

# ASYMPTOTICS OF THE MINIMAL CLADE SIZE AND RELATED FUNCTIONALS OF CERTAIN BETA-COALESCENTS

ARNO SIRI-JÉGOUSSE AND LINGLONG YUAN

**ABSTRACT.** This article shows the asymptotics of distributions of various functionals of the Beta( $2-\alpha, \alpha$ )  $n$ -coalescent process with  $1 < \alpha < 2$  when  $n$  goes to infinity. This process is a Markov process taking values in the set of partitions of  $\{1, \dots, n\}$ , evolving from the initial value  $\{1\}, \dots, \{n\}$  by merging (coalescing) blocks together into one and finally reaching the absorbing state  $\{1, \dots, n\}$ . The minimal clade of 1 is the block which contains 1 at the time of coalescence of the singleton  $\{1\}$ . The limit size of the minimal clade of 1 is provided. To this, we express it as a function of the coalescence time of  $\{1\}$  and sizes of blocks at that time. Another quantity concerning the size of the largest block (at deterministic small time and at the coalescence time of  $\{1\}$ ) is also studied.

## 1. INTRODUCTION AND MAIN RESULTS

Coalescent theory was initiated by Kingman ([23, 24, 25]) to model the genealogical tree of a sample of  $n$  individuals of a certain population. The so-called Kingman  $n$ -coalescent is a continuous-time Markov chain taking values in  $\mathcal{P}_n$ , the set of partitions of  $\mathbb{N}_n = \{1, 2, \dots, n\}$ . It starts from  $n$  singletons  $\{1\}, \{2\}, \dots, \{n\}$  representing  $n$  individuals (or lineages) and at any time, each couple of blocks merges (or coalesces) independently into one block at rate 1. The process reaches almost surely in finite time the absorbing state  $\{1, 2, \dots, n\}$  which is called MRCA (most recent common ancestor). Kingman showed that the genealogy of a sample of size  $n$  in a population evolving according to the Cannings population model of size  $N$  converges in the sense of finite dimensional distribution to the Kingman  $n$ -coalescent when  $N$  goes to  $\infty$ , under some assumptions over the reproduction law in the Cannings model. Roughly speaking, it is required that one individual in the population should not have a lot of progenies so that its children occupy a large ratio of the next generation. This assumption happens to fail in some marine species (see [20], [16], [9], [1]). To model this phenomenon, Pitman ([27]) and Sagitov ([29]) introduced at the same time the  $\Lambda$   $n$ -coalescent, denoted by  $\Pi^{(n)} = (\Pi^{(n)}(t), t \geq 0)$ . It is characterized by a finite measure  $\Lambda$  on  $[0, 1]$ .

The process  $\Pi^{(n)}$  is still a continuous-time Markov chain starting from  $\{1\}, \{2\}, \dots, \{n\}$ , but with the following dynamics: at any time  $t \geq 0$ , if  $\Pi^{(n)}(t)$  has  $b$  blocks ( $b \geq 2$ ), then each  $k$ -tuple ( $2 \leq k \leq b$ ) of blocks coalesces together into one at rate

$$\lambda_{b,k} := \int_0^1 x^{k-2} (1-x)^{b-k} \Lambda(dx). \quad (1)$$

As a consequence, the rate to the next coalescence is

$$g_b := \sum_{k=2}^b \binom{b}{k} \lambda_{b,k}. \quad (2)$$

In particular, the Kingman  $n$ -coalescent is a special  $\Lambda$   $n$ -coalescent with  $\Lambda$  being the Dirac measure at 0.

---

*Date:* July 25, 2018.

*2010 Mathematics Subject Classification.* 60J25, 60F05, 92D15.

*Key words and phrases.* Beta-coalescent, Kingman's paintbox construction, minimal clade size, external branch lengths, largest bloc size, block-counting process.

Linglong Yuan benefited from the support of the "Agence Nationale de la Recherche": ANR MANEGE (ANR-09-BLAN-0215).

Definition (1) is a reformulation of two properties of the process  $\Pi^{(n)}$  (see [27]). The first one is exchangeability. Let  $\rho_n$  be a permutation of  $\mathbb{N}_n$ . The map  $\rho_n$  induces naturally a map  $\bar{\rho}_n$  on  $\mathcal{P}_n$ . Then we have

$$\bar{\rho}_n \circ \Pi^{(n)} \stackrel{(d)}{=} \Pi^{(n)}.$$

The second one is consistency. For any  $2 \leq m \leq n$ , let  $\varrho_{n,m}$  be the natural restriction from  $\mathbb{N}_n$  to  $\mathbb{N}_m$  and  $\bar{\varrho}_{n,m}$  the induced map from  $\mathcal{P}_n$  to  $\mathcal{P}_m$ . Then

$$\bar{\varrho}_{n,m} \circ \Pi^{(n)} \stackrel{(d)}{=} \Pi^{(m)}.$$

These two properties also imply that one can build a projective limit process, the so-called  $\Lambda$ -coalescent, denoted by  $\Pi = (\Pi(t), t \geq 0)$ , taking values in the set  $\mathcal{P}$  of partitions of  $\mathbb{N}$ . For any restriction  $\varrho_n$  from  $\mathbb{N}$  to  $\mathbb{N}_n$  and its induced map  $\bar{\varrho}_n$  from  $\mathcal{P}$  to  $\mathcal{P}_n$ ,

$$\bar{\varrho}_n \circ \Pi \stackrel{(d)}{=} \Pi^{(n)}.$$

The Beta( $2-\alpha, \alpha$ )-coalescent with  $0 < \alpha < 2$  is a special and important example of  $\Lambda$ -coalescents. In this case,  $\Lambda$  is the Beta measure with parameters  $2-\alpha$  and  $\alpha$ . If  $\alpha$  tends to 2, then the limit process obtained is the Kingman coalescent. If  $\alpha = 1$ , the process obtained is the celebrated Bolthausen-Sznitman coalescent ([8]). This article deals with the case  $1 < \alpha < 2$  and, for the sake of simplicity, we will refer to Beta( $2-\alpha, \alpha$ )-coalescent as Beta-coalescent. This class of coalescent processes was introduced by Schweinsberg ([31]) and deeply studied in [2],[3]. In particular, in [3], many results on the small-time behavior of various functionals of the Beta-coalescent are discovered. Meanwhile, many asymptotic studies, motivated by biology, have been developed for the Beta  $n$ -coalescent (see for example [12], [22], [13], [11]), when  $n$  goes to  $\infty$ .

In this paper, we aim to study more asymptotic results on some functionals of the Beta  $n$ -coalescent with  $1 < \alpha < 2$  when  $n$  grows to  $\infty$ . We denote

$$A \sim B,$$

if  $\frac{A}{B}$  tends deterministically or randomly to 1 in the limit, depending on different contexts. Here  $A, B$  can be functions, sequences of real values, random variables. Denote by  $\xrightarrow{a.s.}$  the almost sure convergence and by  $\xrightarrow{P}$  the convergence in probability.

The length of the external branch of individual  $i$ , also called *unicity of individual  $i$*  by biologists ([28]), is denoted by  $T_i^{(n)}$ . It is the coalescence time of  $\{i\}$ , defined as follows

$$T_i^{(n)} := \sup\{t : \{i\} \in \Pi^{(n)}(t)\}.$$

The length of a randomly chosen external branch provides a measure of the genetic variation of the population since it gives some information on the “distance” of an individual to the rest of the sample. Exchangeability of the coalescent implies that

$$T_i^{(n)} \stackrel{(d)}{=} T_1^{(n)}, \quad 1 \leq i \leq n.$$

The law of  $T_1^{(n)}$  has interested many people since the first article [7] dealing with the Kingman coalescent case. We give a short survey of results already discovered.

- Kingman:  $nT_1^{(n)}$  converges in distribution to a random variable with density  $\frac{8}{(2+t)^3} \mathbf{1}_{t \geq 0}$  ([7],[10]).
- Beta-coalescent with  $1 < \alpha < 2$ :  $n^{\alpha-1}T_1^{(n)} \xrightarrow{(d)} T$ , where  $T$  is a random variable with density function  $f_T$ :

$$f_T(t) = \frac{1}{(\alpha-1)\Gamma(\alpha)} \left(1 + \frac{t}{\alpha\Gamma(\alpha)}\right)^{-\frac{\alpha}{\alpha-1}-1} \mathbf{1}_{t \geq 0}. \quad (3)$$

(see [13] where the result is stated in a more general case).

- $\lim_{n \rightarrow \infty} \frac{g_n}{n\mu^{(n)}} = 0$  where  $g_n$  is defined in (2) and  $\mu^{(n)} = \int_{1/n}^1 x^{-1} \Lambda(dx) : \mu^{(n)} T_1^{(n)}$  is asymptotically distributed as an exponential random variable with mean 1 ([33]). This class of processes contains Beta-coalescents with  $0 < \alpha < 1$  (see also [27], [26], [19] for other proofs) and the Bolthausen-Sznitman coalescent (see also [14], [17] for other proofs).

In this paper, we prove the following

**Theorem 1.1.** *Consider a Beta  $n$ -coalescent with  $1 < \alpha < 2$ . For any fixed  $k \in \mathbb{N}$ , as  $n \rightarrow \infty$ ,*

$$n^{\alpha-1} (T_1^{(n)}, \dots, T_k^{(n)}) \xrightarrow{(d)} (T_1, \dots, T_k), \quad (4)$$

where  $(T_i, i \in \mathbb{N})$  are i.i.d. copies of  $T$  with density (3).

A similar result has been proved for Bolthausen-Sznitman coalescent ([14]), but the asymptotic independence is not true for coalescents satisfying  $\int_0^1 x^{-1} \Lambda(dx) < \infty$  ([26]).

Let  $K^{(n)} = (K^{(n)}(t), t \geq 0)$  denote the block-counting process of  $\Pi^{(n)}$ , i.e.,  $K^{(n)}(t)$  stands for the number of blocks of the partition  $\Pi^{(n)}(t)$  for  $t \geq 0$ . Define

$$Q^{(n)} := K^{(n)}((T_1^{(n)})_-) - K^{(n)}(T_1^{(n)}) + 1,$$

where  $(T_1^{(n)})_-$  is the time just prior to  $T_1^{(n)}$ . In other words,  $Q^{(n)}$  is the number of blocks involved in the coalescence event of  $\{1\}$  in  $\Pi^{(n)}$ .

**Theorem 1.2.** *Consider a Beta  $n$ -coalescent with  $1 < \alpha < 2$ .  $Q^{(n)}$  converges in law to a random variable  $Q$  taking values in  $\{2, 3, \dots\}$  such that for any  $k \geq 2$*

$$q_k := \mathbb{P}(Q = k) = \frac{(\alpha - 1)\Gamma(k - \alpha)}{\Gamma(k)\Gamma(2 - \alpha)}. \quad (5)$$

Furthermore,  $Q^{(n)}$  and  $T_1^{(n)}$  are asymptotically independent.

Notice that in the Kingman coalescent,  $Q^{(n)} = Q = 2$  almost surely. The following proposition shows that, for  $\Lambda$ -coalescents satisfying  $\int_{[0,1]} x^{-1} \Lambda(dx) < \infty$ ,  $Q^{(n)}$  converges in probability to infinity.

**Proposition 1.3.** *Consider a  $\Lambda$   $n$ -coalescent with the characteristic measure satisfying  $\int_0^1 x^{-1} \Lambda(dx) < \infty$ , then  $Q^{(n)}$  converges in probability to infinity.*

The above proposition is even true in the Bolthausen-Sznitman coalescent case (see Remark 3.2).

A quantity of interest in biology is the *minimal clade size*. It is the size of the minimal clade of a randomly chosen individual (or of the individual 1, considered in this paper). The minimal clade is the block that contains 1 at the time  $\{1\}$  is coalesced. The size of the minimal clade tells how many individuals share the genealogy with individual 1 after time  $T_1^{(n)}$ . Let us denote the minimal clade size by  $Y^{(n)}$ . In the Kingman case, Blum and François ([7]) showed that

$$\mathbb{P}(Y^{(n)} = k) = \frac{4}{(k+1)k(k-1)}, k = 2, \dots, n-1; \quad \mathbb{P}(Y^{(n)} = n) = \frac{2}{n(n-1)}.$$

Freund and Siri-Jégousse [18] studied the case of the Bolthausen-Sznitman coalescent. In this case

$$\frac{\ln Y^{(n)}}{\ln n} \xrightarrow{(d)} U_{[0,1]},$$

where  $U_{[0,1]}$  is a uniform variable over  $[0, 1]$ . Asymptotics of moments were also found.

We state out our result by at first giving some notations.

- Let  $\mu$  be Slack's probability distribution on  $[0, \infty)$  (see [32]) characterized by its Laplace transform

$$\mathcal{L}_\mu(\lambda) = \int_0^\infty e^{-\lambda x} \mu(dx) = 1 - (1 + \lambda^{1-\alpha})^{-\frac{1}{\alpha-1}}, \quad \lambda \geq 0. \quad (6)$$

- Define a 1-parameter family of laws (not a process)  $(\beta(t), t \geq 0)$  such that for any  $k \geq 1$

$$\mathbb{P}(\beta(t) = k) = \frac{1}{\Gamma(k)} \left( \frac{t}{\alpha\Gamma(\alpha)} \right)^{\frac{k-1}{\alpha-1}} \int_0^\infty e^{-x \left( \frac{t}{\alpha\Gamma(\alpha)} \right)^{\frac{1}{\alpha-1}}} x^k \mu(dx). \quad (7)$$

The following result could be regarded as a consequence of Theorem 1.2.

