Meso-scale approximations of fields around clusters of defects

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In memory of Professor V.V. Zhikov

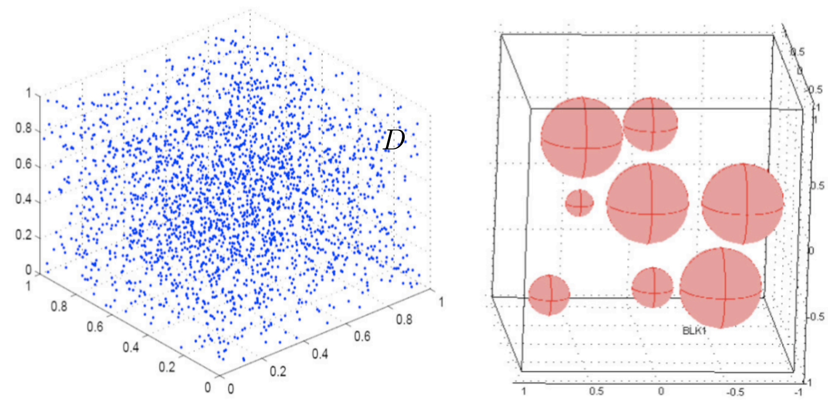
Abstract

We present a review of the recent results on asymptotic approximations of solutions to boundary value value problems in domains with large clusters of small defects. There are no assumptions, which require periodicity within the cluster. The asymptotic approximations, which we discuss here are uniform, and include the boundary layers occurring in the neighbourhood of singularly perturbed boundaries of the domains concerned. The term "meso-scale" is used to describe these approximations, as they go beyond the conventional constraints of the homogenization theory.

1 Introduction

The approach of homogenization for boundary value problems in domains with multiple perforations and for equations with rapidly oscillating coefficients is an important part of the fundamental theory of partial differential equations. The work [1]–[7] has brought seminal results and made an impact in the area homogenization of differential operators and integral functionals and G-convergence [1], [2], two-scale convergence [3], homogenization of degenerate elliptic equations [4], and homogenization techniques for variational problems [5]. These fundamental theoretical developments have led to solutions of many challenging applied problems of high practical importance.

Examples include analysis of challenging problems of elasticity in structured solids, such as periodic structures depending on two geometric parameters, when the homogenization has anon-classical nature [6]. Furthermore, the notion of a critical thickness has been used to characterise a special class of elastic structures in this non-classical homogenization problem. Homogenization of problems of linear elasticity on singular structures is studied in [7].



(a) (b)

Figure 1: Three-dimensional domain containing multiple defects: (a) a large cluster of small inclusions, and (b) a finite cluster of inclusions of finite size

The results of [1]–[7] have made an impact on the abstract theory of homogenization as well as applications in physics, mechanics and engineering of heterogeneous structures including laminates, multiply perforated systems and random composites, as presented in the comprehensive research monograph [2]. The work [2] serves as a fundamental textbook for experts and learners of the homogenization, which covers key topics including G-convergence of differential operators, homogenization of elliptic operators with random coefficients, homogenization and percolation, spectral problems in the homogenization theory and homogenization of nonlinear variational problems.

The powerful asymptotic approach developing the compound asymptotic approximations to solutions of boundary value problem in singularly perturbed domains is presented in the two-volume monograph [10]. The present review is based on the series of papers [11]–[21] developing uniform asymptotic approximations of Green's kernels in singularly perturbed domains as well as the method of meso-scale asymptotic approximations for fields around clusters of many defects of different shapes, sizes and relative positions.

In particular, the article [15] presents the approach leading to uniform asymptotic approximations of singularly perturbed boundary value problems in multiply perforated domains without the homogenization, and it addresses the homogenization limits for the cases when the homogenized operators gain additional terms in their representation. This also provides the link to the classical work [8] and [9] addressing a special case of homogenization referring to "a strange term brought from somewhere else". Meso-scale asymptotic approximations to solutions of transmission and mixed boundary value problems in regions containing many perforations are studied in [17] and [18]. The method has been extended to problems of linear elasticity [19] and to spectral problems for domains containing clusters of small inclusions [21].

The structure of the paper is as follows. We begin by describing the geometry of domain and introduce the notion of meso-scale approximations. Then, a class of homogenization problems are considered from the view point of meso-scale approximations, with two examples concerned with volume and surface clusters, and asymptotic evaluation of the capacity of a volume cluster of small inclusions. The idea of derivation of meso-scale approximations and a formal argument are presented in Section 5. This is followed by meso-scale approximations of Green's functions, outlined in Sections 6 and 7, and the outline of asymptotic results for solutions of eigenvalue problems in Section 8.

2 A domain with a cluster of inclusions

We describe the idea of the method of meso-scale approximations by considering a solution of the Dirichlet problem for Laplace's operator in a three-dimensional singularly perturbed domain D_ε containing a large cluster of small inclusions, as shown in Fig. 1a. For comparison, the Fig. 1b shows a finite cluster of inclusions of finite size, where standard numerical methods, for example FEM or BIE, apply effectively in order to determine physical fields. We note that neither of these cases has periodic arrangements of inclusions, and the shapes and sizes of the inclusions within the cluster may change arbitrarily. It is also noted that the geometry shown in Fig. 1a appears to be problematic for FEM algorithms aiming at pointwise numerical evaluation of physical fields around inclusions within the large cluster.

