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ON THE TOTAL LENGTH OF EXTERNAL BRANCHES FOR BETA-COALESCENTS

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Abstract

In this paper, we consider the Beta$(2-\alpha,\alpha)$-coalescents with $1<\alpha<2$ and study the moments of external branches, in particular the total external branch length $L_{ext}^{(n)}$ of an initial sample of n individuals. For this class of coalescents, it has been proved that $n^{\alpha-1}T^{(n)} \overset{(d)}{\to} T$, where $T^{(n)}$ is the length of an external branch chosen at random, and T is a known non negative random variable. We obtain that for Beta$(2-\alpha,\alpha)$-coalescents with $1<\alpha<2$, $\lim_{n\to+\infty} n^{3\alpha-5}\mathbb{E}[(L_{ext}^{(n)} - n^{2-\alpha}\mathbb{E}[T])^2] = \dfrac{((\alpha-1)\Gamma(\alpha+1))^2\,\Gamma(4-\alpha)}{(3-\alpha)\Gamma(4-2\alpha)}$.

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1. Introduction

1.1. Motivation

In a Wright-Fisher haploid population model with size N, we sample n individuals at present from the total population, and look backward to see the ancestral tree until we get the most recent common ancestor (MRCA). If time is well rescaled and the size N of population becomes large, then the genealogy of the sample of size n converges weakly to the Kingman n-coalescent (see [33],[34]). During the evolution of the population, mutations may occur. We consider the infinite sites model introduced by Kimura [32]. In this model, each mutation is produced at a new site which is never seen before and will never be seen in the future. The neutrality of mutations means that all mutants are equally privileged by the environment. Under the infinite sites model, to detect or reject the neutrality when the genealogy is given by the Kingman coalescent, Fu and Li[22] have proposed a statistical test based on the total mutation numbers on the external branches and internal branches. Mutations on external branches affect only single individuals, so in practice they can be picked out according to the model setting. In this test, the ratio $L_{ext}^{(n)}/L^{(n)}$ between the total external branch length $L_{ext}^{(n)}$ and the total length $L^{(n)}$ measures in some sense the weight of mutations occurred on external branches among all. It then makes the study of these quantities relevant.

For many populations, Kingman coalescent describes the genealogy quite well. But for some others, when descendants of one individual can occupy a big ratio of the next generation with non-negligible probability, it is no more relevant. It is for example the case of some marine species (see [1], [9], [19], [24], [26]). In this case, if time is well rescaled and the size of population becomes large, the ancestral tree converges weakly to the Λ-coalescent which is associated with a finite measure Λ on $[0,1]$. This coalescent allows multiple collisions. It has first been introduced by Pitman[38] and Sagitov[39]. Among Λ-coalescents, a special and important subclass is called Beta(a,b)-coalescents characterized

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by Λ being a Beta distribution $Beta(a,b)$. The most popular ones are those with parameters $2-\alpha$ and α where $\alpha \in (0,2)$.

$Beta$-coalescents arise not only in the context of biology. They also have connections with supercritical Galton-Watson process (see [40]), with continuous-state branching processes (see [6], [2], [20]), with continuous random trees (see [4]). If $\alpha = 1$, we recover the Bolthausen-Sznitman coalescent which appears in the field of spin glasses (see [8], [10]) and is also connected to random recursive trees (see [25]). The Kingman coalescent is also obtained from the $Beta(2-\alpha,\alpha)$-coalescent by letting α tend to 2.

For $Beta(2-\alpha,\alpha)$-coalescents with $1<\alpha<2$, a central limit theorem of the total external branch length $L_{ext}^{(n)}$ is known (see [31]). The aim of this paper is to study its moments. The results obtained can be extended to more general coalescent processes (see [16]). We should say that in this case, the moment method is not able to obtain the right convergence speed in the central limit theorem, which illustrates some limitations of moment calculations.

1.2. Introduction and main results

Let $\mathcal{E}$ be the set of partitions of $\mathbb{N} := \{1,2,3,...\}$ and, for $n \in \mathbb{N}$, $\mathcal{E}_n$ be the set of partitions of $\mathbb{N}_n := \{1,2,\cdots,n\}$. We denote by $\rho^{(n)}$ the natural restriction on $\mathcal{E}_n$: if $1 \le n \le m \le +\infty$ and $\pi = \{A_i\}_{i\in I}$ is a partition of $\mathbb{N}_m$, then $\rho^{(n)}\pi$ is the partition of $\mathbb{N}_n$ defined by $\rho^{(n)}\pi = \{A_i \bigcap \mathbb{N}_n\}_{i\in I}$. For a finite measure Λ on $[0,1]$, we denote by $\Pi = (\Pi_t)_{t\ge 0}$ the Λ-coalescent process introduced independently by Pitman[38] and Sagitov[39]. The process $(\Pi_t)_{t\ge0}$ is a càd-làg continuous time Markovian process taking values in $\mathcal{E}$ with $\Pi_0 = \{\{1\},\{2\},\{3\},...\}$. It is characterized by the càd-làg Λ n-coalescent processes $(\Pi_t^{(n)})_{t\ge0} := (\rho^{(n)}\Pi_t)_{t\ge0}, n \in \mathbb{N}$. For $n \le m \le +\infty$, we have $(\Pi_t^{(n)})_{t\ge0} = (\rho^{(n)}\Pi_t^{(m)})_{t\ge0}$ (where $\Pi^{(+\infty)} = \Pi$).

Let $\nu(dx) = x^{-2}\Lambda(dx)$. For $2 \le a \le b$, we set

$$\lambda_{b,a} = \int_0^1 x^{a-2}(1-x)^{b-a}\Lambda(dx) = \int_0^1 x^a(1-x)^{b-a}\nu(dx).$$

$\Pi^{(n)}$ is a Markovian process with values in $\mathcal{E}_n$, and its transition rates are given by: for $\xi,\eta \in \mathcal{E}_n$, $q_{\xi,\eta} = \lambda_{b,a}$ if η is obtained by merging a of the $b = |\xi|$ blocks of ξ and letting the $b-a$ others unchanged, and $q_{\xi,\eta} = 0$ otherwise. We say that a individuals (or blocks) of ξ have been coalesced in one single individual of η. Remark that the process $\Pi^{(n)}$ is an exchangeable process, which means that, for any permutation τ of $\mathbb{N}_n$, $\tau \circ \Pi^{(n)} \overset{(d)}{=} \Pi^{(n)}$.

The process $\Pi^{(n)}$ finally reaches one block. This final individual is called the most recent common ancestor (MRCA). We denote by $\tau^{(n)}$ the number of collisions it takes for the n individuals to be coalesced to the MRCA.

We define by $R^{(n)} = (R_t^{(n)})_{t\ge0}$ the block counting process of $(\Pi_t^{(n)})_{t\ge0}$: $R_t^{(n)} = |\Pi_t^{(n)}|$, which equals the number of blocks/individuals at time t. Then $R^{(n)}$ is a continuous time Markovian process taking values in $\mathbb{N}_n$, decreasing from n to 1. At state b, for $a = 2,...,b$, each of the $\binom{b}{a}$ groups with a individuals coalesces independently at rate $\lambda_{b,a}$. Hence, the time the process $(R_t^{(n)})_{t\ge0}$ stays at state b is exponential with parameter:

$$g_b = \sum_{a=2}^{b}\binom{b}{a}\lambda_{b,a} = \int_0^1 (1-(1-x)^b - bx(1-x)^{b-1})\nu(dx) = b(b-1)\int_0^1 t(1-t)^{b-2}\rho(t)dt, \tag{1}$$

where $\rho(t) = \int_t^1 \nu(dx)$. We denote by $Y^{(n)} = (Y_k^{(n)})_{k\ge0}$ the discrete time Markov chain associated with $R^{(n)}$. This is a decreasing process from $Y_0^{(n)} = n$ which reaches 1 at the $\tau^{(n)}$-th jump. The probability transitions of the Markov chain $Y^{(n)}$ are given by: for $b \ge 2$, $k \ge 1$ and $1 \le l \le b-1$,

$$p_{b,b-l} := \mathbb{P}(Y_k^{(n)} = b-l | Y_{k-1}^{(n)} = b) = \frac{\binom{b}{l+1}\lambda_{b,l+1}}{g_b}, \tag{2}$$

and 1 is an absorbing state.

We introduce the discrete time process $X_k^{(n)} := Y_{k-1}^{(n)} - Y_k^{(n)}$, $k \geq 1$ with $X_0^{(n)} = 0$. This process counts the number of blocks we lose at the k-th jump. For $i \in \{1, \ldots, n\}$, we define

$$T_i^{(n)} := \inf\left\{t | \{i\} \notin \Pi_t^{(n)}\right\}$$

as the length of the i-th external branch and $T^{(n)}$ the length of a randomly chosen external branch. By exchangeability, $T_i^{(n)} \overset{(d)}{=} T^{(n)}$. We denote by $L_{ext}^{(n)} := \sum_{i=1}^n T_i^{(n)}$ the total external branch length of $\Pi^{(n)}$, and by $L^{(n)}$ the total branch length.

For several measures Λ, many asymptotic results on the external branches and their total external lengths of the Λ n-coalescent are already known.

