Green's kernels for transmission problems in bodies with small inclusions

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In memory of V. B. Lidskii

Abstract

The uniform asymptotic approximation of Green's kernel for the transmission problem of antiplane shear is obtained for domains with small inclusions. The remainder estimates are provided. Numerical simulations are presented to illustrate the effectiveness of the approach.

1 Introduction

Our goal is to obtain a uniform asymptotic approximation of Green's function for a transmission problem of antiplane shear in a domain with small inclusions.

Exact solutions to singularly perturbed problems corresponding to bodies with defects are often unavailable. For complicated geometries involved in problems of this kind, i.e. domains with multiple small perforations, numerical algorithms may become incapable of reaching the required accuracy. Also, when the right-hand sides of such boundary value problems are singular, numerical procedures can suffer from the same deficiencies. In this case, asymptotic solutions to these problems are desirable.

The approximation of Green's kernels for regularly perturbed problems, for the Laplace operator and the biharmonic operator, was first studied by Hadamard in [4]. More recently, the question of uniformity for the approximations of Green's kernels for boundary value problems in domains with singularly and regularly perturbed boundaries, was addressed in [9]. The uniform approximations in [9] were derived using the method of compound asymptotic expansions.

The paper [7] contains the rigorous proofs and remainder estimates for uniform approximations of Green's kernels, given in [9], for $-\Delta$ in an n-dimensional

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domain ($n \geq 2$), with a single small rigid inclusion. Uniform asymptotic formulae of Green's functions for mixed problems of antiplane shear in domains with a small hole or a crack, are discussed in detail in [8]. In [10], an extension of the theory developed in [7] is made to the case of Green's tensors of vector elasticity, for an elastic body with a small inclusion. This was followed by [11], where uniform asymptotics of Green's kernels for planar and three-dimensional elasticity in bodies with multiple rigid inclusions are given. The paper [11] also includes analysis of Green's kernels and numerical simulations for antiplane shear and plane strain.

In the present paper, the new feature of the problem tackled is that on the small inclusions we prescribe transmission conditions (the continuity of tractions and displacements). The inclusions are assumed to be occupied by materials which are different from that of the ambient medium. Compared to previous expositions into the uniform approximation of Green's kernels in [7, 8, 10], where the kernels are approximated in the bodies containing small holes we also must approximate the Green's kernel inside the inclusions. The analysis also brings additional boundary layers when the point force is placed inside the inclusion.

Below, we illustrate one of the main results in this article, for the case when the domain has a single inclusion. Let ω_ε be a small planar inclusion, occupied by a material of shear modulus μ_I, containing the origin $\mathbf{O}$ and which is perfectly bonded to the rest of the matrix $\Omega_\varepsilon \subset \mathbb{R}^2$ whose shear modulus is μ_O. Here, ε is a small positive parameter characterising the normalized size of the inclusion. Consider the antiplane shear Green's function N_ε for the transmission problem inside the perturbed domain $\Omega_\varepsilon \cup \omega_\varepsilon$. We also use $N^{(\Omega)}$, as the Neumann function in the unperturbed domain (without the inclusion) and $R^{(\Omega)}$ as its regular part. We denote by $\mathcal{N}$ the Green's function for the transmission problem, in the unbounded domain with the scaled inclusion containing the origin. Also let us define the vector function $\mathcal{D} = (\mathcal{D}_1, \mathcal{D}_2)^T$, where the components $\mathcal{D}_j$, $j = 1, 2$ are the dipole fields for the scaled inclusion in the unbounded domain.

As one of the results we state

Theorem *The approximation of Green's function for the transmission problem of antiplane shear in* $\Omega_\varepsilon \cup \omega_\varepsilon \subset \mathbb{R}^2$, *is given by*

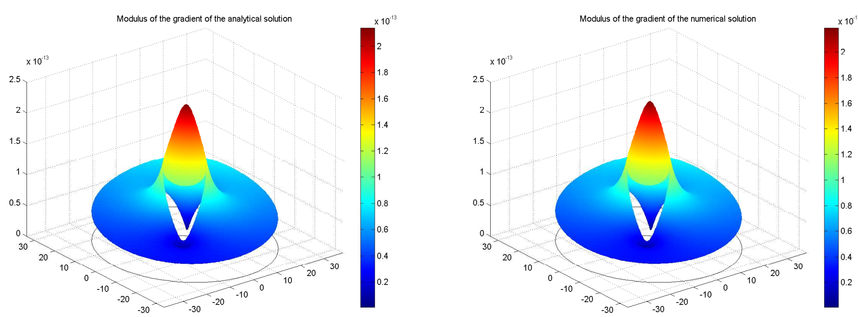
$$\begin{aligned} N_\varepsilon(\mathbf{x}, \mathbf{y}) = N^{(\Omega)}(\mathbf{x}, \mathbf{y}) + \mathcal{N}(\varepsilon^{-1}\mathbf{x}, \varepsilon^{-1}\mathbf{y}) + (2\pi\mu_O)^{-1} \log(\varepsilon^{-1}|\mathbf{x} - \mathbf{y}|) \\ + \varepsilon\mathcal{D}(\varepsilon^{-1}\mathbf{x}) \cdot \nabla_{\mathbf{x}} R^{(\Omega)}(\mathbf{O}, \mathbf{y}) + \varepsilon\mathcal{D}(\varepsilon^{-1}\mathbf{y}) \cdot \nabla_{\mathbf{y}} R^{(\Omega)}(\mathbf{x}, \mathbf{O}) + O(\varepsilon^2) \end{aligned} \tag{1.1}$$

uniformly for $\mathbf{x}, \mathbf{y} \in \Omega_\varepsilon \cup \omega_\varepsilon$.

The structure of the article is as follows. In Section 2, we introduce the main notations that will be adopted throughout the text and define Green's function for the domain containing several inclusions. Section 3 contains the description of model fields used to construct the uniform asymptotic approximations of the Green's function. In Section 4, we state and prove an estimate for solutions to model transmission problems in an unbounded domain with an inclusion.

Solutions to transmission problems in a domain with several small inclusions are studied in Section 5. Results of these sections will aid us in deriving the remainder estimate present in the generalization of approximation (1.1) to the case of multiple inclusions. Asymptotic properties of the boundary layer fields, involving the regular part of Green's function for the transmission problem, in the unbounded domain with an inclusion are investigated in Sections 6. Then, we consider the uniform approximation of Green's function for the transmission problem in the domain with small inclusions in Section 7, and give the formal algorithm together with the remainder estimates. Finally, in Section 8, we demonstrate the effectiveness of our approach and present the numerical simulations comparing the asymptotic formula in Section 7 with the finite element calculations in COMSOL.

The asymptotic formulae obtained in the sequel are readily applicable to numerical simulations. As an example, Figure 1 shows the comparison between an asymptotic approximation and COMSOL computation for the modulus of the gradient of the regular part of Green's function for the transmission problem in a domain with a circular inclusion, for the case when the point force is applied outside the inclusion in a planar body. This Figure represents the regular part of the displacement field produced by a point force inside a Cast Iron disk containing an Aluminum inclusion: Fig. 1a, shows the computations obtained through the formula (1.1) when $\mathbf{y} \in \Omega_\varepsilon$, while Fig. 1b corresponds to the numerical finite element solution produced in COMSOL. The surface plots shown are very similar.



a) b)

Figure 1: a) Modulus of the gradient of the regular part of the Green's function, computed with the aid of the asymptotic formula (1.1) when $\mathbf{y} \in \Omega_\varepsilon$, b) A finite element computation (produced in COMSOL) for the modulus of the gradient of the regular part of the Green's function for the transmission problem in a domain with an inclusion. Here, we consider a circular cylinder of radius 30m containing a circular inclusion of radius 7m, the shear modulus of the inclusion is $\mu_I = 2.6316 \times 10^{10}\text{Nm}^{-2}$ (an Aluminum inclusion), where the shear modulus for the rest of the matrix is $\mu_O = 5.6 \times 10^{10}\text{Nm}^{-2}$. The position of the unit point force is $\mathbf{y} = (10\text{m}, 10\text{m})$, which is quite close to the boundary of the inclusion.

2 Main Notations

Let Ω be a subset of $\mathbb{R}^2$, with smooth boundary $\partial\Omega$ and compact closure $\bar{\Omega}$. Also let $\omega^{(j)} \subset \mathbb{R}^2$, have smooth boundary $\partial\omega^{(j)}$ and compact closure $\bar{\omega}^{(j)}$, whose complement in the infinite plane is $C\bar{\omega}^{(j)} = \mathbb{R}^2 \backslash \bar{\omega}^{(j)}$, $j = 1, \dots, M$. The sets $\omega^{(j)}$, $j = 1, \dots, M$, are assumed to contain the origin $\mathbf{O}$, and the maximum distance between $\mathbf{O}$ and $\partial\omega^{(j)}$ is 1. Let $\omega_\varepsilon^{(j)}$, be a subset of Ω, with centre $\mathbf{O}^{(j)}$, $1 \le j \le M$. We assume that the minimum distance between $\mathbf{O}^{(j)}$ and $\mathbf{O}^{(k)}$, $k \ne j$, $k = 1, \dots, M$ and the minimum distance between $\mathbf{O}^{(j)}$ and $\partial\Omega$ is 1. We relate the domain $\omega_\varepsilon^{(j)}$ to $\omega^{(j)}$ via $\omega_\varepsilon^{(j)} = \{\mathbf{x} : \varepsilon^{-1}(\mathbf{x} - \mathbf{O}^{(j)}) \in \omega^{(j)}\}$, $j = 1, \dots, M$. The perturbed geometry is defined by $\Omega_\varepsilon = \Omega \backslash \overline{\bigcup_j \omega_\varepsilon^{(j)}}$, and we say that this domain is occupied by a material with shear modulus μ_O and the domain $\omega_\varepsilon^{(j)}$ is occupied by a material with shear modulus μ_{I_j}, where $\mu_O, \mu_{I_j} > 0$, $1 \le j \le M$. In the subsequent sections, along with $\mathbf{x}$ and $\mathbf{y}$ we will also use the scaled variables $\boldsymbol{\xi}_j = \varepsilon^{-1}(\mathbf{x} - \mathbf{O}^{(j)})$, $\boldsymbol{\eta}_j = \varepsilon^{-1}(\mathbf{y} - \mathbf{O}^{(j)})$.

By χ_T we mean the characteristic function of the set T, that is

$$\chi_T(\mathbf{x}) = \begin{cases} 1\,, & \text{if } \mathbf{x} \in T\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Our goal is to obtain uniform asymptotics for the Green's function N_ε, of the transmission problem in $\bigcup_j \omega_\varepsilon^{(j)} \cup \Omega_\varepsilon$, which is a solution of

$$\mu_O \Delta_{\mathbf{x}} N_\varepsilon(\mathbf{x}, \mathbf{y}) + \delta(\mathbf{x} - \mathbf{y}) = 0\,, \quad \mathbf{x} \in \Omega_\varepsilon\,, \mathbf{y} \in \bigcup_l \omega_\varepsilon^{(l)} \cup \Omega_\varepsilon\,, \tag{2.1}$$

$$\mu_{I_j} \Delta_{\mathbf{x}} N_\varepsilon(\mathbf{x}, \mathbf{y}) + \delta(\mathbf{x} - \mathbf{y}) = 0\,, \quad \mathbf{x} \in \omega_\varepsilon^{(j)}\,, j = 1, \dots, M, \mathbf{y} \in \bigcup_l \omega_\varepsilon^{(l)} \cup \Omega_\varepsilon\,. \tag{2.2}$$

The normal derivative of N_ε on the exterior boundary satisfies

$$\mu_O \frac{\partial N_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x}, \mathbf{y}) = -\frac{1}{|\partial\Omega|}\,, \quad \mathbf{x} \in \partial\Omega\,, \mathbf{y} \in \bigcup_l \omega_\varepsilon^{(l)} \cup \Omega_\varepsilon\,, \tag{2.3}$$

where $|\partial\Omega|$ is the one-dimensional measure of the set $\partial\Omega$; $\partial/\partial n_{\mathbf{x}} = \boldsymbol{n} \cdot \nabla_{\mathbf{x}}$ is the normal derivative, where $\boldsymbol{n}$ is the unit outward normal.

Assuming that the small inclusions $\omega_\varepsilon^{(j)}$, $j = 1, \dots, M$ are perfectly bonded to the matrix Ω_ε, we write transmission conditions across the interface $\partial\omega_\varepsilon^{(j)}$ in the form

$$\begin{aligned} \mu_O \frac{\partial N_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x}, \mathbf{y})\big|_{\mathbf{x} \in \partial\omega_\varepsilon^{(j)+}} &= \mu_{I_j} \frac{\partial N_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x}, \mathbf{y})\big|_{\mathbf{x} \in \partial\omega_\varepsilon^{(j)-}}\,, \\ N_\varepsilon(\mathbf{x}, \mathbf{y})\big|_{\mathbf{x} \in \partial\omega_\varepsilon^{(j)+}} &= N_\varepsilon(\mathbf{x}, \mathbf{y})\big|_{\mathbf{x} \in \partial\omega_\varepsilon^{(j)-}}\,, \end{aligned} \tag{2.4}$$

for $j = 1, \dots, M$ and $\mathbf{y} \in \bigcup_l \omega_\varepsilon^{(l)} \cup \Omega_\varepsilon$; the notation $\partial\omega_\varepsilon^{(j)\pm}$ indicates the exterior or interior boundary of the set $\omega_\varepsilon^{(j)}$.

The symmetry of N_ε, i.e.

$$N_\varepsilon(\mathbf{x},\mathbf{y}) = N_\varepsilon(\mathbf{y},\mathbf{x}) \ ,$$

is guaranteed by the condition

$$\int_{\partial\Omega} N_\varepsilon(\mathbf{x},\mathbf{y})\, dS_{\mathbf{x}} = 0 \ . \tag{2.5}$$

The proof of symmetry of N_ε, for this problem, is addressed below.

The symmetry of the Green's function N_ε

Proposition 1 *Green's function N_ε for the transmission problem in $\bigcup_{l=1}^M \omega_\varepsilon^{(l)} \cup \Omega_\varepsilon$ satisfies*

$$N_\varepsilon(\mathbf{x},\mathbf{y}) = N_\varepsilon(\mathbf{y},\mathbf{x}) \ . \tag{2.6}$$

Proof. Let

$$\begin{array}{ll} N_\varepsilon^{(O)}(\mathbf{x},\mathbf{y}) = N_\varepsilon(\mathbf{x},\mathbf{y}) \ , & \text{for } \mathbf{x} \in \Omega_\varepsilon, \mathbf{y} \in \bigcup_{l=1}^M \omega_\varepsilon^{(l)} \cup \Omega_\varepsilon \ , \\ N_\varepsilon^{(I_j)}(\mathbf{x},\mathbf{y}) = N_\varepsilon(\mathbf{x},\mathbf{y}) \ , & \text{for } \mathbf{x} \in \omega_\varepsilon^{(j)}, \mathbf{y} \in \bigcup_{l=1}^M \omega_\varepsilon^{(l)} \cup \Omega_\varepsilon \ . \end{array} \tag{2.7}$$

We apply Green's formula for the functions $N_\varepsilon^{(O)}$ and $N_\varepsilon^{(I_j)}$, $j = 1, \dots, M$:

$$\begin{aligned} & \mu_O \int_{\Omega_\varepsilon} \Big\{ N_\varepsilon^{(O)}(\mathbf{z},\mathbf{x}) \Delta_{\mathbf{z}} N_\varepsilon^{(O)}(\mathbf{z},\mathbf{y}) - N_\varepsilon^{(O)}(\mathbf{z},\mathbf{y}) \Delta_{\mathbf{z}} N_\varepsilon^{(O)}(\mathbf{z},\mathbf{x}) \Big\}\, d\mathbf{z} \\ & + \sum_{l=1}^M \mu_{I_l} \int_{\omega_\varepsilon^{(l)}} \Big\{ N_\varepsilon^{(I_l)}(\mathbf{z},\mathbf{x}) \Delta_{\mathbf{z}} N_\varepsilon^{(I_l)}(\mathbf{z},\mathbf{y}) - N_\varepsilon^{(I_l)}(\mathbf{z},\mathbf{y}) \Delta_{\mathbf{z}} N_\varepsilon^{(I_l)}(\mathbf{z},\mathbf{x}) \Big\}\, d\mathbf{z} \\ = \ & \mu_O \int_{\partial\Omega} \Big\{ N_\varepsilon^{(O)}(\mathbf{z},\mathbf{x}) \frac{\partial N_\varepsilon^{(O)}}{\partial n_{\mathbf{z}}}(\mathbf{z},\mathbf{y}) - N_\varepsilon^{(O)}(\mathbf{z},\mathbf{y}) \frac{\partial N_\varepsilon^{(O)}}{\partial n_{\mathbf{z}}}(\mathbf{z},\mathbf{x}) \Big\}\, dS_{\mathbf{z}} \\ & + \sum_{l=1}^M \int_{\partial\omega_\varepsilon^{(l)}} \Bigg\{ N_\varepsilon^{(O)}(\mathbf{z},\mathbf{x}) \left[\mu_{I_l} \frac{\partial N_\varepsilon^{(I_l)}}{\partial n_{\mathbf{z}}}(\mathbf{z},\mathbf{y}) - \mu_O \frac{\partial N_\varepsilon^{(O)}}{\partial n_{\mathbf{z}}}(\mathbf{z},\mathbf{y}) \right] \\ & - N_\varepsilon^{(I_l)}(\mathbf{z},\mathbf{y}) \left[\mu_{I_l} \frac{\partial N_\varepsilon^{(I_l)}}{\partial n_{\mathbf{z}}}(\mathbf{z},\mathbf{x}) - \mu_O \frac{\partial N_\varepsilon^{(O)}}{\partial n_{\mathbf{z}}}(\mathbf{z},\mathbf{x}) \right] \Bigg\}\, dS_{\mathbf{z}} \ , \end{aligned}$$

where the right-hand side is zero as a result of the transmission conditions (2.4), the exterior boundary condition (2.3) and the normalization (2.5). Thus

$$\begin{aligned} 0 = \ & \mu_O \int_{\Omega_\varepsilon} \Big\{ N_\varepsilon^{(O)}(\mathbf{z},\mathbf{x}) \Delta_{\mathbf{z}} N_\varepsilon^{(O)}(\mathbf{z},\mathbf{y}) - N_\varepsilon^{(O)}(\mathbf{z},\mathbf{y}) \Delta_{\mathbf{z}} N_\varepsilon^{(O)}(\mathbf{z},\mathbf{x}) \Big\}\, d\mathbf{z} \\ & + \sum_{l=1}^M \mu_{I_l} \int_{\omega_\varepsilon^{(l)}} \Big\{ N_\varepsilon^{(I_l)}(\mathbf{z},\mathbf{x}) \Delta_{\mathbf{z}} N_\varepsilon^{(I_l)}(\mathbf{z},\mathbf{y}) - N_\varepsilon^{(I_l)}(\mathbf{z},\mathbf{y}) \Delta_{\mathbf{z}} N_\varepsilon^{(I_l)}(\mathbf{z},\mathbf{x}) \Big\}\, d\mathbf{z} \end{aligned} \tag{2.8}$$

The next step involves using the governing equations (2.1) and (2.2) along with the above definitions of $N_\varepsilon^{(O)}$ and $N_\varepsilon^{(I_j)}$, $j = 1, \dots, M$. When $\mathbf{x}, \mathbf{y} \in \Omega_\varepsilon$, (2.8) gives

$$N_\varepsilon^{(O)}(\mathbf{x},\mathbf{y}) = N_\varepsilon^{(O)}(\mathbf{y},\mathbf{x})$$

whereas if $\mathbf{x} \in \Omega_\varepsilon$, $\mathbf{y} \in \omega_\varepsilon^{(j)}$

$$N_\varepsilon^{(O)}(\mathbf{x}, \mathbf{y}) = N_\varepsilon^{(I_j)}(\mathbf{y}, \mathbf{x}) \;, \quad j = 1, \dots, M \;.$$

Similarly, for $\mathbf{x} \in \omega_\varepsilon^{(j)}$, $\mathbf{y} \in \Omega_\varepsilon$, we deduce

$$N_\varepsilon^{(I_j)}(\mathbf{x}, \mathbf{y}) = N_\varepsilon^{(O)}(\mathbf{y}, \mathbf{x}) \;, \quad j = 1, \dots, M \;,$$

and if $\mathbf{x}, \mathbf{y} \in \omega_\varepsilon^{(j)}$

$$N_\varepsilon^{(I_j)}(\mathbf{x}, \mathbf{y}) = N_\varepsilon^{(I_j)}(\mathbf{y}, \mathbf{x}) \;, \quad j = 1, \dots, M \;.$$

Finally, with $\mathbf{x} \in \omega_\varepsilon^{(j)}$, $\mathbf{y} \in \omega_\varepsilon^{(k)}$, $k \neq j$, we have

$$N_\varepsilon^{(I_j)}(\mathbf{x}, \mathbf{y}) = N_\varepsilon^{(I_k)}(\mathbf{y}, \mathbf{x}), \quad 1 \leq j, k \leq M, \quad k \neq j \;.$$

The above relations together with (2.7) lead to (2.6). □

3 Special solutions in model domains

The asymptotic algorithm uses special fields defined in model domains including the unperturbed set and the exterior of a scaled inclusion.