The notation $A_\varepsilon^{(j)}$, $j = 1, \dots, N$, is used for small inclusions within the cluster, as shown in Fig. 2, where it is assumed that their maximum diameter is $\varepsilon \ll 1$, and the number of inclusions is large $N \gg 1$.

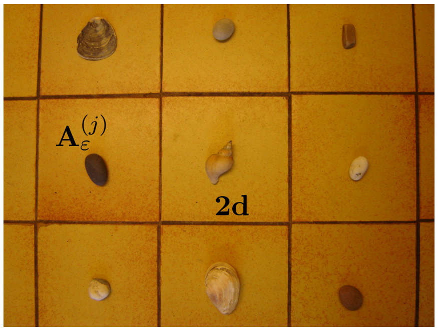


Figure 2: A schematic illustration of a cluster of small inclusions $A_\varepsilon^{(j)}, j = 1, \dots, N$. The distance between the inclusions is much larger compared to their diameter.

The unperturbed simply connected domain is denoted by D with the boundary Γ, and $D_\varepsilon = D \setminus \cup_{j=1}^N A_\varepsilon^{(j)}$, where $A_\varepsilon^{(j)}$ represent compact subsets of D separated from each other and from the exterior boundary Γ by a distance, which is sufficiently large compared to ε.

It is also convenient to introduce a set of points $\{\mathbf{O}^{(j)}\}_{j=1}^N$, which are interior points in $A_\varepsilon^{(j)}$, and a domain ω, such that $\text{diam}(\omega) = 1$, $\omega \subset D$, and $A_\varepsilon^{(j)} \subset \omega$ for all j. If $d = 2^{-1} \min_{i \neq j, 1 \leq i,j \leq N} |\mathbf{O}^{(j)} - \mathbf{O}^{(i)}|$, then we assume that $\varepsilon < c\, d$, where c is a sufficiently small constant. It is also required that dist $(\partial\omega, \Gamma) \geq \ 2d$, and $\text{dist}\Big\{ \bigcup_{j=1}^N A_\varepsilon^{(j)}, \partial\omega \Big\} \geq \ 2d$.

For the case when the number N of small inclusions becomes large, the asymptotic algorithm based on the method of meso-scale asymptotic approximations has been proposed in [15], and it takes into account the "competition" between the small diameter ε of an inclusion and the large number of inclusions within the cluster. The standard approach based on consideration of a singularly perturbed domain with one or two (or a finite number) of inclusions is no longer valid. Also, the approach based on meso-scale approximations goes beyond the range of applications of the homogenization theory as illustrated below.

3 Dirichlet problem: uniform approximation

Here u denotes the variational solution of the Dirichlet problem in the singularly perturbed domain D_ε containing the cluster of small inclusions

$$\begin{aligned} \Delta u(\mathbf{x}) + f(\mathbf{x}) &= 0, \quad \mathbf{x} \in D_\varepsilon, && (1) \\ u(\mathbf{x}) &= 0, \quad \mathbf{x} \in \partial D_\varepsilon. && (2) \end{aligned}$$

The forcing term f is considered as a smooth function with a compact support in $D \setminus \omega$, outside the cluster of small inclusions.

Also, the notation v is used for the solution of the Dirichlet problems in the corresponding limit domain D without inclusions:

$$\begin{aligned} \Delta v(\mathbf{x}) + f(\mathbf{x}) &= 0, \quad \mathbf{x} \in D, && (3) \\ v(\mathbf{x}) &= 0, \quad \mathbf{x} \in \Gamma. && (4) \end{aligned}$$

The notation $P_\varepsilon^{(j)}$ is used for the capacitary potential of the small inclusion $A_\varepsilon^{(j)}$:

$$\begin{aligned} \Delta P_\varepsilon^{(j)}(\mathbf{x}) &= 0, \quad \mathbf{x} \in \mathbb{R}^3 \setminus A_\varepsilon^{(j)}, && (5) \\ P_\varepsilon^{(j)}(\mathbf{x}) &= 1, \quad \mathbf{x} \in \partial A_\varepsilon^{(j)}. && (6) \\ P_\varepsilon^{(j)}(\mathbf{x}) &\to 0, \quad \text{as } |\mathbf{x}| \to \infty, && (7) \end{aligned}$$

Subject to a constraint

$$\varepsilon < c\, d^{7/4}, \tag{8}$$

where c is a sufficiently small absolute constant, the paper [15] presented a *meso-scale uniform approximation* for the solution of the singularly perturbed boundary value problem (1)–(2) in D_ε:

$$u(\mathbf{x}) = v(\mathbf{x}) + \sum_{j=1}^{N} C_j \Big(P_\varepsilon^{(j)}(\mathbf{x}) - \ \mathrm{cap}(A_\varepsilon^{(j)})\ H(\mathbf{x}, \mathbf{O}^{(j)}) \Big) + R(\mathbf{x}), \tag{9}$$

where $\mathrm{cap}(A_\varepsilon^{(j)})$ are capacities of small inclusions within the cluster, H is the regular part of Green's function

$$G(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} - H(\mathbf{x}, \mathbf{y}),$$

and R is the small remainder term.

The formula includes N coefficients C_j, which are evaluated as solutions of the linear algebraic system:

$$v(\mathbf{O}^{(k)}) + C_k + \sum_{1\le j\le N,\ j\ne k} C_j \ \mathrm{cap}(A_\varepsilon^{(j)})\ G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}) = 0, \tag{10}$$

where $k = 1, \dots, N$.