1. If $\Lambda = \delta_0$, Dirac measure on 0, $\Pi^{(n)}$ is the Kingman n-coalescent. Then,
   (a) $nT^{(n)}$ converges in distribution to T which is a random variable with density $f_T(x) = \frac{8}{(2+x)^3}\mathbf{1}_{x\geq 0}$ (See [7], [12], [27]).
   (b) $L_{ext}^{(n)}$ converges in L^2 to 2 (see [22], [18]). A central limit theorem is also proved in [27].
2. If Λ is the uniform probability measure on $[0, 1]$, $\Pi^{(n)}$ is the Bolthausen-Sznitman n-coalescent. Then $(\log n)T^{(n)}$ converges in distribution to an exponential variable with parameter 1 (see [21], [41]). For moment results of $L_{ext}^{(n)}$, we refer to [14] and for central limit theorem, we refer to [30].
3. If $\nu_{-1} = \int_0^1 x^{-1}\Lambda(dx) < +\infty$, which includes the case of the $Beta(2-\alpha, \alpha)$-coalescent with $0 < \alpha < 1$, then
   (a) $T^{(n)}$ converges in distribution to an exponential variable with parameter ν_{-1} (see [23, 37]).
   (b) $L^{(n)}/n$ converges in distribution to a random variable L whose distribution coincides with that of $\int_0^{+\infty} e^{-X_t}dt$, where X_t is a certain subordinator (see page 1405 in [17] and [36]), and $L_{ext}^{(n)}/L^{(n)}$ converges in probability to 1 (see [37]).
4. If Λ is the Beta$(2-\alpha, \alpha)$ measure with $1 < \alpha < 2$, then we get the $Beta(2-\alpha, \alpha)$-coalescents. Note that $n^{\alpha-1}T^{(n)}$ converges in distribution to T which is a random variable with density function (see[15])
$$f_T(x) = \frac{1}{(\alpha-1)\Gamma(\alpha)}(1 + \frac{x}{\alpha\Gamma(\alpha)})^{-\frac{\alpha}{\alpha-1}-1}\mathbf{1}_{x\geq 0}. \tag{3}$$
   For central limit theorems of $L_{ext}^{(n)}$ and $L^{(n)}$, we refer to [31, 29].

In the rest of the paper, we only consider the Beta$(2-\alpha, \alpha)$ coalescents, $1 < \alpha < 2$. In that case, we have
$$\nu(dx) = \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)}x^{-1-\alpha}(1-x)^{\alpha-1}\,dx.$$

T denotes a random variable with density (3). If $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are two real sequences, we define $a_n \sim b_n$ when $\lim_{n\to+\infty} a_n/b_n = 1$ is true.

Theorem 1. *1. The total external branch length $L_{ext}^{(n)}$ satisfies*
$$\lim_{n\to+\infty} n^{3\alpha-5}\mathbb{E}[(L_{ext}^{(n)} - n^{2-\alpha}\mathbb{E}[T])^2] = \Delta(\alpha),$$
where $\mathbb{E}[T] = \alpha(\alpha-1)\Gamma(\alpha)$ and $\Delta(\alpha) = \frac{((\alpha-1)\Gamma(\alpha+1))^2\Gamma(4-\alpha)}{(3-\alpha)\Gamma(4-2\alpha)}$.

2. As a consequence, $n^{\alpha-2}L_{ext}^{(n)} \overset{(L^2)}{\to} \mathbb{E}[T]$.

Remark 1.1.
- For the second part of the theorem, the convergence in probability and almost surely can be found from [4], [5], [3] by Berestycki et al.
- The first part of the theorem gives $n^{(5-3\alpha)/2}$ as the convergence speed for $L_{ext}^{(n)}$ tending to $n^{2-\alpha}\mathbb{E}[T]$ in the sense of second moment. But as shown in [31],

$$\frac{L_{ext}^{(n)} - n^{2-\alpha}\mathbb{E}[T]}{n^{1/\alpha+1-\alpha}} \xrightarrow{(d)} \frac{\alpha(2-\alpha)(\alpha-1)^{1/\alpha+1}\Gamma(\alpha)}{\Gamma(2-\alpha)^{1/\alpha}}\zeta,$$

where ζ is a stable random variable with parameter α. Our moment method fails to get the right speed of convergence in distribution.
To prove this result, the first idea is to write

$$\mathbb{E}[(L_{ext}^{(n)} - n^{2-\alpha}\mathbb{E}[T])^2] = nVar(T_1^{(n)}) + n(n-1)\mathrm{Cov}(T_1^{(n)}, T_2^{(n)}) + (n\mathbb{E}[T_1^{(n)}] - n^{2-\alpha}\mathbb{E}[T])^2. \quad (4)$$

Hence we have to get results on the moments of the external branches. This is given by the next theorems. The first one gives the asymptotic behaviour for the covariance of two external branch lengths.

Theorem 2. *The asymptotic covariance of two external branch lengths is given by:*

$$\lim_{n\to+\infty} n^{3(\alpha-1)} Cov(T_1^{(n)}, T_2^{(n)}) = \frac{\int_0^1((1-x)^{2-\alpha}-1)^2\nu(dx)}{3-\alpha}((\alpha-1)\Gamma(\alpha+1))^3 = \Delta(\alpha).$$

Remark 1.2. $\Delta(\alpha)$ is the limit only in the case of Beta$(2-\alpha,\alpha)$-coalecents, but the result can be extended to more general Λ-coalescent (see [16]).

Notice that $\Delta(\alpha)$ is strictly positive implies that $\mathrm{Cov}(T_1^{(n)}, T_2^{(n)})$ is of order $n^{3-3\alpha}$ and $T_1^{(n)}$, $T_2^{(n)}$ are positively correlated in the limit which is similar to Boltausen-Sznitman coalescent and opposite of Kingman coalescent (negatively correlated) (see [14]). To prove this theorem, we have to give the asymptotic behaviours of $\mathbb{E}[T_1^{(n)}T_2^{(n)}]$ and $\mathbb{E}[T_1^{(n)}]$ (Theorem 4). We also get from Theorem 4 that the third term in (4) satisfies

$$(n\mathbb{E}[T_1^{(n)}] - n^{2-\alpha}\mathbb{E}[T])^2 = O(n^{6-4\alpha}). \quad (5)$$

The second one gives the asymptotic behaviour of moments of one external branch length, hence we can estimate $nVar(T_1^{(n)})$. We then see that $n(n-1)\mathrm{Cov}(T_1^{(n)}, T_2^{(n)})$ is dominant in $\mathbb{E}[(L_{ext}^{(n)} - n^{2-\alpha}\mathbb{E}[T])^2]$ (see (4)). Then we can conclude for Theorem 1.

Theorem 3. *For Beta$(2-\alpha,\alpha)$-coalescent, we have*

1. *If $0 \le \beta < \frac{\alpha}{\alpha-1}$, then* $\lim_{n\to+\infty} \mathbb{E}[(n^{\alpha-1}T_1^{(n)})^\beta] = \mathbb{E}[T^\beta]$.
2. *If $\beta \ge \frac{\alpha}{\alpha-1}$, then* $\lim_{n\to+\infty} \mathbb{E}[(n^{\alpha-1}T_1^{(n)})^\beta] = +\infty$.

1.3. Organization of this paper

In sections 2 and 3, we give estimates of $\mathbb{E}[T_1^{(n)}]$ and $\mathbb{E}[T_1^{(n)}T_2^{(n)}]$ respectively. Both $\mathbb{E}[T_1^{(n)}]$ and $\mathbb{E}[T_1^{(n)}T_2^{(n)}]$ satisfy the same kind of recurrence which allows to get their estimates and they lead to an estimate of $Cov(T_1^{(n)}, T_2^{(n)})$ in section 3. The main tool is Lemma 5.1 given in appendix A. In section 4, we deal with Theorem 3. Section 5 is the appendix where are given some proofs omitted before.

2. First moment of $T_1^{(n)}$ by recursive method

2.1. Preliminaries

For $s > -\alpha$, we define the measure

$$\nu^{(s)}(dx) := (1-x)^s\nu(dx) = \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)}x^{-1-\alpha}(1-x)^{\alpha-1+s}\,dx; \quad (6)$$

The collision rates of the Λ-coalescent associated with the measure $\nu^{(s)}$ is given by

$$g_n^{(s)} := \int_0^1 (1-(1-x)^n - nx(1-x)^{n-1})\nu^{(s)}(dx) \sim \frac{n^\alpha}{\Gamma(\alpha+1)}$$

when n tends to ∞.

We introduce the quantity $\rho^{(s)}(t) := \int_t^1 \nu^{(s)}(dx)$.

Lemma 2.1. *For $s > -\alpha$, we have when t tends to 0:*

1. $\rho^{(s)}(t) = \frac{t^{-\alpha}}{\Gamma(\alpha+1)\Gamma(2-\alpha)} - \frac{(\alpha-1+s)t^{1-\alpha}}{(\alpha-1)\Gamma(\alpha)\Gamma(2-\alpha)} + o(t^{1-\alpha})$;
2. $\int_t^1 \rho^{(s)}(x)dx = \frac{t^{1-\alpha}}{(\alpha-1)\Gamma(\alpha+1)\Gamma(2-\alpha)} + \frac{\int_0^1 x^{-\alpha}((1-x)^{\alpha-1+s}-1)dx}{\Gamma(\alpha)\Gamma(2-\alpha)} - \frac{1}{(\alpha-1)\Gamma(\alpha)\Gamma(2-\alpha)} + O(t^{2-\alpha})$;
3. $\lim_{t\to 0+}\left(\int_t^1 \rho^{(s)}(x)dx - \frac{t^{1-\alpha}}{(\alpha-1)\Gamma(\alpha+1)\Gamma(2-\alpha)}\right)$ *exists, and its value is*

$$C^{(s)} = \frac{\int_0^1 x^{-\alpha}((1-x)^{\alpha-1+s}-1)dx}{\Gamma(\alpha)\Gamma(2-\alpha)} - \frac{1}{(\alpha-1)\Gamma(\alpha)\Gamma(2-\alpha)}.$$

In particular, if $s \geq 1-\alpha$, $C^{(s)} = \frac{\Gamma(\alpha+s)}{\Gamma(s+1)\Gamma(\alpha)(1-\alpha)}$.

