= -e^{2U} {\rm Re} \left(\frac{Y^I\, \hat{q}_I}{r^2} \right) \ , \\ Y'^I &= -N^{IK}\, \frac{\bar{\hat{q}}_K}{r^2} \ , \\ {\rm Im}\left(Y^I\, \hat{q}_I \right) &= 0 \ , \\ \hat{A}_r &= 0 \ . \end{aligned} \tag{669}$$

This yields first-order flow equations for Y^I and for U. Note that these gradient equations are consistent with one another: the latter is a consequence of the former by virtue of (651).

It is convenient to introduce a rescaled version of $Z(X)$, namely

$$Z(Y) = p^I\, F_I(Y) - q_I\, Y^I \ , \tag{670}$$

in terms of which the first-order flow equations become

$$\begin{aligned} r^2\, U' &= e^{2U}\, {\rm Re}\, Z(Y) \ , \\ r^2\, Y'^I &= N^{IK}\, \frac{\partial}{\partial \bar{Y}^K} \bar{Z}(\bar{Y}) \ , \\ {\rm Im}\, Z(Y) &= 0 \ , \\ \hat{A}_r &= 0 \ . \end{aligned} \tag{671}$$

The gradient flow equations for the Y^I can be rewritten as

$$\begin{pmatrix} (Y^I - \bar{Y}^I)' \\ \\ (F_I - \bar{F}_I)' \end{pmatrix} = 2i\, {\rm Im} \begin{pmatrix} N^{IK}\, \hat{q}_K / r^2 \\ \\ \bar{F}_{IK}\, N^{KJ}\, \hat{q}_J / r^2 \end{pmatrix} = -i \begin{pmatrix} p^I / r^2 \\ \\ q_I / r^2 \end{pmatrix} \ , \tag{672}$$

where here $F_I = \partial F(Y)/\partial Y^I$. Each of the vectors appearing in this expression transforms as a symplectic vector under $Sp(2n+2, \mathbb{R})$ transformations. These gradient flow equations can be readily integrated,

$$\begin{pmatrix} Y^I - \bar{Y}^I \\ \\ F_I - \bar{F}_I \end{pmatrix} = i \begin{pmatrix} h^I + p^I / r \\ \\ h_I + q_I / r \end{pmatrix} = i \begin{pmatrix} H^I(r) \\ \\ H_I(r) \end{pmatrix} \ , \tag{673}$$

where (h^I, h_I) denote integration constants. These integration constants are constrained by the third equation in (671), which yields the condition

$$p^I\, h_I - q_I\, h^I = 0\ . \tag{674}$$

The metric factor e^{-2U} is then determined by (650),

$$e^{-2U} = H^I\, F_I(Y) - H_I\, Y^I = H^I\, \bar F_I(\bar Y) - H_I\, \bar Y^I\ , \tag{675}$$

where we used (673). And finally, using (652), it follows that the fourth equation in (671) is automatically satisfied by (673) with (674).

The integrated flow equations (673) and the constraint (674) give rise to BPS black hole solutions [196]. The equations (673) are called attractor equations: the scalar fields Y^I flow to specific values at the horizon of the black hole, irrespective of their asymptotic values at $r = \infty$. These horizon values are entirely determined by the charges carried by the BPS black hole. In the near-horizon region $r \approx 0$, the metric (643) and the scalar fields Y^I take the form (c.f. (645))

$$e^{-2U} = \frac{v_2}{r^2} \quad , \quad Y^I = \frac{Y^I_{\rm hor}}{r}\ , \tag{676}$$

with

$$v_1 = v_2 = Z(Y_{\rm hor}) = \bar Z(\bar Y_{\rm hor}) = p^I F_I(Y_{\rm hor}) - q_I Y^I_{\rm hor}\ , \tag{677}$$

where the horizon values $Y^I_{\rm hor}$ are determined by solving the equations $\mathcal{P}^I = \mathcal{Q}_I = 0$, with

$$\begin{aligned} \mathcal{P}^I &\equiv p^I + i(Y^I_{\rm hor} - \bar Y^I_{\rm hor})\,, \\ \mathcal{Q}_I &\equiv q_I + i\left(F_I(Y_{hor}) - \bar F_I(\bar Y_{\rm hor})\right)\ . \end{aligned} \tag{678}$$

Using (646), one infers the relation

$$Y^I_{\rm hor} = \bar Z(\bar X_{\rm hor})\, X^I_{\rm hor}\ , \tag{679}$$

so that

$$v_1 = v_2 = Z(Y_{\rm hor}) = |Z(X_{\rm hor})|^2\ . \tag{680}$$

The gradient flow equations that we obtained were derived from the reduced action (659) (with q_I replaced by $\hat q_I$) . The equations of motion in four dimensions impose one more condition on the solutions to the field equations derived from the reduced action, namely the so-called Hamiltonian constraint. For a

Lagrangian density $\sqrt{-g}\,(\,\frac{1}{2}\,R\,+\,\mathcal{L}_M)$, it is given by the variation of the action with respect to g^{00},

$$\tfrac{1}{2}\,R_{00}\,+\,\frac{\delta\mathcal{L}_M}{\delta g^{00}}\,-\,\tfrac{1}{2}\,g_{00}\left(\tfrac{1}{2}\,R\,+\,\mathcal{L}_M\right)\,=\,0\;. \tag{681}$$

Then, using the Lagrangian (439) as well as the metric ansatz (643), and replacing the gauge fields by their charges, as in (658), results in

$$r^2\left\{U'^2\;+\;N_{IJ}\,\tilde{X}^{I\prime}\,\bar{\tilde{X}}^{J\prime}+\frac{e^{2U}}{r^4}\,V_{\rm BH}\right\}-2\left[r^2\,U'\right]'=0\;, \tag{682}$$

where $V_{\rm BH}$ denotes the black hole potential (B.129). We rewrite this as

$$r^2\left\{U'^2\;+\;N_{IJ}\,\tilde{X}^{I\prime}\,\bar{\tilde{X}}^{J\prime}-\frac{e^{2U}}{r^4}\,V_{\rm BH}\right\}=2\left[r^2\,U'\right]'-2\frac{e^{2U}}{r^2}\,V_{\rm BH}\;. \tag{683}$$

Using the first-order flow equation (671), one readily verifies that the right hand side of the equation vanishes. This yields the Hamiltonian constraint in the form [193]

$$U'^2\;+\;N_{IJ}\,\tilde{X}^{I\prime}\,\bar{\tilde{X}}^{J\prime}=\frac{e^{2U}}{r^4}\,V_{\rm BH}\;, \tag{684}$$

which is satisfied by virtue of (671). Thus, the Hamiltonian constraint does not lead to any further restriction.

The black hole potential $V_{\rm BH}$ may have several critical points. Critical points $*$ that satisfy $(\mathcal{D}_a Z)|_* = 0\;\forall\, a = 1,\ldots,n$ with $Z|_* \neq 0$ correspond to BPS black hole solutions, whose macroscopic entropy is given by $\mathcal{S}_{\rm macro}(p,q) = \pi v_2 = \pi V_{\rm BH}|_* = \pi|Z(X_{\rm hor})|^2$, c.f. (723). These BPS solutions are obtained by solving the flow equations (671). Critical points satisfying $\mathcal{D}_a Z \neq 0$ do not correspond to BPS solutions. However, if the black hole potential $V_{\rm BH}$ admits a second decomposition in terms of a quantity $W(X)$ (possibly only when restricting to a subset of charges),

$$V_{\rm BH} = g^{a\bar{b}}\mathcal{D}_a W \bar{\mathcal{D}}_{\bar{b}}\bar{W} + |W|^2\;, \tag{685}$$

with $W \neq Z$, such that a critical point that is non-BPS satisfies $(\mathcal{D}_a W)|_* = 0\,\forall\, a = 1,\ldots,n$ with $W|_* \neq 0$, then this non-BPS critical point describes a non-BPS black hole solution that can be obtained by solving first-order flow equations of the form (671), but now with Z replaced by W [197, 198, 199]. The macroscopic entropy of this non-BPS black hole is given by $\mathcal{S}_{\rm macro}(p,q) = \pi v_2 = \pi|W(X_{\rm hor})|^2$. Thus, in certain cases, non-BPS solutions may be obtained by solving first-order flow equations [197, 198, 199].

10.2. Entropy functions for static BPS black holes

The scalar fields supporting an extremal black hole flow to specific values at the horizon. These values are entirely specified by the charges carried by the black hole, and they can be obtained by means of a variational principle based on a so-called entropy function [162, 200].

BPS black holes constitute a subset of extremal black holes, and hence their entropy can be obtained from the entropy function mentioned above. However, their entropy can also be inferred from a so-called BPS entropy function [201, 74] associated with supersymmetry enhancement at the horizon. Both notions of entropy functions give identical results at the semi-classical level. In the following, we review both notions of entropy functions and their relation, with/without higher-curvature terms proportional to the square of the Weyl tensor [172].

10.2.1. Reduced action and entropy function

We consider a local, gauge and general coordinate invariant Lagrangian L that describes a general system of abelian vector gauge fields, scalar and matter fields coupled to gravity, with or without higher-derivative terms. We focus on field configurations in the near-horizon geometry (645). These field configurations have the symmetries of $AdS_2 \times S^2$. We introduce the associated reduced action and derive the entropy function from it.

We denote the scalar and matter fields collectively by u_α. The field strengths ${F_{\mu\nu}}^I$ of the abelian gauge fields A_μ^I are given by (653): they are given in terms of the electric field E^I and the magnetic charge p^I. In the geometry (645), v_1, v_2, E, u_α take constant values, since they are invariant under the $AdS_2 \times S^2$ isometries.

Proceeding as in (657) and (654), we pass from a description based on (p^I, E^I) to a description based of magnetic/electric charges (p^I, q_I),

$$q_I = -4\pi\, v_1 v_2\, \frac{\partial L}{\partial E^I} \ . \tag{686}$$

Defining the reduced Lagrangian by the integral of the full Lagrangian L over S^2,

$$\mathcal{F}(E, p, v, u) = \int d\theta\, d\phi\, \sqrt{-g}\, L \,, \tag{687}$$

we infer

$$q_I = -\frac{\partial \mathcal{F}}{\partial E^I} \ . \tag{688}$$

This reduced Lagrangian does not transform as a function under electric-magnetic duality transformations (198). A quantity that does transform as a function under electric-magnetic duality transformations is the so-called entropy function [162],

$$\mathcal{E}(q,p,v,u) = -\mathcal{F}(E,p,v,u) - E^I q_I \,, \tag{689}$$

which takes the form of a Legendre transform in view of (688). Thus, $\mathcal{E}$ is the analogue of the Hamiltonian density associated with the reduced Lagrangian density (687), as far as the vector fields are concerned. Under electric-magnetic duality, it transforms according to $\tilde{\mathcal{E}}(\tilde{q},\tilde{p},v,u) = \mathcal{E}(q,p,v,u)$.

The constant values of the fields $v_{1,2}$ and u_α are determined by demanding $\mathcal{E}$ to be stationary under variations of v and u,

$$\frac{\partial\mathcal{E}}{\partial v} = \frac{\partial\mathcal{E}}{\partial u} = 0\,. \tag{690}$$

The equations (690) are the attractor equations that determine the values of v and u at the horizon of the black hole. The Wald entropy is directly proportional to the value of $\mathcal{E}$ at the stationary point [162],

$$\mathcal{S}_{\text{macro}}(p,q) \propto \mathcal{E}\Big|_{\text{attractor}}\,. \tag{691}$$

Note that the entropy function need not depend on all the fields at the horizon. The values of some of the fields will then be left unconstrained, but they will not appear in the expression for the Wald entropy.

10.2.2. Entropy function and black hole potential at the two-derivative level

Consider the Maxwell terms in the two-derivative Lagrangian (439), which is part of the Lagrangian describing Poincaré supergravity. The associated reduced Lagrangian (687) reads

$$\mathcal{F} = \tfrac{1}{4}\left\{\frac{iv_1\, p^I(\bar{\mathcal{N}}-\mathcal{N})_{IJ}\, p^J}{4\pi\, v_2} - \frac{4i\pi\, v_2\, E^I(\bar{\mathcal{N}}-\mathcal{N})_{IJ}\, E^J}{v_1}\right\} - \tfrac{1}{2}E^I(\mathcal{N}+\bar{\mathcal{N}})_{IJ}\, p^J \,, \tag{692}$$

and the entropy function (689) is given by (setting $v_1 = v_2$)

$$\mathcal{E} = -\frac{1}{8\pi}\,(q_I - \mathcal{N}_{IK}\, p^K)\,[(\text{Im}\,\mathcal{N})^{-1}]^{IJ}\,(q_J - \bar{\mathcal{N}}_{JL}\, p^L)\,, \tag{693}$$

which equals the black hole potential given in (B.131) [193], up to an overall constant. $\mathcal{E}$ transforms as a function under electric-magnetic duality, as can be verified by noting the transformation property (338) of $\mathcal{N}$.

10.2.3. The BPS entropy function

The isometries of the near-horizon geometry (645) played a crucial role in defining the entropy function (689). On the other hand, when dealing with BPS black holes, it is supersymmetry enhancement at the horizon that plays a crucial role in constraining fields in the near-horizon geometry. This gives rise to a different form of the entropy function for BPS black holes [201, 74], as follows.

We consider $\mathcal{N} = 2$ supergravity theories coupled to vector multiplets, and allow for the presence of higher-order derivative interactions involving the square of the Weyl tensor. As reviewed in section 6.4, the associated Wilsonian effective action is encoded in a holomorphic function $F(X, \hat{A})$ that is homogeneous of degree two under complex rescalings. Introducing rescaled variables (Y^I, Υ), we have

$$F(\lambda Y, \lambda^2 \Upsilon) = \lambda^2 F(Y, \Upsilon) \quad , \quad \lambda \in \mathbb{C}^* \, . \tag{694}$$

Here the Y^I are related to the X^I by a uniform rescaling, and Υ is a complex scalar field related to the square $\hat{A} = 4(T^-_{ab})^2$ by the uniform rescaling, c.f. (709).

At the horizon, the fields Y^I and Υ flow to constant values $Y^I_{\rm hor}$ and $\Upsilon = -64$, with the $Y^I_{\rm hor}$ determined by the BPS attractor equations [36],

$$\mathcal{P}^I = 0 \, , \qquad \mathcal{Q}_I = 0 \, , \tag{695}$$

where

$$\begin{aligned} \mathcal{P}^I &\equiv p^I + i(Y^I - \bar{Y}^I) \, , \\ \mathcal{Q}_I &\equiv q_I + i(F_I(Y, \Upsilon) - \bar{F}_I(\bar{Y}, \bar{\Upsilon})) \, . \end{aligned} \tag{696}$$

These equations are those given in (678), but now in the presence of a chiral background field Υ.

The BPS attractor equations (695) can be obtained from a variational principle based on an entropy function [201, 74]

$$\Sigma(Y, \bar{Y}, p, q) = \mathcal{F}(Y, \bar{Y}, \Upsilon, \bar{\Upsilon}) - q_I(Y^I + \bar{Y}^I) + p^I(F_I + \bar{F}_I) \, , \tag{697}$$

where p^I and q_I couple to the corresponding magneto- and electrostatic potentials at the horizon (c.f. [202]) in a way that is consistent with electric-magnetic duality. The quantity $\mathcal{F}(Y, \bar{Y}, \Upsilon, \bar{\Upsilon})$, which will be denoted as BPS free energy, is defined by

$$\mathcal{F}(Y,\bar{Y},\Upsilon,\bar{\Upsilon}) = -i\left(\bar{Y}^I F_I - Y^I \bar{F}_I\right) - 2i\left(\Upsilon F_\Upsilon - \bar{\Upsilon}\bar{F}_\Upsilon\right) , \tag{698}$$

where $F_\Upsilon = \partial F/\partial\Upsilon$. Also this expression is compatible with electric-magnetic duality, i.e. it transforms as a function under electric-magnetic duality, c.f. (222) [32]. Varying the BPS entropy function Σ with respect to the Y^I, while keeping the charges and Υ fixed, yields the result,

$$\delta\Sigma = \mathcal{P}^I\,\delta(F_I + \bar{F}_I) - \mathcal{Q}_I\,\delta(Y^I + \bar{Y}^I) , \tag{699}$$

where we made use of the homogeneity of the function $F(Y,\Upsilon)$. Assuming that the matrix $N_{IJ} = \mathrm{i}(\bar{F}_{IJ} - F_{IJ})$ is non-degenerate, it follows that stationary points of Σ satisfy the BPS attractor equations (695).

The macroscopic entropy $\mathcal{S}_{\rm macro}$ is equal to the entropy function evaluated at the attractor point, and hence it is Legendre transform of the free energy $\mathcal{F}$. It is given by [36],

$$\mathcal{S}_{\rm macro}(p,q) = \pi\,\Sigma\Big|_{\rm attractor} = \pi\Big[p^I F_I - q_I Y^I - 256\,{\rm Im}\,F_\Upsilon\Big]_{\rm attractor} . \tag{700}$$

Here the term $\pi(p^I F_I - q_I Y^I)|_{\rm attractor}$ equals a quarter of the horizon area (in units where $G_N = 1, \kappa^2 = 8\pi$), i.e. $v_1 = v_2 = (p^I F_I - q_I Y^I)|_{\rm attractor}$. The contribution proportional to F_Υ denotes the deviation from the Bekenstein-Hawking area law, and is subleading in the limit of large charges. In addition, the area also depends on Υ, and hence it also contains subleading terms. In the absence of Υ-dependent terms, the homogeneity of the function $F(Y)$ implies that the area scales quadratically with the charges.

In subsection 10.2.7, we will show that for BPS black holes, the BPS entropy (700) coincides with the one calculated from entropy function (689).

10.2.4. The BPS entropy function, the generalized Hesse potential and its dual

The BPS free energy $\mathcal{F}$ and the BPS entropy function Σ can be expressed in terms of the generalized Hesse potential H and its dual, as follows [74].

The generalized Hesse potential H is expressed in terms of real variables (x^I, y_I) (c.f. (456)),

$$Y^I = x^I + iu^I(x,y,\Upsilon,\bar{\Upsilon}) , \quad F_I = y_I + iv_I(x,y,\Upsilon,\bar{\Upsilon}) , \tag{701}$$

and defined by a Legendre transform with respect to u^I,

$$H(x,y,\Upsilon,\bar{\Upsilon}) = 2\,{\rm Im}\,F(x+iu(x,y,\Upsilon,\bar{\Upsilon}),\Upsilon) - 2y_I u^I(x,y,\Upsilon,\bar{\Upsilon}) . \tag{702}$$

Using the homogeneity relation (214) which, in the present context reads

$$2\,F(Y,\Upsilon) = Y^I\,F_I(Y,\Upsilon) + 2\,\Upsilon\,F_\Upsilon(Y,\Upsilon)\;, \tag{703}$$

one obtains

$$H(x,y,\Upsilon,\bar\Upsilon) = \tfrac{1}{2}\,\mathcal{F}(Y,\bar Y,\Upsilon,\bar\Upsilon)\;. \tag{704}$$

The BPS entropy function Σ can then be expressed as

$$\Sigma(x,y,\Upsilon,\bar\Upsilon) = 2\,H(x,y,\Upsilon,\bar\Upsilon) - 2\,q_I\,x^I + 2\,p^I\,y_I\;. \tag{705}$$

The macroscopic BPS entropy (700) is given by

$$\mathcal{S}_{\rm macro}(p,q) = 2\pi\left(H - x^I\,\frac{\partial H}{\partial x^I} - y_I\,\frac{\partial H}{\partial y_I}\right)_{\rm attractor}\;. \tag{706}$$

Thus, upon extremization, the charges (p^I, q_I) become proportional to the dual affine coordinates, while the BPS entropy is proportional to the dual Hesse potential, evaluated on the background, c.f. (7).

10.2.5. Entropy functions for $\mathcal{N}=2$ supergravity theories

In this section, we follow [172]. We use the normalization $G_N = 1, \kappa^2 = 8\pi$, as in [172].

We consider the Wilsonian effective action describing $\mathcal{N}=2$ vector multiplets coupled to $\mathcal{N}=2$ supergravity, in the presence of interactions proportional to the square of the Weyl multiplet, reviewed in section 6.4. This requires the presence of a second compensating supermultiplet, which we take to be a hypermultiplet. Additional hypermultiplets may also be added, but play a passive role in the following. The relevant Lagrangian L is given by (443), (444) [202]. The components of the Weyl, vector and hypermultiplets are displayed in Tables B.9 and B.14.

We impose spherical symmetry and derive the reduced Lagrangian (687). In a spherically symmetric configuration the field ${T_{ab}}^{ij}$ can be expressed in terms of a single complex scalar w [171],

$$T^-_{\underline{rt}} = -i\,T^-_{\underline{\theta\phi}} = \tfrac{1}{2}\,w\,, \tag{707}$$

where underlined indices denote tangent-space indices. Consequently we have $\hat A = -4w^2$. The field strengths ${F_{\mu\nu}}^I$ of the abelian gauge fields A^I_μ are given in terms of electric fields E^I and magnetic charges p^I, as in (653).

We restrict to a class of solutions by assuming the following consistent set of constraints,

$$R(\mathcal{V})_{\mu\nu}{}^{i}{}_{j} = R(A)_{\mu\nu} = \mathcal{D}_{\mu} X^{I} = \mathcal{D}_{\mu} A_{i}{}^{\alpha} = 0\,, \tag{708}$$

where the first two tensors denote the $SU(2)\times U(1)$ R-symmetry field strengths. These constraints are in accord with those that follow from requiring supersymmetry enhancement at the horizon [202]. Then, since $\hat{B}_{ij}$ is proportional to $R(\mathcal{V})_{\mu\nu}{}^{i}{}_{j}$, this field vanishes as well. Furthermore the auxiliary fields $Y_{ij}{}^{I}$ can be dropped as a result of their equations of motion.

Then, in the AdS_2 background (645), the resulting Lagrangian only depends on the field variables v_1, v_2, w, D, E^I, X^I, χ, which are all constant, and on the magnetic charges. We refer to [172] for the somewhat lengthy expression for the Lagrangian. We trade these field variables for scale invariant variables

$$\begin{aligned}
&Y^I = \tfrac{1}{4} v_2\, \bar{w}\, X^I\,, \quad \Upsilon = \tfrac{1}{16} v_2^2\, \bar{w}^2\, \hat{A} = -\tfrac{1}{4} v_2^2\, |w|^4\,, \quad U = \frac{v_1}{v_2}\,,\\
&\tilde{D} = v_2\, D + \tfrac{2}{3}(U^{-1} - 1)\,, \quad \tilde{\chi} = v_2\, \chi\,.
\end{aligned} \tag{709}$$

Observe that Υ is real and negative, and that $\sqrt{-\Upsilon}$ and U are real and positive. Note also that the hypermultiplets contribute only through the hyperkähler potential χ.

We compute the entropy function (689), adopting the normalization of the Lagrangian used in [172]. Next, we require that $\mathcal{E}$ be stationary with respect to variations of $\tilde{D}$ and $\tilde{\chi}$. This yields $\tilde{D} = 0$, and expresses $\tilde{\chi}$ in terms of the other fields. Upon substitution of these two equations into the entropy function, the expression for $\mathcal{E}$ simplifies considerably [172],

$$\begin{aligned}
\mathcal{E}(Y,\bar{Y},\Upsilon,U) \;=\;& \tfrac{1}{2} U\, \Sigma(Y,\bar{Y},p,q) + \tfrac{1}{2} U\, N^{IJ} (\mathcal{Q}_I - F_{IK}\mathcal{P}^K)\,(\mathcal{Q}_J - \bar{F}_{JL}\mathcal{P}^L)\\
&- \frac{4\mathrm{i}}{\sqrt{-\Upsilon}} (\bar{Y}^I F_I - Y^I \bar{F}_I)(U-1)\\
&- \mathrm{i}(F_\Upsilon - \bar{F}_\Upsilon)\Big[-2U\Upsilon + 32(U + U^{-1} - 2) - 8(1+U)\sqrt{-\Upsilon}\Big]\,.
\end{aligned} \tag{710}$$

This result that is consistent with electric-magnetic duality [171, 172].

The entropy function (710) depends on the variables U, Υ and Y^I, whose values are determined by demanding stationarity of $\mathcal{E}$. These values are the attractor values. The macroscopic entropy is proportional to the entropy function taken at the attractor values,

$$\mathcal{S}_{\rm macro}(p,q) = 2\pi \mathcal{E}\Big|_{\rm attractor}\,. \tag{711}$$

In the following, we will discuss the extremization of $\mathcal{E}$ with respect to these variables, first in the absence of R^2-terms, and then for BPS black holes in the presence of R^2-terms.

10.2.6. Variational equations without R^2-interactions

In the absence of R^2-interactions, the function F does not depend on Υ, so that the entropy function (710) reduces to

$$\begin{aligned}\mathcal{E}(Y,\bar{Y},\Upsilon,U) &= \tfrac{1}{2}U\,\Sigma(Y,\bar{Y},p,q) + \tfrac{1}{2}U\,N^{IJ}(\mathcal{Q}_I - F_{IK}\mathcal{P}^K)\,(\mathcal{Q}_J - \bar{F}_{JL}\mathcal{P}^L) \\ &\quad - \frac{4\mathrm{i}}{\sqrt{-\Upsilon}}(\bar{Y}^I F_I - Y^I \bar{F}_I)(U-1)\,. \end{aligned} \tag{712}$$

Varying (712) with respect to Υ yields

$$U = 1\,. \tag{713}$$

The latter implies that the Ricci scalar of the four-dimensional space-time vanishes. Here we assumed that $\big(\bar{Y}^I F_I - Y^I \bar{F}_I\big)$ is non-vanishing, which is required so that Newton's constant remains finite, c.f. (433). Varying with respect to U yields,

$$\Sigma + \big(\mathcal{Q}_I - F_{IK}\,\mathcal{P}^K\big)\,N^{IJ}\,\big(\mathcal{Q}_J - \bar{F}_{JL}\,\mathcal{P}^L\big) - \frac{8\mathrm{i}}{\sqrt{-\Upsilon}}\,\big(\bar{Y}^I F_I - Y^I \bar{F}_I\big) = 0\,, \tag{714}$$

which determines the value of Υ in terms of the Y^I. This is consistent with the fact that when the function F depends exclusively on the Y^I, the field equation for T_{ab}^- is algebraic, c.f. (438).

The resulting effective entropy function reads

$$\mathcal{E}(Y,\bar{Y},\Upsilon,1) = \tfrac{1}{2}\Sigma(Y,\bar{Y},p,q) + \tfrac{1}{2}N^{IJ}(\mathcal{Q}_I - F_{IK}\mathcal{P}^K)\,(\mathcal{Q}_J - \bar{F}_{JL}\mathcal{P}^L)\,, \tag{715}$$

which is independent of Υ. Note that (715) is homogeneous under uniform rescalings of the charges q_I and p^I and the variables Y^I. This implies that the entropy will be proportional the the square of the charges. Under infinitesimal changes of Y^I and $\bar{Y}^I$ the entropy function (715) changes according to

$$\begin{aligned}\delta\mathcal{E} &= \mathcal{P}^I\,\delta(F_I + \bar{F}_I) - \mathcal{Q}_I\,\delta(Y^I + \bar{Y}^I) \\ &\quad + \tfrac{1}{2}i\,\big(\mathcal{Q}_K - \bar{F}_{KM}\,\mathcal{P}^M\big)\,N^{KI}\,\delta F_{IJ}\,N^{JL}\,\big(\mathcal{Q}_L - \bar{F}_{LN}\,\mathcal{P}^N\big) \\ &\quad - \tfrac{1}{2}i\,\big(\mathcal{Q}_K - F_{KM}\,\mathcal{P}^M\big)\,N^{KI}\,\delta\bar{F}_{IJ}\,N^{JL}\,\big(\mathcal{Q}_L - F_{LN}\,\mathcal{P}^N\big) = 0\,, \end{aligned} \tag{716}$$

where $\delta F_I = F_{IJ}\,\delta Y^J$ and $\delta F_{IJ} = F_{IJK}\,\delta Y^K$. This equation determines the horizon value of the Y^I in terms of the black hole charges (p^I, q_I). Because the function $F(Y)$ is homogeneous of second degree, we have $F_{IJK}Y^K = 0$. Using this relation one deduces from (716) that $\big(\mathcal{Q}_J - F_{JK}\,\mathcal{P}^K\big)\,Y^J = 0$, which is equivalent to

$$i(\bar{Y}^I F_I - Y^I \bar{F}_I) = p^I F_I - q_I Y^I \ . \tag{717}$$

Therefore, at the attractor point, we have

$$\Sigma = i(\bar{Y}^I F_I - Y^I \bar{F}_I) \ . \tag{718}$$

Inserting this result into (714) yields

$$\sqrt{-\Upsilon} = \frac{8\,\Sigma}{\Sigma + N^{IJ}\,(\mathcal{Q}_I - F_{IK}\,\mathcal{P}^K)\,\big(\mathcal{Q}_J - \bar{F}_{JL}\,\mathcal{P}^L\big)}\ , \tag{719}$$

which gives the value of Υ in terms of the attractor values of the Y^I. Using (719) we obtain

$$\mathcal{S}_{\rm macro}(p,q) = 2\pi\,\mathcal{E}\Big|_{\rm attractor} = \frac{8\pi\,\Sigma}{\sqrt{-\Upsilon}}\Big|_{\rm attractor}\ . \tag{720}$$

Observe that, for a BPS black hole, $\mathcal{Q}_I = \mathcal{P}^J = 0$ and $\Upsilon = -64$, so that $\mathcal{S}_{\rm macro} = \pi\,\Sigma|_{\rm attractor}$ in accord with (700).

The entropy function (715) can be written as

$$\mathcal{E} = -q_I(Y^I + \bar{Y}^I) + p^I(F_I + \bar{F}_I) + \tfrac{1}{2}N^{IJ}(q_I - F_{IK}p^K)(q_J - \bar{F}_{JL}p^L) + N_{IJ}Y^I\bar{Y}^J\,, \tag{721}$$

where we used the homogeneity of the function $F(Y)$. Expressing the Y^I as in (679), one obtains (using $N_{IJ}X^I\bar{X}^J = -1$)

$$\mathcal{E} = \tfrac{1}{2}\left(N^{IJ} + 2\,X^I\bar{X}^J\right)(q_I - F_{IK}p^K)(q_J - \bar{F}_{JL}p^L)\,, \tag{722}$$

where F_{IJ} is now the second derivative of $F(X)$ with respect to X^I and X^J. Comparision with the black hole potential (B.129) gives $\mathcal{E} = \frac{1}{2}\,V_{\rm BH}$, and hence,

$$\mathcal{S}_{\rm macro}(p,q) = 2\pi\,\mathcal{E}\Big|_{\rm attractor} = \pi\,V_{\rm BH}\Big|_{\rm attractor} = \pi\,V_{\rm BH}(p,q)\ . \tag{723}$$

10.2.7. BPS black holes with R^2-interactions

In the presence of R^2 interactions, the horizon values of U and Υ for extremal BPS black holes are $U = 1$ and $\Upsilon = -64$ [202]. Inserting these values into (710) results in

$$\mathcal{E}(Y,\bar{Y},-64,1) = \tfrac{1}{2}\Sigma(Y,\bar{Y},p,q) + \tfrac{1}{2}N^{IJ}\left(\mathcal{Q}_I - F_{IK}\,\mathcal{P}^K\right)\left(\mathcal{Q}_J - \bar{F}_{JL}\,\mathcal{P}^L\right) . \tag{724}$$

Observe that the variational principle based on (724) is only consistent with the one based on (710) provided that (724) is supplemented by the extremization equations for U,

$$\begin{aligned} &\Sigma + \left(\mathcal{Q}_I - F_{IK}\,\mathcal{P}^K\right) N^{IJ}\left(\mathcal{Q}_J - \bar{F}_{JL}\,\mathcal{P}^L\right) - \frac{8\mathrm{i}}{\sqrt{-\Upsilon}}(\bar{Y}^I F_I - Y^I \bar{F}_I) \\ &-\mathrm{i}(F_\Upsilon - \bar{F}_\Upsilon)\Big[-4\Upsilon + 64(1-U^{-2}) - 16\sqrt{-\Upsilon}\Big] = 0\,, \end{aligned} \tag{725}$$

and for Υ,

$$\begin{aligned} &U\Sigma - \mathrm{i}(\bar{Y}^I F_I - Y^I \bar{F}_I)\Big[U + 4(-\Upsilon)^{-1/2}(U-1)\Big] \\ &+2\mathrm{i}U\left[\Upsilon F_{I\Upsilon} N^{IJ}(\mathcal{Q}_J - \bar{F}_{JK}\mathcal{P}^K) - \text{h.c.}\right] \\ &+2\mathrm{i}(F_\Upsilon - \bar{F}_\Upsilon)\Big[2U\Upsilon + 4\sqrt{-\Upsilon}(1+U)\Big] = 0\,. \end{aligned} \tag{726}$$

For BPS solutions it can be readily checked that the latter are indeed satisfied. Using $\mathcal{Q}_I = \mathcal{P}^J = 0$, we obtain $\mathcal{S}_{\text{macro}} = \pi\,\Sigma|_{\text{attractor}}$, in accord with (700).

10.3. Large and small BPS black holes: examples

As an application [203, 204] of the above, let us consider BPS black holes in an $\mathcal{N}=2$ supergravity theory coupled to Weyl-square terms, whose Wilsonian action is encoded in the holomorphic function $F(Y,\Upsilon) = F^{(0)}(Y) + F^{(1)}(Y)\,\Upsilon$ given by

$$F(Y,\Upsilon) = -\frac{Y^1 Y^a \eta_{ab} Y^b}{Y^0} + c_1\,\frac{Y^1}{Y^0}\,\Upsilon\ . \tag{727}$$

Here

$$Y^a \eta_{ab} Y^b = Y^2 Y^3 - \sum_{a=4}^{n} (Y^a)^2\,, \qquad a = 2,\ldots,n\,, \tag{728}$$

with real constants η_{ab} and c_1. We define $S = -iY^1/Y^0$.

We introduce the charge vectors N^I and M_I,

$$\begin{aligned} N^I &= (p^0, q_1, p^2, p^3, \ldots, p^n)\,, \\ M_I &= (q_0, -p^1, q_2, q_3, \ldots, q_n)\,. \end{aligned} \tag{729}$$

There are three bilinear charge combinations that are invariant under $SO(n-1,2;\mathbb{Z})$-transformations [205], also referred to as target space duality transformations,

$$\begin{aligned} \langle M,M\rangle &= 2\Big(M_0 M_1 + \tfrac{1}{4} M_a \eta^{ab} M_b\Big) = 2\Big(-q_0 p^1 + \tfrac{1}{4} q_a \eta^{ab} q_b\Big)\ , \\ \langle N,N\rangle &= 2\Big(N^0 N^1 + N^a \eta_{ab} N^b\Big) = 2\Big(p^0 q_1 + p^a \eta_{ab} p^b\Big)\ , \\ M\cdot N &= M_I N^I = q_0 p^0 - q_1 p^1 + q_2 p^2 + \cdots + q_n p^n\ . \end{aligned} \tag{730}$$

For instance, the charge bilinears are clearly invariant under the $SO(n-1,2;\mathbb{Z})$-transformation

$$\begin{aligned} p^0 &\to q_1 \,, & q_0 &\to -p^1 \,, \\ p^1 &\to -q_0 \,, & q_1 &\to p^0 \,, \\ p^a &\to p^a \,, & q_a &\to q_a \,. \end{aligned} \tag{731}$$

10.3.1. Large BPS black holes

Definition 19. Large BPS black holes. *A large single-centre BPS black hole in four dimensions is a dyonic spherically symmetric BPS black hole carrying electric/magnetic charges (q_I, p^I), such that the charge bilinears $\langle M,M\rangle, \langle N,N\rangle$ are positive and $\langle M,M\rangle\langle N,N\rangle - (M\cdot N)^2 \gg 1$.*

This ensures that at the two-derivative level, the black hole has a non-vanishing horizon area A [201], $A = 2\pi(S+\bar{S})\,\langle N,N\rangle$, c.f. (737) and (740) below.