1. The regular part $R^{(\Omega)}$ of the Neumann function $N^{(\Omega)}$ in Ω is defined as a solution of

$$\mu_O \Delta_{\mathbf{x}} R^{(\Omega)}(\mathbf{x}, \mathbf{y}) = 0 \;, \quad \mathbf{x} \in \Omega \;,$$

$$\mu_O \frac{\partial R^{(\Omega)}}{\partial n_{\mathbf{x}}}(\mathbf{x}, \mathbf{y}) = -\frac{\partial}{\partial n_{\mathbf{x}}}((2\pi)^{-1} \log |\mathbf{x} - \mathbf{y}|) + \frac{1}{|\partial\Omega|} \;, \quad \mathbf{x} \in \partial\Omega \;, \tag{3.1}$$

   where $\mathbf{y} \in \Omega$.

   To guarantee the symmetry of $R^{(\Omega)}$ we impose the orthogonality condition

$$\int_{\partial\Omega} R^{(\Omega)}(\mathbf{x}, \mathbf{y}) \, dS_{\mathbf{x}} = -\frac{1}{2\pi\mu_O} \int_{\partial\Omega} \log |\mathbf{x} - \mathbf{y}| \, dS_{\mathbf{x}} \;.$$

   The function $N^{(\Omega)}$ is related to $R^{(\Omega)}$ by

$$N^{(\Omega)}(\mathbf{x}, \mathbf{y}) = -(2\pi\mu_O)^{-1} \log |\mathbf{x} - \mathbf{y}| - R^{(\Omega)}(\mathbf{x}, \mathbf{y}) \;. \tag{3.2}$$

2. The next model field is Green's function for the transmission problem in the domain $C\bar{\omega}^{(j)} \cup \omega^{(j)}$, $j = 1, \dots, M$. This function is denoted by $\mathcal{N}^{(j)}$ and, for $\boldsymbol{\eta} \in C\bar{\omega}^{(j)} \cup \omega^{(j)}$, is subject to

$$\mu_O \Delta_{\boldsymbol{\xi}} \mathcal{N}^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta}) + \delta(\boldsymbol{\xi} - \boldsymbol{\eta}) = 0 \;, \quad \boldsymbol{\xi} \in C\bar{\omega}^{(j)} \;,$$

$$\mu_{I_j}\Delta_{\boldsymbol{\xi}}\mathcal{N}^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta})+\delta(\boldsymbol{\xi}-\boldsymbol{\eta})=0\ ,\quad \boldsymbol{\xi}\in\omega^{(j)}\ ,$$

where the transmission conditions across the interface of the inclusion are given as

$$\begin{gathered}\mu_O\frac{\partial\mathcal{N}^{(j)}}{\partial n_{\boldsymbol{\xi}}}(\boldsymbol{\xi},\boldsymbol{\eta})\big|_{\boldsymbol{\xi}\in\partial\omega^{(j)+}}=\mu_{I_j}\frac{\partial\mathcal{N}^{(j)}}{\partial n_{\boldsymbol{\xi}}}(\boldsymbol{\xi},\boldsymbol{\eta})\big|_{\boldsymbol{\xi}\in\partial\omega^{(j)-}}\ ,\\ \mathcal{N}^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta})\big|_{\boldsymbol{\xi}\in\partial\omega^{(j)+}}=\mathcal{N}^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta})\big|_{\boldsymbol{\xi}\in\partial\omega^{(j)-}}\ ,\end{gathered}\tag{3.3}$$

and at infinity we will prescribe the condition

$$\mathcal{N}^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta})=-(2\pi\mu_O)^{-1}\log|\boldsymbol{\xi}|+c(\boldsymbol{\eta})+O(|\boldsymbol{\xi}|^{-1}),\qquad\text{as}\qquad|\boldsymbol{\xi}|\to\infty\ .\tag{3.4}$$

Symmetry of $\mathcal{N}^{(j)}$. We choose

$$c(\boldsymbol{\eta})\equiv 0\ ,\tag{3.5}$$

and then it can be shown that

$$\mathcal{N}^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta})=\mathcal{N}^{(j)}(\boldsymbol{\eta},\boldsymbol{\xi})\ ,$$

i.e. $\mathcal{N}^{(j)}$ is symmetric.

The proof is analogous to the one of Proposition 1.

We set

$$\begin{gathered}\mathcal{N}^{(j,O)}(\boldsymbol{\xi},\boldsymbol{\eta})=\mathcal{N}^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta})\quad\text{for }\boldsymbol{\eta}\in C\bar{\omega}^{(j)},\boldsymbol{\xi}\in C\bar{\omega}^{(j)}\cup\omega^{(j)}\ ,\\ \mathcal{N}^{(j,I)}(\boldsymbol{\xi},\boldsymbol{\eta})=\mathcal{N}^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta})\quad\text{for }\boldsymbol{\eta}\in\omega^{(j)},\boldsymbol{\xi}\in C\bar{\omega}^{(j)}\cup\omega^{(j)}\ ,\end{gathered}$$

and

$$\begin{gathered}c^{(O)}(\boldsymbol{\eta})=c(\boldsymbol{\eta})\text{ for }\boldsymbol{\eta}\in C\bar{\omega}^{(j)}\ ,\\ c^{(I)}(\boldsymbol{\eta})=c(\boldsymbol{\eta})\text{ otherwise}\ .\end{gathered}$$

Assume $\boldsymbol{\eta}\in C\bar{\omega}^{(j)}$, and let $B_R=\{\boldsymbol{\xi}:|\boldsymbol{\xi}|<R\}$ be a disk, with sufficiently large radius R, so that $\boldsymbol{\eta}\in B_R\backslash\bar{\omega}^{(j)}$. By applying Green's formula to $\mathcal{N}^{(j)}(\mathbf{z},\boldsymbol{\xi})$ and $\mathcal{N}^{(j)}(\mathbf{z},\boldsymbol{\eta})$ in $B_R\backslash\bar{\omega}^{(j)}$ and $\omega^{(j)}$ we obtain

$$\begin{aligned}&\mu_O\int_{B_R\backslash\bar{\omega}^{(j)}}\Big\{\mathcal{N}^{(j)}(\mathbf{z},\boldsymbol{\xi})\Delta_{\mathbf{z}}\mathcal{N}^{(j,O)}(\mathbf{z},\boldsymbol{\eta})-\mathcal{N}^{(j,O)}(\mathbf{z},\boldsymbol{\eta})\Delta_{\mathbf{z}}\mathcal{N}^{(j)}(\mathbf{z},\boldsymbol{\xi})\Big\}\,d\mathbf{z}\\ &+\mu_{I_j}\int_{\omega^{(j)}}\Big\{\mathcal{N}^{(j)}(\mathbf{z},\boldsymbol{\xi})\Delta_{\mathbf{z}}\mathcal{N}^{(j,O)}(\mathbf{z},\boldsymbol{\eta})-\mathcal{N}^{(j)}(\mathbf{z},\boldsymbol{\eta})\Delta_{\mathbf{z}}\mathcal{N}^{(j)}(\mathbf{z},\boldsymbol{\xi})\Big\}\,d\mathbf{z}\\ &=\mu_O\int_{\partial B_R}\Big\{\mathcal{N}^{(j)}(\mathbf{z},\boldsymbol{\xi})\frac{\partial\mathcal{N}^{(j,O)}}{\partial n_{\mathbf{z}}}(\mathbf{z},\boldsymbol{\eta})-\mathcal{N}^{(j,O)}(\mathbf{z},\boldsymbol{\eta})\frac{\partial\mathcal{N}^{(j)}}{\partial n_{\mathbf{z}}}(\mathbf{z},\boldsymbol{\xi})\Big\}\,dS_{\mathbf{z}}\ .\end{aligned}$$

The transmission conditions (3.3) for $\mathcal{N}^{(j)}$ imply that the integral over $\partial\omega^{(j)}$ is zero. When $\boldsymbol{\xi}\in B_R\backslash\bar{\omega}^{(j)}$,

$$\mathcal{N}^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta})-\mathcal{N}^{(j)}(\boldsymbol{\eta},\boldsymbol{\xi})$$

$$=\mu_O\int_{\partial B_R}\Big\{\mathcal{N}^{(j,O)}(\mathbf{z},\boldsymbol{\xi})\frac{\partial\mathcal{N}^{(j,O)}}{\partial n_{\mathbf{z}}}(\mathbf{z},\boldsymbol{\eta})-\mathcal{N}^{(j,O)}(\mathbf{z},\boldsymbol{\eta})\frac{\partial\mathcal{N}^{(j,O)}}{\partial n_{\mathbf{z}}}(\mathbf{z},\boldsymbol{\xi})\Big\}\,dS_{\mathbf{z}}\ .$$

Taking the limit as $R \to \infty$ and employing (3.4), we deduce

$$\begin{aligned}
&\mathcal{N}^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta}) - \mathcal{N}^{(j)}(\boldsymbol{\eta},\boldsymbol{\xi}) \\
&= \lim_{R\to\infty} \mu_O \int_{\partial B_R} \Big\{ \Big(\frac{1}{2\pi\mu_O} \log|\mathbf{z}|^{-1} + c^{(O)}(\boldsymbol{\xi}) + O\left(\frac{1}{R}\right) \Big) \\
&\times \Big(\frac{1}{2\pi\mu_O} \frac{\partial \log|\mathbf{z}|^{-1}}{\partial n_{\mathbf{z}}} + O\left(\frac{1}{R^2}\right) \Big) \\
&- \Big(\frac{1}{2\pi\mu_O} \log|\mathbf{z}|^{-1} + c^{(O)}(\boldsymbol{\eta}) + O\left(\frac{1}{R}\right) \Big) \Big(\frac{1}{2\pi\mu_O} \frac{\partial \log|\mathbf{z}|^{-1}}{\partial n_{\mathbf{z}}} + O\left(\frac{1}{R^2}\right) \Big) \Big\} dS_{\mathbf{z}} \,,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\mathcal{N}^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta}) - \mathcal{N}^{(j)}(\boldsymbol{\eta},\boldsymbol{\xi}) &= (c^{(O)}(\boldsymbol{\eta}) - c^{(O)}(\boldsymbol{\xi})) \lim_{R\to\infty} \int_{\partial B_R} \frac{\partial}{\partial n_{\mathbf{z}}} ((2\pi)^{-1} \log|\mathbf{z}|) dS_{\mathbf{z}} \\
&= c^{(O)}(\boldsymbol{\eta}) - c^{(O)}(\boldsymbol{\xi}) = 0 \,.
\end{aligned}$$

Hence $\mathcal{N}^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta})$ is symmetric for $\boldsymbol{\xi},\boldsymbol{\eta} \in C\bar{\omega}^{(j)}$. In a similar way, it can be shown that

$$\begin{aligned}
\mathcal{N}^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta}) - \mathcal{N}^{(j)}(\boldsymbol{\eta},\boldsymbol{\xi}) &= c^{(I)}(\boldsymbol{\eta}) - c^{(O)}(\boldsymbol{\xi}) \text{ for } \boldsymbol{\xi} \in C\bar{\omega}^{(j)}, \boldsymbol{\eta} \in \omega^{(j)} \,, \\
\mathcal{N}^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta}) - \mathcal{N}^{(j)}(\boldsymbol{\eta},\boldsymbol{\xi}) &= c^{(O)}(\boldsymbol{\eta}) - c^{(I)}(\boldsymbol{\xi}) \text{ for } \boldsymbol{\eta} \in C\bar{\omega}^{(j)}, \boldsymbol{\xi} \in \omega^{(j)} \,, \\
\mathcal{N}^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta}) - \mathcal{N}^{(j)}(\boldsymbol{\eta},\boldsymbol{\xi}) &= c^{(I)}(\boldsymbol{\eta}) - c^{(I)}(\boldsymbol{\xi}) \text{ for } \boldsymbol{\xi} \in \omega^{(j)}, \boldsymbol{\eta} \in \omega^{(j)} \,,
\end{aligned}$$

and the condition (3.5) implies that the above right-hand sides are zero. Thus $\mathcal{N}^{(j)}$ is symmetric.

Regular part of $\mathcal{N}^{(j)}$. Let the function $\mathcal{N}^{(j)}$ have the form

$$\mathcal{N}^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta}) = \chi_{C\bar{\omega}^{(j)}}(\boldsymbol{\eta}) \mathcal{N}^{(j,O)}(\boldsymbol{\xi},\boldsymbol{\eta}) + \chi_{\omega^{(j)}}(\boldsymbol{\eta}) \mathcal{N}^{(j,I)}(\boldsymbol{\xi},\boldsymbol{\eta}) \,,$$

where

$$\begin{aligned}
\mathcal{N}^{(j,O)}(\boldsymbol{\xi},\boldsymbol{\eta}) &= -(2\pi\mu_O)^{-1} \log|\boldsymbol{\xi}-\boldsymbol{\eta}| - h_N^{(j,O)}(\boldsymbol{\xi},\boldsymbol{\eta}) \,, \\
\mathcal{N}^{(j,I)}(\boldsymbol{\xi},\boldsymbol{\eta}) &= -(2\pi\mu_{I_j})^{-1} \log|\boldsymbol{\xi}-\boldsymbol{\eta}| - h_N^{(j,I)}(\boldsymbol{\xi},\boldsymbol{\eta}) \,,
\end{aligned} \tag{3.6}$$

and here $h_N^{(j,O)}$ and $h_N^{(j,I)}$ are the regular parts of $\mathcal{N}^{(j,O)}$ and $\mathcal{N}^{(j,I)}$, $j = 1, \dots, M$, respectively. Moreover, we also set

$$\begin{aligned}
h_N^{(j,O,O)}(\boldsymbol{\xi},\boldsymbol{\eta}) &= h_N^{(j,O)}(\boldsymbol{\xi},\boldsymbol{\eta}) \,, && \text{for } \boldsymbol{\xi},\boldsymbol{\eta} \in C\bar{\omega}^{(j)} \,, \\
h_N^{(j,I,O)}(\boldsymbol{\xi},\boldsymbol{\eta}) &= h_N^{(j,O)}(\boldsymbol{\xi},\boldsymbol{\eta}) \,, && \text{for } \boldsymbol{\xi} \in \omega^{(j)}, \boldsymbol{\eta} \in C\bar{\omega}^{(j)} \,, \text{ and} \\
h_N^{(j,O,I)}(\boldsymbol{\xi},\boldsymbol{\eta}) &= h_N^{(j,I)}(\boldsymbol{\xi},\boldsymbol{\eta}) \,, && \text{for } \boldsymbol{\xi} \in C\bar{\omega}^{(j)}, \boldsymbol{\eta} \in \omega^{(j)} \,, \\
h_N^{(j,I,I)}(\boldsymbol{\xi},\boldsymbol{\eta}) &= h_N^{(j,I)}(\boldsymbol{\xi},\boldsymbol{\eta}) \,, && \text{for } \boldsymbol{\xi},\boldsymbol{\eta} \in \omega^{(j)} \,.
\end{aligned} \tag{3.7}$$

Then, the above definitions for $h_N^{(j,O)}$ and $h_N^{(j,I)}$ lead to this representation for $\mathcal{N}^{(j)}$

$$\begin{aligned}\mathcal{N}^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta}) \quad = \quad & \chi_{C\bar{\omega}^{(j)}}(\boldsymbol{\xi})\chi_{C\bar{\omega}^{(j)}}(\boldsymbol{\eta})\left\{-\frac{1}{2\pi\mu_O}\log|\boldsymbol{\xi}-\boldsymbol{\eta}|-h_N^{(j,O,O)}(\boldsymbol{\xi},\boldsymbol{\eta})\right\} \\ & +\chi_{\omega^{(j)}}(\boldsymbol{\xi})\chi_{C\bar{\omega}^{(j)}}(\boldsymbol{\eta})\left\{-\frac{1}{2\pi\mu_O}\log|\boldsymbol{\xi}-\boldsymbol{\eta}|-h_N^{(j,I,O)}(\boldsymbol{\xi},\boldsymbol{\eta})\right\} \\ & +\chi_{C\bar{\omega}^{(j)}}(\boldsymbol{\xi})\chi_{\omega^{(j)}}(\boldsymbol{\eta})\left\{-\frac{1}{2\pi\mu_{I_j}}\log|\boldsymbol{\xi}-\boldsymbol{\eta}|-h_N^{(j,O,I)}(\boldsymbol{\xi},\boldsymbol{\eta})\right\} \\ & +\chi_{\omega^{(j)}}(\boldsymbol{\xi})\chi_{\omega^{(j)}}(\boldsymbol{\eta})\left\{-\frac{1}{2\pi\mu_{I_j}}\log|\boldsymbol{\xi}-\boldsymbol{\eta}|-h_N^{(j,I,I)}(\boldsymbol{\xi},\boldsymbol{\eta})\right\} \ .\end{aligned}$$

The symmetry of $\mathcal{N}^{(j)}$, then implies the conditions

$$\left.\begin{aligned}h_N^{(j,O,O)}(\boldsymbol{\xi},\boldsymbol{\eta}) = h_N^{(j,O,O)}(\boldsymbol{\eta},\boldsymbol{\xi}) \quad \text{for } \boldsymbol{\xi},\boldsymbol{\eta}\in C\bar{\omega}^{(j)} \ , \\ h_N^{(j,I,O)}(\boldsymbol{\xi},\boldsymbol{\eta}) = h_N^{(j,O,I)}(\boldsymbol{\eta},\boldsymbol{\xi}) + \frac{1}{2\pi}\left\{\frac{1}{\mu_{I_j}}-\frac{1}{\mu_O}\right\}\log|\boldsymbol{\xi}-\boldsymbol{\eta}| \quad \text{for } \boldsymbol{\eta}\in C\bar{\omega}^{(j)}, \boldsymbol{\xi}\in\omega^{(j)} \ , \\ \text{and} \quad h_N^{(j,I,I)}(\boldsymbol{\xi},\boldsymbol{\eta}) = h_N^{(j,I,I)}(\boldsymbol{\eta},\boldsymbol{\xi}) \quad \text{for } \boldsymbol{\xi},\boldsymbol{\eta}\in\omega^{(j)} \ .\end{aligned}\right\} \tag{3.8}$$

3. We also make use of model solutions known as the dipole fields $\mathcal{D}_k^{(j)}$, $k = 1, 2$, $j = 1, \ldots, M$, which play the role of the boundary layers in the asymptotic algorithm. Let $\mathcal{D}^{(j)} = (\mathcal{D}_1^{(j)}, \mathcal{D}_2^{(j)})^T$, where

$$\mathcal{D}^{(j)}(\boldsymbol{\xi}) = \chi_{C\bar{\omega}^{(j)}}(\boldsymbol{\xi})\mathcal{D}^{(j,O)}(\boldsymbol{\xi}) + \chi_{\omega^{(j)}}(\boldsymbol{\xi})\mathcal{D}^{(j,I)}(\boldsymbol{\xi}) \ ,$$

and the vector functions $\mathcal{D}^{(j,O)}$, $\mathcal{D}^{(j,I)}$ solve the problem

$$\mu_O\Delta_{\boldsymbol{\xi}}\mathcal{D}^{(j,O)}(\boldsymbol{\xi}) = \mathbf{O} \ , \quad \boldsymbol{\xi}\in C\bar{\omega}^{(j)} \ ,$$

$$\mu_{I_j}\Delta_{\boldsymbol{\xi}}\mathcal{D}^{(j,I)}(\boldsymbol{\xi}) = \mathbf{O} \ , \quad \boldsymbol{\xi}\in \omega^{(j)} \ .$$

The transmission conditions on the boundary of the inclusion $\omega^{(j)}$ are

$$\mu_{I_j}\frac{\partial\mathcal{D}^{(j,I)}}{\partial n_{\boldsymbol{\xi}}}(\boldsymbol{\xi}) - \mu_O\frac{\partial\mathcal{D}^{(j,O)}}{\partial n_{\boldsymbol{\xi}}}(\boldsymbol{\xi}) = (\mu_{I_j}-\mu_O)\boldsymbol{n}^{(j)} \ , \text{ on } \partial\omega^{(j)} \ , \tag{3.9}$$

$$\mathcal{D}^{(j,O)}(\boldsymbol{\xi}) = \mathcal{D}^{(j,I)}(\boldsymbol{\xi}) \ , \text{ on } \partial\omega^{(j)} \ ,$$

where in (3.9), $\boldsymbol{n}^{(j)}$ is the unit normal to $\omega^{(j)}$. At infinity the vector function $\mathcal{D}^{(j,O)}$ satisfies

$$\mathcal{D}^{(j,O)}(\boldsymbol{\xi}) = O(|\boldsymbol{\xi}|^{-1}), \quad \text{as} \quad |\boldsymbol{\xi}|\to\infty \ . \tag{3.10}$$

4. We introduce the function $\zeta^{(j)}$ which is a solution of

$$\Delta_{\boldsymbol{\xi}} \zeta^{(j)}(\boldsymbol{\xi}) = 0 \,, \quad \boldsymbol{\xi} \in C\bar{\omega}^{(j)} \,,$$

$$\zeta^{(j)}(\boldsymbol{\xi}) = 0 \,, \quad \boldsymbol{\xi} \in \partial\omega^{(j)} \,,$$

$$\zeta^{(j)}(\boldsymbol{\xi}) = (2\pi\mu_O)^{-1} \log|\boldsymbol{\xi}| + \zeta^{(j)}_\infty + O(|\boldsymbol{\xi}|^{-1}) \,, \quad \text{as} \quad |\boldsymbol{\xi}| \to \infty \,,$$

where $\zeta^{(j)}_\infty$ is a constant.

4 An estimate for solutions to transmission problems for antiplane shear in unbounded domains

The next result plays an important role in the asymptotic algorithm. It will allow us to obtain estimates for the boundary layer fields and derive the estimate for a solution of the transmission problem in a domain with multiple inclusions.