While the meso-scale approximation is uniformly applicable in the entire domain D_ε, including the neighbourhood of the singularly perturbed boundary within the cluster of inclusions, and it does not require homogenization, we show in the next section the connection between the two approaches and discuss illustrative examples.

4 Homogenization from the view point of meso-scale approximations

Here we use the meso-scale approximations obtained in [14], [15], [16] and discuss examples to compare these results with homogenization approximations for periodic systems. In particular, we evaluate effective capacity of a large cluster of small inclusions. For volume and surface clusters of inclusions, shown in Figs. 3, 4, we show the connection between pointwise meso-scale approximations and solutions of the homogenized problems.

4.1 Examples of volume and surface clusters of small inclusions

The first example includes a ball-shaped *volume cluster* filled with periodically positioned small inclusions of identical shapes, whereas the second example is concerned with the *surface cluster* of inclusions positioned along the surface of a sphere. In both cases we assume that $D = \mathbb{R}^3$.

In the *first example* we make a stronger assumption than in (8) by taking

$$\varepsilon < cd^3, \tag{11}$$

where c is a d-independent constant; all inclusions $A_\varepsilon^{(j)}$ are obtained by appropriate translations of the same small inclusion A_ε. In this case, $D_\varepsilon = \mathbb{R}^3 \setminus \cup_{j=1}^N A_\varepsilon^{(j)}$, and we also use the quantity μ defined by

$$\mu = \lim_{d\to 0} d^{-3}\mathrm{cap}(A_\varepsilon). \tag{12}$$

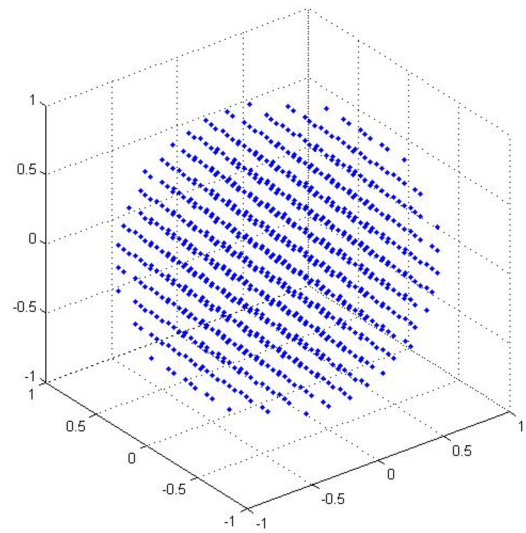


Figure 3: A ball-shaped volume cluster.

The cluster domain ω is the unit ball, and following [8], [9], [15], we refer to a function $\hat{u}$ as a solution of the homogenization problem, as follows

$$\Delta\hat{u}(\mathbf{x}) - \mu\chi(\omega)\hat{u}(\mathbf{x}) + f(\mathbf{x}) = 0, \quad \text{when} \quad \mathbf{x} \in \mathbb{R}^3, \tag{13}$$

$$\hat{u}(\mathbf{x}) \to 0, \quad \text{as} \quad |\mathbf{x}| \to \infty. \tag{14}$$

Here $\chi(\omega)$ is the characteristic function of ω, the forcing term f has its support outside ω, and satisfies the condition

$$\int_{\mathbb{R}^3} f(\mathbf{x})d\mathbf{x} = 0. \tag{15}$$

In this case, the formula (9) is reduced to

$$u(\mathbf{x}) = v(\mathbf{x}) + \sum_{j=1}^{N} C_j P_\varepsilon\Big(\frac{\mathbf{x} - \mathbf{O}^{(j)}}{\varepsilon}\Big) + R(\mathbf{x}), \tag{16}$$

where P_ε is the capacitary potential of the set A_ε.

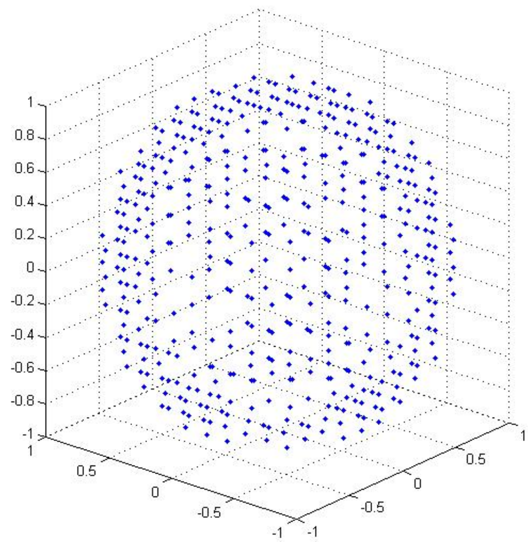


Figure 4: A cluster of inclusions placed on a surface of a sphere.

Consider the following harmonic function incorporating the solutions of the homogenization problem

$$U_d(\mathbf{x}) = v(\mathbf{x}) + \sum_{j=1}^{N} \mathfrak{C}_j P_\varepsilon(\frac{\mathbf{x} - \mathbf{O}^{(j)}}{\varepsilon}), \tag{17}$$

where $\mathfrak{C}_j = -\hat{u}(\mathbf{O}_j)$.