Using (c.f. (696))

$$Y^I - \bar{Y}^I = i\,p^I \quad , \qquad F_I(Y,\Upsilon) - \bar{F}_I(\bar{Y},\bar{\Upsilon}) = i\,q_I \,, \tag{732}$$

one obtains for $I=a$,

$$Y^a = \frac{1}{S+\bar{S}}\Big[-\tfrac{1}{2}\eta^{ab}q_b + i\bar{S}\,p^a\Big]\,, \tag{733}$$

where $\eta^{ab}\,\eta_{bc} = \delta^a_c$. Similarly, one finds

$$\begin{aligned} p^I F_I - q_I Y^I &= i(\bar{Y}^I F_I - Y^I \bar{F}_I) \qquad (734)\\ &= (S+\bar{S})\Big(\bar{Y}^a\eta_{ab}Y^b + \frac{\bar{Y}^0}{Y^0}\Big[-Y^a\eta_{ab}Y^b + c_1\,\Upsilon\Big] + \text{ h.c.}\Big)\,, \end{aligned}$$

as well as

$$\begin{aligned} q_1\,p^0 &= -(Y^0-\bar{Y}^0)(F_1-\bar{F}_1) \\ &= \Big(\frac{\bar{Y}^0}{Y^0}-1\Big)\Big[-Y^a\eta_{ab}Y^b + c_1\,\Upsilon\Big] + \text{ h.c.}\,. \end{aligned} \tag{735}$$

Combining these two equations and using (733) yields

$$p^I F_I - q_I Y^I = (S+\bar{S})\Big(\tfrac{1}{2}\,\langle N,N\rangle + (c_1\,\Upsilon + \text{ h.c.})\Big)\,, \tag{736}$$

where the bilinear charge combination $\langle N,N\rangle$ is defined in (730).

Using (700), we obtain for Wald's entropy (with $\Upsilon = -64$)

$$\mathcal{S}_{\rm macro} = \tfrac{1}{2}\,\pi\,(S+\bar S)\,\Big(\langle N,N\rangle - 512\,c_1\Big)\,, \tag{737}$$

where S is evaluated at the horizon. We now determine its value.

Using (732), one finds that the combinations $S\bar S\,q_1\,p^0 + q_0\,p^1$ and $i(\bar S - S)\,q_1\,p^0 + q_1\,p^1 - q_0\,p^0$ do not explicitly depend on Y^0. This results in the following equations for S,

$$\begin{aligned} S\bar S\,\langle N,N\rangle &= \langle M,M\rangle - 2(S+\bar S)(c_1\,\Upsilon\,S + \text{ h.c.})\,, \\ (S-\bar S)\,\langle N,N\rangle &= 2i\,M\cdot N + 2\,(S+\bar S)\,(c_1\,\Upsilon - \text{ h.c.})\,, \end{aligned} \tag{738}$$

from which one infers the value of S at the horizon in terms of the charges,

$$S = \sqrt{\frac{\langle M,M\rangle\langle N,N\rangle - (M\cdot N)^2}{\langle N,N\rangle\,(\langle N,N\rangle - 512\,c_1)}} \;+\; i\,\frac{M\cdot N}{\langle N,N\rangle}\,. \tag{739}$$

The resulting entropy is expressed in terms of the charges as

$$\mathcal{S}_{\rm macro} = \pi\,\sqrt{\langle M,M\rangle\langle N,N\rangle - (M\cdot N)^2}\,\sqrt{1 - \frac{512\,c_1}{\langle N,N\rangle}}\,. \tag{740}$$

When $c_1 = 0$, this equals one quarter of the area of the horizon.

10.3.2. Small BPS black holes

Definition 20. Small BPS black holes. *A small BPS black hole in four dimensions is a BPS black hole carrying electric/magnetic charges (q_I, p^I) such that the charge combination $\langle M,M\rangle\langle N,N\rangle - (M\cdot N)^2$ vanishes, and such that its macroscopic (Wald) entropy $\mathcal{S}_{\rm macro}$ is, for large charges, given by $\mathcal{S}_{\rm macro} \propto \sqrt{Q^2}$. Here Q^2 denotes a linear combination of charge bilinears.*

At the two-derivative level, a small BPS black hole is a null-singular solution to the equations of motion of $\mathcal{N} = 2$ supergravity theory. For a small BPS black hole to have a non-vanishing area of the event horizon, higher-curvature corrections need to be taken into account [206, 207, 208, 209].[71] When $c_1 \neq 0$, a horizon forms, leading to the cloaking of a null singularity that is present when $c_1 = 0$ [209]. This requires $c_1 < 0$, as we will see below.

In the following, we will consider small black holes with charges $N^I = 0$ in the model (727). To compute the horizon value of S as well as the entropy (700)

[71] For a recent discussion and a different viewpoint, see [210, 211].

of such a small black hole, we proceed s follows. We start by considering a large BPS black hole which is axion free, i.e. one for which $\text{Im}\, S = 0$. We thus set $M \cdot N = 0$ in (739) and in (740), which yields

$$S + \bar{S} = 2\sqrt{\frac{\langle M, M\rangle}{\langle N, N\rangle - 512\, c_1}}\ ,\ \mathcal{S}_{\text{macro}} = \pi\ \sqrt{\langle M, M\rangle\langle N, N\rangle - 512\, c_1\, \langle M, M\rangle} \tag{741}$$

Next, we set $\langle N, N\rangle = 0$ in these expressions, which results in

$$S + \bar{S} = \sqrt{-\langle M, M\rangle/(128\, c_1)} > 0\ \ ,\ \ \mathcal{S}_{\text{macro}} = 2\,\pi\,\sqrt{-128\, c_1\, \langle M, M\rangle}\ . \tag{742}$$

Using (736),

$$\pi\left(p^I F_I - q_I Y^I\right) = -\pi\, 128 c_1\, (S + \bar{S})\ , \tag{743}$$

which equals a quarter of the horizon area and needs to be positive, we infer $c_1 < 0$, and hence $\langle M, M\rangle > 0$. Thus, $\mathcal{S}_{\text{macro}}$ equals one half of the horizon area [209]. Note that S and the entropy only have finite values due to $c_1 \neq 0$.

11. Born-Infeld-dilaton-axion system and F-function

In subsection 4.1 we discussed how to recast point-particle Lagrangians in terms of functions F of the form (157). Here, we will consider the example of a homogenous function $F(x, \bar{x}, \eta)$ of degree 2, with η having scaling weight $m = -2$ (c.f. subsection 4.2) and show that this describes the Born-Infeld-dilaton-axion system in an $AdS_2 \times S^2$ background. We follow [33].

11.1. Homogeneous function F

We consider a function F that depends on three complex scalar fields X^I (with $I = 0, 1, 2$), as well as on an external real parameter η,

$$F(X, \bar{X}, \eta) = -\tfrac{1}{2}\, \frac{X^1 (X^2)^2}{X^0} + 2i\, \Omega(X, \bar{X}, \eta)\ . \tag{744}$$

We demand F to be homogeneous of degree 2 under rescalings $X^I \mapsto \lambda\, X^I$, $\eta \mapsto \lambda^m\, \eta$, with $\lambda \in \mathbb{R}\backslash\{0\}$, as in (169). We leave the scaling weight m arbitrary, for the time being.

Duality transformations are represented by $\text{Sp}(6, \mathbb{R})$ matrices (which are 6×6 matrices of the form (160)) acting on (X^I, F_I), where $F_I = \partial F(X, \bar{X}, \eta)/\partial X^I$. The external parameter η is inert under these transformations.

Let us assume that the model based on (744) is invariant under S-duality as well as under a particular T-duality transformation. These symmetry transformations belong to an $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ subgroup of $\mathrm{Sp}(6,\mathbb{R})$. The first $\mathrm{SL}(2,\mathbb{R})$ subgroup acts as follows on (X^I, F_I),

$$\begin{aligned} X^0 &\mapsto d\,X^0 + c\,X^1\,, & F_0 &\mapsto a\,F_0 - b\,F_1\,, \\ X^1 &\mapsto a\,X^1 + b\,X^0\,, & F_1 &\mapsto d\,F_1 - c\,F_0\,, \\ X^2 &\mapsto d\,X^2 - c\,F_2\,, & F_2 &\mapsto a\,F_2 - b\,X^2\,, \end{aligned} \tag{745}$$

where a, b, c, d are real parameters that satisfy $ad - bc = 1$. This symmetry is referred to as S-duality. Let us describe its action on two complex scalar fields S and T that are given by the scale invariant combinations $S = -iX^1/X^0$ and $T = -iX^2/X^0$. The S-duality transformation (745) acts as

$$S \mapsto \frac{aS - ib}{icS + d} \quad , \quad T \mapsto T + \frac{2i\,c}{\Delta_{\mathrm{S}}\,(Y^0)^2}\,\frac{\partial\Omega}{\partial T} \quad , \quad X^0 \mapsto \Delta_{\mathrm{S}}\,X^0\,, \tag{746}$$

where we view Ω as a function of S, T, X^0 and their complex conjugates, and where

$$\Delta_{\mathrm{S}} = d + ic\,S\,. \tag{747}$$

The second $\mathrm{SL}(2,\mathbb{R})$ subgroup is referred to as T-duality group. Here we focus on a particular T-duality transformation given by

$$\begin{aligned} X^0 &\mapsto F_1\,, & F_0 &\mapsto -X^1\,, \\ X^1 &\mapsto -F_0\,, & F_1 &\mapsto X^0\,, \\ X^2 &\mapsto X^2\,, & F_2 &\mapsto F_2\,, \end{aligned} \tag{748}$$

which results in

$$S \mapsto S + \frac{2}{\Delta_{\mathrm{T}}(X^0)^2}\left[-X^0\frac{\partial\Omega}{\partial X^0} + T\frac{\partial\Omega}{\partial T}\right] \quad , \quad T \mapsto \frac{T}{\Delta_{\mathrm{T}}} \quad , \quad X^0 \mapsto \Delta_{\mathrm{T}}\,X^0\,, \tag{749}$$

where

$$\Delta_{\mathrm{T}} = \tfrac{1}{2}T^2 + \frac{2}{(X^0)^2}\,\frac{\partial\Omega}{\partial S}\,. \tag{750}$$

When a symplectic transformation describes a symmetry of the system, a convenient method for verifying this consists in performing the substitution $X^I \mapsto \tilde{X}^I$ in the derivatives F_I, and checking that this substitution correctly induces the symplectic transformation of F_I. This will impose restrictions on the form of F, and hence also on Ω. Imposing that S-duality (745) constitutes a symmetry of the model (744) results in the following conditions on the transformation behaviour of the derivatives of Ω [35],

$$\begin{aligned}
\left(\frac{\partial\Omega}{\partial T}\right)'_{\rm S} &= \frac{\partial\Omega}{\partial T}\;,\\
\left(\frac{\partial\Omega}{\partial S}\right)'_{\rm S} &= \Delta_{\rm S}{}^2\,\left(\frac{\partial\Omega}{\partial S}\right) + \frac{\partial\left(\Delta_{\rm S}{}^2\right)}{\partial S}\left[-\tfrac{1}{2}X^0\,\frac{\partial\Omega}{\partial X^0} - ic\,2\Delta_{\rm S}\,(X^0)^2\left(\frac{\partial\Omega}{\partial T}\right)^2\right]\;,\\
\left(X^0\,\frac{\partial\Omega}{\partial X^0}\right)'_{\rm S} &= X^0\,\frac{\partial\Omega}{\partial X^0} + \frac{2ic}{\Delta_{\rm S}\,(X^0)^2}\left(\frac{\partial\Omega}{\partial T}\right)^2\;,
\end{aligned}\tag{751}$$

while requiring the particular T-duality transformation (748) to constitute a symmetry imposes the transformation behaviour [35]

$$\begin{aligned}
\left(\frac{\partial\Omega}{\partial S}\right)'_{\rm T} &= \frac{\partial\Omega}{\partial S}\;,\\
\left(\frac{\partial\Omega}{\partial T}\right)'_{\rm T} &= \left(\Delta_{\rm T} - T^2\right)\,\frac{\partial\Omega}{\partial T} + T\,X^0\frac{\partial\Omega}{\partial X^0}\;,\\
\left(X^0\frac{\partial\Omega}{\partial X^0}\right)'_{\rm T} &= X^0\frac{\partial\Omega}{\partial X^0} + \frac{4}{\Delta_{\rm T}\,(X^0)^2}\,\frac{\partial\Omega}{\partial S}\left[-X^0\,\frac{\partial\Omega}{\partial X^0} + T\frac{\partial\Omega}{\partial T}\right]\;.
\end{aligned}\tag{752}$$

Solutions to both (751) and (752) may be constructed iteratively by assuming that Ω possesses a power series expansion in η,

$$\Omega(X,\bar{X},\eta) = \sum_{n=1}^{\infty}\eta^n\,\Omega^{(n)}(X,\bar{X})\;. \tag{753}$$

Note that since Ω and η are real, so are the expansion functions $\Omega^{(n)}$. The latter have to scale as λ^{-mn+2}. Once a solution $\Omega^{(1)}$ to (751) and (752) has been found, the full expression (753) can be constructed by solving (751) and (752) iteratively starting from $\Omega^{(1)}$.

So far, we have not made any assumptions about the scaling weight m in (169). Depending on the choice of m, the expansion (753) will have different properties. For concreteness, let us take $m = -2$, which implies that the expansion functions $\Omega^{(n)}$ in (753) will have to scale as λ^{2n+2}. The lowest function $\Omega^{(1)}$ will therefore scale as λ^4. We make an ansatz for $\Omega^{(1)}$ that is consistent with this scaling behaviour,

$$\Omega^{(1)}(X,\bar{X}) = |X^0|^4\,g(S,T,\bar{S},\bar{T})\;. \tag{754}$$

The equations (751) and (752) require $\Omega^{(1)}$ to be invariant under the S-duality and T-duality transformations given above, and determine it to be given by

$$\Omega^{(1)} = \tfrac{1}{8}\,|X^0|^4\,(S+\bar{S})^2\,|T|^4\;, \tag{755}$$

where we have chosen a particular normalization, for later convenience.

We may now proceed iteratively to determine the higher $\Omega^{(n)}$, solving (751) and (752) order by order in η, using the transformation laws (746) and (749)). Rather than proceeding in this way, we present an exact solution to (751) and (752) that, to lowest order in η, reduces to (755),

$$\begin{aligned}\Omega(X,\bar{X},\eta) \quad = \quad & \tfrac{1}{8}\,\eta^{-2}\Big[\,\sqrt{1-\tfrac{1}{2}\eta^2\,(S+\bar{S})\,(TX^0-\bar{T}\bar{X}^0)^2}\\ & -\sqrt{1-\tfrac{1}{2}\eta^2\,(S+\bar{S})\,(TX^0+\bar{T}\bar{X}^0)^2}\,\Big]^2\;.\end{aligned} \tag{756}$$

It can be verified that (756) satisfies (751) and (752). Note that (756) scales correctly as $\Omega(\lambda X, \lambda\bar{X}, \lambda^{-2}\,\eta) = \lambda^2\,\Omega(X,\bar{X},\eta)$.

In the next subsection, we turn to the interpretation of the function F based on (756).

11.2. Interpretation: the Born-Infeld-dilaton-axion system in an $AdS_2 \times S^2$ background

The function F based on (756) describes a Born-Infeld-dilaton-axion system in an $AdS_2 \times S^2$ background, as we proceed to explain.

We consider the Born-Infeld Lagrangian in the presence of a dilaton-axion field $S = \Phi + i\,B$ [212],

$$L = -g^{-2}\left[\sqrt{|\det[g_{\mu\nu}+g\,\Phi^{1/2}\,F_{\mu\nu}]|}-\sqrt{|\det g_{\mu\nu}|}\right]+\sqrt{|\det g_{\mu\nu}|}\,\tfrac{1}{4}\,B\,F_{\mu\nu}\tilde{F}^{\mu\nu}\;, \tag{757}$$

where here $\tilde{F}_{ab} = \frac{1}{2}\varepsilon_{abcd}\,F^{cd}$ with $\varepsilon_{0123} = 1$. In this Lagrangian, the gauge coupling g appears multiplied by the dilaton field Φ, while the term $BF_{\mu\nu}\tilde{F}^{\mu\nu}$ introduces a scalar field degree of freedom called the axion. The Born-Infeld-dilaton-axion system described by (757) has duality symmetries that will be described below.

Let us consider the system (757) in an $AdS_2 \times S^2$ background

$$\begin{aligned} ds^2 &= v_1\left(-r^2\,dt^2+\frac{dr^2}{r^2}\right)+v_2\left(d\theta^2+\sin^2\theta\,d\phi^2\right)\;,\\ F_{rt} &= v_1\,e \quad,\quad F_{\theta\phi} = v_2\,p\,\sin\theta\;,\end{aligned} \tag{758}$$

i.e., let us restrict to field configurations that have the $SO(2,1) \times SO(3)$ symmetry of $AdS_2 \times S^2$, in which case v_1, v_2, e, p, Φ, B are constants. Integrating over the angular variables and setting $v_1 v_2 \, 4\pi = 1$, for convenience, yields

$$L(e,p,\Phi,B) = -g^{-2}\left[\sqrt{1-g^2\,\Phi\,e^2}\;\sqrt{1+g^2\,\Phi\,p^2}-1\right]+B\,e\,p\,, \tag{759}$$

where we assume $g^2\,\Phi\,e^2 < 1$. To obtain the associated Hamiltonian H,

$$H(p,q,\Phi,B) = q\,e - \mathcal{L}(e,p,\Phi,B)\;, \tag{760}$$

we first compute $q = \partial L/\partial e$,

$$q = e\,\Phi\,\sqrt{\frac{1+g^2\,\Phi\,p^2}{1-g^2\,\Phi\,e^2}} + B\,p\;. \tag{761}$$

Inverting this relation yields

$$e = \frac{q - B\,p}{\sqrt{\Phi^2 + g^2\,\Phi\;[\Phi^2\,p^2 + (q-B\,p)^2]}}\;, \tag{762}$$

and substituting in (760) gives

$$H(p,q,\Phi,B) = g^{-2}\left[\sqrt{1+g^2[\Phi\,p^2+\Phi^{-1}\;(q-B\,p)^2]}-1\right]\;. \tag{763}$$

Then, expressing Φ and B in terms of S and $\bar{S}$ results in

$$H(p,q,S,\bar{S}) = g^{-2}\left[\sqrt{1+2\,g^2\,\Sigma(p,q,S,\bar{S})}-1\right]\;, \tag{764}$$

where

$$\Sigma(p,q,S,\bar{S}) = \frac{q^2 + ip\,q(S-\bar{S}) + p^2\,|S|^2}{S+\bar{S}}\;. \tag{765}$$

The Hamiltonian (764) depends on canonical coordinates (p,q), on an external parameter g^2 as well as on the dilaton-axion field, which describes a background field. We observe that H scales as $H \mapsto \lambda^2 H$ under $(p,q) \mapsto \lambda(p,q)\,,\, g^2 \mapsto \lambda^{-2}\,g^2\,,\, S \mapsto S$, with $\lambda \in \mathbb{R}\backslash\{0\}$. The electric field (762) scales as $e \mapsto \lambda e$.

Let us now return to the reduced Lagrangian (759) and recast it in the form $L = 4\,[\mathrm{Im}F - \Omega]$, c.f. (164), where we introduce the complex variable $x = \frac{1}{2}(p+ie)$, which scales as $x \mapsto \lambda x$. The function F will now depend on the two complex scalar fields x and S,

$$F(x,\bar{x},S,\bar{S},g^2) = F^{(0)}(x,S) + 2i\Omega(x,\bar{x},S,\bar{S},g^2)\;, \tag{766}$$

and is determined as follows. The holomorphic function $F^{(0)}$ encodes all the contributions that are independent of g^2, while Ω, which is real, accounts for all the terms in the reduced Lagrangian that depend on g^2. This yields,

$$\begin{aligned} F^{(0)}(x,S) &= -\tfrac{1}{2} i\, S\, x^2 \,, \\ \Omega(x,\bar{x},S,\bar{S},g^2) &= \tfrac{1}{8}\, g^{-2} \Bigg(\sqrt{1+\tfrac{1}{2} g^2\,(S+\bar{S})\,(x+\bar{x})^2} \\ &\qquad\qquad - \sqrt{1+\tfrac{1}{2} g^2\,(S+\bar{S})\,(x-\bar{x})^2} \Bigg)^2 \,. \end{aligned} \tag{767}$$

Under the scaling $x \mapsto \lambda x$, $g^2 \mapsto \lambda^{-2}\, g^2$, $S \mapsto S$, F scales as $F \mapsto \lambda^2 F$.

Now we note that the function F given in (767) precisely matches the one given in (744) and (756) upon identifying

$$S = -i \frac{X^1}{X^0} \quad , \quad x = X^2 = i\, T\, X^0 \quad , \quad g = \eta \,. \tag{768}$$

The Hamiltonian (764) is invariant under the S- and T-duality transformations discussed in the previous subsection. We proceed to verify this. The external parameter g^2 is inert under these transformations. Using (161) we infer that the canonical pair (p, q) is given by $(2\,\mathrm{Re}\, x, 2\,\mathrm{Re}\, F_x)$. The T-duality transformation (748) leaves (x, F_x) invariant. Since Ω given in (767), or equivalently in (756), satisfies $X^0 \partial\Omega/\partial X^0 = T \partial\Omega/\partial T$, S is inert under (748). Consequently, the Hamiltonian (764) is invariant under the T-duality transformation (748). The S-duality transformation (745),

$$S \mapsto \frac{aS - ib}{icS + d} \,, \tag{769}$$

induces the following transformation of the canonical pair (p, q),

$$\begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \,, \tag{770}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Hence, Σ given in (765) is invariant under S-duality, and so is H.

12. *F*-function for an STU-model

As an application of electric-magnetic duality in a chiral background, discussed in section 4.4.2, let us consider the STU-model of Sen and Vafa (referred to as $N = 2$ Example D in [213]) in the presence of higher curvature interactions proportional to the square of the Weyl tensor. This model possesses duality symmetries which were used recently in [214] to determine the function F. The holomorphic function F takes the form

$$F(X,\hat{A}) = -\frac{X^1 X^2 X^3}{X^0} + 2i\Omega(X,\hat{A}) \;, \tag{771}$$

with $\hat{A}$ given in (444). Note that $\hat{A}$ has scaling weight 2. The model possesses S-, T- and U-duality symmetries $\Gamma_0(2)_S \times \Gamma_0(2)_T \times \Gamma_0(2)_U$ as well as triality symmetry. $\Gamma_0(2)$ is the subgroup of the group $\mathrm{SL}(2,\mathbb{Z})$ defined by restricting its integer-valued matrix elements a, b, c, d (with $ad - bc = 1$) to $a, d \in 2\,\mathbb{Z}+1$, $c \in 2\,\mathbb{Z}$ and $b \in \mathbb{Z}$. Triality symmetry refers to the invariance of the model under exchanges of the scalar fields $S = -iX^1/X^0$, $T = -iX^2/X^0$ and $U = -iX^3/X^0$. The duality and triality symmetries of the model are very restrictive and allow for the determination of the function F. For instance, under S-duality, the derivatives of Ω are required to transform in the following way,

$$\begin{aligned}
\left(\frac{\partial\Omega}{\partial T}\right)'_{\mathrm{S}} &= \frac{\partial\Omega}{\partial T}\;, \qquad \left(\frac{\partial\Omega}{\partial U}\right)'_{\mathrm{S}} = \frac{\partial\Omega}{\partial U}\;,\\
\left(\frac{\partial\Omega}{\partial S}\right)'_{\mathrm{S}} - \Delta_{\mathrm{S}}{}^2\,\frac{\partial\Omega}{\partial S} &= \frac{\partial\Delta_{\mathrm{S}}}{\partial S}\left[-\,\Delta_{\mathrm{S}}\,X^0\,\frac{\partial\Omega}{\partial X^0} - \frac{2}{(X^0)^2}\,\frac{\partial\Delta_{\mathrm{S}}}{\partial S}\,\frac{\partial\Omega}{\partial T}\frac{\partial\Omega}{\partial U}\right]\;,\\
\left(X^0\frac{\partial\Omega}{\partial X^0}\right)'_{\mathrm{S}} &= X^0\frac{\partial\Omega}{\partial X^0} + \frac{4}{\Delta_{\mathrm{S}}\,(X^0)^2}\,\frac{\partial\Delta_{\mathrm{S}}}{\partial S}\,\frac{\partial\Omega}{\partial T}\frac{\partial\Omega}{\partial U}\;.
\end{aligned} \tag{772}$$

Using triality, one obtains similar equations under T- and U-duality.

The equations (772) are non-linear in Ω, and where solved [214] by iteration using the fact that $\Omega(X,\hat{A})$ must be a homogeneous function of second degree, c.f. (213). This was achieved by expanding $\Omega(X,\hat{A})$ in a series expansion in powers of $\hat{A}\,(X^0)^{-2}$ (which has scaling weight zero), with coefficient functions that depend on S, T, U and on an overall factor $\hat{A}$,

$$\Omega(X,\hat{A}) = \hat{A}\left[\gamma\,\ln\frac{(X^0)^2}{\hat{A}} + \omega^{(1)}(S,T,U) + \sum_{n=1}^{\infty}\left(\frac{\hat{A}}{(X^0)^2}\right)^n\omega^{(n+1)}(S,T,U)\right]. \tag{773}$$

Note the presence of the logarithmic term, whose inclusion allowed to implement the duality symmetries of the model, leading to the determination of the gravitational coupling functions $\omega^{(n)}(S,T,U)$ by iteration. Additional important information about the structure of F was gleaned from the Hesse potential for the model and the associated holomorphic anomaly equation. We refer to [214] for a detailed discussion thereof.

Acknowledgements

We would like to thank Vicente Cortés, Bernard de Wit and Swapna Mahapatra for the long-time collaboration which created the work we have been reporting on in this review. We thank Antoine Van Proeyen and Edoardo Lauria for making available to us the draft of a set of Lecture Notes on 'Supergravity and its matter couplings: An introduction to $\mathcal{N} = 2$ in $D = 4, 5, 6$.' The work of GLC was supported by FCT/Portugal through UID/MAT/04459/2019. We thank the referee for helpful comments on the first version of this article.

A. Mathematics background

A.1. Manifolds, group actions, submanifolds, immersions and embeddings

In this article, manifolds M are understood to be smooth, Hausdorff and second countable. The Hausdorff separation property requires that any two points on M can be separated by non-intersecting open neighbourhoods. The second countability property requires that the topology (set of open subsets) is generated by a countable collection of open subsets.

The (left) action

$$G \times M \to M \ , \quad (g, x) \mapsto g \cdot x \tag{A.1}$$

of a group G on a manifold M is called

- *transitive*, if any two $x, y \in M$ are related by the action of G,
- *effective (faithful)*, if every $g \in G$ acts non-trivially on M,
- *free*, if all group elements different from the identity act on M without fixed points,
- *principal (regular, simply transitively)*, if G acts both freely and transitively.
- *proper*, if G is a topological group and $G \times M \to M \times M, (g, x) \mapsto (g \cdot x, x)$ is a proper map in the topological sense, that is, pre-images of compact sets are compact.

Since the orbits of G on M need not all have the same dimension, the *space of orbits* M/G is in general not a manifold. Moreover, even if M/G is a smooth manifold and M is Hausdorff, it can happen that M/G is not Hausdorff. A sufficient condition for M/G to be Hausdorff is that the action of G is proper, which is satisfied in particular for compact groups G . If the action of G is both free and proper, then $M \to M/G$ is a *G-principal bundle*, see A.2. Since the group actions we are interested in involve non-compact groups, we will impose that quotients are Hausdorff as an explicit condition. Actions of Lie groups on manifolds can be described using generating vector fields, see A.8.

The rank of a smooth map $F : M \to N$ between manifolds M, N is the rank of the induced linear map $F_* : T_pM \to T_{F(p)}N$ between tangent space. A smooth map F is called an *immersion (submersion)* if F_* is injective (surjective) at every point, that is if $\text{rank}(F) = \dim M$ ($\text{rank}(F) = \dim N$). A smooth *embedding* is an immersion that is also a topological embedding, that is, a homeomorphism $F : M \to F(M) \subset N$, where $F(M)$ carries the topology induced by N through restriction. Embedded submanifolds are precisely the images of smooth embeddings. An immersed submanifold $S \subset N$ is a subset which is a manifold such that $\iota : S \to N$ is an injective immersion. Immersed submanifold are precisely the images of injective immersions.

Note that the image of an immersion need not be a submanifold, since immersions are not required to be invertible. Thus they can have self-intersection points, for example. Moreover, just requiring an immersion to be invertible does not make it an embedding, because the topology of the image need not agree with the submanifold topology induced by N. However, locally an immersion is an embedding, and if one is interested in local problems one can choose the domain of an immersion small enough, so that it becomes an embedding. This is used frequently in the main part of this review.

As an example consider a smooth immersion which maps the real line onto a 'figure eight' shaped figure in $\mathbb{R}^2$, such that points $x \in \frac{1}{2}\mathbb{Z}$ on the line are mapped to the self-intersection point of the image. Now restrict to an open interval $a < x < b$, equipped with the subspace topology induced by $\mathbb{R}$. For $a < 0, b > 1$ the self-intersection point appears at least twice as an image, and the immersion is not invertible. For $a = 0, b = 1$, the immersion is invertible, but not a topological embedding: if we take a Cauchy sequence accumulating at, say, $a = 0$, this does not converge to a point in the interval, but the image of this sequence will converge to the self-intersection point in the topology induced by $\mathbb{R}^2$. For $0 < a < b < 1$ the topology induced by $\mathbb{R}^2$ is the standard topology of an open one-dimensional interval, and the immersion becomes an embedding.

For further reading we refer to [215], on which this section is partly based.

A.2. Fibre bundles and sections

The material in A.2 – A.6 is mostly standard. Our presentation is based on various sources, including [216, 217, 218].

A smooth *fibre bundle*

$$F \to E \xrightarrow{\pi} M \tag{A.2}$$

is a smooth manifold E which locally looks like the product $M \times F$ of two smooth manifolds, the base M and the fibre F. More precisely, there is a smooth surjective map $\pi \ : E \to M$ such that for all $x \in M$ there exists a neighbourhood U such that $\pi^{-1}(U)$ is diffeomorphic to $U \times F$. Given an open cover $\{U_{(i)} | i \in I\}$ of M a fibre bundle can be described in terms of an atlas with charts $(U_{(i)}, \varphi_{(i)})$ that are glued together consistently by transitions functions

$$\phi_{(ij)} = \varphi_{(i)} \varphi_{(j)}^{-1} \ : U_{(ij)} \times F \to U_{(ij)} \times F \tag{A.3}$$

on overlaps $U_{(ij)} = U_{(i)} \cap U_{(j)}$. The inverse image $F_x = \pi^{-1}(x) \cong F$ of x is called the fibre over $x \in M$. Most of the fibre bundles relevant for us are *vector bundles*, where F is a vector space. Particular cases are the tangent bundle TM, the cotangent bundle T^*M, and tensor bundles

$$TM \otimes \cdots \otimes TM \otimes T^*M \otimes \cdots \otimes T^*M \ . \tag{A.4}$$

A smooth *section* of a fibre bundle is a smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{Id}_M$. In addition to global sections, that is sections defined over all of M, one can consider local sections over domains $U \subset M$. Local sections need not to extend to global sections. By considering all open subsets $U \subset M$ together with all sections of E over subsets U, one obtains the *sheaf of sections* of E. In our applications it will be clear from context whether sections of vector bundles (vector fields, tensor fields) are required to exist locally or globally.

An *affine bundle* modelled on a vector bundle $V \to M$ is a fibre bundle $A \to M$ such that:

- The fibres A_p of A over $p \in M$ are affine spaces over the vector spaces V_p, which are the fibres of the vector bundle V.

- The transition functions of a bundle atlas of A are affine isomorphisms whose linear parts are the transition functions of $V \to M$.

Another important class of bundles are *principal bundles*. For a Lie group G a G-principal bundle P over a manifold M is a manifold P equipped with a principal action of G. Since the G-action on P is free and transitive, each orbit of G on P can be identified with G upon choosing one point on the orbit, which is identified with the unit element. Thus orbits are loosely speaking copies of G where we forget where the unit element is located (similar to passing from a vector space to the associated affine space, or from vector bundles to affine bundles). The base manifold M of the fibre bundle $P \to M$ is the space of orbits, $M = P/G$. A principal bundle is trivial, that is $P = M \times G$ is a product, if and only if P admits a global section (which identifies, in each fibre, which point corresponds to the unit element of the group). By picking a representation $\rho : G \to V$ of G on a vector space V one can associate to the principal bundle G a vector bundle with fibre V and G-action defined by ρ. One then says that the vector bundle is associated to the principal bundle. A $U(1)$ principal bundle is also called a circle bundle. By choosing the representation of $U(1)$ by the action of $SO(2)$ on the complex plane, one obtains an associated complex line bundle, that is a vector bundle with fibre $\mathbb{C}$. We refer to A.12 for more material on complex vector bundles.

A.3. Vector fields and differential forms

A.3.1. Vector fields and frames

Let M be a smooth manifold. Vector fields are denoted $X, Y, \ldots \in \mathfrak{X}(M) = \Gamma(TM)$.[72] The local expansion of a vector field with respect to coordinates x^m is

$$X = X^m \frac{\partial}{\partial x^m} = X^m \partial_m \ . \tag{A.5}$$

Vector fields operate on functions as first order differential operators (directional derivatives):

$$X(f) = X^m \partial_m f \ . \tag{A.6}$$

The *Lie bracket* $[X, Y]$ of two vector fields

$$[X, Y](f) = XY(f) - YX(f) = (X^m(\partial_m Y^n) - Y^m(\partial_m X^n))\, \partial_n f \tag{A.7}$$

[72] Where convenient or required by consistency with the physics literature, we will also use symbols, like $\xi, \eta, \ldots$, or $t, s, \ldots$ for vector fields.

is again a first order differential operator. The Lie bracket gives the space of vector fields the structure of a Lie algebra.

Instead of a *coordinate frame* ∂_m, we can more generally expand a vector field with respect to a local *frame* e_m, that is a set of vector fields which form a basis of T_xM for all $x \in U \subset M$, where U is an open neighbourhood,

$$X = X^m e_m \ . \tag{A.8}$$

The local sections e_m are generators for the Lie algebra of vector fields, $[e_m, e_n] = c^p_{mn} e_p$. A frame $\{e_m\}$ is locally a coordinate frame if and only if $c^p_{mn} = 0$ [216].