Lemma 1 *Let $U^{(j)}$ be a solution of the transmission problem*

$$\mu_O \Delta U^{(j)}(\boldsymbol{\xi}) = 0 \,, \quad \boldsymbol{\xi} \in C\bar{\omega}^{(j)} \,,$$

$$\mu_{I_j} \Delta U^{(j)}(\boldsymbol{\xi}) = 0 \,, \quad \boldsymbol{\xi} \in \omega^{(j)} \,,$$

$$U^{(j)}(\boldsymbol{\xi})\big|_{\boldsymbol{\xi}\in\partial\omega^{(j)+}} = U^{(j)}(\boldsymbol{\xi})\big|_{\boldsymbol{\xi}\in\partial\omega^{(j)-}} \,,$$

$$\mu_O \frac{\partial U^{(j)}}{\partial n}(\boldsymbol{\xi})\big|_{\boldsymbol{\xi}\in\partial\omega^{(j)+}} - \mu_{I_j} \frac{\partial U^{(j)}}{\partial n}(\boldsymbol{\xi})\big|_{\boldsymbol{\xi}\in\partial\omega^{(j)-}} = \varphi^{(j)}(\boldsymbol{\xi}) \,,$$

$$U^{(j)}(\boldsymbol{\xi}) \to 0 \quad as \quad |\boldsymbol{\xi}| \to \infty \,,$$

where $\varphi^{(j)} \in L_\infty(\partial\omega^{(j)})$, $\partial/\partial n$ is the normal derivative on the smooth boundary $\partial\omega^{(j)}$, outward with respect to $\omega^{(j)}$, and

$$\int_{\partial\omega^{(j)}} \varphi^{(j)}(\boldsymbol{\xi})\, dS_{\boldsymbol{\xi}} = 0 \,. \tag{4.1}$$

We also assume that

$$\int_{\partial\omega^{(j)}} U^{(j)}(\boldsymbol{\xi}) \frac{\partial \zeta^{(j)}}{\partial n}(\boldsymbol{\xi})\big|_{\boldsymbol{\xi}\in\partial\omega^{(j)+}} dS_{\boldsymbol{\xi}} = 0 \,,$$

where $\zeta^{(j)}$ is given as solution of Problem 4 of Section 3. Then

$$\sup_{\boldsymbol{\xi}\in C\bar{\omega}^{(j)}\cup\omega^{(j)}} \{(|\boldsymbol{\xi}|+1)|U^{(j)}(\boldsymbol{\xi})|\} \le \text{ const } \|\varphi^{(j)}\|_{L_\infty(\partial\omega^{(j)})} \,, \tag{4.2}$$

where the constant depends on μ_O, μ_{I_j} and $\partial\omega^{(j)}$, for $j = 1, \ldots, M$.

Proof. We note that transmission problems are studied in detail in [1] and [2], in the context of boundary integral equations and their solvability.

Let us first represent the solution $U^{(j)}$ by two functions $U^{(j,O)}$ and $U^{(j,I)}$, harmonic in the domains $C\bar{\omega}^{(j)}$ and $\omega^{(j)}$, respectively. These functions satisfy

$$U^{(j,O)}(\boldsymbol{\xi}) = U^{(j,I)}(\boldsymbol{\xi}) \; , \quad \boldsymbol{\xi} \in \partial\omega^{(j)} \; ,$$

$$\mu_O \frac{\partial U^{(j,O)}}{\partial n}(\boldsymbol{\xi}) - \mu_{I_j} \frac{\partial U^{(j,I)}}{\partial n}(\boldsymbol{\xi}) = \varphi^{(j)}(\boldsymbol{\xi}) \; , \quad \boldsymbol{\xi} \in \partial\omega^{(j)} \; .$$

For the function $U^{(j,O)}$, the condition

$$U^{(j,O)}(\boldsymbol{\xi}) \to 0 \quad \text{as} \quad |\boldsymbol{\xi}| \to \infty \; , \tag{4.3}$$

also holds.

Applying Green's formula to the functions $\mathcal{N}^{(j)}$ (see Problem 2, Section 3), $U^{(j,O)}$ and $U^{(j,I)}$, one obtains

$$U^{(j,O)}(\boldsymbol{\xi}) = -\int_{\partial\omega^{(j)}} \mathcal{N}^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta})\varphi^{(j)}(\boldsymbol{\eta})\, dS_{\boldsymbol{\eta}} \; , \tag{4.4}$$

for $\boldsymbol{\xi} \in C\bar{\omega}^{(j)}$, and

$$U^{(j,I)}(\boldsymbol{\xi}) = -\int_{\partial\omega^{(j)}} \mathcal{N}^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta})\varphi^{(j)}(\boldsymbol{\eta})\, dS_{\boldsymbol{\eta}} \; , \tag{4.5}$$

for $\boldsymbol{\xi} \in \omega^{(j)}$.

First, let $|\boldsymbol{\xi}| \geq 2$, then using the asymptotics for the function $\mathcal{N}^{(j)}$ at infinity, and the condition (4.1), we deduce

$$\begin{aligned}(1+|\boldsymbol{\xi}|)|U^{(j,O)}(\boldsymbol{\xi})| \quad &\leq \quad \text{const}\,\Big((1+|\boldsymbol{\xi}|)|\log|\boldsymbol{\xi}||\Big|\int_{\partial\omega^{(j)}} \varphi^{(j)}(\boldsymbol{\eta})dS_{\boldsymbol{\eta}}\Big| \\ &\qquad +\|\varphi^{(j)}\|_{L_1(\partial\omega^{(j)})}\Big) \\ &\leq \quad \text{const}\|\varphi^{(j)}\|_{L_\infty(\partial\omega^{(j)})} \; . \end{aligned} \tag{4.6}$$

Also by the Cauchy-Schwarz inequality and (4.4)

$$|U^{(j,O)}(\boldsymbol{\xi})| \leq \text{const}\; \|\varphi^{(j)}\|_{L_2(\partial\omega^{(j)})} \leq \text{const}\|\varphi^{(j)}\|_{L_\infty(\partial\omega^{(j)})} \; , \quad \text{for} \quad \boldsymbol{\xi} \in B_3\backslash\bar{\omega}^{(j)} \; , \tag{4.7}$$

where $B_3 = \{\boldsymbol{\xi} : |\boldsymbol{\xi}| < 3\}$.

Similarly the integral representation (4.5) gives

$$|U^{(j,I)}(\boldsymbol{\xi})| \leq \text{const}\; \|\varphi\|_{L_2(\partial\omega^{(j)})} \leq \text{const}\|\varphi^{(j)}\|_{L_\infty(\partial\omega^{(j)})} \; , \quad \text{for} \quad \boldsymbol{\xi} \in \omega^{(j)} \; . \tag{4.8}$$

The combination of (4.6), (4.7) and (4.8), leads to (4.2).

□

As an immediate corollary of Lemma 1, we have an estimate for the dipole fields associated with the scaled inclusion $\omega^{(j)}$, $j = 1, \ldots, M$:

Lemma 2 *For the dipole fields $\mathcal{D}_j^{(i)}$, $j=1,2$, $i=1,\dots,M$,*

$$\sup_{\boldsymbol{\xi}\in C\bar{\omega}^{(i)}\cup\omega^{(i)}}\{(|\boldsymbol{\xi}|+1)|\mathcal{D}_j^{(i)}(\boldsymbol{\xi})|\}\le \text{const} \ ,$$

holds, where the constant in the right-hand side can depend on μ_O, μ_{I_i} and $\omega^{(i)}$.

5 An estimate for the maximum modulus of solutions to transmission problems for antiplane shear in a domain with several small inclusions

Here we obtain an estimate for solutions to transmission problems for antiplane shear in domains with small inclusions. The next lemma will be used in Section 7 incorporating the remainder estimates produced by the approximations of Green's function in a perforated domain.

Lemma 3 *Let u be a function in $L_\infty(\bigcup_{l=1}^M \omega_\varepsilon^{(l)}\cup\Omega_\varepsilon)$ such that ∇u is square integrable in a neighbourhood of $\partial\omega_\varepsilon^{(i)}$, $i=1,\dots,M$. Also, let u be a solution of the transmission problem*

$$\left.\begin{aligned}
&\mu_O\Delta u(\mathbf{x})=0\ ,\quad \mathbf{x}\in\Omega_\varepsilon\ ,\\
&\mu_{I_i}\Delta u(\mathbf{x})=0\ ,\quad \mathbf{x}\in\omega_\varepsilon^{(i)}, i=1,\dots,M\ ,\\
&\mu_O\frac{\partial u}{\partial n}(\mathbf{x})=\psi(\mathbf{x})\ ,\quad \mathbf{x}\in\partial\Omega\ ,\\
&\mu_O\frac{\partial u}{\partial n}(\mathbf{x})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)+}}-\mu_{I_i}\frac{\partial u}{\partial n}(\mathbf{x})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)-}}=\varphi_\varepsilon^{(i)}(\mathbf{x})\ ,\\
&u(\mathbf{x})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)+}}=u(\mathbf{x})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)-}}
\end{aligned}\right\}\tag{5.1}$$

where $\psi\in L_\infty(\partial\Omega)$, $\varphi_\varepsilon^{(i)}\in L_\infty(\partial\omega_\varepsilon^{(i)})$, for $1\le i\le M$,

$$\int_{\partial\Omega}\psi(\mathbf{x})\,dS_{\mathbf{x}}=0\ ,\quad \int_{\partial\omega_\varepsilon^{(i)}}\varphi_\varepsilon^{(i)}(\mathbf{x})\,dS_{\mathbf{x}}=0\ ,$$

and $\varphi_\varepsilon^{(i)}(\mathbf{x})=\varepsilon^{-1}\varphi^{(i)}(\varepsilon^{-1}(\mathbf{x}-\mathbf{O}^{(i)}))$, $i=1,\dots,M$. To provide uniqueness we also assume

$$\int_{\partial\Omega}u(\mathbf{x})dS_{\mathbf{x}}=0\ .$$

Then there exists a positive constant A, independent of ε and such that

$$\|u\|_{L_\infty(\bigcup_{l=1}^M\omega_\varepsilon^{(l)}\cup\Omega_\varepsilon)}\le A\left\{\|\psi\|_{L_\infty(\partial\Omega)}+\varepsilon\max_{1\le k\le M}\|\varphi_\varepsilon^{(k)}\|_{L_\infty(\partial\omega_\varepsilon^{(k)})}\right\}\ .\tag{5.2}$$

Proof. *a) The inverse operators to model problems in Ω and $C\bar{\omega}^{(j)}$, $j = 1, \dots, M$.* Let us introduce the operators

$$\mathbf{N} : \psi \to w \quad \text{and} \quad \mathfrak{N}^{(j)} : \varphi^{(j)} \to v^{(j)} , \tag{5.3}$$

which are the inverse operators of the problems

$$\left.\begin{aligned} &\mu_O \Delta w(\mathbf{x}) = 0 , \quad \mathbf{x} \in \Omega , \\ &\mu_O \frac{\partial w}{\partial n}(\mathbf{x}) = \psi(\mathbf{x}) , \quad \mathbf{x} \in \partial\Omega , \\ &\int_{\partial\Omega} w(\mathbf{x})\, dS_{\mathbf{x}} = 0 , \end{aligned}\right\} \tag{5.4}$$

and

$$\left.\begin{aligned} &\mu_O \Delta v^{(j)}(\boldsymbol{\xi}) = 0 , \quad \boldsymbol{\xi} \in C\bar{\omega}^{(j)} , \\ &\mu_{I_j} \Delta v^{(j)}(\boldsymbol{\xi}) = 0 , \quad \boldsymbol{\xi} \in \omega^{(j)} , \\ &\mu_O \frac{\partial v^{(j)}}{\partial n}(\boldsymbol{\xi})\Big|_{\boldsymbol{\xi} \in \partial\omega^{(j)+}} - \mu_{I_j} \frac{\partial v^{(j)}}{\partial n}(\boldsymbol{\xi})\Big|_{\boldsymbol{\xi} \in \partial\omega^{(j)-}} = \varphi^{(j)}(\boldsymbol{\xi}) , \\ &v^{(j)}(\boldsymbol{\xi})\Big|_{\boldsymbol{\xi} \in \partial\omega^{(j)+}} = v^{(j)}(\boldsymbol{\xi})\Big|_{\boldsymbol{\xi} \in \partial\omega^{(j)-}} , \\ &v^{(j)}(\boldsymbol{\xi}) \to 0 \quad \text{as} \quad |\boldsymbol{\xi}| \to \infty , \end{aligned}\right\} \tag{5.5}$$

where $\psi \in L_\infty(\partial\Omega)$, $\varphi \in L_\infty(\partial\omega^{(j)})$, $j = 1, \dots, M$, also

$$\int_{\partial\omega^{(j)}} \varphi^{(j)}(\boldsymbol{\xi})\, dS_{\boldsymbol{\xi}} = 0 \quad \text{and} \quad \int_{\partial\Omega} \psi(\mathbf{x})\, dS_{\mathbf{x}} = 0 .$$

In scaled coordinates $\boldsymbol{\xi}_j = \varepsilon^{-1}(\mathbf{x} - \mathbf{O}^{(j)})$, $j = 1, \dots, M$, the operator $\mathfrak{N}_\varepsilon^{(j)}$ is defined by

$$(\mathfrak{N}_\varepsilon^{(j)} \varphi_\varepsilon^{(j)})(\mathbf{x}) = (\mathfrak{N}^{(j)} \varphi^{(j)})(\boldsymbol{\xi}_j) ,$$

where $\varphi_\varepsilon^{(j)}(\mathbf{x}) = \varepsilon^{-1} \varphi^{(j)}(\varepsilon^{-1}(\mathbf{x} - \mathbf{O}^{(j)}))$.

b) An estimate for solutions to the model Neumann problem in Ω. Let $N^{(\Omega)}$ denote the Neumann function (3.2) in Ω.

Then an application of Green's formula to $N^{(\Omega)}(\mathbf{x}, \mathbf{y})$ and $w(\mathbf{y})$ yields the representation for w

$$w(\mathbf{x}) = \int_{\partial\Omega} N^{(\Omega)}(\mathbf{y}, \mathbf{x}) \psi(\mathbf{y})\, dS_{\mathbf{y}} + \frac{1}{|\partial\Omega|} \int_{\partial\Omega} w(\mathbf{y})\, dS_{\mathbf{y}} . \tag{5.6}$$

The solution w of the Neumann problem in Ω is subject to the orthogonality condition in problem (5.4), and hence the last term on the right-hand side of (5.6) is zero, so that

$$w(\mathbf{x}) = \int_{\partial\Omega} N^{(\Omega)}(\mathbf{y}, \mathbf{x}) \psi(\mathbf{y})\, dS_{\mathbf{y}} .$$

From this we obtain the estimate

$$\sup_{\Omega} |w| \leq \text{const} \sup_{\partial\Omega} |\psi| \ . \tag{5.7}$$

c) The case of the homogeneous boundary condition on $\partial\Omega$. When the right-hand side of the Neumann condition on $\partial\Omega$ in (5.1) is zero we look for a solution of the form

$$u_1 = \sum_{j=1}^{M} \mathfrak{N}_\varepsilon^{(j)} g_\varepsilon^{(j)} - \mathbf{N}\Big(\mathrm{Tr}_{\partial\Omega}\, \mu_O \sum_{j=1}^{M} \frac{\partial}{\partial n}(\mathfrak{N}_\varepsilon^{(j)} g_\varepsilon^{(j)})\Big) \ , \tag{5.8}$$

where $g_\varepsilon^{(j)}$ is an unknown function defined on $\partial\omega_\varepsilon^{(j)}$ such that

$$\int_{\partial\omega^{(j)}} g^{(j)}(\boldsymbol{\xi}_j)\, dS_{\boldsymbol{\xi}_j} = 0 \ ,$$

and $g^{(j)}(\boldsymbol{\xi}_j) = \varepsilon g_\varepsilon^{(j)}(\mathbf{x})$. The function u_1 is harmonic inside $\bigcup_{l=1}^{M} \omega_\varepsilon^{(l)} \cup \Omega_\varepsilon$, and is continuous across the boundaries of the small inclusions. On $\partial\Omega$, u_1 satisfies

$$\mu_O \frac{\partial u_1}{\partial n}(\mathbf{x}) = 0 \ .$$

Computing the jump in tractions of u_1 on $\partial\omega_\varepsilon^{(m)}$, we obtain

$$\varphi_\varepsilon^{(m)}(\mathbf{x}) = \mu_O \frac{\partial u_1}{\partial n}(\mathbf{x})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(m)+}} - \mu_{I_m} \frac{\partial u_1}{\partial n}(\mathbf{x})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(m)-}} = g_\varepsilon^{(m)} + (S_\varepsilon^{(m)} g_\varepsilon)(\mathbf{x}) \ .$$

where $g_\varepsilon(\mathbf{x}) = (g^{(1)}(\mathbf{x}), \dots, g^{(M)}(\mathbf{x}))^T$ and

$$\begin{aligned} (S_\varepsilon^{(m)} g_\varepsilon)(\mathbf{x}) \quad &= \quad (\mu_O - \mu_{I_m})\Big[\frac{\partial}{\partial n}\Big\{ \sum_{\substack{j\neq m \\ 1\leq j\leq M}} \mathfrak{N}_\varepsilon^{(j)} g_\varepsilon^{(j)} \Big\}\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(m)}} \\ &\quad - \frac{\partial}{\partial n}\Big\{\mathbf{N}\Big(\mathrm{Tr}_{\partial\Omega}\, \mu_O \sum_{j=1}^{M} \frac{\partial}{\partial n}(\mathfrak{N}_\varepsilon^{(j)} g_\varepsilon^{(j)})\Big)\Big\}\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(m)}}\Big] \end{aligned} \tag{5.9}$$

Let $B^{(m)}$ denote a disk centered at $\mathbf{O}^{(m)}$ and containing $\omega_\varepsilon^{(m)}$. Using a local estimate for solutions of Laplace's equation, along with Lemma 1 and the definition of $g_\varepsilon^{(j)}$ above, we obtain

$$\begin{aligned} \Big\|\frac{\partial}{\partial n}\Big\{ \sum_{\substack{j\neq m \\ 1\leq j\leq M}} \mathfrak{N}_\varepsilon^{(j)} g_\varepsilon^{(j)} \Big\}\Big\|_{L_\infty(\partial\omega_\varepsilon^{(m)})} \quad &\leq \quad \text{const} \Big\| \sum_{\substack{j\neq m \\ 1\leq j\leq M}} \mathfrak{N}_\varepsilon^{(j)} g_\varepsilon^{(j)} \Big\|_{L_\infty(B^{(m)})} \\ &\leq \quad \text{const} \sum_{\substack{j\neq m \\ 1\leq j\leq M}} \varepsilon^2 \|g_\varepsilon^{(j)}\|_{L_\infty(\partial\omega_\varepsilon^{(j)})} \end{aligned} \tag{5.10}$$

Then, from the local estimate for harmonic functions, we can also assert the inequality

$$\begin{aligned} & \Big\| \frac{\partial}{\partial n} \Big\{ \mathbf{N}\Big(\mathrm{Tr}_{\partial\Omega}\, \mu_O \sum_{j=1}^{M} \frac{\partial}{\partial n} (\mathfrak{N}_\varepsilon^{(j)} g_\varepsilon^{(j)}) \Big) \Big\} \Big\|_{L_\infty(\partial\omega_\varepsilon^{(m)})} \\ \leq \quad & \mathrm{const}\ \Big\| \mathbf{N}\Big(\mathrm{Tr}_{\partial\Omega}\, \mu_O \sum_{j=1}^{M} \frac{\partial}{\partial n} (\mathfrak{N}_\varepsilon^{(j)} g_\varepsilon^{(j)}) \Big) \Big\|_{L_\infty(B^{(m)})} \\ \leq \quad & \mathrm{const}\ \sum_{j=1}^{M} \Big\| \frac{\partial}{\partial n} (\mathfrak{N}_\varepsilon^{(j)} g_\varepsilon^{(j)}) \Big\|_{L_\infty(\partial\Omega)} \leq \mathrm{const}\ \sum_{j=1}^{M} \varepsilon^2 \| g_\varepsilon^{(j)} \|_{L_\infty(\partial\omega_\varepsilon^{(j)})} \ , \end{aligned}$$

where in moving to the second inequality we used estimate (5.7), and then Lemma 1 brings us to the last inequality.