It has been verified in [15, 16] that

$$|C_k - \mathfrak{C}_k| = O(d), \quad k = 1, \dots, N.$$

The function U_d can be used as an approximation of u, with the remainder term being of order $O(d)$. Although this is inferior to (16), the convenience of using the homogenized field $\hat{u}(\mathbf{x})$ in the representation of coefficients in (17) is notable.

In the *second example*, which is concerned with a surface cluster, we allow for a weaker constraint on ε, compared to (11)

$$\varepsilon < cd^2, \tag{18}$$

where c is a d-independent constant; this assumption is still stronger compared to (8). In this case, the centres of all inclusions $A_\varepsilon^{(j)}$ are located on the unit sphere γ, as shown in Fig. 4, the inclusions have the same shape and are obtained by appropriate translations of the same small inclusion A_ε. For a regular tessellation of the unit sphere, into N cells of equal area $\mathcal{D}_N^2 = 4\pi/N$, we assume that each j-th cell contains the centre $\mathbf{O}^{(j)}$ of one inclusion $A_\varepsilon^{(j)}$. Taking the limit as $N \to \infty$

$$\mu_\gamma = \lim_{N\to\infty} \mathcal{D}_N^{-2} \text{cap}(A_\varepsilon), \tag{19}$$

we deduce that the homogenized field $\hat{u}_\gamma$, approximating in this case the solution of the singularly perturbed problem in D_ε has the form:

$$\hat{u}_\gamma(\mathbf{x}) = v(\mathbf{x}) - \mu_\gamma \int_{\gamma=\{|\mathbf{y}|=1\}} \frac{v(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|} d\mathbf{y}. \tag{20}$$

In turn, the approximation (17) is also in place with $\mathfrak{C}_j = -v(\mathbf{O}_j)$.

4.2 Capacity of a volume cluster of small inclusions

Let $\Xi_{\varepsilon,N} = \cup_{j=1}^N A_\varepsilon^{(j)}$ be a volume cluster of small inclusions $A_\varepsilon^{(j)}$ in $\mathbb{R}^3$.To evaluate the capacity of this cluster, we introduce a function $U_\Xi(\mathbf{x})$, which is harmonic and satisfies the following conditions

$$U_\Xi(\mathbf{x}) = 1, \quad \text{when} \;\; \mathbf{x} \in \partial(\mathbb{R}^3 \setminus \overline{\Xi}_{\varepsilon,N}), \tag{21}$$

and

$$U_\Xi(\mathbf{x}) \to 0, \quad \text{as} \;\; |\mathbf{x}| \to \infty. \tag{22}$$

The capacity of the cloud is defined by

$$\text{cap } \Xi_{\varepsilon,N} = \int_{\mathbb{R}^3\setminus\overline{\Xi}_{\varepsilon,N}} |\nabla U_\Xi(\mathbf{x})|^2 d\mathbf{x} = \int_{\mathbb{R}^3\setminus\overline{\Xi}_{\varepsilon,N}} |\nabla(U_\Xi(\mathbf{x})-1)|^2 d\mathbf{x}. \tag{23}$$

Similar to (16) we write

$$U_\Xi(\mathbf{x}) = \sum_{j=1}^N C_j P_\varepsilon^{(j)}(\frac{\mathbf{x}-\mathbf{O}^{(j)}}{\varepsilon}) + R_{\varepsilon,N}, \tag{24}$$

where the vector of coefficients $\mathbf{C} = (C_1, \ldots, C_N)$ is

$$\mathbf{C} = \mathbf{E}(\mathbf{I} + \boldsymbol{\mathcal{D}}\mathbf{S})^{-1}; \tag{25}$$

here $\mathbf{E} = (1, \ldots, 1)$ is the N-dimensional row vector whose entries are equal to 1, and the matrices $\mathbf{S}$ and $\boldsymbol{\mathcal{D}}$ are

$$\mathbf{S} = \Big\{(1 - \delta_{ik}) G(\mathbf{O}^{(k)}, \mathbf{O}^{(i)})\Big\}_{i,k=1}^{N}, \tag{26}$$

and

$$\boldsymbol{\mathcal{D}} = \text{diag } \{\text{cap}(A_\varepsilon^{(1)}), \ldots, \text{cap}(A_\varepsilon^{(N)})\}, \tag{27}$$

Then the capacity of the cloud of inclusions of identical shape, replicating an inclusion A_ε, is approximated by the formula

$$\text{cap } \Xi_{\varepsilon,N} \sim \text{cap}(A_\varepsilon) \ \mathbf{E}(\mathbf{I} + \text{cap}(A_\varepsilon)\mathbf{S})^{-1}\mathbf{E}^T. \tag{28}$$

The derivation of this formula is as follows:

$$\text{cap } \Xi_{\varepsilon,N} \sim \lim_{R\to\infty} \int_{B_R \setminus \Xi_{\varepsilon,N}} |\nabla(U_\Xi(\mathbf{x}) - 1)|^2 d\mathbf{x}$$

$$= \lim_{R\to\infty} \int_{\partial B_R} (U_\Xi(\mathbf{x}) - 1) \frac{\partial U_\Xi(\mathbf{x})}{\partial n} ds_X$$

$$= -\sum_{j=1}^{N} C_j \lim_{R\to\infty} \int_{\partial B_R} \frac{\partial}{\partial r}\Big(\frac{\text{cap}(A_\varepsilon^{(j)})}{4\pi|\mathbf{x} - \mathbf{O}^{(j)}|}\Big) ds_X$$

$$= \sum_{j=1}^{N} C_j \text{cap}(A_\varepsilon^{(j)}) = \mathbf{E}\big(\mathbf{I} + \boldsymbol{\mathcal{D}}\mathbf{S}\big)^{-1} \boldsymbol{\mathcal{D}}\mathbf{E}^T.$$

Here B_R is a ball of radius R centred at the origin. In particular, when all the holes have the same shape, the above formula is reduced to (28).