Proof. The result for $\rho^{(s)}(t)$ is straightforward since

$$\rho^{(s)}(t) = \int_t^1 \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} x^{-1-\alpha}(1-x)^{\alpha-1}\, dx.$$

For $\int_t^1 \rho^{(s)}(x)dx$, using integration by parts, we have

$$\begin{aligned}\int_t^1 \rho^{(s)}(x)dx &= -t\rho^{(s)}(t) + \frac{\int_t^1 x^{-\alpha}(1-x)^{\alpha-1+s}dx}{\Gamma(\alpha)\Gamma(2-\alpha)} \\ &= -\frac{t^{1-\alpha}}{\alpha\Gamma(\alpha)\Gamma(2-\alpha)} + \frac{\int_t^1 (x^{-\alpha}(1-x)^{\alpha-1+s}-1)dx}{\Gamma(\alpha)\Gamma(2-\alpha)} + \frac{\int_t^1 x^{-\alpha}dx}{\Gamma(\alpha)\Gamma(2-\alpha)} + O(t^{2-\alpha}) \\ &= \frac{t^{1-\alpha}}{(\alpha-1)\Gamma(\alpha+1)\Gamma(2-\alpha)} + \frac{\int_0^1 x^{-\alpha}((1-x)^{\alpha-1+s}-1)dx}{\Gamma(\alpha)\Gamma(2-\alpha)} - \frac{1}{(\alpha-1)\Gamma(\alpha)\Gamma(2-\alpha)} + O(t^{2-\alpha}),\end{aligned}$$

which gives also the existence and the first definition of $C^{(s)}$.

If $s = 1-\alpha$, $C^{(s)} = \frac{1}{(1-\alpha)\Gamma(\alpha)\Gamma(2-\alpha)}$. If $s > 1-\alpha$, using again integration by parts obtains $C^{(s)} = \frac{\int_0^1 x^{-\alpha}((1-x)^{\alpha-1+s}-1)dx}{\Gamma(\alpha)\Gamma(2-\alpha)} - \frac{1}{(\alpha-1)\Gamma(\alpha)\Gamma(2-\alpha)} = \frac{\Gamma(\alpha+s)}{\Gamma(s+1)\Gamma(\alpha)(1-\alpha)}$.

We then define two values $A := \displaystyle\int_0^1 ((1-x)^{1-\alpha} - 1 - (\alpha-1)x)\nu^{(1)}(dx)$, $B := \displaystyle\int_0^1 ((1-x)^{2(1-\alpha)} - 1 - 2(\alpha-1)x)\nu^{(2)}(dx)$, which will be used many times later.

Lemma 2.2. *If A, B are defined as above, then $A = \alpha(\alpha^2-\alpha-1)\Gamma(\alpha-1)$ and $B = \frac{1}{(\alpha-1)}(\frac{\Gamma(4-\alpha)}{\Gamma(4-2\alpha)} + (\alpha^2-\alpha-1)\Gamma(\alpha+2))$.*

Proof. Using integration by parts two times,

$$\begin{aligned}A &= \frac{\alpha}{\Gamma(2-\alpha)}\frac{1}{\alpha(\alpha-1)}\int_0^1 x^{1-\alpha}\left(-\alpha(\alpha-1)(1-x)^{\alpha-2} + 2\alpha(\alpha-1)(1-x)^{\alpha-1} - \alpha(\alpha-1)^2 x(1-x)^{\alpha-2}\right)dx \\ &= \frac{1}{\Gamma(2-\alpha)(\alpha-1)}\left(-\Gamma(\alpha+1)\Gamma(2-\alpha) + 2(\alpha-1)\Gamma(\alpha+1)\Gamma(2-\alpha) - (\alpha-1)\Gamma(3-\alpha)\Gamma(\alpha+1)\right) \\ &= \alpha(\alpha^2-\alpha-1)\Gamma(\alpha-1).\end{aligned}$$

In the same way, one gets $B = \frac{1}{(\alpha-1)}(\frac{\Gamma(4-\alpha)}{\Gamma(4-2\alpha)} + (\alpha^2-\alpha-1)\Gamma(\alpha+2))$.

2.2. The main result

Theorem 4.

$$\mathbb{E}[T_1^{(n)}] = (\alpha-1)\Gamma(\alpha+1)n^{1-\alpha} + \frac{(\alpha-1)^2(\Gamma(\alpha+1))^2}{2-\alpha}\Big(A+(\alpha-1)C^{(1)}-C^{(0)}\Big)n^{2(1-\alpha)} + o(n^{2(1-\alpha)}).$$

The idea is to use the recurrence satisfied by $\mathbb{E}[T_1^{(n)}]$(see [14]):

$$\mathbb{E}[T_1^{(n)}] = \frac{1}{g_n} + \sum_{k=2}^{n-1} p_{n,k}\frac{k-1}{n}\mathbb{E}[T_1^{(k)}]. \tag{7}$$

Let $L = (\alpha-1)\Gamma(\alpha+1)$ and $Q = \frac{(\alpha-1)^2(\Gamma(\alpha+1))^2}{2-\alpha}(A+(\alpha-1)C^{(1)}-C^{(0)})$. We transform the recurrence (7) to

$$\begin{aligned}\Big(\mathbb{E}[n^{\alpha-1}T_1^{(n)}]-L\Big)n^{\alpha-1}-Q &= \left(\frac{n^{\alpha-1}}{g_n}-(1-\sum_{k=2}^{n-1}p_{n,k}\frac{k-1}{n}(\frac{n}{k})^{\alpha-1})L\right)n^{\alpha-1}-Q(1-\sum_{k=2}^{n-1}p_{n,k}\frac{k-1}{n}(\frac{n}{k})^{2(\alpha-1)})\\ &\quad+\sum_{k=2}^{n-1}(\frac{n}{k})^{2(\alpha-1)}p_{n,k}\frac{k-1}{n}\Big(k^{\alpha-1}(\mathbb{E}[k^{\alpha-1}T_1^{(k)}]-L)-Q\Big). \end{aligned}\tag{8}$$

Hence we get a recurrence

$$a_n = b_n + \sum_{k=2}^{n-1} q_{n,k}a_k, \tag{9}$$

with

$$a_n = \Big(\mathbb{E}[n^{\alpha-1}T_1^{(n)}]-L\Big)n^{\alpha-1}-Q,$$

$$b_n = \left(\frac{n^{\alpha-1}}{g_n}-(1-\sum_{k=2}^{n-1}p_{n,k}\frac{k-1}{n}(\frac{n}{k})^{\alpha-1})L\right)n^{\alpha-1}-Q(1-\sum_{k=2}^{n-1}p_{n,k}\frac{k-1}{n}(\frac{n}{k})^{2(\alpha-1)}),$$

$$q_{n,k} = (\frac{n}{k})^{2(\alpha-1)}p_{n,k}\frac{k-1}{n}.$$

With this notations, the theorem can be written $\lim_{n\to+\infty} a_n = 0$. It is then natural to study the behaviour of b_n when n tends to ∞. To this aim, we should get asymptotics of $1/g_n$, and $\sum_{k=2}^{n-1}p_{n,k}\frac{(k-1)_l}{(n)_l}(\frac{n}{k})^r$, $r\geq 0$ and $l\in\mathbb{N}$, where $(n)_l$ is (the same for $(k-1)_l$):

$$(n)_l = \begin{cases} n(n-1)(n-2)\cdots(n-l+1) & \text{if } n\geq l\geq 1,\\ 0 & \text{if } l>n\geq 1.\end{cases}$$

2.2.1. *Asymptotics of* $1/g_n$ For any $c,d\in\mathbb{R}$, we have

$$\frac{\Gamma(n+c)}{\Gamma(n+d)} = n^{c-d}(1+(c-d)\frac{c+d-1}{2}n^{-1}+O(n^{-2})). \tag{10}$$

This is a straightforward consequence of Stirling's formula:

$$\Gamma(z) = \sqrt{2\pi}z^{z-1/2}e^{-z}(1+\frac{1}{12z}+O(\frac{1}{z^2})), z>0. \tag{11}$$

Then we can proceed to: For any real numbers a and $b>-1$,

$$\int_0^1 (1-t)^{n+a}t^b dx = \frac{\Gamma(n+a+1)\Gamma(b+1)}{\Gamma(n+a+b+2)} = \Gamma(b+1)n^{-1-b}\left(1+(-1-b)\frac{b+2a+2}{2}n^{-1}+O(n^{-2})\right). \tag{12}$$

Using (12), we get the following lemma.

Lemma 2.3. *For Beta$(2-\alpha,\alpha)$-coalescents, we have*

$$g_n = \frac{n^\alpha}{\Gamma(\alpha+1)} - \Big(\frac{\alpha(\alpha-1)}{2\Gamma(\alpha+1)} + \frac{2-\alpha}{\Gamma(\alpha)}\Big)n^{\alpha-1} + o(n^{\alpha-1}),$$

and

$$\frac{1}{g_n} = \Gamma(\alpha+1)\left(1 + (-\alpha^2/2 + 3\alpha/2)n^{-1} + o(n^{-1})\right) n^{-\alpha}. \tag{13}$$

Proof. It is straightforward using Lemma 2.1 and $g_n^{(s)} = n(n-1)\int_0^1 t(1-t)^{n-2}\rho^{(s)}(t)dt$ for any $s > -\alpha$.