The expression of a Lie bracket with respect to frame is:

$$[X, Y] = (X^m e_m(Y^p) - Y^m e_m(X^p) + X^m Y^n c^p_{mn})\, e_p \ . \tag{A.9}$$

A.3.2. Differential forms, dual frames, exterior derivative

Given a frame $\{e_m\}$, the dual *co-frame* $\{e^m\}$, which forms a basis for the one-forms $\omega \in \Omega^1(M) = \Gamma(T^*M)$ is defined by $e^m(e_n) = \delta^m_n$. In the following 'choosing a frame' (or co-frame) always means that we choose a dual pair $\{e^m, e_n\}$. Given a coordinate system, the coordinate differentials dx^m form the frame dual to the coordinate vector fields ∂_m. A co-frame is locally a coordinate co-frame if $de^m = 0$. The expansion of a one-form in a coordinate co-frame is

$$\omega = \omega_m dx^m \ . \tag{A.10}$$

The *wedge product* of one-forms is defined by

$$\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha \ . \tag{A.11}$$

Our convention for the components of a p-form $\omega \in \Omega^p(M) = \Gamma(\Lambda^p T^*M)$ is

$$\omega = \frac{1}{p!} \omega_{m_1 \cdots m_p} dx^{m_1} \wedge \cdots \wedge dx^{m_p} \ . \tag{A.12}$$

Therefore the evaluation of a p-form on vector fields gives:

$$\omega(X_1, \ldots, X_p) = \omega_{m_1 \cdots m_p} X_1^{m_1} \cdots X_p^{m_p} \ . \tag{A.13}$$

A.3.3. Exterior derivative and dual Lie algebra structure of co-frames

The coordinate expression for the *exterior derivative* $d\omega \in \Omega^{p+1}(M)$ of a p-form ω is:

$$d\omega = \frac{1}{p!} \partial_m \omega_{m_1 \cdots m_p} dx^m \wedge dx^{m_1} \wedge \cdots dx^{m_p} \Leftrightarrow (d\omega)_{m m_1 \cdots m_p} = (p+1) \partial_{[m} \omega_{m_1 \cdots m_p]} \ . \tag{A.14}$$

Note that we distinguish by brackets between the component $(d\omega)_{mm_1\cdots m_p}$ of the form $d\omega$ (a notation used by physicists) and the exterior derivative $(d\omega_{m_1\cdots m_p})$ of the component $\omega_{m_1\cdots m_p}$ regarded as a function (a notation used by mathematicians),

$$d\omega_{m_1\cdots m_p} = \partial_m \omega_{m_1\cdots m_p} dx^m \ . \tag{A.15}$$

Our convention for the *antisymmetrization symbol* $[\cdots]$ is such that it includes a weight factor $1/p!$:

$$T_{[m_1\cdots m_p]} = \frac{1}{p!} \sum_{\sigma\in S_p} (-1)^{\mathrm{sign}(\sigma)} T_{\sigma(m_1)\cdots\sigma(m_p)} \ , \tag{A.16}$$

where S_p is the permutation group of p objects.

The generators e^m of a co-frame satisfy the *dual Lie algebra*, $de^m = -\frac{1}{2} c^m_{np} e^n \wedge e^p$.

The exterior derivative is a natural map $\Omega^p(M) \to \Omega^{p+1}(M)$ in the sense that it commutes with pullbacks of smooth maps $f : M \to N$, that is

$$f^* d\omega = d(f^*\omega) \ . \tag{A.17}$$

A.3.4. Interior product and contraction

The *interior product* ι_X between a vector field $X \in \mathfrak{X}(M)$ and a p-form $\omega \in \Omega^p(M)$ is defined by substituting X into the first argument of the form, that is by contraction over the first index:

$$(\iota_X\omega)(X_1, \ldots, X_{p-1}) = \omega(X, X_1, \ldots, X_{p-1}) \Leftrightarrow (\iota_X\omega)_{m_1\cdots m_{p-1}} = X^m \omega_{mm_1\cdots m_{p-1}} \ . \tag{A.18}$$

We will often write $\omega(X, \cdot) := \iota_X\omega(\cdot)$.

A.3.5. Lie derivatives

The *Lie derivative* $L_X T$ of a tensor field $T \in \mathcal{T}^p_q(M) := \Gamma(\bigotimes^p TM \otimes \bigotimes^q T^*M)$ with respect to a vector field X is a directional derivative which is defined using the flow of the vector field X. The Lie derivative is additive and satisfies the Leibniz rule,

$$L_X(T+S) = L_X T + L_X S \ , \quad L_X(T\otimes S) = L_X T \otimes S + T \otimes L_X S \ , \tag{A.19}$$

where T, S are tensor fields. To compute the components $(L_X T)^{m_1\cdots m_p}{}_{n_1\cdots n_q}$ of the Lie derivative $L_X T$ of a tensor field T it is therefore sufficient to know the action of L_X on functions f, coordinate vector fields ∂_p and coordinate differentials dx^p:

$$L_X f = X^m \partial_m f \ , \quad L_X \partial_p = -(\partial_p X^n)\partial_n \ , \quad L_X dx^p = (\partial_n X^p) dx^n \ . \qquad \text{(A.20)}$$

For vector fields Y and one-forms ω one obtains:

$$\begin{aligned} L_X Y &= [X,Y] = (X^m \partial_m Y^n - Y^m \partial_m X^n)\partial_n \ , \\ L_X \omega &= i_X d\omega + d(i_X \omega) = (X^m \partial_m \omega_n + \omega_m \partial_n X^m) dx^n \ . \end{aligned}$$

The second formula remains valid for p-forms, and is known as Cartan's magic formula

$$L_X \omega = i_X d\omega + d(i_X \omega) \ , \quad X \in \mathfrak{X}(M) \ , \quad \omega \in \Omega^p(M) \ . \qquad \text{(A.21)}$$

For computations it is useful to note that

$$L_X f = X(f) = df(X) \ . \qquad \text{(A.22)}$$

A.4. Pseudo-Riemannian manifolds

A *pseudo-Riemannian* manifold is a manifold equipped with a symmetric, non-degenerate rank two co-tensor field, called the metric. Pseudo-Riemannian manifolds are also referred to as *semi-Riemannian* manifolds.

Our convention for the *symmetrized tensor product* of one-forms is

$$\alpha\beta = \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha) \ . \qquad \text{(A.23)}$$

Therefore the local expression for the metric is

$$g = g_{mn} dx^m dx^n = \frac{1}{2} g_{mn} (dx^m \otimes dx^n + dx^n \otimes dx^m) \ . \qquad \text{(A.24)}$$

The metric provides a natural isomorphism between vector fields and one forms. We use the 'musical' notation:

$$X = X^m \partial_m \Rightarrow X^\flat = X_m dx^m \ , \quad X_m = g_{mn} X^n \ , \qquad \text{(A.25)}$$

$$\omega = \omega_m dx^m \Rightarrow \omega^\sharp = \omega^m \partial_m \ , \quad \omega^m = g^{mn} \omega_n \ , \qquad \text{(A.26)}$$

where g^{mn} are the components of the matrix inverse of g_{mn}.

We do not require that the metric is positive definite, and consider general signatures (t,s), where t is the number of time-like and s the number of space-like dimensions. Since we adopt a 'mostly plus convention', t is the number of negative eigenvalues, and s the number of positive eigenvalues of the matrix g_{mn}. For completeness we define that a Riemannian manifold is a pseudo-Riemannian manifold with definite signature.

A.5. Connections

A.5.1. Connections on the tangent bundle

A *connection* ∇ on TM (also called a connection on M, or an affine or linear connection on TM) is a bilinear map[73]

$$\nabla\ : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)\ : (X,Y) \mapsto \nabla_X Y\ , \tag{A.27}$$

which satisfies

$$\nabla_{fX} Y = f\nabla_X Y\ , \quad \nabla_X(fY) = X(f)Y + f\nabla_X Y\ , \tag{A.28}$$

for all $f \in C^\infty(M)$. The *covariant derivative*

$$\nabla_X\ : \mathfrak{X}(M) \to \mathfrak{X}(M)\ :\ \ Y \mapsto \nabla_X Y \tag{A.29}$$

is extended to general tensor fields,

$$\nabla_X\ : \mathcal{T}^p_q(M) \to \mathcal{T}^p_q(M) \tag{A.30}$$

by imposing linearity and the Leibnitz rule in $\mathcal{T}^p_q(M)$ and $C^\infty(M)$-linearity in X.

We remark that in the literature the expressions 'covariant derivative' and 'connection' are used variably for ∇ and ∇_X. If one needs to distinguish ∇ from ∇_X, then the first is called the absolute covariant derivative and the second the directional covariant derivative.

The *connection coefficients* γ^p_{mn} and *connection one-form* $\omega^p_n = \gamma^p_{mn} e^m$ with respect to a frame are defined by

$$\nabla_{e_m} e_n = \gamma^p_{mn} e_p \tag{A.31}$$

or in terms of the dual frame

$$\nabla_{e_p} e^m = -\gamma^m_{pn} e^n\ . \tag{A.32}$$

If the frame e_m is a coordinate frame, the connection coefficients are denoted Γ^p_{mn}:

$$\nabla_{\partial_m} \partial_n = \Gamma^p_{mn} \partial_p\ . \tag{A.33}$$

[73] Alternatively, one can view ∇ as a map $\mathfrak{X}(M) \to \Omega^1(M) \otimes \mathfrak{X}(M)$ which assigns to a vector field X the vector-valued one-form ∇X.

The *torsion* and *curvature* of a connection are the following multilinear maps

$$T^{\nabla}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \,, \tag{A.34}$$

$$R^{\nabla}_{X,Y} Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \,, \tag{A.35}$$

where $X, Y, Z \in \mathfrak{X}(M)$. The torsion and curvature tensor are defined by

$$T(\alpha, X, Y) = \alpha(T^{\nabla}(X,Y)) \,, \tag{A.36}$$

$$R(\alpha, Z, X, Y) = \alpha(R^{\nabla}_{X,Y} Z) \tag{A.37}$$

where $\alpha \in \Omega^1(M)$. The components with respect to a frame are:

$$T^m{}_{np} = T(e^m, e_n, e_p) = \gamma^m_{np} - \gamma^m_{pn} - c^m_{np} \,, \tag{A.38}$$

$$R^m{}_{npq} = R(e^m, e_n, e_p, e_q) = e_p(\gamma^m_{qn}) - e_q(\gamma^m_{pn}) \tag{A.39}$$

$$+\gamma^m_{pa}\gamma^a_{qn} - \gamma^m_{qa}\gamma^a_{pn} - c^a_{pq}\gamma^m_{an} \,. \tag{A.40}$$

For a coordinate frame these expression reduce to

$$T^m{}_{pq} = \Gamma^m_{pq} - \Gamma^m_{qp} \,, \tag{A.41}$$

$$R^m{}_{npq} = \partial_p \Gamma^m_{qn} - \partial_q \Gamma^m_{pn} + \Gamma^m_{pa}\Gamma^a_{qn} - \Gamma^m_{qa}\Gamma^a_{pn} \,.$$

A.5.2. The Levi-Civita Connection

The *Levi-Connection* D on a Riemannian manifold (M, g) is the unique connection on the tangent bundle TM which is both metric (compatible) and torsion free:

$$D_X g = 0 \,, \quad T^D(X,Y) = 0 \,, \quad \forall X, Y \in \mathfrak{X}(M) \,. \tag{A.42}$$

Our conventions for the Levi-Civita connection and the Christoffel symbols are summarized in B.2.

A.5.3. Flat, torsion-free connections and affine manifolds

If a connection is flat, $R^{\nabla} = 0$, it is possible to choose a frame consisting of parallel vector fields [216], i.e.

$$\nabla_{e_m} e_n = 0 \Rightarrow \gamma^p_{mn} = 0 \,. \tag{A.43}$$

If the connection is in addition torsion-free, then this parallel frame is a coordinate frame, since

$$\left. \begin{array}{l} T(e^m, e_n, e_p) = 0 \\ \gamma^m_{np} = 0 \end{array} \right\} \Rightarrow c^m_{np} = 0 \,. \tag{A.44}$$

Alternatively, we note that the expression for the torsion tensor with respect to a frame is

$$T = e_m \otimes de^m + e_n \otimes \omega^n_m \otimes e^m \ . \tag{A.45}$$

If ∇ is flat, we can choose a basis of parallel sections, so that $\omega^n_m = 0$, and then

$$\left.\begin{array}{l} T = 0 \\ \\ \omega^n_m = 0 \end{array}\right\} \Rightarrow de^m = 0 \Rightarrow e^m = dq^m \ , \tag{A.46}$$

where q^m are local functions that provide coordinates underlying the parallel frame. Such coordinates are called ∇-affine coordinates and are unique up to affine transformations. The condition on a coordinate system to be affine is $\nabla dq^m = 0$, that is, that the coordinates define a parallel co-frame.

If a manifold admits a flat, torsion-free connection, it can be covered with ∇-affine coordinate charts which are related by affine transition functions. Such an atlas is called an *affine structure*. A manifold M equipped with a flat, torsion-free connection ∇ is called an *affine manifold*.

A.5.4. Connections on vector bundles

Let $E \to M$ be a vector bundle over a manifold M. A *connection* on E is a map

$$\nabla \ : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E) \ , \quad (X, s) \mapsto \nabla_X s \ , \tag{A.47}$$

which is linear and satisfies the product rule with respect to sections $s \in \Gamma(E)$, while being $C^\infty(M)$-linear with respect to vector fields $X \in \mathfrak{X}(M)$.

Let $E \to M$ be a vector bundle with connection ∇, and let D be a linear connection on M. If $s \in \Gamma(E)$ is a section of E, then ∇s is a section of $T^*M \otimes E$. One can then use the connection induced by D and ∇ to define the *second covariant derivative* $\nabla^2 s$, which is a section of $T^*M \otimes T^*M \otimes E$:

$$\nabla^2 s(X, Y) = \nabla_X(\nabla_Y s) - \nabla_{D_X Y} s \ . \tag{A.48}$$

Alternative notations are $\nabla^2_{X,Y} s$ or $(\nabla^2 s)_{X,Y}$.

If $E = TM$, denoting the connection induced by D on tensor bundles again by D, we obtain the following formula for the second covariant derivative of a vector field:

$$D^2_{X,Y} Z = D_X(D_Y Z) - D_{D_X Y} Z \ . \tag{A.49}$$

In local coordinates, the relevant expression are, using the notation $D_m = D_{\partial_m}$:

$$\begin{aligned}(D^2_{X,Y}Z)^p &= X^m Y^n D_m D_n Z^p \ , &\text{(A.50)}\\ (D_X(D_Y Z))^p &= X^m D_m(Y^n D_n Z^p) \ , &\text{(A.51)}\\ (D_{D_X Y}Z)^p &= X^m (D_m Y^n) D_n Z^p \ . &\text{(A.52)}\end{aligned}$$

We can define the *Hessian* Ddf of a function f with respect to the linear connection D:

$$\begin{aligned}Ddf(X,Y) &= XY(f) - (D_X Y)f = X^m D_m(Y^n \partial_n f) - X^m (D_m Y^n)\partial_n f\\ &= X^m Y^n D_m \partial_n f \ . \qquad \text{(A.53)}\end{aligned}$$

If the connection D is torsion-free,

$$D_X Y - D_Y X = [X,Y] \ , \qquad \text{(A.54)}$$

then the Hessian is symmetric, and the curvature of D can be written

$$R^D_{X,Y}Z = [D_X, D_Y]Z - D_{[X,Y]}Z = D^2_{X,Y}Z - D^2_{Y,X}Z \ . \qquad \text{(A.55)}$$

For the bundle $\Omega^p(M,E) = \Gamma(\Lambda^p T^*M \otimes E)$ of vector-valued p-forms, one defines the *exterior covariant derivative*

$$d_\nabla \ : \Omega^p(M,E) \to \Omega^{p+1}(M,E) \qquad \text{(A.56)}$$

by its action on sections of E. For a basis $\{s_a\}$ of sections one sets

$$d_\nabla s_a := \nabla s_a = \omega_a^b \otimes s_b \ , \qquad \text{(A.57)}$$

where ω_a^b is the connection one-form of ∇. The exterior covariant derivative of a general section $s = f^a s_a \in \Omega^0(M,E) = \Gamma(E)$ is determined by the product rule

$$d_\nabla s = df^a \otimes s_a + f^a \omega_a^b \otimes s_b \ . \qquad \text{(A.58)}$$

The extension of d_∇ to forms of degree $p > 0$ is uniquely determined by linearity and the product rule:

$$d_\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^{\deg(\alpha)} \alpha \wedge d_\nabla s \ , \ \ \alpha \in \Omega^p(M) \ . \qquad \text{(A.59)}$$

The exterior covariant derivative of a vector valued p-form $\rho \in \Omega^p(M,E)$ can be expressed in terms of the covariant derivative by

$$\begin{aligned}(d_\nabla \rho)(X_0, \ldots, X_p) &= \sum_{l=0}^{p} (-1)^l \nabla_{X_l}(\rho(\ldots, \hat{X}_l, \ldots)) \qquad \text{(A.60)}\\ &\quad + \sum_{i<j} (-1)^{i+j} \rho([X_i, X_j], \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots) \ ,\end{aligned}$$

where $X_0, \ldots, X_p$ are vector fields, and where $\hat{X}$ indicates that the vector field $\hat{X}$ is omitted as an argument. The second exterior derivative of a section $s \in \Gamma(E)$ is related to the curvature of the connection ∇ by

$$d_\nabla^2 s(X,Y) = R^\nabla_{X,Y} \ , \quad \forall X, Y \in \mathfrak{X}(M) \ , s \in \Gamma(E) \ . \tag{A.61}$$

Thus d_∇ satisfies $d_\nabla^2 = 0$ if and only if the connection is flat. If this is the case, a version of the Poincaré lemma holds which allows to write a d_∇-closed vector-valued p-form locally as the d_∇ derivative of a vector valued $(p-1)$-form. In general the Bianchi identity $d_\nabla R_\nabla = 0$ for the curvature implies that $d_\nabla^3 = 0$. We refer to [219], [220] for more details on the exterior covariant derivative.

In the case when $E = TM$, ∇ is a connection on TM and we can define its torsion. It is useful to note that the torsion tensor can be expressed as

$$T^\nabla = d_\nabla \mathrm{Id} \ , \tag{A.62}$$

where $\mathrm{Id} = e^m \otimes e_m \in \Gamma(\mathrm{End}(TM)) \simeq \Gamma(T^*M \otimes TM) \simeq \Omega^1(M, TM)$ is the identity endomorphism on TM, regarded as a vector-valued one-form. Equation (A.62) can be verified using that

$$d_\nabla \left(e^a \otimes e_a\right) = de^a \otimes e_a + e^a \wedge \omega_a^b \otimes e_b \tag{A.63}$$

and evaluating both sides of the equation on vector fields, that is by showing that $T^\nabla(X,Y) = (d_\nabla \mathrm{Id})(X,Y)$. Instead of general vector fields X, Y, one can choose $X = e_a$, $Y = e_b$ with arbitrary a, b, thus comparing the components with respect to a frame.

A.6. Pull-back bundles

If $f : M \to N$ is a smooth map between smooth manifolds M, N, then one can pull back any vector bundle $\pi_E : E \to N$ to a vector bundle $f^*E \to M$ over M, called the *pull-back bundle of M by f*, which is constructed as follows:

- The total space of f^*E is

$$f^*E := \{(m, e) \in M \times E | f(m) = \pi_E(e)\} \tag{A.64}$$

- The bundle projection is the restriction of the canonical projection $\pi_1 : M \times E \to M$ to f^*E:

$$\pi_{f^*E}(m, e) = m \ . \tag{A.65}$$

By construction the fibres of f^*E are mapped to fibres of E, more precisely $(f^*E)_m \cong E_{f(m)}$ for all $m \in M$. By restricting the canonical projection $\pi_2 : M \times E \to E$ to f^*E we obtain the so-called covering morphism

$$F : f^*E \to E \ : (m,e) \mapsto F(m,e) = e \ , \tag{A.66}$$

which completes the commutative diagram

$$\begin{array}{ccc} f^*E & \xrightarrow{F} & E \\ \big\downarrow{\scriptstyle \pi_{f^*E}} & & \big\downarrow{\scriptstyle \pi_E} \\ M & \xrightarrow{f} & N \end{array} \tag{A.67}$$

The pull-back $f^*s \in \Gamma(f^*E)$ of a section $s \in \Gamma(E)$ is defined by

$$(f^*s)(m) = s(f(m)) \ . \tag{A.68}$$

We can also pull back a connection D on E to a connection f^*D on f^*E . This pull-back connection is defined by

$$(f^*D)_X f^*s := D_{df\,X}\, s \ , \tag{A.69}$$

for all vector fields X on M.

A.7. The Frobenius theorem, hypersurfaces, and hypersurface orthogonal vector fields

This section is partly based on [216] and on [221], Appendix B.

A p-dimensional *distribution* $V = \cup_{x\in M} V_x$ on the tangent bundle TM of a smooth manifold is a map

$$M \ni x \mapsto V_x \subset T_xM \ , \tag{A.70}$$

where V_x is a p-dimensional subspace of T_xM. A distribution is called smooth if it depends smoothly on p. This means that for each $x \in M$ there exists a neighbourhood U and p linearly independent smooth vector fields defined on U which span V_x for $x \in U$. One may then ask whether there exist on M smooth p-dimensional submanifolds which are tangent to V. Such submanifolds are called the *integral manifolds* of the distribution, and provide a *foliation* of M, that is a disjoint decomposition into submanifolds, called the leafs of the foliation.

According to the Frobenius theorem a distribution is integrable if and only if it is involutive, that is if the Lie bracket of any two tangent vector fields is again a tangent vector field, for all points $x \in M$. Distributions which possess integral manifolds are called (Frobenius-)integrable.

The Frobenius theorem can be given a dual formulation in terms of differential forms. Given a distribution $V \subset TM$ one can consider the dual distribution $V^* \subset T^*M$ on the cotangent bundle defined by

$$\omega \in V^* \Leftrightarrow \omega(X) = 0 \ , \ \ \forall X \in V \ . \tag{A.71}$$

For differential forms, the integrability condition is

$$d\omega = \sum_i \alpha_{(i)} \wedge \beta_{(i)} \ , \tag{A.72}$$

where $\alpha_{(i)} \in V^*$, and where $\beta_{(i)} \in \Omega^1(M)$.

A vector field ξ is called *hypersurface orthogonal* if it is orthogonal to a foliation of M by hypersurfaces. This is equivalent to the statement that the distribution $V = \langle \xi \rangle^{\perp}$ is Frobenius integrable. The dual distribution V^* on the cotangent bundle is spanned by the one-form $\xi^\flat$, that is $V^* = (\langle \xi \rangle^{\perp})^* = \langle \xi^\flat \rangle$, because $\xi^\flat(\cdot) = g(\xi, \cdot)$. Specializing the dual version of the Frobenius theorem to the case of a hypersurface distribution we obtain

$$d\xi^\flat = \xi^\flat \wedge \beta \ , \tag{A.73}$$

for some one-form β, where we used that the distribution V^* is one-dimensional. This equation is equivalent to

$$\xi^\flat \wedge d\xi^\flat = 0 \Leftrightarrow \xi_{[m} \partial_n \xi_{p]} = 0 \ , \tag{A.74}$$

which is the standard criterion used in the literature for verifying the hypersurface orthogonality of a vector field. Note that due to the antisymmetrization the expression $\xi_{[m} \partial_n \xi_{p]}$ is covariant, since we can replace ∂_n by any torsion free covariant derivative. Also note that the integrability condition is satisfied in particular if the vector field is closed, that is if $d\xi^\flat = 0$.

Foliations by hypersurfaces can be described locally as level sets of a function $F : M \to \mathbb{R}$:

$$M \simeq \cup_{\mathfrak{c} \in \mathbb{R}} \{ x \in M | F(x) = \mathfrak{c} \} \ . \tag{A.75}$$

The standard normal vector field to such a foliation is $n = \text{grad}(F) = (dF)^\sharp$, with components $n^m = g^{mn}\partial_n F$. Tangent vectors t to the foliation are characterized by any of the following relations:

$$g_{mn}n^m t^n = g(n,t) = 0 = dF(t) = t^m \partial_m F \ . \tag{A.76}$$

The most general vector field ξ normal to the foliation can differ from the standard normal n by a function $f : M \to \mathbb{R}$, that is $\xi = f(dF)^\sharp$. Such a vector field clearly satisfies the integrability condition we derived earlier, since $\xi^\flat = fdF$. The standard normal vector field n is distinguished by being 'closed', more precisely by $dn^\flat = 0$. This is a stronger condition than Frobenius integrability.

A.8. Integral curves, one-parameter groups and quotient manifolds

A one-dimensional distribution on the tangent bundle is always integrable, because the integrability condition becomes trivial. Such a distribution defines a smooth vector field X, and its integrability corresponds to the existence of a family of so-called integral curves, whose tangent vectors are given by X. The integral curve $C_{x_0} : t \mapsto x(t)$ through a given point $p \in M$ with coordinate x_0 is found by solving the initial value problem

$$\frac{dx}{dt} = X(t) \ , \quad t \in I \subset \mathbb{R} \ , \quad x(0) = x_0 \ . \tag{A.77}$$

The flow of the vector field X is defined by

$$\sigma : I \times M \to M \ , \quad (t,x) \mapsto \sigma(t,x) = x(t) \tag{A.78}$$

where $x(t) = \sigma_x(t)$ is the integral curve of X with initial condition $x(0) = x_0$.

Further defining

$$\sigma_t \ : \quad M \to M \ , \quad x(0) = \sigma_0(x) = \sigma(x,0) \mapsto \sigma(x,t) = \sigma_t(x) = x(t) \tag{A.79}$$

we see that σ_t moves the points of M along the integral curves of X. Since $\sigma_{s+t} = \sigma_t \circ \sigma_s$ and $\sigma_0 = \text{Id}$, these transformations form a group, called the one-parameter transformation group generated by X. If this action is a globally defined group action of $G = U(1)$ or $G = \mathbb{R}$ on M, then the integral curves are called the orbits of G, and denoted $\langle X \rangle$. As already discussed in A.1, the space of orbits, denoted $M/\langle X \rangle = M/G$, need not be a manifold, in particular it need not satisfy the Hausdorff separation axiom. However, in many cases, including those relevant for this review, the quotient is a (Hausdorff) manifold, and various structures, such as the metric, complex or symplectic structure project to the quotient manifolds. Quotient manifolds can also be defined with respect to the action of higher-dimensional groups. Examples relevant for this review are the action of the group $\mathbb{C}^*$ on CASK manifolds and the action of the group $\mathbb{H}^*$ on hyper-Kähler manifolds.

A.9. Metric cones and metric products

In this section we elaborate on some standard definitions, and in particular adapt them to the pseudo-Riemannian setting.

If $(\mathcal{H}, h)$ is a pseudo-Riemannian manifold, then the *metric cone* (or *Riemannian cone*) (M, g) over $(\mathcal{H}, h)$ is the manifold $M = \mathbb{R}^{>0} \times \mathcal{H}$ equipped with the metric

$$g = \pm d\rho^2 + \rho^2 h \ . \tag{A.80}$$

We note that $\xi = \partial_\rho$ is a closed homothetic Killing vector field:

$$L_\xi g = 2g \ , \quad d\xi^\flat = 0 \ . \tag{A.81}$$

Since ξ is closed, it is gradient vector field:

$$\xi^\flat = dH \Leftrightarrow \xi = \mathrm{grad} H \tag{A.82}$$

or, in local coordinates x^m on M:

$$\xi_m = \partial_m H \Leftrightarrow \xi^m = g^{mn}\partial_n H \ . \tag{A.83}$$

M is foliated by the level surfaces $H = \mathfrak{c}$ which are orthogonal to ξ, and $\mathcal{H}$ can be identified with the hypersurface $H = 1$.

The two equations (A.82) are the symmetric and anti-symmetric part of

$$D\xi = \mathrm{Id}_{TM} \Leftrightarrow D_m \xi_n = g_{mn} \ , \tag{A.84}$$

where D is the Levi-Civita connection of g. This equation provides a local characterization of a metric cone:

Remark 10. Let (M, g) be a pseudo-Riemannian manifold of dimension $n+1$, equipped with a vector field ξ, which is nowhere isotropic, that is $g(\xi, \xi) \neq 0$ everywhere, and which satisfies

$$D\xi = \mathrm{Id}_{TM} \ . \tag{A.85}$$

Then there exist local coordinates (r, x^i), $i = 1, \ldots, n$ such that metric g takes the form

$$g = \pm dr^2 + r^2 h_{ij} dx^i dx^j \ , \tag{A.86}$$

where h_{ij} only depend on the coordinates x^i.

This is a special case of the standard form of an *n-conical Riemannian metric*, which we derive in section 2.3.

If $(\mathcal{H}_1, h_1)$ and $(\mathcal{H}_2, h_2)$ are two pseudo-Riemannian manifolds, their *metric product* or *Riemannian product* (M, g) is defined by $M = \mathcal{H}_1 \times \mathcal{H}_2$ equipped with the product metric

$$g = h_{\mathcal{H}_1} + h_{\mathcal{H}_2} \; . \tag{A.87}$$

In local coordinates (x^m, y^i) on $\mathcal{H}_1 \times \mathcal{H}_2$, this takes the form

$$g = (h_1)_{mn}(x) dx^m dx^n + (h_2)_{ij}(y) dy^i dy^j \; . \tag{A.88}$$

In applications we encounter product manifolds of the special form

$$M = \mathbb{R} \times \mathcal{H} \cong \mathbb{R}^{>0} \times \mathcal{H} \; , \tag{A.89}$$

for which the metric takes the form

$$g = \pm d\rho^2 + h_{ij} dx^i dx^j = \pm \frac{dr^2}{r^2} + h_{ij} dx^i dx^j \; , \tag{A.90}$$

where the coordinates r, ρ are related by $r = e^\rho$. The vector field $\xi = \partial_\rho = r\partial_r$ is a Killing vector field, $L_\xi g = 0$, which is closed $d\xi^\flat = 0$, and therefore hypersurface orthogonal, and which in addition has constant norm $g(\xi, \xi) = \pm 1$. The manifold M is foliated by hypersurfaces where $\rho =$ const., and all these hypersurfaces are isometric to each other and to $(\mathcal{H}, h)$.

The Killing equation can be combined with the closed-ness condition to

$$D\xi = 0 \; . \tag{A.91}$$

Note that this equation does not by itself imply that a metric g locally takes the form (A.90) of a product. This requires in addition that the norm of ξ is constant, so that surfaces of constant ρ are isometric to each other. The proof that this is sufficient to bring the metric to the form (A.90) is given in section 2.3.

A.10. Affine hyperspheres and centroaffine hypersurfaces

Here we review some facts about *affine hyperspheres* and *centroaffine hypersurfaces*, following [43, 44, 143]. Consider $\mathbb{R}^{m+1}$ equipped with the standard connection ∂ (given by the partial derivative with respect to standard linear coordinates), and the standard volume form vol, which is parallel with respect to ∂. Let M be a connected manifold which is immersed as a hypersurface

$$\varphi : M \to \mathbb{R}^{m+1} . \tag{A.92}$$

Assume that there exists a vector field ξ which is transversal along M. Then $\mathrm{vol}_M = \mathrm{vol}(\xi, \cdots)$ is a volume form on M, and by decomposing

$$\begin{aligned} \partial_X Y &= \nabla_X Y + g(X,Y)\xi , & \text{(A.93)} \\ \partial_X \xi &= SX + \theta(X)\xi , & \text{(A.94)} \end{aligned}$$

where X, Y are tangent to M, one obtains on M: (i) a torsion-free connection ∇, (ii) a symmetric co-tensor g, (iii) an endomorphism field S and (iv) a one-form θ. If g is non-degenerate, it defines a pseudo-Riemannian metric on M. It can be shown that once the orientation of M has been fixed there is a unique choice for ξ, called the *affine normal* such that the induced volume form vol_M of M coincides with the volume form defined by the metric g. If ξ is chosen to be the affine normal, then $\theta = 0$ and S can be expressed in terms of the so-called Blaschke data (g, ∇).

There are two special cases:

1. A hypersurface is called a *parabolic* (or *improper*) *affine hypersphere* if the affine normals are parallel, $\partial\xi = 0$, and thus only intersect 'at ∞.' One can show that
$$\partial\xi = 0 \Leftrightarrow S = 0 \Leftrightarrow \nabla \ \text{ flat} . \tag{A.95}$$
Thus parabolic affine hyperspheres carry a flat torsion-free connection.
2. A hypersurface is called a *proper affine hypersphere* if the lines generated by the affine normals intersect at a point $p \in \mathbb{R}^{m+1}$. For a proper affine hypersphere $S = \lambda \mathrm{Id}$, $\lambda \in \mathbb{R}^*$.

The ASK manifolds of four-dimensional vector multiplets are parabolic affine hyperspheres with additional structure, called *special parabolic hyperspheres*, see section 5.1.3.

The PSR manifolds of five-dimensional vector multiplets coupled to supergravity, which are discussed in section 2.5.2, are, in general, not (proper) affine hyperspheres, but *centroaffine hypersurfaces*. According to section 1.1 of [143] a hypersurface immersion $\varphi : M \to \mathbb{R}^{m+1}$ is called a *centroaffine hypersurface* *immersion* if the position vector field ξ is transversal to the image of M. The equation

$$\partial_X Y = \nabla_X Y + g(X,Y)\xi \ , \tag{A.96}$$

for $X, Y \in \mathfrak{X}(M)$ induces on M a connection ∇, a symmetric tensor field g, and a ∇-parallel volume form $\mathrm{vol}_M = \det(\xi, \ldots)$. The data $(\nabla, g, \mathrm{vol}_M)$ are called the induced centroaffine data on M. The hypersurface M is called *non-degenerate* if g is non-degenerate, *definite* if g is definite, *elliptic* if g is negative definite, and *hyperbolic* if g is positive definite. Every homogeneous function defines a centroaffine hypersurface embedding, and every centroaffine hypersurface immersion is locally generated by a homogeneous function. Centroaffine structures can be characterized intrinsically: a *centroaffine manifold* $(M, \nabla, g, \mathrm{vol}_M)$ is a manifold equipped with a torsion-free connection ∇, a pseudo-Riemannian metric g and a volume form vol_M, subject to three compatibility conditions: (i) the volume form is ∇-parallel, (ii) the cubic form $C := \nabla g$ is completely symmetric, and (iii) the curvature tensor R of ∇ is given by

$$R(X,Y)Z = -(g(Y,Z)X - g(X,Z))Y) \tag{A.97}$$

for $X, Y, Z \in \mathfrak{X}(M)$. By Theorem 1.6 of [143] a centroaffine immersion $\varphi : M \to \mathbb{R}^{m+1}$ induces on M the structure of a centroaffine manifold. Conversely, every connected and simply connected centroaffine manifold can be realized as a centroaffine immersion, which is unique up to $SL(m+1, \mathbb{R})$ transformations. Note that in contradistinction to affine hyperspheres, the position vector field ξ of a centroaffine hypersurfaces is in general not the affine normal of M.[74]

PSR manifolds, which are the scalar manifolds of five-dimensional vector multiplets coupled to supergravity, were discussed in section 2.5.2. We now review how they fit into the theory of centroaffine hypersurfaces, following section 2.1 of [143]. A PSR manifold is a smooth hypersurface $\bar{M} \cong \mathcal{H} \subset \mathbb{R}^{m+1}$, which is realized as the level set $\mathcal{V} = 1$ of a homogeneous cubic polynomial $\mathcal{V}$, such that $\partial^2 \mathcal{V}$ is negative definite on $T\mathcal{H}$. This induces a centroaffine structure $(\nabla, g, \mathrm{vol}_{\bar{M}})$ on $\bar{M}$.