**Theorem 1.4.** *Consider a Beta  $n$ -coalescent with  $1 < \alpha < 2$ . Let  $(\beta_i(t), t \geq 0)_{i \geq 1}$  be i.i.d. copies of  $(\beta(t), t \geq 0)$  and  $Q, T$  be random variables defined respectively in (5) and (3). Assume that  $(\beta_i(t), t \geq 0)_{i \geq 1}, Q, T$  are all independent. Then*

$$Y^{(n)} \xrightarrow{(d)} Y = 1 + \sum_{i=1}^{Q-1} \beta_i(T). \quad (8)$$

The law of  $Y$  can be described as follows: for any  $l \geq 2$ ,

$$\mathbb{P}(Y = l) = \int_0^\infty \sum_{k=2}^l q_k \sum_{i_1 + \dots + i_{k-1} = l-1} \left( \prod_{j=1}^{k-1} \mathbb{P}(\beta(t) = i_j) \right) f_T(t) dt.$$

Next, we establish a close relation between the random variable  $Q$  and the family  $(\beta(t), t \geq 0)$ . Notice that  $\lim_{t \rightarrow 0^+} \mathbb{P}(\beta(t) = 1) = 1$ .

**Proposition 1.5.** 1) For any  $k \geq 2$ ,

$$q_k = (\alpha - 1)\Gamma(\alpha) \lim_{t \rightarrow 0^+} \frac{\mathbb{P}(\beta(t) = k)}{t}. \quad (9)$$

2) The Laplace transform of  $Q$  is

$$\mathbb{E}[e^{-\lambda Q}] = \lim_{t \rightarrow 0^+} \mathbb{E}\left[ (\alpha - 1)\Gamma(\alpha) \frac{e^{-\lambda\beta(t)} \mathbf{1}_{\beta(t) \geq 2}}{t} \right] = e^{-\lambda} \left( 1 - (1 - e^{-\lambda})^{\alpha-1} \right) \quad (10)$$

for any  $\lambda \geq 0$ .

The law of  $Y$  looks quite complicated, which may harm the applicability of the result. However the clarification given below could at some point improve the situation.

**Corollary 1.6.** *If  $k$  tends to  $\infty$ , one has  $\mathbb{P}(Y > k) \sim \frac{\int_0^\infty t^{\alpha-1} f_T(t) dt}{((\alpha-1)\Gamma(\alpha))^{\alpha-1} \Gamma(1-(\alpha-1)^2)} k^{-(\alpha-1)^2}$ .*

If  $\alpha$  goes to 1,  $k^{-(\alpha-1)^2}$  goes to 1. This is consistent with the Bolthausen-Sznitman case where  $Y = \infty$  almost surely. If  $\alpha$  tends to 2,  $k^{-(\alpha-1)^2}$  goes to  $k^{-1}$ . This is in fact not consistent with the law of  $Y$  in the Kingman case. The corollary reveals some kind of “discontinuity” between the Beta-coalescent and the Kingman coalescent.

The size of the block containing one specific integer evolves in an increasing way at different speed. It is clear that at time  $(T_1^{(n)})_-$  the block containing 1 is still of size 1 while other blocks could have grown quite a lot. One way to measure this speed is to consider the size of the largest block at time  $T_1^{(n)}$ . We denote this variable by  $\tilde{W}^{(n)}$ . The bigger  $\tilde{W}^{(n)}$  is, the more inhomogeneous the speed is. To study  $\tilde{W}^{(n)}$ , we first consider the size of the largest block at any time  $t$ , denoted by  $W^{(n)}(t)$ . In this way, we have

$$\tilde{W}^{(n)} = W^{(n)}(T_1^{(n)}).$$

**Theorem 1.7.** *Consider a Beta  $n$ -coalescent with  $1 < \alpha < 2$ ,*

$$\frac{W^{(n)}((\alpha - 1)\alpha\Gamma(\alpha)n^{1-\alpha}t)}{n^{\frac{1}{\alpha}}} \xrightarrow{(d)} W(t), \quad (11)$$

where  $W(t)$  is a positive random variable with a type-2 Gumbel law, i.e., for any  $x \geq 0$ ,  $\mathbb{P}(W(t) \leq x) = e^{-x - \alpha \frac{(\alpha-1)t}{\Gamma(2-\alpha)}}$ .

The methodology employed to prove the above theorem is similar to that used in the proof of Proposition 1.6 in [3], although there are some small differences.

The following result about  $\tilde{W}^{(n)}$  happens to be a straightforward consequence of the above theorem.

**Corollary 1.8.** *As  $n$  tends to  $\infty$ ,*

$$\frac{\tilde{W}^{(n)}}{n^{\frac{1}{\alpha}}} \xrightarrow{(d)} \tilde{W}, \quad (12)$$

where  $\tilde{W}$  is a positive random variable such that for any  $x \geq 0$ ,

$$\mathbb{P}(\tilde{W} \leq x) = \int_0^\infty e^{-x^{-\alpha} \frac{t}{\alpha \Gamma(\alpha) \Gamma(2-\alpha)}} f_T(t) dt.$$

This paper is organized as follows. In Section 2, we study external branch lengths and the block-counting process in small time and prove Theorem 1.1. In Section 3, we focus on the way of coalescing an external branch and prove Theorem 1.2, Proposition 1.3, Theorem 1.4, Proposition 1.5 and Corollary 1.6. Section 4 is devoted to the size of the largest block and Theorem 1.7 and Corollary 1.8 are proved.

## 2. EXTERNAL BRANCH LENGTHS

**2.1. Ranked  $\Lambda$ -coalescent.** Assume from now on that  $1 < \alpha < 2$ . Let  $\Pi = (\Pi(t), t \geq 0)$  be the Beta-coalescent and denote by  $K = (K(t), t > 0)$  the block-counting process of  $\Pi$ , i.e.,  $K(t)$  stands for the number of blocks of  $\Pi(t)$ . It is known that  $\Pi$  is coming down from infinity: for any  $t > 0$ ,  $K(t)$  is finite almost surely ([30]). Recall that for any  $t \geq 0$ ,  $\Pi(t)$  is an exchangeable random partition of  $\mathbb{N}$ . Applying Kingman's paintbox theorem on exchangeable random partitions ([23]), almost surely, for every block  $B \in \Pi(t)$ , there exists the following limit which is called the asymptotic frequency of  $B$ :

$$\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{i=1}^m \mathbf{1}_{i \in B}.$$

Furthermore, when  $t > 0$ , the sum of all asymptotic frequencies equals 1 ([27]). When  $t = 0$ , every block is a singleton and hence has the asymptotic frequency 0. Pitman ([27]) shows that almost surely for all  $t \geq 0$ , every block in  $\Pi(t)$  has the asymptotic frequency. Hence if  $t > 0$ , one can reorder all the asymptotic frequencies in a non-increasing way to define a sequence  $\Theta(t) = \{\theta_1(t), \theta_2(t), \dots, \theta_{K(t)}(t)\}$  where  $\theta_1(t) \geq \theta_2(t) \geq \dots \geq \theta_{K(t)}(t)$  and  $\sum_{i=1}^{K(t)} \theta_i(t) = 1$ . At time  $t = 0$ , since every block has asymptotic frequency 0, one can naturally set  $\Theta(0) = \{0, 0, \dots\}$ . Then the process  $\Theta = (\Theta(t), t \geq 0)$  is well defined and called the *ranked  $\Lambda$ -coalescent*.

Given  $\Theta(t)$  with  $t > 0$ , one can recover the distribution of  $\Pi(t)$  using again Kingman's paintbox theorem. Let us at first divide  $[0, 1]$  into  $K(t)$  intervals such that the lengths of intervals correspond one to one to the elements of  $\Theta(t)$ . Then we throw individuals  $1, 2, \dots$  uniformly and independently into  $[0, 1]$ . Finally, all individuals within one interval form a block and this procedure provides a random exchangeable partition which has the same law as  $\Pi(t)$ . It is of course possible, thanks to the consistency property, to build the restricted partition  $\Pi^{(n)}(t)$  using the same procedure but throwing nothing but  $n$  particles instead of an infinity. This construction will be the key point of our proofs.

**2.2. Properties of the ranked  $\Lambda$ -coalescent.** Let  $K(t, x) := \#\{i : \theta_i(t) \leq x\}$  for any  $x \in [0, 1]$ . Let  $\zeta(t)$  be a size-biased picking of  $\Theta(t)$ , i.e.,  $\zeta(t)$  is a discrete random variable such that

$$\mathbb{P}(\zeta(t) = \theta_i(t) | \Theta(t)) = \theta_i(t) \times \#\{j : \theta_j(t) = \theta_i(t), 1 \leq j \leq K(t)\}, \quad 1 \leq i \leq K(t). \quad (13)$$

One can construct or regard  $\zeta(t)$  in the following way: Suppose that  $[0, 1]$  is divided into  $K(t)$  intervals whose lengths are in one-to-one correspondence to the elements of  $\Theta(t)$ . We throw a particle uniformly and independently over  $[0, 1]$  and  $\zeta(t)$  is the length of the interval containing this particle.

Recall the measure  $\mu$  defined in (6). It is easy to check that

$$\int_0^\infty y \mu(dy) = \frac{d\mathcal{L}_\mu(\lambda)}{d\lambda} \Big|_{\lambda=0} = 1. \quad (14)$$

**Proposition 2.1.** *We have*

$$\lim_{t \rightarrow 0^+} \sup_{x \geq 0} \left| \mathbb{P}(\zeta(t) \leq t^{\frac{1}{\alpha-1}} x | \Theta(t)) - \int_0^{x(\alpha\Gamma(\alpha))^{\frac{1}{\alpha-1}}} y \mu(dy) \right| = 0, \text{ a.s.} \quad (15)$$

*Proof.* In order to simplify the notations, let us denote  $f(t, x) = \mathbb{P}(\zeta(t) \leq t^{\frac{1}{\alpha-1}} x | \Theta(t))$  and  $f(x) = \int_0^{x(\alpha\Gamma(\alpha))^{\frac{1}{\alpha-1}}} y \mu(dy)$ . Let

$$S_t = \sup_{x \geq 0} \left| t^{\frac{1}{\alpha-1}} K \left( t, t^{\frac{1}{\alpha-1}} x \right) - (\alpha\Gamma(\alpha))^{\frac{1}{\alpha-1}} \mu \left( [0, x(\alpha\Gamma(\alpha))^{\frac{1}{\alpha-1}}] \right) \right|.$$

It is shown in Theorem 1.4 of [3] that

$$\lim_{t \rightarrow 0^+} S_t = 0, \quad \text{a.s.} \quad (16)$$

Observe that

$$\begin{aligned} f(t, x) &= \sum_{i=0}^{K(t)} \theta_i(t) \mathbf{1}_{\{\theta_i(t) \leq t^{\frac{1}{\alpha-1}} x\}} \\ &= \sum_{j=0}^{n-1} \sum_{i=0}^{K(t)} \theta_i(t) \mathbf{1}_{\{t^{\frac{1}{\alpha-1}} \frac{jx}{n} < \theta_i(t) \leq t^{\frac{1}{\alpha-1}} \frac{(j+1)x}{n}\}}. \end{aligned} \quad (17)$$

Then

$$f(t, x) \geq I_1^{(n)} := \sum_{j=0}^{n-1} \sum_{i=0}^{K(t)} t^{\frac{1}{\alpha-1}} \frac{jx}{n} \mathbf{1}_{\{t^{\frac{1}{\alpha-1}} \frac{jx}{n} < \theta_i(t) \leq t^{\frac{1}{\alpha-1}} \frac{(j+1)x}{n}\}}$$

and

$$f(t, x) \leq I_2^{(n)} := \sum_{j=0}^{n-1} \sum_{i=0}^{K(t)} t^{\frac{1}{\alpha-1}} \frac{(j+1)x}{n} \mathbf{1}_{\{t^{\frac{1}{\alpha-1}} \frac{jx}{n} < \theta_i(t) \leq t^{\frac{1}{\alpha-1}} \frac{(j+1)x}{n}\}}.$$

For  $n$  fixed and applying (16), one gets for  $t \rightarrow 0^+$

$$I_1^{(n)} \xrightarrow{\text{a.s.}} \sum_{j=0}^{n-1} \frac{jx}{n} (\alpha\Gamma(\alpha))^{\frac{1}{\alpha-1}} \mu \left( \left( \frac{jx}{n} (\alpha\Gamma(\alpha))^{\frac{1}{\alpha-1}}, \frac{(j+1)x}{n} (\alpha\Gamma(\alpha))^{\frac{1}{\alpha-1}} \right) \right),$$

and

$$I_2^{(n)} \xrightarrow{\text{a.s.}} \sum_{j=0}^{n-1} \frac{(j+1)x}{n} (\alpha\Gamma(\alpha))^{\frac{1}{\alpha-1}} \mu \left( \left( \frac{jx}{n} (\alpha\Gamma(\alpha))^{\frac{1}{\alpha-1}}, \frac{(j+1)x}{n} (\alpha\Gamma(\alpha))^{\frac{1}{\alpha-1}} \right) \right).$$

The above two limit values converge to  $f(x)$  as  $n$  goes to  $\infty$ . Then we can conclude.  $\square$

It is straightforward to see that

**Corollary 2.2.** *For any  $f \in C_b^0[0, \infty)$  and  $c \geq 0$ ,  $M \in \mathbb{R}_+ \cup \{\infty\}$ ,*

$$\mathbb{E} \left[ f \left( ct^{-\frac{1}{\alpha-1}} \zeta(t) \right) \mathbf{1}_{\{0 \leq ct^{-\frac{1}{\alpha-1}} \zeta(t) \leq M\}} | \Theta(t) \right] \xrightarrow{\text{a.s.}} \int_0^{Mc^{-1}(\alpha\Gamma(\alpha))^{\frac{1}{\alpha-1}}} f \left( c(\alpha\Gamma(\alpha))^{-\frac{1}{\alpha-1}} y \right) y \mu(dy)$$

when  $t \rightarrow 0^+$ .