2.2.2. *Calculus of* $\sum_{k=2}^{n-1} p_{n,k}\frac{(k-1)_l}{(n)_l}(\frac{n}{k})^r$

Lemma 2.4. *Consider any Λ-coalescent process, associated with measure ν. Let $l \in \{1, 2, \cdots, n-2\}$ fixed. Then for any real function f:*

$$\sum_{k=2}^{n-1} p_{n,k}\frac{(k-1)_l}{(n)_l}f(k) = \mathbb{E}\Big[\frac{(n-1-X_1^{(n)})_l}{(n)_l}\Big]\mathbb{E}^{\nu^{(l)}}[f(n - X_1^{(n-l)})],$$

where $\mathbb{E}^{\nu^{(l)}}[*]$ means that the Λ-coalescent is associated with the measure $\nu^{(l)}$.

Proof. Recall the definitions of g_n and $p_{n,k}$(see (1), (2)). We have

$$\begin{aligned}
\sum_{k=2}^{n-1} p_{n,k}\frac{(k-1)_l}{(n)_l} &= \sum_{k=l+1}^{n-1} \frac{\int_0^1 \binom{n-l}{n-k+1} x^{n-k+1}(1-x)^{k-1}\nu(dx)}{g_n} \\
&= \sum_{k=l+1}^{n-1} \frac{\int_0^1 \binom{n-l}{n-k+1} x^{n-k+1}(1-x)^{k-1-l}\nu^{(l)}(dx)}{g_n} \\
&= \sum_{k=1}^{n-1-l} \frac{\int_0^1 \binom{n-l}{n-k-l+1} x^{n-k-l+1}(1-x)^{k-1}\nu^{(l)}(dx)}{g_n} = \frac{g_{n-l}^{(l)}}{g_n}.
\end{aligned} \tag{14}$$

Then,

$$\begin{aligned}
\sum_{k=2}^{n-1} p_{n,k}\frac{(k-1)_l}{(n)_l}f(k) &= \left(\sum_{k=2}^{n-1} p_{n,k}\frac{(k-1)_l}{(n)_l}\right)\frac{\sum_{k=2}^{n-1} p_{n,k}\frac{(k-1)_l}{(n)_l}f(k)}{\sum_{k=2}^{n-1} p_{n,k}\frac{(k-1)_l}{(n)_l}} \\
&= \mathbb{E}\Big[\frac{(n-1-X_1^{(n)})_l}{(n)_l}\Big]\frac{\sum_{k=l+1}^{n-1}\int_0^1 \binom{n-l}{n-k+1} x^{n-k+1}(1-x)^{k-1-l}f(k)\nu^{(l)}(dx)}{g_{n-l}^{(l)}} \\
&= \mathbb{E}\Big[\frac{(n-1-X_1^{(n)})_l}{(n)_l}\Big]\frac{\sum_{k=1}^{n-1-l}\int_0^1 \binom{n-l}{n-k-l+1} x^{n-k-l+1}(1-x)^{k-1}f(k+l)\nu^{(l)}(dx)}{g_{n-l}^{(l)}} \\
&= \mathbb{E}\Big[\frac{(n-1-X_1^{(n)})_l}{(n)_l}\Big]\mathbb{E}^{\nu^{(l)}}[f(Y_1^{(n-l)}+l)] = \mathbb{E}\Big[\frac{(n-1-X_1^{(n)})_l}{(n)_l}\Big]\mathbb{E}^{\nu^{(l)}}[f(n-X_1^{(n-l)})].
\end{aligned}$$

This achieves the proof of the lemma.

In consequence,

$$\sum_{k=2}^{n-1} p_{n,k}\frac{(k-1)_l}{(n)_l}\Big(\frac{n}{k}\Big)^r = \mathbb{E}\Big[\frac{(n-1-X_1^{(n)})_l}{(n)_l}\Big]\mathbb{E}^{\nu^{(l)}}\Big[\Big(\frac{n}{n-X_1^{(n-l)}}\Big)^r\Big]. \tag{15}$$

We have to study $\mathbb{E}[\frac{(n-1-X_1^{(n)})_l}{(n)_l}]$ and $\mathbb{E}^{\nu^{(l)}}[(\frac{n}{n-X_1^{(n-l)}})^r]$. The latter is given by Proposition 5.1 in appendix A. The following lemma studies the former.

Lemma 2.5. *Consider a Beta$(2-\alpha,\alpha)$ n-coalescent. Let $l \in \{1,2,\cdots,n-2\}$ fixed. We have*

$$\mathbb{E}[\frac{(n-1-X_1^{(n)})_l}{(n)_l}] = 1 - \frac{l\alpha}{n(\alpha-1)} + \Gamma(\alpha+1)\left(\sum_{j=2}^{l}\binom{l}{j}(-1)^j\int_0^1 x^j\nu(dx) - C^{(0)}l\right)n^{-\alpha} + o(n^{-\alpha}),$$

Proof. We have

$$\mathbb{E}[\frac{(n-1-X_1^{(n)})_l}{(n)_l}] = \mathbb{E}[1 - \sum_{i=0}^{l-1}\frac{X_1^{(n)}+1}{n-i} + \sum_{j=2}^{l}\sum_{i_1,\cdots,i_j \text{ all different}}(-1)^j\frac{(X_1^{(n)}+1)^j}{(n-i_1)(n-i_2)\cdots(n-i_j)}].$$

For $\mathbb{E}[\sum_{i=0}^{l-1}\frac{X_1^{(n)}+1}{n-i}]$, we use Lemma 5.2 in appendix B. While using Lemme 5.3, we get

$$\begin{aligned}&\mathbb{E}[\sum_{j=2}^{l}\sum_{i_1,\cdots,i_j \text{ all different}}(-1)^j\frac{(X_1^{(n)}+1)^j}{(n-i_1)(n-i_2)\cdots(n-i_j)}]\\ =&n^{-\alpha}\Gamma(\alpha+1)\sum_{j=2}^{l}\binom{l}{j}(-1)^j\int_0^1 x^j\nu(dx) + O(n^{-\min\{1+\alpha,j\}}).\end{aligned}$$

Then we conclude.

Now we can give the estimate of $\sum_{k=2}^{n-1}p_{n,k}\frac{(k-1)_l}{(n)_l}(\frac{n}{k})^r$ using (15), Lemma 2.5 and Proposition 5.1.

Proposition 2.1. *Consider a Beta$(2-\alpha,\alpha)$ n-coalescent. Let $l \in \{1,2,\cdots,n-2\}$ and $r \in [0,\alpha+l)$ fixed. We have*

$$\begin{aligned}&\sum_{k=2}^{n-1}p_{n,k}\frac{(k-1)_l}{(n)_l}(\frac{n}{k})^r\\ =&1 + \frac{(r-l\alpha)}{n(\alpha-1)} + \Gamma(\alpha+1)\left(\int_0^1((1-x)^{-r}-1-rx)\nu^{(l)}(dx) + \sum_{j=2}^{l}\binom{l}{j}(-1)^j\int_0^1 x^j\nu(dx) + rC^{(l)} - lC^{(0)}\right)n^{-\alpha}\\ &+ o(n^{-\alpha}).\end{aligned}$$

2.3. Proof of Theorem 4.

Recall the transformation (8) and the associated recurrence (9). The aim is to prove that $\lim_{n\to+\infty} a_n = 0$ for a_n in (9). Using Proposition 2.1, we get

$$1 - \sum_{k=2}^{n-1}p_{n,k}\frac{k-1}{n}(\frac{n}{k})^{\alpha-1} = \frac{1}{n(\alpha-1)} - \Gamma(\alpha+1)\left(A + (\alpha-1)C^{(1)} - C^{(0)}\right)n^{-\alpha} + o(n^{-\alpha}),$$

and

$$1 - \sum_{k=2}^{n-1}p_{n,k}\frac{k-1}{n}(\frac{n}{k})^{2(\alpha-1)} = \frac{2-\alpha}{n(\alpha-1)} + O(n^{-\alpha}).$$

Hence we deduce that $b_n = o(n^{-1})$.

Let $\varepsilon > 0$ such that $2(\alpha-1)+\epsilon < \alpha$. We have $1 - \sum_{k=2}^{n-1}p_{n,k}\frac{k-1}{n}(\frac{n}{k})^{2(\alpha-1)+\varepsilon} = O(n^{-1}) > 0$. The recurrence (9) satisfies the assumptions of Lemma 5.1 which leads to $\lim_{n\to+\infty} a_n = 0$. Then we can conclude.