Then, the preceding estimate and (5.9), (5.10) lead to

$$\| S_\varepsilon^{(m)} g_\varepsilon \|_{L_\infty(\partial\omega_\varepsilon^{(m)})} \leq \mathrm{const}\ \varepsilon^2 \sum_{j=1}^{M} \| g_\varepsilon^{(j)} \|_{L_\infty(\partial\omega_\varepsilon^{(j)})} \ .$$

Hence, from the smallness of $S_\varepsilon^{(m)}$, we can write

$$g_\varepsilon = (\mathbf{I} + \mathbf{S}_\varepsilon)^{-1} \varphi_\varepsilon \ ,$$

where $g_\varepsilon(\mathbf{x}) = (g_\varepsilon^{(1)}(\mathbf{x}), \dots, g_\varepsilon^{(M)}(\mathbf{x}))^T$, $\varphi_\varepsilon(\mathbf{x}) = (\varphi_\varepsilon^{(1)}(\mathbf{x}), \dots, \varphi_\varepsilon^{(M)}(\mathbf{x}))^T$, $\mathbf{S}_\varepsilon$ is a matrix whose rows are $S_\varepsilon^{(1)}, \dots, S_\varepsilon^{(M)}$, and

$$\| g_\varepsilon^{(j)} \|_{L_\infty(\partial\omega_\varepsilon^{(j)})} \leq \mathrm{const}\ \max_{1 \leq k \leq M} \| \varphi_\varepsilon^{(k)} \|_{L_\infty(\partial\omega_\varepsilon^{(k)})} \ . \tag{5.11}$$

From (5.8), together with (5.7) and Lemma 1, we obtain

$$\| u_1 \|_{L_\infty(\bigcup_{l=1}^{M} \omega_\varepsilon^{(l)} \cup \Omega_\varepsilon)} \leq \mathrm{const}\ \sum_{j=1}^{M} \varepsilon \| g_\varepsilon^{(j)} \|_{L_\infty(\partial\omega_\varepsilon^{(j)})} \ .$$

Now, this and (5.11) give

$$\| u_1 \|_{L_\infty(\bigcup_{l=1}^{M} \omega_\varepsilon^{(l)} \cup \Omega_\varepsilon)} \leq \mathrm{const}\ \varepsilon \max_{1 \leq k \leq M} \| \varphi_\varepsilon^{(k)} \|_{L_\infty(\partial\omega_\varepsilon^{(k)})} \ . \tag{5.12}$$

d) The case of continuous tractions on $\partial\omega_\varepsilon^{(j)}$, $j = 1, \dots, M$. In this situation we look for the solution u_2 in the form

$$u_2 = \mathbf{N}\psi + v \ . \tag{5.13}$$

Then, v is a solution of (5.1) with the boundary conditions

$$\mu_O \frac{\partial v}{\partial n}(\mathbf{x}) = 0 \ , \ \text{on } \partial\Omega \ ,$$

and

$$v(\mathbf{x})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(j)+}} = v(\mathbf{x})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(j)-}}$$

$$\mu_O \frac{\partial v}{\partial n}(\mathbf{x})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(j)+}} - \mu_{I_j}\frac{\partial v}{\partial n}(\mathbf{x})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(j)-}} = (\mu_{I_j}-\mu_O)\frac{\partial \mathbf{N}\psi}{\partial n}(\mathbf{x})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(j)}}$$

for $1 \le j \le M$, where the right-hand sides of the above traction conditions on $\partial\omega_\varepsilon^{(j)}$, $j = 1, \dots, M$ are self balanced, so by part c) of the present proof

$$\begin{aligned}\|v\|_{L_\infty(\bigcup_{l=1}^M \omega_\varepsilon^{(l)}\cup\Omega_\varepsilon)} &\le \text{const } \varepsilon \max_{1\le k\le M} \Big\|\frac{\partial \mathbf{N}\psi}{\partial n}(\mathbf{x})\Big\|_{L_\infty(\partial\omega_\varepsilon^{(k)})}\\ &\le \text{const } \varepsilon \max_{1\le k\le M} \big\|\mathbf{N}\psi\big\|_{L_\infty(B^{(k)})}\\ &\le \text{const } \varepsilon\|\psi\|_{L_\infty(\partial\Omega)} .\end{aligned}$$

This inequality, (5.7) and (5.13), give

$$\|u_2\|_{L_\infty(\bigcup_{l=1}^M \omega_\varepsilon^{(l)}\cup\Omega_\varepsilon)} \le \text{const } \|\psi\|_{L_\infty(\partial\Omega)} . \tag{5.14}$$

Finally, we obtain (5.2) through the combination of (5.12) and (5.14). □

6 An asymptotic approximation for the regular part of the Green's function $\mathcal{N}^{(j)}$ at infinity

In this section, we shall prove a result which concerns the asymptotics of $h_N^{(j,O)}$ and $h_N^{(j,I)}$, $j = 1, \dots, M$, (see (3.6)), at infinity.

Lemma 4 *For $|\boldsymbol{\xi}| > 2$. Then*

a)

$$h_N^{(j,O)}(\boldsymbol{\xi},\boldsymbol{\eta}) = -\mathcal{D}^{(j,O)}(\boldsymbol{\eta})\cdot\nabla_{\boldsymbol{\xi}}((2\pi\mu_O)^{-1}\log|\boldsymbol{\xi}|^{-1}) + O(|\boldsymbol{\xi}|^{-2}(|\boldsymbol{\eta}|+1)^{-1}) ,$$

for $\boldsymbol{\eta} \in C\bar{\omega}^{(j)}$,

b)

$$\begin{aligned} h_N^{(j,I)}(\boldsymbol{\xi},\boldsymbol{\eta}) &= -(\mu_{I_j}^{-1}-\mu_O^{-1})\{(2\pi)^{-1}\log|\boldsymbol{\xi}| - \boldsymbol{\eta}\cdot\nabla_{\boldsymbol{\xi}}((2\pi)^{-1}\log|\boldsymbol{\xi}|)\}\\ &\quad -\mathcal{D}^{(j,I)}(\boldsymbol{\eta})\cdot\nabla_{\boldsymbol{\xi}}((2\pi\mu_O)^{-1}\log|\boldsymbol{\xi}|^{-1}) + O(|\boldsymbol{\xi}|^{-2}) .\end{aligned}$$

for $\boldsymbol{\eta} \in \omega^{(j)}$.

Proof. First we study the asymptotic behaviour of the functions $h_N^{(j,O,O)}$, $h_N^{(j,I,O)}$, $h_N^{(j,O,I)}$ introduced in Problem 2 of Section 3, in the neighbourhood of infinity. We show that for $|\boldsymbol{\xi}| > 2$, a)

$$h_N^{(j,O,O)}(\boldsymbol{\xi},\boldsymbol{\eta}) = -\mathcal{D}^{(j,O)}(\boldsymbol{\eta})\cdot\nabla_{\boldsymbol{\xi}}((2\pi\mu_O)^{-1}\log|\boldsymbol{\xi}|^{-1}) + O(|\boldsymbol{\xi}|^{-2}(|\boldsymbol{\eta}|+1)^{-1}) \;, \tag{6.1}$$

for $\boldsymbol{\eta}\in C\bar{\omega}^{(j)}$,

b)

$$h_N^{(j,I,O)}(\boldsymbol{\eta},\boldsymbol{\xi}) = -\mathcal{D}^{(j,I)}(\boldsymbol{\eta})\cdot\nabla_{\boldsymbol{\xi}}((2\pi\mu_O)^{-1}\log|\boldsymbol{\xi}|^{-1}) + O(|\boldsymbol{\xi}|^{-2}) \;, \tag{6.2}$$

for $\boldsymbol{\eta}\in\omega^{(j)}$,

c)

$$\begin{aligned} h_N^{(j,O,I)}(\boldsymbol{\xi},\boldsymbol{\eta}) &= -(\mu_{I_j}^{-1}-\mu_O^{-1})\{(2\pi)^{-1}\log|\boldsymbol{\xi}| - \boldsymbol{\eta}\cdot\nabla_{\boldsymbol{\xi}}((2\pi)^{-1}\log|\boldsymbol{\xi}|)\} \\ &\quad -\mathcal{D}^{(j,I)}(\boldsymbol{\eta})\cdot\nabla_{\boldsymbol{\xi}}((2\pi\mu_O)^{-1}\log|\boldsymbol{\xi}|^{-1}) + O(|\boldsymbol{\xi}|^{-2}) \;, \end{aligned} \tag{6.3}$$

for $\boldsymbol{\eta}\in\omega^{(j)}$.

i) Auxiliary functions $h_N^{(j)}$ *and* $\Upsilon^{(j)}$*.* Let us introduce the function $h_N^{(j)}$ as

$$h_N^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta}) = \chi_{C\bar{\omega}^{(j)}}(\boldsymbol{\eta})h_N^{(j,O)}(\boldsymbol{\xi},\boldsymbol{\eta}) + \chi_{\omega^{(j)}}(\boldsymbol{\eta})h_N^{(j,I)}(\boldsymbol{\xi},\boldsymbol{\eta}) \;. \tag{6.4}$$

From Problem 2 of Section 3, when $\boldsymbol{\eta}\in C\bar{\omega}^{(j)}\cup\omega^{(j)}$, $h_N^{(j)}$ is a solution of the transmission problem

$$\mu_O\Delta_{\boldsymbol{\xi}}h_N^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta}) = 0 \;, \quad \boldsymbol{\xi}\in C\bar{\omega}^{(j)} \;,$$

$$\mu_{I_j}\Delta_{\boldsymbol{\xi}}h_N^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta}) = 0 \;, \quad \boldsymbol{\xi}\in\omega^{(j)} \;,$$

$$\begin{aligned} &\mu_{I_j}\frac{\partial h_N^{(j)}}{\partial n_{\boldsymbol{\xi}}}(\boldsymbol{\xi},\boldsymbol{\eta})\Big|_{\boldsymbol{\xi}\in\partial\omega^{(j)-}} - \mu_O\frac{\partial h_N^{(j)}}{\partial n_{\boldsymbol{\xi}}}(\boldsymbol{\xi},\boldsymbol{\eta})\Big|_{\boldsymbol{\xi}\in\partial\omega^{(j)+}} \\ &= -\frac{(\mu_{I_j}-\mu_O)}{2\pi}\left(\frac{\chi_{C\bar{\omega}^{(j)}}(\boldsymbol{\eta})}{\mu_O} + \frac{\chi_{\omega^{(j)}}(\boldsymbol{\eta})}{\mu_{I_j}}\right)\frac{\partial}{\partial n_{\boldsymbol{\xi}}}(\log|\boldsymbol{\xi}-\boldsymbol{\eta}|) \;, \end{aligned} \tag{6.5}$$

$$h_N^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta})\Big|_{\boldsymbol{\xi}\in\partial\omega^{(j)+}} = h_N^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta})\Big|_{\boldsymbol{\xi}\in\partial\omega^{(j)-}} \;, \tag{6.6}$$

$$h_N^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta}) = -(\mu_{I_j}^{-1}-\mu_O^{-1})\chi_{\omega^{(j)}}(\boldsymbol{\eta})(2\pi)^{-1}\log|\boldsymbol{\xi}| + O(|\boldsymbol{\xi}|^{-1}), \quad \text{as} \quad |\boldsymbol{\xi}|\to\infty \;.$$

We introduce one more auxiliary vector function $\Upsilon^{(j)}(\boldsymbol{\xi}) = \{\Upsilon_i^{(j)}(\boldsymbol{\xi})\}_{i=1}^2$, defined by

$$\Upsilon^{(j)}(\boldsymbol{\xi}) = \boldsymbol{\xi} - \mathcal{D}^{(j)}(\boldsymbol{\xi}) \;. \tag{6.7}$$

It solves the transmission problem

$$\left.\begin{aligned}
\mu_O \Delta_{\boldsymbol{\xi}} \Upsilon^{(j)}(\boldsymbol{\xi}) = \mathbf{O}\,, \quad \boldsymbol{\xi} \in C\bar{\omega}^{(j)}\,,\\
\mu_{I_j} \Delta_{\boldsymbol{\xi}} \Upsilon^{(j)}(\boldsymbol{\xi}) = \mathbf{O}\,, \quad \boldsymbol{\xi} \in \omega^{(j)}\,,\\
\mu_{I_j} \frac{\partial \Upsilon^{(j)}}{\partial n_{\boldsymbol{\xi}}}(\boldsymbol{\xi})\big|_{\boldsymbol{\xi}\in\partial\omega^{(j)-}} = \mu_O \frac{\partial \Upsilon^{(j)}}{\partial n_{\boldsymbol{\xi}}}(\boldsymbol{\xi})\big|_{\boldsymbol{\xi}\in\partial\omega^{(j)+}}\,,\\
\Upsilon^{(j)}(\boldsymbol{\xi})\big|_{\boldsymbol{\xi}\in\partial\omega^{(j)+}} = \Upsilon^{(j)}(\boldsymbol{\xi})\big|_{\boldsymbol{\xi}\in\partial\omega^{(j)-}}\,,\\
\Upsilon^{(j)}(\boldsymbol{\xi}) = \boldsymbol{\xi} + O(|\boldsymbol{\xi}|^{-1}), \quad \text{as} \quad |\boldsymbol{\xi}| \to \infty\,,
\end{aligned}\right\} \tag{6.8}$$

which is consistent with Problem 3 of Section 3. One can also represent $\Upsilon^{(j)}$, as

$$\Upsilon^{(j)}(\boldsymbol{\xi}) = \chi_{C\bar{\omega}^{(j)}}(\boldsymbol{\xi})\Upsilon^{(j,O)}(\boldsymbol{\xi}) + \chi_{\omega^{(j)}}(\boldsymbol{\xi})\Upsilon^{(j,I)}(\boldsymbol{\xi})\,.$$

ii) The asymptotics of $h_N^{(j,O,O)}$ *at infinity.* For $|\boldsymbol{\xi}| > 2$, $\boldsymbol{\eta} \in C\bar{\omega}^{(j)}$, we have from Lemma 2 in [6], that the function $h_N^{(j,O,O)}$ defined in (6.4), has the asymptotic representation

$$h_N^{(j,O,O)}(\boldsymbol{\xi},\boldsymbol{\eta}) = \mathcal{C}^{(j,O)}(\boldsymbol{\eta}) \cdot \nabla_{\boldsymbol{\xi}}((2\pi)^{-1}\log|\boldsymbol{\xi}|^{-1}) + r_N^{(j,O,O)}(\boldsymbol{\xi},\boldsymbol{\eta})\,, \tag{6.9}$$

where the remainder $r_N^{(j,O,O)}(\boldsymbol{\xi},\boldsymbol{\eta})$ satisfies

$$|r_N^{(j,O,O)}(\boldsymbol{\xi},\boldsymbol{\eta})| \le \text{const}\ (1+|\boldsymbol{\eta}|)^{-1}|\boldsymbol{\xi}|^{-2} \quad \text{for } |\boldsymbol{\xi}|>2, \boldsymbol{\eta}\in C\bar{\omega}^{(j)}\,.$$

The vector function $\mathcal{C}^{(j,O)}(\boldsymbol{\eta}) = \{\mathcal{C}_i^{(j,O)}(\boldsymbol{\eta})\}_{i=1}^2$ is evaluated below.

Evaluation of $\mathcal{C}^{(j,O)}(\boldsymbol{\eta})$. Let B_R be a disk centered at the origin with a sufficiently large radius R. We apply Green's formula to $h_N^{(j)}$ and $\Upsilon_i^{(j)}$ in the domain $B_R \backslash \bar{\omega}^{(j)} \cup \omega^{(j)}$

$$\begin{aligned}
0 &= \mu_O \int_{B_R\backslash\bar{\omega}^{(j)}} \Big\{\Upsilon_i^{(j)}(\boldsymbol{\xi})\Delta_{\boldsymbol{\xi}} h_N^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta}) - h_N^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta})\Delta_{\boldsymbol{\xi}}\Upsilon_i^{(j)}(\boldsymbol{\xi})\Big\}\, d\boldsymbol{\xi}\\
&\quad +\mu_{I_j} \int_{\omega^{(j)}} \Big\{\Upsilon_i^{(j)}(\boldsymbol{\xi})\Delta_{\boldsymbol{\xi}} h_N^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta}) - h_N^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta})\Delta_{\boldsymbol{\xi}}\Upsilon_i^{(j)}(\boldsymbol{\xi})\Big\}\, d\boldsymbol{\xi}\\
&= \mu_O \int_{\partial B_R} \left\{\Upsilon_i^{(j)}(\boldsymbol{\xi})\frac{\partial h_N^{(j)}}{\partial n_{\boldsymbol{\xi}}}(\boldsymbol{\xi},\boldsymbol{\eta}) - h_N^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta})\frac{\partial \Upsilon_i^{(j)}}{\partial n_{\boldsymbol{\xi}}}(\boldsymbol{\xi})\right\} dS_{\boldsymbol{\xi}}\\
&\quad + \int_{\partial\omega^{(j)}} \left[\Upsilon_i^{(j)}(\boldsymbol{\xi})\left\{\mu_{I_j}\frac{\partial h_N^{(j)}}{\partial n_{\boldsymbol{\xi}}}(\boldsymbol{\xi},\boldsymbol{\eta})\big|_{\partial\omega^{(j)-}} - \mu_O\frac{\partial h_N^{(j)}}{\partial n_{\boldsymbol{\xi}}}(\boldsymbol{\xi},\boldsymbol{\eta})\big|_{\partial\omega^{(j)+}}\right\}\right.\\
&\quad \left. -h_N^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta})\left\{\mu_{I_j}\frac{\partial \Upsilon_i^{(j)}}{\partial n_{\boldsymbol{\xi}}}(\boldsymbol{\xi})\big|_{\partial\omega^{(j)-}} - \mu_O\frac{\partial \Upsilon_i^{(j)}}{\partial n_{\boldsymbol{\xi}}}(\boldsymbol{\xi})\big|_{\partial\omega^{(j)+}}\right\}\right] dS_{\boldsymbol{\xi}}\,,
\end{aligned} \tag{6.10}$$

where while combining the integrals over $\partial\omega^{(j)+}$ and $\partial\omega^{(j)-}$ we have used the continuity conditions for the functions $h_N^{(j)}$ and $\Upsilon_i^{(j)}$, respectively (see (6.6) and problem (6.8)). With the use of the transmission conditions for $\Upsilon_i^{(j)}$ and $h_N^{(j)}$, when $\boldsymbol{\eta} \in C\bar{\omega}^{(j)}$, this equality becomes

$$\begin{aligned} 0 &= \mu_O \int_{\partial B_R} \left\{ \Upsilon_i^{(j)}(\boldsymbol{\xi}) \frac{\partial h_N^{(j)}}{\partial n_{\boldsymbol{\xi}}}(\boldsymbol{\xi}, \boldsymbol{\eta}) - h_N^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta}) \frac{\partial \Upsilon_i^{(j)}}{\partial n_{\boldsymbol{\xi}}}(\boldsymbol{\xi}) \right\} dS_{\boldsymbol{\xi}} \\ &\quad - \frac{(\mu_{I_j} - \mu_O)}{\mu_O} \int_{\partial\omega^{(j)}} \Upsilon_i^{(j)}(\boldsymbol{\xi}) \frac{\partial}{\partial n_{\boldsymbol{\xi}}} ((2\pi)^{-1} \log|\boldsymbol{\xi} - \boldsymbol{\eta}|)\, dS_{\boldsymbol{\xi}}\ , \end{aligned} \tag{6.11}$$

for $\boldsymbol{\eta} \in C\bar{\omega}^{(j)}$. The last integral, owing to (6.7), is equal to

$$\begin{aligned} &\int_{\partial\omega^{(j)}} \Upsilon_i^{(j)}(\boldsymbol{\xi}) \frac{\partial}{\partial n_{\boldsymbol{\xi}}} ((2\pi)^{-1} \log|\boldsymbol{\xi} - \boldsymbol{\eta}|)\, dS_{\boldsymbol{\xi}} \\ = &\int_{\partial\omega^{(j)}} \left\{ \xi_i \frac{\partial}{\partial n_{\boldsymbol{\xi}}} ((2\pi)^{-1} \log|\boldsymbol{\xi} - \boldsymbol{\eta}|) - \mathcal{D}_i^{(j)}(\boldsymbol{\xi}) \frac{\partial}{\partial n_{\boldsymbol{\xi}}} ((2\pi)^{-1} \log|\boldsymbol{\xi} - \boldsymbol{\eta}|) \right\} dS_{\boldsymbol{\xi}} \\ = &\int_{\partial\omega^{(j)}} \left\{ (2\pi)^{-1} \log|\boldsymbol{\xi} - \boldsymbol{\eta}| \frac{\partial \xi_i}{\partial n_{\boldsymbol{\xi}}} - \mathcal{D}_i^{(j)}(\boldsymbol{\xi}) \frac{\partial}{\partial n_{\boldsymbol{\xi}}} ((2\pi)^{-1} \log|\boldsymbol{\xi} - \boldsymbol{\eta}|) \right\} dS_{\boldsymbol{\xi}}, \end{aligned} \tag{6.12}$$

where $\boldsymbol{\eta} \in C\bar{\omega}^{(j)}$. Now, using the jump in tractions for the dipole fields, stated in Problem 3 of Section 3, we can write

$$\begin{aligned} &\frac{(\mu_{I_j} - \mu_O)}{\mu_O} \int_{\partial\omega^{(j)}} \Upsilon_i^{(j)}(\boldsymbol{\xi}) \frac{\partial}{\partial n_{\boldsymbol{\xi}}} ((2\pi)^{-1} \log|\boldsymbol{\xi} - \boldsymbol{\eta}|^{-1})\, dS_{\boldsymbol{\xi}} \\ = &\frac{1}{\mu_O} \int_{\partial\omega^{(j)}} \left\{ (2\pi)^{-1} \log|\boldsymbol{\xi} - \boldsymbol{\eta}|^{-1} \left[\mu_{I_j} \frac{\partial \mathcal{D}_i^{(j,I)}}{\partial n_{\boldsymbol{\xi}}}(\boldsymbol{\xi}) - \mu_O \frac{\partial \mathcal{D}_i^{(j,O)}}{\partial n_{\boldsymbol{\xi}}}(\boldsymbol{\xi}) \right] \right. \\ &\left. -(\mu_{I_j} - \mu_O) \mathcal{D}_i^{(j)}(\boldsymbol{\xi}) \frac{\partial}{\partial n_{\boldsymbol{\xi}}} ((2\pi)^{-1} \log|\boldsymbol{\xi} - \boldsymbol{\eta}|^{-1}) \right\} dS_{\boldsymbol{\xi}}\ . \end{aligned}$$