**2.3. External branches.** We start the proof of Theorem 1.1 with a simpler version.

**Proposition 2.3.** *Let  $\{T_i^{(n)}, 1 \leq i \leq k\}$  and  $T$  be as in Theorem 1.1. The following almost sure convergence holds as  $n$  goes to  $\infty$ :*

$$\mathbb{P}(n^{\alpha-1}T_1^{(n)} > t, n^{\alpha-1}T_2^{(n)} > t, \dots, n^{\alpha-1}T_k^{(n)} > t | \Theta(n^{1-\alpha}t)) \xrightarrow{a.s.} \mathbb{P}(T > t)^k \quad (18)$$

for any  $t \geq 0$ . As a consequence,

$$n^{\alpha-1}T_1^{(n)} \xrightarrow{(d)} T. \quad (19)$$

*Remark 2.1.* The convergence (19) has already been obtained in [13] using two different methods.

*Proof.* For the sake of simplicity in notations, let  $t_n = n^{1-\alpha}t$ . Let us build  $\Pi^{(n)}(t)$  from  $\Theta(t)$  and the paintbox construction (using  $n$  particles). We now prove (18) for  $k = 2$ . The proof for  $k > 2$  and  $k = 1$  follows similarly. Let  $\bar{\zeta}(t_n)$  be an independent copy of  $\zeta(t_n)$ , conditionally on  $\Theta(t_n)$ . Then,

$$\begin{aligned} & \mathbb{P}(n^{\alpha-1}T_1^{(n)} > t, n^{\alpha-1}T_2^{(n)} > t | \Theta(t_n)) \\ &= \sum_{i,j=1, i \neq j}^{K(t_n)} \theta_i(t_n)\theta_j(t_n) \left(1 - \theta_i(t_n) - \theta_j(t_n)\right)^{n-2} \\ &= \sum_{i,j=1}^{K(t_n)} \theta_i(t_n)\theta_j(t_n) \left(1 - \theta_i(t_n) - \theta_j(t_n)\right)^{n-2} - \sum_{i=1}^{K(t_n)} \theta_i(t_n)^2 \left(1 - 2\theta_i(t_n)\right)^{n-2} \\ &= \mathbb{E}\left[\left(1 - \zeta(t_n) - \bar{\zeta}(t_n)\right)^{n-2} | \Theta(t_n)\right] - \mathbb{E}\left[\zeta(t_n) \left(1 - 2\zeta(t_n)\right)^{n-2} | \Theta(t_n)\right], \end{aligned}$$

Using Corollary 2.2, the second term converges almost surely to 0. Let  $M$  be a real positive number and write the first term as

$$\mathbb{E}\left[\left(1 - \zeta(t_n) - \bar{\zeta}(t_n)\right)^{n-2} | \Theta(t_n)\right] = I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \mathbb{E}\left[\left(1 - \zeta(t_n) - \bar{\zeta}(t_n)\right)^{n-2} \mathbf{1}_{\{\zeta(t_n) \leq Mn^{-1}, \bar{\zeta}(t_n) \leq Mn^{-1}\}} | \Theta(t_n)\right], \\ I_2 &= \mathbb{E}\left[\left(1 - \zeta(t_n) - \bar{\zeta}(t_n)\right)^{n-2} \mathbf{1}_{\{\zeta(t_n) \leq Mn^{-1}, \bar{\zeta}(t_n) \leq Mn^{-1}\}^c} | \Theta(t_n)\right]. \end{aligned}$$

By Proposition 2.1,

$$I_2 \leq 1 - \mathbb{P}(\bar{\zeta}(t_n) \leq Mn^{-1} | \Theta(t_n)) \xrightarrow{a.s.} 1 - \left(1 - \int_{Mt}^{\infty} \frac{1}{t^{1-\alpha}} y \mu(dy)\right)^2.$$

The limit value goes to 0 as  $M$  tends to  $\infty$ . For  $I_1$ , notice that  $x \mapsto (1 - n^{-1}x)^{n-2}$  converges uniformly to  $x \mapsto e^{-x}$  for  $0 \leq x \leq 2M$  as  $n$  tends to  $\infty$ . Then

$$I_1 - \mathbb{E}[\exp(-n\zeta(t_n) - n\bar{\zeta}(t_n)) \mathbf{1}_{\{\zeta(t_n) \leq Mn^{-1}, \bar{\zeta}(t_n) \leq Mn^{-1}\}} | \Theta(t_n)] \xrightarrow{a.s.} 0.$$

Now, thanks to Corollary 2.2, we get

$$\begin{aligned} & \mathbb{E}[\exp(-n\zeta(t_n) - n\bar{\zeta}(t_n)) \mathbf{1}_{\{\zeta(t_n) \leq Mn^{-1}, \bar{\zeta}(t_n) \leq Mn^{-1}\}} | \Theta(t_n)] \\ &= \mathbb{E}[\exp(-n\zeta(t_n)) \mathbf{1}_{\{\zeta(t_n) \leq Mn^{-1}\}} \times \exp(-n\bar{\zeta}(t_n)) \mathbf{1}_{\{\bar{\zeta}(t_n) \leq Mn^{-1}\}} | \Theta(t_n)] \\ & \xrightarrow{a.s.} \left( \int_0^{M^{-1}(\frac{\alpha\Gamma(\alpha)}{t})^{\frac{1}{\alpha-1}}} e^{-(\frac{\alpha\Gamma(\alpha)}{t})^{\frac{1}{\alpha-1}} y} y \mu(dy) \right)^2 \\ & \xrightarrow{M \rightarrow \infty} \left(1 + \frac{t}{\alpha\Gamma(\alpha)}\right)^{-2\frac{\alpha}{\alpha-1}}. \end{aligned}$$

Then we can conclude. □

**2.4. The block-counting process in small time.** Recall that  $K^{(n)} = (K^{(n)}(t), t > 0)$  and  $K = (K(t), t > 0)$  are respectively the block-counting processes of  $\Pi^{(n)}$  and  $\Pi$ .

**Lemma 2.4.** *Let  $t > 0$  and  $t_n = n^{1-\alpha}t$ . We have*

$$\mathbb{E}[K^{(n)}(t_n)|\Theta(t_n)] = \sum_{i=1}^{K(t_n)} 1 - \left(1 - \theta_i(t_n)\right)^n. \quad (20)$$

and

$$\begin{aligned} \text{Var}(K^{(n)}(t_n)|\Theta(t_n)) &= \sum_{i=1}^{K(t_n)} \left(1 - \theta_i(t_n)\right)^n \left(1 - (1 - \theta_i(t_n))^n\right) \\ &+ \sum_{i,j=1, i \neq j}^{K(t_n)} \left(1 - \theta_i(t_n) - \theta_j(t_n)\right)^n - \left(1 - \theta_i(t_n)\right)^n \left(1 - \theta_j(t_n)\right)^n. \end{aligned} \quad (21)$$

Furthermore,

$$\frac{\mathbb{E}[K^{(n)}(t_n)|\Theta(t_n)]}{n} \xrightarrow{a.s.} \left(1 + \frac{t}{\alpha\Gamma(\alpha)}\right)^{-\frac{1}{\alpha-1}}, \quad n \rightarrow \infty. \quad (22)$$

and

$$\frac{\text{Var}(K^{(n)}(t_n)|\Theta(t_n))}{n} \xrightarrow{a.s.} \left(2^{1-\alpha} + \frac{t}{\alpha\Gamma(\alpha)}\right)^{-\frac{1}{\alpha-1}} - \left(1 + \frac{t}{\alpha\Gamma(\alpha)}\right)^{-\frac{1}{\alpha-1}}, \quad n \rightarrow \infty. \quad (23)$$

*Remark 2.2.* It can be deduced from (22) and (23) that

$$\frac{K^{(n)}(t_n)}{n\left(1 + \frac{t}{\alpha\Gamma(\alpha)}\right)^{-\frac{1}{\alpha-1}}} \xrightarrow{P} 1,$$

whereas, interestingly, due to Proposition 2.1 or Theorem 1.1 of [3],

$$\frac{K(t_n)}{n\left(\frac{t}{\alpha\Gamma(\alpha)}\right)^{-\frac{1}{\alpha-1}}} \xrightarrow{a.s.} 1. \quad (24)$$

*Proof.* The equalities (20) and (21) come directly from (4.1) and (4.2) in [21]. The arguments to prove (22) and (23) include (24) and those used in the proof of Proposition 2.1. To be more clear, we just show the proof of (22) and leave the other to the readers.

$$\begin{aligned} \frac{\mathbb{E}[K^{(n)}(t_n)|\Theta(t_n)]}{n} &= \frac{K(t_n)}{n} - \sum_{i=0}^{K(t_n)} n^{-1}(1 - \theta_i)^n \\ &\leq \frac{K(t_n)}{n} - \sum_{j=0}^{n-1} n^{-1}\left(1 - \frac{j+1}{n}\right)^n \left(K\left(t_n, \frac{j+1}{n}\right) - K\left(t_n, \frac{j}{n}\right)\right). \end{aligned}$$

In the same way,

$$\frac{\mathbb{E}[K^{(n)}(t_n)|\Theta(t_n)]}{n} \geq \frac{K(t_n)}{n} - \sum_{j=0}^{n-1} n^{-1}\left(1 - \frac{j}{n}\right)^n \left(K\left(t_n, \frac{j+1}{n}\right) - K\left(t_n, \frac{j}{n}\right)\right)$$

While (16) shows that

$$\begin{aligned} &\sup_{0 \leq j \leq n-1} \left| t_n^{\frac{1}{\alpha-1}} \left( K\left(t_n, \frac{j+1}{n}\right) - K\left(t_n, \frac{j}{n}\right) \right) - \alpha\Gamma(\alpha)^{\frac{1}{\alpha-1}} \mu\left(\left(\frac{\alpha\Gamma(\alpha)}{t}\right)^{\frac{1}{\alpha-1}} \frac{j}{n}, \left(\frac{\alpha\Gamma(\alpha)}{t}\right)^{\frac{1}{\alpha-1}} \frac{j+1}{n}\right) \right| \\ &= \sup_{0 \leq j \leq n-1} \left| n^{-1} t_n^{\frac{1}{\alpha-1}} \left( K\left(t_n, \frac{j+1}{n}\right) - K\left(t_n, \frac{j}{n}\right) \right) - \alpha\Gamma(\alpha)^{\frac{1}{\alpha-1}} \mu\left(\left(\frac{\alpha\Gamma(\alpha)}{t}\right)^{\frac{1}{\alpha-1}} \frac{j}{n}, \left(\frac{\alpha\Gamma(\alpha)}{t}\right)^{\frac{1}{\alpha-1}} \frac{j+1}{n}\right) \right| \\ &\leq 2S_{t_n} \xrightarrow{a.s.} 0. \end{aligned}$$



Notice also that

$$\lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} \left(1 - \frac{j}{n}\right)^n = \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} \left(1 - \frac{j+1}{n}\right)^n = \sum_{j=1}^{\infty} e^{-j} < \infty.$$

Hence using (24), almost surely,

$$\begin{aligned} \frac{\mathbb{E}[K^{(n)}(t_n) | \Theta(t_n)]}{n} &\sim \left(\frac{\alpha\Gamma(\alpha)}{t}\right)^{\frac{1}{\alpha-1}} \left(1 - \sum_{j=0}^{n-1} \left(1 - \frac{j}{n}\right)^n \mu\left(\left(\frac{\alpha\Gamma(\alpha)}{t}\right)^{\frac{1}{\alpha-1}} \frac{j}{n}, \left(\frac{\alpha\Gamma(\alpha)}{t}\right)^{\frac{1}{\alpha-1}} \frac{j+1}{n}\right)\right) \\ &\xrightarrow{n \rightarrow \infty} \left(\frac{\alpha\Gamma(\alpha)}{t}\right)^{\frac{1}{\alpha-1}} \left(1 - \int_0^{\infty} e^{-y\left(\frac{\alpha\Gamma(\alpha)}{t}\right)^{\frac{1}{\alpha-1}}} \mu(dy)\right) \\ &= \left(1 + \frac{t}{\alpha\Gamma(\alpha)}\right)^{-\frac{1}{\alpha-1}}. \end{aligned}$$

□

We are now able to prove our first result.