According to definition 2.2 of [143] an *intrinsic projective special real manifold* is a centroaffine manifold $(\bar{M}, \nabla, g, \mathrm{vol}_{\bar{M}})$ with a positive definite metric g

[74]We thank the referee for pointing this out to us.

such that the covariant derivative of the cubic form $C = \nabla g$ is given by

$$(\nabla_X C)(Y, Z, W) = g(X, Y)g(Z, W) + g(X, Z)g(W, Y) + g(X, W)g(Y, Z) \quad \text{(A.98)}$$

for all $X, Y, Z, W \in \mathfrak{X}(\bar{M})$.

Theorem 2.3 of [143] relates the extrinsic and intrinsic definitions of PSR manifolds. The induced centroaffine structure on a PSR manifold gives it the structure of an intrinsic PSR manifold, and any connected and simply connected intrinsic PSR manifold can be realized by an immersion $\varphi : \bar{M} \to \mathbb{R}^{m+1}$ which is unique up to $SL(m+1, \mathbb{R})$ transformations.

A.11. Complex manifolds

An *almost complex manifold* (M, J) is a real manifold M together with an almost complex structure J. An *almost complex structure* J is a section of $\mathrm{End}(TM) \simeq TM \otimes T^*M$, which satisfies $J^2 = -\mathbb{1}_{TM}$. Note that an almost complex manifold is always of even dimension. A *complex manifold* N of complex dimension n is a manifold which is locally biholomorphic to $\mathbb{C}^n$. A complex manifold automatically carries an almost complex structure (with additional properties, see below) which is called its complex structure. In terms of local holomorphic coordinates $z^i = x^i + iy^i$, the complex structure acts on TM as

$$JX_i = Y_i \ , \quad JY_i = -X_i \quad \text{(A.99)}$$

where X_i, Y_i is the coordinate frame

$$X_i = \frac{\partial}{\partial x^i} \ , \quad Y_i = \frac{\partial}{\partial y^i} \ . \quad \text{(A.100)}$$

X_i, Y_i is called a holomorphic frame on (M, J).

As a consequence of the Newlander-Nirenberg theorem, an almost complex manifold (M, J) is a complex manifold if and only if the Nijenhuis tensor (or torsion tensor) associated to J, defined by

$$N_J(X, Y) := 2\left([JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]\right) \ , \quad \text{(A.101)}$$

vanishes. An almost complex structure with vanishing torsion tensor is called an integrable almost complex structure, or simply a complex structure.

For further reading we refer to [115], on which A.11 – A.13 are mostly based.

A.12. Complex vector bundles

A *complex vector bundle* E over a manifold M is a vector bundle whose fibres are complex vector spaces. A one-dimensional complex vector bundle is called a *complex line bundle*. A *Hermitian metric* γ on E is a family of Hermitian scalar products γ_x on the fibres E_x, which varies smoothly with $x \in M$. Our convention for Hermitian forms is that they are complex linear in the first and complex anti-linear in the second argument. A *Hermitian vector bundle* (E, M, γ) is a complex vector bundle (E, M) equipped with a Hermitian metric. A connection D on a Hermitian vector bundle is called *metric compatible*, or *metric*, or *Hermitian* if

$$d\left(\gamma(s,t)\right) = \gamma(Ds,t) + \gamma(s,Dt) \tag{A.102}$$

for all sections s, t.

A *holomorphic vector bundle* E is a complex vector bundle over a complex manifold M such that the projection $\pi : E \to M$ is holomorphic. Every complex manifold comes equipped with a standard holomorphic vector bundle, the tangent bundle TM equipped with the complex structure J. Another canonical complex vector bundle over M is the *complexified tangent bundle* $T_{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C}$, equipped with the complex linear extension of J. The complexified tangent bundle can then be split into the eigen-distributions of J, called the holomorphic and anti-holomorphic tangent bundle,

$$T_{\mathbb{C}}M = T^{(1,0)}M + T^{(0,1)}M \ . \tag{A.103}$$

The maps

$$TM \to T^{(1,0)}M \ : X \mapsto \frac{1}{2}(X - iJX) \ , \quad TM \to T^{(0,1)}M \ : X \mapsto \frac{1}{2}(X + iJX) \ , \tag{A.104}$$

are complex linear and complex anti-linear isomorphisms, respectively, of complex vector bundles. Since TM is a holomorphic vector bundle over M, so is $T^{(1,0)}M$, but the smooth complex vector bundle $T^{(0,1)}M$ is not a holomorphic vector bundle in a natural way.

A *complex vector field* Z is a section of $T_{\mathbb{C}}M$ and can be decomposed into its $(1,0)$ and $(0,1)$ parts

$$Z^{(1,0)} = \frac{1}{2}\left(Z - iJZ\right) \ , \quad Z^{(0,1)} = \frac{1}{2}\left(Z + iJZ\right) \ . \tag{A.105}$$

Given local holomorphic coordinates $z^i = x^i + iy^i$ we can define local complex frames

$$Z_i = \frac{\partial}{\partial z^i} = \frac{1}{2}(X_i - iY_i) \ , \quad \overline{Z}_i = \frac{\partial}{\partial \bar{z}^i} = \frac{1}{2}(X_i + iY_i) \ , \tag{A.106}$$

on $T^{(1,0)}M$ and $T^{(0,1)}M$, where $X_i = \frac{\partial}{\partial x^i}, Y_i = JX_i = \frac{\partial}{\partial y^i}$ is a coordinate frame on TM.

Like the complexified tangent bundle, all associated complex tensor bundles admit decompositions into 'holomorphic' and 'anti-holomorphic' components. For example complex n-forms can be decomposed into (p,q)-forms, $p+q=n$, $\Omega^n(M) = \bigoplus_{p+q=n} \Omega^{p,q}(M)$. The de-Rham differential can be decomposed as

$$d = \partial + \bar{\partial} \ , \quad \partial = d^{(1,0)} \ , \ \ \bar{\partial} = d^{(0,1)} \ . \tag{A.107}$$

If the complex structure is integrable, then

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M) \ , \quad \bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M) \ , \tag{A.108}$$

and since $\partial^2 = 0 = \bar{\partial}^2$, the de-Rham cohomology admits a refinement called Dolbeault cohomology:

$$H^n(M) = \sum_{n=p+q} H^{p,q}_{\bar{\partial}}(M) \ . \tag{A.109}$$

A connection D on a holomorphic vector bundle is called a *holomorphic connection* if it is compatible with the holomorphic structure, that is if $\pi^{0,1}Ds = \overline{\partial}s = 0$ for all holomorphic sections s, where $\pi^{0,1}$ is the projection onto the anti-holomorphic co-tangent bundle, and where $\overline{\partial} = \pi^{0,1}d$ is the standard anti-holomorphic partial derivative, i.e. the anti-holomorphic projection of the exterior derivative d. Equivalently, the $(0,1)$-part of the connection one-form vanishes, $\omega^{0,1} = \pi^{0,1}\omega = 0$. Equivalently, for holomorphic sections s the covariant derivative along a complex vector field of type $(0,1)$ vanishes, $D_{\bar{X}}s = 0$ for all $X \in \Gamma(T^{(1,0)}M)$ and $s \in \Gamma_{\text{holom}}(E)$.

On a holomorphic Hermitian vector bundle there is a unique connection, called the *Chern connection*, which is simultaneously Hermitian and holomorphic. As an example consider the trivial holomorphic Hermitian vector bundle $\mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n$, where the Hermitian metric γ is defined by choosing a Hermitian inner product on $\mathbb{C}^m$. This vector bundle carries a canonical flat connection d which is defined by the standard partial derivative, that is by declaring that any frame defined by a basis (e_i) of $\mathbb{C}^m$ is parallel, $d_X e_i = 0$ for all complex vector fields X on $\mathbb{C}^n$. The covariant derivative $d_X v$ of a section $v(P) = v^i(P)e_i$, $P \in M$ along a complex vector field $X = X^a \partial_a + X^{\bar{a}} \partial_{\bar{a}} \in \Gamma(T_{\mathbb{C}}\mathbb{C}^n)$ is

$$d_X v = X(v^i)e_i = (X^a \partial_a v^i + X^{\bar{a}} \partial_{\bar{a}} v^i)e_i = \partial_X v + \overline{\partial}_X v \ . \tag{A.110}$$

The connection d is manifestly holomorphic, and it is also Hermitian since

$$d_X \gamma(v, w) = X\gamma(v, w) = \gamma(d_X v, w) + \gamma(v, d_{\bar{X}} w) \ . \tag{A.111}$$

A.13. Hermitian manifolds

An *(almost) Hermitian manifold* (M, J, g) is an (almost) complex manifold (M, J) equipped with a J-invariant pseudo-Riemannian metric g,

$$(J^* g)(X, Y) = g(JX, JY) = g(X, Y) \ , \quad \forall X, Y \in \mathfrak{X}(M) \ . \tag{A.112}$$

Note that we allow the Riemannian metric to be indefinite. Such manifolds are often called (almost) pseudo-Hermitian. Also note that the positive and negative eigenvalues of an (almost) Hermitian metric always come in pairs. One therefore says that a pseudo Hermitian metric has *complex signature* (m, n) if the underlying Riemannian metric has real signature $(2m, 2n)$.

The metric g can be extended complex-linearly to the complexified tangent bundle $T_{\mathbb{C}}M$. The resulting complex bilinear form has the following properties, where Z, W are complex vector fields:

$$g(Z, W) = g(W, Z) \ , \tag{A.113}$$

$$g(\bar{Z}, \bar{W}) = \overline{g(Z, W)} \ , \tag{A.114}$$

$$g(Z, W) = 0 \ , \text{if} \quad Z, W \in \Gamma(T^{(1,0)}M) \ , \tag{A.115}$$

$$g(\bar{Z}, Z) > 0 \ , \text{unless} \ \ Z = 0 \ . \tag{A.116}$$

Assume that J is integrable, and let $z^i = x^i + iy^i$ be local complex coordinates, with associated holomorphic frame $Z_i = \frac{1}{2}(X_i - iY_i)$. Then the components of the metric are

$$g_{jk} \ := \ 2g(Z_j, Z_k) = 0 \ , \tag{A.117}$$

$$g_{\bar{j}\bar{k}} \ := \ 2g(\overline{Z}_j, \overline{Z}_k) = 0 \ , \tag{A.118}$$

$$g_{j\bar{k}} \ := \ 2g(Z_j, \overline{Z}_k) = 2g(\overline{Z}_k, Z_j) = g_{\bar{k}j} \ , \tag{A.119}$$

$$\overline{g_{j\bar{k}}} \ = \ g_{\bar{j}k} = g_{k\bar{j}} \ . \tag{A.120}$$

Here we used properties of the complex-linear extension of the metric to $T_{\mathbb{C}}M$, and choose the normalization for later convenience. Note that the coefficients can be arranged as a Hermitian matrix. The metric can be written

$$g = g_{j\bar{k}} dz^j d\bar{z}^k = \frac{1}{2} g_{j\bar{k}} \left(dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j \right) . \tag{A.121}$$

Note that $g(Z_i, \bar{Z}_j) = \frac{1}{2} g_{i\bar{j}}$, which explains our normalization of $g_{i\bar{j}}$.

Given a metric and a compatible (almost) complex structure, one defines the *fundamental two-form* ω by

$$\omega(X, Y) := g(X, JY) . \tag{A.122}$$

The coefficients of the fundamental two-form with respect to the holomorphic frame Z_i are

$$\begin{aligned} \omega_{i\bar{j}} &= 2\omega(Z_i, \bar{Z}_j) = 2g(Z_i, J\bar{Z}_j) = 2g(Z_i, -i\bar{Z}_j) = -ig_{i\bar{j}} \\ \omega_{\bar{j}i} &= 2\omega(\bar{Z}_j, Z_i) = 2g(\bar{Z}_j, JZ_i) = ig_{\bar{j}i} = ig_{i\bar{j}} = -\omega_{i\bar{j}} . \end{aligned} \tag{A.123}$$

Therefore the fundamental two-form has the expansion

$$\omega = -\frac{i}{2} g_{i\bar{j}} \left(dz^i \otimes d\bar{z}^j - d\bar{z}^j \otimes dz^i \right) = -\frac{i}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^j = \frac{1}{2} \omega_{i\bar{j}} dz^i \wedge \bar{z}^j . \tag{A.124}$$

The fundamental two-form is non-degenerate. Given ω and J we can therefore solve for the metric using that

$$g(X, Y) = \omega(JX, Y) \tag{A.125}$$

Moreover the complex structure

$$J \in \Gamma(\mathrm{End}(TM)) \ : TM \to TM \tag{A.126}$$

is determined by g and ω as

$$J = g^{-1}\omega \in \Gamma(T^*M \otimes TM) \cong \Gamma(\mathrm{End}(TM)) , \tag{A.127}$$

where the map $g^{-1}\omega$ is defined by

$$Y = (g^{-1}\omega)(X) \Leftrightarrow g(\cdot, Y) = \omega(\cdot, X) . \tag{A.128}$$

Thus any two of the three compatible data (g, ω, J) suffice to determine the third. To provide the corresponding local formulae, we introduce the components of the inverse metric by

$$g^{i\bar{k}} g_{\bar{k}j} = \delta^i_j , \quad g^{\bar{i}k} g_{k\bar{j}} = \delta^{\bar{i}}_{\bar{j}} . \tag{A.129}$$

The components (A.123) of the fundamental form are determined by antisymmetry. These relations are consistent with complex conjugation, $\overline{\omega_{i\bar{j}}} = \omega_{\bar{i}j}$. Evaluating $J^m_{\ n} = g^{mp}\omega_{pn}$ in complex coordinates, we obtain the components of the complex structure:

$$J^i{}_j = g^{i\bar{k}}\omega_{\bar{k}j} = i\delta^i_j \ , \quad J^{\bar{i}}{}_{\bar{j}} = g^{\bar{i}k}\omega_{k\bar{j}} = -i\delta^{\bar{i}}_{\bar{j}} \ . \tag{A.130}$$

The metric g and the fundamental form ω can be combined into a Hermitian form γ, which defines a Hermitian metric on the complex vector bundle TM. Its components with respect to the holomorphic frame $Z_i = \frac{\partial}{\partial z^i}$ are

$$\gamma = g_{i\bar{j}} dz^i \otimes d\bar{z}^j = g + i\omega \ . \tag{A.131}$$

We remark that our conventions differ from [115], on which A.11, A.12, A.13, and A.15 are partly based. In particular, we avoid a factor $\frac{1}{2}$ between the coefficients of the metric g on M and the Hermitian metric γ on the complex vector bundle TM, we include a factor $\frac{1}{2}$ in the definition of the symmetrized tensor product, we define ω in terms of g, J with a relative minus sign, and we take Hermitian forms complex anti-linear in the second rather than in the first argument.

A.14. Symplectic manifolds

A *symplectic manifold* (M, ω) is a real manifold equipped with a closed non-degenerate two-form ω, called the symplectic form. Symplectic manifolds are even dimensional. The tangent spaces (T_pM, ω_p) are symplectic vector spaces isomorphic to $\mathbb{R}^{2n}$ with its standard symplectic form ω. Let W be a linear subspace, $\iota : W \to V$ be the canonical embedding, and

$$W^\perp = \{v \in V | \omega(v, w) = 0 \ , \ \forall w \in W\} \tag{A.132}$$

be the 'symplectically perpendicular' subspace. Then

- $W \subset V$ is called *isotropic* if $W \subset W^\perp$. This implies $\dim W \leq \frac{1}{2} \dim V$ and $\iota^*\omega$ is totally degenerate, $\iota^*\omega = 0$.
- $W \subset V$ is called *co-isotropic* if $W^\perp \subset W$. This implies $\dim W \geq \frac{1}{2} \dim V$ and $W/W^\perp$ inherits a symplectic structure from V.
- $W \subset V$ is called *Lagrangian* if it is isotropic and co-isotropic, that is if $W^\perp = W$. This implies $\dim W = \frac{1}{2} \dim V$ and W is an isotropic subspace of maximal dimension.

- $W \subset V$ is called *symplectic* if $W \cap W^\perp = \{0\}$.

Consider the following example of a co-isotropic subspace. Let $\{\xi, \eta, X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ be a basis of $V \cong \mathbb{R}^{2n+2}$, such that

$$\omega(\xi, \eta) = 1 \ , \quad \omega(X_i, Y_j) = \omega_{ij} \ , \tag{A.133}$$

with all other components determined by antisymmetry, or else being zero. Define W as the linear subspace $W = \langle \eta, X_1, \ldots, X_n, Y_1, \ldots, Y_n \rangle$. Then $W^\perp = \ker(\iota^*\omega) = \langle \eta \rangle \subset W$, so that W is co-isotropic. The quotient $\bar{W} := W/W^\perp$ is defined by the equivalence relation

$$w \sim w' \Leftrightarrow w - w' = \alpha\eta \ . \tag{A.134}$$

The projection map onto the quotient is

$$\pi : W \to \bar{W} \ , \quad w \mapsto \bar{w} = \pi(w) \ , \tag{A.135}$$

where $\bar{w} = \pi(w)$ denotes the equivalence class of w with respect to (A.134). On $\bar{W}$ we can define a two-form $\bar{\omega}$ by

$$\bar{\omega}(\bar{X}, \bar{Y}) = (\pi^*\bar{\omega})(X, Y) = (\iota^*\omega)(X, Y) \ , \tag{A.136}$$

which is non-degenerate because we have factored out the kernel of $\iota^*\omega$. Choosing the basis $\{\bar{X}_1, \ldots, \bar{X}_n, \bar{Y}_1, \ldots, \bar{Y}_n\}$ for $\bar{W}$, the components of $\bar{\omega}$ are

$$\bar{\omega}_{ij} = \bar{\omega}(\bar{X}_i, \bar{Y}_j) = \omega_{ij} \ . \tag{A.137}$$

A submanifold $\iota : S \to M$ is called a(n) *isotropic, co-isotropic, Lagrangian and symplectic submanifold*, respectively, if all its tangent spaces are isotropic, co-isotropic, Lagrangian and symplectic, respectively. The pullback $\iota^*\omega$ of the symplectic form is thus totally degenerate on isotropic and symplectic submanifolds, and isotropic submanifolds have maximal dimension $\frac{1}{2} \dim M$.

An immersion $\iota : S \to M$ is called a *Lagrangian immersion* if its image is a Lagrangian submanifold. A vector field X on (M, ω) is called *a Hamiltonian vector field* if

$$\omega(X, \cdot) = -dH(\cdot) \tag{A.138}$$

for a function H, called the Hamiltonian or moment(um) map(ping) of X.

Example of a symplectic quotient. We now give a simple example of a *symplectic quotient* (or *symplectic reduction*), which is useful for understanding the complex version of the superconformal quotient relating affine conical to projective special Kähler manifolds. Let (M, ω) be a symplectic manifold, and let X be a Hamiltonian vector field which generates a $U(1)$-action on M. The level surfaces $\mathcal{H}_{\mathfrak{c}} = \{H = \mathfrak{c}\} = H^{-1}(\mathfrak{c})$ of the moment map H[75] are invariant under the action of X, since $L_X H = dH(X) = -\omega(X, X) = 0$. We assume that the resulting $U(1)$-action on $\mathcal{H}_c$ is such that the orbit space $\bar{M} = \mathcal{H}_{\mathfrak{c}}/\langle X \rangle = \mathcal{H}_{\mathfrak{c}}/U(1)$ is a smooth manifold. We note that any vector field T which is tangent to $\mathcal{H}_{\mathfrak{c}}$ must be symplectically perpendicular to X, that is

$$\omega(X, T) = -dH(T) = 0 \; . \tag{A.139}$$

In particular X itself is tangent to $\mathcal{H}_{\mathfrak{c}}$. We choose a vector field ξ transversal to $\mathcal{H}_{\mathfrak{c}}$ by imposing the condition $\omega(\xi, X) = dH(\xi) = 1$. Thus in a coordinate system where we use H as one of the coordinates, $\xi = \partial_H$. The restriction $\omega_{\mathfrak{c}} := \iota_{\mathfrak{c}}^* \omega$ of ω to the immersed hypersurface $\iota_{\mathfrak{c}} : \mathcal{H}_{\mathfrak{c}} \to M$ is degenerate. From the above it is clear that its kernel is spanned by X, therefore $\mathcal{H}_{\mathfrak{c}}$ is a co-isotropic submanifold. The two-form $\omega_{\mathfrak{c}}$ is invariant under X,

$$L_X \omega_{\mathfrak{c}} = d(\omega_{\mathfrak{c}}(X, \cdot)) + (d\omega_{\mathfrak{c}})(X, \cdot, \cdot) = 0 \; , \tag{A.140}$$

because $\omega_{\mathfrak{c}}(X, \cdot) = 0$ and $d\omega_{\mathfrak{c}} = d\iota_{\mathfrak{c}}^* \omega = \iota_{\mathfrak{c}}^* d\omega = 0$. Since $\omega_{\mathfrak{c}}$ is also transversal to the action of X (that is, its components in the X-direction vanish), $\omega_{\mathfrak{c}}$ can be projected to the quotient $\bar{M} = \mathcal{H}_{\mathfrak{c}}/U(1)$ to define a two-form $\bar{\omega}$ by $\pi^* \bar{\omega} = \omega_{\mathfrak{c}}$, where $\pi : \mathcal{H}_{\mathfrak{c}} \to \bar{M}$ is the projection onto the quotient. Since we take a quotient with respect to the kernel of $\omega_{\mathfrak{c}}$, the two-form $\bar{\omega}$ is non-degenerate. It is also smooth because all maps entering into its construction are by assumption smooth. To verify that $\bar{\omega}$ is closed we note that $d\omega = 0$ implies

$$0 = d\omega_{\mathfrak{c}} = d(\pi^* \bar{\omega}) = \pi^* d\bar{\omega} \; . \tag{A.141}$$

Since the projection map is surjective, every tangent vector $\bar{X}$ of $\bar{M}$ can be lifted to a tangent vector X of $\mathcal{H}_{\mathfrak{c}}$. Therefore

$$d\bar{\omega}(\bar{X}, \bar{Y}) = (\pi^* d\bar{\omega})(X, Y) = 0 \; , \tag{A.142}$$

for all $\bar{X}, \bar{Y} \in \mathfrak{X}(\bar{M})$, and thus $d\bar{\omega} = 0$. This shows that $(\mathcal{H}_{\mathfrak{c}}/U(1), \bar{\omega})$ is a symplectic manifold. The construction by which it is obtained from (M, ω)

[75] Here $H^{-1}(\mathfrak{c})$ denotes the inverse image of $\mathfrak{c}$ under H, that is, the level set. This notation is common in the literature about symplectic quotients.

is called a symplectic quotient, denoted $M//U(1)$. Symplectic quotients can more generally be defined for symplectic actions of Lie groups G on symplectic manifolds, and are denoted $M//G$ [222].

A.15. Kähler manifolds

An *(almost) Kähler manifold* (M, g, ω) is an (almost) Hermitian manifold (M, J, g) where the fundamental two-form is closed. We will restrict ourselves to Kähler manifolds, that is to the case where J is integrable and (M, J) is a complex manifold. As for Hermitian manifolds we include cases where the metric is indefinite. We remark that for Hermitian manifolds the condition $d\omega = 0$ is equivalent to J being parallel with respect to the Levi-Civita connection, $DJ = 0$. The fundamental form of a Kähler manifold is called its *Kähler form*. Note that (M, ω) is a symplectic manifold. Thus Kähler manifolds are pseudo-Riemannian manifolds which simultaneously admit a compatible complex structure and a compatible symplectic structure. Another equivalent characterization of a Kähler manifold is that the Chern connection of the Hermitian metric $\gamma = g + i\omega$ on TM is equal to the Levi-Civita connection D of g. Evaluating the condition $d\omega = 0$ in local holomorphic coordinates we obtain the integrability condition

$$\partial_l g_{j\bar{k}} = \partial_j g_{l\bar{k}} \ , \tag{A.143}$$

or, equivalently,

$$\partial_{\bar{l}} g_{j\bar{k}} = \partial_{\bar{k}} g_{j\bar{l}} \ . \tag{A.144}$$

Another equivalent characterization is the local existence of a Kähler potential K, that is of a smooth real function such that

$$\omega = -\frac{i}{2}\partial\bar{\partial}K \ . \tag{A.145}$$

This follows by combining Poincaré's lemma with the decomposition of forms into types ($\partial\bar{\partial}$-lemma): locally $\omega = d\alpha$, where $\alpha = \beta + \bar{\beta}$, with $\beta \in \Omega^{1,0}(M)$. Since $\omega \in \Omega^{1,1}(M)$ and $d = \partial + \bar{\partial}$, where $\partial^2 = 0 = \bar{\partial}^2$, and where $\partial, \bar{\partial}$ act consistently with type (since we assume that the complex structure is integrable):

$$d\alpha = \partial(\beta + \bar{\beta}) + \bar{\partial}(\beta + \bar{\beta}) \in \Omega^{1,1}(M) \Rightarrow \partial\beta = 0 \ , \ \omega = \bar{\partial}\beta + \partial\bar{\beta} \ , \ \bar{\partial}\bar{\beta} = 0 \ . \tag{A.146}$$

Hence $\beta = \partial\varphi$ by the ∂-version of the Poincaré lemma, where φ is a smooth complex function. Therefore

$$\omega = \partial\bar{\partial}(\bar{\varphi} - \varphi) = -\frac{i}{2}\partial\bar{\partial}K \; , \tag{A.147}$$

where $K = -2i(\varphi - \bar{\varphi})$. This provides a real potential for the metric,

$$g_{i\bar{k}} = \partial_i \partial_{\bar{k}} K \; , \tag{A.148}$$

called the Kähler potential. Note that the Kähler potential is only determined up to adding the real part of a harmonic function, since K and $K + f + \bar{f}$ with $\bar{\partial} f = 0$ define the same metric. For further reading on Kähler manifolds we refer to [115] on which this section is partly based.

Since a Kähler manifold is in particular a symplectic manifold, one can apply symplectic reduction. If the symplectic group action is in addition holomorphic and isometric, it preserves the extra structures which distinguish a Kähler manifold from a symplectic manifold, and the quotient carries an induced Kähler structure. The Hamiltonian vector fields generating such a group action must be holomorphic Killing vector fields. Symplectic quotients of Kähler manifolds by symplectic, holomorphic and isometric group actions are called *Kähler quotients* [223]. One uses the same notation $M//G$ as for symplectic quotients.

A.16. Contact manifolds

A one-form θ on a manifold M of odd dimension $2n + 1$ is called a *contact form* if the $(2n+1)$-form $\theta \wedge (d\theta)^n$ is a volume, that is, if it is nowhere vanishing,

$$(\theta \wedge d\theta \wedge \cdots \wedge d\theta)_p \neq 0 \; , \quad \forall p \in M \; . \tag{A.149}$$

A *contact manifold* (M, θ) is an odd-dimensional manifold equipped with a contact form. A *contact structure* on an odd-dimensional manifold M is defined by the choice of a hyperplane distribution $V = \cup_{\in M} V_x$ on its tangent bundle TM, which is maximally non-integrable, that is non-integrable at every point.

To relate the concepts of contact form and contact structure, we note that the kernel $\ker(\theta)$ of the one-form θ defines a hyperplane distribution on TM. By the dual version of the Frobenius theorem, the integrability condition for this distribution is $\theta \wedge d\theta = 0$, which implies $\theta \wedge (d\theta)^n = 0$. Thus by definition a contact distribution is not integrable, and in fact maximally non-integrable, since the integrability condition does not hold at any point of the manifold.

Consequently, a contact form determines a contact structure. Since any two one-forms θ, θ', which differ by multiplication with a nowhere vanishing function f, $\theta' = f\theta$ have the same kernel, a contact structure corresponds to an equivalence class of contact forms. Since $\theta \wedge (d\theta)^n$ is nowhere vanishing, the kernel of $d\theta$ defines a one-dimensional distribution on TM which is complementary to the contact distribution, that is $TM = \ker(\theta) \oplus \ker(d\theta)$.

To each contact form there is an associated vector field, called the *Reeb vector field* R, which is the unique vector field on M such that

$$\theta(R) = 1 \ , \quad d\theta(R, \cdot) = 0 \ . \tag{A.150}$$

Thus R spans the kernel of $d\theta$ and extends any given frame on $V = \ker(\theta)$ to a frame on TM.

Contact manifolds can be regarded as the odd-dimensional analogues of symplectic manifolds. Contact and symplectic manifolds can be related by constructions which change the number of dimensions by one.

The symplectification of a contact manifold. Let (M, θ) be a contact manifold of dimension $2n+1$. Consider the cone $\mathbb{R}^{>0} \times M$ over M with coordinate r on $\mathbb{R}^{>0}$. Then $(\mathbb{R}^{>0} \times M, \omega)$, with $\omega = r^2 d\theta + 2rdr \wedge \theta$ is a symplectic manifold, because $d\omega = 2rdr \wedge d\theta - 2rdr \wedge d\theta = 0$, and because ω is non-degenerate, as can be verified using a frame consisting of ∂_r and a frame for M. Note that we have seen above that the kernels of θ and $d\theta$ define complementary distributions on TM. Using the variable ρ, defined by $r^2 = e^\rho$, we can write the cone in 'product form': $(\mathbb{R}^{>0} \times M, \omega) \cong (\mathbb{R} \times M, \omega')$, where $\omega' = d(e^\rho \theta)$. In this parametrization we see that the symplectic form is exact. The symplectic manifold $(\mathbb{R} \times M, d(e^\rho\theta)) \cong (\mathbb{R}^{>0}, r^2 d\theta + 2rdr \wedge \theta)$ is called the *symplectification* (or symplectization) of the contact manifold (M, θ).

Legendrian submanifolds. If θ is a contact form on a manifold M of dimension $2n+1$, then $d\theta_{|V}$ is a symplectic form on the contact distribution $V = \ker\theta$. Therefore a subdistribution $L \subset V$ can only be integrable if it isotropic with respect to $d\theta_{|V}$. This implies that $2 \dim L \leq \dim M - 1 = 2n$, that is $\dim L \leq n$. Integral manifolds of dimension n which saturate this bound are called *Legendrian submanifolds*, and are the counterparts of Lagrangian submanifolds in symplectic geometry. In particular, the Legendrian submanifolds of a contact manifold lift to Lagrangian submanifolds of its symplectification. An immersion $\iota : \mathcal{H} \to M$ into a contact manifold (M, θ) is called a *Legendrian* *immersion* if the image of $\mathcal{H}$ is a Legendrian submanifold.

For further reading on contact geometry we refer to [224].

A.17. Sasakian Manifolds

The following section is based on various sources, including [151, 124, 46, 47].

Kähler manifolds can be thought of as symplectic manifolds with an additional pseudo-Riemannian metric subject to compatibility conditions, which determine a complex structure. Sasakian manifolds are the 'contact analogue' of Kähler manifolds, that is contact manifolds equipped with a metric which satisfies certain compatibility conditions. One way to characterize Sasakian manifolds is by requiring that their symplectification is Kähler: A *Sasakian manifold* (S, θ, g) is a contact manifold (S, θ) equipped with a (pseudo-)Riemannian metric g, such that the Riemannian cone $(M, g_M) = (S \times \mathbb{R}^{>0}, r^2 g_S + dr^2)$ is a Kähler manifold with Kähler form $\omega = r^2 d\theta + 2r dr \wedge \theta$. Comparing to the previous section we see that the Riemannian cone is indeed the symplectification of the contact manifold (M, θ). We remark that the complex structure J relates the homothetic Killing vector field $\xi = r\partial_r$ to the Reeb vector field $R = -J\xi$.

If in addition the Reeb vector field generates a $U(1)$-action on S such that $\bar{M} = S/U(1)$ is a smooth manifold, then $\bar{M}$ is the Kähler quotient of M, $\bar{M} = M//U(1) = S/U(1)$. Moreover, if a Kähler manifold M admits a homothetic Killing vector field ξ, which satisfies $D\xi = \mathrm{Id}$, then M is a Riemannian cone over the Sasakian $S = \{g(\xi, \xi) = 1\}$. If the quotient $M/\mathbb{C}^*$ by the holomorphic and homothetic action generated by $\xi, J\xi$ defines a smooth manifold, this manifold is precisely the symplectic quotient with respect to the action of $J\xi$. Finally, given a Kähler manifold $\bar{M}$ we can construct a 'complex cone' or 'conical Kähler manifold' M as the total space of a $\mathbb{C}^*$ bundle over $\bar{M}$, such that $\bar{M} = M//\mathbb{C}$.[76]

A.18. Complex symplectic manifolds and complex contact manifolds

The concepts of symplectic and contact geometry, which we have formulated for real manifolds, can be formulated analogously for complex manifolds. We illustrate this by examples.

The vector space $V = T^*\mathbb{C}^n \cong \mathbb{C}^{2n}$ equipped with the complex symplectic form $\Omega = dz^i \wedge dw_i$ is the standard example for a complex symplectic vector space of complex dimension $2n$. Its projectivization $P(V')$, where

[76] Another natural name for M would be 'Kähler cone' in analogy to Riemannian cone, but Kähler cone is also use for the cone of Kähler structures on a Calabi-Yau manifold.

$V' = \{(z,w) \in \mathbb{C}^{2n} | (z,w) \neq (0,0)\}$, is the space $V'/\sim$, where $\sim$ denotes the equivalence relation

$$(z,w) \sim (z',w') \Leftrightarrow (z',w') = \lambda(z,w) \ , \quad \exists \lambda \in \mathbb{C}^* \ . \tag{A.151}$$

$P(V')$ is a complex contact space with complex symplectification V. In special geometry projective special Kähler manifolds $\bar{M}$ can be realized as holomorphic Legendrian immersions into $P(V')$, which lift to holomorphic Lagrangian immersion of the corresponding conical affine special Kähler manifold M into V.

A.19. Some groups and their actions

This section is based on [46, 47]. The Heisenberg group $\mathrm{Heis}_{2n+1}(\mathbb{R})$ is the nilpotent Lie group obtained as a central extension of the translation group $\mathbb{R}^{2n}$, with group law

$$(s,v) \circ (s',v') = \left(s + s' + \frac{1}{2}\Omega(v,v'), v + v' \right) \ , \tag{A.152}$$

where $s, s' \in \mathbb{R}$ are central, $v, v' \in \mathbb{R}^n$ are translations, and where Ω is the standard symplectic form on $\mathbb{R}^{2n}$. The standard generators p_i, q_i, z, $i = 1, \ldots, n$ for the Lie algebra $\mathfrak{heis}_{2n+1}(\mathbb{R})$ satisfy

$$[p_i, q_j] = \delta_{ij} z \ , \tag{A.153}$$

with all other commutators vanishing. The group $G = \mathrm{Sp}(\mathbb{R}^{2n}) \ltimes \mathrm{Heis}_{2n+1}(\mathbb{R})$ is the semi-direct extension of the real Heisenberg group by its group $\mathrm{Sp}(\mathbb{R}^{2n})$ of automorphisms, with group law

$$g \cdot g' = \left(MM', s + s' + \frac{1}{2}\Omega(v, Mv'), v + Mv' \right) \ , \tag{A.154}$$

where $M, M' \in \mathrm{Sp}(\mathbb{R}^{2n})$ and $(s,v) \in \mathrm{Heis}_{2n+1}(\mathbb{R})$. We use the same notation $g = (M,s,v)$ for elements of the complexification $G_{\mathbb{C}} = \mathrm{Sp}(\mathbb{C}^{2n}) \ltimes \mathrm{Heis}_{2n+1}(\mathbb{C})$. The quotient map

$$G_{\mathbb{C}} \to \mathrm{Aff}_{\mathrm{Sp}(\mathbb{C}^{2n})} = G_{\mathbb{C}}/Z(G_{\mathbb{C}}) \ : (M,s,v) \mapsto (M,v) \tag{A.155}$$

induces an affine representation $\bar{\rho}$ of $G_{\mathbb{C}}$, whose restriction to the real subgroup G provides an affine representation of $\mathrm{Aff}_{\mathrm{Sp}(\mathbb{R}^{2n})}(\mathbb{R}^{2n})$.