Using the continuity of the displacements for the dipole fields (see Problem 3, Section 3), we write the above right-hand side as

$$\begin{aligned} &\frac{\mu_{I_j}}{\mu_O} \int_{\partial\omega^{(j)}} \left\{ (2\pi)^{-1} \log|\boldsymbol{\xi} - \boldsymbol{\eta}|^{-1} \frac{\partial \mathcal{D}_i^{(j,I)}}{\partial n_{\boldsymbol{\xi}}}(\boldsymbol{\xi}) - \mathcal{D}_i^{(j,I)}(\boldsymbol{\xi}) \frac{\partial}{\partial n_{\boldsymbol{\xi}}} ((2\pi)^{-1} \log|\boldsymbol{\xi} - \boldsymbol{\eta}|^{-1}) \right\} dS_{\boldsymbol{\xi}} \\ &- \int_{\partial\omega^{(j)}} \left\{ (2\pi)^{-1} \log|\boldsymbol{\xi} - \boldsymbol{\eta}|^{-1} \frac{\partial \mathcal{D}_i^{(j,O)}}{\partial n_{\boldsymbol{\xi}}}(\boldsymbol{\xi}) - \mathcal{D}_i^{(j,O)}(\boldsymbol{\xi}) \frac{\partial}{\partial n_{\boldsymbol{\xi}}} ((2\pi)^{-1} \log|\boldsymbol{\xi} - \boldsymbol{\eta}|^{-1}) \right\} dS_{\boldsymbol{\xi}}\ . \end{aligned}$$

Then, upon applying Green's formula to $\mathcal{D}_i^{(j,O)}$, $\mathcal{D}_i^{(j,I)}$ in $B_R \backslash \bar{\omega}^{(j)}$, $\omega^{(j)}$, respectively, with the fundamental solution of $-\Delta$, and using the definition of $\Upsilon^{(j)}$, we obtain the above in the form

$$\begin{aligned} &\mathcal{D}_i^{(j,O)}(\boldsymbol{\eta}) - \int_{\partial B_R} \left\{ (2\pi)^{-1} \log|\boldsymbol{\xi} - \boldsymbol{\eta}|^{-1} \frac{\partial \mathcal{D}_i^{(j,O)}}{\partial n_{\boldsymbol{\xi}}}(\boldsymbol{\xi}) \right. \\ &\left. -\mathcal{D}_i^{(j,O)}(\boldsymbol{\xi}) \frac{\partial}{\partial n_{\boldsymbol{\xi}}} ((2\pi)^{-1} \log|\boldsymbol{\xi} - \boldsymbol{\eta}|^{-1}) \right\} dS_{\boldsymbol{\xi}} \end{aligned}$$

where $\boldsymbol{\eta} \in C\bar{\omega}^{(j)}$ and by (3.10) the integral over ∂B_R decays as $R \to \infty$ like $O(|\log R|/R)$. Therefore, we have

$$\frac{(\mu_{I_j} - \mu_O)}{\mu_O} \int_{\partial \omega^{(j)}} \Upsilon_i^{(j)}(\boldsymbol{\xi}) \frac{\partial}{\partial n_{\boldsymbol{\xi}}}((2\pi)^{-1} \log |\boldsymbol{\xi} - \boldsymbol{\eta}|^{-1})\, dS_{\boldsymbol{\xi}} = \mathcal{D}_i^{(j,O)}(\boldsymbol{\eta})$$

for $\boldsymbol{\eta} \in C\bar{\omega}^{(j)}$. Equation (6.11), with substitution of the preceding equality, leads to

$$-\mathcal{D}_i^{(j,O)}(\boldsymbol{\eta}) = \mu_O \int_{\partial B_R} \left\{ \Upsilon_i^{(j)}(\boldsymbol{\xi}) \frac{\partial h_N^{(j)}}{\partial n_{\boldsymbol{\xi}}}(\boldsymbol{\xi}, \boldsymbol{\eta}) - h_N^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta}) \frac{\partial \Upsilon_i^{(j)}}{\partial n_{\boldsymbol{\xi}}}(\boldsymbol{\xi}) \right\} dS_{\boldsymbol{\xi}} \,. \tag{6.13}$$

We now aim to determine the vector function $\mathcal{C}^{(j,O)}(\boldsymbol{\eta}) = \{\mathcal{C}_i^{(j,O)}(\boldsymbol{\eta})\}_{i=1}^2$ in (6.9). Taking the limit $R \to \infty$ in (6.13) and using (6.9) (where $h_N^{(j,O,O)}(\boldsymbol{\xi}, \boldsymbol{\eta}) = h_N^{(j)}(\boldsymbol{\xi}, \boldsymbol{\eta})$ for $\boldsymbol{\xi}, \boldsymbol{\eta} \in C\bar{\omega}^{(j)}$), we have

$$\begin{aligned} -\mathcal{D}_i^{(j,O)}(\boldsymbol{\eta}) &= \lim_{R\to\infty} \mu_O \mathcal{C}^{(j,O)}(\boldsymbol{\eta}) \cdot \int_{\partial B_R} \Big\{ \xi_i \frac{\partial}{\partial n_{\boldsymbol{\xi}}} (\nabla_{\boldsymbol{\xi}}((2\pi)^{-1} \log |\boldsymbol{\xi}|^{-1})) \\ &\quad - \nabla_{\boldsymbol{\xi}}((2\pi)^{-1} \log |\boldsymbol{\xi}|^{-1}) \frac{\partial \xi_i}{\partial n_{\boldsymbol{\xi}}} \Big\} dS_{\boldsymbol{\xi}} \\ &= \mu_O \mathcal{C}_i^{(j,O)}(\boldsymbol{\eta}) \,. \end{aligned}$$

Thus, we have derived that

$$\mathcal{C}^{(j,O)}(\boldsymbol{\eta}) = -\frac{\mathcal{D}^{(j,O)}(\boldsymbol{\eta})}{\mu_O} \,,$$

which corresponds to the leading order term of the function $h_N^{(j,O,O)}$, stated in the current lemma in (6.1).

iii) The asymptotics of $h^{(j,I,O)}$ at infinity. We have already shown that for $|\boldsymbol{\xi}| \geq 2$, $\boldsymbol{\eta} \in C\bar{\omega}^{(j)}$

$$h_N^{(j,O,O)}(\boldsymbol{\xi}, \boldsymbol{\eta}) = -\mathcal{D}^{(j,O)}(\boldsymbol{\eta}) \cdot \nabla_{\boldsymbol{\xi}}((2\pi\mu_O)^{-1} \log |\boldsymbol{\xi}|^{-1}) + r_N^{(j,O,O)}(\boldsymbol{\xi}, \boldsymbol{\eta}) \,. \tag{6.14}$$

In order to deduce the leading order term of $h^{(j,I,O)}$ at infinity, we recall the relations (3.8). First, since $h_N^{(j,O,O)}$ is symmetric for $\boldsymbol{\xi}, \boldsymbol{\eta} \in C\bar{\omega}^{(j)}$ we have the above asymptotic representation also holds for $h_N^{(j,O,O)}(\boldsymbol{\eta}, \boldsymbol{\xi})$. Next we allow $\boldsymbol{\eta}$ to approach the boundary of the inclusion $\omega^{(j)}$. For $\boldsymbol{\eta} \in \partial \omega^{(j)}$ we have

$$\mathcal{D}^{(j,O)}(\boldsymbol{\eta}) = \mathcal{D}^{(j,I)}(\boldsymbol{\eta}) \,,$$

$$h_N^{(j,O,O)}(\boldsymbol{\eta}, \boldsymbol{\xi}) = h_N^{(j,I,O)}(\boldsymbol{\eta}, \boldsymbol{\xi}).$$

Therefore, allowing $\boldsymbol{\eta} \in \omega^{(j)}$, for $|\boldsymbol{\xi}| > 2$, we arrive at

$$h_N^{(j,I,O)}(\boldsymbol{\eta}, \boldsymbol{\xi}) = -\mathcal{D}^{(j,I)}(\boldsymbol{\eta}) \cdot \nabla_{\boldsymbol{\xi}}((2\pi\mu_O)^{-1} \log |\boldsymbol{\xi}|^{-1}) + r_N^{(j,O,I)}(\boldsymbol{\xi}, \boldsymbol{\eta}) \,, \tag{6.15}$$

where $r_N^{(j,O,I)}$ is the remainder term and subject to its smallness the leading order part of (6.2) has been formally deduced.

Remainder estimate. Consider the function

$$r_N^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta}) = \chi_{C\bar{\omega}^{(j)}}(\boldsymbol{\eta}) r_N^{(j,O,O)}(\boldsymbol{\xi},\boldsymbol{\eta}) + \chi_{\omega^{(j)}}(\boldsymbol{\eta}) r_N^{(j,O,I)}(\boldsymbol{\xi},\boldsymbol{\eta})$$

which by (6.14) and (6.15) is

$$\begin{aligned} r_N^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta}) &= \chi_{C\bar{\omega}^{(j)}}(\boldsymbol{\eta})\{h_N^{(j,O,O)}(\boldsymbol{\eta},\boldsymbol{\xi}) + \mathcal{D}^{(j,O)}(\boldsymbol{\eta})\cdot\nabla_{\boldsymbol{\xi}}((2\pi\mu_O)^{-1}\log|\boldsymbol{\xi}|^{-1})\} \\ &\quad + \chi_{\omega^{(j)}}(\boldsymbol{\eta})\{h_N^{(j,I,O)}(\boldsymbol{\eta},\boldsymbol{\xi}) + \mathcal{D}^{(j,I)}(\boldsymbol{\eta})\cdot\nabla_{\boldsymbol{\xi}}((2\pi\mu_O)^{-1}\log|\boldsymbol{\xi}|^{-1})\}\ . \end{aligned}$$

Let $|\boldsymbol{\xi}| > 2$ and write the problem for $r_N^{(j)}$ with respect to $\boldsymbol{\eta}$ as follows

$$\mu_O \Delta_{\boldsymbol{\eta}} r_N^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta}) = 0\ , \quad \boldsymbol{\eta}\in C\bar{\omega}^{(j)}\ ,$$

$$\mu_{I_j}\Delta_{\boldsymbol{\eta}} r_N^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta}) = 0\ , \quad \boldsymbol{\eta}\in \omega^{(j)}\ ,$$

$$\begin{aligned} &\mu_{I_j}\frac{\partial r_N^{(j)}}{\partial n_{\boldsymbol{\eta}}}(\boldsymbol{\xi},\boldsymbol{\eta})\Big|_{\boldsymbol{\eta}\in\partial\omega^{(j)-}} - \mu_O\frac{\partial r_N^{(j)}}{\partial n_{\boldsymbol{\eta}}}(\boldsymbol{\xi},\boldsymbol{\eta})\Big|_{\boldsymbol{\eta}\in\partial\omega^{(j)+}} \\ &= \frac{(\mu_{I_j}-\mu_O)}{\mu_O}\left\{\frac{\partial}{\partial n_{\boldsymbol{\xi}}}((2\pi)^{-1}\log|\boldsymbol{\xi}-\boldsymbol{\eta}|) - \frac{\partial}{\partial n_{\boldsymbol{\xi}}}((2\pi)^{-1}\log|\boldsymbol{\xi}|)\right\}, \end{aligned} \quad (6.16)$$

$$r_N^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta})\Big|_{\boldsymbol{\eta}\in\partial\omega^{(j)+}} = r_N^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta})\Big|_{\boldsymbol{\eta}\in\partial\omega^{(j)-}}\ ,$$

$$r_N^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta}) \to \mathbf{O}, \quad \text{as} \quad |\boldsymbol{\eta}|\to\infty\ .$$

We have that the right-hand side of (6.16) is self balanced and

$$\begin{aligned} &\left|\frac{(\mu_{I_j}-\mu_O)}{\mu_O}\left\{\frac{\partial}{\partial n_{\boldsymbol{\xi}}}((2\pi)^{-1}\log|\boldsymbol{\xi}-\boldsymbol{\eta}|) - \frac{\partial}{\partial n_{\boldsymbol{\xi}}}((2\pi)^{-1}\log|\boldsymbol{\xi}|)\right\}\right|\ , \\ \le\ &\text{const}\ |\boldsymbol{\eta}||\boldsymbol{\xi}|^{-2} \le \text{const}\ |\boldsymbol{\xi}|^{-2}\ , \end{aligned}$$

where $\boldsymbol{\eta}\in\partial\omega^{(j)}$, $|\boldsymbol{\eta}|\le 1$ and $|\boldsymbol{\xi}|>2$. Now an application of Lemma 1, leads to the estimate for $r_N^{(j)}$

$$|r_N^{(j)}(\boldsymbol{\xi},\boldsymbol{\eta})| \le \text{const}\ |\boldsymbol{\xi}|^{-2}(|\boldsymbol{\eta}|+1)^{-1}\ ,$$

for $|\boldsymbol{\xi}|>2$, $\boldsymbol{\eta}\in C\bar{\omega}^{(j)}\cup\omega^{(j)}$.

iv) The asymptotics of $h_N^{(j,O,I)}$. We once again refer to (3.8), for the relation

$$h_N^{(j,O,I)}(\boldsymbol{\xi},\boldsymbol{\eta}) = h_N^{(j,I,O)}(\boldsymbol{\eta},\boldsymbol{\xi}) - \frac{1}{2\pi}\left\{\frac{1}{\mu_{I_j}} - \frac{1}{\mu_O}\right\}\log|\boldsymbol{\xi}-\boldsymbol{\eta}| \quad \text{for } \boldsymbol{\xi}\in C\bar{\omega}^{(j)}, \boldsymbol{\eta}\in\omega^{(j)}\ .$$

For $|\boldsymbol{\xi}|>2$, $\boldsymbol{\eta}\in\omega^{(j)}$, this can be rewritten as

$$h_N^{(j,O,I)}(\boldsymbol{\xi},\boldsymbol{\eta}) = h_N^{(j,I,O)}(\boldsymbol{\eta},\boldsymbol{\xi}) - \frac{1}{2\pi}\left\{\frac{1}{\mu_{I_j}} - \frac{1}{\mu_O}\right\}\{\log|\boldsymbol{\xi}| - \boldsymbol{\eta}\cdot\nabla_{\boldsymbol{\xi}}(\log|\boldsymbol{\xi}|)\} + \left(\frac{1}{|\boldsymbol{\xi}|^2}\right)\ .$$

By combining this with (6.2) we obtain (6.3). □

The proof of Lemma 4 is then completed by applying (3.7), (6.1) and (6.3).

7 Uniform approximation of N_ε in a domain with multiple inclusions

The aim of the current section is to present the uniform asymptotic approximation for N_ε in a domain with several inclusions.

Theorem 1 *The approximation of Green's function for the transmission problem of antiplane shear in* $\bigcup_l \omega_\varepsilon^{(l)} \cup \Omega_\varepsilon \subset \mathbb{R}^2$*, is given by*

$$\boxed{\begin{aligned} N_\varepsilon(\mathbf{x},\mathbf{y}) &= N^{(\Omega)}(\mathbf{x},\mathbf{y}) + \sum_{j=1}^{M} \mathcal{N}^{(j)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) + M(2\pi\mu_O)^{-1}\log(\varepsilon^{-1}|\mathbf{x}-\mathbf{y}|) \\ &+\varepsilon \sum_{j=1}^{M} \left\{ \mathcal{D}^{(j)}(\boldsymbol{\xi}_j)\cdot\nabla_{\mathbf{x}} R^{(\Omega)}(\mathbf{O}^{(j)},\mathbf{y}) + \mathcal{D}^{(j)}(\boldsymbol{\eta}_j)\cdot\nabla_{\mathbf{y}} R^{(\Omega)}(\mathbf{x},\mathbf{O}^{(j)})\right\} + O(\varepsilon^2)\end{aligned}} \tag{7.1}$$

uniformly for $\mathbf{x},\mathbf{y} \in \bigcup_l \omega_\varepsilon^{(l)} \cup \Omega_\varepsilon$.

Proof. Formal asymptotic algorithm. First, we give a plausible argument concerning the representation of N_ε. We propose the function N_ε to be given in the form

$$N_\varepsilon(\mathbf{x},\mathbf{y}) = -\frac{1}{2\pi}\left(\frac{\chi_{\Omega_\varepsilon}(\mathbf{y})}{\mu_O} + \sum_{l=1}^{M}\frac{\chi_{\omega_\varepsilon^{(l)}}(\mathbf{y})}{\mu_{I_l}}\right)\log|\mathbf{x}-\mathbf{y}| - R_\varepsilon(\mathbf{x},\mathbf{y}) , \tag{7.2}$$

where for $\mathbf{y} \in \bigcup_{l=1}^M \omega_\varepsilon^{(l)} \cup \Omega_\varepsilon$, R_ε is a solution of

$$\mu_O \Delta_{\mathbf{x}} R_\varepsilon(\mathbf{x},\mathbf{y}) = 0 , \quad \mathbf{x} \in \Omega_\varepsilon , \tag{7.3}$$

$$\mu_{I_i}\Delta_{\mathbf{x}} R_\varepsilon(\mathbf{x},\mathbf{y}) = 0 , \quad \mathbf{x} \in \omega_\varepsilon^{(i)} , i = 1,\dots,M , \tag{7.4}$$

$$\mu_O\frac{\partial R_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y}) = -\frac{\mu_O}{2\pi}\left(\frac{\chi_{\Omega_\varepsilon}(\mathbf{y})}{\mu_O} + \sum_{l=1}^{M}\frac{\chi_{\omega_\varepsilon^{(l)}}(\mathbf{y})}{\mu_{I_l}}\right)\frac{\partial(\log|\mathbf{x}-\mathbf{y}|)}{\partial n_{\mathbf{x}}} + \frac{1}{|\partial\Omega|} , \quad \mathbf{x} \in \partial\Omega , \tag{7.5}$$

and for $i = 1,\dots,M$, R_ε satisfies the transmission conditions

$$\begin{aligned} &\mu_{I_i}\frac{\partial R_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)-}} - \mu_O\frac{\partial R_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)+}} \\ = \; & -\frac{\mu_{I_i}-\mu_O}{2\pi}\left(\frac{\chi_{\Omega_\varepsilon}(\mathbf{y})}{\mu_O} + \sum_{l=1}^{M}\frac{\chi_{\omega_\varepsilon^{(l)}}(\mathbf{y})}{\mu_{I_l}}\right)\frac{\partial(\log|\mathbf{x}-\mathbf{y}|)}{\partial n_{\mathbf{x}}} , \end{aligned} \tag{7.6}$$

$$R_\varepsilon(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)+}} = R_\varepsilon(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)-}} . \tag{7.7}$$

In the above problem we have that R_ε is subject to the orthogonality condition

$$\int_{\partial\Omega} R_\varepsilon(\mathbf{x},\mathbf{y})dS_{\mathbf{x}} = -\frac{1}{2\pi}\left(\frac{\chi_{\Omega_\varepsilon}(\mathbf{y})}{\mu_O} + \sum_{l=1}^{M}\frac{\chi_{\omega_\varepsilon^{(l)}}(\mathbf{y})}{\mu_{I_l}}\right)\int_{\partial\Omega}\log|\mathbf{x}-\mathbf{y}|\,dS_{\mathbf{x}} . \tag{7.8}$$

First order approximation for R_ε. If we allow $\mathbf{y}$ to be located inside one of the M inclusions, then we assume $\mathbf{y} \in \omega_\varepsilon^{(m)}$, where m is fixed, $1 \le m \le M$. We first rewrite the boundary conditions for R_ε.