*Proof of Theorem 1.1.* We will prove only the version for  $k = 2$ . For any  $0 \leq t_1 \leq t_2$ , write

$$\begin{aligned} &\mathbb{P}(n^{\alpha-1}T_1^{(n)} > t_1, n^{\alpha-1}T_2^{(n)} > t_2) \\ &= \mathbb{P}(n^{\alpha-1}T_1^{(n)} > t_1, n^{\alpha-1}T_2^{(n)} > t_1) \mathbb{P}(n^{\alpha-1}T_2^{(n)} > t_2 | n^{\alpha-1}T_1^{(n)} > t_1, n^{\alpha-1}T_2^{(n)} > t_1). \end{aligned}$$

Proposition 2.3 gives that the first term of the above product has limit value

$$\mathbb{P}(T > t_1)^2 = \left(1 + \frac{t_1}{\alpha\Gamma(\alpha)}\right)^{-\frac{2}{\alpha-1}}.$$

Lemma 2.4 implies that conditional on  $\{n^{\alpha-1}T_1^{(n)} > t_1, n^{\alpha-1}T_2^{(n)} > t_1\}$ , the random variable  $\frac{K^{(n)}(n^{1-\alpha}t_1)}{n}$  converges in probability to  $\left(1 + \frac{t_1}{\alpha\Gamma(\alpha)}\right)^{-\frac{1}{\alpha-1}}$ . For any  $j \geq 2$ , let  $\tilde{T}_1^{(j)}$  be independent of  $\Pi^{(n)}$  and have the same law as  $T_1^{(j)}$ . Using the Markov property of  $\Pi^{(n)}$ , one obtains

$$\begin{aligned} &\mathbb{P}(n^{\alpha-1}T_2^{(n)} > t_2 | n^{\alpha-1}T_1^{(n)} > t_1, n^{\alpha-1}T_2^{(n)} > t_1) \\ &= \mathbb{P}(n^{\alpha-1}\tilde{T}_1^{(K^{(n)}(n^{1-\alpha}t_1))} > t_2 - t_1 | n^{\alpha-1}T_1^{(n)} > t_1, n^{\alpha-1}T_2^{(n)} > t_1) \\ &\xrightarrow{P} \mathbb{P}\left(T > (t_2 - t_1) \left(1 + \frac{t_1}{\alpha\Gamma(\alpha)}\right)^{-1}\right) \\ &= \left(1 + \frac{t_1}{\alpha\Gamma(\alpha)}\right)^{\frac{1}{\alpha-1}} \left(1 + \frac{t_2}{\alpha\Gamma(\alpha)}\right)^{-\frac{1}{\alpha-1}} \end{aligned}$$

when  $n$  tends to  $\infty$ . Then we can conclude. □

### 3. THE WAY OF COALESCING AN EXTERNAL BRANCH

**3.1. The size of the jump.** Let us look at the random variable  $Q^{(n)}$ .

*Proof of Theorem 1.2.* Assume that at some time  $t$ ,  $K^{(n)}(t) = b$  and  $\{1\} \in \Pi^{(n)}(t)$ . The coalescence of  $\{1\}$  with some other  $k-1$  blocks happens at rate

$$\lambda_{1,b,k} := \int_0^1 \binom{b-1}{k-1} x^k (1-x)^{b-k} x^{-2} \Lambda(dx) = \frac{\Gamma(k-\alpha)\Gamma(b-k+\alpha)}{\Gamma(\alpha)\Gamma(2-\alpha)\Gamma(k)\Gamma(b-k+1)}.$$

The total rate at which the singleton  $\{1\}$  participates in a coalescence event is

$$\begin{aligned} g_{1,b} &:= \int_0^1 \sum_{k=2}^b \binom{b-1}{k-1} x^k (1-x)^{b-k} x^{-2} \Lambda(dx) \\ &= \int_0^1 (1 - (1-x)^{b-1}) x^{-1} \Lambda(dx) \\ &= \int_0^1 (b-1)(1-t)^{b-2} \rho_1(t) dt, \end{aligned} \tag{25}$$

where

$$\rho_1(t) = \int_t^1 x^{-1} \Lambda(dx) \sim \frac{t^{1-\alpha}}{(\alpha-1)\Gamma(\alpha)\Gamma(2-\alpha)}$$

when  $t$  tends to  $0+$ . We get, thanks to Stirling's formula,

$$g_{1,b} \sim \frac{b^{\alpha-1}}{(\alpha-1)\Gamma(\alpha)}$$

when  $b$  tends to  $\infty$ . If the next coalescence after  $t$  involves  $\{1\}$ , then using the strong Markov property of  $\Pi^{(n)}$ , the probability for  $\{1\}$  to coalesce with some other  $k-1$  blocks is

$$\frac{\lambda_{1,b,k}}{g_{1,b}} \sim q_k = \frac{\Gamma(k-\alpha)(\alpha-1)}{\Gamma(k)\Gamma(2-\alpha)} \tag{26}$$

when  $b$  tends to  $\infty$ . In this way, if we know the value  $K^{(n)}((T_1^{(n)})_-)$ , then we can obtain the probability for  $\{1\}$  to coalesce with  $k-1$  blocks. Notice that  $\frac{K^{(n)}((T_1^{(n)})_-)}{n}$  converges in distribution to  $(1 + \frac{T}{\alpha\Gamma(\alpha)})^{-\frac{1}{\alpha-1}}$  (see Corollary 5.3 of [13] or implicitly from Theorem 1.1 and Lemma 2.4), one can get the following, due to (26),

$$\mathbb{P}(Q^{(n)} = k) = \mathbb{E}\left[\frac{\lambda_{1,K^{(n)}((T_1^{(n)})_-),k}}{g_{1,K^{(n)}((T_1^{(n)})_-)}}\right] \longrightarrow q_k \tag{27}$$

when  $n$  tends to  $\infty$ .

The asymptotic independence of  $T_1^{(n)}$  and  $Q^{(n)}$  is clear, since  $Q^{(n)}$  only depends on  $K^{(n)}((T_1^{(n)})_-)$  which tends to  $\infty$  in probability when  $n$  goes to  $\infty$ . Then we can conclude.  $\square$

*Remark 3.1.* Following the same arguments, Theorem 1.2 is still valid for the more general class of coalescents satisfying the following condition when  $t$  tends to 0:

$$\int_t^1 x^{-2} \Lambda(dx) \sim Ct^{-\alpha}, C > 0.$$

*Remark 3.2.* We can use similar arguments in the Bolthausen-Sznitman case to get that  $\mathbb{P}(Q^{(n)} = k) \rightarrow 0$  for any  $k \in \mathbb{N}$ . The result actually remains true for the more general class where

$$\int_t^1 x^{-2} \Lambda(dx) \sim Ct^{-1}, C > 0.$$

*Proof of Proposition 1.5.* 1) Recall that  $q_k = \frac{\Gamma(k-\alpha)(\alpha-1)}{\Gamma(k)\Gamma(2-\alpha)}$ . It then suffices to prove that

$$\lim_{t \rightarrow 0+} \frac{\mathbb{P}(\beta(t) = k)}{t} = \frac{\Gamma(k-\alpha)}{\Gamma(k)\Gamma(\alpha)\Gamma(2-\alpha)}.$$

To simplify the notations, let  $t_\alpha = \left(\frac{t}{\alpha\Gamma(\alpha)}\right)^{\frac{1}{\alpha-1}}$  and  $\rho_x = \mu([x, \infty))$ . Recall (7) and let  $0 < \eta < \frac{2-\alpha}{2}$ , then for any  $k \geq 2$ ,

$$\begin{aligned}\mathbb{P}(\beta(t) = k) &= \frac{t_\alpha^{k-1}}{\Gamma(k)} \int_0^\infty e^{-xt_\alpha} x^k \mu(dx) \\ &= \frac{t_\alpha^{k-1}}{\Gamma(k)} \int_0^\infty \rho_x e^{-xt_\alpha} x^{k-1} (k - xt_\alpha) dx \\ &= I_1 + I_2,\end{aligned}$$

where

$$\begin{aligned}I_1 &= \frac{t_\alpha^{k-1}}{\Gamma(k)} \int_0^{t_\alpha^{-\eta}} \rho_x e^{-xt_\alpha} x^{k-1} (k - xt_\alpha) dx, \\ I_2 &= \frac{t_\alpha^{k-1}}{\Gamma(k)} \int_{t_\alpha^{-\eta}}^\infty \rho_x e^{-xt_\alpha} x^{k-1} (k - xt_\alpha) dx.\end{aligned}$$

For  $t$  small enough, it is easy to get  $I_1 \leq \frac{k}{\Gamma(k)} t_\alpha^{k(1-\eta)-1} = o(t)$ . To deal with  $I_2$ , recall from Equation (33) of [3] that

$$\rho_x = \mu([x, \infty)) \sim \frac{x^{-\alpha}}{\Gamma(2-\alpha)} \quad (28)$$

when  $x$  goes to  $\infty$ . Notice that  $t_\alpha^{-\eta}$  goes to  $\infty$  when  $t$  tends to  $0+$ . Let  $0 < \varepsilon < 1$ , then for  $t$  small enough, we have

$$1 - \varepsilon < \frac{\rho_x}{x^{-\alpha}/\Gamma(2-\alpha)} \leq 1 + \varepsilon, \text{ for all } x \geq t_\alpha^{-\eta}.$$

Since  $\varepsilon$  can be arbitrarily small, using a change of variable  $y = xt_\alpha$ , one gets

$$t_\alpha^{1-\alpha} I_2 \rightarrow \frac{1}{\Gamma(k)\Gamma(2-\alpha)} \int_0^\infty e^{-y} y^{k-1-\alpha} (k-y) dy = \frac{\alpha\Gamma(k-\alpha)}{\Gamma(k)\Gamma(2-\alpha)}$$

when  $t \rightarrow 0+$ . Then we can obtain (9).

2) A simple calculation shows that  $\frac{\mathbb{P}(\beta(t) \geq 2)}{t}$  converges to  $\frac{1}{(\alpha-1)\Gamma(\alpha)}$  when  $t$  tends to  $0+$ . Hence the first equality of (10) holds, using the dominated convergence theorem. For the second equality, the formulas (7) and (6) imply that

$$\begin{aligned}\mathbb{E}[e^{-\lambda\beta(t)}] &= \sum_{k \geq 1} e^{-\lambda k} \mathbb{P}(\beta(t) = k) \\ &= e^{-\lambda} \int_0^\infty e^{(e^{-\lambda}-1)xt_\alpha} x \mu(dx) \\ &= e^{-\lambda} (1 + (t_\alpha(1 - e^{-\lambda}))^{\alpha-1})^{\frac{1}{1-\alpha}}.\end{aligned} \quad (29)$$

Meanwhile, using the same arguments

$$\mathbb{P}(\beta(t) = 1) = (1 + t_\alpha^{\alpha-1})^{\frac{1}{1-\alpha}}.$$

Then we can obtain (10). □

Let us consider the case of coalescents satisfying  $\int_0^1 x^{-1} \Lambda(dx) < \infty$ .

*Proof of Proposition 1.3.* In this case, the process  $\Pi^{(n)}$  can be constructed using a subordinator. This construction can be found on page 7 of [19] and the original idea is in [27]. Let  $\nu(dx) = x^{-2} \Lambda(dx)$  and  $\tilde{\nu}$  be the push-forward of  $\nu$  by the transformation  $x \rightarrow -\ln(1-x)$ . Let  $(\tilde{S}_t, t \geq 0)$  be a subordinator with Lévy measure  $\tilde{\nu}$  and  $S_t = e^{-\tilde{S}_t}$ . Then  $(S_t, t \geq 0)$  is a non-increasing positive pure-jump process with  $S_0 = 1$ . Put individuals  $1, 2, \dots, n$  uniformly and independently over  $(0, 1]$ . Let  $t_1$  be the first time when  $(S_{t_1}, S_{t_1-}]$  contains at least one individual, then we set  $\Pi^{(n)}(s) = \{\{1\}, \{2\}, \dots, \{n\}\}$  for  $0 \leq s < t_1$ . We regroup the individuals located in  $(S_{t_1}, S_{t_1-}]$  into one block and let  $\Pi^{(n)}(t_1)$  be the set of this block and the rest singletons. The block is then put uniformly and independently into

$(0, S_{t_1}]$ . Find the next time  $t_2$ , such that  $(S_{t_2}, S_{t_2-}]$  contains at least one block (singleton or not) and we regroup all blocks in this interval into one bigger block which will be again put independently into  $(0, S_{t_2}]$ . In general, at each time  $t$ ,  $\Pi^{(n)}(t)$  is the set of the blocks located in  $(0, S_t]$  and also the union of blocks located in  $(S_t, S_{t-}]$ . This operation can be iterated until reaching the MRCA and the process resulted has the same law as  $\Pi^{(n)}$ .

Notice that this construction is consistent, i.e., if we add a  $n + 1$ -th individual to get  $\Pi^{(n+1)}$ , the structure of  $\Pi^{(n)}$  is conserved. To see how many blocks will merge with  $\{1\}$  in the limit, we assume that  $\{1\}$  is put into  $(S_{t_1}, S_{t_1-}]$  at a certain time. As  $n$  goes to  $\infty$ , the number of singletons put into the same interval also goes to  $\infty$  with probability 1. According to the construction, all singletons put for the first time into the same interval will be coalesced together. Hence  $Q^{(n)}$  converges in probability to infinity.  $\square$

**3.2. Minimal clade size.** Let  $(s_i^{(n)}(t), 1 \leq i \leq K^{(n)}(t))$  be the increasing sequence of the smallest elements of blocks of  $\Pi^{(n)}(t)$ . We have the following lemma.

