Multi-Dimensional Normal Approximation of Heavy-Tailed Moving Averages

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Abstract

In this paper we extend the refined second-order Poincaré inequality in [2] from a one-dimensional to a multi-dimensional setting. Its proof is based on a multivariate version of the Malliavin-Stein method for normal approximation on Poisson spaces. We also present an application to partial sums of vector-valued functionals of heavy-tailed moving averages. The extension we develop is not only in the co-domain of the functional, but also in its domain. Such a set-up has previously not been explored in the framework of stable moving average processes. It can potentially capture probabilistic properties which cannot be described solely by the one-dimensional marginals, but instead require the joint distribution.

Key words: Central limit theorem, heavy-tailed moving average, Lévy process, Malliavin-Stein method, Poisson random measure, second-order Poincaré inequality

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1 Introduction

In recent decades the combination of Malliavin calculus and Stein's method for normal approximation has led to a plethora of Gaussian limit theorems in fields ranging from stochastic geometry, over cosmology to statistics. Classically, the assumptions require third or fourth moment conditions which makes the Malliavin-Stein method unsuitable for distributions with heavier tails. However, in [2] a careful differentiation between small and large values has led to a refined so-called second-order Poincaré inequality for Poisson functionals, which allows to circumvent these difficulties to a certain extent. Based on the approach in [13] the principal goal of this paper is to obtain a multivariate extension of

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the central results in [2]. This opens the possibility to capture properties of the underlying process not accessible solely by the one-dimensional marginal distributions.

We shall now define the heavy-tailed moving average model to which we are going to apply our general multivariate central limit theorem. Let $L = (L_t)_{t \in \mathbb{R}}$ be a two-sided Lévy process with no Gaussian component and Lévy measure ν. We assume that the latter admits a Lebesgue density $w : \mathbb{R} \to \mathbb{R}$ such that

$$|w(x)| \leq C\,|x|^{-1-\beta} \tag{1}$$

for all $x \neq 0$, some $\beta \in (0, 2)$ and a constant $C > 0$. Hence, the distribution of L_1 exhibits β-stable tails, see [19]. Consider then for each $i \in \{1, \ldots, m\}$, $m \in \mathbb{N}$, the process

$$X_t^i := \int_{\mathbb{R}} g_i(t - s)\,\mathrm{d}L_s \qquad (t \in \mathbb{R}) \tag{2}$$

for some measurable function $g_i : \mathbb{R} \to \mathbb{R}$. Necessary and sufficient conditions for the integral to exists are given in [15] and if L is symmetric around zero, i.e., if $-L_1$ and L_1 are identically distributed, then we mention that a sufficient condition is $\int_{\mathbb{R}} |g_i(s)|^\beta\,\mathrm{d}s < \infty$.

The main examples of kernels g_i we consider satisfy a power-law behaviour around zero and at infinity. Henceforth we shall assume for all $i \in \{1, \ldots, m\}$ the existence of a constant $K > 0$ together with exponents $\alpha_i > 0$ and $\kappa_i \in \mathbb{R}$ such that

$$|g_i(x)| \leq K(x^{\kappa_i}\mathbb{1}_{[0,a_i)}(x) + x^{-\alpha_i}\mathbb{1}_{[a_i,\infty)}(x)) \tag{3}$$

for all $x \in \mathbb{R}$, where $a_i > 0$ are suitable splitting points, which may alter the constant K. Without loss of generality we choose $a_i = 1$ for all $i \in \{1, \ldots, m\}$ and let K stand for the corresponding constant. Note that this in particular implies that $g_i(x) = 0$ for $x < 0$, consequently we will only consider *casual* moving averages.

The main objects of interest in this paper are rescaled partial sums of multi-dimensional functionals of the joint distribution $X_s = (X_s^1, \ldots, X_s^m)$, namely

$$V_n(X; f) = \frac{1}{\sqrt{n}} \sum_{s=1}^{n} (f(X_s^1, \ldots, X_s^m) - \mathbb{E}[f(X_0^1, \ldots, X_0^m)]), \tag{4}$$

where $f : \mathbb{R}^m \to \mathbb{R}^d$ is a suitable Borel-measurable function, with d being some positive integer. Observe that $V_n(X; f)$ is a d-dimensional random vector and for convenience we shall denote by $V_n^i(X; f)$ its ith coordinate. We remark that in the one dimensional case $d = m = 1$ the distributional convergence of $V_n(X; f)$, as $n \to \infty$, is studied for general functions f in [1] and here the so-called Appell rank of f is seen to play an important role. The results in that paper also imply that one cannot in general expect convergence in distribution after rescaling with the factor $\sqrt{n}$ as in (4) or a Gaussian limiting distribution if the memory of the processes are too long, i.e., if the α_i are too close to 0. We shall see that if the tails are not too heavy and the memory is not too long, which in our case means that $\alpha_i\beta > 2$, we do in fact have convergence in distribution of $V_n(X; f)$ to a Gaussian random variable and we shall discuss the *speed* of this convergence by considering an appropriate metric on the space of probability laws on $\mathbb{R}^d$, see Section 2 below. To conclude such a result,

2

we could also in principle rely on a multivariate second-order Poincaré inequality for random vectors of Poisson functionals in [18]. But as already observed in the one-dimensional case, the existing bounds are not suitable for the application to Lévy driven moving averages just described. In fact, in this specific situation the bounds in [18] do not even tend to zero, as n increases. Against this background, we will develop in this paper a refined multivariate second-order Poincaré inequality for general random vectors of Poisson functionals, which is more adapted to our situation and allows us to distinguish carefully between small and large values. We believe that this result is of independent interest as well. This eventually paves the way to the central limit theory for the random vectors $V_n(X; f)$.

One motivation for the extension of the theory from [2] to a multivariate set-up is the fact that important properties of random processes, such as self-similarity, are determined by the finite dimensional distributions of X, but not by the one-dimensional marginals. The one-dimensional theory, i.e., the case $m = d = 1$, could so-far capture only probabilistic properties of the distribution of X_1. In the case of the linear fractional stable motion this made the estimation of the Hurst parameter problematic if jointly estimated with the scale parameter and the stability index β, see [9]. We will come back to such applications of the results we develop in this paper in a separate work. However, we would like to mention finally that the case $m = 1$ and general d has been considered in the seminal paper [14].

2 Main results

2.1 A refined multivariate second-order Poincaré inequality

Consider a measurable space $(S, \mathcal{S})$ equipped with a σ-finite measure μ. Let η be a Poisson process on $(S, \mathcal{S})$ with intensity measure μ. This means that η is a collection of random variables of the form $\eta(B)$, $B \in \mathcal{S}$, with the properties that

(i) for each $B \in \mathcal{S}$ with $\mu(B) < \infty$ the random variable $\eta(B)$ is Poisson distributed with mean $\mu(B)$,

(ii) for $m \in \mathbb{N}$ and pairwise disjoint $B_1, \ldots, B_m \in \mathcal{S}$ with $\mu(B_1), \ldots, \mu(B_m) < \infty$ the random variables $\eta(B_1), \ldots, \eta(B_m)$ are independent.

We can and will regard η as a random function from an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathcal{N}$, the space of all integer-valued σ-finite measures on $(S, \mathcal{S})$. The set $\mathcal{N}$ is equipped with the evaluation σ-algebra, i.e. the σ-algebra generated by the evaluation mappings $\mu \mapsto \mu(A)$, $A \in \mathcal{S}$.