In particular, for the ball-shaped volume cluster discussed in the example above and shown in Fig. 3, the homogenization approximation leads to a simple asymptotic representation of the capacity of the cluster

$$\text{cap } \Xi_{\varepsilon,N} \sim 1 - \mu^{-1/2} \tanh(\mu^{1/2}), \tag{29}$$

where μ is the same as in (12).

5 The idea of derivation of a meso-scale approximation

Let the solution u of (1), (2) be written as

$$u(\mathbf{x}) = v(\mathbf{x}) + R^{(1)}(\mathbf{x}), \tag{30}$$

where v solves the Dirichlet problem in the unperturbed domain D, whereas the function $R^{(1)}$ is harmonic in D_ε and satisfies the boundary conditions

$$R^{(1)}(\mathbf{x}) = 0 \quad \text{when} \quad \mathbf{x} \in \Gamma, \tag{31}$$

and

$$R^{(1)}(\mathbf{x}) = -v(\mathbf{x}) = -v(\mathbf{O}^{(k)}) + O(\varepsilon) \quad \text{when} \quad \mathbf{x} \in \partial A_\varepsilon^{(k)}. \tag{32}$$

The function $R^{(1)}$ is approximated by

$$R^{(1)}(\mathbf{x}) \sim \sum_{j=1}^{N} C_j \Big(P_\varepsilon^{(j)}(\mathbf{x}) - \ \text{cap}(A_\varepsilon^{(j)}) \ H(\mathbf{x}, \mathbf{O}^{(j)}) \Big), \tag{33}$$

where C_j are unknown constant coefficients, and H is the regular part of Green's function in D.

For all $\mathbf{x} \in \Gamma, \ j = 1, \dots, N$, the following approximation holds

$$P^{(j)}(\mathbf{x}) - \ \text{cap}(A_\varepsilon^{(j)}) \ H(\mathbf{x}, \mathbf{O}^{(j)}) = O(\varepsilon \ \text{cap}(A_\varepsilon^{(j)}) |\mathbf{x} - \mathbf{O}^{(j)}|^{-2}). \tag{34}$$

On the boundary of a small inclusion $A_\varepsilon^{(k)} \quad (k = 1, \dots, N)$ we have

$$\begin{aligned} & v(\mathbf{O}^{(k)}) + O(\varepsilon) + C_k(1 + O(\varepsilon)) \\ & + \sum_{1 \le j \le N, \ j \ne k} C_j \Big(\text{cap}(A_\varepsilon^{(j)}) \ G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}) \\ & + O(\varepsilon \ \text{cap}(A_\varepsilon^{(j)}) |\mathbf{x} - \mathbf{O}^{(j)}|^{-2}) \Big) = 0, \end{aligned} \tag{35}$$

for all $\mathbf{x} \in \partial A_\varepsilon^{(k)}$.

Hence, the constant coefficients $C_j, \ j = 1, \dots, N$, are chosen to satisfy the system of linear algebraic equations

$$\begin{aligned} & v(\mathbf{O}^{(k)}) + C_k \\ & + \sum_{1 \le j \le N, \ j \ne k} C_j \ \text{cap}(A_\varepsilon^{(j)}) \ G(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}) = 0, \ k = 1, \dots, N. \end{aligned} \tag{36}$$

Then within certain constraints on the small parameters ε and d, it is shown that the above system of algebraic equations is solvable and that the harmonic function

$$R^{(2)}(\mathbf{x}) = R^{(1)}(\mathbf{x}) - \sum_{j=1}^{N} C_j \Big(P_\varepsilon^{(j)}(\mathbf{x}) - \ \text{cap}(A_\varepsilon^{(j)}) \ H(\mathbf{x}, \mathbf{O}^{(j)}) \Big)$$

is small on ∂D_ε. Further application of the maximum principle for harmonic functions leads to an estimate of the remainder $R^{(2)}$ in D_ε.

Hence, the solution (30) takes the form

$$u(\mathbf{x}) = v(\mathbf{x}) + \sum_{j=1}^{N} C_j \Big(P_\varepsilon^{(j)}(\mathbf{x}) - \ \mathrm{cap}(A_\varepsilon^{(j)}) \ H(\mathbf{x}, \mathbf{O}^{(j)}) \Big) + R^{(2)}(\mathbf{x}), \quad (37)$$

where C_j are obtained from the algebraic system (36).