On the complex vector space $\mathbb{C}^{2n}$ we choose Darboux coordinates (X^I, W_I), such that the complex symplectic form is $\Omega = dX^I \wedge dW_I$. We can embed $\mathbb{C}^{2n}$

into $\mathbb{C}^{2n+2} = \mathbb{C}^2 \oplus \mathbb{C}^{2n}$ with standard coordinates (X^0, W_0, X^i, W_i). A linear representation $\rho : G_{\mathbb{C}} \to Sp(\mathbb{C}^{2n+2})$ is defined by

$$g = (M, s, v) \mapsto \rho(x) = \begin{pmatrix} 1 & 0 & 0 \\ -2s & 1 & \hat{v}^T \\ v & 0 & M \end{pmatrix} , \quad \hat{v} := M^T \Omega_0 v = \Omega_0 M^{-1} v \ , \tag{A.156}$$

where Ω_0 is the standard representation matrix for the symplectic form on $\mathbb{C}^{2n}$. According to Proposition 3.2.2 of [47] this is a faithful representation which induces the affine representation $\bar{\rho} : G_{\mathbb{C}} \to \mathrm{Aff}_{\mathrm{Sp}(\mathbb{C}^{2n})}(\mathbb{C}^{2n})$ because it preserves the affine hyperplane $\{X^0 = 1\} \subset \mathbb{C}^{2n+2}$ and the distribution ∂_{W_0}. The orbit space $\{X^0 = 1\}/\langle \partial_{W_0} \rangle$ is the symplectic reduction of $\mathbb{C}^{2n+2}$ with respect to the holomorphic Hamiltonian group action generated by ∂_{W_0}, and ρ induces $\bar{\rho}$ under this quotient. Similarly, the real symplectic affine space $\mathbb{R}^{2n}$ is the symplectic reduction of the real symplectic vector space $\mathbb{R}^{2n+2}$, with $G_{\mathbb{C}}$ replaced by its real subgroup G.

Finally we define the group $G_{SK} = \mathrm{Sp}(\mathbb{R}^{2n}) \ltimes \mathrm{Heis}_{2n+1}(\mathbb{C}) \subset G_{\mathbb{C}}$. Note that $G \subset G_{SK}$ and that G_{SK} is a central extension of $\bar{\rho}(G_{SK}) = \mathrm{Aff}_{\mathrm{Sp}(\mathbb{R}^{2n})}(\mathbb{C}^{2n}) = Sp(\mathbb{R}^{2n}) \ltimes \mathbb{C}^{2n}$. The latter group acts simply transitively on Kählerian Lagrangian immersions of simply connected ASK manifolds, in other words, it is the duality group of ASK geometry.

A.20. Para-complex geometry

Here we collect some definitions and statements about para-complex geometry. More details can be found in [19, 78, 45, 17].

A *para-complex structure* J on a finite-dimensional real vector space V is a non-trivial involution $J \in \mathrm{End}(V)$, $J \neq \mathrm{Id}$, $J^2 = \mathrm{Id}$, such that the eigenspaces $V^{\pm} := \ker(\mathrm{Id} \mp J)$ of J are of the same dimension. A *para-complex vector space* (V, J) is a real vector space V endowed with a para-complex structure J. A homomorphism of para-complex vector spaces is a linear map $\Phi : (V, J) \to (V', J')$ such that $\Phi \circ J = J' \circ \Phi$. Para-complex vector spaces have even dimension and admit bases $e_i^{\pm}$ such that $J e_i^{\pm} = \pm e_i^{\pm}$. It is easy to see that for $\dim_{\mathbb{R}} V = 2n$ a para-complex structure is invariant under the group $\mathrm{Aut}(V, J) := \{L \in GL(V) | LJL^{-1}\}$, where

$$\mathrm{Aut}(V, J) \cong GL(n, \mathbb{R}) \times GL(n, \mathbb{R}) \subset GL(V) \cong GL(2n, \mathbb{R}) \ . \tag{A.157}$$

An *almost para-complex structure* on a smooth manifold M is an endomorphism field $J \in \mathrm{End}(TM) : p \mapsto J_p$ such that J_p is a para-complex structure on T_pM for all $p \in M$. An *almost para-complex manifold* (M, J) is a smooth manifold M endowed with an almost para-complex structure.

If one relaxes the condition that the eigenspaces of J_p have equal dimension, one obtains the concept of an *almost product structure.* Thus almost para-complex structures are almost product structure where the dimensions of the eigendistributions 'balance.' This creates many analogies with almost complex manifolds.

An almost para-complex structure J is called *integrable* if the eigendistributions $T^{\pm}M := \ker(\mathrm{Id} \mp J)$ are both integrable. An integrable almost para-complex structure is called a *para-complex structure.* A *para-complex manifold* (M, J) is a manifold M endowed with a para-complex structure J. The Frobenius theorem implies that an almost para-complex structure is integrable if and only if its Nijenhuis tensor $N_J(X,Y) = [X,Y]+[JX,JY]-J[X,JY]-J[JX,Y]$ vanishes for all vector fields X, Y on M.

A smooth map $\Phi : (M, J_M) \to (N, J_N)$ between para-complex manifolds is called a *para-holomorphic map* if $d\Phi J_M = J_N d\Phi$.

It can be shown that the integrability of the almost para-complex structure J is equivalent to the existence of local para-complex coordinate systems. This uses the algebra C of *para-complex numbers*, which are also known as split complex numbers of hyperbolic complex numbers. As a real algebra C is generated by 1 and the symbol e, subject to the relation $e^2 = 1$. The map

$$\bar{\cdot} : C \to C\ , x + ey \mapsto x - ey\ , \quad x, y \in \mathbb{R} \tag{A.158}$$

is called para-complex conjugation and is a C-antilinear involution, which allows to regard x, y as the real and imaginary part of $z = x + ey$. The algebra C has zero-divisors, its group of invertible elements is isomorphic to $O(1,1)$ and has four connected components separated by the light cone $z\bar{z} = x^2 - y^2 = \pm 1$. The algebra C and the free C-module C^n are para-complex vector spaces of real dimensions 2 and $2n$, respectively, with a para-complex structure given by multiplication with e. One can show that a smooth manifold M endowed with an atlas of C^n-valued coordinate maps related by para-holomorphic coordinate transformations admits an integrable para-complex structure. Conversely, any real manifold with an integrable para-complex structure admits a para-complex atlas.

Remark 11. For almost para-complex manifolds it is interesting to consider the case where only one of the eigendistributions $T^{\pm}M$ is integrable. This has applications in particular in doubled/generalized geometry. Here we focus on the case were both eigendistributions are integrable, which is relevant for Euclidean special geometry.

A para-holomorphic map $\Phi : (M, J) \to C$ is called a para-holomorphic function.

A *para-holomorphic vector bundle* of rank r is a smooth real vector bundle $W \to M$ of rank $2r$ whose total space W and base M are para-complex manifolds and whose projection π is a para-holomorphic map. On a para-holomorphic vector bundle we have a canonical splitting $W = W^+ \oplus W^-$ induced by the para-complex structure. The tangent bundle $TM \to M$ over any para-complex manifold M is a para-holomorphic vector bundle. The splitting $TM = T^+M \oplus T^-M$ can be used to define a real version of Dolbeault cohomology on any para-complex manifold.

The *para-complexified tangent bundle* $T_C M = TM \otimes C$ can be equipped with the C-linear extension of the para-complex structure J. It decomposes canonically into eigenbundles of J with eigenvalues $\pm e$,

$$T_C M = T^{1,0}M \oplus T^{0,1}M \ . \tag{A.159}$$

There is a canonical isomorphism

$$TM \xrightarrow{\cong} T^{1,0}M \ , \quad X \mapsto \frac{1}{2}(X + eJX) \tag{A.160}$$

of real vector bundles which is compatible with the para-complex structures on the fibre. C-valued differential forms admit a decomposition into types in analogy with complex-valued differential forms on complex manifolds, which allows to define a para-complex version of Dolbeault cohomology.

A *para-Hermitian vector space* (V, J, g) is a para-complex vector space (V, J), equipped with a pseudo-Euclidean scalar product g for which J is an anti-isometry,

$$J^*g = g(J\cdot, J\cdot) = -g \ . \tag{A.161}$$

Then g is called a *para-Hermitian scalar product*, and (J, g) a *para-Hermitian structure* on V. A para-Hermitian scalar product always has neutral signature.

The *standard para-Hermitian structure on* $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ is given by

$$Ie_i^{\pm} = \pm e_i^{\pm}\ , \quad g(e_i^{\pm}, e_j^{\pm}) = 0\ , \quad g(e_i^{\pm}, e_j^{\mp}) = \delta_{ij}\ , \tag{A.162}$$

where $e_i^+ = e_i \oplus 0$, and $e_i^- = 0 \oplus e_i$.

The *standard para-Hermitian structure on* $C^n = \mathbb{R}^n \oplus e\mathbb{R}^n$ with basis $e_i, f_i := ee_i$ is given by

$$Je_i = f_i\ , \quad Jf_i = e_i\ , \quad g(e_i, e_j) = -g(f_i, f_j) = \delta_{ij}\ . \tag{A.163}$$

Any two para-Hermitian vector spaces of the same dimension are isomorphic. Using any of the two standard realizations $\mathbb{R}^{2n}$ or C^n, it is straightforward to show that the *para-unitary group*

$$U^{\pi}(V) := \mathrm{Aut}(V, J, g) = \{L \in GL(V) | LJL^{-1} = J\ , L^* g = g\} \tag{A.164}$$

of a para-Hermitian vector space of real dimension $2n$ is

$$U^{\pi}(V) \cong GL(n, \mathbb{R}) \subset \mathrm{Aut}(V, J) \cong GL(n, \mathbb{R}) \times GL(n, \mathbb{R}) \subset GL(V) \cong GL(2n, \mathbb{R})\ . \tag{A.165}$$

Note that J itself is not an element of the para-unitary group, though it is an element of the para-unitary Lie algebra.

An (*almost*) *para-Hermitian manifold* (M, J, g) is an (almost) para-complex manifold (M, J) endowed with a pseudo-Riemannian metric g such that $J^* g = g(J\cdot, J\cdot) = -g$. The two-form $\omega = g(\cdot, J\cdot) = -g(J\cdot, \cdot)$ is called the *fundamental two-form* of the (almost) para-Hermitian manifold (M, J, g). Compared to [19] we have changed the sign of ω to be consistent with our conventions. Note that it is essential that J is an anti-isometry, and not an isometry, for ω to be antisymmetric.

A *para-Kähler manifold* is an almost para-Hermitian manifold (M, J, g) such that J is parallel with respect to the Levi-Civita connection, $DJ = 0$. Note that $DJ = 0$ implies both $d\omega = 0$ and the integrability condition $N_J = 0$. Alternatively, a para-Kähler manifold is a para-Hermitian manifold with closed fundamental form. The symplectic form ω is called the *para-Kähler form.* It can be shown that for a para-Kähler metric there exists around any point a real valued function K, called a *para-Kähler potential*, such that the coefficients of ω and g are given by the mixed second derivatives with respect to para-holomorphic coordinates.

An *affine special para-Kähler manifold* (M, J, g, ∇) is a para-Kähler manifold (M, J, g) endowed with a flat, torsion-free connection such that ∇ is symplectic, i.e. $\nabla\omega = 0$, and such that $d_\nabla J = 0$. One can show that any simply connected affine special para-Kähler manifold can be realized by a para-Kählerian Lagrangian immersion $\phi : M \to V$ into the standard para-complex vector space $V = T^*C^n \cong C^{2n}$ endowed with the C-valued symplectic form $\Omega = dX^I \wedge dW_I$, standard para-complex structure I_V, para-complex conjugation τ and para-Hermitian form $\gamma = g + e\omega = e\Omega(\cdot, \tau\cdot)$. For a generic choice of para-complex symplectic coordinates X^I, W_I, the image of ϕ is the graph of a map $C^n \to C^n$, and therefore ϕ has a para-holomorphic prepotential F, i.e. $\phi = dF$.

A *conical affine special para-Kähler manifold* (M, J, g, ∇, ξ) is an affine special para-Kähler manifold (M, J, g, ∇) endowed with a vector field ξ such that

$$\nabla\xi = D\xi = \mathrm{Id} \; , \tag{A.166}$$

where D is the Levi-Civita connection.

One can show that near any point $p \in M$ there exist coordinates $(q^a) = (x^I, y_I)$ such that

$$\xi = q^a\partial_a = x^I\partial_{x^I} + y_I\partial_{y_I} \; . \tag{A.167}$$

Such coordinates are unique up to linear symplectic transformations, and are called conical special real coordinates. On conical special para-Kähler manifolds it is understood that 'special coordinates' means 'conical special coordinates.' A para-holomorphic immersion $\phi \to V = C^{2n}$ is called a *conical para-holomorphic immersion* if the position vector field $\xi^V = p \in V \cong T_pV$ is tangent along ϕ, that is, if $\xi^V \in d\phi_p T_p M$. Every simply connected conical special para-Kähler manifold can be realized by a conical para-Kählerian Lagrangian immersion, which is unique up to linear symplectic transformations. The corresponding para-holomorphic prepotential can be chosen to be homogeneous of degree two.

The vector fields ξ and $J\xi$ generate an infinitesimal C^*-action on the conical affine special para-Kähler manifold M. To be able to take a quotient which defines a para-Kähler manifold, one needs to make additional assumptions. A conical affine special para-Kähler manifold (M, J, g, ∇, ξ) is called a *regular conical affine special Kähler manifold* if the norm $g(\xi, \xi)$ of ξ does not vanish on M and if the quotient map $\pi \; : M \to \bar{M} = M/C^*$ is a para-holomorphic submersion onto a Hausdorff manifold. Under these assumptions, the symmetric tensor field

$$\tilde{g}^{(0)} = -\partial\bar{\partial}\left(-e(X^I \bar{F}_I - F_I \bar{X}^I)\right) \ , \tag{A.168}$$

which projects onto the orbit space $\bar{M}$, induces a para-Kähler metric $\bar{g}$ on $\bar{M}$, such that $\tilde{g}^{(0)} = \pi^* \bar{g}$. A *projective special para-Kähler manifold* $(\bar{M}, \bar{J}, \bar{g})$ is a para-Kähler manifold that can be realized locally as the quotient of a regular conical affine special para-Kähler manifold M by its C^*-action.

With a proper choice of conventions, local formulae for affine special Kähler and affine special para-Kähler manifolds are related by the replacement $i \to e$. Therefore one can use an ε-complex terminology which employs the notation $i_\varepsilon = e, i$ for $\varepsilon = \pm 1$. All statements in this section remain true when omitting 'para' or replacing it by 'ε-' and applying the appropriate substitutions for e.

A.21. ε-quaternionic geometries

This section is based on [19, 78, 45, 17].

Hypermultiples contains four real scalars and their scalar geometries are related to the algebra $\mathbb{H}_{-1} = \mathbb{H}$ of the quaternions, or, for Euclidean space-time signature, to the algebra $\mathbb{H}_1$ of para-quaternions. We treat both cases in parallel by writing $\mathbb{H}_\varepsilon$, where $\varepsilon = \pm 1$.

The algebra $\mathbb{H}_\varepsilon$ of ε-quaternions is the four-dimensional real algebra generated by three ε-complex units i_1, i_2, i_3, which pairwise anticommute and satisfy the ε-quaternionic algebra

$$i_1^2 = i_2^2 = -\varepsilon i_3^2 = \varepsilon \ , \quad i_1 i_2 = i_3 . \tag{A.169}$$

An *ε-quaternionic structure* on a real vector space of dimension $4n$ is a Lie subalgebra $Q \subset \mathrm{End}(V)$ spanned by three pairwise anticommuting endomorphisms J_1, J_2, J_3 which satisfy the ε-quaternionic algebra (A.169). The Lie group generated by the Lie algebra generated by J_α, $\alpha = 1, 2, 3$ is

$$Sp_\varepsilon(1) = \begin{cases} SU(2) \cong Sp(1) \ , & \text{if } \ \varepsilon = -1 \ , \\ SU(1,1) \cong Sp(2, \mathbb{R}) \cong SL(2, \mathbb{R}) \ , & \text{if } \ \varepsilon = 1 \ . \end{cases} \tag{A.170}$$

Our notation for symplectic groups is such that $Sp(2n, \mathbb{R}) = Sp(\mathbb{R}^{2n})$, $Sp(2n, \mathbb{C}) = Sp(\mathbb{C}^{2n})$ and

$$Sp(n) = Sp(2n, \mathbb{C}) \cap U(2n) \ , \quad Sp(k, l) = Sp(2n, \mathbb{C}) \cap U(2k, 2l) \ . \tag{A.171}$$

In particular, $Sp(1) = Sp(2,\mathbb{C}) \cap U(2) = SU(2)$ is the group often denoted $USp(2)$ in the physics literature. Also note that $Sp(1,1) = Sp(2n,\mathbb{C}) \cap U(1,1) = SU(1,1) \cong Sp(2,\mathbb{R})$.

While there are various types of ε-quaternionic geometries, we will only use two types which can be viewed as generalizations of ε-Kähler geometry. The first is realized by rigid hypermultiplets.

An *ε-hyper-Kähler* manifold (ε-HK manifold) is a pseudo-Riemannian manifold (N, g) of dimension $4n = 4k+4l$ whose holonomy group $\mathrm{Hol}(N)$ is contained in $Sp_\varepsilon(k,l)$, where

$$Sp_\varepsilon(k,l) = \begin{cases} Sp(k,l) \subset SO(4k,4l)\ , & \text{if } \ \varepsilon = -1\ , \\ Sp(2n,\mathbb{R}) \subset SO(2n,2n)\ , & \text{if } \ \varepsilon = 1\ . \end{cases} \tag{A.172}$$

An ε-Kähler manifold has three pairwise anticommuting integrable ε-complex structures I_α, such that $\omega_\alpha = g(I_\alpha \cdot, \cdot)$ are antisymmetric and closed, and therefore form an $Sp_\varepsilon(1)$-triplet of ε-Kähler forms. The ε-Kähler metric g admits ε-Kähler potentials with respect to any of the three ε-complex structures, though in general there is no 'ε-hyper-Kähler potential,' that is a potential which is ε-Kähler with respect to all three ε-complex structures simultaneously. It is useful to note that the closed-ness of the three forms ω_α implies the integrability of the ε-complex structures [223]. The construction of symplectic and Kähler quotients has been extended to the so-called hyper-Kähler quotient [223], which can be adapted to para-Kähler manifolds.

Hypermultiplets coupled to supergravity display another type of ε-quaternionic geometry. An *ε-quaternionic Kähler manifold* (ε-QK manifold) of real dimension $4n = 4k + 4l > 4$ is a pseudo-Riemannian manifold (N, g) whose holonomy group $\mathrm{Hol}(N)$ is contained in $Sp_\varepsilon(1) \cdot Sp_\varepsilon(k,l)$. An *$\varepsilon$-quaternionic Kähler manifold of real dimension 4* is an Einstein manifold equipped with an ε-quaternionic structure under which the curvature tensor is invariant. In this definition it is assumed implicitly that the ε-HK case is excluded, that is that the holonomy group is not contained in $Sp_\varepsilon(k,l)$. Due to the presence of the additional factor $Sp_\varepsilon(1)$, an ε-QK manifold need not admit any global ε-complex structure, and in particular need not be ε-Kähler. Instead it possesses an ε-quaternionic structure, that is the tangent bundle TN carries a fibre-wise ε-quaternionic structure, which is parallel with respect to a torsion-free connection (here: the Levi-Civita connection). In addition the locally defined ε-complex structures

J_α are skew with respect to the metric g, and the distribution spanned by them is parallel with respect to the Levi-Civita connection. Note that only the distribution $\langle J_\alpha | \alpha = 1, 2, 3 \rangle$ is invariant under parallel transport, while the individual structures undergo $Sp_\varepsilon(1)$-transformations which mix them. Using the locally defined fundamental forms $\omega_\alpha = g(J_\alpha \cdot, \cdot)$ one can define the four-form $\Lambda = \sum_{\alpha=1}^{3} \omega_\alpha \wedge \omega_\alpha$, which is globally defined and closed. It is useful to know that for manifolds of dimension $4n \geq 12$ the closed-ness of the four-form Λ already implies that the manifold ε-QK (for $\varepsilon = -1$ this is known from [225, 226]). The 'generic' definition given for dimension $4n > 4$ is not satisfactory for dimension 4, since $\mathrm{Hol}(N) \subset Sp_\varepsilon(1) \cdot Sp_\varepsilon(1)$ only implies that N is orientable: $\mathrm{Hol}(N) \subset SU(2) \cdot SU(2) \cong SO(4)$, or $\mathrm{Hol}(N) \subset SL(2, \mathbb{R}) \cdot SL(2, \mathbb{R}) \cong SO(2, 2)$. The property of the curvature tensor used in the above definition for dimension $4n = 4$ is non-trivial and natural, since it follows for dimension $4n > 4$ from the 'generic' definition.

Every ε-QK manifold of dimension $4n$ can be obtained as the quotient of a conical ε-HK manifold of dimension $4n + 4$ by the action of the invertible ε-quaternions $\mathbb{H}_\varepsilon^*$. Here 'conical ε-HK manifold' is defined analogously to CASR and CASK manifolds, and every such manifold defines a ε-QK manifold. In the physics literature conical ε-HK manifolds are usually called ε-HK cones, while in the mathematical literature they are known, for $\varepsilon = -1$, as the Swann bundle associated to a QK manifold [225, 226]. One interesting property of ε-HK cones is that they admit an ε-HK-potential, that is a potential which is an ε-Kähler potential for all three ε-complex structures simultaneously. For the case $\varepsilon = -1$ we have encountered the HK potential χ in the context of the superconformal construction of the four-dimensional Poincaré supergravity Lagrangian. We mention for completeness that there also is a quotient construction which relates QK manifolds to QK manifolds, called quaternionic reduction or quaternionic quotient [227, 228].

B. Physics background

B.1. Non-linear sigma models and maps between manifolds

Supergravity theories with scalars involve non-linear sigma models coupled to gravity. Sigma models are theories of massless scalars on a pseudo-Riemannian space-time (N, h), which are valued in another pseudo-Riemannian manifold (M, g), called the target space. More precisely, scalar fields are components of a map

$$f : (N, h) \to (M, g) \tag{B.1}$$

between two pseudo-Riemannian manifolds. When expressed in local coordinates $x = (x^1, \ldots, x^n)$ on N and $\varphi = (\varphi^1, \ldots, \varphi^m)$ on M, scalar fields become real-valued local functions on space-time, which are the pull-backs to space-time of the composition of the map f with coordinate maps. In this section we explain the relation between the global geometrical description in terms of the map f and the local description used in the physics literature in some detail. We remark that we do not aim for the highest degree of generality. In particular, one can define scalar fields more generally as sections of a pseudo-Riemannian submersion $\pi : P \to N$. We refer to [229] for a detailed discussion of this generalization and its potential implications. We also remark that both N and M can have various signatures, so that it makes sense to discuss sigma models in the general pseudo-Riemannian set-up. Space-time N has Lorentzian signature, but in the Euclidean formulation of quantum field theories it is replaced by an 'Euclidean' manifold, that is a Riemannian manifold (pseudo-Riemannian manifold of definite signature). Also, string dualities and the idea that space-time signature might be dynamical in quantum gravity motivate the study of exotic space-times with multiple time-like directions. Similarly, in standard cases the target manifold has positive signature, to ensure that all scalar fields have positive signature. However, dimensional reduction over time, which is used frequently to find stationary solutions, sometimes leads to indefinite signature target spaces. Moreover, supersymmetric theories on Euclidean space-times sometimes also require target spaces of indefinite signature.

The standard action of a non-linear sigma model coupled to gravity is the sum of the Einstein-Hilbert action and of the energy functional (or Dirichlet functional) for a map between two pseudo-Riemannian manifolds (N, h) and (M, g),

$$S[h, f] = \int d\,\mathrm{vol}_h \left(\frac{1}{2} R[h] - \langle df, df \rangle \right) . \tag{B.2}$$

Here $R[h]$ and $d\,\mathrm{vol}_h$ are the Ricci scalar and the volume form of (N, h). Since we have coupled the sigma model to gravity, the metric h is a dynamical field, while the metric g is fixed and part of the definition of the model. The vector valued one-form $df \in \Omega^1(N, f^*TM)$ is the differential of the map $f : N \to M$, and $\langle \cdot, \cdot \rangle$ is the scalar product induced by the metrics h and g on the vector bundle $T^*N \otimes f^*TM$ over N whose fibre over $p \in N$ is $T_p^*N \otimes T_{f(p)}M$.

We introduce the following coordinate maps:

$$\begin{aligned} \psi : N \supset V \to \mathcal{V} \subset \mathbb{R}^n \ , \qquad & p \mapsto \psi(p) = (x^1(p), \ldots, x^n(p)) \ , \\ \varphi : M \supset U \to \mathcal{U} \subset \mathbb{R}^m \ , \qquad & q \mapsto \varphi(q) = (\varphi^1(q), \ldots, \varphi^m(q)) \ . \end{aligned} \tag{B.3}$$

By restricting f to V and composing with the coordinate maps we obtain a local representation of f as a vector-valued function ϕ,

$$\begin{aligned} \phi = \varphi \circ f \circ \psi^{-1} \ : \mathbb{R}^n \supset \mathcal{V} \quad & \to \quad \mathcal{U} \subset \mathbb{R}^m \ , \\ x \quad & \mapsto \quad \phi(x) := (\phi^a(x^\mu)) := (\varphi^a(f(\psi^{-1}(x^\mu)))) \ . \end{aligned} \tag{B.4}$$

The physical scalar fields as defined in the physics literature are the components $\phi^a(x)$ of the map $f : N \to M$ with respect to the local coordinates $\{x^\mu\}, \{\varphi^a\}$.

Each of the above maps has a differential, which assigns to each point of its domain a linear map between the tangent spaces of domain and target:

$$df \ : p \quad \mapsto \quad df_p \ : T_pN \to T_{f(p)}M \ , \tag{B.5}$$

$$d\varphi \ : q \quad \mapsto \quad d\varphi_q \ : T_qM \to T_{\varphi(q)}\mathcal{U} \ , \tag{B.6}$$

$$d\psi \ : p \quad \mapsto \quad d\psi_p \ : T_pN \to T_{\psi(p)}\mathcal{V} \ . \tag{B.7}$$

The linear maps $d\varphi_q$ and $d\psi_p$ are invertible at all points. The differential of the local coordinate expression $d\phi : x \mapsto d\phi_x$ of df at the point x is

$$d\phi_x = (d\varphi \circ df \circ (d\psi)^{-1})_x \ : T_x\mathcal{V} \cong \mathbb{R}^n \to T_{\phi(x)}\mathcal{U} \cong \mathbb{R}^m \ .$$

Since $d\phi_x \in \mathrm{Hom}(T_x\mathcal{V}, T_{\phi(x)}\mathcal{U}) \cong T_x^*\mathcal{V} \otimes T_{\phi(x)}\mathcal{U}$, we interpret $d\phi \in \Gamma(\mathcal{V}, T^*\mathcal{V} \otimes \phi^*T\mathcal{U}) = \Omega^1(\mathcal{V}, \phi^*T\mathcal{U})$ as a vector-valued one-form on $\mathcal{V}$,

$$d\phi = \frac{\partial \phi^a}{\partial x^\mu} dx^\mu \otimes \partial_a \in \Omega^1(\mathcal{V}, \phi^*T\mathcal{U}) \ . \tag{B.8}$$

The local coordinate expression for the metric g restricted to $U \cong \mathcal{U}$ is

$$g_{\mathcal{U}} = g_{ab}(\varphi) d\varphi^a d\varphi^b \ . \tag{B.9}$$

Using that the pull-back is given by $\phi^a(x) = \varphi^a(f(\psi^{-1}(x)))$, the corresponding expression for the pull-back metric f^*g is

$$\phi^*g = g_{ab}(\phi(x)) d\phi^a(x) d\phi^b(x) = g_{ab}(\phi(x)) \partial_\mu \phi^a \partial_\nu \phi^b dx^\mu dx^\nu \ .$$

The local expression for $\langle df, df\rangle$ is

$$\langle df, df\rangle = \langle d\phi, d\phi\rangle = h^{\mu\nu}(x)\left(g_{ab}(\phi(x))\partial_\mu\phi^a\partial_\nu\phi^b\right) = \mathrm{tr}_h(f^*g)\ , \tag{B.10}$$

where tr_h is the trace defined by contraction with the metric h, and where f^*g is the pullback by f to N of the metric g. The Lagrangian $\mathcal{L}$ is defined by

$$S = \int d\,\mathrm{vol}_h \mathcal{L}\ . \tag{B.11}$$

In local coordinates it takes the form

$$\mathcal{L}_{|\mathcal{V}} = \frac{1}{2}R[h] - g_{ab}(\phi(x))\partial_\mu\phi^a\partial^\mu\phi^b\ , \tag{B.12}$$

and the corresponding equations of motion are

$$R[h]_{\mu\nu} - \frac{1}{2}R[h]h_{\mu\nu} = T_{\mu\nu}\ , \tag{B.13}$$

$$\Delta_h\phi^a + \Gamma^a{}_{bc}\partial_\mu\phi^b\partial^\mu\phi^c = 0\ , \tag{B.14}$$

where $R[h]_{\mu\nu}$ is the Ricci tensor of (N,h). We denote by Δ_h the pseudo-Riemannian Laplace operator

$$\Delta_h = \mathrm{tr}_h(D^{(h)}D^{(h)}) = h^{\mu\nu}D^{(h)}_\mu D^{(h)}_\nu\ , \tag{B.15}$$

where $D^{(h)}$ is the Levi-Civita connection on (N,h). $\Gamma^a{}_{bc}$ are the Christoffel symbols with respect to the Levi-Civita connection on (M,g). Finally

$$T_{\mu\nu} := \frac{-2}{\sqrt{|\det h|}}\frac{\delta\mathcal{L}_{\mathrm{Matter}}}{\delta h^{\mu\nu}} = 2g_{ab}\partial_\mu\phi^a\partial_\nu\phi^b - h_{\mu\nu}g_{ab}\partial_\rho\phi^a\partial^\rho\phi^b \tag{B.16}$$

is the energy momentum tensor, which is proportional to the variation of the matter Lagrangian

$$\mathcal{L}_{\mathrm{Matter}} = -\sqrt{|\det(h)|}g_{ab}\partial_\mu\phi^a\partial^\mu\phi^b \tag{B.17}$$

with respect to the metric h.

The coordinate-free version of the equations of motion is:

$$\mathrm{Ric}[h] - \frac{1}{2}R[h]h = T\ ,\ \ \text{where } T := 2f^*g - \langle df, df\rangle h\ , \tag{B.18}$$

$$\mathrm{tr}_h Ddf = 0\ , \tag{B.19}$$

where D is the covariant derivative on $T^*N \otimes f^*TM$ induced by the Levi-Civita connections on (N,h) and (M,g). Equation (B.19) is the equation satisfied by a harmonic map $f : (N,h) \to (M,g)$ between two pseudo-Riemannian manifolds.

To obtain the local coordinate form of $\mathrm{tr}_h Ddf$, we start with Ddf and evaluate it in local coordinates:

$$D_\mu \partial_\nu \phi^a = D_\mu^{(h)} \partial_\nu \phi^a + \partial_\mu \phi^b \Gamma^a{}_{bc} \partial_\nu \phi^c \ , \tag{B.20}$$

where $D^{(h)}$ is the Levi-Civita connection on (N, h), and where $\partial_\mu \phi^b \Gamma^a{}_{bc}$ is the pullback by f to N of the connection coefficients $\Gamma^a{}_{bc}$ of the connection on M. Taking the trace using the metric h we obtain

$$\mathrm{tr}_h(Ddf) = h^{\mu\nu} \left(D_\mu^{(h)} \partial_\nu \phi^a + \Gamma^a{}_{bc} \partial_\mu \phi^b \partial_\nu \phi^c \right) = \Delta_h \phi^a + \Gamma^a{}_{bc} \partial_\mu \phi^b \partial^\mu \phi^c \ . \tag{B.21}$$

We remark that this expression does not require the existence of a metric on M: the metric g is not used explicitly, and instead of the Levi-Civita connection we could use any other connection on M.

B.2. Notation and Conventions

Our notation and conventions for space-times with Minkowski signature in four and in five dimensions are as follows.

We denote space-time indices by $\mu, \nu, \ldots$, and local Lorentz indices by $a, b, \cdots = 0, 1, 2, \ldots$. Indices $i, j, k, \cdots = 1, 2$ are reserved for $SU(2)_R$ indices.

Our (anti)symmetrization conventions are

$$[ab] = \tfrac{1}{2}(ab - ba) \quad , \quad (ab) = \tfrac{1}{2}(ab + ba) \tag{B.22}$$

and (c.f. (A.23) and (A.11))

$$\begin{aligned} da\, db &= \tfrac{1}{2} \left(da \otimes db + db \otimes da \right) \ , \\ da \wedge db &= da \otimes db - db \otimes da \ . \end{aligned} \tag{B.23}$$

We take the Lorentz metric η_{ab} to have signature $(- + + \cdots +)$. We denote the vielbein by $e_\mu{}^a$, and its inverse by $e_a{}^\mu$,

$$e_\mu{}^a \, e_a{}^\nu = \delta_\mu{}^\nu \quad , \quad e_a{}^\nu \, e_\nu{}^b = \delta_a{}^b \ . \tag{B.24}$$

The space-time metric $g_{\mu\nu}$ and the Lorentz metric η_{ab} are related by

$$g_{\mu\nu} = e_\mu{}^a \, \eta_{ab} \, e_\nu{}^b \ . \tag{B.25}$$

The Christoffel symbols of the Levi-Civita connection read

$$\Gamma^\mu{}_{\nu\rho} = \tfrac{1}{2} g^{\mu\lambda} \left(2\, \partial_{(\nu} g_{\rho)\lambda} - \partial_\lambda g_{\nu\rho} \right) \ . \tag{B.26}$$

We note

$$\Gamma^{\rho}{}_{\rho\mu} = \tfrac{1}{2}\, g^{\lambda\rho}\partial_{\mu}\, g_{\lambda\rho} = \tfrac{1}{2}\,\partial_{\mu} \ln|g| \;, \tag{B.27}$$

where $g = \det g_{\mu\nu}$.

The Riemann tensor is (c.f. (A.41))

$$R_{\mu\nu}{}^{\rho}{}_{\sigma} = 2\partial_{[\mu}\Gamma^{\rho}{}_{\nu]\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\,\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\,\Gamma^{\lambda}{}_{\mu\sigma} \;. \tag{B.28}$$

We raise and lower space-time indices by contracting with the space-time metric, i.e.

$$R_{\mu\nu\rho\sigma} = g_{\rho\lambda}\, R_{\mu\nu}{}^{\lambda}{}_{\sigma} \;. \tag{B.29}$$

The Riemann tensor satisfies the pair exchange property $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$.