To each Poisson process η we associate the Hilbert space $\mathcal{L}_\eta^2(\mathbb{P})$ consisting of all square integrable Poisson functionals F, i.e., those random variables for which there exists a function $\phi : \mathcal{N} \to \mathbb{R}$ such that almost surely $F = \phi(\eta) \in \mathcal{L}^2(\mathbb{P})$. Finally, we introduce the notion of the Malliavin derivative in a Poisson setting, which is also known as the add-one-cost operator. For each $z \in S$ and $F = \phi(\eta) \in \mathcal{L}_\eta^2(\mathbb{P})$ we define $D_z F$ as

$$D_z F := \phi(\eta + \delta_z) - \phi(\eta)$$

and note that DF is a bi-measurable map from $\Omega \times S$ to $\mathbb{R}$. In a straightforward way this

definition extends to vector-valued Poisson functionals. Indeed, consider $F = (F_1, \ldots, F_d)$ where each F_i lies in $\mathcal{L}^2_\eta(\mathbb{P})$, then the Malliavin derivative $D_z F$ at $z \in S$ is given by

$$D_z F = (D_z F_1, \ldots, D_z F_d).$$

Similarly to $D_z F$ we may introduce the iterated Malliavin derivative $D^2 F$ of F by putting

$$D^2_{z_1, z_2} F := D_{z_1}(D_{z_2} F) = D_{z_2}(D_{z_1} F), \qquad z_1, z_2 \in S.$$

For further background material on Poisson processes we refer to the treatments in [4, 6, 12], for the Malliavin formalism on Poisson spaces we refer to Section 3.1 below.

To measure the distance between (the laws of) two random vectors X and Y taking values in $\mathbb{R}^d$ we use the so-called d_3-distance, see [13]. To introduce it, assume that $\mathbb{E}[\|X\|^2_{\mathbb{R}^d}], \mathbb{E}[\|Y\|^2_{\mathbb{R}^d}] < \infty$, where $\|\cdot\|_{\mathbb{R}^d}$ stands for the Euclidean norm in $\mathbb{R}^d$. The d_3-distance between (the laws of the) random vectors X and Y, denoted by $d_3(X, Y)$, is given by

$$d_3(X, Y) := \sup_{\varphi \in \mathcal{H}_3} \left| \mathbb{E}[\varphi(X)] - \mathbb{E}[\varphi(Y)] \right|,$$

where the class $\mathcal{H}_3$ of test functions indicates the collection of all thrice differentiable functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ (i.e., $\varphi \in \mathcal{C}^3(\mathbb{R}^d, \mathbb{R})$) such that $\|\varphi''\|_\infty \leq 1$ and $\|\varphi'''\|_\infty \leq 1$, where

$$\|\varphi''\|_\infty := \max_{1 \leq i, j \leq d} \sup_{x \in \mathbb{R}^d} \left| \frac{\partial^2}{\partial x_i \partial x_j} \varphi(x) \right|, \qquad \|\varphi'''\|_\infty := \max_{1 \leq i, j, k \leq d} \sup_{x \in \mathbb{R}^d} \left| \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} \varphi(x) \right|.$$

We can now formulate our multivariate second-order Poincaré inequality, which generalises [2, Theorem 3.1] and refines [18, Theorem 1.1]. Its proof, which is given in Section 4 below, is based on the Malliavin-Stein technique for normal approximation of random vectors of Poisson functionals. For two Poisson functionals $F, G \in \mathcal{L}^2_\eta(\mathbb{P})$ we define the quantities

$$\gamma_1^2(F, G) := 3 \int_{S^3} \mathbb{E}\big[(D^2_{z_1, z_3} F)^2 (D^2_{z_2, z_3} F)^2\big]^{1/2} \mathbb{E}\big[(D_{z_1} G)^2 (D_{z_2} G)^2\big]^{1/2} \mu^3(\mathrm{d}z_1, \mathrm{d}z_2, \mathrm{d}z_3),$$

$$\gamma_2^2(F, G) := \int_{S^3} \mathbb{E}\big[(D^2_{z_1, z_3} F)^2 (D^2_{z_2, z_3} F)^2\big]^{1/2} \mathbb{E}\big[(D^2_{z_1, z_3} G)^2 (D^2_{z_2, z_3} G)^2\big]^{1/2} \mu^3(\mathrm{d}z_1, \mathrm{d}z_2, \mathrm{d}z_3).$$

Moreover, for $x, y \in \mathbb{R}$ we denote by $x \wedge y = \min\{x, y\}$ the minimum of x and y.

Theorem 2.1 *Let $d \geq 1$ and assume that $F_1, \ldots, F_d \in \mathcal{L}^2_\eta(\mathbb{P})$ satisfy $DF_i \in \mathcal{L}^2(\mathbb{P} \otimes \mu)$ and $\mathbb{E}[F_i] = 0$ for all $i \in \{1, \ldots, d\}$. Let $\sigma_{ik} := \mathbb{E}[F_i F_k]$ and define the covariance matrix $\Sigma^2 = (\sigma_{ik})_{i,k=1}^d$. Let $Y \sim N_d(0, \Sigma^2)$ be a centred Gaussian random vector with covariance matrix Σ^2 and put $F := (F_1, \ldots, F_d)$. Then*

$$d_3(F, Y) \leq \sum_{i,k=1}^d \big(\gamma_1(F_i, F_k) + \gamma_2(F_i, F_k)\big) + \gamma_3,$$

where the term γ_3 is defined as

$$\gamma_3 := \sum_{i,j,k=1}^d \int_S \mathbb{E}\big[|D_z F_j D_z F_k|^{3/2} \wedge \|D_z F\|^{3/2}_{\mathbb{R}^d}\big]^{2/3} \mathbb{E}\big[|D_z F_i|^3\big]^{1/3} \mu(\mathrm{d}z). \tag{5}$$

4

Remark 2.2

(i) The difference between Theorem 2.1 and [18, Theorem 1.1] lies in the term γ_3. We emphasise that the bound in [18] does not lead to a meaningful error bound in the application to heavy-tailed moving averages we consider in the next section as the corresponding γ_3-term in [18] would diverge. Similarly to the univariate case, the bound provided by Theorem 2.1 is much more suitable for our purposes as it leads to a reasonable error bound, which tends to zero, as the number of observations n there tends to infinity.

(ii) It is in principal possible to derive error bounds as in Theorem 2.1 for probability metrics different from the d_3-metric. Namely, assuming in addition that the covariance matrix Σ^2 is *positive definite*, one can deal with the d_2-distance used in [13] and even with the convex distance introduced and studied in [18]. Since the corresponding error bounds for these notions of distance become rather long and technical, we refrain from presenting results in this direction. Moreover, in our application in the next section it seems in general rather difficult to check whether or not the covariance matrix is positive definite. This is another reason for us considering only the d_3-distance.

(iii) We would like to point out that quantitative central limit theorems for random vectors of Poisson functionals having a *finite* Wiener-Itô chaos expansion with respect to the d_3-distance were obtained [7]. Specifically, random vectors of so-called Poisson U-statistics were considered in [7] together with applications in stochastic geometry to Poisson process of k-dimensional flat in $\mathbb{R}^n$.

2.2 Asymptotic normality of multivariate heavy-tailed moving averages

Here, we present our application of the refined multivariate second-order Poincaré inequality formulated in the previous section. For this recall the set-up described in the introduction. Especially, recall the definition of the random processes $(X_t^i)_{t \in \mathbb{R}}$, $i \in \{1, \ldots, m\}$ from (2). Also recall that the exponents α_i control the memory of the processes X^i. Given the limit theory for heavy-tailed moving averages as developed in [8] it comes as no surprise that the smallest such α_i will be of dominating importance. Hence, we define

$$\underline{\alpha} = \min\{\alpha_1, \ldots, \alpha_m\}.$$

Finally, by $\mathcal{C}_b^2(\mathbb{R}^m, \mathbb{R}^d)$ we denote the space of bounded functions $f : \mathbb{R}^m \to \mathbb{R}^d$ which are twice continuously differentiable and have all partial derivatives up to order two bounded by some constant.