3. Estimate of $\mathbb{E}[T_1^{(n)}T_2^{(n)}]$ and proof of Theorem 2

Using Theorem 1.1 in [14], we have

$$\mathbb{E}[T_1^{(n)}T_2^{(n)}] = \frac{2\mathbb{E}[T_1^{(n)}]}{g_n} + \sum_{k=2}^{n-1} p_{n,k}\frac{(k-1)_2}{(n)_2}\mathbb{E}[T_1^{(k)}T_2^{(k)}]. \tag{16}$$

As a consequence of (13) and Theorem 4, we have

$$\frac{2\mathbb{E}[T_1^{(n)}]}{g_n} = 2(\Gamma(\alpha+1))^2 n^{1-2\alpha}\left(\alpha - 1 + \frac{(\alpha-1)^2\Gamma(\alpha+1)}{2-\alpha}(A + (\alpha-1)C^{(1)} - C^{(0)})n^{1-\alpha}\right) + o(n^{2-3\alpha}).$$

Using the recurrence method described in the previous section, a direct calculation gives that

$$\begin{aligned}
&\mathbb{E}[T_1^{(n)}T_2^{(n)}]\\
&= ((\alpha-1)\Gamma(\alpha+1))^2 n^{2(1-\alpha)}\\
&\quad + \frac{\alpha-1}{3-\alpha}((\alpha-1)\Gamma(\alpha+1))^3\left(B + 2(\alpha-1)C^{(2)} + 1 - 2C^{(0)} + \frac{2}{2-\alpha}(A + (\alpha-1)C^{(1)} - C^{(0)})\right)n^{3(1-\alpha)}\\
&\quad + o(n^{3(1-\alpha)}).
\end{aligned}$$

Now together with Theorem 4, we can get the estimate of $\mathrm{Cov}(T_1^{(n)}, T_2^{(n)})$.

$$\mathrm{Cov}(T_1^{(n)}, T_2^{(n)}) = \frac{((\alpha-1)\Gamma(\alpha+1))^3}{3-\alpha}\left(B - 2A + 2(\alpha-1)(C^{(2)} - C^{(1)}) + 1\right)n^{3(1-\alpha)} + o(n^{3(1-\alpha)}).$$

Then

$$\Delta(\alpha) = \frac{((\alpha-1)\Gamma(\alpha+1))^3}{3-\alpha}\left(B - 2A + 2(\alpha-1)(C^{(2)} - C^{(1)}) + 1\right). \tag{17}$$

It is straightforward to see that $\Delta(\alpha) = \frac{((\alpha-1)\Gamma(\alpha+1))^2\Gamma(4-\alpha)}{(3-\alpha)\Gamma(4-2\alpha)}$ by recalling the values of $A, B, C^{(1)}$ and $C^{(2)}$. We prove then that $\Delta(\alpha) = \frac{\int_0^1((1-x)^{2-\alpha}-1)^2\nu(dx)}{3-\alpha}((\alpha-1)\Gamma(\alpha+1))^3$. Notice that

$$B - 2A = \int_0^1\left((1-x)^{2(2-\alpha)} - 2(1-x)^{2-\alpha} + 1 - x^2 + 2(\alpha-1)x^2(1-x)\right)\nu(dx).$$

By definition,

$$C^{(2)} - C^{(1)} = \lim_{t\to+\infty}\int_t^1(\rho^{(2)}(x) - \rho^{(1)}(x))dx = \lim_{t\to0}\int_t^1 x(\nu^{(2)}(dx) - \nu^{(1)}(dx)) = \int_0^1 -x^2(1-x)\nu(dx),$$

and $\int_0^1 x^2\nu(dx) = 1$. Then it allows to conclude.

4. Proof of Theorem 3

Notice that $n^{\alpha-1}T_1^{(n)} \xrightarrow{(d)} T$ and if $\beta \geq \frac{\alpha}{\alpha-1}$, one gets $\mathbb{E}[T^\beta] = +\infty$, hence $\mathbb{E}[(n^{\alpha-1}T_1^{(n)})^\beta]$ converges to $+\infty$ (see Lemma 4.11 of [28]). If $0 \leq \beta_1 < \beta_2 < \frac{\alpha}{\alpha-1}$ and $(\mathbb{E}[(n^{\alpha-1}T_1^{(n)})^{\beta_2}], n \geq 2)$ is bounded. Then $((n^{\alpha-1}T_1^{(n)})^{\beta_1}, n \geq 2)$ is uniformly integrable (see Lemma 4.11 of [28] and Problem 14 in section 8.3 [11]). Then we need only to prove that for $\beta \in [2, \frac{\alpha}{\alpha-1})$, $(\mathbb{E}[(n^{\alpha-1}T_1^{(n)})^\beta], n \geq 2)$ is bounded.

We will prove by induction on n that there exists a constant $C > 0$ such that for all $n \geq 2$, $(\mathbb{E}[n^{\alpha-1}T_1^{(n)}])^\beta \leq C$. We first assume that, for all $2 \leq k \leq n-1$, $(\mathbb{E}[k^{\alpha-1}T_1^{(k)}])^\beta \leq C$ and then will prove that (if C is large enough) $(\mathbb{E}[n^{\alpha-1}T_1^{(n)}])^\beta \leq C$.

Writing the decomposition of $T_1^{(n)}$ at the first coalescence, we have

$$T_1^{(n)} = \frac{e_0}{g_n} + \sum_{k=2}^{n-1} \mathbf{1}_{\{H_{n,k}\}} \bar{T}_1^{(k)},$$

where:

- $H_{n,k}$ is the event: {From n individuals, we have k individuals after the first coalescence, and individual 1 is not involved in this collision}, $2 \leq k \leq n-1$;
- e_0 is a unit exponential random variable, $\bar{T}_1^{(k)} \overset{(d)}{=} T_1^{(k)}$, and all these random variables e_0, $\bar{T}_1^{(k)}$, $\mathbf{1}_{\{H_{n,k}\}}$ are independent. One notices that $\mathbb{P}(H_{n,k}) = p_{n,k}\frac{k-1}{n}$ (see (7)).

Using Lemma 5.6 in Appendix D, we have the following inequality.

$$\mathbb{E}[(T_1^{(n)})^\beta] = \mathbb{E}[((\frac{e_0}{g_n} + \sum_{k=2}^{n-1} \mathbf{1}_{\{H_{n,k}\}} \bar{T}_1^{(k)}))^\beta] \leq I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4} \tag{18}$$

where

$$I_{n,1} = \mathbb{E}[(\frac{e_0}{g_n})^\beta], \quad I_{n,2} = \mathbb{E}[(\sum_{k=2}^{n-1} \mathbf{1}_{\{H_{n,k}\}} \bar{T}_1^{(k)})^\beta],$$
$$I_{n,3} = \mathbb{E}[\beta 2^{\beta-1} \frac{e_0}{g_n} (\sum_{k=2}^{n-1} \mathbf{1}_{\{H_{n,k}\}} \bar{T}_1^{(k)})^{\beta-1}] \text{ and } I_{n,4} = \mathbb{E}[\beta 2^{\beta-1} (\frac{e_0}{g_n})^{\beta-1} \sum_{k=2}^{n-1} \mathbf{1}_{\{H_{n,k}\}} \bar{T}_1^{(k)}].$$

We first bound $I_{n,1}$. Recall that $g_n \sim \dfrac{n^\alpha}{\Gamma(\alpha+1)}$. Hence there exists a constant $K_1 > 0$ (which depends on β) such that for any $n \geq 2$,

$$n^{(\alpha-1)\beta} I_{n,1} \leq \frac{K_1}{n}. \tag{19}$$

We now consider $I_{n,2}$. Notice that $(\alpha-1)\beta < \alpha+1$. Hence, using Proposition 2.1, we have

$$n^{(\alpha-1)\beta} I_{n,2} = n^{-(\alpha-1)\beta} \sum_{k=2}^{n-1} p_{n,k} \frac{k-1}{n} (\frac{n}{k})^{(\alpha-1)\beta} \mathbb{E}[(k^{\alpha-1} T_1^{(k)})^\beta] \tag{20}$$
$$\leq C \sum_{k=2}^{n-1} p_{n,k} \frac{k-1}{n} (\frac{n}{k})^{(\alpha-1)\beta} \tag{21}$$
$$= C(1 - \frac{\alpha - (\alpha-1)\beta}{n(\alpha-1)} + o(n^{-1})) \qquad \leq C(1 - \frac{\alpha-(\alpha-1)\beta}{2n(\alpha-1)}), \tag{22}$$

for $n \geq N$, where N is a fixed positive integer.

We now proceed to $I_{n,3}$. Notice that for $2 \leq k \leq n-1$,

$$\mathbb{E}[(k^{\alpha-1} T_1^{(k)})^{\beta-1}] \leq (\mathbb{E}[(k^{\alpha-1} T_1^{(k)})^\beta])^{\frac{\beta-1}{\beta}} \leq C^{\frac{\beta-1}{\beta}}.$$

Hence we have

$$\begin{aligned}
n^{(\alpha-1)\beta}I_{n,3} &= n^{(\alpha-1)\beta}\mathbb{E}[\beta 2^{\beta-1}\frac{e_0}{g_n}\sum_{k=2}^{n-1}\mathbf{1}_{\{H_{n,k}\}}(\bar{T}_1^{(k)})^{\beta-1}] \\
&\le C^{\frac{\beta-1}{\beta}}\beta 2^{\beta-1}n^{\alpha-1}g_n^{-1}\sum_{k=2}^{n-1}p_{n,k}\frac{k-1}{n}\left(\frac{n}{k}\right)^{(\alpha-1)(\beta-1)} \\
&= C^{\frac{\beta-1}{\beta}}n^{\alpha-1}\beta 2^{\beta-1}g_n^{-1}(1-\frac{\alpha-(\alpha-1)(\beta-1)}{n(\alpha-1)}+o(n^{-1})) \\
&\le \frac{C^{\frac{\beta-1}{\beta}}K_2}{n},
\end{aligned} \tag{23}$$

where K_2 is a positive constant. In the second equality, we have used Proposition 2.1.

While for any $n \ge 2$,

$$\begin{aligned}
n^{(\alpha-1)\beta}I_{n,4} &= n^{(\alpha-1)\beta}\mathbb{E}[\beta 2^{\beta-1}(\frac{e_0}{g_n})^{\beta-1}\sum_{k=2}^{n-1}\mathbf{1}_{\{H_{n,k}\}}\bar{T}_1^{(k)}] \\
&\le \beta 2^{\beta-1}\mathbb{E}[e_0^{\beta-1}](g_n)^{1-\beta}n^{(\alpha-1)(\beta-1)}\mathbb{E}[n^{\alpha-1}T_1^{(n)}] \\
&\le \frac{K_3}{n^{\beta-1}} \le \frac{K_3}{n},
\end{aligned} \tag{24}$$

where K_3 is a positive constant. We have used Lemma 4 to bound $\mathbb{E}[n^{\alpha-1}T_1^{(n)}]$.