It is convenient to define the matrices $\mathbf{S}$ and $\boldsymbol{\mathcal{D}}$ as follows:

$$\mathbf{S} = \Big\{ (1 - \delta_{ik}) G(\mathbf{O}^{(k)}, \mathbf{O}^{(i)}) \Big\}_{i,k=1}^{N}, \quad (38)$$

and

$$\boldsymbol{\mathcal{D}} = \mathrm{diag}\ \{\mathrm{cap}(A_\varepsilon^{(1)}), \dots, \mathrm{cap}(A_\varepsilon^{(N)})\}, \quad (39)$$

where G is Green's function of the unperturbed domain D. Then the coefficients C_j in the formula (37) can be placed as components of the vector $\mathbf{C} = (C_1, \dots, C_N)^T$ and evaluated as

$$\mathbf{C} = -(\mathbf{I} + \mathbf{S}\boldsymbol{\mathcal{D}})^{-1}\mathbf{V}, \quad (40)$$

where

$$\mathbf{V} = (v(\mathbf{O}^{(1)}), \dots, v(\mathbf{O}^{(N)}))^T. \quad (41)$$

The solvability of the algebraic system (36) is analysed in [15, 16] and the individual estimates for the coefficients C_j are given by

Lemma 1 *Let the small parameters ε and d satisfy*

$$\varepsilon < c d^2, \quad (42)$$

where c is a sufficiently small absolute constant. Then the components C_j of vector $\mathbf{C}$ in (40) allow for the estimate

$$|C_k| \le c \max_{1 \le j \le N} |v(\mathbf{O}^{(j)})|. \quad (43)$$

Meso-scale uniform approximation of u. As proved in [15, 16], the following uniform asymptotic approximation of the solution u holds:

Theorem 1 *Let the parameters ε and d satisfy the inequality*

$$\varepsilon < c\ d^{7/4}, \quad (44)$$

where c is a sufficiently small absolute constant.

Then the matrix $\mathbf{I} + \mathbf{S}\boldsymbol{\mathcal{D}}$, defined according to (38), (39)*, is invertible, and the solution $u(\mathbf{x})$ to the boundary value problem* (1)–(2) *is defined by the asymptotic formula*

$$u(\mathbf{x}) = v(\mathbf{x}) + \sum_{j=1}^{N} C_j \Big(P_\varepsilon^{(j)}(\mathbf{x}) - \ \mathrm{cap}(A_\varepsilon^{(j)}) \ H(\mathbf{x}, \mathbf{O}^{(j)}) \Big) + R(\mathbf{x}), \qquad (45)$$

where the column vector $\mathbf{C} = (C_1, \dots, C_N)^T$ is given by (40) *and the remainder $R(\mathbf{x})$ is a function harmonic in Ω_N, which satisfies the estimate*

$$|R(\mathbf{x})| \ \le C \Big\{ \varepsilon \|\nabla v\|_{L_\infty(\omega)} + \varepsilon^2 d^{-7/2} \|v\|_{L_\infty(\omega)} \Big\}. \qquad (46)$$

6 Meso-scale approximation of Green's function

Let $G_\varepsilon(\mathbf{x}, \mathbf{y})$ be Green's function of the Dirichlet problem for the operator $-\Delta$ in D_ε containing a cluster of many inclusions. The case of several spherical inclusions is discussed in [22]. The work [15, 16, 18] includes a uniform asymptotic approximation of $G_\varepsilon(\mathbf{x}, \mathbf{y})$. In addition to the solution of the algebraic system similar to (36), it also requires several models fields: Green's functions $g^{(j)}(\mathbf{x}, \mathbf{y}) = (4\pi|\mathbf{x} - \mathbf{y}|)^{-1} - h^{(j)}(\mathbf{x}, \mathbf{y})$ of the Dirichlet problem for the operator $-\Delta$ in $\mathbb{R}^3 \setminus A_\varepsilon^{(j)}$, $j = 1, \dots, N$, and its regular part $h^{(j)}$. The asymptotic formula for Green's function $G_\varepsilon(\mathbf{x}, \mathbf{y})$ is given by

Theorem 2 *Let the small parameters ε and d satisfy the inequality $\varepsilon < c\, d^2$, where c is a sufficiently small absolute constant. Then*

$$G_\varepsilon(\mathbf{x}, \mathbf{y}) = G(\mathbf{x}, \mathbf{y}) - \sum_{j=1}^{N} \Big\{ h^{(j)}(\mathbf{x}, \mathbf{y}) \qquad (47)$$

$$- P_\varepsilon^{(j)}(\mathbf{y}) H(\mathbf{x}, \mathbf{O}^{(j)}) - P_\varepsilon^{(j)}(\mathbf{x}) H(\mathbf{O}^{(j)}, \mathbf{y}) + \ \mathrm{cap}(A_\varepsilon^{(j)}) H(\mathbf{x}, \mathbf{O}^{(j)}) H(\mathbf{O}^{(j)}, \mathbf{y})$$

$$+ H(\mathbf{O}^{(j)}, \mathbf{O}^{(j)}) \ T^{(j)}(\mathbf{x}) T^{(j)}(\mathbf{y}) - \sum_{i=1}^{N} \mathcal{C}_{ij} T^{(i)}(\mathbf{x}) T^{(j)}(\mathbf{y}) \Big\} + \mathcal{R}(\mathbf{x}, \mathbf{y}),$$

where

$$T^{(j)}(\mathbf{y}) = P_\varepsilon^{(j)}(\mathbf{y}) - \ \mathrm{cap}(A_\varepsilon^{(j)}) H(\mathbf{O}^{(j)}, \mathbf{y}), \qquad (48)$$

with the capacitary potentials $P^{(j)}$ and the regular part H of Green's function G of Ω being the same as in Section 3*. The matrix $\boldsymbol{\mathcal{C}} = (\mathcal{C}_{ij})_{i,j=1}^N$ is defined by*

$$\boldsymbol{\mathcal{C}} = (\mathbf{I} + \mathbf{S}\boldsymbol{\mathcal{D}})^{-1} \mathbf{S}, \qquad (49)$$

where $\mathbf{S}$ *and* $\boldsymbol{\mathcal{D}}$ *are the same as in* (38), (39). *The remainder* $\mathcal{R}(\mathbf{x}, \mathbf{y})$ *is a harmonic function, both in* $\mathbf{x}$ *and* $\mathbf{y}$, *and satisfies the estimate*

$$|\mathcal{R}(\mathbf{x}, \mathbf{y})| \le \text{const } \varepsilon d^{-2} \tag{50}$$

uniformly with respect to $\mathbf{x}$ *and* $\mathbf{y}$ *in* Ω_N.