Theorem 2.3 *Fix $d, m \geq 1$. Let (X_t^i), $i = 1, \ldots, m$, be moving averages as in (2) with Lévy measure having density w satisfying (1) for some $\beta \in (0, 2)$ and kernels g_i which satisfy (3) with $\alpha_i \beta > 2$ and $\kappa_i > -1/\beta$. Let a function $f = (f_1, \ldots, f_d) \in \mathcal{C}_b^2(\mathbb{R}^m, \mathbb{R}^d)$ be given and consider $V_n(X; f)$ as at (4) based on f and $X = (X^1, \ldots, X^m)$. Let $\Sigma_n = \mathrm{Cov}(V_n(X; f))^{1/2}$ denote a non-negative definite square root of the covariance matrix $\mathrm{Cov}(V_n(X; f))$ of the d-dimensional random vector $V_n(X; f)$. Then $\Sigma_n \to \Sigma = (\Sigma_{i,j})_{i,j=1}^d$, as $n \to \infty$, where, for*

$i, j \in \{1, \ldots, d\}$,

$$\Sigma_{i,j}^2 = \sum_{s=0}^{\infty} \mathrm{Cov}(f_i(X_s^1, \ldots, X_s^m), f_j(X_0^1, \ldots, X_0^m))$$
$$+ \sum_{s=1}^{\infty} \mathrm{Cov}(f_i(X_0^1, \ldots, X_0^m), f_j(X_s^1, \ldots, X_s^m)). \tag{6}$$

Moreover, $V_n(X; f)$ converges in distribution, as $n \to \infty$, to a d-dimensional centred Gaussian random vector $Y \sim N_d(0, \Sigma^2)$ with covariance matrix Σ^2. More precisely, there exists a constant $C > 0$ such that

$$d_3(V_n(X; f), Y) \leq C \begin{cases} n^{-1/2}, & \text{if } \underline{\alpha}\beta > 3, \\ n^{-1/2} \log(n), & \text{if } \underline{\alpha}\beta = 3, \\ n^{(2-\underline{\alpha}\beta)/2}, & \text{if } 2 < \underline{\alpha}\beta < 3. \end{cases}$$

Remark 2.4

(i) We remark that in the special case $d = m = 1$ the order for the d_3-distance provided by Theorem 2.1 is precisely the same as that for the Wasserstein distance in [2].

(ii) Even for particular functions $f = (f_1, \ldots, f_d)$, such as trigonometric functions, it seems to be a rather demanding task to check whether the covariance matrix Σ^2 is positive definite or not. Note in this context that even in the one-dimensional case $d = m = 1$ the question of whether the asymptotic variance constant is strictly positive or not is generally difficult. This is the reason why we are working with the d_3-distance in this paper, since more refined probability metric usually require positive definiteness of the covariance matrix, see Remark 2.2.

(iii) It is straightforward to modify the proof of Theorem 2.3 to the situation where $X = (X_1, \ldots, X_m)$ for some fixed moving average $(X_t)_{t \in \mathbb{R}}$ as in (2) and where the kernel g satisfy
$$|g(x)| \leq K(x^{\kappa} \mathbb{1}_{[0,a)}(x) + x^{-\alpha} \mathbb{1}_{[a,\infty)}(x))$$
for some constants $a, \alpha, K > 0$ and $\kappa \in \mathbb{R}$ such that $\alpha\beta > 2$ and $\kappa > -1/\beta$. In this case the kernel of $X^i = X_i$ is simply $g_i = g(i + \cdot)$. Choosing an appropriate functional f in $V_n(X; f)$, such as the empirical characteristic function of X, opens up the possibility of inference on $(X_t)_{t \in \mathbb{R}}$ based on not only the marginal distribution X_1 as in much of the previous literature, but also on the joint distribution $(X_1, \ldots, X_m)$.

As in [2], Theorem 2.1 can be applied to particular processes (X_t^i). We mention here the linear fractional stable noises, which may be regarded as heavy-tailed extensions of a fractional Brownian motion. Let L be a β-stable Lévy process with $\beta \in (0, 2)$ and put

$$X_t^i := Y_t - Y_{t-1} \qquad \text{for} \qquad Y_t^i := \int_{-\infty}^t \left[(t-s)_+^{H_i - 1/\beta} - (-s)_+^{H_i - 1/\beta}\right] \mathrm{d}L_s,$$

where $H_1, \ldots, H_m \in (0, 1)$ (if $\beta = 1$ we additionally suppose that L is symmetric). In this case, $\alpha_i = 1 - H_i + 1/\beta$ for all $i \in \{1, \ldots, m\}$ and the condition $\underline{\alpha}\beta > 2$ translates into

$\beta \in (1,2)$ and $\max\{H_1, \ldots, H_m\} < 1 - \frac{1}{\beta}$. Note that since $\beta > 1$ we automatically have that $\underline{\alpha}\beta < 3$. In this set-up the bound in Theorem 2.1 reads as follows:

$$d_3(V_n(X;f),Y) \leq C\, n^{1/2-\beta(1-\max\{H_1,\ldots,H_m\})/2}.$$

As a second application we mention a stable Ornstein–Uhlenbeck process. Again, for a β-stable Lévy process L with $\beta \in (0,2)$ define for $i \in \{1, \ldots, m\}$,

$$X_t^i := \int_{-\infty}^t e^{-\lambda_i(t-s)} \,\mathrm{d}L_s,$$

where $\lambda_1, \ldots, \lambda_m > 0$. In this case, the parameters $\alpha_1, \ldots, \alpha_m$ may be arbitrary and the error bound in Theorem 2.1 reduces to

$$d_3(V_n(X;f),Y) \leq C\, n^{-1/2}.$$

In a similar spirit, one my consider multivariate quantitative central limit theorems for functionals of linear fractional Lévy noises or of stable fractional ARIMA processes, see [2] for the corresponding one-dimensional situations.

3   Background material

3.1   Malliavin calculus on Poisson spaces

To take advantage of the powerful Malliavin-Stein method we need to recall some background material regarding the Malliavin formalism on Poisson spaces. For further details we refer to [4, 6, 11].

Throughout this section η denotes a Poisson process with intensity measure μ defined on some measurable space $(S, \mathcal{S})$ and over some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We start by recalling that any $F \in \mathcal{L}_\eta^2(\mathbb{P})$ admits a chaos expansion (with convergence in $\mathcal{L}^2(\mathbb{P})$). That is,

$$F = \sum_{n=0}^{\infty} I_n(f_n), \tag{7}$$

where I_n denotes the nth order Wiener-Itô integral with respect to the compensated Poisson process $\eta - \mu$ and the kernels $f_n \in \mathcal{L}^2(\mu^n)$ are symmetric functions (i.e., they are invariant under permutations of its variables). Especially, $I_0(c) = c$ for all $c \in \mathbb{R}$.

The Kabanov-Skorohod integral δ is defined for a subclass of random processes $u \in \mathcal{L}^2(\mathbb{P} \otimes \mu)$ having chaotic decomposition

$$u(z) = \sum_{n=0}^{\infty} I_n(h_n(\,\cdot\,, z)),$$

where for each $z \in S$ the function $h_n(\,\cdot\,, z)$ is symmetric and belongs to $\mathcal{L}^2(\mu^n)$. Denoting by $\tilde{h}$ the canonical symmetrisation of a function $h : S^n \to \mathbb{R}$, i.e.,

$$\tilde{h}(z_1, \ldots, z_n) = \frac{1}{n!} \sum_{\sigma \in S_n} h(z_{\sigma(1)}, \ldots, z_{\sigma(n)}),$$

7

with S_n being the group of all permutations of $\{1,\dots,n\}$, we put

$$\delta(u) := \sum_{n=0}^{\infty} I_{n+1}(\tilde{h}_n),$$

whenever $\sum_{n=0}^{\infty}(n+1)\,\|\tilde{h}_n\|^2_{\mathcal{L}^2(\mu^{n+1})} < \infty$ (we indicate this by writing $u \in \operatorname{dom}\delta$), where $\|\cdot\|_{\mathcal{L}^2(\mu^{n+1})}$ denotes the usual $\mathcal{L}^2$-norm with respect to μ^{n+1}.