Using (18),(19),(20),(23),(24), we have proved that for any $n, n \ge N$, if there exists $C > 0$ such that for all $2 \le k \le n-1$, $\mathbb{E}[\left(k^{\alpha-1}T_1^{(k)}\right)^{\beta}] \le C$, then

$$\mathbb{E}[(n^{\alpha-1}T_1^{(n)})^{\beta}] \le \frac{C+\left(K_1 - C^{\frac{\alpha-(\alpha-1)\beta}{2(\alpha-1)}} + C^{\frac{\beta-1}{\beta}}K_2 + K_3\right)}{n}. \tag{25}$$

Let C large enough such that

$$K_1 - C\frac{\alpha-(\alpha-1)\beta}{2(\alpha-1)} + C^{\frac{\beta-1}{\beta}}K_2 + K_3 < 0, \tag{26}$$

Then $\mathbb{E}[(n^{\alpha-1}T_1^{(n)})^{\beta}] \le C$, which allows to conclude.

5. Appendix

A) The main recurrence tool

Lemma 5.1. *We consider the recurrence $a_n = b_n + \sum_{k=1}^{n-1} q_{n,k}a_k$. We assume that $b_n = o(n^{-1})$ and that there exist $\varepsilon > 0$ and $C > 0$ such that $1 - \sum_{k=1}^{n-1} q_{n,k}(\frac{n}{k})^{\varepsilon} \ge Cn^{-1}$ for n large enough. Then $\lim_{n\to+\infty} a_n = 0$.*

Proof. Let $(\bar{c}_n)_{n\ge1}$ be an increasing sequence such that

$$\lim_{n\to+\infty}\bar{c}_n = +\infty; \lim_{n\to+\infty} nb_n\bar{c}_n = 0.$$

Define another sequence $(c_n)_{n\ge1}$ by: $c_1 = \bar{c}_1$. For $n \ge 1$,

$$c_{n+1} = \min\{c_n(\frac{n+1}{n})^{\varepsilon}, \bar{c}_{n+1}\},$$

Then we have $\lim_{n\to+\infty} c_n = +\infty, c_nb_n = o(n^{-1})$ and for any $1 \leq k \leq n-1$, $\frac{c_n}{c_k} \leq (\frac{n}{k})^{\varepsilon}$. In consequence, $1-\sum_{k=1}^{n-1} q_{n,k}\frac{c_n}{c_k} \geq Cn^{-1}$ for n large enough. Let $n_1 > 0$ such that for $n > n_1$, we have $1-\sum_{k=1}^{n-1} q_{n,k}\frac{c_n}{c_k} > \frac{C}{n}$ and $c_nb_n < \frac{C}{2n}$ and pick a number C' such that $C' > \max\{1, c_ka_k; 1 \leq k \leq n_1\}$. We transform the original recurrence to

$$c_na_n = c_nb_n + \sum_{k=1}^{n-1}\left(q_{n,k}\frac{c_n}{c_k}\right)c_ka_k.$$

Then $c_{n_1+1}a_{n_1+1} \leq \frac{C}{2(n_1+1)} + (1-\frac{C}{n_1+1})C' \leq C'$. By induction, we prove that the sequence $(c_na_n)_{n\geq 1}$ is bounded by C'. Since c_n tends to the infinity, we get $\lim_{n\to+\infty} a_n = 0$.

Remark 5.1. We refer to [35] for a rather detailed survey on this kind of recurrence relationships.

B) Asymptotic behaviours of $X_1^{(n)}$

Lemma 5.2. *Consider the coalescent process with related measure $\nu^{(s)}$ where $s > -\alpha$. Then*

$$\mathbb{E}^{\nu^{(s)}}[X_1^{(n)}] = \frac{1}{\alpha-1} + \Gamma(\alpha+1)C^{(s)}n^{1-\alpha} + o(n^{1-\alpha}),$$

Proof. We have (see [13]):

$$\mathbb{E}^{\nu^{(s)}}[X_1^{(n)}] = \frac{\int_0^1(1-t)^{n-2}(\int_t^1\rho^{(s)}(r)dr)dt}{\int_0^1(1-t)^{n-2}t\rho^{(s)}(t)dt}$$

Lemma 2.1 gives the developments of $\rho^{(s)}(t)$ and $\int_t^1\rho^{(s)}(r)dr$. Using (10), we get

$$\int_0^1(1-t)^{n-2}(\int_t^1\rho^{(s)}(r)dr)dt = \frac{n^{\alpha-2}}{(\alpha-1)\Gamma(\alpha+1)} + C^{(s)}n^{-1} + o(n^{-1}),$$

and $\int_0^1(1-t)^{n-2}t\rho^{(s)}(t)dt = \frac{n^{\alpha-2}}{\Gamma(\alpha+1)} + O(n^{\alpha-3})$. Then we can conclude.

Lemma 5.3. *If $s > -\alpha$ and $k \geq 2$,*

$$\mathbb{E}^{\nu^{(s)}}[(\frac{X_1^{(n)}}{n})^k] = \Gamma(\alpha+1)\int_0^1 x^k\nu^{(s)}(dx)n^{-\alpha} + O(n^{-\min\{1+\alpha,k\}}).$$

Proof. Let $B_{n,x}$ denote a binomial random variable with parameter $(n,x), n\geq 2, 0\leq x\leq 1$. Recall that for $2\leq i\leq n$, $\mathbb{P}^{\nu^{(s)}}(X_1^{(n)} = i-1) = \int_0^1\binom{n}{i}x^i(1-x)^{n-i}\nu^{(s)}(dx)/g_n^{(s)} = \int_0^1\mathbb{P}(B_{n,x}=i)\nu^{(s)}(dx)/g_n^{(s)}$. Here $\mathbb{P}^{\nu^{(s)}}$ means that $X_1^{(n)}$ is related to the coalescent process with measure $\nu^{(s)}$.

$$\begin{aligned}\mathbb{E}^{\nu^{(s)}}[(\frac{X_1^{(n)}}{n})^k] &= \int_0^1\mathbb{E}[(\frac{B_{n,x}-1}{n})^k\mathbf{1}_{B_{n,x}\geq 1}])\nu^{(s)}(dx)/g_n^{(s)}\\ &= \int_0^1 n^{-k}\mathbb{E}[(B_{n,x}^k - B_{n,x})\\ &\qquad + \sum_{i=1}^{k-1}\binom{k}{i}(-1)^i(B_{n,x}^{k-i}-B_{n,x}) + (-1)^k(1-B_{n,x})\mathbf{1}_{B_{n,x}\geq 1})]\nu^{(s)}(dx)/g_n^{(s)}.\end{aligned}$$

Using Lemma 5.4 in Appendix C, we get $\mathbb{E}[(B_{n,x}^k - B_{n,x})] = (nx)^k + O(n^{k-1})x^2$. Then

$$\begin{aligned}\mathbb{E}^{\nu^{(s)}}[(\frac{X_1^{(n)}}{n})^k] &= \int_0^1 n^{-k}\left((nx)^k + O(n^{k-1})x^2\right)\nu^{(s)}(dx)/g_n^{(s)} + n^{-k}\int_0^1(-1)^k(1-nx-(1-x)^n)\nu^{(s)}(dx)/g_n^{(s)}\\ &= \Gamma(\alpha+1)\int_0^1 x^k\nu^{(s)}(dx)n^{-\alpha} + O(n^{-\min\{1+\alpha,k\}}).\end{aligned}$$

In the second equality, we have used $g_n^{(s)} \sim \frac{n^\alpha}{\Gamma(\alpha+1)}$ and also the fact that $\int_0^1 (1-nx-(1-x)^n)\nu^{(s)}(dx) \le g_n^{(s)} = \int_0^1 (1-nx(1-x)^{n-1} - (1-x)^n)\nu^{(s)}(dx)$. This achieves the proof.

Proposition 5.1. *For $s \in \mathbb{N}\bigcup\{0\}$ and $0 \le r < \alpha + s$, we have*

$$\mathbb{E}^{\nu^{(s)}}[\left(\frac{n}{n-X_1^{(n-s)}}\right)^r] = 1+\frac{r}{n(\alpha-1)}+\Gamma(\alpha+1)\left(\int_0^1((1-x)^{-r}-1-rx)\nu^{(s)}(dx)+rC^{(s)}\right)n^{-\alpha}+o(n^{-\alpha}).$$

Proof. By Taylor expansion formula, for $m \ge 2$ and $n \ge s+2$, we have,

$$\mathbb{E}^{\nu^{(s)}}[\left(\frac{n}{n-X_1^{(n-s)}}\right)^r] = \mathbb{E}^{\nu^{(s)}}[\left(\frac{1}{1-\frac{X_1^{(n-s)}}{n}}\right)^r]$$

$$= \mathbb{E}^{\nu^{(s)}}[1+r\frac{X_1^{(n-s)}}{n}+\sum_{k=2}^m \frac{\Gamma(k+r)}{\Gamma(r)k!}(\frac{X_1^{(n-s)}}{n})^k + \frac{\Gamma(m+1+r)}{\Gamma(r)m!}\int_0^{\frac{X_1^{(n-s)}}{n}}(1-t)^{-r-m-1}(\frac{X_1^{(n-s)}}{n}-t)^m dt].$$