It is noted that the coefficients, represented as components of the matrix $\boldsymbol{\mathcal{C}}$ can be estimated as follows.

Lemma 2 *Let the small parameters* ε *and* d *obey the inequality* $\varepsilon < c\ d^2$, *where* c *is a sufficiently small absolute constant. Then the matrix* $\boldsymbol{\mathcal{C}}$ *in* (49) *satisfies the estimate*

$$\|\boldsymbol{\mathcal{C}}\|_{\mathbb{R}^N \to \mathbb{R}^N} \le c d^{-3}. \tag{51}$$

7 Homogenized coefficients in meso-scale approximations of Green's functions

For a bounded three-dimensional domain D, we assume that a large number of points $\mathbf{O}_1, \mathbf{O}_2, ..., \mathbf{O}_N$ are packed in a cubic three-dimensional array. Assuming that all inclusions have identical shape, and d, which is proportional to the minimum of the distance between neighbouring inclusions, is a small parameter as in Section 2, we consider the case where $\lim_{\varepsilon \to 0}\ \text{cap}(A_\varepsilon)/d^3 = \mu = \text{const}$.

Then Green's function of the Dirichlet problem in the perforated domain is approximated by the asymptotic formula:

$$G_\varepsilon(\mathbf{x}, \mathbf{y}) = G(\mathbf{x}, \mathbf{y}) - \sum_{j=1}^{N} h^{(j)}(\mathbf{x}, \mathbf{y}) + \sum_{j=1}^{N} \{P_\varepsilon^{(j)}(\mathbf{y}) H(\mathbf{x}, \mathbf{O}_j) + P_\varepsilon^{(j)}(\mathbf{x}) H(\mathbf{O}_j, \mathbf{y})\}$$

$$-\text{cap}(A_\varepsilon)\ \sum_{j=1}^{N} H(\mathbf{x}, \mathbf{O}_j) H(\mathbf{O}_j, \mathbf{y}) - \sum_{j=1}^{N} H(\mathbf{O}_j, \mathbf{O}_j) T^{(j)}(\mathbf{x}) T^{(j)}(\mathbf{y})$$

$$+ \sum_{j=1}^{N} \sum_{i \ne j} G_{hom}(\mathbf{O}_j, \mathbf{O}_i) T^{(i)}(\mathbf{y}) T^{(j)}(\mathbf{x}) + O(d^{-1}\varepsilon), \tag{52}$$

where $T^{(j)}$ are the same as in (48), and $G_{hom}(\mathbf{x}, \mathbf{y})$ is the Green's function of the "homogenized" operator $\Delta - \mu$ in D.

It is worthwhile to note that the formula (52) provides a pointwise approximation of G_ε in D_ε. It includes the discrete values of the homogenized Green's function, and in comparison with the classical homogenization theory the approximation (52) satisfies the same equation as the perturbed function G_ε in D_ε. An important feature of this approximation is its uniformity with respect to $\mathbf{x}$ and $\mathbf{y}$ in D_ε.

8 Eigenvalues for domains containing clusters of small inclusions

Eigenvalue problems in domains with defects are important in the framework of applications in acoustics, as well as defect detection and non-destructive testing of structures.

Asymptotic analysis of spectral problems for the Laplacian in domains containing a small ball-shaped or disk-shaped inclusions was published in a series of papers [23]–[28].

Fundamental results obtained for spectral problems in the homogenization theory are included in the monograph [2]. In particular, [2] incorporates analysis of spectral properties of stratified media, as well as the study of density of states for random elliptic operators.

The analysis of spectral problems in domains with a finite number of small voids of arbitrary shapes is presented in [10]. The method of compound asymptotic approximations has been implemented for evaluation of the first eigenvalue and the corresponding eigenfunction for the Laplacian in a domain with a small defect.

The recent paper [21] has extended the analysis of eigenvalue problems to the case of large clusters of inclusions by employing the approach of mesoscale asymptotic approximations

We now outline the asymptotic representations for the first eigenvalue and the corresponding eigenfunction for the Laplacian in the domain containing a cluster of small inclusions, with the Dirichlet boundary conditions on their surfaces, and the homogeneous Neumann boundary condition on the exterior boundary of the domain.