Next, we shall define the two operators $L : \operatorname{dom}L \to \mathcal{L}^2_{\eta}(\mathbb{P})$ and $L^{-1} : \mathcal{L}^2_{\eta}(\mathbb{P}) \to \mathcal{L}^2_{\eta}(\mathbb{P})$, where $\operatorname{dom}L$ denotes the class of Poisson functionals $F \in \mathcal{L}^2_{\eta}(\mathbb{P})$ with chaos expansion as in (7) satisfying $\sum_{n=1}^{\infty} n^2 n!\, \|f_n\|^2_{\mathcal{L}^2(\mu^n)} < \infty$. Then, we define

$$LF := -\sum_{n=1}^{\infty} n I_n(f_n).$$

Similarly, the pseudo-inverse L^{-1} of L acts on centred $F \in \mathcal{L}^2_{\eta}(\mathbb{P})$ with chaotic expansion (7) as follows:

$$L^{-1}F := \sum_{n=1}^{\infty} \frac{1}{n} I_n(f_n).$$

Finally, we recall that for $F \in \mathcal{L}^2_{\eta}(\mathbb{P})$ with chaotic expansion (7) satisfying $\sum_{n=0}^{\infty}(n+1)!\,\|f_n\|^2_{\mathcal{L}^2(\mu^n)} < \infty$ the Malliavin derivative admits the representation

$$D_z F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\,\cdot\,,z)), \qquad z \in S.$$

Using these definitions and representations, one may prove the following crucial formulas and relationships of Malliavin calculus, which also play a prominent role in our approach:

(1) $LL^{-1}F = F$ if F is centred.

(2) $LF = -\delta DF$ for $F \in \operatorname{dom}L$.

(3) $\mathbb{E}[F\delta(u)] = \mathbb{E}[\int_S (D_z F)u(z)\,\mu(\mathrm{d}z)]$, when $u \in \operatorname{dom}\delta$.

3.2 Multivariate normal approximation by Stein's method

Stein's method for multivariate normal approximation is a powerful device to prove quantitative multivariate central limit theorems. The proof of Theorem 2.1 is based on the following result, which is known as Stein's Lemma (see [10, Lemma 4.1.3]). To present it, let us recall that the Hilbert-Schmidt inner product between two $d \times d$ matrices $A = (a_{ik})$ and $B = (b_{ik})$ is defined as

$$\langle A, B\rangle_{\mathrm{HS}} = \operatorname{Tr}(B^{\top}A) = \sum_{i,k=1}^{d} b_{ki}a_{ki}.$$

Moreover, for a differentiable function $\varphi : \mathbb{R}^d \to \mathbb{R}$ we shall write $\nabla\varphi$ for the gradient and $\nabla^2\varphi$ for the Hessian of φ. Also, we let $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{R}^d}$ denote the Euclidean scalar product in $\mathbb{R}^d$.

8

Lemma 3.1 (Stein's Lemma) *Let $\Sigma^2 \in \mathbb{R}^{d \times d}$ be a positive semi-definite matrix and Y be a d-dimensional random vector. Then $Y \sim N_d(0, \Sigma^2)$ if and only if it for all twice continuously differentiable functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ with bounded derivatives one has that*

$$\mathbb{E}[\langle Y, \nabla\varphi(Y)\rangle_{\mathbb{R}^d} - \langle \Sigma^2, \nabla^2\varphi(Y)\rangle_{\mathrm{HS}}] = 0.$$

4   Proof of Theorem 2.1

By definition of the d_3-distance we need to prove that

$$|\mathbb{E}[\varphi(Y)] - \mathbb{E}[\varphi(F)]| \leq \sum_{i,k=1}^{d} (\gamma_1(F_i, F_k) + \gamma_2(F_i, F_k)) + \gamma_3$$

for every function $\varphi \in \mathcal{H}_3$. For this, we may assume that Y and F are independent. We start out by applying the interpolation technique already demonstrated in [13]. Consider the function $\Psi : [0,1] \to \mathbb{R}$ given by

$$\Psi(t) := \mathbb{E}[\varphi(\sqrt{1-t}F + \sqrt{t}Y)], \qquad t \in [0,1].$$

Note that from the mean value theorem it follows that

$$|\mathbb{E}[\varphi(Y)] - \mathbb{E}[\varphi(F)]| = |\Psi(1) - \Psi(0)| \leq \sup_{t \in (0,1)} |\Psi'(t)|.$$

Hence it is enough to consider Ψ', which is given by

$$\Psi'(t) = \mathbb{E}\big[\langle \nabla\varphi(\sqrt{1-t}F + \sqrt{t}Y), \tfrac{1}{2\sqrt{t}}Y - \tfrac{1}{2\sqrt{1-t}}F\rangle_{\mathbb{R}^d}\big] =: \frac{1}{2\sqrt{t}}T_1 - \frac{1}{2\sqrt{1-t}}T_2.$$

We consider the two terms T_1 and T_2 separately. For T_1 it follows first by independence of F and Y and Stein's Lemma (used on the function $y \mapsto \varphi(\sqrt{1-t}a + \sqrt{t}y)$) and then dividing by $\sqrt{t}$ that

$$\begin{aligned}
T_1 &= \mathbb{E}[\langle \nabla\varphi(\sqrt{1-t}F + \sqrt{t}Y), Y\rangle_{\mathbb{R}^d}] \\
&= \mathbb{E}\big[\mathbb{E}[\langle \nabla\varphi(\sqrt{1-t}a + \sqrt{t}Y), Y\rangle_{\mathbb{R}^d}]\,|_{a=F}\big] \\
&= \sqrt{t}\,\mathbb{E}\big[\mathbb{E}[\langle \Sigma^2, \nabla^2\varphi(\sqrt{1-t}a + \sqrt{t}Y)\rangle_{\mathrm{HS}}]\,|_{a=F}\big].
\end{aligned}$$

Let $\partial_i f$ denote the derivative of f in the ith coordinate. We have by independence of F and Y and the Malliavin rules (1)–(3) rephrased at the end of Section 3.1 that

$$T_2 = \mathbb{E}[\langle \nabla\varphi(\sqrt{1-t}F + \sqrt{t}Y), F\rangle_{\mathbb{R}^d}] = \sum_{i=1}^{d} \mathbb{E}\big[\mathbb{E}[\partial_i\varphi(\sqrt{1-t}F + \sqrt{t}a)F_i]\,|_{a=Y}\big]$$

$$= \sum_{i=1}^{d} \mathbb{E}\big[\mathbb{E}[\partial_i\varphi(\sqrt{1-t}F + \sqrt{t}a)L(L^{-1}F_i)]\,|_{a=Y}\big]$$

$$= -\sum_{i=1}^{d} \mathbb{E}\big[\mathbb{E}[\partial_i\varphi(\sqrt{1-t}F + \sqrt{t}a)\delta(DL^{-1}F_i)]\,|_{a=Y}\big]$$

$$= \sum_{i=1}^{d} \mathbb{E}\big[\mathbb{E}[\langle D\partial_i\varphi(\sqrt{1-t}F + \sqrt{t}a), -DL^{-1}(F_i)\rangle_{\mathcal{L}^2(\mu)}]\,|_{a=Y}\big].$$