Using Lemma 5.2 and Lemma 5.3, we have for $m \ge 2$,

$$\lim_{n\to+\infty} n^\alpha \mathbb{E}^{\nu^{(s)}}[\sum_{k=2}^m \frac{\Gamma(k+r)}{\Gamma(r)k!}(\frac{X_1^{(n-s)}}{n})^k] = \Gamma(\alpha+1)\sum_{k=2}^m \frac{\Gamma(k+r)}{\Gamma(r)k!}\int_0^1 x^k \nu^{(s)}(x).$$

In consequence,

$$\lim_{m\to+\infty}\lim_{n\to+\infty} n^\alpha \mathbb{E}^{\nu^{(s)}}[\sum_{k=2}^m \frac{\Gamma(k+r)}{\Gamma(r)k!}(\frac{X_1^{(n-s)}}{n})^k] = \Gamma(\alpha+1)\int_0^1((1-x)^{-r}-1-rx)\nu^{(s)}(dx).$$

It remains to estimate $\frac{\Gamma(m+1+r)}{\Gamma(r)m!}\mathbb{E}^{\nu^{(s)}}[\int_0^{\frac{X_1^{(n-s)}}{n}}(1-t)^{-r-m-1}(\frac{X_1^{(n-s)}}{n}-t)^m dt]$, which is the sum of two terms $P_1(m,n,s,y)$ and $P_2(m,n,s,y)$, with $0<y<1$, defined by

$$P_1(m,n,s,y) = \frac{\Gamma(m+1+r)}{\Gamma(r)m!}\mathbb{E}^{\nu^{(s)}}[\int_0^{\frac{X_1^{(n-s)}}{n}}(1-t)^{-r-m-1}(\frac{X_1^{(n-s)}}{n}-t)^m dt \mathbf{1}_{X_1^{(n-s)}\ge ny}],$$

$$P_2(m,n,s,y) = \frac{\Gamma(m+1+r)}{\Gamma(r)m!}\mathbb{E}^{\nu^{(s)}}[\int_0^{\frac{X_1^{(n-s)}}{n}}(1-t)^{-r-m-1}(\frac{X_1^{(n-s)}}{n}-t)^m dt \mathbf{1}_{X_1^{(n-s)}< ny}].$$

We first focus on $P_1(m,n,s,y)$. By Proposition 5.2 in Appendix C, we have

$$\begin{aligned} P_1(m,n,s,y) \le & \mathbb{E}^{\nu^{(s)}}[\left(\frac{n}{n-X_1^{(n-s)}}\right)^r \mathbf{1}_{X_1^{(n-s)}\ge ny}] \\ \le & \mathbb{E}^{\nu^{(s)}}[\left(\frac{n-s}{n-s-X_1^{(n-s)}}\right)^r \mathbf{1}_{X_1^{(n-s)}\ge (n-s)y}] \\ \le & n^{-\alpha}K_4 y^{-\alpha}(1-y)^{\bar{r}-r}, \end{aligned} \tag{27}$$

where $\bar{r} \in (r, \alpha+s)$ and K_4 is a number depending only on $\bar{r}$ and $\nu^{(s)}$ (it is important that it does not depend on y).

We now give an upper bound for $P_2(m,n,s,y)$. We have

$$n^\alpha P_2(m,n,s,y) = n^\alpha \frac{\Gamma(m+1+r)}{\Gamma(r)m!}\mathbb{E}^{\nu^{(s)}}[\int_0^{\frac{X_1^{(n-s)}}{n}}(1-t)^{-r-1}(\frac{X_1^{(n-s)}/n-t}{1-t})^m dt \mathbf{1}_{X_1^{(n-s)}<ny}].$$

For $t\in[0,x)$ with $0<x\leq 1$, we have $\frac{x-t}{1-t}\leq x$. Then $\int_0^{\frac{X_1^{(n-s)}}{n}}(\frac{X_1^{(n-s)}/n-t}{1-t})^m dt \leq (\frac{X_1^{(n-s)}}{n})^{m+1}$.

Hence, using Lemma 5.3, for $m>2$,

$$\begin{aligned} n^\alpha P_2(m,n,s,y) &\leq n^\alpha \frac{\Gamma(m+1+r)}{\Gamma(r)m!}(1-y)^{-r-1}\mathbb{E}[(X_1^{(n-s)}/n)^{m+1}] \\ &= (1-y)^{-r-1}\frac{\Gamma(m+1+r)}{\Gamma(r)m!}\left(\Gamma(\alpha+1)\int_0^1 x^{m+1}\nu^{(s)}(dx)+O(n^{-1})\right). \end{aligned}$$

Using Lemme 5.5 in Appendix C, we have

$$\int_0^1 x^{m+1}\nu^{(s)}(dx) = \int_0^1 x^{m+1}(1-x)^{\bar r}\nu^{(-\bar r+s)}(dx)\leq K_5 m^{-\bar r},$$

where K_5 is a positive real number depending only on $\bar r$ and $\nu^{(s)}$.

Notice that $\frac{\Gamma(m+r+1)}{\Gamma(r)m!}\sim\frac{m^r}{\Gamma(r)}$. Hence

$$P_2(m,n,s,y)\leq n^{-\alpha}(1-s)^{-r-1}m^r(O(m^{-\bar r})+o(n^{-1})). \tag{28}$$

Combining (27) and (28), we deduce that

$$\lim_{m\to+\infty}\limsup_{n\to+\infty} n^\alpha(P_1(m,n,s,y)+P_2(m,n,s,y))=0.$$

This convergence together with Lemma 5.2 and 5.3 yield this proposition.

C) Some necessary results for Appendix B

Lemma 5.4. *Let $B_{n,x}$ be a binomial random variable with parameter $(n,x), n\geq 2, 0\leq x\leq 1$. Let k be an integer such that $2\leq k\leq n$. Then*

$$nx+n(n-1)\cdots(n-k+1)x^k\leq \mathbb{E}[B_{n,x}^k]\leq (nx)^k+\binom{k}{2}n^{k-1}x^2,$$

Proof. Write $B_{n,x}=Y_1+\cdots+Y_n$, where $Y_1,\cdots,Y_n$ are independent Bernoulli random variables. Let $S:=\{\{i_1,\cdots,i_k\};1\leq i_1,\cdots,i_k\leq n\}$. Then

$$\begin{aligned} \mathbb{E}[\sum_{\{i_1,\cdots,i_k\}\in S_1} Y_{i_1}\cdots Y_{i_k}]+\mathbb{E}[\sum_{\{i_1,\cdots,i_k\}\in S_3} Y_{i_1}\cdots Y_{i_k}] &\leq \mathbb{E}[(B_{n,x})^k] \\ &\leq \mathbb{E}[\sum_{\{i_1,\cdots,i_k\}\in S_2} Y_{i_1}\cdots Y_{i_k}]+\mathbb{E}[\sum_{\{i_1,\cdots,i_k\}\in S_3} Y_{i_1}\cdots Y_{i_k}], \end{aligned}$$

where

1. $S_1:=\{\{i_1,\cdots,i_n\}\in A; i_1=\cdots=i_k\}$. Then $\mathbb{E}[\sum_{\{i_1,\cdots,i_k\}\in S_1}Y_{i_1}\cdots Y_{i_k}]=nx$.
2. $S_2:=\{\{i_1,\cdots,i_n\}\in A;\exists 1\leq p<q\leq k, i_p=i_q\}$. Then $\mathbb{E}[\sum_{\{i_1,\cdots,i_k\}\in S_2}Y_{i_1}\cdots Y_{i_k}]\leq \binom{k}{2}n^{k-1}x^2$.
3. $S_3:=\{\{i_1,\cdots,i_n\}\in A;\forall 1\leq p<q\leq k, i_p\neq i_q\}$. Then $\mathbb{E}[\sum_{\{i_1,\cdots,i_k\}\in S_3}Y_{i_1}\cdots Y_{i_k}]=n(n-1)\cdots(n-k+1)x^k$.

Then we can conclude.

Lemma 5.5. *Consider any Λ-coalescent such that $\rho(t) = Ct^{-\alpha} + o(t^{-\alpha})$. Then for every $s \geq 0$, $n \geq 2$, $\int_0^1 x^n (1-x)^s \nu(dx) \leq K_6 n^{-s}$, where K_6 is a positive constant which depends only on s and ν.*

Proof. It is clear that there exists $K_7 > 0$ such that $\rho(t) \leq K_7 t^{-\alpha}$, for all $0 < t \leq 1$. Then

$$\begin{aligned}\int_0^1 x^n (1-x)^s \nu(dx) &= \int_0^1 \rho(t)(n-(n+s)t)t^{n-1}(1-t)^{s-1}dt \\ &\leq \int_0^1 \rho(t)(n-nt)t^{n-1}(1-t)^{s-1}dt \\ &\leq nK_7 \int_0^1 t^{n-1-\alpha}(1-t)^s dt = nK_7 \frac{\Gamma(n-\alpha)\Gamma(s+1)}{\Gamma(n-\alpha+s+1)} \leq K_6 n^{-s},\end{aligned}$$

for some K_6 which only depends on K_7 and s. This achieves the proof of the lemma.