The boundary condition for R_ε when $\mathbf{x} \in \partial\Omega$, $\mathbf{y} \in \Omega_\varepsilon \cup \omega_\varepsilon^{(m)}$, is equivalent to

$$\mu_O \frac{\partial R_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x}, \mathbf{y}) = \chi_{\Omega_\varepsilon}(\mathbf{y}) \left\{ -\frac{1}{2\pi} \frac{\partial (\log |\mathbf{x} - \mathbf{y}|)}{\partial n_{\mathbf{x}}} + \frac{1}{|\partial\Omega|} \right\}$$

$$+\chi_{\omega_\varepsilon^{(m)}}(\mathbf{y}) \left\{ -\frac{\mu_O}{2\pi} \frac{\partial}{\partial n_{\mathbf{x}}} \left[\left\{ \frac{1}{\mu_{I_m}} - \frac{1}{\mu_O} \right\} \log |\mathbf{x} - \mathbf{y}| + \frac{1}{\mu_O} \log |\mathbf{x} - \mathbf{y}| \right] + \frac{1}{|\partial\Omega|} \right\} .$$

Using scaled variables we can also rewrite the transmission conditions (7.6) on $\partial\omega_\varepsilon^{(q)}$, $q \ne m$, as

$$\mu_{I_q} \frac{\partial R_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x}, \mathbf{y})\big|_{\mathbf{x} \in \partial\omega_\varepsilon^{(q)-}} - \mu_O \frac{\partial R_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x}, \mathbf{y})\big|_{\mathbf{x} \in \partial\omega_\varepsilon^{(q)+}}$$

$$= -\chi_{\Omega_\varepsilon}(\mathbf{y}) \frac{\mu_{I_q} - \mu_O}{2\pi\mu_O} \frac{\partial (\log |\boldsymbol{\xi}_q - \boldsymbol{\eta}_q|)}{\partial n_{\mathbf{x}}}$$

$$+\chi_{\omega_\varepsilon^{(m)}}(\mathbf{y}) \Big\{ -\frac{\mu_{I_q} - \mu_O}{2\pi} \frac{\partial}{\partial n_{\mathbf{x}}} \left[\left(\frac{1}{\mu_{I_m}} - \frac{1}{\mu_O} \right) \log |\mathbf{x} - \mathbf{y}| + \frac{1}{\mu_O} \log |\boldsymbol{\xi}_q - \boldsymbol{\eta}_q| \right] \Big\} ,$$

where $\mathbf{y} \in \Omega_\varepsilon \cup \omega_\varepsilon^{(m)}$. Finally, the transmission condition (7.6) on the m^{th} inclusion becomes

$$\mu_{I_m} \frac{\partial R_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x}, \mathbf{y})\big|_{\mathbf{x} \in \partial\omega_\varepsilon^{(m)-}} - \mu_O \frac{\partial R_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x}, \mathbf{y})\big|_{\mathbf{x} \in \partial\omega_\varepsilon^{(m)+}}$$

$$= \quad -\frac{\mu_{I_m} - \mu_O}{2\pi} \left(\frac{\chi_{\Omega_\varepsilon}(\mathbf{y})}{\mu_O} + \frac{\chi_{\omega_\varepsilon^{(m)}}(\mathbf{y})}{\mu_{I_m}} \right) \frac{\partial (\log |\boldsymbol{\xi}_m - \boldsymbol{\eta}_m|)}{\partial n_{\mathbf{x}}} ,$$

for $\mathbf{y} \in \Omega_\varepsilon \cup \omega_\varepsilon^{(m)}$. We therefore have the representation

$$\begin{aligned} R_\varepsilon(\mathbf{x}, \mathbf{y}) = \chi_{\Omega_\varepsilon}(\mathbf{y}) \Big\{ R^{(\Omega)}(\mathbf{x}, \mathbf{y}) + \sum_{k=1}^{M} h_N^{(k,O)}(\boldsymbol{\xi}_k, \boldsymbol{\eta}_k) \Big\} \\ +\chi_{\omega_\varepsilon^{(m)}}(\mathbf{y}) \Big\{ R^{(\Omega)}(\mathbf{x}, \mathbf{y}) + h_N^{(m,I)}(\boldsymbol{\xi}_m, \boldsymbol{\eta}_m) \\ + \sum_{\substack{k \ne m \\ 1 \le k \le M}} h_N^{(k,O)}(\boldsymbol{\xi}_k, \boldsymbol{\eta}_k) \Big\} + Z_\varepsilon(\mathbf{x}, \mathbf{y}) . \end{aligned} \tag{7.9}$$

The construction of boundary layer terms. The function Z_ε, given in (7.9), is harmonic for $\mathbf{x} \in \bigcup_{l=1}^{M} \omega_\varepsilon^{(l)} \cup \Omega_\varepsilon$, $\mathbf{y} \in \Omega_\varepsilon \cup \omega_\varepsilon^{(m)}$ and is continuous across the boundaries of the small inclusions. The normal derivative of Z_ε on the exterior boundary is

$$\mu_O \frac{\partial Z_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y}) = -\mu_O \chi_{\Omega_\varepsilon}(\mathbf{y}) \sum_{k=1}^{M} \frac{\partial h_N^{(k,O)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_k, \boldsymbol{\eta}_k)$$

$$-\mu_O \chi_{\omega_\varepsilon^{(m)}}(\mathbf{y}) \frac{\partial}{\partial n_{\mathbf{x}}}\Big[\frac{1}{2\pi}\left(\frac{1}{\mu_{I_m}} - \frac{1}{\mu_O}\right)\log|\mathbf{x}-\mathbf{y}|$$

$$+h_N^{(m,I)}(\boldsymbol{\xi}_m, \boldsymbol{\eta}_m) + \sum_{\substack{k\neq m\\ 1\leq k\leq M}} h_N^{(k,O)}(\boldsymbol{\xi}_k, \boldsymbol{\eta}_k)\Big]\,,$$

for $\mathbf{x} \in \partial\Omega$, and the traction transmission conditions for Z_ε on the small inclusions are

$$\begin{aligned} & \mu_{I_m} \frac{\partial Z_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(m)-}} - \mu_O \frac{\partial Z_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(m)+}} \\ = \;& -(\mu_{I_m} - \mu_O)(\chi_{\Omega_\varepsilon}(\mathbf{y}) + \chi_{\omega_\varepsilon^{(m)}}(\mathbf{y}))\frac{\partial}{\partial n_{\mathbf{x}}}\Big\{R^{(\Omega)}(\mathbf{x},\mathbf{y}) + \sum_{\substack{k\neq m\\ 1\leq k\leq M}} h_N^{(k,O)}(\boldsymbol{\xi}_k, \boldsymbol{\eta}_k)\Big\}\,, \end{aligned}$$

and

$$\mu_{I_q} \frac{\partial Z_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(q)-}} - \mu_O \frac{\partial Z_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(q)+}}$$

$$= -(\mu_{I_q} - \mu_O)\Big\{\chi_{\Omega_\varepsilon}(\mathbf{y})\frac{\partial}{\partial n_{\mathbf{x}}}\Big[R^{(\Omega)}(\mathbf{x},\mathbf{y}) + \sum_{\substack{k\neq q\\ 1\leq k\leq M}} h_N^{(k,O)}(\boldsymbol{\xi}_k, \boldsymbol{\eta}_k)\Big]$$

$$+\chi_{\omega_\varepsilon^{(m)}}(\mathbf{y})\frac{\partial}{\partial n_{\mathbf{x}}}\Big[\frac{1}{2\pi}\left(\frac{1}{\mu_{I_m}} - \frac{1}{\mu_O}\right)\log|\mathbf{x}-\mathbf{y}| + R^{(\Omega)}(\mathbf{x},\mathbf{y})$$

$$+h_N^{(m,I)}(\boldsymbol{\xi}_m, \boldsymbol{\eta}_m) + \sum_{\substack{k\neq m\\ k\neq q\\ 1\leq k\leq M}} h_N^{(k,O)}(\boldsymbol{\xi}_k, \boldsymbol{\eta}_k)\Big]\Big\}\,,$$

where $1 \leq q \leq M$, $q \neq m$ and $\mathbf{y} \in \Omega_\varepsilon \cup \omega_\varepsilon^{(m)}$. By expanding the first order derivatives of the logarithmic term near $\chi_{\omega_\varepsilon^{(m)}}(\mathbf{y})$, about $\mathbf{y} = \mathbf{O}^{(m)}$ up to $O(\varepsilon^2)$, in the exterior boundary condition and the traction condition on $\partial\omega_\varepsilon^{(q)}$, $q \neq m$, we obtain

$$\mu_O \frac{\partial Z_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y}) = -\mu_O \chi_{\Omega_\varepsilon}(\mathbf{y}) \sum_{k=1}^{M} \frac{\partial h_N^{(k,O)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_k, \boldsymbol{\eta}_k)$$

$$-\mu_O \chi_{\omega_\varepsilon^{(m)}}(\mathbf{y}) \Big\{ \frac{\partial}{\partial n_{\mathbf{x}}} \Big[\frac{1}{2\pi} \left(\frac{1}{\mu_{I_m}} - \frac{1}{\mu_O} \right) \Big(\log|\mathbf{x} - \mathbf{O}^{(m)}|$$

$$- \frac{(\mathbf{y} - \mathbf{O}^{(m)}) \cdot (\mathbf{x} - \mathbf{O}^{(m)})}{|\mathbf{x} - \mathbf{O}^{(m)}|^2} \Big) + h_N^{(m,I)}(\boldsymbol{\xi}_m, \boldsymbol{\eta}_m) + \sum_{\substack{k \neq m \\ 1 \leq k \leq M}} h_N^{(k,O)}(\boldsymbol{\xi}_k, \boldsymbol{\eta}_k) \Big] + O(\varepsilon^2) \Big\} ,$$

for $\mathbf{x} \in \partial\Omega$, and

$$\mu_{I_q} \frac{\partial Z_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x} \in \partial\omega_\varepsilon^{(q)-}} - \mu_O \frac{\partial Z_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x} \in \partial\omega_\varepsilon^{(q)+}}$$

$$= -(\mu_{I_q} - \mu_O) \Big[\chi_{\Omega_\varepsilon}(\mathbf{y}) \frac{\partial}{\partial n_{\mathbf{x}}} \Big\{ R^{(\Omega)}(\mathbf{x},\mathbf{y}) + \sum_{\substack{k \neq q \\ 1 \leq k \leq M}} h_N^{(k,O)}(\boldsymbol{\xi}_k, \boldsymbol{\eta}_k) \Big\}$$

$$+\chi_{\omega_\varepsilon^{(m)}}(\mathbf{y}) \Big\{ \frac{\partial}{\partial n_{\mathbf{x}}} \Big[\frac{1}{2\pi} \left(\frac{1}{\mu_{I_m}} - \frac{1}{\mu_O} \right) \Big(\log|\mathbf{x} - \mathbf{O}^{(m)}| - \frac{(\mathbf{y} - \mathbf{O}^{(m)}) \cdot (\mathbf{x} - \mathbf{O}^{(m)})}{|\mathbf{x} - \mathbf{O}^{(m)}|^2} \Big)$$

$$+R^{(\Omega)}(\mathbf{x},\mathbf{y}) + h_N^{(m,I)}(\boldsymbol{\xi}_m, \boldsymbol{\eta}_m) + \sum_{\substack{k \neq m \\ k \neq q \\ 1 \leq k \leq M}} h_N^{(k,O)}(\boldsymbol{\xi}_k, \boldsymbol{\eta}_k) \Big] + O(\varepsilon^2) \Big\} \Big] ,$$

where $\mathbf{y} \in \Omega_\varepsilon \cup \omega_\varepsilon^{(m)}$. Next we apply Lemma 4, in order to rewrite the above boundary condition on $\partial\Omega$ for Z_ε as

$$\mu_O \frac{\partial Z_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y}) = -\mu_O \chi_{\Omega_\varepsilon}(\mathbf{y}) \sum_{k=1}^{M} \frac{\partial}{\partial n_{\mathbf{x}}} \Big\{ \frac{\varepsilon \mathcal{D}^{(k,O)}(\boldsymbol{\eta}_k)}{2\pi\mu_O} \cdot \frac{\mathbf{x} - \mathbf{O}^{(k)}}{|\mathbf{x} - \mathbf{O}^{(k)}|^2}$$

$$+O\Big(\sum_{k=1}^{M} \varepsilon^3 (|\mathbf{y} - \mathbf{O}^{(k)}| + \varepsilon)^{-1} \Big) \Big\} + \chi_{\omega_\varepsilon^{(m)}}(\mathbf{y}) \Big\{ -\mu_O \frac{\partial}{\partial n_{\mathbf{x}}} \Big[\frac{\varepsilon \mathcal{D}^{(m,I)}(\boldsymbol{\eta}_m)}{2\pi\mu_O} \cdot \frac{\mathbf{x} - \mathbf{O}^{(m)}}{|\mathbf{x} - \mathbf{O}^{(m)}|^2}$$

$$+(\mu_{I_m}^{-1} - \mu_O^{-1}) \log\varepsilon + \sum_{\substack{k \neq m \\ 1 \leq k \leq M}} \frac{\varepsilon \mathcal{D}^{(k,O)}(\boldsymbol{\eta}_k)}{2\pi\mu_O} \cdot \frac{\mathbf{x} - \mathbf{O}^{(k)}}{|\mathbf{x} - \mathbf{O}^{(k)}|^2} \Big] + O(\varepsilon^2) \Big\} ,$$

for $\mathbf{x} \in \partial\Omega$, $\mathbf{y} \in \Omega_\varepsilon \cup \omega_\varepsilon^{(m)}$. The same lemma in combination with the Taylor expansion about $\mathbf{x} = \mathbf{O}^{(m)}$ of the first order derivatives for the function $R^{(\Omega)}$

leads to

$$\mu_{I_m} \frac{\partial Z_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(m)-}} - \mu_O \frac{\partial Z_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(m)+}}$$

$$= -(\chi_{\Omega_\varepsilon}(\mathbf{y}) + \chi_{\omega_\varepsilon^{(m)}}(\mathbf{y}))(\mu_{I_m} - \mu_O)\boldsymbol{n}^{(m)} \cdot \nabla_{\mathbf{x}} R^{(\Omega)}(\mathbf{O}^{(m)},\mathbf{y}) + O(\varepsilon) \;, \quad \mathbf{y} \in \Omega_\varepsilon \cup \omega_\varepsilon^{(m)} \;,$$

where $\boldsymbol{n}^{(m)}$ is the unit outward normal to the inclusion $\omega_\varepsilon^{(m)}$. Similarly, on the q^{th} inclusion, $1 \le q \le M$, $q \ne m$, we derive

$$\mu_{I_q} \frac{\partial Z_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(q)-}} - \mu_O \frac{\partial Z_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(q)+}}$$

$$= -(\chi_{\Omega_\varepsilon}(\mathbf{y}) + \chi_{\omega_\varepsilon^{(m)}}(\mathbf{y}))(\mu_{I_q} - \mu_O)\boldsymbol{n}^{(q)} \cdot \nabla_{\mathbf{x}} R^{(\Omega)}(\mathbf{O}^{(q)},\mathbf{y}) + O(\varepsilon) \;, \quad \mathbf{y} \in \Omega_\varepsilon \cup \omega_\varepsilon^{(m)} \;.$$

Then, we approximate Z_ε by

$$Z_\varepsilon(\mathbf{x},\mathbf{y}) = -(\chi_{\Omega_\varepsilon}(\mathbf{y}) + \chi_{\omega_\varepsilon^{(m)}}(\mathbf{y})) \sum_{\substack{k\ne m\\ 1\le k\le M}} \varepsilon\{\mathcal{D}^{(k,O)}(\boldsymbol{\eta}_k) \cdot \nabla_{\mathbf{y}} R^{(\Omega)}(\mathbf{x},\mathbf{O}^{(k)})$$

$$+\mathcal{D}^{(k)}(\boldsymbol{\xi}_k) \cdot \nabla_{\mathbf{x}} R^{(\Omega)}(\mathbf{O}^{(k)},\mathbf{y})\} - \varepsilon\{\chi_{\Omega_\varepsilon}(\mathbf{y})\mathcal{D}^{(m,O)}(\boldsymbol{\eta}_m) \cdot \nabla_{\mathbf{y}} R^{(\Omega)}(\mathbf{x},\mathbf{O}^{(m)})$$

$$+(\chi_{\Omega_\varepsilon}(\mathbf{y}) + \chi_{\omega_\varepsilon^{(m)}}(\mathbf{y}))\mathcal{D}^{(m)}(\boldsymbol{\xi}_m) \cdot \nabla_{\mathbf{x}} R^{(\Omega)}(\mathbf{O}^{(m)},\mathbf{y})\}$$

$$-\chi_{\omega_\varepsilon^{(m)}}(\mathbf{y})\{\varepsilon\mathcal{D}^{(m,I)}(\boldsymbol{\eta}_m) \cdot \nabla_{\mathbf{y}} R^{(\Omega)}(\mathbf{x},\mathbf{O}^{(m)}) + (\mu_{I_m}^{-1} - \mu_O^{-1})\log\varepsilon\} - r_\varepsilon^{(m)}(\mathbf{x},\mathbf{y}) \;. \tag{7.10}$$

Combined formula for N_ε. The substitution of (7.9), (7.10) into (7.2), for $\mathbf{y} \in \Omega_\varepsilon \cup \omega_\varepsilon^{(m)}$, yields

$$N_\varepsilon(\mathbf{x},\mathbf{y}) = \chi_{\Omega_\varepsilon}(\mathbf{y}) \Bigg\{ -(2\pi\mu_O)^{-1} \log|\mathbf{x}-\mathbf{y}| - R^{(\Omega)}(\mathbf{x},\mathbf{y}) - \sum_{k=1}^{M} h_N^{(k,O)}(\boldsymbol{\xi}_k, \boldsymbol{\eta}_k)$$

$$+\varepsilon \sum_{k=1}^{M} \big\{ \mathcal{D}^{(k)}(\boldsymbol{\xi}_k) \cdot \nabla_{\mathbf{x}} R^{(\Omega)}(\mathbf{O}^{(k)},\mathbf{y}) + \mathcal{D}^{(k,O)}(\boldsymbol{\eta}_k) \cdot \nabla_{\mathbf{y}} R^{(\Omega)}(\mathbf{x},\mathbf{O}^{(k)}) \big\} \Bigg\}$$

$$+\chi_{\omega_\varepsilon^{(m)}}(\mathbf{y}) \Bigg\{ -(2\pi\mu_{I_m})^{-1} \log|\mathbf{x}-\mathbf{y}| - R^{(\Omega)}(\mathbf{x},\mathbf{y}) - h_N^{(m,I)}(\boldsymbol{\xi}_m, \boldsymbol{\eta}_m)$$

$$- \sum_{\substack{k\neq m \\ 1\leq k\leq M}} h_N^{(k,O)}(\boldsymbol{\xi}_k, \boldsymbol{\eta}_k) + (\mu_{I_m}^{-1} - \mu_O^{-1})(2\pi)^{-1} \log\varepsilon + \varepsilon\, \mathcal{D}^{(m)}(\boldsymbol{\xi}_m) \cdot \nabla_{\mathbf{x}} R^{(\Omega)}(\mathbf{O}^{(m)},\mathbf{y})$$

$$+\varepsilon \mathcal{D}^{(m,I)}(\boldsymbol{\eta}_m) \cdot \nabla_{\mathbf{y}} R^{(\Omega)}(\mathbf{x},\mathbf{O}^{(m)}) + \varepsilon \sum_{\substack{k\neq m \\ 1\leq k\leq M}} \mathcal{D}^{(k)}(\boldsymbol{\xi}_k) \cdot \nabla_{\mathbf{x}} R^{(\Omega)}(\mathbf{O}^{(k)},\mathbf{y})$$

$$+\mathcal{D}^{(k,O)}(\boldsymbol{\eta}_k) \cdot \nabla_{\mathbf{y}} R^{(\Omega)}(\mathbf{x},\mathbf{O}^{(k)}) \Bigg\} + r_\varepsilon^{(m)}(\mathbf{x},\mathbf{y})\ . \tag{7.11}$$

Then, by the definitions of $N^{(\Omega)}$, $\mathcal{N}^{(m,O)}$ and $\mathcal{N}^{(m,I)}$, this is equivalent to

$$N_\varepsilon(\mathbf{x},\mathbf{y}) = \chi_{\Omega_\varepsilon}(\mathbf{y}) \Bigg\{ N^{(\Omega)}(\mathbf{x},\mathbf{y}) + \sum_{k=1}^{M} \mathcal{N}^{(k,O)}(\boldsymbol{\xi}_k, \boldsymbol{\eta}_k) + M(2\pi\mu_O)^{-1} \log(\varepsilon^{-1}|\mathbf{x}-\mathbf{y}|)$$

$$+\varepsilon \sum_{k=1}^{M} \big\{ \mathcal{D}^{(k)}(\boldsymbol{\xi}_k) \cdot \nabla_{\mathbf{x}} R^{(\Omega)}(\mathbf{O}^{(k)},\mathbf{y}) + \mathcal{D}^{(k,O)}(\boldsymbol{\eta}_k) \cdot \nabla_{\mathbf{y}} R^{(\Omega)}(\mathbf{x},\mathbf{O}^{(k)}) \big\} \Bigg\}$$

$$+\sum_{j=1}^{M} \chi_{\omega_\varepsilon^{(j)}}(\mathbf{y}) \Bigg\{ N^{(\Omega)}(\mathbf{x},\mathbf{y}) + \mathcal{N}^{(j,I)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) + \sum_{\substack{k\neq j \\ 1\leq k\leq M}} \mathcal{N}^{(k,O)}(\boldsymbol{\xi}_k, \boldsymbol{\eta}_k)$$

$$+M(2\pi\mu_O)^{-1} \log(\varepsilon^{-1}|\mathbf{x}-\mathbf{y}|) + \varepsilon \sum_{k=1}^{M} \mathcal{D}^{(k)}(\boldsymbol{\xi}_k) \cdot \nabla_{\mathbf{x}} R^{(\Omega)}(\mathbf{O}^{(k)},\mathbf{y})$$

$$+\varepsilon \mathcal{D}^{(j,I)}(\boldsymbol{\eta}_j) \cdot \nabla_{\mathbf{y}} R^{(\Omega)}(\mathbf{x},\mathbf{O}^{(j)}) + \varepsilon \sum_{\substack{k\neq j \\ 1\leq k\leq M}} \mathcal{D}^{(k,O)}(\boldsymbol{\eta}_k) \cdot \nabla_{\mathbf{y}} R^{(\Omega)}(\mathbf{x},\mathbf{O}^{(k)}) \Bigg\} + r_\varepsilon(\mathbf{x},\mathbf{y})\ .$$

Remainder estimates for the approximation of N_ε. We represent r_ε as

$$r_\varepsilon(\mathbf{x},\mathbf{y}) = \chi_{\Omega_\varepsilon}(\mathbf{y})\mathfrak{M}_\varepsilon(\mathbf{x},\mathbf{y}) + \sum_{j=1}^{M} \chi_{\omega_\varepsilon^{(p)}}(\mathbf{y})\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y}) \ , \tag{7.12}$$

where $\mathfrak{M}_\varepsilon$ and $\mathfrak{h}_\varepsilon^{(j)}$, $j = 1, \dots, M$ are defined below. In what follows, we estimate $\mathfrak{M}_\varepsilon$, and $\mathfrak{h}_\varepsilon^{(j)}$, $j = 1, \dots, M$, in order to estimate r_ε.