**Lemma 3.1.** *For any  $t > 0$  and  $k \in \mathbb{N}$ , let  $t_n = n^{1-\alpha}t$  and define the event  $E_{n,k} = \{s_1^{(n)}(t_n) = 1, \dots, s_k^{(n)}(t_n) = k\}$ . Then*

$$\mathbb{P}(E_{n,k} | \Theta(t_n)) \xrightarrow{a.s.} 1$$

as  $n$  tends to  $\infty$ .

*Proof.* We only need to prove that the probability for individuals 1 and 2 to be in the same block of  $\Pi^{(n)}(t_n)$  tends to 0. The case of  $k \geq 3$  follows in the same way. Let us write this event  $\{1 \stackrel{t_n}{\sim} 2\}$ . Then

$$\begin{aligned} \mathbb{P}(1 \stackrel{t_n}{\sim} 2 | \Theta(t_n)) &= \sum_{i=1}^{K(t_n)} \theta_i(t_n)^2 \\ &= \mathbb{E}[\zeta(t_n) | \Theta(t_n)] \xrightarrow{a.s.} 0, \end{aligned}$$

where the convergence is due to Corollary 2.2.  $\square$

**Theorem 3.2.** *Let  $t > 0$  and  $t_n = n^{1-\alpha}t$ . For  $1 \leq i \leq K^{(n)}(t)$ , let  $K_i^{(n)}(t)$  be the size of the block containing  $s_i^{(n)}(t)$ . Then for any  $k \in \mathbb{N}$  and  $(r_1, \dots, r_k) \in \mathbb{N}^k$ , as  $n \rightarrow \infty$ ,*

$$\mathbb{P}(K_1^{(n)}(t_n) = r_1, \dots, K_k^{(n)}(t_n) = r_k | \Theta(t_n)) \xrightarrow{a.s.} \prod_{i=1}^k \mathbb{P}(\beta(t) = r_i) \quad (30)$$

where  $\beta(t)$  is defined in (7).

*Proof.* Let  $t_n = n^{1-\alpha}t$ . Define the event  $E_{n,r} = \{K_1^{(n)}(t_n) = r_1, \dots, K_k^{(n)}(t_n) = r_k\}$  and recall  $E_{n,k}$  defined in Lemma 3.1. Let  $(\varsigma_i(t_n), 1 \leq i \leq k)$  be  $k$  independent copies of  $\varsigma(t_n)$  which is defined in (13), conditional on  $\Theta(t_n)$ . For  $1 \leq i \leq k$ ,  $\varsigma_i(t_n)$  denotes the size of the subinterval into which  $i$  is thrown in the paintbox construction of  $\Pi^{(n)}(t_n)$  with  $n \geq k$ . Due to Lemma 3.1, for  $n$  large enough we can almost surely approach  $\mathbb{P}(E_{n,r} | \Theta(t_n))$  by

$$\mathbb{P}(E_{n,r} | E_{n,k}, \Theta(t_n)) = \mathbb{E}\left[\binom{n-k}{r_1-1, \dots, r_k-1} \prod_{j=1}^k (\varsigma_j(t_n))^{r_j-1} \left(1 - \sum_{j=1}^k \varsigma_j(t_n)\right)^{n-\sum_{j=1}^k r_j} | E_{n,k}, \Theta(t_n)\right].$$

The Lemma 3.1 implies again that the difference

$$\begin{aligned} &\mathbb{E}\left[\binom{n-k}{r_1-1, \dots, r_k-1} \prod_{j=1}^k (\varsigma_j(t_n))^{r_j-1} \left(1 - \sum_{j=1}^k \varsigma_j(t_n)\right)^{n-\sum_{j=1}^k r_j} | E_{n,k}, \Theta(t_n)\right] \\ &- \mathbb{E}\left[\binom{n-k}{r_1-1, \dots, r_k-1} \prod_{j=1}^k (\varsigma_j(t_n))^{r_j-1} \left(1 - \sum_{j=1}^k \varsigma_j(t_n)\right)^{n-\sum_{j=1}^k r_j} | \Theta(t_n)\right] \end{aligned}$$

converges almost surely to 0. We then obtain

$$\begin{aligned} & \mathbb{E}\left[\binom{n-k}{r_1-1, \dots, r_k-1} \prod_{j=1}^k (\zeta_j(t_n))^{r_j-1} \left(1 - \sum_{j=1}^k \zeta_j(t_n)\right)^{n-\sum_{j=1}^k r_j} \mid \Theta(t_n)\right] \\ &= \mathbb{E}\left[\prod_{j=1}^k \frac{1}{(r_j-1)!} (n\zeta_j(t_n))^{r_j-1} e^{-n\zeta_j(t_n)} \mid \Theta(t_n)\right] + o_{a.s.}(1), \end{aligned}$$

where  $o_{a.s.}(1)$  is a term converging almost surely to 0 when  $n$  tends to  $\infty$ . The result thus follows from Corollary 2.2.  $\square$

We next show how the external branch of 1 is connected to the whole process. Let  $\Pi^{(2,n)}$  be the restriction of  $\Pi^{(n)}$  from  $\mathbb{N}_n$  to  $\{2, 3, \dots, n\}$ . By consistency and exchangeability of  $\Pi^{(n)}$ ,  $\Pi^{(2,n)}$  has the same law as  $\Pi^{(n-1)}$  except for the integer notations. Given  $\Pi^{(2,n)}$ , one can attach  $\{1\}$  to  $\Pi^{(2,n)}$  following the *recursive construction* introduced in [13]. One thing important is that  $n^{\alpha-1}T_1^{(n)}$  and  $\Pi^{(2,n)}$  are asymptotically independent. The following lemma is given in the proof of Theorem 5.2 of [13].

**Lemma 3.3.** *Let  $t \geq 0$ . As  $n$  tends to  $\infty$ ,*

$$\mathbb{P}(n^{\alpha-1}T_1^{(n)} \geq t \mid \Pi^{(n,2)}) \xrightarrow{P} \left(1 + \frac{t}{\alpha\Gamma(\alpha)}\right)^{-\frac{\alpha}{\alpha-1}}. \quad (31)$$

Now we are able to deal with the minimal clade size.

*Proof of Theorem 1.4.* Recall that  $Q^{(n)}$  is the number of blocks involved in the coalescence of  $\{1\}$ . Then  $Y^{(n)}$  is just the sum of the  $Q^{(n)}$  block sizes (one of these blocks is  $\{1\}$ ). It suffices to determine the size of each block involved in the coalescence event. By exchangeability of the coalescent, the  $Q^{(n)} - 1$  blocks not being  $\{1\}$  will be chosen randomly at time  $(T_1^{(n)})_-$ . Hence by the strong Markov property of  $\Pi^{(n)}$ , the joint distribution of the sizes of the randomly chosen  $Q^{(n)} - 1$  blocks has the same law as the distribution of  $(K_2^{(n)}((T_1^{(n)})_-), \dots, K_{Q^{(n)}}^{(n)}((T_1^{(n)})_-))$ . Hence

$$Y^{(n)} \stackrel{(d)}{=} 1 + \sum_{i=2}^{Q^{(n)}} K_i^{(n)}((T_1^{(n)})_-).$$

Let  $K^{(2,n)} = (K^{(2,n)}(t), t \geq 0)$  be the block-counting process of  $\Pi^{(2,n)}$  and  $(K_1^{(2,n)}(t), K_2^{(2,n)}(t), \dots, K_{K^{(2,n)}(t)}^{(2,n)}(t))$  be the vector of the block sizes of  $\Pi^{(2,n)}(t)$ , increasingly ordered by their least elements. Notice that if  $t < T_1^{(n)}$ ,  $K_i^{(n)}(t) = K_{i-1}^{(2,n)}(t)$  for  $1 \leq i \leq K^{(n)}(t)$ . Therefore

$$Y^{(n)} \stackrel{(d)}{=} 1 + \sum_{i=1}^{Q^{(n)}-1} K_i^{(2,n)}((T_1^{(n)})_-). \quad (32)$$

The formula (27) shows that the law of  $Q^{(n)}$  is uniquely determined by  $K^{(2,n)}((T_1^{(n)})_-)$ . As long as  $K^{(2,n)}((T_1^{(n)})_-)$  goes to  $\infty$ ,  $Q^{(n)}$  converges in law to a distribution which depends only on  $\alpha$ . While (22) and (19) imply that the variable  $K^{(2,n)}((T_1^{(n)})_-)$  goes to  $\infty$  with probability 1. Hence  $Q^{(n)}$  is asymptotically independent of  $(T_1^{(n)}, K^{(2,n)}((T_1^{(n)})_-))$ . Furthermore, Lemma 3.3 gives that  $T_1^{(n)}$  and  $K^{(2,n)}((T_1^{(n)})_-)$  are asymptotically independent. In total,  $Q^{(n)}$ ,  $T_1^{(n)}$  and  $(K^{(2,n)}((T_1^{(n)})_-))$  are all asymptotically independent. In the limit, using Theorem 3.2,

$$Y^{(n)} \xrightarrow{(d)} Y \stackrel{(d)}{=} 1 + \sum_{i=1}^{Q-1} \beta_i(T),$$

where  $Q, T, (\beta_i(t))_{i \in \mathbb{N}}$  are all independent and follow respectively the limit laws of  $Q^{(n)}, T_1^{(n)}, (K_i^{(2,n)}(t))_{i \in \mathbb{N}}$  for fixed  $t \geq 0$ . Then we can conclude.  $\square$

*Proof of Corollary 1.6.* Consider the Laplace transform of  $Y$ . For any  $\lambda > 0$ , using (29)

$$\begin{aligned}\mathbb{E}[e^{-\lambda Y}] &= e^{-\lambda} \mathbb{E}[(\mathbb{E}[e^{-\lambda \beta(T)}])^{Q-1}] \\ &= e^{-\lambda} \mathbb{E}\left[\left(e^{-\lambda} (1 + (T_\alpha(1 - e^{-\lambda}))^{\alpha-1})^{\frac{1}{1-\alpha}}\right)^{Q-1}\right]\end{aligned}$$

where  $T_\alpha = \left(\frac{T}{\alpha\Gamma(\alpha)}\right)^{\frac{1}{\alpha-1}}$ . Denote  $\Delta := e^{-\lambda}(1 + (T_\alpha(1 - e^{-\lambda}))^{\alpha-1})^{\frac{1}{1-\alpha}}$ . Using (10), one gets

$$\begin{aligned}\mathbb{E}[e^{-\lambda Y}] &= \mathbb{E}[e^{-\lambda}(1 - (1 - \Delta)^{\alpha-1})] \\ &= I_1 + I_2,\end{aligned}$$

where  $I_1 = \mathbb{E}[e^{-\lambda Y} \mathbf{1}_{T_\alpha > \lambda^{-\frac{1}{2}}}]$ ,  $I_2 = \mathbb{E}[e^{-\lambda Y} \mathbf{1}_{T_\alpha \leq \lambda^{-\frac{1}{2}}}]$ . The density (3) of  $T$  implies, when  $\lambda \rightarrow 0+$

$$I_1 = O(\lambda^{\frac{\alpha}{2}}) = o(\lambda^{(\alpha-1)^2}). \quad (33)$$

Notice that there exists  $C_{34} > 0$  such that for any  $0 < \varepsilon < 1$ , if  $\lambda$  is small enough, we have

$$|\Delta \mathbf{1}_{T_\alpha \leq \lambda^{-\frac{1}{2}}} - (1 + \frac{\alpha}{1-\alpha}(T_\alpha \lambda)^{\alpha-1}) \mathbf{1}_{T_\alpha \leq \lambda^{-\frac{1}{2}}}| \leq \varepsilon (T_\alpha \lambda)^{\alpha-1} \mathbf{1}_{T_\alpha \leq \lambda^{-\frac{1}{2}}} + C_{34} \lambda. \quad (34)$$

Letting  $\lambda \rightarrow 0+$  and using (33), (34), one obtains

$$\mathbb{E}[e^{-\lambda Y}] = 1 - \left(\frac{\alpha}{\alpha-1}\right)^{\alpha-1} \lambda^{(\alpha-1)^2} \mathbb{E}[T_\alpha^{(\alpha-1)^2}] + o(\lambda^{(\alpha-1)^2}).$$

Thanks to Lemma 5.4 of [3] or Theorem 8.1.6 of [5], we get

$$\mathbb{P}(Y > k) \sim \frac{\left(\frac{\alpha}{\alpha-1}\right)^{\alpha-1} \mathbb{E}[T_\alpha^{(\alpha-1)^2}]}{\Gamma(1 - (\alpha-1)^2)} k^{-(\alpha-1)^2} = \frac{\int_0^\infty t^{\alpha-1} f_T(t) dt}{((\alpha-1)\Gamma(\alpha))^{\alpha-1} \Gamma(1 - (\alpha-1)^2)} k^{-(\alpha-1)^2}$$

when  $k \rightarrow \infty$ . □

#### 4. THE LARGEST BLOCK

In this section, we aim to prove Theorem 1.7 and Corollary 1.8. We start with a technical lemma.

**Lemma 4.1.** *Let  $k > 0$  and  $X$  be a random variable distributed according to  $\mu$ . Define  $\mathcal{X}$  such that conditional on  $X$ ,  $\mathcal{X}$  is a Poisson variable with parameter  $\frac{X}{k}$ . Then for any  $x > 0$ ,*

$$\lim_{n \rightarrow \infty} n \mathbb{P}(\mathcal{X} \geq xn^{\frac{1}{\alpha}}) = \frac{(kx)^{-\alpha}}{\Gamma(2-\alpha)}.$$

*Proof.* First of all, let us consider two technical results. Let  $M = \lfloor xn^{\frac{1}{\alpha}} \rfloor$ .