Proposition 5.2. *Let $s > -\alpha$ and $0 \leq r < \alpha + s$, $\bar{r} \in (r, \alpha + s)$. Then there exists a constant K_{11} depending only on $\bar{r}$ and s such that for all $y \in (0,1)$, $n \geq 2$,*

$$\mathbb{E}^{\nu^{(s)}}[\left(\frac{n}{n - X_1^{(n)}}\right)^r \mathbf{1}_{X_1^{(n)} \geq ny}] \leq n^{-\alpha} K_{11} y^{-\alpha}(1-y)^{\bar{r}-r}.$$

Proof. Define $\lceil x \rceil = \min\{m \in \mathbb{Z}; m \geq x\}$. We have

$$\mathbb{E}^{\nu^{(s)}}[\left(\frac{n}{n - X_1^{(n)}}\right)^r \mathbf{1}_{X_1^{(n)} \geq ny}] = \sum_{k=\lceil ny \rceil}^{n-1} \int_0^1 \binom{n}{k+1} x^{k+1}(1-x)^{n-k-1}(\frac{n}{n-k})^r \nu^{(s)}(dx)/g_n^{(s)}.$$

Using (10), there exist two positive constants K_8, K_9 such that for all $k \in \{1, 2, \ldots, n-1\}$,

$$K_8 \frac{\Gamma(n+1+r)}{\Gamma(k+2)\Gamma(n-k+r)} \leq \binom{n}{k+1}(\frac{n}{n-k})^r \leq K_9 \frac{\Gamma(n+1+r)}{\Gamma(k+2)\Gamma(n-k+r)}.$$

Moreover using integration by parts, for $1 \leq l \leq n-1$ and $0 \leq x \leq 1$, we have:

$$\begin{aligned}&\sum_{k=l}^{n-1} \frac{\Gamma(n+1+r)}{\Gamma(k+2)\Gamma(n-k+r)} x^{k+1}(1-x)^{n-k-1+r} \\ =&\frac{\Gamma(n+1+r)}{\Gamma(l+1)\Gamma(n-l+r)} \int_0^x t^l (1-t)^{n-l+r-1} dt + \frac{\Gamma(n+1+r)}{\Gamma(n+1)\Gamma(1+r)} x^n (1-x)^r - \frac{\Gamma(n+1+r)}{\Gamma(n)\Gamma(1+r)} \int_0^x t^{n-1}(1-t)^r dt.\end{aligned} \tag{29}$$

Lemma 2.1 says $\rho^{(-\bar{r}+s)}(t) = \frac{t^{-\alpha}}{\Gamma(2-\alpha)\Gamma(\alpha+1)} + o(t^{-\alpha})$. Then there exists $K_{10} > 0$, such that $\rho^{(-\bar{r}+s)}(t) \leq K_{10} t^{-\alpha}$ for all $t \in (0,1]$.

$$\begin{aligned}
&\mathbb{E}^{\nu^{(s)}}[\left(\frac{n}{n-X_1^{(n)}}\right)^{\bar{r}}\mathbf{1}_{X_1^{(n)}\geq ny}]\\
&=\sum_{k=\lceil ny\rceil}^{n-1}\frac{\int_0^1\binom{n}{k+1}(\frac{n}{n-k})^{\bar{r}}x^{k+1}(1-x)^{n-k-1}\nu^{(s)}(dx)}{g_n^{(s)}}=\sum_{k=\lceil ny\rceil}^{n-1}\frac{\int_0^1\binom{n}{k+1}(\frac{n}{n-k})^{\bar{r}}x^{k+1}(1-x)^{n-k-1+\bar{r}}\nu^{(-\bar{r}+s)}(dx)}{g_n^{(s)}}\\
&\leq K_9\frac{\int_0^1\frac{\Gamma(n+1+\bar{r})}{\Gamma(\lceil ny\rceil+1)\Gamma(n-\lceil ny\rceil+\bar{r})}\int_0^x t^{\lceil ny\rceil}(1-t)^{n-\lceil ny\rceil+\bar{r}-1}dt\nu^{(-\bar{r}+s)}(dx)}{g_n^{(s)}}+K_9\frac{\int_0^1\frac{\Gamma(n+1+\bar{r})}{\Gamma(n+1)\Gamma(1+\bar{r})}x^n(1-x)^{\bar{r}}\nu^{(-\bar{r}+s)}(dx)}{g_n^{(s)}}\\
&\leq K_9\frac{\int_0^1\frac{\Gamma(n+1+\bar{r})}{\Gamma(\lceil ny\rceil+1)\Gamma(n-\lceil ny\rceil+\bar{r})}\rho^{(-\bar{r}+s)}(t)t^{\lceil ny\rceil}(1-t)^{n-\lceil ny\rceil+\bar{r}-1}dt}{g_n^{(s)}}+K_9\frac{\int_0^1\frac{\Gamma(n+1+\bar{r})}{\Gamma(n+1)\Gamma(1+\bar{r})}x^n(1-x)^{\bar{r}}\nu^{(-\bar{r}+s)}(dx)}{g_n^{(s)}}\\
&\leq K_9K_{10}\frac{\frac{\Gamma(n+1+\bar{r})\Gamma(\lceil ny\rceil+1-\alpha)}{\Gamma(\lceil ny\rceil+1)\Gamma(n+1+\bar{r}-\alpha)}}{g_n^{(s)}}+K_6K_9\frac{\frac{\Gamma(n+1+\bar{r})}{\Gamma(n+1)\Gamma(1+\bar{r})}n^{-\bar{r}}}{g_n^{(s)}}\\
&\leq K_{11}s^{-\alpha}n^{-\alpha},
\end{aligned}$$

where for the first inequality, we use (29) with $l=\lceil ny\rceil$, in the second inequality, we have used an argument of integration by parts and for the third inequality, we bound $\rho^{(-\bar{r}+s)}(x)$ by $K_{10}x^{-\alpha}$ and we also use Lemma 5.5. For the last inequality, we use (10). Here K_{11} is a constant which depends only on $\bar{r}$ and $\nu^{(s)}$. Then we get

$$\mathbb{E}^{\nu^{(s)}}[\left(\frac{n}{n-X_1^{(n)}}\right)^{r}\mathbf{1}_{X_1^{(n)}\geq ny}]\leq(1-y)^{\bar{r}-r}\mathbb{E}^{\nu^{(s)}}[\left(\frac{n}{n-X_1^{(n)}}\right)^{\bar{r}}\mathbf{1}_{X_1^{(n)}\geq ny}]\leq K_{11}y^{-\alpha}(1-y)^{\bar{r}-r}n^{-\alpha},$$

which achieves the proof of the lemma.

Remark 5.2. If $r\geq\alpha+s$, this lemma is false. Assume that $s=0,r\geq\alpha$ and for any fixed $0<y<1$, $n\geq\frac{1}{1-y}$, we have $ny\leq n-1$ and it follows that

$$\begin{aligned}
&P_1(m,n,s,y)\\
&\geq\mathbb{E}[\left(\left(\frac{n}{n-X_1^{(n)}}\right)^{r}-1-r\frac{X_1^{(n)}}{n}-\sum_{k=2}^{m}\frac{\prod_{i=0}^{k-1}(r+i)}{k!}(\frac{X_1^{(n)}}{n})^k\right)\mathbf{1}_{X_1^{(n)}=n-1}]\\
&=\mathbb{P}(X_1^{(n)}=n-1)\left(n^r-1-r\frac{n-1}{n}-\sum_{k=2}^{m}\frac{\prod_{i=0}^{k-1}(r+i)}{k!}(\frac{n-1}{n})^k\right)\\
&=\frac{\int_0^1x^n\nu(dx)}{g_n}\left(n^r-1-r\frac{n-1}{n}-\sum_{k=2}^{m}\frac{\prod_{i=0}^{k-1}(r+i)}{k!}(\frac{n-1}{n})^k\right)\\
&\sim Cn^{-2\alpha}\left(n^r-1-r\frac{n-1}{n}-\sum_{k=2}^{m}\frac{\prod_{i=0}^{k-1}(r+i)}{k!}(\frac{n-1}{n})^k\right)
\end{aligned}$$

where C is a positive number. Then

$$\liminf_{n\to+\infty}n^{\alpha}P_1(m,n,s,y)\geq C,\forall 0<y<1.$$

Hence this remark justifies the constraint $0\leq r<\alpha+s$.

D) Results that are used to prove Theorem 3.

Lemma 5.6. *Let $a>0,b>0$, $\beta\geq1$. Then* $0<(a+b)^{\beta}\leq a^{\beta}+b^{\beta}+\beta2^{\beta-1}ab^{\beta-1}+\beta2^{\beta-1}ba^{\beta-1}$.

Proof. If $0 \leq m \leq 1$, then

$$(1+m)^\beta \leq 1 + \beta 2^{\beta-1} m \leq 1 + m^\beta + \beta 2^{\beta-1} m + \beta 2^{\beta-1} m^{\beta-1}.$$

We use that the function $m \mapsto (1+m)^\beta$ is convex and that $\beta 2^{\beta-1}$ is the derivative of $(1+m)^\beta$ at $m = 1$.

If $1 < m$, then

$$(1+m)^\beta = m^\beta (1 + \frac{1}{m})^\beta \leq (m)^\beta (1 + \beta 2^{\beta-1} \frac{1}{m}) \leq 1 + m^\beta + \beta 2^{\beta-1} m + \beta 2^{\beta-1} m^{\beta-1}.$$

Hence for all $m \geq 0$,

$$(1+m)^\beta \leq 1 + m^\beta + \beta 2^{\beta-1} m + \beta 2^{\beta-1} m^{\beta-1}.$$

Then for all $a > 0, b > 0$,

$$(a+b)^\beta = a^\beta (1 + \frac{b}{a})^\beta \leq a^\beta (1 + (\frac{b}{a})^\beta + \beta 2^{\beta-1} \frac{b}{a} + \beta 2^{\beta-1} (\frac{b}{a})^{\beta-1}) = a^\beta + b^\beta + \beta 2^{\beta-1} a b^{\beta-1} + \beta 2^{\beta-1} b a^{\beta-1}.$$

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