We define covariant derivatives (c.f. (A.49))

$$\begin{aligned} D_{\mu}V_{\nu} &= \partial_{\mu}V_{\nu} - \Gamma^{\lambda}{}_{\mu\nu}\, V_{\lambda} \;, \\ D_{\mu}V^{\nu} &= \partial_{\mu}V^{\nu} + \Gamma^{\nu}{}_{\mu\lambda}\, V^{\lambda} \;. \end{aligned} \tag{B.30}$$

We have

$$[D_{\mu}, D_{\nu}]V_{\rho} = -R_{\mu\nu}{}^{\lambda}{}_{\rho}\, V_{\lambda} \;. \tag{B.31}$$

We define the Ricci tensor by

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = R_{\lambda\mu}{}^{\lambda}{}_{\nu} = R_{\mu}{}^{\lambda}{}_{\nu\lambda} \;. \tag{B.32}$$

It satisfies the property

$$R_{\mu\nu} = R_{\nu\mu} \;. \tag{B.33}$$

The Ricci scalar is

$$R = g^{\mu\nu}\, R_{\mu\nu} \;. \tag{B.34}$$

With these conventions, the kinetic terms for physical fields in a gravitational action take the form (we set $\kappa^2 = 8\pi G_N = 1$)

$$L = \tfrac{1}{2}R - \tfrac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \;. \tag{B.35}$$

We define covariant derivatives of vectors V^a by

$$\mathcal{D}_{\mu}V^{a} = \partial_{\mu}V^{a} + \omega_{\mu}{}^{ab}\, V_{b} \;, \tag{B.36}$$

where $\omega_{\mu}{}^{ab}$ denotes the spin connection,

$$\omega_{\mu}{}^{ab} = 2e^{\nu[a}\,\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}\, e^{b]\sigma}\, e_{\mu c}\partial_{\nu}e_{\sigma}{}^{c} \;, \tag{B.37}$$

and satisfies the compatibility requirement

$$0 = \mathcal{D}_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a + \omega_\mu{}^{ab}\, e_{\nu b} - \Gamma^\rho{}_{\mu\nu}\, e_\rho{}^a \,. \tag{B.38}$$

Defining

$$\Omega_{ab}{}^c = 2 e_{[a}^{\;\mu} e_{b]}^{\;\nu} \partial_\mu e_\nu{}^c, \tag{B.39}$$

we obtain

$$\omega_{a\,bc} = \tfrac{1}{2}\left(\Omega_{abc} + \Omega_{cab} - \Omega_{bca}\right) \,. \tag{B.40}$$

The associated Riemann tensor reads

$$R_{\mu\nu}{}^{ab} = 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\,\omega_{[\mu}{}^{ac}\,\omega_{\nu]c}{}^b \,, \tag{B.41}$$

and it is related to the one in (B.28) by

$$R_{\mu\nu}{}^\rho{}_\sigma = R_{\mu\nu}{}^{ab}\, e_a{}^\rho\, e_{\sigma b} \,. \tag{B.42}$$

We define the completely antisymmetric Levi-Civita tensor as follows. In four space-time dimensions, we take

$$\begin{aligned} \varepsilon^{\mu\nu\lambda\sigma} &= e\,\varepsilon^{abcd}\, e_a{}^\mu\, e_b{}^\nu\, e_c{}^\lambda\, e_d{}^\sigma \quad , \quad \varepsilon_{0123} = 1 \,, \\ \varepsilon_{\mu\nu\lambda\sigma} &= e^{-1}\,\varepsilon_{abcd}\, e_\mu{}^a\, e_\nu{}^b\, e_\lambda{}^c\, e_\sigma{}^d \,, \end{aligned} \tag{B.43}$$

where $e^{-1} = |g|^{-1/2}$. Similarly, in five space-time dimensions we take

$$\begin{aligned} \varepsilon^{\mu\nu\lambda\sigma\rho} &= e\,\varepsilon^{abcde}\, e_a{}^\mu\, e_b{}^\nu\, e_c{}^\lambda\, e_d{}^\sigma\, e_e{}^\rho \quad , \quad \varepsilon_{01235} = 1 \,, \\ \varepsilon_{\mu\nu\lambda\sigma\rho} &= e^{-1}\,\varepsilon_{abcde}\, e_\mu{}^a\, e_\nu{}^b\, e_\lambda{}^c\, e_\sigma{}^d\, e_\rho{}^e \,, \end{aligned} \tag{B.44}$$

where $e^{-1} = |g|^{-1/2}$.

In four dimensions, we define the dual of an antisymmetric tensor field F_{ab} by

$$\tilde{F}_{ab} = -\tfrac{i}{2}\,\varepsilon_{abcd}\, F^{cd} \,. \tag{B.45}$$

We denote the selfdual part of F_{ab} by F^+_{ab}, and the anti-selfdual part by F^-_{ab},

$$F^\pm_{ab} = \tfrac{1}{2}\left(F_{ab} \pm \tilde{F}_{ab}\right) \,. \tag{B.46}$$

In four dimensions, in the context of $\mathcal{N} = 2$ special geometry, we will encounter the $SU(2)_R$ valued selfdual tensor field T_{abij} and the $SU(2)_R$ valued anti-selfdual tensor field T^{ij}_{ab}. Accordingly, we introduce the notation

$$\begin{aligned} T^+_{ab} &= \tfrac{1}{2}\,\varepsilon^{ij}\, T_{abij} \,, \\ T^-_{ab} &= \tfrac{1}{2}\,\varepsilon_{ij}\, T^{ij}_{ab} \,, \end{aligned} \tag{B.47}$$

where the Levi-Civita symbol $\varepsilon_{ij} = -\varepsilon_{ji}$ satisfies

$$\varepsilon_{ij}\varepsilon^{jk} = -\delta_i{}^k \ , \tag{B.48}$$

with

$$\varepsilon_{12} = \varepsilon^{12} = 1 \tag{B.49}$$

and $\varepsilon_{ij}\varepsilon^{ji} = -2$. Under Hermitian conjugation (h.c.), selfdual becomes anti-selfdual and vice-versa. Any $SU(2)_R$ index i changes position under h.c., for instance

$$(T_{ab\,ij})^* = T_{ab}^{ij} \, . \tag{B.50}$$

B.3. Jacobians

The Jacobians for the coordinate transformations (458) take the form

$$\frac{D(x,u,\Upsilon,\bar{\Upsilon})}{D(x,y,\Upsilon,\bar{\Upsilon})} = \begin{pmatrix} \mathbb{1} & 0 & 0 & 0 \\ \frac{\partial u}{\partial x}\Big|_y & \frac{\partial u}{\partial y}\Big|_x & \frac{\partial u}{\partial \Upsilon}\Big|_{x,y} & \frac{\partial u}{\partial \bar{\Upsilon}}\Big|_{x,y} \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix} \tag{B.51}$$

and

$$\frac{D(x,y,\Upsilon,\bar{\Upsilon})}{D(x,u,\Upsilon,\bar{\Upsilon})} = \begin{pmatrix} \mathbb{1} & 0 & 0 & 0 \\ \frac{\partial y}{\partial x}\Big|_u & \frac{\partial y}{\partial u}\Big|_x & \frac{\partial y}{\partial \Upsilon}\Big|_{x,u} & \frac{\partial y}{\partial \bar{\Upsilon}}\Big|_{x,u} \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix} \, . \tag{B.52}$$

By the chain rule it is straightforward to evaluate

$$\frac{D(x,y,\Upsilon,\bar{\Upsilon})}{D(x,u,\Upsilon,\bar{\Upsilon})} = \begin{pmatrix} \mathbb{1} & 0 & 0 & 0 \\ \frac{1}{2}R & -\frac{1}{2}N & \frac{1}{2}F_{I\Upsilon} & \frac{1}{2}\bar{F}_{I\Upsilon} \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix} \, , \tag{B.53}$$

where $2F_{IJ} = R_{IJ} + iN_{IJ}$. This matrix can easily be inverted,

$$\frac{D(x,u,\Upsilon,\bar{\Upsilon})}{D(x,y,\Upsilon,\bar{\Upsilon})} = \begin{pmatrix} \mathbb{1} & 0 & 0 & 0 \\ N^{-1}R & -2N^{-1} & N^{-1}F_{I\Upsilon} & N^{-1}\bar{F}_{I\Upsilon} \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix} . \tag{B.54}$$

In order to transform the Kähler metric (452) to special real coordinates (c.f. (460)), the following relations are useful,

$$\frac{\partial H}{\partial x^I} = 2v_I \ , \quad \frac{\partial H}{\partial y_I} = -2u^I \ . \tag{B.55}$$

Moreover, using the chain rule, one computes

$$\begin{aligned} \left.\frac{\partial v_I}{\partial x^J}\right|_y &= \frac{1}{2}\left(N + RN^{-1}R\right)_{IJ} \ , \\ \left.\frac{\partial v_I}{\partial y_J}\right|_x &= -\left.\frac{\partial u^J}{\partial x^I}\right|_y = 2\left(N^{-1}\right)^{IJ} \ , \\ \left.\frac{\partial v_I}{\partial u^J}\right|_x &= \frac{1}{2}R_{IJ} \ . \end{aligned} \tag{B.56}$$

The Jacobians for the coordinate transformations (503) are given by

$$\frac{D(x,y,\Upsilon,\bar{\Upsilon})}{D(x,u,\Upsilon,\bar{\Upsilon})} = \begin{pmatrix} \mathbb{1} & 0 & 0 & 0 \\ \frac{1}{2}R_+ & -\frac{1}{2}N_- & \frac{1}{2}(F_{I\Upsilon} + \bar{F}_{\bar{I}\Upsilon}) & \frac{1}{2}(\bar{F}_{\bar{I}\bar{\Upsilon}} + F_{I\bar{\Upsilon}}) \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix} \tag{B.57}$$

and

$$\frac{D(x,u,\Upsilon,\bar{\Upsilon})}{D(x,y,\Upsilon,\bar{\Upsilon})} = \begin{pmatrix} \mathbb{1} & 0 & 0 & 0 \\ N_-^{-1}R_+ & -2N_-^{-1} & N_-^{-1}(F_{I\Upsilon} + \bar{F}_{\bar{I}\Upsilon}) & N_-^{-1}(\bar{F}_{\bar{I}\bar{\Upsilon}} + F_{I\bar{\Upsilon}}) \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix} . \tag{B.58}$$

This reduces to the results for the Jacobians (B.53) and (B.54) when switching off the non-holomorphic deformation.

B.4. Superconformal formalism in four dimensions

The idea behind the superconformal approach to supergravity consists in using the superconformal symmetry as a powerful tool for constructing matter-coupled theories with local Poincaré supersymmetry, and in doing so to gain insights into the structure of Poincaré supergravity [230, 231, 232, 31, 25]. We refer to [22] for a recent detailed discussion.

To illustrate this construction, we begin by reviewing the formulation of Einstein gravity in four dimensions based on the bosonic conformal algebra.

B.4.1. Gravity as a conformal gauge theory

Consider the following action in four dimensions,

$$S = \int d^4x \sqrt{-g} \left(\tfrac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) + \tfrac{1}{12} R \phi^2 \right) , \tag{B.59}$$

where $\phi(x)$ denotes a real scalar field. Note that the sign of the kinetic energy term of the scalar field is opposite from the one of a physical scalar field, c.f. (B.35). This Lagrangian is invariant under local scale transformations, also called local dilatations or local Weyl transformations, given by

$$\phi \mapsto e^{\lambda_D} \phi \quad , \quad g_{\mu\nu} \mapsto e^{-2\lambda_D} g_{\mu\nu} . \tag{B.60}$$

Here, $\lambda_D(x)$ denotes the local parameter of Weyl transformations.

The field ϕ is called a compensating field (or, compensator), because it compensates for the non-invariance of the Einstein-Hilbert term under local scale transformations caused by the transformation properties of the metric, thus resulting in a Weyl-invariant action. We can eliminate the compensating field ϕ by performing the gauge-fixing

$$\phi \mapsto e^{\lambda_D} \phi \equiv \frac{\sqrt{6}}{\kappa^2} . \tag{B.61}$$

Inserting this into (B.59) results in the Einstein-Hilbert action,

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} R . \tag{B.62}$$

Here $\kappa^2 = 8\pi G_N$, where G_N denotes the Newton's constant. Thus, the Einstein-Hilbert action can be obtained by starting from an action that possesses invariance under local scale transformations due to the presence of a compensating field, and then eliminating the compensating field by going to a particular Weyl gauge. That is, Einstein gravity emerges from a theory that is initially invariant under transformations associated with the generators of the bosonic conformal algebra. We review this algebra next.

B.4.2. The bosonic conformal algebra

The bosonic conformal algebra in four dimensions is isomorphic to $so(4,2)$, and contains generators P_a, M_{ab}, K_a, D associated with translations, Lorentz transformations, special conformal transformations and dilations, respectively. These generators satisfy the algebra (we only give the commutators that are non-vanishing)

$$\begin{aligned} \big[M_{ab}, M_{cd}\big] &= 4\,\eta_{[a[c}M_{d]b]} = \eta_{ac}M_{db} - \eta_{bc}M_{da} - \eta_{ad}M_{cb} + \eta_{bd}M_{ca}\ , \\ \big[P_a, M_{bc}\big] &= 2\,\eta_{a[b}P_{c]}\ , \\ \big[K_a, M_{bc}\big] &= 2\,\eta_{a[b}K_{c]}\ , \\ \big[P_a, K_b\big] &= 2\,(\eta_{ab}\,D + M_{ab})\ , \\ \big[D, P_a\big] &= P_a\ , \\ \big[D, K_a\big] &= -K_a\ . \end{aligned} \tag{B.63}$$

To each of these generators, we assign a local parameter, as well as a gauge field. This is summarized in the Table B.6 below.

generator	P_a	M_{ab}	K_a	D
parameter	ξ^a	λ^{ab}	λ_K^a	λ_D
gauge field	${e_\mu}^a$	ω_μ^{ab}	${f_\mu}^a$	b_μ
Weyl weight w	-1	0	1	0

Table B.6: The bosonic conformal algebra: generators, local parameters, gauge fields, Weyl weights.

The translations P_a, which are gauged by e_μ^a, play a special rule, and will be considered separately. Under infinitesimal conformal transformations generated by M_{ab}, K_a, D, the gauge fields transform as follows,

$$
\begin{aligned}
\delta e_\mu{}^a &= -\lambda^{ab}\, e_{\mu b} - \lambda_D\, e_\mu{}^a \ , \\
\delta \omega_\mu^{ab} &= \partial_\mu \lambda^{ab} + 2\omega_{\mu c}{}^{[a}\, \lambda^{b]c} - 4\lambda_K^{[a}\, e_\mu{}^{b]} \ , \\
\delta f_\mu{}^a &= \partial_\mu \lambda_K^a - b_\mu\, \lambda_K^a + \omega_\mu{}^{ab}\, \lambda_{Kb} - \lambda^{ab}\, f_{\mu b} + \lambda_D\, f_\mu{}^a \ , \\
\delta b_\mu &= \partial_\mu \lambda_D + 2\lambda_K^a\, e_{\mu a} \ .
\end{aligned} \tag{B.64}
$$

The commutators of two transformations (B.64) yield a realization of the conformal algebra. The transformation behaviour under dilatations is specified by the Weyl weight w of each of the gauge fields. The vielbein has weight $w = -1$, the field $f_\mu{}^a$ has weight $w = 1$, while the other gauge fields have $w = 0$. Note that all the gauge fields, with the exception of the vielbein, transform under special conformal transformations.

Next, we introduce a field strength for each of the generators of the conformal algebra. These field strengths, of the form $R_{\mu\nu}{}^A$, transform covariantly under conformal transformations. They are given by

$$
\begin{aligned}
R_{\mu\nu}{}^a(P) &= 2\left(\partial_{[\mu} + b_{[\mu}\right) e_{\nu]}{}^a + 2\omega_{[\mu}{}^{ab}\, e_{\nu]b} = 2\,\mathcal{D}_{[\mu} e_{\nu]}{}^a \ , \\
R_{\mu\nu}{}^{ab}(M) &= 2\partial_{[\mu} \omega_{\nu]}{}^{ab} + 2\omega_{[\mu}{}^a{}_c\, \omega_{\nu]}{}^{cb} + 8 f_{[\mu}{}^{[a}\, e_{\nu]}{}^{b]} \ , \\
R_{\mu\nu}{}^a(K) &= 2\left(\partial_{[\mu} - b_{[\mu}\right) f_{\nu]}{}^a + 2\omega_{[\mu}{}^{ab}\, f_{\nu]b} \ , \\
R_{\mu\nu}(D) &= 2\partial_{[\mu} b_{\nu]} - 4 f_{[\mu}{}^a\, e_{\nu]a} \ .
\end{aligned} \tag{B.65}
$$

It can checked that these field strengths transform covariantly under the transformations (B.64).

Since we are interested in the construction of Einstein gravity as a gauge fixed version of a gravitational theory that is invariant under conformal transformations, not all of the gauge fields associated to the conformal algebra can describe independent gauge fields. To be able to identify the translations generated by P_a with space-time diffeomorphisms, one needs to impose a constraint on the associated field strength $R_{\mu\nu}{}^a(P)$, so as to ensure that the translation gauge field e_μ^a becomes a vielbein field (frame) over space-time. In addition, the gauge field $f_\mu{}^a$ for special conformal transformations needs to be eliminated as an independent gauge field. This is achieved by imposing the constraints

$$
\begin{aligned}
R_{\mu\nu}{}^a(P) &= 0 \ , \\
R_{\mu\nu}{}^{ab}(M)\, e_b{}^\nu &= 0 \ .
\end{aligned} \tag{B.66}
$$

In this way, two of the gauge fields, namely the spin connection ω_μ^{ab} and the gauge field $f_\mu{}^a$ for special conformal transformations, become composite fields,

$$\begin{aligned} \omega_\mu^{ab} &= \omega_\mu^{ab}(e) + 2e_\mu{}^{[a}\, e^{b]\nu}\, b_\nu \ , \\ f_\mu{}^a &= -\tfrac{1}{4}\, R_\mu{}^a + \tfrac{1}{24} e_\mu{}^a\, R \ , \end{aligned} \tag{B.67}$$

with $\omega_\mu^{ab}(e)$ given by (B.37). The constraint $R_{\mu\nu}{}^a(P) = 2\,\mathcal{D}_{[\mu} e_{\nu]}{}^a = 0$ is the condition for metric compatibility, but now in the presence of the dilational connection b_μ. Note that the Riemann tensor computed from the spin connection (B.67) does not have the pair exchange property mentioned below (B.29). To obtain the relation for $f_\mu{}^a$, we expressed the second constraint in (B.66) as

$$\Big(R_{\mu\nu}{}^{ab} + 8 f_{[\mu}{}^{[a}\, e_{\nu]}{}^{b]}\Big)\, e_b{}^\nu = 0 \ , \tag{B.68}$$

where $R_{\mu\nu}{}^{ab}$ is the Riemann tensor constructed out of the spin connection ω_μ^{ab} that also contains the gauge field b_μ, c.f. (B.67). Then, using the definitions for the Ricci tensor[77] and the Ricci scalar,

$$R_{\mu\nu} = R_{\mu\rho\nu}{}^\rho \quad , \quad R = g^{\mu\nu}\, R_{\mu\nu} \ , \tag{B.69}$$

we obtain

$$R_\mu{}^a + 2\,(2 f_\mu{}^a + f_\nu{}^\nu\, e_\mu{}^a) = 0 \ , \tag{B.70}$$

where $f_\nu{}^\nu = f_\nu{}^a\, e_a{}^\nu$. Contracting this relation with $e_a{}^\mu$ gives

$$f_\mu{}^\mu = -\tfrac{1}{12}\, R \ , \tag{B.71}$$

Inserting this into (B.70) gives the relation in (B.67).

As a check of (B.67), one verifies that when inserting the transformation law for $e_\mu{}^a$ and for b_μ into (B.67), one correctly reproduces the transformation laws for ω_μ^{ab} and $f_\mu{}^a$ given in (B.64).

Upon imposing the constraints (B.66), the independent gauge fields in (B.67) are the vielbein $e_\mu{}^a$ and the gauge field for dilations b_μ. Inspection of the transformation law for the field b_μ given in (B.64) shows that the value of b_μ can be arbitrarily changed by performing a special conformal transformation. Therefore, we fix b_μ to the value

$$b_\mu = 0 \quad , \quad \mathrm{K-gauge} \ , \tag{B.72}$$

[77] We use the last equation given in (B.32).

by means of a special conformal transformation. Since this represents a gauge-fixing of special conformal transformations with gauge parameter $\lambda_{K\mu} \equiv \lambda_K^a \, e_a{}^\mu$, this is called the K-gauge. In this gauge, special conformal transformations are no longer independent transformations. Inspection of (B.64) shows that in order to stay in the K-gauge (B.72), the allowed residual special conformal transformations are

$$\lambda_{K\mu} = -\tfrac{1}{2} \, \partial_\mu \lambda_D \, . \tag{B.73}$$

B.4.3. Weyl multiplet

The extension of the above to supergravity is called the superconformal approach to supergravity [230, 231, 232, 31, 25]. The standard superconformal approach to $\mathcal{N} = 2$ supergravity in four dimensions is based on the Weyl multiplet. In its standard formulation, the Weyl multiplet is a supermultiplet with 24+24 bosonic and fermionic off-shell degrees of freedom.[78] Let us briefly describe this multiplet.

The $\mathcal{N} = 2$ superconformal algebra contains the following bosonic generators: it contains the bosonic conformal algebra discussed in the previous subsection as well as two bosonic generators T and $U_i{}^j$ that generate $U(1)_R$ and $SU(2)_R$ R-symmetry transformations, respectively. As before, we assign a local parameter and a gauge field to each of these bosonic generators. The gauge fields associated with $U(1)_R$ and $SU(2)_R$ R-symmetry transformations will be denoted by $(A_\mu, \mathcal{V}_\mu{}^i{}_j)$. This is summarized in Table B.7 below.

bosonic generator	P_a	M_{ab}	K_a	D	T	$U^j{}_i$
parameter	ξ^a	λ^{ab}	λ_K^a	λ_D	λ_T	$\lambda^i{}_j$
gauge field	$e_\mu{}^a$	ω_μ^{ab}	$f_\mu{}^a$	b_μ	A_μ	$\mathcal{V}_\mu{}^i{}_j$

Table B.7: The $\mathcal{N} = 2$ bosonic subalgebra: generators, local parameters, gauge fields.

The bosonic components of the Weyl multiplet are given by the gauge fields displayed in table B.7, together with a complex anti-selfdual tensor field T_{ab}^-

[78] Recently, a new Weyl multiplet was constructed in [233], called the dilaton Weyl multiplet, with $24 + 24$ off-shell degrees of freedom.

and a real scalar field D:

$$(e_\mu{}^a, \omega_\mu^{ab}, f_\mu{}^a, b_\mu, A_\mu, \mathcal{V}_\mu{}^i{}_j, T^-_{ab}, D) \ . \tag{B.74}$$

These describe 24 independent bosonic degrees of freedom, as depicted in Table B.8.

field	subtraction by gauge transformations	number of degrees of freedom left
$e_\mu{}^a$	P_a, M_{ab}, D	$16 - (4 + 6 + 1) = 5$
$\omega_\mu{}^{ab}$		composite field
$f_\mu{}^a$		composite field
b_μ	K_a	0
A_μ	$U(1)_R$	$4 - 1 = 3$
$\mathcal{V}_\mu{}^i{}_j$	$SU(2)_R$	$12 - 3 = 9$
T^-_{ab}		6
D		1

Table B.8: Counting of bosonic off-shell degrees of freedom: $5 + 9 + 3 + 6 + 1 = 24$.

The component fields of the Weyl multiplet carry a Weyl weight w and a chiral ($U(1)_R$) weight c. This is summarized for the bosonic components in table B.9.

field	$e_\mu{}^a$	b_μ	A_μ	$\mathcal{V}_\mu{}^i{}_j$	T^-_{ab}	D	$\omega_\mu{}^{ab}$	$f_\mu{}^a$
w	-1	0	0	0	1	2	0	1
c	0	0	0	0	-1	0	0	0

Table B.9: Weyl and chiral weights (w and c, respectively) of the Weyl multiplet bosonic component fields.

As indicated in table B.9, the gauge fields $\omega_\mu{}^{ab}$ and $f_\mu{}^a$ are composite fields. Their expressions are obtained by imposing constraints, as in (B.66). While we still impose $R_{\mu\nu}{}^a(P) = 0$, which results in the expression for the spin connection given in (B.67), we impose the following constraint on the curvature $R_{\mu\nu}{}^{ab}(M)$, taking into account that there are additional fields in the Weyl multiplet,[79]

$$R_{ac}{}^{bc}(M) + i\tilde{R}_a{}^b(T) - \tfrac{1}{4} T^-_{ac}\, T^{+bc} + \tfrac{3}{2}\, \delta_a{}^b D = 0 \,, \tag{B.75}$$

where $\tilde{R}^{ab}(T)$ denotes the dual of the $U(1)_R$ field strength $R_{ab}(T)$, c.f. (B.45), where $R_{ab}(T) = e_a{}^\mu e_b{}^\nu\, R_{\mu\nu}(T)$ with

$$R_{\mu\nu}(T) = 2\partial_{[\mu} A_{\nu]} \,. \tag{B.76}$$

Note that all the terms in the linear combination (B.75) have Weyl weight 2.

The constraint (B.75) results in

$$R_a{}^b + 2\left(2 f_a{}^b + f_c{}^c\, \delta_a{}^b\right) + i\tilde{R}_a{}^b(T) - \tfrac{1}{4} T^-_{ac}\, T^{+bc} + \tfrac{3}{2}\, \delta_a{}^b D = 0 \,, \tag{B.77}$$

where $R_a{}^b = R_{ac}{}^{bc}$, with $R_{\mu\nu}{}^{ab}$ the Riemann tensor constructed out of the spin connection ω_μ^{ab} that also contains the gauge field b_μ, c.f. (B.67). Then, contracting (B.77) gives

$$R + 12 f_a{}^a + 6D = 0 \,, \tag{B.78}$$

where $R = R_a{}^a$. Therefore, we infer

$$f_a{}^a = -\tfrac{1}{12}\, R - \tfrac{1}{2}\, D \,. \tag{B.79}$$

Inserting this into (B.77) gives the relation

$$f_\mu{}^a = \tfrac{1}{2}\left(-\tfrac{1}{2} R_\mu{}^a - \tfrac{1}{4}\left(D - \tfrac{1}{3}\, R\right) e_\mu{}^a - \tfrac{1}{2} i\tilde{R}_\mu{}^a(T) + \tfrac{1}{8} T^-_{\mu b} T^{ab+}\right) \,. \tag{B.80}$$

B.4.4. Covariant derivatives

In the superconformal approach one introduces covariant derivatives $\mathcal{D}_\mu$ and D_μ. The first one, $\mathcal{D}_\mu$, denotes a covariant derivative with respect to Lorentz, dilatations, $U(1)_R$ and $SU(2)_R$ transformations. The second one, D_μ, denotes a covariant derivative with respect to these transformations as well as with respect to special conformal transformations,[80] and it is used to construct actions that are invariant under superconformal transformations. Let us illustrate this.

[79] Note that there are additional fermionic terms in this expression which we have suppressed.
[80] Here, D_μ should not be confused with the Levi-Civita connection (B.30).

Consider a scalar field ϕ with Weyl weight w and chiral weight c. It transforms as

$$\begin{aligned} \delta_D\phi &= w\,\lambda_D\,\phi\,, \\ \delta_T\phi &= ic\,\lambda_T\,\phi \end{aligned} \tag{B.81}$$

under infinitesimal dilatational and $U(1)_R$ transformations. The associated covariant derivative of ϕ is

$$\mathcal{D}_\mu\phi = (\partial_\mu - w\,b_\mu - ic\,A_\mu)\,\phi\,. \tag{B.82}$$

Note that $D_\mu\phi = \mathcal{D}_\mu\phi$. Since the dilational connection b_μ transforms as in (B.64) under special conformal transformations, $\mathcal{D}^a\phi$ undergoes a K-transformation,

$$\delta_K\mathcal{D}^a\phi = -2w\,\lambda_K^a\phi\,, \tag{B.83}$$

that needs to be compensated for when constructing an invariant action. To this end, consider evaluating

$$D_\mu D^a\phi = D_\mu\mathcal{D}^a\phi = \mathcal{D}_\mu\mathcal{D}^a\phi + 2w\,f_\mu{}^a\,\phi\,, \tag{B.84}$$

where

$$\mathcal{D}_\mu\mathcal{D}^a\phi = \partial_\mu\mathcal{D}^a\phi - (w+1)b_\mu\mathcal{D}^a\phi - ic\,A_\mu\mathcal{D}^a\phi + \omega_\mu{}^{ab}\mathcal{D}_b\phi\,. \tag{B.85}$$

Here we used that the covariant derivative $\mathcal{D}_\mu$ of a vector V^a of Weyl weight w and chiral weight c is

$$\mathcal{D}_\mu V^a = \partial_\mu V^a - w\,b_\mu V^a - ic\,A_\mu V^a + \omega_\mu{}^{ab}V_b\,, \tag{B.86}$$

c.f. (B.36). Then, under K-transformations, $D_\mu D^\mu\phi$ transforms as

$$\delta_K\left(D_\mu D^\mu\phi\right) = 4(1-w)\lambda_K^a\,\mathcal{D}_a\phi\,. \tag{B.87}$$

Choosing $w=1$ renders $D_\mu D^\mu\phi$ invariant under K-transformations. Then, the quantity $e\phi D_\mu D^\mu\phi$, which has Weyl weight zero, is invariant under both K-transformations and under local dilations. It can thus be used as a Lagrangian that is invariant under the transformations associated with the bosonic conformal algebra discussed earlier. It contains the term $\phi^2\,f_\mu{}^\mu \propto \phi^2 R$, as in (B.59).

Similarly, consider evaluating $D_\mu D_c T^+_{ab}$, where T^+_{ab} has Weyl and chiral weights $w=c=1$, so that

$$\mathcal{D}_\nu T^{ab+} = (\partial_\nu - b_\nu - iA_\nu)\,T^{ab+} + \omega_\nu{}^{ad}T_d{}^{b+} + \omega_\nu{}^{bd}T^a{}_d{}^+\,. \tag{B.88}$$

Taking into account that both b_ν and $\omega_\nu{}^{ab}$ transform under K-transformations, c.f. (B.64), we infer

$$\delta_K \mathcal{D}_\nu T^{ab+} = -2\lambda_{K\nu}\, T^{ab+} - 4\lambda_K^{[a}\, e^{d]}{}_\nu\, T_d{}^{b+} - 4\lambda_K^{[b}\, e^{d]}{}_\nu\, T^a{}_d{}^+ \,. \tag{B.89}$$

This needs to be compensated for in $D_\mu D_\nu T^+_{ab}$,

$$\begin{aligned} D_\mu D_c T^+_{ab} &= \mathcal{D}_\mu \mathcal{D}_c T^+_{ab} + 2 f_{\mu c} T^+_{ab} + 4 f_\mu{}^{[a}\, \delta^{d]}_c\, T_d{}^{b+} + 4 f_\mu{}^{[b}\, \delta^{d]}_c\, T^a{}_d{}^+ \\ &= \mathcal{D}_\mu \mathcal{D}_c T^+_{ab} + 2 f_{\mu c} T^+_{ab} - 4 f_{\mu[a}\, T^+_{b]c} + 4 f_\mu{}^d\, \eta_{c[a} T^+_{b]d} \,. \end{aligned} \tag{B.90}$$

Hence

$$D_\mu D^c T^+_{cb} = \mathcal{D}_\mu \mathcal{D}^c T^+_{cb} - 2 f_\mu{}^c T^+_{cb} \,. \tag{B.91}$$

It follows that

$$T^{ab-} D_a D^c T^+_{cb} = T^{ab-} \mathcal{D}_a \mathcal{D}^c T^+_{cb} - 2 f_a{}^c\, T^{ab-} T^+_{cb} \,. \tag{B.92}$$

This relation will be used in the main text.

B.4.5. Vector multiplets

The field content of a four-dimensional abelian vector multiplet is given by a complex scalar field X, an abelian gauge field[81] A_μ, an $SU(2)_R$ triplet of scalar fields Y_{ij}, and an $SU(2)$ doublet of chiral fermions Ω_i, i.e. $(X, \Omega_i, A_\mu, Y^{ij})$, where Y_{ij} is a symmetric matrix satisfying the reality condition

$$Y_{ij} = \varepsilon_{ik}\, \varepsilon_{jl}\, Y^{kl} \quad , \quad Y^{ij} = (Y_{ij})^* \,. \tag{B.93}$$

Here, $i = 1, 2$ is an $SU(2)_R$ index. Thus, off-shell, an abelian vector multiplet has eight bosonic and eight fermionic real degrees of freedom.

The component fields of a vector multiplet carry a Weyl weight w and a chiral weight c. This is summarized for the bosonic components in Table B.10.

B.4.6. Hypermultiplets

The bosonic degrees of freedom of r hypermultiplets are described by $4r$ real scalar fields ϕ^A $(A = 1, \ldots, 4r)$ that can be conveniently described in terms of local sections $A_i{}^\alpha(\phi)$ of an $\mathrm{Sp}(r) \times \mathrm{Sp}(1)$ bundle $(\alpha = 1, \ldots, 2r; i = 1, 2)$ [28]. In the main text we set $r = n_H + 1$. The hypermultiplets provide one of the

[81]Not to be confused with the $U(1)$ gauge field in the Weyl multiplet (B.74).

	vector multiplet			hyper-multiplet
field	X^I	A_μ^I	Y_{ij}^I	A_i^α
w	1	0	2	1
c	-1	0	0	0

Table B.10: Weyl and chiral weights (w and c, respectively) of the vector and hypermultiplet bosonic component fields.

compensating multiplets for obtaining Poincaré supergravity. In this review, we will not be concerned with physical hypermultiplets, and hence set $n_H = 0$.

The hyper-Kähler potential χ and the covariant derivative $\mathcal{D}_\mu A_i{}^\alpha(\phi)$ are defined by

$$\begin{aligned} \varepsilon_{ij}\,\chi &= \bar{\Omega}_{\alpha\beta} A_i{}^\alpha A_j{}^\beta \,, \\ \mathcal{D}_\mu A_i{}^\alpha &= \partial_\mu A_i{}^\alpha - b_\mu A_i{}^\alpha + \tfrac{1}{2} V_{\mu i}{}^j A_j{}^\alpha + \partial_\mu \phi^A\, \Gamma_A{}^\alpha{}_\beta\, A_i{}^\beta \,, \end{aligned} \tag{B.94}$$

in accordance with the Weyl weight given in Table B.10. The connection $\Gamma_A{}^\alpha{}_\beta$ takes values in $\mathrm{sp}(n_H + 1)$, and $\bar{\Omega}_{\alpha\beta}$ is a covariantly constant antisymmetric tensor [28].