1) Using Stirling's formula for  $M!$  and a change of variable, we get that for any  $0 < \beta < 1$ ,

$$\begin{aligned}\int_0^{M\beta} e^{-t} \frac{t^M}{M!} dt &= \int_0^{M\beta} e^{M-t} \left(\frac{t}{M}\right)^M (2\pi M)^{-\frac{1}{2}} (1 + O(M^{-1})) dt \\ &= \int_0^\beta e^{M(1-t+\ln t)} \left(\frac{M}{2\pi}\right)^{\frac{1}{2}} (1 + O(M^{-1})) dt \\ &= O(e^{M(1-\beta+\ln \beta)} M^{\frac{1}{2}}).\end{aligned} \quad (35)$$

The last equality is due to the fact that  $1 - t + \ln t$  is negative and increasing for  $t \in (0, 1)$ .

2) If  $\beta > 1$ , then

$$\int_{M\beta}^\infty e^{-t} \frac{t^M}{M!} dt = \int_\beta^\infty e^{M(1-t+\ln t)} \left(\frac{M}{2\pi}\right)^{\frac{1}{2}} (1 + O(M^{-1})) dt.$$

Notice that  $1 - t + \ln t$  is strictly decreasing and concave over  $[\beta, \infty]$ , then there exists a positive number  $\varepsilon$  such that  $1 - t + \ln t \leq -\varepsilon t$  for any  $t \geq \beta$ . Therefore,

$$\int_{M\beta}^{\infty} e^{-t} \frac{t^M}{M!} dt \leq \int_{\beta}^{\infty} e^{-\varepsilon Mt} \left(\frac{M}{2\pi}\right)^{1/2} (1 + O(M^{-1})) dt = O(e^{-\varepsilon M\beta} M^{-1/2}). \quad (36)$$

Now we can turn to the study of  $\mathcal{X}$ . Thanks to successive integrations by parts,

$$\mathbb{P}(\mathcal{X} \geq M + 1) = \mathbb{E}\left[\int_0^{\frac{X}{k}} e^{-t} \frac{t^M}{M!} dt\right]. \quad (37)$$

Let  $0 < \beta_1 < 1$  and  $\beta_2 > 1$ , then we have

$$\mathbb{P}(\mathcal{X} \geq M + 1) = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \mathbb{E}\left[\int_0^{\frac{X}{k}} e^{-t} \frac{t^M}{M!} dt \mathbf{1}_{\{X < kM\beta_1\}}\right], \\ I_2 &= \mathbb{E}\left[\int_0^{\frac{X}{k}} e^{-t} \frac{t^M}{M!} dt \mathbf{1}_{\{kM\beta_1 \leq X \leq kM\beta_2\}}\right], \\ I_3 &= \mathbb{E}\left[\int_0^{\frac{X}{k}} e^{-t} \frac{t^M}{M!} dt \mathbf{1}_{\{X > kM\beta_2\}}\right]. \end{aligned}$$

Now let  $n$  tend to infinity. By (35), we get

$$0 \leq nI_1 \leq n\mathbb{P}(X < kM\beta_1) \int_0^{M\beta_1} e^{-t} \frac{t^M}{M!} dt \longrightarrow 0. \quad (38)$$

It is easy to verify that  $\int_0^{\infty} e^{-t} \frac{t^M}{M!} dt = 1$  for any integer  $M \geq 0$ . Then using together (28) and (36), we obtain

$$\lim_{n \rightarrow \infty} nI_3 = \lim_{n \rightarrow \infty} n\mathbb{P}(X > kM\beta_2) = \frac{(kx\beta_2)^{-\alpha}}{\Gamma(2 - \alpha)}. \quad (39)$$

In the same way, we have

$$0 \leq nI_2 \leq n\mathbb{P}(kM\beta_1 \leq X \leq kM\beta_2) \longrightarrow \frac{(kx\beta_1)^{-\alpha}}{\Gamma(2 - \alpha)} - \frac{(kx\beta_2)^{-\alpha}}{\Gamma(2 - \alpha)}, \quad n \rightarrow \infty. \quad (40)$$

If  $\beta_1$  and  $\beta_2$  are close enough to 1,  $nI_2$  can be bounded by an arbitrarily small positive number for  $n$  large enough. Combining (38), (39) and (40), we conclude this lemma.  $\square$

To prove Theorem 1.7, we will use classical relations between Beta-coalescents and continuous-state branching processes (CSBPs) developed in [6] (see also Section 2 of [3]). We give a short summary to provide a minimal set of tools. A continuous-state branching process  $(Z(t), t \geq 0)$  is a  $[0, \infty]$ -valued Markov process (in continuous time) whose transition functions  $p_t(x, \cdot)$  satisfy the branching property

$$p_t(x + y, \cdot) = p_t(x, \cdot) * p_t(y, \cdot), \quad \text{for all } x, y \geq 0.$$

For each  $t \geq 0$ , there exists a function  $u_t : [0, \infty) \rightarrow \mathbb{R}$  such that

$$\mathbb{E}[e^{-\lambda Z(t)} | Z(0) = a] = e^{-au_t(\lambda)}. \quad (41)$$

If almost surely, the process has no instantaneous jump to infinity, the function  $u_t$  satisfies the following differential equation

$$\frac{\partial u_t(\lambda)}{\partial t} = -\Psi(u_t(\lambda)),$$

where  $\Psi : [0, \infty) \rightarrow \mathbb{R}$  is a function of the form

$$\Psi(u) = \gamma u + \beta u^2 + \int_0^{\infty} (e^{-xu} - 1 + xu \mathbf{1}_{\{x \leq 1\}}) \pi(dx),$$

where  $\gamma \in \mathbb{R}, \beta \geq 0$  and  $\pi$  is a Lévy measure on  $(0, \infty)$  satisfying  $\int_0^{\infty} (1 \wedge x^2) \pi(dx) < \infty$ . The function  $\Psi$  is called the branching mechanism of the CSBP.

As explained in [4], a CSBP can be extended to a two-parameter random process  $(Z(t, a), t \geq 0, a \geq 0)$  with  $Z(0, a) = a$ . For fixed  $t$ ,  $(Z(t, a), a \geq 0)$  turns out to be a subordinator with Laplace exponent  $\lambda \mapsto u_t(\lambda)$  thanks to (41).

There exists a measure-valued process  $(M_t, t \geq 0)$  taking values in the set of finite measures on  $[0, 1]$  which characterizes  $(Z(t, a), t \geq 0, 0 \leq a \leq 1)$ . More precisely,  $(M_t([0, a]), t \geq 0, 0 \leq a \leq 1)$  has the same finite-dimensional distributions as  $(Z(t, a), t \geq 0, 0 \leq a \leq 1)$ . Hence  $(M_t([0, a]), 0 \leq a \leq 1)$  is a subordinator with Laplace exponent  $\lambda \mapsto u_t(\lambda)$  and  $Z(t) = M_t([0, 1])$  is a CSBP with branching mechanism  $\Psi$  started at  $M_0([0, 1]) = 1$ . In particular, if the branching mechanism is  $\Psi(\lambda) = \lambda^\alpha$  and hence Lévy measure is given by  $\pi(dx) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}x^{-1-\alpha}dx$ , for all  $t > 0$ ,  $M_t$  consists only of finite number of atoms. For the construction of  $(M_t([0, a]), t \geq 0, 0 \leq a \leq 1)$ , we refer to [2, 6, 15].

A deep relation has been revealed in [6] between the Beta-coalescent and the CSBP with branching mechanism  $\Psi(\lambda) = \lambda^\alpha$  and Lévy measure  $\pi(dx) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}x^{-1-\alpha}dx$ . The relationship is described by the following two lemmas which are respectively Lemma 2.1 and 2.2 of [3] and will be important in the sequel.

To save notations, from now on,  $(Z(t), t \geq 0)$  always denotes a continuous state branching process  $(Z(t, 1), t \geq 0)$ .

**Lemma 4.2.** *Assume  $(Z(t), t \geq 0)$  is a CSBP with branching mechanism  $\Psi(\lambda) = \lambda^\alpha$  and let  $(M_t, t \geq 0)$  be its associated measure-valued process. If  $(\Pi(t), t \geq 0)$  is a Beta-coalescent and  $(\Theta(t), t \geq 0)$  is the associated ranked coalescent, then for all  $t > 0$ , the distribution of  $\Theta(t)$  is the same as the distribution of the sizes of the atoms of the measure  $\frac{M_{R^{-1}(t)}}{Z(R^{-1}(t))}$ , ranked in decreasing order. Here  $R(t) = (\alpha - 1)\alpha\Gamma(\alpha) \int_0^t Z(s)^{1-\alpha} ds$  and  $R^{-1}(t) = \inf\{s : R(s) > t\}$ .*

**Lemma 4.3.** *Assume  $\Psi(\lambda) = \lambda^\alpha$ . For any  $t \geq 0$ , let  $D(t)$  be the number of atoms of  $M_t$ , and let  $J(t) = (J_1(t), \dots, J_{D(t)}(t))$  be the sizes of the atoms of  $M_t$ , ranked in decreasing order. Then  $D(t)$  is Poisson with mean  $\gamma_t = ((\alpha - 1)t)^{-\frac{1}{\alpha-1}}$ . Moreover, conditional on  $D(t) = k$ , the distribution of  $J(t)$  is the same as the distribution of  $(\gamma_t^{-1}X_1, \dots, \gamma_t^{-1}X_k)$  where  $X_1, \dots, X_k$  are obtained by picking  $k$  i.i.d. random variables with distribution  $\mu$  and then ranking them in decreasing order.*

*Remark 4.1.* From the relation between  $(M_t, t \geq 0)$  and  $(Z(t, a), t \geq 0, 0 \leq a \leq 1)$  and also the fact that for all  $t > 0$ ,  $M_t$  consists only of finite number of atoms (the number is actually  $D(t)$ ), for a given  $t > 0$ , there exist  $0 \leq a_1, \dots, a_{D(t)} \leq 1$  such that  $\{Z(t, a_1) - Z(t, a_1-), \dots, Z(t, a_{D(t)}) - Z(t, a_{D(t)}-)\}$  are exactly the values of the atoms of  $M_t$ . By the strong Markov property of  $(Z(t, a), t \geq 0, 0 \leq a \leq 1)$ , for  $s \geq t$ , the jumping at  $s$  can only happen at the points  $\{(s, a_1), \dots, (s, a_{D(t)})\}$ . Therefore,  $D(t)$  decreases on  $t$ .

The idea of the proof of Theorem 1.7 is as follows: Let  $t_n = n^{1-\alpha}t$ . Lemma 4.2 shows that  $\Theta(t_n)$  has the same law as  $\frac{M_{R^{-1}(t_n)}}{Z(R^{-1}(t_n))}$ . Moreover it is proved in Lemma 4.2 of [3] that  $\frac{R^{-1}(t_n)}{t_n} \xrightarrow{P} \frac{1}{(\alpha-1)\alpha\Gamma(\alpha)}$ , as  $n$  goes to  $\infty$ . Hence one can compare the block sizes at time  $t_n$  to those at time  $R^{-1}((\alpha-1)\alpha\Gamma(\alpha)t_n)$ . To this, we use the paintbox construction and the closeness between the measures  $\frac{M_{t_n}}{Z(t_n)}$  and  $\frac{M_{R^{-1}((\alpha-1)\alpha\Gamma(\alpha)t_n)}}{Z(R^{-1}((\alpha-1)\alpha\Gamma(\alpha)t_n))}$ . This idea can be executed through two steps.

**1) Analysis of the largest block size at time  $t_n$  with the measure  $\frac{M_{t_n}}{Z(t_n)}$ :** If  $D(t_n) \neq 0$ , let  $\bar{J}_i(t_n) = \frac{J_i(t_n)}{Z(t_n)}$  for  $1 \leq i \leq D(t_n)$ . Let  $\{d_1(t_n), \dots, d_{D(t_n)}(t_n)\}$  be an interval partition of  $[0, 1]$  such that the Lebesgue measure of  $d_i(t_n)$  is  $\bar{J}_i(t_n)$ . Build a partition of  $\mathbb{N}_n$  from a paintbox associated with  $\{d_1(t_n), \dots, d_{D(t_n)}(t_n)\}$ . Let  $N_i$  be the number of integers in  $d_i(t_n)$  and  $N = \max\{N_i : 1 \leq i \leq D(t_n)\}$ .

