A probabilistic interpretation of the Gaussian binomial coefficients

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7 March 2017

Abstract

We give a stand-alone simple proof of a probabilistic interpretation of the Gaussian binomial coefficients by conditioning a random walk to hit a given lattice point at a given time.

Introduction

The Gaussian binomial coefficients [4] are generalizations of classical binomial coefficients and are usually defined as

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-m+1})}{(1-q)(1-q^2)\cdots(1-q^m)}.$$

The term "generalization" is justified, e.g., by the fact that $\lim_{q\to 1}\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]_q = \binom{n}{m}$, which becomes obvious if we divide each term in the numerator and denominator of the last display by $1-q$ and expand the ratio into a power series with finitely many terms. The Gaussian binomial coefficients turn out to be polynomial functions of the variable q and satisfy many analogs of the usual properties of binomial coefficients. We refer, e.g., to the textbook of Kac and Cheung [5].

Originally, they appeared in combinatorics, so it is not surprising that they are nowadays very important in random polymer models which have strong connections to algebraic combinatorics; see, for example, the recent work on the q-weighted version of the Robinson-Schensted algorithm introduced by O'Connell and Pei [6]. In the study of random graphs, Gaussian binomial coefficients are present, for instance, in the distributions of the sizes of the transitive closure and transitive reduction of node 1 in a random acyclic digraph with n nodes, see [3] and [2]. Another application is in integer-valued random matrices; see, for example, [1] where the distribution of the m-rank of a random matrix is expressed in terms of these coefficients.

The purpose of this note is to give a short proof of a probabilistic interpretation of the Gaussian coefficients which, not surprisingly, is very similar to their combinatorial interpretation, given by Pólya [7], as counting the number of nondecreasing paths in a rectangle in the 2-dimensional integer lattice that leave a fixed area below them. The probabilistic proof given below (Theorem 1) is different than Pólya's. The note is stand-alone in that everything discussed is proved, including Heine's formula (see (4) below) that is needed at the end of the proof of Theorem 1. The probabilistic interpretation gives a natural meaning to several identities and properties satisfied by the coefficients (see end remarks).

*Supported by Swedish Research Council grant 2013-4688

The statement and proof

Consider nondecreasing paths in the standard 2-dimensional integer lattice $\mathbb{Z}^2$, that is, finite or infinite sequences $x_0, x_1, \dots$ of elements of $\mathbb{Z}^2$ such that $\eta_i = x_i - x_{i-1}$ is either e_1 or e_2, where $e_1 = (1,0)$, $e_2 = (0,1)$, the standard unit vectors. Let r, s be nonnegative integers. By a *random nondecreasing path* from $(0,0)$ to (r,s) we mean a finite nondecreasing path that starts at $x_0 = (0,0)$ and ends at $x_{r+s} = (r,s)$, and that is chosen *uniformly at random* among the set of all such paths. Since there are $\binom{r+s}{r}$ such paths, the increments sequence $(\eta_1, \dots, \eta_n)$ is assigned probability equal to $\binom{r+s}{r}^{-1}$.

Theorem 1. *Consider a random nondecreasing path from $(0,0)$ to (r,s). This path splits the rectangle $[0,r] \times [0,s]$ into two regions. Let $A_{r,s}$ be the area of the region under the path. Then*

$$\mathbb{E} q^{A_{r,s}} = \begin{bmatrix} r+s \\ r \end{bmatrix}_q \Big/ \binom{r+s}{r}.$$

Proof. Toss a fair coin independently and let e_1 represent heads and e_2 tails. Denote by $\xi_1, \xi_2, \dots$ the successive outcomes, a random independent sequence with $\mathbb{P}(\xi_t = e_i) = 1/2$, $i = 1, 2$, $t \geq 1$. Let $X_0 = 0$, $X_t = \xi_1 + \cdots + \xi_t$, $t \geq 1$. If it takes T_{r+1} coin tosses until the $(r+1)$-th head occurs for the first time then, conditional on the event that we have seen s tails up to T_{r+1}, the sequence $(\xi_1, \dots, \xi_{T_{r+1}-1})$ (of length $T_{r+1} - 1 = s + (r+1) - 1 = s + r$, under the conditioning) has uniform distribution. Thus, conditional on the same event, the path $(X_0, X_1, \dots, X_{T_{r+1}-1})$ is a random nondecreasing path from $(0,0)$ to (r,s). Let $N_i(t)$ be the number of heads/tails seen up to the t-th toss:

$$N_i(t) := \sum_{k=1}^{t} \mathbf{1}\{\xi_k = e_i\}, \quad i = 1, 2,$$

and consider the stopping times

$$T_0 := 0, \quad T_m := \inf\{t \geq 1 : N_1(t) = m\}, \ m \geq 1.$$

Since $T_1, T_2, \dots$ is an increasing sequence of stopping times in i.i.d. Bernoulli trials, the random variables $Z_i = N_2(T_{i+1}) - N_2(T_i)$, $i = 0, 1, 2, \dots$, are i.i.d. geometric: $\mathbb{P}(Z_i = j) = (1/2)^{j+1}$, $j \geq 0$, and so $\mathbb{E}\theta^{Z_i} = \frac{1}{2}(1 - \theta/2)^{-1}$, $|\theta| < 2$. Simply putting it, the Z_i count the number of up-steps of the path between two successive right-steps and so

$$V = N_2(T_{r+1}) = \sum_{i=0}^{r} Z_i$$

is the total number of up-steps up to T_{r+1}. The distribution of V is

$$\mathbb{P}(V = s) = \big(\frac{1}{2}\big)^{r+s+1} \binom{r+s}{r}, \quad s \geq 0. \tag{1}$$