Consider now the function $\varphi_i^{t,a} : \mathbb{R}^d \to \mathbb{R}$ defined by

$$\varphi_i^{t,a}(x) := \partial_i\varphi(\sqrt{1-t}x + \sqrt{t}a).$$

By Taylor expansion we can write

$$D_z\varphi_i^{t,a}(F) = \sum_{k=1}^{d} \partial_k\varphi_i^{t,a}(F)(D_zF_k) + R_i^a(D_zF)$$

for any $z \in \mathbb{R}^d$, where the remainder term $R_i^a(D_zF) = \sum_{j,k=1}^{d} R_{i,j,k}^a(D_zF_k, D_zF_j)$ satisfies the estimate

$$
\begin{aligned}
|R_{i,j,k}^a(x,y)| &\leq \frac{1}{2}\,|xy| \max_{k,l} \sup_{x\in\mathbb{R}^d} \big|\partial_{k,l}\varphi_i^{t,a}(x)\big| \\
&\leq \frac{1}{2}\,|xy|(1-t) \max_{k,l} \sup_{x\in\mathbb{R}^d} \big|\partial_{i,k,l}\varphi(\sqrt{1-t}x + \sqrt{t}a)\big| \qquad (8)\\
&\leq \frac{1}{2}(1-t)\,|xy|.
\end{aligned}
$$

Here, we have used the definition of the class $\mathcal{H}_3$. On the other hand, the remainder term also satisfies the inequality

$$
\begin{aligned}
\Big|D_z\varphi_i^{t,a}(F) - \sum_{k=1}^{d} \partial_k\varphi_i^{t,a}(F)(D_zF_k)\Big| &\leq |D_z\varphi_i^{t,a}(F)| + |\langle\nabla\varphi_i^{t,a}(F), D_zF\rangle_{\mathbb{R}^d}| \\
&\leq 2\,\|\nabla\varphi_i^{t,a}(F)\|_{\mathbb{R}^d}\,\|D_zF\|_{\mathbb{R}^d} \qquad (9)\\
&\leq 2\sqrt{1-t}\,\|D_zF\|_{\mathbb{R}^d},
\end{aligned}
$$

where we used again the mean value theorem and the Cauchy-Schwarz inequality. We may thus rewrite T_2 as

$$
\begin{aligned}
T_2 &= \sum_{i,k=1}^{d} \mathbb{E}\big[\mathbb{E}[\langle\partial_k\varphi_i^{t,a}(F)(DF_k), -DL^{-1}(F_i)\rangle_{\mathcal{L}^2(\mu)}]\,|_{a=Y}\big] \\
&\quad + \sum_{i=1}^{d} \mathbb{E}\big[\mathbb{E}[\langle R_i^a(DF), -DL^{-1}(F_i)\rangle_{\mathcal{L}^2(\mu)}]\,|_{a=Y}\big] \\
&= \sqrt{1-t}\sum_{i,k=1}^{d} \mathbb{E}\big[\partial_{k,i}\varphi(\sqrt{1-t}F + \sqrt{t}Y)\langle DF_k, -DL^{-1}(F_i)\rangle_{\mathcal{L}^2(\mu)}\big]
\end{aligned}
$$

10

$$+ \sum_{i=1}^{d} \mathbb{E}\big[\mathbb{E}[\langle R_i^a(DF), -DL^{-1}(F_i)\rangle_{\mathcal{L}^2(\mu)}]\,|_{a=Y}\big].$$

From this together with the Cauchy-Schwarz inequality and the bounds (8) and (9) it follows that

$$|\mathbb{E}[\varphi(Y)] - \mathbb{E}[\varphi(F)]| \leq \sup_{t \in (0,1)} |\Psi'(t)|$$

$$\leq \sup_{t \in (0,1)} \frac{1}{2} \sum_{i,k=1}^{d} \mathbb{E}\Big[\big|\partial_{i,k}\varphi(\sqrt{1-t}F + \sqrt{t}X)\big|\,\big|\sigma_{ik} - \langle DF_k, -DL^{-1}(F_i)\rangle_{\mathcal{L}^2(\mu)}\big|\Big]$$

$$+ \sup_{t \in (0,1)} \frac{1}{2\sqrt{1-t}} \sum_{i=1}^{d} \mathbb{E}\big[|\langle R_i^a(DF), -DL^{-1}(F_i)\rangle_{\mathcal{L}^2(\mu)}|\,|_{a=Y}\big]$$

$$\leq \frac{1}{2} \sum_{i,k=1}^{d} \mathbb{E}\big[|\sigma_{ik} - \langle DF_k, -DL^{-1}F_i\rangle_{\mathcal{L}^2(\mu)}|\big]$$

$$+ \sum_{i,j,k=1}^{d} \int_S \mathbb{E}[(|D_z F_j D_z F_k| \wedge \|D_z F\|_{\mathbb{R}^d})\,|D_z L^{-1}F_i|]\,\mu(\mathrm{d}z).$$

Applying now Proposition 4.1 in [5] to the first of these terms yields the inequality

$$\sum_{i,k=1}^{d} \mathbb{E}\Big[\big|\sigma_{ik} - \langle DF_k, -DL^{-1}F_i\rangle_{\mathcal{L}^2(\mu)}\big|\Big] \leq 2 \sum_{i,k=1}^{d} (\gamma_{1,i,k} + \gamma_{2,i,k}).$$

For the remainder term we deduce by Hölder's inequality with exponents 3 and 3/2 that

$$\int_S \mathbb{E}\big[(|D_z F_j D_z F_k| \wedge \|D_z F\|_{\mathbb{R}^d})\,|D_z L^{-1}F_i|\big]\,\mu(\mathrm{d}z)$$

$$\leq \int_S \mathbb{E}\big[(|D_z F_j D_z F_k| \wedge \|D_z F\|_{\mathbb{R}^d})^{3/2}\big]^{2/3} \mathbb{E}\big[|D_z L^{-1}F_i|^3\big]^{1/3}\,\mu(\mathrm{d}z)$$

$$\leq \int_S \mathbb{E}\big[|D_z F_j D_z F_k|^{3/2} \wedge \|D_z F\|_{\mathbb{R}^d}^{3/2}\big]^{2/3} \mathbb{E}\big[|D_z F_i|^3\big]^{1/3}\,\mu(\mathrm{d}z),$$

where we also used the contraction inequality $\mathbb{E}[|D_z L^{-1}F_i|^p] \leq \mathbb{E}[|D_z F_i|^p]$ from [5, Lemma 3.4], which is valid for all $p \geq 1$ and $z \in \mathbb{R}^d$. This completes the proof of Theorem 2.1. $\qquad\square$

5   Proof of Theorem 2.3

In order to apply Theorem 2.1 we need to ensure that the processes (X_t^i) can be represented in terms of a Poisson process. Indeed, following [16] and [2] we can represent X^i as the integral

$$X_t^i = \int_{\mathbb{R}^2} g_i(t-s)x\big(\eta(\mathrm{d}s, \mathrm{d}x) - \tau(g_i(t-s)x)\,\mathrm{d}s\,\nu(\mathrm{d}x)\big) + \tilde{b}_i,$$

11

with

$$\tilde{b}_i := \int_{\mathbb{R}} \left(g_i(s)b + \int_{\mathbb{R}} (\tau(xg_i(s)) - g_i(s)\tau(x)) \, \nu(\mathrm{d}x) \right) \mathrm{d}s,$$

and where η is a Poisson process on $\mathbb{R}^2$ with intensity measure $\mu(\mathrm{d}s, \mathrm{d}x) := \mathrm{d}s \, \nu(\mathrm{d}x)$. Here, ν is the Lévy measure of L, b the shift parameter in the characteristic triple for L_1 and τ is a truncation function, cf. (8.3)–(8.4) in [17].

In what follows, C will denote a strictly positive constant whose value might change from occasion to occasion.