**Lemma 4.4.** *Let  $x > 0$ . Then*

1)

$$\lim_{n \rightarrow \infty} \mathbb{P}(N \leq xn^{1/\alpha}) = \exp\left(-\frac{(\alpha-1)tx^{-\alpha}}{\Gamma(2-\alpha)}\right).$$



2) Let  $0 < y < x$ . Then

$$\lim_{n \rightarrow \infty} \mathbb{P}(\exists i : J_i(t_n) < n^{\frac{1-\alpha}{\alpha}} y, N_i \geq xn^{\frac{1}{\alpha}}) = 0. \quad (42)$$

*Proof.* 1) It is well known that if we throw a Poisson number of parameter  $nZ(t_n)$  on  $[0, 1]$ , the number of integers in  $d_i(t_n)$ , denoted by  $\mathcal{N}_i$ , is a Poisson variable of parameter  $nJ_i(t_n)$ . Conditional on all  $J_i(t_n)$ 's, all  $\mathcal{N}_i$ 's are independent. Let  $\mathcal{N}$  be the maximum of all  $\mathcal{N}_i$ 's. Then, using Lemmas 4.1 and 4.3,

$$\mathbb{P}(\mathcal{N} \leq xn^{1/\alpha}) = \mathbb{E}[\prod_{i=1}^{D(t_n)} \mathbb{P}(\mathcal{N}_i \leq xn^{1/\alpha})] \longrightarrow \exp(-\gamma_t^{1-\alpha} \frac{x^{-\alpha}}{\Gamma(2-\alpha)}) = \exp(-\frac{(\alpha-1)tx^{-\alpha}}{\Gamma(2-\alpha)}), \quad n \rightarrow \infty.$$

Lemma 4.3 implies that  $Z(t_n)$  tends in probability to 1 as  $n$  goes to infinity. Hence  $N$  and  $\mathcal{N}$  are close in the limit and standard comparison techniques allows to conclude.

2) As  $Z(t_n)$  converges to 1, (42) is equivalent to

$$\lim_{n \rightarrow \infty} \mathbb{P}(\exists i : J_i(t_n) < n^{\frac{1-\alpha}{\alpha}} y, \mathcal{N}_i \geq xn^{\frac{1}{\alpha}}) = 0.$$

Let  $\tilde{\mathcal{N}} = \max\{\mathcal{N}_i : J_i(t_n) < n^{\frac{1-\alpha}{\alpha}} y\}$ . It is necessary and sufficient to show that  $\lim_{n \rightarrow \infty} \mathbb{P}(\tilde{\mathcal{N}} \geq xn^{\frac{1}{\alpha}}) = 0$ . Notice that conditional on  $J_i(t_n)$ ,  $\mathcal{N}_i$  is a Poisson variable with parameter  $nJ_i(t_n)$ . Let  $\{P_1(yn^{\frac{1}{\alpha}}), P_2(yn^{\frac{1}{\alpha}}), \dots\}$  be a sequence of i.i.d. Poisson variables with parameter  $yn^{\frac{1}{\alpha}}$  and also independent of  $D(t_n)$ . Then

$$\mathbb{P}(\tilde{\mathcal{N}} \geq xn^{\frac{1}{\alpha}}) \leq \mathbb{P}\left(\max\{P_i(yn^{\frac{1}{\alpha}}) : 1 \leq i \leq D(t_n)\} \geq xn^{\frac{1}{\alpha}}\right) = 1 - \mathbb{E}[(\mathbb{P}(P_1(yn^{\frac{1}{\alpha}}) < xn^{\frac{1}{\alpha}}))^{D(t_n)}].$$

Using (37) and (35), one gets

$$\mathbb{P}(P_1(yn^{\frac{1}{\alpha}}) < xn^{\frac{1}{\alpha}}) = 1 - o\left(\frac{1}{n}\right).$$

Meanwhile, Lemma 4.3 tells that  $\frac{D(t_n)}{n}$  converges in probability to  $\gamma_t$  as  $n$  goes to infinity. Hence we can conclude.  $\square$

*Remark 4.2.* The key point to prove (42) is that  $Z(t_n)$  converges to 1 in probability,  $\frac{D(t_n)}{n}$  is asymptotically bounded by a positive value from above. The distribution of  $\{J_i(t_n)\}_{1 \leq i \leq D(t_n)}$  is not necessary to know. One can still find (42) true if we replace  $t_n$  by a random time and conditions for  $Z(t_n)$  and  $D(t_n)$  are satisfied at the same time.

**2) A tool lemma for the transfer from  $\frac{M_{t_n}}{Z(t_n)}$  to  $\frac{M_{R^{-1}((\alpha-1)\alpha\Gamma(\alpha)t_n)}}{Z(R^{-1}((\alpha-1)\alpha\Gamma(\alpha)t_n))}$  :** Let  $(A_1, \dots, A_k)$  and  $(B_1, \dots, B_k)$  be two partitions of  $[0, 1]$  with  $k \geq 1$ . We throw away  $n$  particles uniformly and independently on  $[0, 1]$  and regroup those within the same intervals of  $(B_1, \dots, B_k)$ , which gives a sequence of  $k$  numbers  $(N_{B_1}, \dots, N_{B_k})$  such that  $N_{B_i}$  is the number of particles located in  $B_i$ . We can obtain the law of this sequence in another way using  $(A_1, \dots, A_k)$ : We throw  $n$  particles uniformly and independently on  $[0, 1]$ . Let  $I := \{i : 1 \leq i \leq n, l(A_i) \leq l(B_i)\}$ , where  $l(\cdot)$  denotes the Lebesgue measure. If a particle falls in  $A_i$  where  $i \in I$ , then move this particle to  $B_i$ . If a particle falls in  $A_i$  where  $i \in I^c$ , then do the following: we attach to this particle an independent Bernoulli variable with parameter  $\frac{l(B_i)}{l(A_i)}$ . If the Bernoulli variable gives 1, then the particle is put into  $B_i$ . Otherwise, this particle will be put into  $B_j$  for  $j \in I$  with probability

$$\frac{l(B_j) - l(A_j)}{\sum_{j \in I} (l(B_j) - l(A_j))}. \quad (43)$$

We denote by  $N_{A_i}^B$  the new amount of particles in  $B_i$ . We have the the following result.

**Lemma 4.5.** *The following identity in law holds.*

$$(N_{A_1}^B, \dots, N_{A_k}^B) \stackrel{(d)}{=} (N_{B_1}, \dots, N_{B_k}).$$

*Proof.* Notice that only the measure of each element of  $(A_1, \dots, A_k)$  and  $(B_1, \dots, B_k)$  matters, one can always assume that  $[0, 1]$  is divided in a way that  $A_i$  is contained in  $B_i$  for  $i \in I$  and  $B_i$  is contained in  $A_i$  for  $i \in I^c$ . So if a particle is located in  $A_i$  for  $i \in I$ , it is also located in  $B_i$ . But if a particle is located in  $A_i$  for  $i \in I^c$ , with probability  $\frac{l(B_i)}{l(A_i)}$  it is located in  $B_i$ . Assume that this particle is not located in  $B_i$ , then it must be in  $\cup_{j \in I} B_j/A_j$ . Using the uniformity of the throw, this particle falls in  $B_j$  with probability (43).  $\square$

The above two steps allow to start the proof of Theorem 1.7. But before that, let us just recall some technical results from [3]. Let  $\varepsilon > 0$ ,  $t > 0$  and  $t_n = n^{1-\alpha}t$ . Let  $t_- = (1 - \varepsilon)t_n$ ,  $t_+ = (1 + \varepsilon)t_n$  and  $t_* = (\alpha - 1)\alpha\Gamma(\alpha)t_n$ . Define the event  $B_{1,t} := \{t_- \leq R^{-1}(t_*) \leq t_+\}$ . It can be found in Lemma 4.2 of [3] that there exists a constant  $C_{44}$  such that

$$\mathbb{P}(B_{1,t}) \geq 1 - C_{44}t_*\varepsilon^{-\alpha}. \quad (44)$$

Also from Lemma 5.1 of [3], there exists a constant  $C_{45}$  such that for all  $a > 0$ ,  $t > 0$  and  $\eta > 0$ ,

$$q(a, t, \eta) = \mathbb{P}\left(\sup_{0 \leq s \leq t} |Z(s, a) - a| > \eta\right) \leq C_{45}(a + \eta)t\eta^{-\alpha}. \quad (45)$$

Thus, if we define  $B_{2,t} := \{1 - n^{\frac{1-\alpha}{2\alpha}} \leq Z(s) \leq 1 + n^{\frac{1-\alpha}{2\alpha}}, \forall s \in [t_-, t_+]\}$ , we can obtain that

$$\mathbb{P}(B_{2,t}) \geq 1 - C_{45}t(1 + \varepsilon)(1 + n^{\frac{1-\alpha}{2\alpha}})n^{\frac{1-\alpha}{2}}. \quad (46)$$

*Proof of Theorem 1.7.* Lemma 4.2 tells us that for any  $s \geq 0$ , we have

$$\frac{M_{R^{-1}(s)}}{Z(R^{-1}(s))} \stackrel{(d)}{=} \Theta(s).$$

Let  $\pi$  be the partition of  $\mathbb{N}_n$  obtained from a paintbox associated with  $\frac{M_{R^{-1}(s)}}{Z(R^{-1}(s))}$ . Then  $\pi \stackrel{(d)}{=} \Pi^{(n)}(s)$ . If  $R^{-1}(s) \geq t_-$ , we can as well at first build a partition from a paintbox associated with  $\frac{M_{t_-}}{Z(t_-)}$  and then use Lemma 4.5 to get  $\pi$ . This kind of construction is the key of this proof.

For  $s \geq t_-$ , one builds a partition of  $\mathbb{N}_n$  from a paintbox associated with  $(\frac{m_i(s)}{Z(s)}, 1 \leq i \leq D(t_-))$ . We denote this partition by  $V^{(n)}(s) = (V_1(s), V_2(s), \dots, V_{D(t_-)}(s))$ . Let  $I_i^{(n)}(s)$  be the number of particles in  $V_i^{(n)}(s)$ .

For  $s \geq t_-$ , let  $M^{(n)}(s) = \sup\{I_i^{(n)}(s), 1 \leq i \leq D(t_-)\}$  be the size of the largest block of  $V^{(n)}(s)$ . Let  $x > 0$  and  $B_{3,t} = \{\exists k : I_k^n(t_-) \geq xn^{\frac{1}{\alpha}}, J_k(t_-) \geq n^{\frac{2(1-\alpha)}{\alpha}}, \sup_{t_- \leq s \leq t_+} |m_k(s) - J_k(t_-)| \leq \varepsilon J_k(t_-)\}$ .

On the event  $B_{3,t}$ , we have that  $M^{(n)}(t_-) \geq xn^{\frac{1}{\alpha}}$ . Conditional on  $B_{1,t}$  we can build the partition  $V^{(n)}(R^{-1}(t_*))$  from a paintbox associated to the partition  $Z(t_-)^{-1}(J_1(t_-), \dots, J_{D(t_-)}(t_-))$  and Lemma 4.5. Let  $B(m, p)$  be a binomial variable with parameters  $m \geq 2$  and  $0 \leq p \leq 1$ . Lemma 4.5 implies

$$\begin{aligned} & \mathbb{P}\left(M^{(n)}(R^{-1}(t_*)) \geq (1 - 2\varepsilon)xn^{\frac{1}{\alpha}} | B_{1,t} \cap B_{2,t} \cap B_{3,t}\right) \\ & \geq \mathbb{P}\left(B\left(\lceil xn^{\frac{1}{\alpha}} \rceil, \frac{m_k(R^{-1}(t_*))Z(t_-)}{J_k(t_-)Z(R^{-1}(t_*))} \wedge 1\right) \geq (1 - 2\varepsilon)xn^{\frac{1}{\alpha}} | B_{1,t} \cap B_{2,t} \cap B_{3,t}\right) \\ & \geq \mathbb{P}\left(B\left(\lceil xn^{\frac{1}{\alpha}} \rceil, (1 - \varepsilon)\frac{1 - n^{\frac{1-\alpha}{2\alpha}}}{1 + n^{\frac{1-\alpha}{2\alpha}}}\right) \geq (1 - 2\varepsilon)xn^{\frac{1}{\alpha}}\right) \\ & = \mathbb{P}\left((xn^{\frac{1}{\alpha}})^{-1}B\left(\lceil xn^{\frac{1}{\alpha}} \rceil, (1 - \varepsilon)\frac{1 - n^{\frac{1-\alpha}{2\alpha}}}{1 + n^{\frac{1-\alpha}{2\alpha}}}\right) \geq (1 - \varepsilon) - \varepsilon\right). \end{aligned}$$

A law of large numbers argument implies that

$$\mathbb{P}\left(M^{(n)}(R^{-1}(t_*)) \geq (1 - 2\varepsilon)xn^{\frac{1}{\alpha}} | B_{1,t} \cap B_{2,t} \cap B_{3,t}\right) \geq 1 - \varepsilon \quad (47)$$

for  $n$  large enough. Now observe that

$$\begin{aligned}
\mathbb{P}(B_{3,t}) &= \mathbb{P}(\exists k : I_k^n(t_-) \geq xn^{\frac{1}{\alpha}}, J_k(t_-) \geq n^{\frac{2(1-\alpha)}{\alpha}}) \\
&\quad \times \mathbb{P}\left(\sup_{t_- \leq s \leq t_+} |m_k(s) - J_k(t_-)| \leq \varepsilon J_k(t_-) \mid \{\exists k : I_k^n(t_-) \geq xn^{\frac{1}{\alpha}}, J_k(t_-) \geq n^{\frac{2(1-\alpha)}{\alpha}}\}\right) \\
&= \mathbb{P}(\exists k : I_k^n(t_-) \geq xn^{\frac{1}{\alpha}}, J_k(t_-) \geq n^{\frac{2(1-\alpha)}{\alpha}}) \\
&\quad \times (1 - \mathbb{E}[q(J_k(t_-), t_+ - t_-, \varepsilon J_k(t_-)) \mid \{\exists k : I_k^n(t_-) \geq xn^{\frac{1}{\alpha}}, J_k(t_-) \geq n^{\frac{2(1-\alpha)}{\alpha}}\}]) \\
&\geq \mathbb{P}(\exists k : I_k^n(t_-) \geq xn^{\frac{1}{\alpha}}, J_k(t_-) \geq n^{\frac{2(1-\alpha)}{\alpha}}) (1 - 2tC_{45} n^{\frac{(1-\alpha)(2-\alpha)}{\alpha}} (1 + \varepsilon) \varepsilon^{1-\alpha}).
\end{aligned}$$