Remainder estimate for the function $\mathfrak{M}_\varepsilon(\mathbf{x},\mathbf{y})$. First, let $\mathbf{y} \in \Omega_\varepsilon$. According to (7.11), the function $\mathfrak{M}_\varepsilon$ is a solution of

$$\mu_O \Delta_{\mathbf{x}} \mathfrak{M}_\varepsilon(\mathbf{x},\mathbf{y}) = 0, \quad \mathbf{x} \in \Omega_\varepsilon \ ,$$
$$\mu_{I_i} \Delta_{\mathbf{x}} \mathfrak{M}_\varepsilon(\mathbf{x},\mathbf{y}) = 0, \quad \mathbf{x} \in \omega_\varepsilon^{(i)}, i = 1, \dots, M \ ,$$

and on the exterior contour is subject to

$$\mu_O \frac{\partial \mathfrak{M}_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y}) = -\varepsilon \mu_O \sum_{j=1}^{M} \frac{\partial \mathcal{D}^{(j,O)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_j) \cdot \nabla_{\mathbf{x}} R^{(\Omega)}(\mathbf{O}^{(j)},\mathbf{y})$$
$$+\mu_O \sum_{j=1}^{M} \left\{ \frac{\partial h_N^{(j,O)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j) - \varepsilon \mathcal{D}^{(j,O)}(\boldsymbol{\eta}_j) \cdot \frac{\partial}{\partial n_{\mathbf{x}}} \nabla_{\mathbf{y}} R^{(\Omega)}(\mathbf{x},\mathbf{O}^{(j)}) \right\}, \quad \mathbf{x} \in \partial\Omega \ , \tag{7.13}$$

with transmission conditions on $\partial\omega_\varepsilon^{(i)}$ of the form

$$\mu_{I_i} \frac{\partial \mathfrak{M}_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)-}} - \mu_O \frac{\partial \mathfrak{M}_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)+}}$$
$$= (\mu_{I_i} - \mu_O)\frac{\partial R^{(\Omega)}}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y}) + (\mu_{I_i} - \mu_O) \sum_{\substack{j\neq i \\ 1\le j\le M}} \frac{\partial h_N^{(j,O)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)$$
$$-\varepsilon \sum_{j=1}^{M} \left\{ \mu_{I_i} \frac{\partial \mathcal{D}^{(j)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_j)\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)-}} - \mu_O \frac{\partial \mathcal{D}^{(j)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_j)\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)+}} \right\} \cdot \nabla_{\mathbf{x}} R^{(\Omega)}(\mathbf{O}^{(j)},\mathbf{y}) \ ,$$
$$-(\mu_{I_i} - \mu_O)\varepsilon \sum_{j=1}^{M} \mathcal{D}^{(j,O)}(\boldsymbol{\eta}_j) \cdot \nabla_{\mathbf{y}} \frac{\partial R^{(\Omega)}}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{O}^{(j)}) \ ,$$
$$\mathfrak{M}_\varepsilon(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)-}} = \mathfrak{M}_\varepsilon(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)+}} \ , \tag{7.14}$$

for $1 \le i \le M$, where $\mathbf{y} \in \Omega_\varepsilon$ and

$$\int_{\partial\Omega} \mathfrak{M}_\varepsilon(\mathbf{x},\mathbf{y}) dS_{\mathbf{x}} = 0 \ . \tag{7.15}$$

Before estimating the boundary conditions we observe that

$$\int_{\partial\Omega} \mu_O \frac{\partial \mathfrak{M}_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\, dS_{\mathbf{x}} = 0 \ , \tag{7.16}$$

$$\int_{\partial\omega_\varepsilon^{(i)}} \left(\mu_{I_i} \frac{\partial \mathfrak{M}_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)-}} - \mu_O \frac{\partial \mathfrak{M}_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)+}} \right) dS_{\mathbf{x}} = 0, \tag{7.17}$$

for $i = 1, \dots, M$.

Estimation of the right-hand side of (7.13) *on* $\partial\Omega$. Since $\mathbf{x} \in \partial\Omega$, $|\mathbf{x} - \mathbf{O}^{(j)}| \geq 1$ for $j = 1, \dots, M$, and by Lemma 2, the estimate

$$\varepsilon \mu_O \Big| \sum_{j=1}^{M} \frac{\partial \mathcal{D}^{(j,O)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_j) \cdot \nabla_{\mathbf{x}} R^{(\Omega)}(\mathbf{O}^{(j)}, \mathbf{y}) \Big| \leq \text{const } \varepsilon^2 ,$$

holds for $\mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega_\varepsilon$. Using Lemma 4a), one has for $\mathbf{x} \in \partial\Omega$,

$$\begin{aligned}
& \mu_O \Big| \frac{\partial h_N^{(j,O)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) - \varepsilon \mathcal{D}^{(j,O)}(\boldsymbol{\eta}_j) \cdot \frac{\partial}{\partial n_{\mathbf{x}}} \nabla_{\mathbf{y}} R^{(\Omega)}(\mathbf{x}, \mathbf{O}^{(j)}) \Big| \\
= \; & \mu_O \Big| \frac{\partial h_N^{(j,O)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) - \frac{\varepsilon \mathcal{D}^{(j,O)}(\boldsymbol{\eta}_j)}{2\pi \mu_O} \cdot \frac{\partial}{\partial n_{\mathbf{x}}} \Big(\frac{\mathbf{x} - \mathbf{O}^{(j)}}{|\mathbf{x} - \mathbf{O}^{(j)}|^2} \Big) \Big| \\
\leq \; & \text{const } \varepsilon^3 |\mathbf{x} - \mathbf{O}^{(j)}|^{-2} (|\mathbf{y} - \mathbf{O}^{(j)}| + \varepsilon)^{-1} \leq \text{const } \varepsilon^3 (|\mathbf{y} - \mathbf{O}^{(j)}| + \varepsilon)^{-1}
\end{aligned}$$

where we have also made use of the boundary condition (3.1) for $R^{(\Omega)}$. The previous two inequalities then give

$$\mu_O \Big| \frac{\partial \mathfrak{M}_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x}, \mathbf{y}) \Big| \leq \text{const } \varepsilon^2 \quad \mathbf{x} \in \partial\Omega, \mathbf{y} \in \Omega_\varepsilon . \tag{7.18}$$

Estimate for the right-hand side of (7.14) *on* $\partial\omega_\varepsilon^{(i)}$, $i = 1, \dots, M$. The regular part $R^{(\Omega)}$ of $N^{(\Omega)}$ is smooth for $\mathbf{x}, \mathbf{y} \in \Omega$, and we can expand the first order derivatives of this function about the centre of the small inclusion $\omega_\varepsilon^{(i)}$ to obtain

$$\begin{aligned}
& \Big| (\mu_{I_i} - \mu_O) \frac{\partial R^{(\Omega)}}{\partial n_{\mathbf{x}}}(\mathbf{x}, \mathbf{y}) - \varepsilon \Big\{ \mu_{I_i} \frac{\partial \mathcal{D}^{(i,I)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_i) \big|_{\mathbf{x} \in \partial\omega_\varepsilon^{(i)-}} \\
& - \mu_O \frac{\partial \mathcal{D}^{(i,O)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_i) \big|_{\mathbf{x} \in \partial\omega_\varepsilon^{(i)+}} \Big\} \cdot \nabla_{\mathbf{x}} R^{(\Omega)}(\mathbf{O}^{(i)}, \mathbf{y}) \Big| \\
= \; & \Big| (\mu_{I_i} - \mu_O) \boldsymbol{n}^{(i)} \cdot (\nabla_{\mathbf{x}} R^{(\Omega)}(\mathbf{x}, \mathbf{y}) - \nabla_{\mathbf{x}} R^{(\Omega)}(\mathbf{O}^{(i)}, \mathbf{y})) \Big| \\
\leq \; & \text{const } \varepsilon \quad \mathbf{x} \in \partial\omega_\varepsilon^{(i)}, \mathbf{y} \in \Omega_\varepsilon .
\end{aligned} \tag{7.19}$$

With the use of Lemma 2, we have

$$\begin{aligned}
& \varepsilon \Big| \sum_{\substack{j \neq i \\ 1 \leq j \leq M}} \Big\{ \mu_{I_i} \frac{\partial \mathcal{D}^{(j,O)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_j) \big|_{\mathbf{x} \in \partial\omega_\varepsilon^{(i)-}} - \mu_O \frac{\partial \mathcal{D}^{(j,O)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_j) \big|_{\mathbf{x} \in \partial\omega_\varepsilon^{(i)+}} \Big\} \cdot \nabla_{\mathbf{x}} R^{(\Omega)}(\mathbf{O}^{(j)}, \mathbf{y}) \Big| \\
& \leq \text{const } \varepsilon^2, \quad \text{for } \mathbf{x} \in \partial\omega_\varepsilon^{(i)}, \mathbf{y} \in \Omega_\varepsilon ,
\end{aligned} \tag{7.20}$$

and the same lemma also gives

$$\begin{aligned}&\Big|\varepsilon\sum_{j=1}^{M}\mathcal{D}^{(j,O)}(\boldsymbol{\eta}_j)\cdot\frac{\partial\nabla_{\mathbf{y}}R^{(\Omega)}}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{O}^{(j)})\Big|\\ \leq\quad&\text{const}\ \sum_{j=1}^{M}\varepsilon^2(|\mathbf{y}-\mathbf{O}^{(j)}|+\varepsilon)^{-1}\ ,\qquad\text{for }\mathbf{x}\in\partial\omega_\varepsilon^{(i)},\mathbf{y}\in\Omega_\varepsilon\ .\end{aligned}\tag{7.21}$$

Due to Lemma 4a),

$$\Big|\sum_{\substack{j\neq i\\1\leq j\leq M}}\frac{\partial h_N^{(j,O)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi},\boldsymbol{\eta})\Big|\leq\text{const}\sum_{\substack{j\neq i\\1\leq j\leq M}}\varepsilon^2(|\mathbf{y}-\mathbf{O}^{(j)}|+\varepsilon)^{-1},\qquad\text{for }\mathbf{x}\in\partial\omega_\varepsilon^{(i)},\mathbf{y}\in\Omega_\varepsilon\ .$$

Therefore, this estimate with (7.19), (7.20) and (7.21) lead to

$$\begin{aligned}&\Big|\mu_{I_i}\frac{\partial\mathfrak{M}_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)-}}-\mu_O\frac{\partial\mathfrak{M}_\varepsilon}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)+}}\Big|\\ \leq\quad&\text{const}\ \varepsilon,\quad\mathbf{x}\in\partial\omega_\varepsilon^{(i)},i=1,\dots,M,\mathbf{y}\in\Omega_\varepsilon\ .\end{aligned}$$

Then, by Lemma 3 and the preceding estimate with (7.15)–(7.18), we obtain

$$|\mathfrak{M}_\varepsilon(\mathbf{x},\mathbf{y})|\leq\text{const}\ \varepsilon^2,\quad\mathbf{x}\in\bigcup_{l=1}^{M}\omega_\varepsilon^{(l)}\cup\Omega_\varepsilon,\mathbf{y}\in\Omega_\varepsilon\ .\tag{7.22}$$

Remainder estimate for $\mathfrak{h}_\varepsilon^{(j)}$, $j=1,\dots,M$. Let $\mathbf{y}\in\omega_\varepsilon^{(j)}$, where j is fixed, $1\leq j\leq M$. Then by (7.1), the remainder term $\mathfrak{h}_\varepsilon^{(j)}$ solves

$$\mu_O\Delta_{\mathbf{x}}\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})=0,\quad\mathbf{x}\in\Omega_\varepsilon\ ,$$

$$\mu_{I_i}\Delta_{\mathbf{x}}\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})=0,\quad\mathbf{x}\in\omega_\varepsilon^{(i)},i=1,\dots,M\ ,$$

and satisfies on the exterior boundary

$$\begin{aligned}\mu_O\frac{\partial\mathfrak{h}_\varepsilon^{(j)}}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})=-\mu_O\Big\{&\frac{\partial\mathcal{N}^{(j,I)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)+\frac{\partial}{\partial n_{\mathbf{x}}}((2\pi\mu_O)^{-1}\log(\varepsilon^{-1}|\mathbf{x}-\mathbf{y}|))\\ &-\sum_{\substack{k\neq m\\1\leq k\leq M}}\frac{\partial h_N^{(k,O)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_k,\boldsymbol{\eta}_k)+\varepsilon\frac{\partial\mathcal{D}^{(j,O)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_j)\cdot\nabla_{\mathbf{x}}R^{(\Omega)}(\mathbf{O}^{(j)},\mathbf{y})\\ &+\varepsilon\mathcal{D}^{(j,I)}(\boldsymbol{\eta}_j)\cdot\frac{\partial}{\partial n_{\mathbf{x}}}\nabla_{\mathbf{y}}R^{(\Omega)}(\mathbf{x},\mathbf{O}^{(j)})+\varepsilon\sum_{\substack{k\neq j\\1\leq k\leq M}}\Big[\frac{\partial\mathcal{D}^{(k,O)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_k)\cdot\nabla_{\mathbf{x}}R^{(\Omega)}(\mathbf{O}^{(k)},\mathbf{y})\\ &+\mathcal{D}^{(k,O)}(\boldsymbol{\eta}_k)\cdot\frac{\partial}{\partial n_{\mathbf{x}}}\nabla_{\mathbf{y}}R^{(\Omega)}(\mathbf{x},\mathbf{O}^{(j)})\Big]\Big\},\quad\mathbf{x}\in\partial\Omega\ .\end{aligned}\tag{7.23}$$

Also, on the boundary of the i^{th} inclusion, $\mathfrak{h}_\varepsilon^{(j)}$ is subject to

$$\mu_{I_i}\frac{\partial \mathfrak{h}_\varepsilon^{(j)}}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)-}} - \mu_O\frac{\partial \mathfrak{h}_\varepsilon^{(j)}}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)+}}$$

$$= (\mu_{I_i}-\mu_O)\Bigg[\frac{\partial R^{(\Omega)}}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y}) - \varepsilon\mathcal{D}^{(j,I)}(\boldsymbol{\eta}_j)\cdot\frac{\partial}{\partial n_{\mathbf{x}}}\nabla_{\mathbf{y}}R^{(\Omega)}(\mathbf{x},\mathbf{O}^{(j)})$$

$$-\varepsilon\sum_{\substack{k\neq j\\1\le k\le M}}\mathcal{D}^{(k,O)}(\boldsymbol{\eta}_k)\cdot\frac{\partial}{\partial n_{\mathbf{x}}}\nabla_{\mathbf{y}}R^{(\Omega)}(\mathbf{x},\mathbf{O}^{(k)}) + \sum_{\substack{k\neq j\\1\le k\le M}}\Big((1-\delta_{ik})\frac{\partial h_N^{(k,O)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_k,\boldsymbol{\eta}_k)$$

$$-\delta_{ik}(2\pi\mu_O)^{-1}\frac{\partial}{\partial n_{\mathbf{x}}}(\log(\varepsilon^{-1}|\mathbf{x}-\mathbf{y}|))\Big) - (1-\delta_{ij})\frac{\partial \mathcal{N}^{(j,I)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)\Bigg]$$

$$-\varepsilon\left\{\mu_{I_i}\frac{\partial \mathcal{D}^{(j)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_j)\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)-}} - \mu_O\frac{\partial \mathcal{D}^{(j)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_j)\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)+}}\right\}\cdot\nabla R^{(\Omega)}(\mathbf{O}^{(j)},\mathbf{y})$$

$$-\varepsilon\sum_{\substack{k\neq j\\1\le k\le M}}\left\{\mu_{I_i}\frac{\partial \mathcal{D}^{(k)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_k)\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)-}} - \mu_O\frac{\partial \mathcal{D}^{(k)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_k)\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)+}}\right\}\cdot\nabla_{\mathbf{x}}R^{(\Omega)}(\mathbf{O}^{(k)},\mathbf{y})\ ,$$

$$\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)-}} = \mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)+}}\ , \tag{7.24}$$

for $i=1,\dots,M$, where $\mathbf{y}\in\omega_\varepsilon^{(j)}$ and

$$\int_{\partial\Omega}\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})dS_{\mathbf{x}} = 0\ . \tag{7.25}$$

From this problem we can see

$$\int_{\partial\Omega}\mu_O\frac{\partial \mathfrak{h}_\varepsilon^{(j)}}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\,dS_{\mathbf{x}} = 0\ , \tag{7.26}$$

$$\int_{\partial\omega_\varepsilon^{(j)}}\left(\mu_{I_j}\frac{\partial \mathfrak{h}_\varepsilon^{(j)}}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(j)-}} - \mu_O\frac{\partial \mathfrak{h}_\varepsilon^{(j)}}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(j)+}}\right)dS_{\mathbf{x}} = 0\ , \tag{7.27}$$

for $i=1,\dots,M$.

Before estimating the discrepancies in the boundary conditions, we note for $\mathbf{y}\in\omega_\varepsilon^{(j)}$, by Lemma 2

$$\Big|\sum_{\substack{k\neq j\\1\le k\le M}}\mathcal{D}^{(k,O)}(\boldsymbol{\eta}_k)\cdot\frac{\partial}{\partial n_{\mathbf{x}}}\nabla_{\mathbf{y}}R^{(\Omega)}(\mathbf{x},\mathbf{O}^{(k)})\Big| \le \text{const}\ \varepsilon, \quad \text{holds for } \mathbf{x}\in\partial\Omega_\varepsilon\ , \tag{7.28}$$

whereas the same lemma in conjunction with Lemma 4a), leads to

$$\Big|\frac{\partial h_N^{(k,O)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_k,\boldsymbol{\eta}_k)\Big| \le \text{const}\ \varepsilon^2\ , \quad \text{for } \mathbf{x}\in\bigcup_{\substack{i\neq k\\1\le i\le M}}\partial\omega_\varepsilon^{(i)}\cup\partial\Omega\ . \tag{7.29}$$

Estimate of the right-hand side of (7.23) *on* $\partial\Omega$. The definition of $\mathcal{N}^{(j,I)}$ in Section 3 and the boundary condition (3.1) for $R^{(\Omega)}$, give us

$$\begin{aligned}
&\mu_O\Big|\frac{\partial}{\partial n_{\mathbf{x}}}\Big\{\mathcal{N}^{(j,I)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)+(2\pi\mu_O)^{-1}\log(\varepsilon^{-1}|\mathbf{x}-\mathbf{y}|)+\varepsilon\mathcal{D}^{(j,I)}(\boldsymbol{\eta}_j)\cdot\nabla_{\mathbf{y}}R^{(\Omega)}(\mathbf{x},\mathbf{O}^{(j)})\Big\}\Big|\\
=\;&\mu_O\Big|\frac{\partial}{\partial n_{\mathbf{x}}}\{-(2\pi)^{-1}(\mu_{I_j}^{-1}-\mu_O^{-1})\log|\mathbf{x}-\mathbf{y}|-h_N^{(j,I)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)\\
&+\varepsilon(2\pi\mu_O)^{-1}\mathcal{D}^{(j,I)}(\boldsymbol{\eta}_j)\cdot(\mathbf{x}-\mathbf{O}^{(j)})|\mathbf{x}-\mathbf{O}^{(j)}|^{-2}\Big|\quad \text{for } \mathbf{x}\in\partial\Omega,\mathbf{y}\in\omega_\varepsilon^{(j)}\;.
\end{aligned}$$

Using the asymptotics of $h_N^{(j,I)}$ at infinity contained in Lemma 4b), we obtain

$$\begin{aligned}
&\mu_O\Big|\frac{\partial}{\partial n_{\mathbf{x}}}\Big\{\mathcal{N}^{(j,I)}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)+(2\pi\mu_O)^{-1}\log(\varepsilon^{-1}|\mathbf{x}-\mathbf{y}|)+\varepsilon\mathcal{D}^{(j,I)}(\boldsymbol{\eta}_j)\cdot\nabla_{\mathbf{y}}R^{(\Omega)}(\mathbf{x},\mathbf{O}^{(j)})\Big\}\Big|\\
\le\;&\text{const}\;\varepsilon^2\;,\quad \text{for } \mathbf{x}\in\partial\Omega,\mathbf{y}\in\omega_\varepsilon^{(j)}\;.
\end{aligned}\tag{7.30}$$

Then from Lemma 2 we have

$$\begin{aligned}
&\varepsilon\mu_O\Big|\frac{\partial}{\partial n_{\mathbf{x}}}\Big\{\mathcal{D}^{(j,O)}(\boldsymbol{\xi}_j)\cdot\nabla_{\mathbf{x}}R^{(\Omega)}(\mathbf{O}^{(j)},\mathbf{y})+\sum_{\substack{k\neq j\\1\le k\le M}}\mathcal{D}^{(k,O)}(\boldsymbol{\xi}_k)\cdot\nabla_{\mathbf{x}}N^{(\Omega)}(\mathbf{O}^{(k)},\mathbf{y})\Big\}\Big|\\
\le\;&\text{const}\;\varepsilon^2,\quad \mathbf{x}\in\partial\Omega,\mathbf{y}\in\omega_\varepsilon^{(j)}\;.
\end{aligned}$$

Therefore, this along with (7.28), (7.29) and (7.30) yield

$$\mu_O\Big|\frac{\partial\mathfrak{h}_\varepsilon^{(j)}}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\Big|\le\text{const}\;\varepsilon^2\;,\quad \mathbf{x}\in\partial\Omega,\mathbf{y}\in\omega_\varepsilon^{(j)}\;.\tag{7.31}$$