5.1 Estimating the Malliavin derivative

We start out by deriving simple estimates on the Malliavin derivative. By definition of the terms γ_1, γ_2, γ_3 introduced in Section 2.1 it is sufficient to consider the Malliavin derivatives of each of the coordinates of $f = (f_1, \ldots, f_d)$ separately. So, let $i \in \{1, \ldots, d\}$ and $z_j = (x_j, t_j) \in \mathbb{R}^2$ for $j \in \{1, 2\}$ be given. Define for $z = (x, t) \in \mathbb{R}^2$ the vector $\delta_s(z)$, for $s \in \mathbb{R}$, as

$$\delta_s(z) := x(g_1(s-t), \ldots, g_m(s-t)) \in \mathbb{R}^m. \tag{10}$$

The mean value theorem together with the Cauchy-Schwarz inequality and the assumption that $f_i \in \mathcal{C}_b^2(\mathbb{R}^m, \mathbb{R})$ then yield the existence of a constant $C > 0$ such that

$$\begin{aligned} |D_{z_1} f_i(X_s^1, \ldots, X_s^m)| &= |f_i((X_s^1, \ldots, X_s^m) + \delta_s(z_1)) - f_i(X_s^1, \ldots, X_s^m)| \\ &\leq C(1 \wedge \|\delta_s(z_1)\|_{\mathbb{R}^m}). \end{aligned} \tag{11}$$

Similarly, we deduce again by the mean value theorem and boundedness of f_i and its derivatives the following inequality for the iterated Malliavin derivative:

$$\begin{aligned} |D_{z_1,z_2}^2 f_i(X_s^1, \ldots, X_s^m)| &= \big|f_i((X_s^1, \ldots, X_s^m) + \delta_s(z_1) + \delta_s(z_2)) \\ &\quad - f_i((X_s^1, \ldots, X_s^m) + \delta_s(z_1)) \\ &\quad - f_i((X_s^1, \ldots, X_s^m) + \delta_s(z_2)) + f_i(X_s^1, \ldots, X_s^m)\big| \\ &\leq C(1 \wedge \|\delta_s(z_1)\|_{\mathbb{R}^m})(1 \wedge \|\delta_s(z_2)\|_{\mathbb{R}^m}). \end{aligned} \tag{12}$$

Note that the estimates (11) and (12) are purely deterministic and allow us to replace stochastic terms by deterministic estimates of the underlying kernels. This confirms in another context that many properties of moving averages can be deduced solely from the driving spectral density, see, for example, [3].

5.2 Analysing the asymptotic covariance matrix

Define for each $k \in \mathbb{Z}$ and $i, j \in \{1, \ldots, m\}$ the integral

$$\rho_{i,j,k} := \int_{\mathbb{R}} |g_i(x)g_j(x+k)|^{\beta/2} \, \mathrm{d}x \tag{13}$$

and observe that $\rho_{i,j,k} = \rho_{j,i,-k}$. Now, $\rho_{i,j,k}$ is closely related to the asymptotic covariances, which motivates the following technical lemma, which in turn leads to our assumption

that $\alpha_i\beta > 2$ for any $i \in \{1, \ldots, m\}$. In what follows we write $x \vee y := \max\{x, y\}$ for the maximum of $x, y \in \mathbb{R}$.

Lemma 5.1 *Let $k \in \mathbb{N}$ and $i, j \in \{1, \ldots, m\}$. Then there is a constant $C > 0$ such that*

$$\rho_{i,j,k} \leq Ck^{-(\alpha_i \wedge \alpha_j)\beta/2}.$$

Proof. The same technique as in the proof of [2, Lemma 4.1] yields the bound $\rho_{i,j,k} \leq Ck^{-\alpha_j\beta/2}$ and to obtain a bound symmetric in i and j observe that obviously

$$\rho_{i,j,k} \leq C(k^{-\alpha_i\beta/2} \vee k^{-\alpha_j\beta/2}) = Ck^{-(\alpha_i \wedge \alpha_j)\beta/2}.$$

This completes the argument. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Proposition 5.2 *The series defining $\Sigma_{i,j}^2$ in (6) is absolutely convergent and we have that $\Sigma_n^2 \to \Sigma^2$, as $n \to \infty$. In particular, $\Sigma_n \to \Sigma$.*

Proof. First, we prove that the series in (6) converges absolutely. By symmetry it is enough to show that

$$\sum_{s=1}^{\infty} |\mathrm{Cov}(f_i(X_s^1, \ldots, X_s^m), f_j(X_0^1, \ldots, X_0^m))| < \infty \quad \text{for all } i, j \in \{1, \ldots, d\}.$$

To this end, we let $\widetilde{\eta}$ be a Poisson process on $[0, 1] \times \mathbb{R}^2$ with intensity measure $\mathrm{d}u\,\mu(\mathrm{d}z)$ with the property that $\eta = \widetilde{\eta}([0, 1] \times \cdot)$. Using now the covariance identity for Poisson functionals from [6, Theorem 5.1] we conclude that

$$\mathrm{Cov}(f_i(X_s^1, \ldots, X_s^m), f_j(X_0^1, \ldots, X_0^m))$$
$$= \mathbb{E}\Big[\int_0^1 \Big(\int_{\mathbb{R}} \mathbb{E}[D_z f_i(X_s^1, \ldots, X_s^m) \mid \mathcal{G}_u]\mathbb{E}[D_z f_j(X_0^1, \ldots, X_0^m) \mid \mathcal{G}_u]\,\mu(\mathrm{d}z)\Big)\,\mathrm{d}u\Big],$$

where $\mathcal{G}_u$ is the σ-algebra generated by the restriction of the Poisson process $\widetilde{\eta}$ to $[0, u] \times \mathbb{R}^2$. Applying the Cauchy-Schwarz inequality, our assumption (1) on ν and (11) implies that

$$|\mathrm{Cov}(f_i(X_s^1, \ldots, X_s^m), f_j(X_0^1, \ldots, X_0^m))|$$
$$\leq \int_0^1 \Big(\int_{\mathbb{R}^2} \mathbb{E}\big[|\mathbb{E}[D_z f_i(X_s^1, \ldots, X_s^m) \mid \mathcal{G}_u]\mathbb{E}[D_z f_j(X_0^1, \ldots, X_0^m) \mid \mathcal{G}_u]|\big]\,\mu(\mathrm{d}z)\Big)\,\mathrm{d}u$$
$$\leq \int_{\mathbb{R}^2} \mathbb{E}\big[|D_z f_i(X_s^1, \ldots, X_s^m)|^2\big]^{1/2}\mathbb{E}\big[|D_z f_j(X_0^1, \ldots, X_0^m)|^2\big]^{1/2}\,\mu(\mathrm{d}z)$$
$$\leq C \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} (1 \wedge |x|^2\,\|(g_\ell(s - t))_{\ell=1}^m\|_{\mathbb{R}^m}\,\|(g_\ell(-t))_{\ell=1}^m\|_{\mathbb{R}^m})\,|x|^{-1-\beta}\,\mathrm{d}x\Big)\,\mathrm{d}t$$
$$= C \int_{\mathbb{R}} \|(g_\ell(s - t))_{\ell=1}^m\|_{\mathbb{R}^m}^{\beta/2}\,\|(g_\ell(-t))_{\ell=1}^m\|_{\mathbb{R}^m}^{\beta/2}\,\mathrm{d}t$$
$$\leq C \sum_{k,\ell=1}^{m} \int_{\mathbb{R}} |g_\ell(s - t)g_k(-t)|^{\beta/2}\,\mathrm{d}t$$