B.5. Superconformal formalism in five dimensions

B.5.1. Weyl multiplet

The superconformal approach to $\mathcal{N} = 2$ supergravity in five space-time dimensions [20, 234, 235, 29, 233] is based on the Weyl multiplet. In its standard formulation, the Weyl multiplet in five dimensions is a supermultiplet with 32+32 bosonic and fermionic off-shell degrees of freedom. When reduced to four space-time dimensions [101], it decomposes into the Weyl multiplet in four dimensions with 24+24 bosonic and fermionic off-shell degrees of freedom, and a vector multiplet with 8+8 bosonic and fermionic off-shell degrees of freedom.

The algebra underlying the superconformal approach is the $\mathcal{N} = 2$ superconformal algebra. In five dimensions, this superalgebra contains the bosonic generators $P_a, M_{ab}, K_a, D, U_i{}^j$ associated with translations, Lorentz transformations, special conformal transformations, dilations and $SU(2)_R$ R-symmetry transformations, respectively. One assigns a local parameter and a gauge field to each of these bosonic generators. The gauge fields associated with $SU(2)_R$ R-symmetry transformations will be denoted by $\mathcal{V}_\mu{}^i{}_j$, which is an anti-hermitian, traceless matrix in the indices i, j. This is summarized in Table B.11 below.

bosonic generator	P_a	M_{ab}	K_a	D	$U^j{}_i$
parameter	ξ^a	λ^{ab}	λ_K^a	λ_D	$\lambda^i{}_j$
gauge field	$e_\mu{}^a$	ω_μ^{ab}	$f_\mu{}^a$	b_μ	$\mathcal{V}_\mu{}^i{}_j$

Table B.11: The $\mathcal{N} = 2$ bosonic subalgebra: generators, local parameters, gauge fields.

The bosonic components of the Weyl multiplet are given by the gauge fields displayed in Table B.12 together with a real anti-symmetric tensor field T_{ab} and a real scalar field D:

$$(e_\mu{}^a, \omega_\mu^{ab}, f_\mu{}^a, b_\mu, \mathcal{V}_\mu{}^i{}_j, T_{ab}, D) \ . \tag{B.95}$$

These describe 32 independent bosonic degrees of freedom.

field	subtraction by gauge transformations	number of degrees of freedom left
$e_\mu{}^a$	P_a, M_{ab}, D	$25 - (5 + 10 + 1) = 9$
$\omega_\mu{}^{ab}$		composite field
$f_\mu{}^a$		composite field
b_μ	K_a	0
$\mathcal{V}_\mu{}^i{}_j$	$SU(2)_R$	$15 - 3 = 12$
T_{ab}		10
D		1

Table B.12: Counting of bosonic off-shell degrees of freedom: $9 + 12 + 10 + 1 = 32$.

The component fields of the Weyl multiplet carry a Weyl weight w. This is summarized for the bosonic components in Table B.13.

field	$e_\mu{}^a$	b_μ	$\mathcal{V}_\mu{}^i{}_j$	T_{ab}	D	$\omega_\mu{}^{ab}$	$f_\mu{}^a$
w	-1	0	0	1	2	0	1

Table B.13: Weyl weights of the Weyl multiplet bosonic component fields [23].

As indicated in Table B.12, the gauge fields $\omega_\mu{}^{ab}$ and $f_\mu{}^a$ are composite fields. Their expressions are obtained by imposing constraints on the associated field strengths $R_{\mu\nu}{}^a(P)$ and $R_{\mu\nu}{}^{ab}(M)$ [23],[82]

$$\begin{aligned} R_{\mu\nu}{}^a(P) &= 2\mathcal{D}_{[\mu}e_{\nu]}{}^a = 0 \,, \\ e_a{}^\mu \, R_{\mu\nu}{}^{ab}(M) &= e_a{}^\mu \left(2\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b + 8e_{[\mu}{}^{[a} f_{\nu]}{}^{b]}\right) = 0 \,. \end{aligned} \tag{B.96}$$

Here, the covariant derivative $\mathcal{D}_\mu$ of a vector V^a of Weyl weight w is

$$\mathcal{D}_\mu V^a = \partial_\mu V^a - w\, b_\mu V^a + \omega_\mu{}^{ab} V_b \,. \tag{B.97}$$

We infer from (B.96),

$$\begin{aligned} f_a{}^a &= -\frac{1}{16} R \,, \\ f_\mu{}^a &= \frac{1}{6}\left(-R_\mu{}^a + \frac{1}{8} e_\mu{}^a R\right) \,. \end{aligned} \tag{B.98}$$

	vector multiplet			hyper-multiplet
field	σ^I	A_μ^I	Y_{ij}^I	$A_i{}^\alpha$
w	1	0	2	$\frac{3}{2}$

Table B.14: Weyl weights w of the vector and hypermultiplet bosonic component fields.

B.5.2. Vector multiplets

The field content of a five-dimensional abelian vector multiplet is given by a real scalar field σ, an abelian gauge field A_μ, an $SU(2)_R$ triplet of scalar

[82] Note that our definition of $R_{\mu\nu}{}^{ab}(M)$ differs from the one in [23] by an overall minus sign.

fields Y_{ij}, and an $SU(2)_R$ doublet of symplectic Majorana fermions λ^i, i.e. $(\sigma, \lambda^i, A_\mu, Y^{ij})$, where Y^{ij} is a symmetric matrix satisfying the reality condition

$$Y_{ij} = \varepsilon_{ik}\,\varepsilon_{jl}\,Y^{kl} \quad , \quad Y^{ij} = (Y_{ij})^* \; . \tag{B.99}$$

Here, $i = 1, 2$ is an $SU(2)_R$ index. Thus, off-shell, an abelian vector multiplet has eight bosonic and eight fermionic real degrees of freedom.

The component fields of a vector multiplet carry a Weyl weight w. This is summarized for the bosonic components in Table B.14.

B.5.3. Hypermultiplets

As we mentioned in B.4.6, the bosonic degrees of freedom of r hypermultiplets are described by $4r$ real scalar fields ϕ^A $(A = 1, \ldots, 4r)$ that can be conveniently described in terms of local sections $A_i{}^\alpha(\phi)$ of an $\mathrm{Sp}(r) \times \mathrm{Sp}(1)$ bundle $(\alpha = 1, \ldots, 2r; i = 1, 2)$ [28]. In the main text we set $r = n_H + 1$.

In five space-time dimensions, the hyper-Kähler potential χ and the covariant derivative $\mathcal{D}_\mu A_i{}^\alpha(\phi)$ are defined by

$$\begin{aligned} \varepsilon_{ij}\,\chi &= \Omega_{\alpha\beta} A_i{}^\alpha A_j{}^\beta \, , \\ \mathcal{D}_\mu A_i{}^\alpha &= \partial_\mu A_i{}^\alpha - \tfrac{3}{2} b_\mu A_i{}^\alpha + \tfrac{1}{2} V_{\mu i}{}^j A_j{}^\alpha + \partial_\mu \phi^A\, \Gamma_A{}^\alpha{}_\beta\, A_i{}^\beta \, , \end{aligned} \tag{B.100}$$

in accordance with the Weyl weight given in Table B.14. The connection $\Gamma_A{}^\alpha{}_\beta$ takes values in $\mathrm{sp}(n_H + 1)$, and $\Omega_{\alpha\beta}$ is a covariantly constant antisymmetric tensor [23].

B.6. Special holomorphic coordinates

As discussed in subsection 5.3, the PSK manifold $(\bar{M}, g_{\bar{M}})$ can be obtained by a superconformal quotient of a regular CASK manifold (M, g_M). As mentioned in subsection 5.4.1, one may choose special holomorphic coordinates $z^a = X^a/X^0$ $(a = 1, \ldots, n)$ on the PSK manifold $(\bar{M}, g_{\bar{M}})$. Here, we provide a few more details on the relation of these coordinates to the special holomorphic coordinates X^I $(I = 0, \ldots, n)$ on the CASK manifold (M, g_M). We give various conversion formulae that facilitate the construction of the space-time two-derivative Lagrangian for the z^a when viewed as components of a map $\mathcal{Z} : N \to \bar{M}$ from space-time N into the PSK manifold $\bar{M}$.

The superconformal quotient proceeds by first restricting the X^I to the hypersurface

$$i\left(\bar{X}^I F_I - \bar{F}_I X^I\right) = 1 \; . \tag{B.101}$$

Setting

$$(X^I, F_I) = \frac{(X^I(z), F_I(z))}{||(X^I(z), F_I(z))||} \; , \tag{B.102}$$

the constraint (B.101) imposes that (X^I, F_I) has unit norm. Here

$$||(X^I(z), F_I(z))|| = \sqrt{|i\left(\bar{X}^I(\bar{z})\, F_I(z) - \bar{F}_I(\bar{z})\, X^I(z)\right)|} \; . \tag{B.103}$$

As discussed in subsection 5.4.1, the vector $(X^I(z), F_I(z))$ denotes the components of the holomorphic section $s^*\phi : \bar{M} \to \mathcal{U}^{\bar{M}}$ of the line bundle $\mathcal{U}^{\bar{M}} \to \bar{M}$, which depends holomorphically on z^a. The norm of the vector $(X^I(z), F_I(z))$ yields the Kähler potential $K(z, \bar{z})$ of $g_{\bar{M}}$,

$$e^{-K(z,\bar{z})} = i\left(\bar{X}^I(\bar{z})\, F_I(z) - \bar{F}_I(\bar{z})\, X^I(z)\right) \; , \tag{B.104}$$

so that

$$X^I = e^{\frac{1}{2}K(z,\bar{z})}\, X^I(z) \; . \tag{B.105}$$

The $\mathbb{C}^*$-action

$$X^I(z) \mapsto e^{-f(z)}\, X^I(z) \tag{B.106}$$

induces the Kähler transformation

$$K \mapsto K + f + \bar{f} \tag{B.107}$$

on the Kähler potential, while on the symplectic vector $(X^I, F_I(X))$ it induces the $U(1)$-transformation

$$(X^I, F_I(X)) \mapsto \mathrm{e}^{-\frac{1}{2}(f-\bar{f})}\, (X^I, F_I(X)) \; . \tag{B.108}$$

The Kähler potential (B.104) can be written as

$$e^{-K(z,\bar{z})} = |X^0(z)|^2 \left(-N_{IJ}\, Z^I\, \bar{Z}^J\right) \; , \tag{B.109}$$

with $Z^I(z) = (Z^0, Z^a) = (1, z^a)$, and

$$N_{IJ} = -i\left(F_{IJ} - \bar{F}_{IJ}\right) \; . \tag{B.110}$$

Using the homogeneity of $F(X)$,

$$F(X) = \left(X^0\right)^2\, \mathcal{F}(z) \; , \tag{B.111}$$

we get

$$F_0 = X^0\left(2\mathcal{F}(z) - z^a\, \mathcal{F}_a\right) \; , \tag{B.112}$$

where $\mathcal{F}_a = \partial \mathcal{F}/\partial z^a$. Using

$$\begin{aligned} F_{00} &= 2\mathcal{F} - 2z^a\,\mathcal{F}_a + z^a\,z^b\,\mathcal{F}_{ab}\ , \\ F_{0b} &= \mathcal{F}_b - z^a\,\mathcal{F}_{ab}\ , \\ F_{ab} &= \mathcal{F}_{ab}\ , \end{aligned} \tag{B.113}$$

where $\mathcal{F}_{ab} = \partial^2\mathcal{F}/\partial z^a \partial z^b$, we obtain

$$-N_{IJ}\,Z^I\,\bar{Z}^J = i\left[2\left(\mathcal{F} - \bar{\mathcal{F}}\right) - (z^a - \bar{z}^a)\left(\mathcal{F}_a + \bar{\mathcal{F}}_a\right)\right]\ , \tag{B.114}$$

and hence

$$e^{-K(z,\bar{z})} = i\,|X^0(z)|^2\,\left[2\left(\mathcal{F} - \bar{\mathcal{F}}\right) - (z^a - \bar{z}^a)\left(\mathcal{F}_a + \bar{\mathcal{F}}_a\right)\right]\ . \tag{B.115}$$

The metric $g_{\bar{M}}$ on the PSK manifold is, locally, given by

$$g_{a\bar{b}} = \frac{\partial^2 K(z,\bar{z})}{\partial z^a\,\partial \bar{z}^b}\ . \tag{B.116}$$

Next, we relate the PSK metric (B.116) to the CASK metric (B.110). Differentiating e^{-K} yields

$$\begin{aligned} \partial_a\partial_{\bar{b}}e^{-K} &= \left[-g_{a\bar{b}} + \partial_a K\,\partial_{\bar{b}}K\right]e^{-K} \qquad \text{(B.117)} \\ &= i\,|X^0(z)|^2\,\left(\mathcal{F}_{ab} - \bar{\mathcal{F}}_{ab}\right) - \Big[\partial_a \ln X^0(z)\,\partial_{\bar{b}}\ln\bar{X}^0(\bar{z}) \\ &\qquad -\partial_a \ln X^0(z)\,\partial_{\bar{b}}K - \partial_a K\,\partial_{\bar{b}}\ln\bar{X}^0(\bar{z})\Big]e^{-K}\ . \end{aligned}$$

Using (B.113) we have

$$N_{ab} = -i\left(\mathcal{F}_{ab} - \bar{\mathcal{F}}_{ab}\right)\ , \tag{B.118}$$

and hence we infer from (B.117) that

$$g_{a\bar{b}} = N_{ab}\,|X^0|^2 + \frac{1}{|X^0(z)|^2}\mathcal{D}_a X^0(z)\mathcal{D}_{\bar{b}}\,\bar{X}^0(\bar{z})\ , \tag{B.119}$$

where

$$\mathcal{D}_a X^0(z) = \partial_a X^0(z) - iA^h_a\,X^0(z) = \partial_a X^0(z) + \partial_a K\,X^0(z) \tag{B.120}$$

denotes the connection given in (364), i.e. the covariant derivative under the transformation (B.106).

Next, using the connection given in (367),

$$\mathcal{D}_a X^I = \partial_a X^I + \tfrac{1}{2}\partial_a K\,X^I\ , \tag{B.121}$$

we introduce the space-time covariant derivative

$$\mathcal{D}_\mu X^I = \partial_\mu X^I + iA_\mu X^I = \partial_\mu X^I + \tfrac{1}{2}\left(\partial_a K\, \partial_\mu z^a - \partial_{\bar{a}} K \partial_\mu \bar{z}^a\right) X^I \ , \tag{B.122}$$

which is a covariant derivative for $U(1)$ transformations (B.108). Observe that

$$\mathcal{D}_\mu X^0 = \mathrm{e}^{K/2}\, \mathcal{D}_a X^0(z)\, \partial_\mu z^a \ . \tag{B.123}$$

Now we evaluate the $U(1)$ invariant combination $N_{IJ}\, \mathcal{D}_\mu X^I\, \mathcal{D}^\mu \bar{X}^J$ subject to the constraint (B.101),

$$\begin{aligned} N_{IJ}\, \mathcal{D}_\mu X^I\, \mathcal{D}^\mu \bar{X}^J|_{-N_{IJ}X^I\bar{X}^J=1} &= |X^0|^2\, N_{ab}\, \partial_\mu z^a\, \partial^\mu \bar{z}^b - \frac{1}{|X^0|^2}\, \mathcal{D}_\mu X^0\, \mathcal{D}^\mu \bar{X}^0 \\ &+ \frac{X^0}{\bar{X}^0}\, N_{aJ} \bar{X}^J\, \partial_\mu z^a\, \mathcal{D}^\mu \bar{X}^0 + \frac{\bar{X}^0}{X^0}\, N_{Ia} X^I\, \partial_\mu \bar{z}^a\, \mathcal{D}^\mu X^0 \ . \end{aligned} \tag{B.124}$$

Using

$$X^0\, \bar{X}^J\, N_{aJ} = \frac{1}{X^0(z)}\, \mathcal{D}_a X^0(z) \ , \tag{B.125}$$

as well as (B.119) and (B.123) we establish

$$N_{IJ}\, \mathcal{D}_\mu X^I\, \mathcal{D}^\mu \bar{X}^J|_{-N_{IJ}X^I\bar{X}^J=1} = g_{a\bar{b}}\, \partial_\mu z^a\, \partial^\mu \bar{z}^b \ . \tag{B.126}$$

We close with the following useful relations. First, we note the relation [236]

$$N^{IJ} = g^{a\bar{b}}\, \mathcal{D}_a X^I\, \bar{\mathcal{D}}_{\bar{b}} \bar{X}^J - X^I\, \bar{X}^J \ . \tag{B.127}$$

Then, we recall the definition of $\mathcal{N}_{IJ}$ in (440), and we note the relations

$$\begin{aligned} \mathcal{N}_{IJ}\, X^J &= F_I \ , \\ -\tfrac{1}{2}\left[(\mathrm{Im}\mathcal{N})^{-1}\right]^{IJ} &= N^{IJ} + X^I\, \bar{X}^J + X^J\, \bar{X}^I \ . \end{aligned} \tag{B.128}$$

B.7. The black hole potential

We consider the Maxwell terms in the two-derivative Lagrangian (439), and define $\mu_{IJ} = \mathrm{Im}\,\mathcal{N}_{IJ}$ and $\nu_{IJ} = \mathrm{Re}\,\mathcal{N}_{IJ}$.

The black hole potential in four dimensions is defined by [156],

$$V_{\mathrm{BH}} = g^{a\bar{b}} \mathcal{D}_a Z \bar{\mathcal{D}}_{\bar{b}} \bar{Z} + |Z|^2 = \left(N^{IJ} + 2X^I \bar{X}^J\right) \hat{q}_I\, \bar{\hat{q}}_J \ , \tag{B.129}$$

where

$$Z(X) = p^I\, F_I(X) - q_I\, X^I = -\hat{q}_I\, X^I \quad , \quad \hat{q}_I = q_I - F_{IJ}\, p^J \ . \tag{B.130}$$

Here, (p^I, q_I) denote magnetic/electric charges as in (644). The black hole potential transforms as a function under symplectic transformations (198).

Using (B.128), the black hole potential can also be written as

$$V_{\rm BH} = -\frac{1}{2}\,(q_I - \mathcal{N}_{IK}\,p^K)\,[(\mathrm{Im}\,\mathcal{N})^{-1}]^{IJ}\,(q_J - \bar{\mathcal{N}}_{JL}\,p^L)\ . \tag{B.131}$$

This equals [156, 193]

$$V_{\rm BH} = -\tfrac{1}{2}\,(p \quad q)\begin{pmatrix} \mu + \nu\mu^{-1}\nu & -\nu\mu^{-1} \\ -\mu^{-1}\nu & \mu^{-1} \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}\ , \tag{B.132}$$

where we have suppressed the indices I, J for notational simplicity. The black hole potential can be expressed [127] in terms of the tensor field $\hat{H}_{ab}$ defined in (335),

$$V_{\rm BH} = -\tfrac{1}{2}Q^a\,\hat{H}_{ab}\,Q^b\ , \tag{B.133}$$

where $Q^a = (p^I, q_I)^T$.

Extrema of the black hole potential $V_{\rm BH}$ may either correspond to BPS black holes or to non-BPS black holes. If an extremum satisfies $\mathcal{D}_a Z = 0\ \forall\, a = 1, \dots, n$ with $Z \neq 0$, then it corresponds to a BPS black hole [156]. Conversely, if $\mathcal{D}_a Z \neq 0$ at the extremum, then the black hole is non-supersymmetric.

B.8. Wald's entropy

In a general classical theory of gravity with higher-curvature terms, based on a diffeomorphism invariant Lagrangian, the entropy of a stationary black hole is computed using Wald's definition of black hole entropy [237, 238, 239, 240]. If the higher-curvature terms involve the Riemann tensor, but not derivatives of the Riemann tensor, Wald's entropy is given by

$$\mathcal{S}_{\rm macro} = -\tfrac{1}{4}\int_{\Sigma_{\rm hor}} \frac{\partial L}{\partial R_{\mu\nu\rho\sigma}}\,\varepsilon_{\mu\nu}\varepsilon_{\rho\sigma}\ , \tag{B.134}$$

where $\varepsilon_{\mu\nu}$ denotes the bi-normal tensor associated with a cross-section of the Killing horizon $\Sigma_{\rm hor}$, normalized such that $\varepsilon_{\mu\nu}\varepsilon^{\mu\nu} = -2$. In tangent space indices, the non-vanishing components are $\varepsilon_{01} = \pm 1$. We have normalized (B.134) in such a way that when $L = \frac{1}{2}\,R$, we obtain the area law $\mathcal{S}_{\rm macro} = A/4$.

References

[1] H. Shima, The geometry of Hessian structures, World Scientific, 2007.

[2] D. V. Alekseevsky, V. Cortés, C. Devchand, A. Van Proeyen, Flows on quaternionic-kähler and very special real manifolds, Commun. Math. Phys. 238 (2003) 525–543. arXiv:hep-th/0109094.

[3] D. V. Alekseevsky, V. Cortés, Geometric construction of the r-map: from affine special real to special Kähler manifolds, Commun. Math. Phys. 291 (2009) 579–590. arXiv:0811.1658.

[4] V. Cortés, X. Han, T. Mohaupt, Completeness in supergravity constructions, Commun. Math. Phys. 311 (2012) 191. arXiv:1101.5103.

[5] T. Mohaupt, O. Vaughan, Developments in special geometry, J.Phys.Conf.Ser. 343 (2012) 012078. arXiv:1112.2873.

[6] T. Mohaupt, O. Vaughan, Non-extremal black holes from the generalised r-map, Springer Proc. Phys. 144 (2013) 233–254. arXiv:1208.4302, doi:10.1007/978-3-319-00215-6_6.

[7] O. Vaughan, The r-map, the c-map and black hole solutions, Ph.D. thesis, University of Liverpool (2012).

[8] E. E. Donets, A. Pashnev, V. O. Rivelles, D. P. Sorokin, M. Tsulaia, N=4 superconformal mechanics and the potential structure of AdS spaces, Phys. Lett. B484 (2000) 337–346. arXiv:hep-th/0004019, doi:10.1016/S0370-2693(00)00670-5.

[9] E. E. Donets, A. Pashnev, J. J. Rosales, M. M. Tsulaia, N=4 supersymmetric multidimensional quantum mechanics, partial SUSY breaking and superconformal quantum mechanics, Phys. Rev. D61 (2000) 043512. arXiv:hep-th/9907224, doi:10.1103/PhysRevD.61.043512.

[10] N. Kozyrev, S. Krivonos, O. Lechtenfeld, A. Nersessian, A. Sutulin, $\mathcal{N}=4$ supersymmetric mechanics on curved spaces, Phys. Rev. D97 (8) (2018) 085015. arXiv:1711.08734, doi:10.1103/PhysRevD.97.085015.

[11] N. Kozyrev, S. Krivonos, O. Lechtenfeld, A. Nersessian, A. Sutulin, Curved Witten-Dijkgraaf-Verlinde-Verlinde equation and $\mathcal{N}=4$ mechanics, Phys. Rev. D96 (10) (2017) 101702. arXiv:1710.00884, doi:10.1103/PhysRevD.96.101702.

[12] S. Krivonos, A. Nersessian, H. Shmavonyan, Geometry and integrability in $\mathcal{N}=8$ supersymmetric mechanics, arXiv:1908.06490.

[13] S. Bellucci, S. Krivonos, A. Nersessian, N=8 supersymmetric mechanics on special Kahler manifolds, Phys. Lett. B605 (2005) 181–184. arXiv:hep-th/0410029, doi:10.1016/j.physletb.2004.11.023.

[14] N. Kozyrev, The curved WDVV equations and superfields, J. Phys. Conf. Ser. 1194 (1) (2019) 012061. arXiv:1812.01406, doi:10.1088/1742-6596/1194/1/012061.

[15] G. W. Gibbons, P. Rychenkova, Cones, triSasakian structures and superconformal invariance, Phys. Lett. B443 (1998) 138–142. arXiv:hep-th/9809158, doi:10.1016/S0370-2693(98)01287-8.

[16] D. Errington, T. Mohaupt, O. Vaughan, Non-extremal black hole solutions from the c-map, JHEP 05 (2014) 052. arXiv:1408.0923, doi:10.1007/JHEP05(2015)052.

[17] V. Cortés, P. Dempster, T. Mohaupt, O. Vaughan, Special Geometry of Euclidean Supersymmetry IV: the local c-map, JHEP 10 (2015) 066. arXiv:1507.04620, doi:10.1007/JHEP10(2015)066.

[18] M. Gunaydin, G. Sierra, P. K. Townsend, The Geometry of N=2 Maxwell-Einstein Supergravity and Jordan Algebras, Nucl. Phys. B242 (1984) 244–268. doi:10.1016/0550-3213(84)90142-1.

[19] V. Cortés, C. Mayer, T. Mohaupt, F. Saueressig, Special geometry of Euclidean supersymmetry. I: Vector multiplets, JHEP 03 (2004) 028. arXiv:hep-th/0312001.

[20] E. Bergshoeff, T. de Wit, R. Halbersma, S. Cucu, M. Derix, A. Van Proeyen, Weyl multiplets of N=2 conformal supergravity in five-dimensions, JHEP 06 (2001) 051. arXiv:hep-th/0104113, doi:10.1088/1126-6708/2001/06/051.

[21] L. Gall, T. Mohaupt, Five-dimensional vector multiplets in arbitrary signature, JHEP 09 (2018) 053. arXiv:1805.06312, doi:10.1007/JHEP09(2018)053.

[22] D. Z. Freedman, A. Van Proeyen, Supergravity, Cambridge Univ. Press, Cambridge, UK, 2012.
URL http://www.cambridge.org/mw/academic/subjects/physics/theoretical-physics-and-mathematical-physics/supergravity?format=AR

[23] B. de Wit, S. Katmadas, Near-Horizon Analysis of D=5 BPS Black Holes and Rings, JHEP 02 (2010) 056. arXiv:0910.4907, doi:10.1007/JHEP02(2010)056.

[24] J. Bagger, E. Witten, Matter Couplings in N=2 Supergravity, Nucl. Phys. B222 (1983) 1. doi:10.1016/0550-3213(83)90605-3.

[25] B. de Wit, P. G. Lauwers, A. Van Proeyen, Lagrangians of N=2 Supergravity - Matter Systems, Nucl. Phys. B255 (1985) 569–608. doi:10.1016/0550-3213(85)90154-3.

[26] K. Galicki, Geometry of the scalar couplings in N=2 supergravity models, Class. Quant. Grav. 9 (1992) 27–40. doi:10.1088/0264-9381/9/1/006.

[27] B. de Wit, B. Kleijn, S. Vandoren, Rigid N = 2 superconformal hypermultiplets, arXiv:hep-th/9808160.

[28] B. de Wit, B. Kleijn, S. Vandoren, Superconformal hypermultiplets, Nucl. Phys. B568 (2000) 475–502. arXiv:hep-th/9909228, doi:10.1016/S0550-3213(99)00726-9.

[29] K. Hanaki, K. Ohashi, Y. Tachikawa, Supersymmetric Completion of an R^2 term in Five-dimensional Supergravity, Prog. Theor. Phys. 117 (2007) 533. arXiv:hep-th/0611329, doi:10.1143/PTP.117.533.

[30] M. K. Gaillard, B. Zumino, Duality Rotations for Interacting Fields, Nucl. Phys. B193 (1981) 221–244. doi:10.1016/0550-3213(81)90527-7.

[31] B. de Wit, A. Van Proeyen, Potentials and Symmetries of General Gauged N=2 Supergravity: Yang-Mills Models, Nucl. Phys. B245 (1984) 89–117. doi:10.1016/0550-3213(84)90425-5.

[32] B. de Wit, N=2 electric - magnetic duality in a chiral background, Nucl. Phys. Proc. Suppl. 49 (1996) 191–200, [,191(1996)]. arXiv:hep-th/9602060, doi:10.1016/0920-5632(96)00335-0.

[33] G. L. Cardoso, B. de Wit, S. Mahapatra, Non-holomorphic deformations of special geometry and their applications, Springer Proc. Phys. 144 (2013) 1–58. arXiv:1206.0577, doi:10.1007/978-3-319-00215-6_1.

[34] G. L. Cardoso, B. de Wit, S. Mahapatra, Deformations of special geometry: in search of the topological string, JHEP 09 (2014) 096. arXiv:1406.5478, doi:10.1007/JHEP09(2014)096.

[35] G. L. Cardoso, B. de Wit, S. Mahapatra, Subleading and non-holomorphic corrections to N=2 BPS black hole entropy, JHEP 02 (2009) 006. arXiv:0808.2627, doi:10.1088/1126-6708/2009/02/006.

[36] G. L. Cardoso, B. de Wit, T. Mohaupt, Corrections to macroscopic supersymmetric black hole entropy, Phys. Lett. B451 (1999) 309–316. arXiv:hep-th/9812082, doi:10.1016/S0370-2693(99)00227-0.

[37] G. L. Cardoso, B. de Wit, S. Mahapatra, BPS black holes, the Hesse potential, and the topological string, JHEP 06 (2010) 052. arXiv:1003.1970, doi:10.1007/JHEP06(2010)052.

[38] V. Kaplunovsky, J. Louis, Field dependent gauge couplings in locally supersymmetric effective quantum field theories, Nucl. Phys. B422 (1994) 57–124. arXiv:hep-th/9402005, doi:10.1016/0550-3213(94)00150-2.

[39] B. de Wit, F. Vanderseypen, A. Van Proeyen, Symmetry structure of special geometries, Nucl. Phys. B400 (1993) 463–524. arXiv:hep-th/9210068, doi:10.1016/0550-3213(93)90413-J.

[40] B. de Wit, A. Van Proeyen, Potentials and Symmetries of General Gauged N=2 Supergravity: Yang-Mills Models, Nucl. Phys. B245 (1984) 89. doi:10.1016/0550-3213(84)90425-5.

[41] D. S. Freed, Special kaehler manifolds, Commun. Math. Phys. 203 (1999) 31–52. arXiv:hep-th/9712042.

[42] D. V. Alekseevsky, V. Cortés, C. Devchand, Special complex manifolds, J. Geom. Phys. 42 (2002) 85–105. arXiv:math/9910091.

[43] O. Baues, V. Cortés, Realisation of special Kähler manifolds as parabolic spheres, Proc. Amer. Math. Soc. 129 (8) (2001) 2403–2407. `arXiv:math/9911079`.

[44] V. Cortés, A Holomorphic representation formula for parabolic hyperspheres, Banach Center Publ. 57 (2002) 11–16. `arXiv:math/0107037`.

[45] V. Cortés, T. Mohaupt, Special Geometry of Euclidean Supersymmetry III: the local r-map, instantons and black holes, JHEP 07 (2009) 066. `arXiv:0905.2844`, `doi:10.1088/1126-6708/2009/07/066`.

[46] V. Cortés, P.-S. Dietrich, T. Mohaupt, ASK/PSK-correspondence and the r-map, Lett. Math. Phys. 108 (5) (2018) 1279–1306. `arXiv:1702.02400`, `doi:10.1007/s11005-017-1032-1`.

[47] P.-S. Dieterich, The affine special kahler/projective special kaehler correspondence and related constructions, Ph.D. thesis, Universität Hamburg, Fachbereich Mathematik (2017).

[48] V. Cortés, Homogeneous special geometry, Transform. Groups 1 (4) (1996) 337–373.

[49] V. Cortés, On Hyper-Kähler manifolds associated to Lagrangian Kähler submanifolds of T^*C^n, Trans. Amer. Math. Soc. 350 (8) (1998) 3193–3205.

[50] N. Hitchin, The moduli space of complex Lagrangian submanifolds, `arXiv:math/9901069`.

[51] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré, General matter coupled N=2 supergravity, Nucl. Phys. B476 (1996) 397–417. `arXiv:hep-th/9603004`, `doi:10.1016/0550-3213(96)00344-6`.

[52] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré, T. Magri, N = 2 supergravity and N = 2 super Yang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map, J. Geom. Phys. 23 (1997) 111–189. `arXiv:hep-th/9605032`.

[53] B. Craps, F. Roose, W. Troost, A. Van Proeyen, What is special Kaehler geometry?, Nucl. Phys. B503 (1997) 565–613. `arXiv:hep-th/9703082`, `doi:10.1016/S0550-3213(97)00408-2`.

[54] M. A. Lledo, O. Macia, A. Van Proeyen, V. S. Varadarajan, Special geometry for arbitrary signatures, IRMA Lect. Math. Theor. Phys. 16 (2010) 85–147. arXiv:hep-th/0612210.

[55] A. Strominger, Special Geometry, Commun. Math. Phys. 133 (1990) 163–180. doi:10.1007/BF02096559.

[56] L. Castellani, R. D'Auria, P. Fre, Supergravity and superstrings: A Geometric perspective. Vol. 2: Supergravity, World Scientific, 1991.

[57] L. Castellani, R. D'Auria, S. Ferrara, Special geometry without special coordinates, Class. Quant. Grav. 7 (1990) 1767–1790. doi:10.1088/0264-9381/7/10/009.

[58] L. Castellani, R. D'Auria, S. Ferrara, Special Kahler geometry: an intrinsic formulation from N=2 space-time supersymmetry, Phys. Lett. B241 (1990) 57. doi:10.1016/0370-2693(90)91486-U.

[59] A. Ceresole, R. D'Auria, S. Ferrara, A. Van Proeyen, On electromagnetic duality in locally supersymmetric N=2 Yang-Mills theory, arXiv:hep-th/9412200.

[60] S. Ferrara, O. Macia, Observations on the Darboux coordinates for rigid special geometry, JHEP 05 (2006) 008. arXiv:hep-th/0602262, doi:10.1088/1126-6708/2006/05/008.

[61] S. Ferrara, O. Macia, Real symplectic formulation of local special geometry, Phys. Lett. B637 (2006) 102–106. arXiv:hep-th/0603111, doi:10.1016/j.physletb.2006.04.010.

[62] E. Witten, J. Bagger, Quantization of Newton's Constant in Certain Supergravity Theories, Phys. Lett. 115B (1982) 202–206. doi:10.1016/0370-2693(82)90644-X.

[63] V. Cortés, C. I. Lazaroiu, C. S. Shahbazi, $\mathcal{N} = 1$ Geometric Supergravity and chiral triples on Riemann surfaces , Communications in Mathematical Physics (2019), arXiv:1810.12353, doi:10.1007/s00220-019-03476-7.

[64] M. B. Green, S. J. H., E. Witten, Superstring theory (2 vols), Cambridge University Press, 1987.

[65] P. Candelas, Lectures on complex manifolds, in: L. Alvarez-Gaume (Ed.), Superstrings '87, 1987.

[66] T. Hubsch, Calabi-Yau Manifolds, World Scientific, 1991.

[67] R. Kallosh, T. Mohaupt, M. Shmakova, Excision of singularities by stringy domain walls, J.Math.Phys. 42 (2001) 3071–3081. arXiv:hep-th/0010271, doi:10.1063/1.1373424.

[68] C. Mayer, T. Mohaupt, The Kähler cone as cosmic censor, Class. Quant. Grav. 21 (2004) 1879–1896. arXiv:hep-th/0312008, doi:10.1088/0264-9381/21/7/010.

[69] E. Witten, Phase Transitions In M-Theory And F-Theory, Nucl. Phys. B471 (1996) 195–216. arXiv:hep-th/9603150, doi:10.1016/0550-3213(96)00212-X.

[70] D. V. Alekseevsky, S. Marchiafava, Hermitian and kähler submanifolds of a quaternionic kähler manifold, Osaka J. Math. 38 (4) (2001) 869.

[71] P. Fayet, Fermi-Bose Hypersymmetry, Nucl. Phys. B113 (1976) 135. doi:10.1016/0550-3213(76)90458-2.