Using Lemma 4.4,

$$\begin{aligned}
\mathbb{P}(\exists k : I_k^n(t_-) \geq xn^{\frac{1}{\alpha}}, J_k(t_-) \geq n^{\frac{2(1-\alpha)}{\alpha}}) &\sim \mathbb{P}(\exists k : I_k^n(t_-) \geq xn^{\frac{1}{\alpha}}) = \mathbb{P}(M^{(n)}(t_-) \geq n^{\frac{1}{\alpha}}x) \\
&\sim 1 - \exp(-(1 - \varepsilon) \frac{(\alpha - 1)tx^{-\alpha}}{\Gamma(2 - \alpha)}).
\end{aligned}$$

In consequence,

$$\liminf_{n \rightarrow \infty} \mathbb{P}(B_{3,t}) \geq 1 - \exp(-(1 - \varepsilon) \frac{(\alpha - 1)tx^{-\alpha}}{\Gamma(2 - \alpha)})$$

when  $n$  tends to  $\infty$ . Then, thanks to (44) and (46), we deduce that

$$\liminf_{n \rightarrow \infty} \mathbb{P}(B_{1,t} \cap B_{2,t} \cap B_{3,t}) \geq 1 - \exp(-(1 - \varepsilon) \frac{(\alpha - 1)tx^{-\alpha}}{\Gamma(2 - \alpha)}).$$

Combining the latter with (47), we obtain

$$\liminf_{n \rightarrow \infty} \mathbb{P}\left(M^{(n)}(R^{-1}(t_*)) \geq (1 - 2\varepsilon)xn^{\frac{1}{\alpha}}\right) \geq 1 - \exp(-(1 - \varepsilon) \frac{(\alpha - 1)tx^{-\alpha}}{\Gamma(2 - \alpha)}). \quad (48)$$

Next, we seek to find an upper bound for  $\mathbb{P}\left(M^{(n)}(R^{-1}(t_*)) \geq xn^{\frac{1}{\alpha}}\right)$ . Conditional on  $B_{1,t}$ , we construct  $V^{(n)}(t_+)$  from  $V^{(n)}(R^{-1}(t_*))$  using the method in Lemma 4.5. Let

$$B_{4,t} = B_{1,t} \cap \{\exists k : I_k^{(n)}(R^{-1}(t_*)) \geq xn^{\frac{1}{\alpha}}, m_k(R^{-1}(t_*)) \geq n^{\frac{2(1-\alpha)}{\alpha}}, \sup_{R^{-1}(t_*) \leq s \leq t_+} \frac{|m_k(s) - m_k(R^{-1}(t_*))|}{m_k(R^{-1}(t_*))} \leq \varepsilon\}.$$

Similarly as for the lower bound,

$$\begin{aligned}
&\mathbb{P}\left(M^{(n)}(t_+) \geq (1 - 2\varepsilon)xn^{\frac{1}{\alpha}} \mid B_{2,t} \cap B_{4,t}\right) \\
&\geq \mathbb{P}\left(B\left(\lceil xn^{\frac{1}{\alpha}} \rceil, \frac{Z(R^{-1}(t_*))m_k(t_+)}{Z(t_+)m_k(R^{-1}(t_*))} \wedge 1\right) \geq (1 - 2\varepsilon)xn^{\frac{1}{\alpha}} \mid B_{2,t} \cap B_{4,t}\right) \\
&\geq \mathbb{P}\left(B\left(\lceil xn^{\frac{1}{\alpha}} \rceil, (1 - \varepsilon) \frac{1 - n^{(1-\alpha)/\alpha}}{1 + n^{(1-\alpha)/\alpha}}\right) \geq (1 - 2\varepsilon)xn^{\frac{1}{\alpha}}\right) \rightarrow 1.
\end{aligned} \quad (49)$$

Using the strong Markov property of the CSBP and (45), we have

$$\mathbb{P}(B_{4,t}) = \mathbb{P}(B_{1,t} \cap \{\exists k : I_k^{(n)}(R^{-1}(t_*)) \geq xn^{\frac{1}{\alpha}}, m_k(R^{-1}(t_*)) \geq n^{\frac{2(1-\alpha)}{\alpha}}\}) \quad (50)$$

$$\times (1 - 2tC_{45} n^{\frac{(1-\alpha)(2-\alpha)}{\alpha}} (1 + \varepsilon) \varepsilon^{1-\alpha}) \quad (51)$$

Notice that using (45), in the sense of convergence of probability

$$\lim_{n \rightarrow \infty} \sup_{t_- \leq s \leq t_+} Z(s) = \lim_{n \rightarrow \infty} \inf_{t_- \leq s \leq t_+} Z(s) = 1$$

Together with (44), we get the following convergence in probability

$$\lim_{n \rightarrow \infty} Z(R^{-1}(t_*)) = 1.$$

Remark 4.1 tells that  $D(t)$  is decreasing on  $t$ . Under  $B_{1,t}$ ,  $D(t_-) \leq D(R^{-1}(t_*)) \leq D(t_+)$ . It is then easy to deduce that  $\frac{D(R^{-1}(t_*))}{n}$  is asymptotically bounded from above by a certain positive number. Now we can apply Remark 4.2 and get

$$\mathbb{P}(B_{4,t}) = \mathbb{P}(\exists k : I_k^{(n)}(R^{-1}(t_*)) \geq xn^{\frac{1}{\alpha}}) + o(1) = \mathbb{P}(M^{(n)}(R^{-1}(t_*)) \geq xn^{\frac{1}{\alpha}}) + o(1). \quad (52)$$

Using (49), (46) and (52)

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \mathbb{P}(M^{(n)}(R^{-1}(t_*)) \geq xn^{\frac{1}{\alpha}}) \\ & \leq \lim_{n \rightarrow \infty} \mathbb{P}(M^{(n)}(t_+) \geq (1 - 2\varepsilon)xn^{\frac{1}{\alpha}}) \\ & = 1 - \exp(-(x(1 - 2\varepsilon))^{-\alpha} \frac{(\alpha - 1)t(1 + \varepsilon)}{\Gamma(2 - \alpha)}). \end{aligned} \quad (53)$$

Since  $\varepsilon$  can be arbitrarily small, (48) and (53) allow to conclude.  $\square$

Finally, observe that Corollary 1.8 is obtained from a combination of Lemma 3.3 and Theorem 1.7.

#### REFERENCES

- [1] E. Árnason. Mitochondrial cytochrome b dna variation in the high-fecundity atlantic cod: trans-atlantic clines and shallow gene genealogy. *Genetics*, 166(4):1871–1885, 2004.
- [2] J. Berestycki, N. Berestycki and J. Schweinsberg. Beta-coalescents and continuous stable random trees. *Ann. Probab.*, 35(5):1835–1887, 2007.
- [3] J. Berestycki, N. Berestycki and J. Schweinsberg. Small-time behavior of Beta-coalescents. *Ann. Inst. H. Poincaré Probab. Stat.*, 44(2):214–238, 2008.
- [4] J. Bertoin and J-F. Le Gall. The Bolthausen-Sznitman coalescent and the genealogy of continuous-state branching processes. *Probab. Theory Related Fields*, 117(2):249–266, 2000.
- [5] N.H. Bingham, C.M. Goldie and J.L. Teugels. Regular variation- *Encyclopedia of Mathematics and its Applications*, 27. Cambridge University Press, Cambridge. MR898871. 1987.
- [6] M. Birkner, J. Blath, M. Capaldo, A. M. Etheridge, M. Möhle, J. Schweinsberg and A. Wakolbinger. Alpha-stable branching and Beta-coalescents. *Electron. J. Probab.*, 10(9):303–325, 2005.
- [7] M.G.B. Blum and O. François. Minimal clade size and external branch length under the neutral coalescent. *Adv. in Appl. Probab.*, 37(3):647–662, 2005.
- [8] E. Bolthausen and A. Sznitman. On Ruelle’s probability cascades and an abstract cavity method. *Commun. Math. Phys.*, 197(2):247–276, 1998.
- [9] J. Boom, E. Boulding and A. Beckenbach. Mitochondrial dna variation in introduced populations of pacific oyster, *crassostrea gigas*, in british columbia. *Canadian Journal of Fisheries and Aquatic Sciences*, 51(7):1608–1614, 1994.
- [10] A. Caliebe, R. Neininger, M. Krawczak and U. Röslér. On the length distribution of external branches in coalescence trees: genetic diversity within species. *Theoret. Population Biol.*, 72(2):245–252, 2007.
- [11] I. Dahmer, G. Kersting and A. Wakolbinger. The total external branch length of Beta-coalescents. *arXiv preprint arXiv:1212.6070*. To appear in *Combinatorics, Probability and Computing*.
- [12] J-F. Delmas, J-S. Dhersin and A. Siri-Jégousse. Asymptotic results on the length of coalescent trees. *Ann. Appl. Probab.*, 18(3):997–1025, 2008.
- [13] J-S. Dhersin, F. Freund, A. Siri-Jégousse and L. Yuan. On the length of an external branch in the beta-coalescent. *Stoch. Proc. Appl.*, 123:1691–1715, 2013.
- [14] J-S. Dhersin and M. Möhle. On the external branches of coalescents with multiple collisions. *Electron. J. Probab.*, 18(40):1–11, 2013.
- [15] P. Donnelly and T.G. Kurtz. Particle representations for measure-valued population models. *Ann. Probab.*, 27(1):166–205, 1999.
- [16] B. Eldon and J. Wakeley. Coalescent processes when the distribution of offspring number among individuals is highly skewed. *Genetics*, 172:2621–2633, 2006.
- [17] F. Freund and M. Möhle. On the time back to the most recent common ancestor and the external branch length of the Bolthausen-Sznitman coalescent. *Markov Process. Related Fields*, 15(3):387–416, 2009.
- [18] F. Freund and A. Siri-Jégousse. Minimal clade size in the Bolthausen-Sznitman coalescent. To appear in *J. Appl. Probab.*.
- [19] A. Gnedin, A. Iksanov and M. Möhle. On asymptotics of exchangeable coalescents with multiple collisions. *J. Appl. Probab.*, 45:1186–1195, 2008.
- [20] D. Hedgecock. Does variance in reproductive success limit effective population sizes of marine organisms? In *Genetics and Evolution of Aquatic Organisms*, Chapman and Hall, London, 1222–1344, 1994.
- [21] H-K. Hwang and S. Janson. Local limit theorems for finite and infinite urn models. *Ann. Probab.*, 36(3):992–1022, 2008.

- [22] G. Kersting. The asymptotic distribution of the length of beta-coalescent trees. *Ann. Appl. Probab.*, 22(5):2086–2107, 2012.
- [23] J.F.C. Kingman. The coalescent. *Stoch. Proc. Appl.*, 13(3):235–248, 1982.
- [24] J.F.C. Kingman. Exchangeability and the evolution of large populations. In *Exchangeability in probability and statistics (Rome, 1981)*, North-Holland, Amsterdam, 97–112, 1982.
- [25] J.F.C. Kingman. On the genealogy of large populations. *J. Appl. Probab.*, 19(1):27–43, 1982.
- [26] M. Möhle. Asymptotic results for coalescent processes without proper frequencies and applications to the two-parameter Poisson-Dirichlet coalescent. *Stoch. Proc. Appl.*, 120(11):2159–2173, 2010.
- [27] J. Pitman. Coalescents with multiple collisions. *Ann. Probab.*, 27(4):1870–1902, 1999.
- [28] E. Rauch and Y. Bar-Yam. Theory predicts the uneven distribution of genetic diversity within species. *Nature*, 431:449–452, 2004.
- [29] S. Sagitov. The general coalescent with asynchronous mergers of ancestral lines. *J. Appl. Probab.*, 36(4):1116–1125, 1999.
- [30] J. Schweinsberg. A necessary and sufficient condition for the  $\Lambda$ -coalescent to come down from infinity. *Electron. Comm. Probab.*, 5:1–11, 2000.
- [31] J. Schweinsberg. Coalescent processes obtained from supercritical Galton-Watson processes. *Stoch. Proc. Appl.*, 106(1):107–139, 2003.
- [32] R. Slack. A branching process with mean one and possibly infinite variance. *Probab. Theory Related Fields*, 9(2):139–145, 1968.
- [33] L. Yuan. On the measure division construction of  $\Lambda$ -coalescents. *arXiv preprint arXiv:1302.1083*, 2013. To appear in *Markov Process. Related Fields*.

UNIVERSIDAD DE GUANAJUATO, DEPARTAMENTO DE MATEMÁTICAS. CALLE JALISCO S/N COL. MINERAL DE VALENCIANA 36240 GUANAJUATO, GUANAJUATO MEXICO.

*E-mail address:* arno@cimat.mx

UNIVERSITÉ PARIS 13, SORBONNE PARIS CITÉ, LAGA, CNRS, UMR 7539, F-93430, VILLETANEUSE, FRANCE.

*E-mail address:* yuan@math.univ-paris13.fr