13

$$= C \sum_{k,\ell=1}^{m} \rho_{k,\ell,s}$$

$$\leq C s^{-\alpha\beta/2},$$

where the last inequality follows from Lemma 5.1. Since $\alpha\beta > 2$ by assumption the series in (6) converges absolutely. To deduce the convergence $\Sigma_n^2 \to \Sigma^2$ we use the stationarity of the sequence $(X_t^1, \ldots, X_t^m)$, $t \in \mathbb{R}$, to see that, for any $i, j \in \{1, \ldots, d\}$,

$$\mathrm{Cov}(V_n^i(X; f), V_n^j(X; f))$$

$$= n^{-1} \sum_{s,t=1}^{n} \mathrm{Cov}(f_i(X_s^1, \ldots, X_s^m), f_j(X_t^1, \ldots, X_t^m))$$

$$= n^{-1} \sum_{\substack{s,t=1 \\ s \geq t}}^{n} \mathrm{Cov}(f_i(X_{s-t}^1, \ldots, X_{s-t}^m), f_j(X_0^1, \ldots, X_0^m))$$

$$+ n^{-1} \sum_{\substack{s,t=1 \\ s < t}}^{n} \mathrm{Cov}(f_i(X_0^1, \ldots, X_0^m), f_j(X_{t-s}^1, \ldots, X_{t-s}^m))$$

$$= \sum_{k=0}^{n-1} (1 - \tfrac{k}{n}) \, \mathrm{Cov}(f_i(X_k^1, \ldots, X_k^m), f_j(X_0^1, \ldots, X_0^m))$$

$$+ \sum_{k=1}^{n-1} (1 - \tfrac{k}{n}) \, \mathrm{Cov}(f_i(X_0^1, \ldots, X_0^m), f_j(X_k^1, \ldots, X_k^m)) \longrightarrow \Sigma_{i,j}^2,$$

as $n \to \infty$, where the convergence follows by Lebesgue's dominated convergence theorem together with the absolute convergence of the series defining the limit $\Sigma_{i,j}^2$. Finally, the last claim simply follows by continuity of the square root. $\qquad \square$

5.3 Bounding $d_3(V_n, Y)$

Recall for $i, k \in \{1, \ldots, m\}$ the definition of the quantities $\gamma_1(F_i, F_k)$ and $\gamma_2(F_i, F_k)$ from Section 2.1, which are applied with $F_i = V_n^i(X; f)$ and $F_k = V_n^k(X; f)$. According to Theorem 2.1 we have that for any $n \in \mathbb{N}$,

$$d_3(V_n(X; f), Y) \leq \sum_{i,k=1}^{d} (\gamma_1(F_i, F_k) + \gamma_2(F_i, F_k)) + \gamma_3,$$

where γ_3 is defined at (5). We consider each of these terms separately in the following three lemmas. Let us point to the fact that the sum will converge at a speed of order $1/\sqrt{n}$, whereas the γ_3-term will generally converge at a lower speed, depending on the parameters α and β. It is also this last term that requires the stronger assumption (3) rather than just $\sum_{u=0}^{\infty} \rho_{i,j,u} < \infty$ for all $i, j \in \{1, \ldots, m\}$. Indeed, as a product, in γ_3 we carefully have to distinguish between small and large values, where the latter are non-negligible for heavy-tailed moving averages.

14

Lemma 5.3 *There exists a constant $C > 0$ such that $\gamma_1(F_i, F_k) \leq Cn^{-1/2}$ for any $i, k \in \{1, \ldots, m\}$.*

Proof. To simplify the notation put $V_n^i := V_n^i(X; f)$ and recall that

$$\gamma_1^2(F_i, F_k) = 3 \int_{(\mathbb{R}^2)^3} \mathbb{E}\big[(D_{z_1,z_3}^2 V_n^i)^2 (D_{z_2,z_3}^2 V_n^i)^2\big]^{1/2}$$
$$\times \mathbb{E}\big[(D_{z_1} V_n^k)^2 (D_{z_2} V_n^k)^2\big]^{1/2} \mu^3(\mathrm{d}z_1, \mathrm{d}z_2, \mathrm{d}z_3).$$

If $z_i = (x_i, t_i) \in \mathbb{R}^2$ for $i \in \{1, 2, 3\}$, the integrand can be bounded using (11) and (12) as follows:

$$\mathbb{E}\big[(D_{z_1,z_3}^2 V_n^i)^2 (D_{z_2,z_3}^2 V_n^i)^2\big]^{1/2} \mathbb{E}\big[(D_{z_1} V_n^k)^2 (D_{z_2} V_n^k)^2\big]^{1/2}$$

$$\leq \frac{C}{n^2}\Big(\sum_{s_1=1}^n (1 \wedge \|\delta_{s_1}(z_1)\|_{\mathbb{R}^m})(1 \wedge \|\delta_{s_1}(z_3)\|_{\mathbb{R}^m})\Big)$$

$$\times \Big(\sum_{s_2=1}^n (1 \wedge \|\delta_{s_2}(z_2)\|_{\mathbb{R}^m})(1 \wedge \|\delta_{s_2}(z_3)\|_{\mathbb{R}^m})\Big)$$

$$\times \Big(\sum_{s_3=1}^n (1 \wedge \|\delta_{s_3}(z_1)\|_{\mathbb{R}^m})\Big)\Big(\sum_{s_4=1}^n (1 \wedge \|\delta_{s_4}(z_2)\|_{\mathbb{R}^m})\Big)$$

$$\leq \frac{C}{n^2} \sum_{s_1,\ldots,s_4=1}^n \big[(1 \wedge \|\delta_{s_1}(z_1)\|_{\mathbb{R}^m} \|\delta_{s_3}(z_1)\|_{\mathbb{R}^m})(1 \wedge \|\delta_{s_2}(z_2)\|_{\mathbb{R}^m} \|\delta_{s_4}(z_2)\|_{\mathbb{R}^m})$$

$$\times (1 \wedge \|\delta_{s_1}(z_3)\|_{\mathbb{R}^m} \|\delta_{s_2}(z_3)\|_{\mathbb{R}^m})\big]$$

$$\leq \frac{C}{n^2} \sum_{s_1,\ldots,s_4=1}^n \sum_{j_1,\ldots,j_6=1}^m (1 \wedge x_1^2 |g_{j_1}(s_1 - t_1)g_{j_2}(s_3 - t_1)|)$$

$$\times (1 \wedge x_2^2 |g_{j_3}(s_2 - t_2)g_{j_4}(s_4 - t_2)|)(1 \wedge x_3^2 |g_{j_5}(s_1 - t_3)g_{j_6}(s_2 - t_3)|).$$

Using the substitution $u_i = x_i^2 y_i$ for $y_i > 0$, $i \in \{1, 2, 3\}$, one easily verifies the relation

$$\int_{\mathbb{R}^3} (1 \wedge x_1^2 y_1)(1 \wedge x_2^2 y_2)(1 \wedge x_3^2 y_3) |x_1 x_2 x_3|^{-1-\beta} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3 = C y_1^{\beta/2} y_2^{\beta/2} y_3^{\beta/2},$$

for $\beta \in (0, 2)$. This yields the bound

$$\gamma_1^2(F_i, F_k) \leq \frac{C}{n^2} \sum_{j_1,\ldots,j_6=1}^m \sum_{s_1,\ldots,s_4=1}^n \Big(\int_{\mathbb{R}} |g_{j_1}(s_1 - t_1)g_{j_2}(s_3 - t_1)|^{\beta/2} \, \mathrm{d}t_1$$

$$\times \int_{\mathbb{R}} |g_{j_3}(s_2 - t_2)g_{j_4}(s_4 - t_2)|^{\beta/2} \, \mathrm{d}t_2 \int_{\mathbb{R}} |g_{j_5}(s_1 - t_3)g_{j_6}(s_2 - t_3)|^{\beta/2} \, \mathrm{d}t_3\Big)$$

$$= \frac{C}{n^2} \sum_{j_1,\ldots,j_6=1}^m \sum_{s_1,\ldots,s_4=1}^n \rho_{j_1,j_2,s_3-s_1} \rho_{j_3,j_4,s_4-s_2} \rho_{j_5,j_6,s_2-s_1}$$

$$\leq \frac{C}{n} \sum_{j_1,\ldots,j_6=1}^m \Big(\sum_{u=-n}^n \rho_{j_1,j_2,u}\Big)\Big(\sum_{u=-n}^n \rho_{j_3,j_4,u}\Big)\Big(\sum_{u=-n}^n \rho_{j_5,j_6,u}\Big) \leq \frac{C}{n},$$

15

where the penultimate inequality follows by substitution and the last inequality is due to Lemma 5.1, and where we used that $\sum_{u=0}^{\infty} \rho_{j,\ell,u} < \infty$ for all $j, \ell \in \{1, \ldots, m\}$. $\qquad \square$