First, we consider a finite cluster, i.e. N is assumed to be finite, inclusions are assumed to be separated by a finite distance from each other, and $d = O(1)$. The points $\mathbf{O}^{(1)}, \mathbf{O}^{(2)}, \dots, \mathbf{O}^{(N)}$ are also assumed to be separated by a finite distance from the exterior boundary $\Gamma = \partial D$. In this case, the first eigenvalue λ and the corresponding eigenfunction u solve the problem:

$$\Delta_{\mathbf{x}} u(\mathbf{x}) + \lambda u(\mathbf{x}) = 0 \ , \quad \mathbf{x} \in D_\varepsilon \ , \tag{53}$$

$$\frac{\partial u}{\partial n}(\mathbf{x}) = 0 \ , \quad \mathbf{x} \in \Gamma \ , \tag{54}$$

$$u(\mathbf{x}) = 0 \ , \quad \mathbf{x} \in \partial A_\varepsilon^{(j)}, \quad 1 \leq j \leq N \ . \tag{55}$$

We also use here the notion $\mathcal{G}(\mathbf{x}, \mathbf{y})$ of the Neumann function and the harmonic function $\mathcal{H}(\mathbf{x}, \mathbf{y}) = (4\pi|\mathbf{x} - \mathbf{y}|)^{-1} - \mathcal{G}(\mathbf{x}, \mathbf{y})$ for the limit domain D.

According to the method of compound asymptotic expansions [10], [21] applied to the dilute cluster of small inclusions, the first eigenvalue and the corresponding eigenfunction are approximated as follows:

Theorem 3 *The first eigenvalue λ corresponding to the problem* (53)–(55) *in D_ε is evaluated in the form*

$$\lambda = \frac{1}{|D|} \sum_{j=1}^{N} \text{cap}(A_\varepsilon^{(j)}) + O(\varepsilon^2) \ . \tag{56}$$

Theorem 4 *The asymptotic approximation of the eigenfunction u, corresponding to the first eigenvalue of* (53)–(55) *in D_ε, is given by*

$$\begin{aligned} u_\varepsilon(\mathbf{x}) \, = \, & 1 - \sum_{j=1}^{N} \Gamma^{(j)} \text{cap}(A_\varepsilon^{(j)}) \\ & - \sum_{j=1}^{N} \{ P_\varepsilon^{(j)}(\mathbf{x}) - \text{cap}(A_\varepsilon^{(j)}) \mathcal{H}(\mathbf{x}, \mathbf{O}^{(j)}) \} + R(\mathbf{x}) \ , \end{aligned}$$

where R is the remainder term satisfying

$$\|R\|_{L_2(D_\varepsilon)} \leq \text{Const } \varepsilon^2 \ ,$$

and

$$\Gamma^{(j)} = \frac{1}{4\pi|D|} \int_D \frac{d\mathbf{z}}{|\mathbf{z} - \mathbf{O}^{(j)}|} \ . \tag{57}$$

When the number of inclusions in the cluster increases, and hence the quantity d becomes a small parameter, we consider an additional constraint $\varepsilon < c\, d^3$, and follow the results of the paper [21]:

Theorem 5 *When $N \gg 1$, it is assumed that*

$$\varepsilon < c\, d^3 \tag{58}$$

where c is a sufficiently small constant. Then the asymptotic approximation of the eigenfunction u, which is a solution of (53)–(55) *in D_ε, is given by*

$$u(\mathbf{x}) = 1 + \sum_{j=1}^{N} C_j \Gamma^{(j)} \mathrm{cap}(A_\varepsilon^{(j)}) + \sum_{j=1}^{N} C_j \{P_\varepsilon^{(j)}(\mathbf{x}) - \mathrm{cap}(A_\varepsilon^{(j)}) \mathcal{H}(\mathbf{x}, \mathbf{O}^{(j)})\} + R(\mathbf{x}) \ , \qquad (59)$$

where R is the remainder term, and the coefficients C_k, $1 \le k \le N$, satisfy the solvable algebraic system

$$1 + C_k(1 - \mathrm{cap}(A_\varepsilon^{(k)})\{H(\mathbf{O}^{(k)}, \mathbf{O}^{(k)}) - \Gamma^{(k)}\}) + \sum_{\substack{j \ne k \\ 1 \le j \le N}} C_j \mathrm{cap}(A_\varepsilon^{(j)}) \Big\{ \mathcal{G}(\mathbf{O}^{(k)}, \mathbf{O}^{(j)}) + \Gamma^{(j)} \Big\} = 0 \ , \quad 1 \le k \le N \ . \quad (60)$$

Here R satisfies the estimate

$$\|R\|_{L_2(\Omega_N)} \le \mathrm{Const}\ \varepsilon^2 d^{-6} \ . \qquad (61)$$

The corresponding first eigenvalue is then defined in the form:

Theorem 6 *Let the small parameters ε and d satisfy* (58) *Then the first eigenvalue λ corresponding to the eigenfunction u admits the approximation*

$$\lambda = -\frac{1}{|D|} \sum_{j=1}^{N} C_j \mathrm{cap}(A_\varepsilon^{(j)}) + O(\varepsilon^2 d^{-6}) \ . \qquad (62)$$

The paper [21] also incorporates the higher-order approximations for the first eigenvalue and the corresponding eigenfunction for the case of a cluster with many small inclusions.

It is noted that the approximations included in Theorems 3 and 4 are constructed for the case when the inclusions within the cluster are separated by a finite distance, and these approximations are not designed to be used in the case when the inclusions are close together and when their numbers become large. On the other hand, the asymptotic formulae (59)–(61) and (62) cover the scenarios of a domain containing a cluster containing many small inclusions. The uniform approximation of Theorem 4 does not require the solution of an auxiliary algebraic system of equations. On the other hand, Theorem 5 does require the solutions C_j, $1 \le j \le N$, to the system (60), which incorporates the data about the shape, size and relative position of small inclusions within the cluster.

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