The area $A = A_{r,s}$ under the path $(X_0, X_1, \dots, X_{T_{r+1}-1})$ is then

$$A = rZ_0 + (r-1)Z_1 + \cdots + Z_{r-1}.$$

Letting q, θ be variable with, say, $|q|, |\theta| < 2$, we have

$$\begin{aligned} \mathbb{E}\, q^A \theta^V &= \mathbb{E}[(q^r\theta)^{Z_0} (q^{r-1}\theta)^{Z_1} \cdots (q\theta)^{Z_{r-1}} \theta^{Z_r}] \\ &= \big(\frac{1}{2}\big)^{r+1} \frac{1}{1 - \frac{q^r\theta}{2}} \frac{1}{1 - \frac{q^{r-1}\theta}{2}} \cdots \frac{1}{1 - \frac{q\theta}{2}} \frac{1}{1 - \frac{\theta}{2}} = \big(\frac{1}{2}\big)^{r+1} \sum_{s=0}^{\infty} C_{r,s}(q) (\theta/2)^s, \end{aligned} \tag{2}$$

where the $C_{r,s}(q)$ are defined by the right-hand side as coefficients in the Taylor expansion in the variable $\theta/2$. On the other hand,

$$\mathbb{E}\, q^A \theta^V = \sum_{s=0}^{\infty} \theta^s \mathbb{P}(V=s)\mathbb{E}[q^A | V=s]. \tag{3}$$

Equating coefficients in (2) and (3), also taking into account (1), gives

$$\mathbb{E}[q^A | V=s] = C_{r,s}(q) \Big/ \binom{r+s}{r}.$$

It remains to show that the $C_{r,s}(q)$ are Gaussian binomial coefficients. To this end, we prove that if

$$F_r(x) := \prod_{j=0}^{r} \frac{1}{1-q^j x} = \sum_{s=0}^{\infty} C_{r,s}(q)\, x^s, \tag{4}$$

then the recurrence relation

$$C_{r,s}(q) = C_{r,s-1}(q)\, \frac{1-q^{r+s}}{1-q^s}, \quad s \geq 1, \tag{5}$$

holds. This follows quite easily from the observation that

$$(1-q^{r+1}x)F_r(qx) = (1-x)F_r(x).$$

Indeed, if, in this identity, we replace $F_r(qx)$ and $F_r(x)$ by their series, from the right-hand side of (4), and equate coefficients of similar powers, we obtain (5). Since, clearly, $C_{r,0}(q) = F_r(0) = 1$, we can iterate (5) to obtain

$$C_{r,s}(q) = \frac{1-q^{r+s}}{1-q^s} \frac{1-q^{r+s-1}}{1-q^{s-1}} \cdots \frac{1-q^{r+1}}{1-q} = \begin{bmatrix} r+s \\ r \end{bmatrix}_q .$$

This completes the proof. $\square$

Remarks

1. Since $\left[{r+s \atop r}\right]_q$ is proportional to $\mathbb{E}q^{A_{r,s}}$ we have that $\left[{r+s \atop r}\right]_q$ is a polynomial in q.
2. Formula (4) with $C_{r,s}(q)$ the Gaussian binomial coefficients is known as Heine's formula [5]. When $q = 1$ it corresponds to the Taylor series (Newton's formula) $(1-x)^{-r} = \sum_{s\geq 0} \binom{-r}{s}(-x)^s = \sum_{s\geq 0} \binom{r+s}{r} x^s$.
3. By symmetry, the area above the random nondecreasing path has the same distribution as the area below, i.e., the random variables $A_{r,s}$ and $rs - A_{r,s}$ have the same distribution. This is equivalent to the identity $\left[{r+s \atop r}\right]_q = q^{rs} \left[{r+s \atop r}\right]_{1/q}$.
4. By the definition of the random variable $A_{r,s}$ as the area under a random nondecreasing path from $(0,0)$ to (r,s) we see, by conditioning on the last edge of this path, that $A_{r,s}$ is in distribution equal to $A_{r,s-1}$ with probability $s/(r+s)$ or to $A_{r-1,s} + s$ with probability $r/(r+s)$. Using then the result of Theorem 1, the well-known recursion $\left[{r+s \atop r}\right]_q = \left[{r+s-1 \atop r}\right]_q + q^s \left[{r+s-1 \atop r-1}\right]_q$ follows.

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