Estimate for the right-hand side of (7.24) *on* $\partial\omega_\varepsilon^{(i)}$, $i=1,\ldots,M$. Consider first the situation when $i=j$. Since $R^{(\Omega)}$ is smooth for $\mathbf{x},\mathbf{y}\in\Omega$, we can take the Taylor expansion of its first order derivatives about $\mathbf{x}=\mathbf{O}^{(j)}$ to obtain

$$\begin{aligned}
&\Big|(\mu_{I_j}-\mu_O)\frac{\partial R^{(\Omega)}}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})-\varepsilon\Big\{\mu_{I_j}\frac{\partial\mathcal{D}^{(j)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_j)\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(j)-}}\\
&-\mu_O\frac{\partial\mathcal{D}^{(j)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_j)\big|_{\mathbf{x}\in\partial\omega^{(j)+}}\Big\}\cdot\nabla R^{(\Omega)}(\mathbf{O}^{(j)},\mathbf{y})\Big|\\
=\;&|\mu_{I_j}-\mu_O|\big|\boldsymbol{n}^{(j)}\cdot\nabla_{\mathbf{x}}R^{(\Omega)}(\mathbf{x},\mathbf{y})-\boldsymbol{n}^{(j)}\cdot\nabla_{\mathbf{x}}R^{(\Omega)}(\mathbf{O}^{(j)},\mathbf{y})\big|\\
\le\;&\text{const}\;\varepsilon,\quad \mathbf{x}\in\partial\omega_\varepsilon^{(j)},\mathbf{y}\in\omega_\varepsilon^{(j)}\;,
\end{aligned}\tag{7.32}$$

where the boundary condition (3.9) for the dipole fields of the inclusion $\partial\omega_\varepsilon^{(i)}$ was also used. Then, owing to Lemma 2 for $i=j$ one has

$$\begin{aligned}
&\Big|\varepsilon(\mu_{I_j}-\mu_O)\mathcal{D}^{(j,I)}(\boldsymbol{\eta}_j)\cdot\frac{\partial}{\partial n_{\mathbf{x}}}\nabla_{\mathbf{y}}R^{(\Omega)}(\mathbf{x},\mathbf{O}^{(j)})\\
&-\varepsilon\sum_{\substack{k\neq j\\1\le k\le M}}\Big\{\mu_{I_j}\frac{\partial\mathcal{D}^{(k,O)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_k)\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(j)-}}-\mu_O\frac{\partial\mathcal{D}^{(k,O)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_k)\big|_{\mathbf{x}\in\partial\omega^{(j)+}}\Big\}\cdot\nabla_{\mathbf{x}}R^{(\Omega)}(\mathbf{O}^{(k)},\mathbf{y})\Big|\\
\le\;&\text{const}\;\varepsilon,\quad \mathbf{x}\in\partial\omega_\varepsilon^{(j)},\mathbf{y}\in\omega_\varepsilon^{(j)}\;.
\end{aligned}$$

This and (7.28), (7.29), (7.32) lead to the estimate

$$\Big|\mu_{I_j}\frac{\partial \mathfrak{h}_\varepsilon^{(j)}}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(j)-}}-\mu_O\frac{\partial \mathfrak{h}_\varepsilon^{(j)}}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(j)+}}\Big|\le \text{const}\ \varepsilon,\quad \mathbf{y}\in\omega_\varepsilon^{(j)}\ . \tag{7.33}$$

It remains to consider the case $i\neq j$. In this situation, we require Lemma 4b) to obtain

$$\begin{aligned}
&\Big|\frac{\partial \mathcal{N}^{(j,I)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)+(2\pi\mu_O)^{-1}\frac{\partial}{\partial n_{\mathbf{x}}}(\log(\varepsilon^{-1}|\mathbf{x}-\mathbf{y}|))\\
&+\varepsilon\mathcal{D}^{(j,I)}(\boldsymbol{\eta}_j)\cdot\frac{\partial}{\partial n_{\mathbf{x}}}\nabla_{\mathbf{y}}R^{(\Omega)}(\mathbf{x},\mathbf{O}^{(j)})\Big|\\
=\ &\varepsilon\Big|\mathcal{D}^{(j,I)}(\boldsymbol{\eta}_j)\cdot\Big\{\frac{\partial}{\partial n_{\mathbf{x}}}\nabla_{\mathbf{x}}((2\pi\mu_O)^{-1}\log|\mathbf{x}-\mathbf{O}^{(j)}|^{-1})-\frac{\partial}{\partial n_{\mathbf{x}}}\nabla_{\mathbf{y}}R^{(\Omega)}(\mathbf{x},\mathbf{O}^{(j)})\Big\}\Big|\\
\le\ &\text{const}\ \varepsilon\ ,\quad \mathbf{x}\in\partial\omega_\varepsilon^{(i)},\mathbf{y}\in\omega_\varepsilon^{(j)},i\neq j\ .
\end{aligned} \tag{7.34}$$

The boundary conditions for the dipole fields (3.9) give

$$\begin{aligned}
&\Big|(\mu_{I_i}-\mu_O)\frac{\partial R^{(\Omega)}}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\\
&-\varepsilon\Big\{\mu_{I_i}\frac{\partial \mathcal{D}^{(i)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_i)\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)-}}-\mu_O\frac{\partial \mathcal{D}^{(i)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_i)\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)+}}\Big\}\cdot\nabla_{\mathbf{x}}R^{(\Omega)}(\mathbf{O}^{(i)},\mathbf{y})\Big|\\
\le\ &|\mu_{I_i}-\mu_O||\boldsymbol{n}^{(i)}\cdot\nabla_{\mathbf{x}}R^{(\Omega)}(\mathbf{x},\mathbf{y})-\nabla_{\mathbf{x}}R^{(\Omega)}(\mathbf{O}^{(i)},\mathbf{y})|\\
\le\ &\text{const}\ \varepsilon,\quad \mathbf{x}\in\partial\omega_\varepsilon^{(i)},\mathbf{y}\in\omega_\varepsilon^{(j)},i\neq j\ .
\end{aligned} \tag{7.35}$$

Then, Lemma 2, allows one to deduce

$$\begin{aligned}
&\varepsilon\Big|\Big\{\mu_{I_i}\frac{\partial \mathcal{D}^{(j,O)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_j)\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)-}}-\mu_O\frac{\partial \mathcal{D}^{(j,O)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_j)\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)+}}\Big\}\cdot\nabla_{\mathbf{y}}R^{(\Omega)}(\mathbf{O}^{(j)},\mathbf{y})\\
&+\sum_{\substack{k\neq j\\ k\neq i\\ 1\le k\le M}}\Big\{\mu_{I_i}\frac{\partial \mathcal{D}^{(k,O)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_k)\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)-}}-\mu_O\frac{\partial \mathcal{D}^{(k,O)}}{\partial n_{\mathbf{x}}}(\boldsymbol{\xi}_k)\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)+}}\Big\}\cdot\nabla_{\mathbf{x}}R^{(\Omega)}(\mathbf{O}^{(k)},\mathbf{y})\Big|\\
&\qquad\le \text{const}\ \varepsilon^2,\quad \mathbf{x}\in\partial\omega_\varepsilon^{(i)},\mathbf{y}\in\omega_\varepsilon^{(j)},i\neq j\ .
\end{aligned}$$

Thus, from the preceding inequality and (7.28), (7.29), (7.34), (7.35), we have

$$\Big|\mu_{I_i}\frac{\partial \mathfrak{h}_\varepsilon^{(j)}}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)-}}-\mu_O\frac{\partial \mathfrak{h}_\varepsilon^{(j)}}{\partial n_{\mathbf{x}}}(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}\in\partial\omega_\varepsilon^{(i)+}}\Big|\le \text{const}\ \varepsilon,\quad \mathbf{y}\in\omega_\varepsilon^{(j)},i\neq j\ .$$

We can conclude from this estimate, the conditions (7.25)–(7.27), inequalities (7.31), (7.33), and Lemma 3 that

$$\Big|\mathfrak{h}_\varepsilon^{(j)}(\mathbf{x},\mathbf{y})\Big|\le \text{const}\ \varepsilon,\quad \mathbf{x}\in\bigcup_{l=1}^{M}\omega_\varepsilon^{(l)}\cup\Omega_\varepsilon,\mathbf{y}\in\omega_\varepsilon^{(j)},j=1,\dots,M\ .$$

Finally, the above estimate for $\mathfrak{h}_\varepsilon^{(j)}$, $j = 1, \dots, M$ and (7.22), combined with (7.12) complete the proof. $\square$

8 Numerical simulations

In the current section, we implement the asymptotic formulae derived in Section 7 for Green's function N_ε in numerical simulations. The numerical computations are carried out for the regular part R_ε of Green's function in $\bigcup_l \omega_\varepsilon^{(l)} \cup \Omega_\varepsilon$ for the transmission problem. In other words, let this regular part be given by the formula

$$R_\varepsilon(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi}\left(\frac{\chi_{\Omega_\varepsilon}(\mathbf{y})}{\mu_O} + \sum_{l=1}^{M} \frac{\chi_{\omega_\varepsilon^{(l)}}(\mathbf{y})}{\mu_{I_l}}\right) \log|\mathbf{x}-\mathbf{y}| - N_\varepsilon(\mathbf{x}, \mathbf{y})\ . \tag{8.1}$$

Then in accordance with the boundary value problem for N_ε given in Section 2, the function R_ε, which we choose to consider for our numerical schemes, when $\mathbf{y} \in \Omega_\varepsilon$ is a solution of the problem (7.3)–(7.8).

8.1 Asymptotic formulae for R_ε

From formula (8.1) and Theorem 1, we can immediately state the asymptotic formulae for the regular part R_ε that will be used in the examples below. When $\mathbf{y} \in \Omega_\varepsilon$ we have from (7.1), that R_ε admits the asymptotic representation

$$R_\varepsilon(\mathbf{x}, \mathbf{y}) = R^{(\Omega)}(\mathbf{x}, \mathbf{y}) + \sum_{j=1}^{M} h_N^{(j,O)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j)$$

$$-\varepsilon \sum_{j=1}^{M} \left\{\mathcal{D}^{(j)}(\boldsymbol{\xi}_j) \cdot \nabla_{\mathbf{x}} R^{(\Omega)}(\mathbf{O}^{(j)}, \mathbf{y}) + \mathcal{D}^{(j,O)}(\boldsymbol{\eta}_j) \cdot \nabla_{\mathbf{y}} R^{(\Omega)}(\mathbf{x}, \mathbf{O}^{(j)})\right\} + O(\varepsilon^2)\ . \tag{8.2}$$

When $\mathbf{y} \in \omega_\varepsilon^{(j)}$ where j is fixed, $j = 1, \dots, M$, R_ε has the form

$$R_\varepsilon(\mathbf{x}, \mathbf{y}) = R^{(\Omega)}(\mathbf{x}, \mathbf{y}) + h_N^{(j,I)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) + \sum_{\substack{k \neq j \\ 1 \le k \le M}} h_N^{(k,O)}(\boldsymbol{\xi}_k, \boldsymbol{\eta}_k)$$

$$+(2\pi)^{-1}(\mu_O^{-1} - \mu_{I_j}^{-1}) \log\varepsilon - \varepsilon \mathcal{D}^{(j,I)}(\boldsymbol{\eta}_j) \cdot \nabla_{\mathbf{y}} R^{(\Omega)}(\mathbf{x}, \mathbf{O}^{(j)})$$

$$-\varepsilon \sum_{\substack{k \neq j \\ 1 \le k \le M}} \mathcal{D}^{(k,O)}(\boldsymbol{\eta}_k) \cdot \nabla_{\mathbf{y}} R^{(\Omega)}(\mathbf{x}, \mathbf{O}^{(k)}) - \varepsilon \sum_{p=1}^{M} \mathcal{D}^{(p)}(\boldsymbol{\xi}_p) \cdot \nabla_{\mathbf{x}} R^{(\Omega)}(\mathbf{O}^{(p)}, \mathbf{y}) + O(\varepsilon^2)\ . \tag{8.3}$$

Before proceeding with examples where these formulae will be implemented, we first discuss the numerical settings.

8.2 Numerical settings: Description of the geometry and physical parameters

Let Ω be a disk of radius 150m, with centre at the origin, and occupied by a material with shear modulus $\mu_O = 5.6 \times 10^{10}\text{Nm}^{-2}$, which is that of Cast Iron. We set the number of inclusions $M = 6$, and assume that the $\omega_\varepsilon^{(j)}$, $j = 1, \dots, 6$, are circular. We summarize the data corresponding to the inclusions in Table 1.

Inclusion	Centre	Radius (m)	Shear Modulus ($\times 10^{10}\text{Nm}^{-2}$)	Material
$\omega_\varepsilon^{(1)}$	(-90m, 40m)	27	2.6316	Aluminum
$\omega_\varepsilon^{(2)}$	(-50m,-50m)	24	4.0741	Copper
$\omega_\varepsilon^{(3)}$	(-30m, 10m)	9	7.7519	Iron
$\omega_\varepsilon^{(4)}$	(20m, 70m)	19.5	7.5188	High Strength Alloy Steel
$\omega_\varepsilon^{(5)}$	(50m, 0m)	22.5	8.0078	Steel AISI 4348
$\omega_\varepsilon^{(6)}$	(70m, -80m)	15	9.0496	Nimonic Alloy 90

Table 1: Data for the inclusions $\omega_\varepsilon^{(j)}$, $j = 1, \dots, 6$.

8.3 Model solutions used in the numerical simulations

Neumann's function for the disk Ω

In our examples, we need the Neumann function $N^{(\Omega)}$ for a disk of radius R ($R = 150$m for our demonstrations), which is given by

$$N^{(\Omega)}(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi\mu_O} \log|\mathbf{x} - \mathbf{y}| - \frac{1}{2\pi\mu_O} \log\left(\left| \mathbf{x} - \frac{R^2}{|\mathbf{y}|^2}\mathbf{y} \right| |\mathbf{y}| \right) ,$$

and the regular part $R^{(\Omega)}$ of this function is defined by the formula

$$R^{(\Omega)}(\mathbf{x}, \mathbf{y}) = (2\pi\mu_O)^{-1} \log|\mathbf{x} - \mathbf{y}| - N^{(\Omega)}(\mathbf{x}, \mathbf{y}) .$$

The regular part of Green's function for the transmission problem in a plane with an inclusion at the origin

Now, we state the form of the regular part $h_N^{(j)}$ of the Green's function for the transmission problem in the infinite plane with a circular inclusion at the origin of radius a_j. The solution is constructed using the involution procedure which is discussed in [5]. The representation of this function, is dependent on the position of the point $\boldsymbol{\eta}_j$. When $\boldsymbol{\eta}_j \in C\bar{\omega}^{(j)}$

$$h^{(j,O)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) = -\frac{1}{2\pi\mu_O} \frac{\mu_{I_j} - \mu_O}{\mu_{I_j} + \mu_O} \log\left(\frac{1}{|\boldsymbol{\xi}_j|} \left| \boldsymbol{\xi}_j - \frac{a_j^2}{|\boldsymbol{\eta}_j|^2}\boldsymbol{\eta}_j \right| \right) \quad \text{for} \quad \boldsymbol{\xi}_j \in C\bar{\omega}^{(j)} ,$$

and

$$h^{(j,O)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) = \frac{1}{2\pi\mu_O} \frac{\mu_{I_j} - \mu_O}{\mu_{I_j} + \mu_O} (\log|\boldsymbol{\xi}_j - \boldsymbol{\eta}_j|^{-1} + \log|\boldsymbol{\eta}_j|) \ , \quad \boldsymbol{\xi}_j \in \omega^{(j)} \ .$$

For the case $\boldsymbol{\eta}_j \in \omega^{(j)}$, $\boldsymbol{\xi}_j \in C\bar{\omega}^{(j)}$

$$h^{(j,I)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) = \frac{1}{2\pi} \frac{\mu_{I_j} - \mu_O}{\mu_{I_j} + \mu_O} \left(\frac{1}{\mu_{I_j}} \log|\boldsymbol{\xi}_j - \boldsymbol{\eta}_j| + \frac{1}{\mu_O} \log|\boldsymbol{\xi}_j| \right) \ ,$$

and for $\boldsymbol{\eta}_j, \boldsymbol{\xi}_j \in \omega^{(j)}$ we have

$$h^{(j,I)}(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) = \frac{1}{2\pi} \frac{\mu_{I_j} - \mu_O}{\mu_{I_j} + \mu_O} \left(\frac{1}{\mu_{I_j}} \log \left(\frac{|\boldsymbol{\eta}_j|}{a_j} \left| \boldsymbol{\xi}_j - \frac{a_j^2}{|\boldsymbol{\eta}_j|^2} \boldsymbol{\eta}_j \right| \right) + \frac{1}{\mu_O} \log a_j \right) \ .$$

The dipole fields for the circular inclusion in the infinite plane

Here, we give the vector function $\mathcal{D}^{(j)}$ whose components are the dipole fields for the circular inclusion of radius a_j in the infinite plane and

$$\mathcal{D}^{(j)}(\boldsymbol{\xi}_j) = \chi_{C\bar{\omega}^{(j)}}(\boldsymbol{\xi}_j)\mathcal{D}^{(j,O)}(\boldsymbol{\xi}_j) + \chi_{\omega^{(j)}}(\boldsymbol{\xi}_j)\mathcal{D}^{(j,I)}(\boldsymbol{\xi}_j) \ .$$

We have

$$\mathcal{D}^{(j,O)}(\boldsymbol{\xi}_j) = \frac{\mu_{I_j} - \mu_O}{\mu_{I_j} + \mu_O} \frac{a_j^2 \, \boldsymbol{\xi}_j}{|\boldsymbol{\xi}_j|^2} \ , \quad \text{for } \boldsymbol{\xi}_j \in C\bar{\omega}^{(j)} \ ,$$

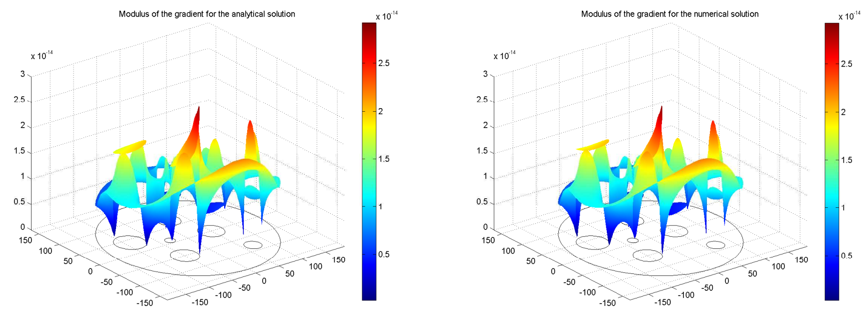
and

$$\mathcal{D}^{(j,I)}(\boldsymbol{\xi}_j) = \frac{\mu_{I_j} - \mu_O}{\mu_{I_j} + \mu_O} \boldsymbol{\xi}_j \ , \quad \text{for } \boldsymbol{\xi}_j \in \omega^{(j)} \ .$$

8.4 Example 1

The case of the force applied outside the inclusions

For our first example, we look at the case when $\mathbf{y} \in \Omega_\varepsilon$. We therefore base our computations on the asymptotic formula (8.2) when comparing with those of COMSOL. The coordinates of the point force are given as $\mathbf{y} = (-10\text{m}, -80\text{m})$. We plot the modulus of the gradient of the regular part R_ε in Figure 2a) according to the analytical formulae (8.2). Figure 2b) is the same quantity computed using the method of finite elements in COMSOL. Both figures are very similar, the maximum absolute error between these computations is 7.666×10^{-16} occurring on the exterior boundary.



a) b)

Figure 2: a) Computations based on asymptotic formula (8.2) and b) Numerical solution for the absolute value of the gradient of the regular part of Green's function for the transmission problem. Here $\mathbf{y} = (-10\text{m}, -80\text{m})$ and the mesh contains 44784 elements. The plots are practically indistinguishable.

8.5 Example 2

The case of the force positioned inside an inclusion

In the second example, we aim to compare the computations produced by formula (8.3) with those generated by COMSOL. Now the point force is assumed to be situated at $\mathbf{y} = (60\text{m}, 0\text{m})$, in the inclusion $\omega_\varepsilon^{(5)}$, containing the Steel AISI 4340. Figure 3a), gives the surface plot for the modulus of the gradient of the regular part, provided by formula (8.3). The numerical solution given in COMSOL is shown in Figure 3b). The maximum absolute error here is 7.98×10^{-16}, which occurs on the boundary of the inclusion $\omega_\varepsilon^{(1)}$. We conclude that the asymptotic formulae and numerical computations are in a good agreement with each other.

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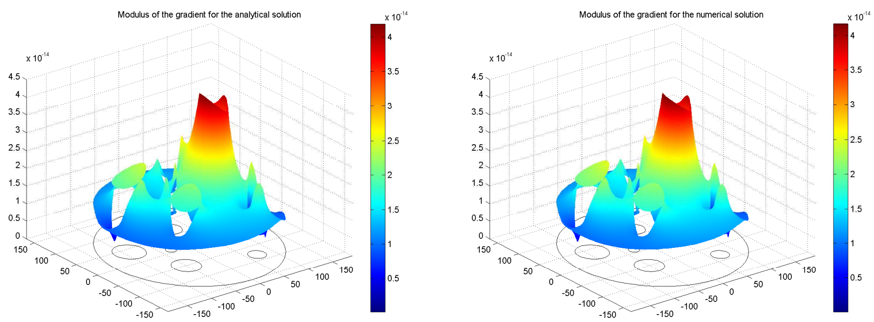

a) b)

Figure 3: a) Computations based on asymptotic formula (8.3) and b) Numerical solution for the absolute value of the gradient of the regular part of Green's function for the transmission problem. Here $\mathbf{y} = (60\text{m}, 0\text{m})$ and lies in the Steel AISI 4340 inclusion. The mesh contains 44784 elements. There is once again a good similarity between the surface plots.

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