[72] G. L. Cardoso, T. Mohaupt, Hessian geometry and the holomorphic anomaly, JHEP 02 (2016) 161. arXiv:1511.06658, doi:10.1007/JHEP02(2016)161.

[73] G. L. Cardoso, A. Veliz-Osorio, On the sigma-model of deformed special geometry, Nucl. Phys. B872 (2013) 228–252. arXiv:1212.4364, doi:10.1016/j.nuclphysb.2013.04.001.

[74] G. L. Cardoso, B. de Wit, J. Kappeli, T. Mohaupt, Black hole partition functions and duality, JHEP 03 (2006) 074. arXiv:hep-th/0601108, doi:10.1088/1126-6708/2006/03/074.

[75] M. Bershadsky, S. Cecotti, H. Ooguri, C. Vafa, Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes, Commun. Math. Phys. 165 (1994) 311–428. arXiv:hep-th/9309140, doi:10.1007/BF02099774.

[76] T. W. Grimm, A. Klemm, M. Marino, M. Weiss, Direct Integration of the Topological String, JHEP 08 (2007) 058. arXiv:hep-th/0702187, doi:10.1088/1126-6708/2007/08/058.

[77] M. Aganagic, V. Bouchard, A. Klemm, Topological Strings and (Almost) Modular Forms, Commun. Math. Phys. 277 (2008) 771–819. arXiv:hep-th/0607100, doi:10.1007/s00220-007-0383-3.

[78] V. Cortés, C. Mayer, T. Mohaupt, F. Saueressig, Special geometry of Euclidean supersymmetry. II: Hypermultiplets and the c-map, JHEP 06 (2005) 025. arXiv:hep-th/0503094.

[79] I. Vaisman, On the geometry of double field theory, J. Math. Phys. 53 (2012) 033509. arXiv:1203.0836, doi:10.1063/1.3694739.

[80] I. Vaisman, Towards a double field theory on para-Hermitian manifolds, J. Math. Phys. 54 (2013) 123507. arXiv:1209.0152, doi:10.1063/1.4848777.

[81] L. Freidel, F. J. Rudolph, D. Svoboda, Generalised Kinematics for Double Field Theory, JHEP 11 (2017) 175. arXiv:1706.07089, doi:10.1007/JHEP11(2017)175.

[82] L. Freidel, F. J. Rudolph, D. Svoboda, A Unique Connection for Born Geometry, arXiv:1806.05992.

[83] V. E. Marotta, R. J. Szabo, Para-Hermitian Geometry, Dualities and Generalized Flux Backgrounds, arXiv:1810.03953.

[84] V. Cruceanu, P. Fortuny, P. M. Gadea, A survey of paracomplex geometry, Rocky Mountain J Math 26 (1996) 83–115.

[85] J. B. Gutowski, W. Sabra, Euclidean N=2 Supergravity, Phys. Lett. B718 (2012) 610–614. arXiv:1209.2029, doi:10.1016/j.physletb.2012.10.050.

[86] H. Nicolai, A Possible constructive approach to super ϕ^3 in four-dimensions. 1. Euclidean formulation of the model, Nucl. Phys. B140 (1978) 294.

[87] P. van Nieuwenhuizen, A. Waldron, A continuous Wick rotation for spinor fields and supersymmetry in Euclidean space, arXiv:hep-th/9611043.

[88] P. van Nieuwenhuizen, A. Waldron, On euclidean spinors and wick rotations, Phys. Lett. B389 (1996) 29–36. arXiv:hep-th/9608174.

[89] U. Theis, P. Van Nieuwenhuizen, Ward identities for N = 2 rigid and local supersymmetry in Euclidean space, Class. Quant. Grav. 18 (2001) 5469–5486. arXiv:hep-th/0108204, doi:10.1088/0264-9381/18/24/311.

[90] G. W. Gibbons, M. B. Green, M. J. Perry, Instantons and Seven-Branes in Type IIB Superstring Theory, Phys. Lett. B370 (1996) 37–44. arXiv:hep-th/9511080, doi:10.1016/0370-2693(95)01565-5.

[91] T. Mohaupt, K. Waite, Euclidean Actions, Instantons, Solitons and Supersymmetry, J. Phys. A44 (2011) 175403. arXiv:1011.6301, doi:10.1088/1751-8113/44/17/175403.

[92] D. Gal'tsov, O. Rytchkov, Generating branes via sigma-models, Phys. Rev. D 58 (1998) 122001.

[93] K. S. Stelle, BPS branes in supergravity, lecture notes (1998). arXiv:hep-th/9803116.

[94] C. Hull, Duality and the signature of space-time, JHEP 9811 (1998) 017. arXiv:hep-th/9807127.

[95] C. M. Hull, Timelike T-duality, de Sitter space, large N gauge theories and topological field theory, JHEP 07 (1998) 021. arXiv:hep-th/9806146.

[96] C. M. Hull, R. R. Khuri, Branes, times and dualities, Nucl.Phys. B536 (1998) 219–244. arXiv:hep-th/9808069, doi:10.1016/S0550-3213(98)00691-9.

[97] E. Bergshoeff, A. Van Proeyen, The Many faces of $OSp(1|32)$, Class. Quant. Grav. 17 (2000) 3277–3304. arXiv:hep-th/0003261, doi:10.1088/0264-9381/17/16/312.

[98] E. A. Bergshoeff, J. Hartong, A. Ploegh, J. Rosseel, D. Van den Bleeken, Pseudo-supersymmetry and a tale of alternate realities, JHEP 07 (2007) 067. arXiv:arXiv:0704.3559[hep-th].

[99] W. Sabra, Special geometry and space–time signature, Phys. Lett. B773 (2017) 191–195. arXiv:1706.05162, doi:10.1016/j.physletb.2017.08.021.

[100] V. Cortés, L. Gall, T. Mohaupt, Four-dimensional vector multiplets in arbitrary signature, arXiv:1907.12067.

[101] N. Banerjee, B. de Wit, S. Katmadas, The Off-Shell 4D/5D Connection, JHEP 03 (2012) 061. arXiv:1112.5371, doi:10.1007/JHEP03(2012)061.

[102] B. de Wit, V. Reys, Euclidean supergravity, JHEP 12 (2017) 011. arXiv:1706.04973, doi:10.1007/JHEP12(2017)011.

[103] I. Antoniadis, S. Ferrara, T. R. Taylor, N=2 Heterotic Superstring and its Dual Theory in Five Dimensions, Nucl. Phys. B460 (1996) 489–505. arXiv:hep-th/9511108, doi:10.1016/0550-3213(95)00659-1.

[104] J. Louis, T. Mohaupt, M. Zagermann, Effective actions near singularities, JHEP 02 (2003) 053. arXiv:hep-th/0301125.

[105] M. T. Grisaru, A. E. M. van de Ven, D. Zanon, Four Loop Divergences for the N=1 Supersymmetric Nonlinear Sigma Model in Two-Dimensions, Nucl. Phys. B277 (1986) 409–428. doi:10.1016/0550-3213(86)90449-9.

[106] S. Hosono, A. Klemm, S. Theisen, S.-T. Yau, Mirror symmetry, mirror map and applications to Calabi-Yau hypersurfaces, Commun. Math. Phys. 167 (1995) 301–350. arXiv:hep-th/9308122, doi:10.1007/BF02100589.

[107] A. Ceresole, R. D'Auria, S. Ferrara, A. Van Proeyen, Duality transformations in supersymmetric Yang-Mills theories coupled to supergravity, Nucl. Phys. B444 (1995) 92–124. arXiv:hep-th/9502072, doi:10.1016/0550-3213(95)00175-R.

[108] B. de Wit, V. Kaplunovsky, J. Louis, D. Lust, Perturbative couplings of vector multiplets in n=2 heterotic string vacua, Nucl. Phys. B451 (1995) 53–95. arXiv:hep-th/9504006.

[109] I. Antoniadis, S. Ferrara, E. Gava, K. S. Narain, T. R. Taylor, Perturbative prepotential and monodromies in N=2 heterotic superstring,

Nucl. Phys. B447 (1995) 35–61. arXiv:hep-th/9504034, doi:10.1016/0550-3213(95)00240-S.

[110] J. A. Harvey, G. W. Moore, Algebras, BPS states, and strings, Nucl.Phys. B463 (1996) 315–368. arXiv:hep-th/9510182, doi:10.1016/0550-3213(95)00605-2.

[111] S. Cecotti, S. Ferrara, L. Girardello, Geometry of Type II Superstrings and the Moduli of Superconformal Field Theories, Int. J. Mod. Phys. A4 (1989) 2475. doi:10.1142/S0217751X89000972.

[112] D. Kaledin, Hyperkaehler structures on total spaces of holomorphic cotangent bundles, arXiv:alg-geom/9710029.

[113] J. Gates, S. James, S. M. Kuzenko, 4D N = 2 supersymmetric off-shell sigma models on the cotangent bundles of Kaehler manifolds, Fortsch. Phys. 48 (2000) 115–118. arXiv:hep-th/9903013.

[114] S. Ferrara, S. Sabharwal, Quaternionic Manifolds for Type II Superstring Vacua of Calabi-Yau Spaces, Nucl. Phys. B332 (1990) 317. doi:10.1016/0550-3213(90)90097-W.

[115] W. Ballmann, Lectures on Kähler Manifolds, European Mathematical Society, 2006.

[116] P. S. Aspinwall, Compactification, geometry and duality: N = 2, arXiv:hep-th/0001001.

[117] U. Theis, S. Vandoren, Instantons in the double-tensor multiplet, JHEP 09 (2002) 059. arXiv:hep-th/0208145.

[118] B. de Wit, F. Saueressig, Off-shell N = 2 tensor supermultiplets, JHEP 09 (2006) 062. arXiv:hep-th/0606148.

[119] M. Rocek, C. Vafa, S. Vandoren, Hypermultiplets and topological strings, JHEP 02 (2006) 062. arXiv:hep-th/0512206, doi:10.1088/1126-6708/2006/02/062.

[120] B. de Wit, F. Saueressig, Tensor supermultiplets and toric quaternion-Kaehler geometry, Fortsch. Phys. 55 (2007) 699–704. arXiv:hep-th/0701223, doi:10.1002/prop.200610371.

[121] N. Banerjee, B. de Wit, S. Katmadas, The off-shell c-map, JHEP 01 (2016) 156. arXiv:1512.06686, doi:10.1007/JHEP01(2016)156.

[122] A. Haydys, Hyper-kähler and quaternionic kähler manifolds with s^1-symmetries, J. Geom. Phys. 58 (2008) 293–306.

[123] S. Alexandrov, D. Persson, B. Pioline, Wall-crossing, Rogers dilogarithm, and the QK/HK correspondence, JHEP 1112 (2011) 027. arXiv:1110.0466, doi:10.1007/JHEP12(2011)027.

[124] D. V. Alekseevsky, V. Cortés, T. Mohaupt, Conification of Kähler and hyper-Kähler manifolds, Commun. Math. Phys. 324 (2013) 637–655. arXiv:1205.2964, doi:10.1007/s00220-013-1812-0.

[125] D. V. Alekseevsky, V. Cortés, M. Dyckmanns, T. Mohaupt, Quaternionic Kähler metrics associated with special Kähler manifolds, J.Geom.Phys. 92 (2015) 271–287. arXiv:1305.3549, doi:10.1016/j.geomphys.2014.12.012.

[126] M. Dyckmanns, O. Vaughan, The para-HK/QK correspondence, J. Geom. Phys. 116 (2017) 244–257. arXiv:1601.05001, doi:10.1016/j.geomphys.2017.01.024.

[127] T. Mohaupt, O. Vaughan, The Hesse potential, the c-map and black hole solutions, JHEP 07 (2012) 163. arXiv:1112.2876, doi:10.1007/JHEP07(2012)163.

[128] V. Cortés, J. Louis, P. Smyth, H. Triendl, On certain Kähler quotients of quaternionic Kähler manifolds, Commun.Math.Phys. 317 (2013) 787–816. arXiv:1111.0679, doi:10.1007/s00220-012-1541-9.

[129] P. Dempster, D. Errington, T. Mohaupt, Nernst branes from special geometry, JHEP 05 (2015) 079. arXiv:1501.07863, doi:10.1007/JHEP05(2015)079.

[130] P. Dempster, D. Errington, J. Gutowski, T. Mohaupt, Five-dimensional Nernst branes from special geometry, JHEP 11 (2016) 114. arXiv:1609.05062, doi:10.1007/JHEP11(2016)114.

[131] J. Gutowski, T. Mohaupt, G. Pope, From static to cosmological solutions of $\mathcal{N} = 2$ supergravity, JHEP 08 (2019) 172. arXiv:1905.09167, doi:10.1007/JHEP08(2019)172.

[132] T. Mohaupt, O. Vaughan, Non-extremal Black Holes, Harmonic Functions, and Attractor Equations, Class. Quant. Grav. 27 (2010) 235008. arXiv:1006.3439, doi:10.1088/0264-9381/27/23/235008.

[133] N. Hitchin, Quaternionic Kähler moduli spaces, in: Riemannian topology and geometric structures on manifolds, Vol. 271 of Progr. Math., Birkhäuser Boston, Boston, MA, 2009, pp. 49–61. doi:10.1007/978-0-8176-4743-8_3.

[134] T. van der Aalst, A geometric interpretation of the c-map, Master's thesis, Universiteit Utrecht, Utrecht Master Thesis 2008 (2008).

[135] I. Antoniadis, R. Minasian, S. Theisen, P. Vanhove, String loop corrections to the universal hypermultiplet, Class.Quant.Grav. 20 (2003) 5079–5102. arXiv:hep-th/0307268, doi:10.1088/0264-9381/20/23/009.

[136] D. Robles-Llana, F. Saueressig, S. Vandoren, String loop corrected hypermultiplet moduli spaces, JHEP 0603 (2006) 081. arXiv:hep-th/0602164, doi:10.1088/1126-6708/2006/03/081.

[137] B. de Wit, A. Van Proeyen, Symmetries of dual quaternionic manifolds, Phys.Lett. B252 (1990) 221–229. doi:10.1016/0370-2693(90)90864-3.

[138] B. de Wit, A. Van Proeyen, Isometries of special manifolds (1995). arXiv:hep-th/9505097.

[139] M. Gunaydin, A. Neitzke, O. Pavlyk, B. Pioline, Quasi-conformal actions, quaternionic discrete series and twistors: SU(2,1) and $G_2(2)$, Commun. Math. Phys. 283 (2008) 169–226. arXiv:0707.1669, doi:10.1007/s00220-008-0563-9.

[140] S. Bellucci, S. Ferrara, M. Gunaydin, A. Marrani, SAM Lectures on Extremal Black Holes in d=4 Extended Supergravity, arXiv:0905.3739.

[141] M. Gunaydin, Lectures on Spectrum Generating Symmetries and U-duality in Supergravity, Extremal Black Holes, Quantum Attractors and Harmonic Superspace, arXiv:0908.0374.

[142] B. de Wit, A. Van Proeyen, Special geometry, cubic polynomials and homogeneous quaternionic spaces, Commun. Math. Phys. 149 (1992) 307–334. arXiv:hep-th/9112027, doi:10.1007/BF02097627.

[143] V. Cortés, M. Nardmann, S. Suhr, Completeness of hyperbolic centroaffine hypersurfaces, Comm. Anal. Geom 24 (1) (2014) 59–92. arXiv:1407.3251.

[144] V. Cortés, M. Dyckmanns, D. Lindemann, Classification of complete projective special real surfaces, Proc. London Math. Soc. 109 (2) (2013) 353–381. arXiv:1302.4570.

[145] V. Cortés, M. Dyckmanns, M. Jüngling, D. Lindemann, A class of cubic hypersurfaces and quaternionic Kähler manifolds of co-homogeneity one, arXiv:1701.07882.

[146] D. V. Alekseevsky, V. Cortés, Classification of stationary compact homogeneous special pseudo-kähler manifolds of semi-simple groups, Proceedings of the London Mathematical Society (2000) 211–230.

[147] V. Cortés, M. Dyckmanns, S. Suhr, Completeness of projective special Kähler and quaternionic Kähler manifolds, in: Special Metrics and group actions in geometry, Vol. 23 of INdAM, Springer, 2017, pp. 81–106. arXiv:1607.07232.

[148] D. V. Alekseevsky, Classification of quaternionic spaces with transitive solvable group of motions, Izv. Akad. Nauk SSSR Ser. Mat. 39 (2) (1975) 315–362, 472.

[149] V. Cortés, Alekseevskian spaces, Differential Geom. Appl. 6 (2) (1996) 129–168. doi:10.1016/0926-2245(96)89146-7.

[150] V. Cortés, D. V. Alekseevsky, Classification of pseudo-riemannian symmetric spaces of quaternionic kähler type, Amer. Math. Soc. Transl. (2) 213 (2005) 33–62.

[151] O. Baues, V. Cortés, Proper affine hypersheres which fiber over projective special kähler manifolds, Asian J. Math 7 (115-132).

[152] S. Cecotti, Homogeneous Kahler manifolds and T algebras in N=2 supergravity and superstrings, Commun.Math.Phys. 124 (1989) 23–55. doi:10.1007/BF01218467.

[153] S. Alexandrov, Twistor Approach to String Compactifications: a Review, Phys. Rept. 522 (2013) 1–57. arXiv:1111.2892, doi:10.1016/j.physrep.2012.09.005.

[154] S. W. Hawking, G. F. R. Ellis, The Large Scale Structure of Space-Time, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 2011. doi:10.1017/CBO9780511524646.

[155] S. Ferrara, R. Kallosh, A. Strominger, N=2 extremal black holes, Phys. Rev. D52 (1995) R5412–R5416. arXiv:hep-th/9508072, doi:10.1103/PhysRevD.52.R5412.

[156] S. Ferrara, R. Kallosh, Supersymmetry and attractors, Phys. Rev. D54 (1996) 1514–1524. arXiv:hep-th/9602136, doi:10.1103/PhysRevD.54.1514.

[157] S. Ferrara, R. Kallosh, Universality of supersymmetric attractors, Phys. Rev. D54 (1996) 1525–1534. arXiv:hep-th/9603090, doi:10.1103/PhysRevD.54.1525.

[158] G. W. Gibbons, D. Kastor, L. A. J. London, P. K. Townsend, J. H. Traschen, Supersymmetric selfgravitating solitons, Nucl. Phys. B416 (1994) 850–880. arXiv:hep-th/9310118, doi:10.1016/0550-3213(94)90558-4.

[159] A. H. Chamseddine, W. A. Sabra, Metrics admitting Killing spinors in five-dimensions, Phys. Lett. B426 (1998) 36–42. arXiv:hep-th/9801161, doi:10.1016/S0370-2693(98)00277-9.

[160] G. L. Cardoso, A. Ceresole, G. Dall'Agata, J. M. Oberreuter, J. Perz, First-order flow equations for extremal black holes in very special geometry, JHEP 10 (2007) 063. arXiv:0706.3373, doi:10.1088/1126-6708/2007/10/063.

[161] F. Larsen, The Attractor Mechanism in Five Dimensions, Lect. Notes Phys. 755 (2008) 249–281. arXiv:hep-th/0608191.

[162] A. Sen, Black hole entropy function and the attractor mechanism in higher derivative gravity, JHEP 09 (2005) 038. arXiv:hep-th/0506177, doi:10.1088/1126-6708/2005/09/038.

[163] A. Castro, J. L. Davis, P. Kraus, F. Larsen, 5D Black Holes and Strings with Higher Derivatives, JHEP 06 (2007) 007. arXiv:hep-th/0703087, doi:10.1088/1126-6708/2007/06/007.

[164] A. Castro, J. L. Davis, P. Kraus, F. Larsen, Precision Entropy of Spinning Black Holes, JHEP 09 (2007) 003. arXiv:0705.1847, doi:10.1088/1126-6708/2007/09/003.

[165] A. Castro, J. L. Davis, P. Kraus, F. Larsen, String Theory Effects on Five-Dimensional Black Hole Physics, Int. J. Mod. Phys. A23 (2008) 613–691. arXiv:0801.1863, doi:10.1142/S0217751X08039724.

[166] T. Ortin, A Simple Derivation of Supersymmetric Extremal Black Hole Attractors, Phys. Lett. B700 (2011) 261–264. arXiv:1103.2738, doi:10.1016/j.physletb.2011.05.012.

[167] G. L. Cardoso, J. M. Oberreuter, J. Perz, Entropy function for rotating extremal black holes in very special geometry, JHEP 05 (2007) 025. arXiv:hep-th/0701176, doi:10.1088/1126-6708/2007/05/025.

[168] K. P. Tod, All Metrics Admitting Supercovariantly Constant Spinors, Phys. Lett. 121B (1983) 241–244. doi:10.1016/0370-2693(83)90797-9.

[169] K. P. Tod, More on supercovariantly constant spinors, Class. Quant. Grav. 12 (1995) 1801–1820. doi:10.1088/0264-9381/12/7/020.

[170] K. Goldstein, N. Iizuka, R. P. Jena, S. P. Trivedi, Non-supersymmetric attractors, Phys. Rev. D72 (2005) 124021. arXiv:hep-th/0507096, doi:10.1103/PhysRevD.72.124021.

[171] B. Sahoo, A. Sen, Higher derivative corrections to non-supersymmetric extremal black holes in N=2 supergravity, JHEP 09 (2006) 029. arXiv:hep-th/0603149, doi:10.1088/1126-6708/2006/09/029.

[172] G. L. Cardoso, B. de Wit, S. Mahapatra, Black hole entropy functions and attractor equations, JHEP 03 (2007) 085. arXiv:hep-th/0612225, doi:10.1088/1126-6708/2007/03/085.

[173] T. Mohaupt, Black hole entropy, special geometry and strings, Fortsch. Phys. 49 (2001) 3–161. arXiv:hep-th/0007195, doi:10.1002/1521-3978(200102)49:1/3<3::AID-PROP3>3.0.CO;2-#.

[174] T. Mohaupt, Supersymmetric black holes in string theory, Fortsch. Phys. 55 (2007) 519–544. arXiv:hep-th/0703035, doi:10.1002/prop.200610382.

[175] B. de Wit, BPS black holes, Nucl. Phys. Proc. Suppl. 171 (2007) 16–38. arXiv:0704.1452, doi:10.1016/j.nuclphysbps.2007.06.004.

[176] A. Sen, Black Hole Entropy Function, Attractors and Precision Counting of Microstates, Gen. Rel. Grav. 40 (2008) 2249–2431. arXiv:0708.1270, doi:10.1007/s10714-008-0626-4.

[177] T. Mohaupt, From Special Geometry to Black Hole Partition Functions, Springer Proc. Phys. 134 (2010) 165–241. arXiv:0812.4239, doi:10.1007/978-3-642-10736-8_4.

[178] G. W. Moore, Attractors and arithmetic. arXiv:hep-th/9807056.

[179] G. W. Moore, Arithmetic and attractors. arXiv:hep-th/9807087.

[180] A. Dabholkar, S. Murthy, D. Zagier, Quantum Black Holes, Wall Crossing, and Mock Modular Forms. arXiv:1208.4074.

[181] F. Denef, Supergravity flows and D-brane stability, JHEP 08 (2000) 050. arXiv:hep-th/0005049, doi:10.1088/1126-6708/2000/08/050.

[182] H. Ooguri, A. Strominger, C. Vafa, Black hole attractors and the topological string, Phys. Rev. D70 (2004) 106007. arXiv:hep-th/0405146, doi:10.1103/PhysRevD.70.106007.

[183] D. Gaiotto, A. Strominger, X. Yin, 5D black rings and 4D black holes, JHEP 02 (2006) 023. arXiv:hep-th/0504126, doi:10.1088/1126-6708/2006/02/023.

[184] K. Behrndt, G. L. Cardoso, S. Mahapatra, Exploring the relation between 4-D and 5-D BPS solutions, Nucl. Phys. B732 (2006) 200–223. arXiv:hep-th/0506251, doi:10.1016/j.nuclphysb.2005.10.026.

[185] A. Ceresole, S. Ferrara, A. Marrani, 4d/5d Correspondence for the Black Hole Potential and its Critical Points, Class. Quant. Grav. 24 (2007) 5651–5666. arXiv:0707.0964, doi:10.1088/0264-9381/24/22/023.

[186] J. Bellorin, P. Meessen, T. Ortin, Supersymmetry, attractors and cosmic censorship, Nucl. Phys. B762 (2007) 229–255. arXiv:hep-th/0606201, doi:10.1016/j.nuclphysb.2006.11.004.

[187] D. Astefanesei, K. Goldstein, R. P. Jena, A. Sen, S. P. Trivedi, Rotating attractors, JHEP 10 (2006) 058. arXiv:hep-th/0606244, doi:10.1088/1126-6708/2006/10/058.

[188] A. Sen, Quantum Entropy Function from AdS(2)/CFT(1) Correspondence, Int. J. Mod. Phys. A24 (2009) 4225–4244. arXiv:0809.3304, doi:10.1142/S0217751X09045893.

[189] A. Dabholkar, J. Gomes, S. Murthy, Quantum black holes, localization and the topological string, JHEP 06 (2011) 019. arXiv:1012.0265, doi:10.1007/JHEP06(2011)019.

[190] S. Nampuri, M. Soroush, New Perspectives on Attractor Flows and Trees from CFT. arXiv:1009.5768.

[191] M. C. Camara, G. L. Cardoso, T. Mohaupt, S. Nampuri, A Riemann-Hilbert approach to rotating attractors, JHEP 06 (2017) 123. arXiv:1703.10366, doi:10.1007/JHEP06(2017)123.

[192] K. Goldstein, V. Jejjala, J. Mashiyane, James, S. Nampuri, Generalized Hot Attractors, JHEP 03 (2019) 188. arXiv:1811.04963, doi:10.1007/JHEP03(2019)188.

[193] S. Ferrara, G. W. Gibbons, R. Kallosh, Black holes and critical points in moduli space, Nucl. Phys. B500 (1997) 75–93. arXiv:hep-th/9702103, doi:10.1016/S0550-3213(97)00324-6.

[194] G. Dall'Agata, A. Gnecchi, Flow equations and attractors for black holes in N = 2 U(1) gauged supergravity, JHEP 03 (2011) 037. arXiv:1012.3756, doi:10.1007/JHEP03(2011)037.

[195] S. Barisch, G. L. Cardoso, M. Haack, S. Nampuri, N. A. Obers, Nernst branes in gauged supergravity, JHEP 11 (2011) 090. arXiv:1108.0296, doi:10.1007/JHEP11(2011)090.

[196] K. Behrndt, D. Lust, W. A. Sabra, Stationary solutions of N=2 supergravity, Nucl. Phys. B510 (1998) 264–288. arXiv:hep-th/9705169, doi:10.1016/S0550-3213(97)00633-0,10.1016/S0550-3213(98)81014-6.

[197] A. Ceresole, G. Dall'Agata, Flow Equations for Non-BPS Extremal Black Holes, JHEP 03 (2007) 110. arXiv:hep-th/0702088, doi:10.1088/1126-6708/2007/03/110.

[198] A. Ceresole, G. Dall'Agata, S. Ferrara, A. Yeranyan, First order flows for N=2 extremal black holes and duality invariants, Nucl. Phys. B824 (2010) 239–253. arXiv:0908.1110, doi:10.1016/j.nuclphysb.2009.09.003.

[199] G. Dall'Agata, Black holes in supergravity: flow equations and duality, Springer Proc. Phys. 142 (2013) 1–45. arXiv:1106.2611, doi:10.1007/978-3-642-31380-6_1.

[200] A. Sen, Entropy function for heterotic black holes, JHEP 03 (2006) 008. arXiv:hep-th/0508042, doi:10.1088/1126-6708/2006/03/008.

[201] K. Behrndt, G. L. Cardoso, B. de Wit, R. Kallosh, D. Lust, T. Mohaupt, Classical and quantum N=2 supersymmetric black holes, Nucl. Phys. B488 (1997) 236–260. arXiv:hep-th/9610105, doi:10.1016/S0550-3213(97)00028-X.

[202] G. L. Cardoso, B. de Wit, J. Kappeli, T. Mohaupt, Stationary BPS solutions in N=2 supergravity with R^2 interactions, JHEP 12 (2000) 019. arXiv:hep-th/0009234, doi:10.1088/1126-6708/2000/12/019.

[203] G. L. Cardoso, B. de Wit, T. Mohaupt, Macroscopic entropy formulae and nonholomorphic corrections for supersymmetric black holes, Nucl. Phys. B567 (2000) 87–110. arXiv:hep-th/9906094, doi:10.1016/S0550-3213(99)00560-X.

[204] G. L. Cardoso, B. de Wit, J. Kappeli, T. Mohaupt, Asymptotic degeneracy of dyonic N = 4 string states and black hole entropy, JHEP 12 (2004) 075. arXiv:hep-th/0412287, doi:10.1088/1126-6708/2004/12/075.

[205] G. L. Cardoso, D. Lust, T. Mohaupt, Modular symmetries of N=2 black holes, Phys. Lett. B388 (1996) 266–272. arXiv:hep-th/9608099, doi:10.1016/S0370-2693(96)01138-0.

[206] A. Sen, Extremal black holes and elementary string states, Mod. Phys. Lett. A10 (1995) 2081–2094. arXiv:hep-th/9504147, doi:10.1142/S0217732395002234.

[207] A. Dabholkar, Exact counting of black hole microstates, Phys. Rev. Lett. 94 (2005) 241301. arXiv:hep-th/0409148, doi:10.1103/PhysRevLett.94.241301.

[208] A. Sen, Stretching the horizon of a higher dimensional small black hole, JHEP 07 (2005) 073. arXiv:hep-th/0505122, doi:10.1088/1126-6708/2005/07/073.

[209] A. Dabholkar, R. Kallosh, A. Maloney, A Stringy cloak for a classical singularity, JHEP 12 (2004) 059. arXiv:hep-th/0410076, doi:10.1088/1126-6708/2004/12/059.

[210] P. A. Cano, P. F. Ramirez, A. Ruiperez, The small black hole illusion. arXiv:1808.10449.

[211] F. Faedo, P. F. Ramirez, Exact charges from heterotic black holes. arXiv:1906.12287.

[212] G. W. Gibbons, D. A. Rasheed, Sl(2,R) invariance of nonlinear electrodynamics coupled to an axion and a dilaton, Phys. Lett. B365 (1996) 46–50. arXiv:hep-th/9509141, doi:10.1016/0370-2693(95)01272-9.

[213] A. Sen, C. Vafa, Dual pairs of type II string compactification, Nucl. Phys. B455 (1995) 165–187. arXiv:hep-th/9508064, doi:10.1016/0550-3213(95)00498-H.

[214] G. L. Cardoso, B. de Wit, S. Mahapatra, Exact results for an STU-model. arXiv:1907.04077.

[215] J. M. Lee, Introduction to Smooth Manifolds, Springer, 2003.

[216] T. Frankel, The Geometry of Physics, Cambridge University Press, 2004.

[217] S. Gallot, D. Hulin, J. Lafontaine, Riemannian Geometry, Springer, 2004.

[218] Y. Choquet-Bruhat, C. DeWitt-Morette, M. Dillard-Bleick, Analysis, Manifolds and Physics, North Holland, 1977.

[219] R. L. Fernandes, Differential Geometry, Lecture Notes, UIUC (2013).

[220] A. L. Besse, Einstein Manifolds, Springer, 2002.

[221] R. M. Wald, General Relativity, The University of Chicago Press, 1984.

[222] J. Marsden, A. Weinstein, Reduction of symplectic manifolds with symmetry, Rept. Math. Phys. 5 (1) (1974) 121–130. doi:10.1016/0034-4877(74)90021-4.

[223] N. J. Hitchin, A. Karlhede, U. Lindstrom, M. Rocek, Hyperkahler Metrics and Supersymmetry, Commun. Math. Phys. 108 (1987) 535. doi:10.1007/BF01214418.

[224] G. Rudolf, M. Schmidt, Differential Geometry and Mathematical Physics, Springer, 2013.

[225] A. Swann, Hyperkähler and quanternionic kähler geometry, Ph.D. thesis, Oxford (1990).

[226] A. Swann, Hyperkähler and quanternionic kähler geometry, Math. Ann. 289 (1991) 421–450.

[227] K. Galicki, A Generalization of the Momentum Mapping Construction for Quaternionic Kahler Manifolds, Commun. Math. Phys. 108 (1987) 117. doi:10.1007/BF01210705.

[228] K. Galicki, H. B. Lawson, Quaternionic reduction and quaternionic orbifolds, Math. Ann. 282 (1988) 1.

[229] C. I. Lazaroiu, C. S. Shahbazi, Section sigma models coupled to symplectic duality bundles on Lorentzian four-manifolds, J. Geom. Phys. 128 (2018) 58–86. arXiv:1711.05651, doi:10.1016/j.geomphys.2018.02.003.

[230] B. de Wit, J. W. van Holten, A. Van Proeyen, Transformation Rules of N=2 Supergravity Multiplets, Nucl. Phys. B167 (1980) 186. doi:10.1016/0550-3213(80)90125-X.

[231] E. Bergshoeff, M. de Roo, B. de Wit, Extended Conformal Supergravity, Nucl. Phys. B182 (1981) 173–204. doi:10.1016/0550-3213(81)90465-X.

[232] B. de Wit, J. W. van Holten, A. Van Proeyen, Structure of N=2 Supergravity, Nucl. Phys. B184 (1981) 77, [Erratum: Nucl. Phys.B222,516(1983)]. doi:10.1016/0550-3213(83)90548-5,10.1016/0550-3213(81)90211-X.

[233] D. Butter, S. Hegde, I. Lodato, B. Sahoo, $N = 2$ dilaton Weyl multiplet in 4D supergravity, JHEP 03 (2018) 154. arXiv:1712.05365, doi:10.1007/JHEP03(2018)154.

[234] T. Fujita, K. Ohashi, Superconformal tensor calculus in five-dimensions, Prog. Theor. Phys. 106 (2001) 221–247. arXiv:hep-th/0104130, doi:10.1143/PTP.106.221.

[235] E. Bergshoeff, S. Cucu, T. de Wit, J. Gheerardyn, S. Vandoren, A. Van Proeyen, N = 2 supergravity in five-dimensions revisited, Class. Quant. Grav. 21 (2004) 3015–3042, [Class. Quant. Grav. 23, 7149 (2006)]. arXiv:hep-th/0403045, doi:10.1088/0264-9381/23/23/C01, 10.1088/0264-9381/21/12/013.

[236] B. de Wit, N=2 symplectic reparametrizations in a chiral background, Fortsch. Phys. 44 (1996) 529–538. arXiv:hep-th/9603191, doi:10.1002/prop.2190440606.

[237] R. M. Wald, Black hole entropy is the Noether charge, Phys. Rev. D48 (8) (1993) R3427–R3431. arXiv:gr-qc/9307038, doi:10.1103/PhysRevD.48.R3427.

[238] T. Jacobson, G. Kang, R. C. Myers, On black hole entropy, Phys. Rev. D49 (1994) 6587–6598. arXiv:gr-qc/9312023, doi:10.1103/PhysRevD.49.6587.

[239] V. Iyer, R. M. Wald, Some properties of Noether charge and a proposal for dynamical black hole entropy, Phys. Rev. D50 (1994) 846–864. arXiv:gr-qc/9403028, doi:10.1103/PhysRevD.50.846.

[240] G. L. Cardoso, B. de Wit, T. Mohaupt, Deviations from the area law for supersymmetric black holes, Fortsch. Phys. 48 (2000) 49–64. arXiv:hep-th/9904005.