Lemma 5.4 *There exists a constant $C > 0$ such that $\gamma_2(F_i, F_k) \leq Cn^{-1/2}$ for all $i, k \in \{1, \ldots, m\}$ and $n \in \mathbb{N}$.*

Proof. Using (12) we conclude that the integrand in the definition of $\gamma_2(F_i, F_k)$ is bounded as follows:

$$
\mathbb{E}\big[(D_{z_1,z_3}^2 V_n^i)(D_{z_2,z_3}^2 V_n^i)\big]^{1/2} \mathbb{E}\big[(D_{z_1,z_3}^2 V_n^k)(D_{z_2,z_3}^2 V_n^k)\big]^{1/2}
$$

$$
\leq \frac{C}{n^2} \sum_{s_1,\ldots,s_4=1}^{n} (1 \wedge \|\delta_{s_1}(z_1)\|_{\mathbb{R}^m} \|\delta_{s_3}(z_1)\|_{\mathbb{R}^m})(1 \wedge \|\delta_{s_2}(z_2)\|_{\mathbb{R}^m} \|\delta_{s_4}(z_2)\|_{\mathbb{R}^m})
$$

$$
\times (1 \wedge \|\delta_{s_1}(z_3)\|_{\mathbb{R}^m} \|\delta_{s_2}(z_3)\|_{\mathbb{R}^m} \|\delta_{s_3}(z_3)\|_{\mathbb{R}^m} \|\delta_{s_4}(z_3)\|_{\mathbb{R}^m})
$$

$$
\leq \frac{C}{n^2} \sum_{s_1,\ldots,s_4=1}^{n} \sum_{j_1,\ldots,j_8=1}^{m} (1 \wedge x_1^2 |g_{j_1}(s_1 - t_1)g_{j_2}(s_3 - t_1)|)
$$

$$
\times (1 \wedge x_2^2 |g_{j_3}(s_2 - t_2)g_{j_4}(s_4 - t_2)|)
$$

$$
\times (1 \wedge x_3^4 |g_{j_5}(s_1 - t_3)g_{j_6}(s_2 - t_3)g_{j_7}(s_3 - t_3)g_{j_8}(s_4 - t_3)|).
$$

Moreover, as in the proof of the previous lemma we have that

$$
\int_{\mathbb{R}^3} (1 \wedge x_1^2 y_1)(1 \wedge x_2^2 y_2)(1 \wedge x_3^4 y_3) |x_1 x_2 x_3|^{-1-\beta} \, dx_1 \, dx_2 \, dx_3 = C y_1^{\beta/2} y_2^{\beta/2} y_3^{\beta/4}
$$

for $\beta \in (0,2)$ and real numbers $y_1, y_2, y_3 > 0$. This implies that

$$
\gamma_2^2(F_i, F_k) \leq \frac{C}{n^2} \sum_{j_1,\ldots,j_8=1}^{m} \sum_{s_1,\ldots,s_4=1}^{n} \int_{\mathbb{R}} |g_{j_1}(s_1 - t_1)g_{j_2}(s_3 - t_1)|^{\beta/2} \, dt_1
$$

$$
\times \int_{\mathbb{R}} |g_{j_3}(s_2 - t_2)g_{j_4}(s_4 - t_2)|^{\beta/2} \, dt_2
$$

$$
\times \int_{\mathbb{R}} |g_{j_5}(s_1 - t_3)g_{j_6}(s_2 - t_3)g_{j_7}(s_3 - t_3)g_{j_8}(s_4 - t_3)|^{\beta/4} \, dt_3
$$

$$
\leq \frac{C}{n^2} \sum_{j_1,\ldots,j_8=1}^{m} \sum_{s_1,\ldots,s_4=1}^{n} \rho_{j_1,j_2,s_3-s_1} \rho_{j_3,j_4,s_4-s_2} (\rho_{j_5,j_6,s_2-s_1} + \rho_{j_7,j_8,s_4-s_3})
$$

$$
\leq \frac{C}{n},
$$

where the last inequality follows as in Lemma 5.3 and the penultimate inequality follows immediately from the fact that $|xy| \leq x^2 + y^2$ for all $x, y \in \mathbb{R}$. $\qquad \square$

Finally, we consider the crucial term γ_3.

Lemma 5.5 *There exists a constant $C > 0$ such that, for all $n \in \mathbb{N}$,*

$$
\gamma_3 \leq C \begin{cases} n^{-1/2}, & \text{if } \underline{\alpha}\beta > 3, \\ n^{-1/2} \log(n), & \text{if } \underline{\alpha}\beta = 3, \\ n^{(2-\underline{\alpha}\beta)/2}, & \text{if } 2 < \underline{\alpha}\beta < 3. \end{cases}
$$

16

Proof. Recall the definition of $\delta_s(z) = (\delta_s^1(z), \ldots, \delta_s^m(z))$ from (10) and define for $i \in \{1, \ldots, m\}$,

$$A_n^i(z) := \frac{1}{\sqrt{n}} \sum_{s=1}^n (1 \wedge |\delta_s^i(z)|).$$

By (11) and the sub-additivity of the minimum it follows that

$$\gamma_3 \leq C \int_{\mathbb{R}^2} \left(n^{-1/2} \sum_{s=1}^n 1 \wedge \|\delta_s(z)\|_{\mathbb{R}^m} \right)^2 \wedge \left(n^{-1/2} \sum_{s=1}^n 1 \wedge \|\delta_s(z)\|_{\mathbb{R}^m} \right)^3 \mu(\mathrm{d}z)$$

$$\leq \sum_{i,j=1}^m \int_{\mathbb{R}^2} (A_n^i(z)^2 \wedge A_n^j(z)^3) \, \mu(\mathrm{d}z).$$

From this point on, we can literally follow the proof of Lemma 4.6 in [2]. In fact, this shows that for any $p \in [0,2]$, $q > 2$ and $i, j \in \{1, \ldots, m\}$, one has that

$$\int_{\mathbb{R}^2} (A_n^i(z)^p \wedge A_n^j(z)^q) \, \mu(\mathrm{d}z) \leq C \begin{cases} n^{1-q/2}, & \text{if } \underline{\alpha}\beta > q, \\ n^{1-q/2} \log(n), & \text{if } \underline{\alpha}\beta = q, \\ n^{(2-\underline{\alpha}\beta)/2}, & \text{if } 2 < \underline{\alpha}\beta < 3. \end{cases}$$

Indeed, these bounds rely solely on the tail behaviour (in terms of the α_i's) of the kernels g_i, where $\underline{\alpha}$ reflects the *weakest* behaviour, and the power behaviour (in terms of the κ_i's) around 0, all of which satisfies the condition $\kappa_i > -1/\beta$. This completes the argument. $\square$

Proof of Theorem 2.3. According to Theorem 2.1 we have that for any $n \in \mathbb{N}$,

$$d_3(V_n(X; f), Y) \leq \sum_{i,k=1}^d (\gamma_1(F_i, F_k) + \gamma_2(F_i, F_k)) + \gamma_3.$$

Using now Lemmas 5.3, 5.4 and 5.5 we see that

$$d_3(V_n(X; f), Y) \leq C(n^{-1/2} + n^{-1/2}) + C \begin{cases} n^{-1/2}, & \text{if } \underline{\alpha}\beta > 3, \\ n^{-1/2} \log(n), & \text{if } \underline{\alpha}\beta = 3, \\ n^{(2-\underline{\alpha}\beta)/2}, & \text{if } 2 < \underline{\alpha}\beta < 3, \end{cases}$$

$$\leq C \begin{cases} n^{-1/2}, & \text{if } \underline{\alpha}\beta > 3, \\ n^{-1/2} \log(n), & \text{if } \underline{\alpha}\beta = 3, \\ n^{(2-\underline{\alpha}\beta)/2}, & \text{if } 2 < \underline{\alpha}\beta < 3. \end{cases}$$

This completes the argument. $\square$

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