EXTINCTION TIME OF NON-MARKOVIAN SELF-SIMILAR PROCESSES, PERSISTENCE, ANNIHILATION OF JUMPS AND THE FRÉCHET DISTRIBUTION

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Abstract. We start by providing an explicit characterization and analytical properties, including the persistence phenomena, of the distribution of the extinction time $\mathbb{T}$ of a class of non-Markovian self-similar stochastic processes with two-sided jumps that we introduce as a stochastic time-change of Markovian self-similar processes. For a suitably chosen time-change, we observe, for classes with two-sided jumps, the following surprising facts. On the one hand, all the $\mathbb{T}$'s within a class have the same law which we identify in a simple form for all classes and reduces, in the spectrally positive case, to the Fréchet distribution. On the other hand, each of its distribution corresponds to the law of an extinction time of a single Markov process without positive jumps, leaving the interpretation that the time-change has annihilated the effect of positive jumps. The example of the non-Markovian processes associated to Lévy stable processes is detailed.

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1. Introduction and main results

The aim of this paper is to characterize explicitly and derive analytical properties of the distribution of the positive random variable

$$\mathbb{T} = \inf\{t > 0;\ \mathbb{X}_t \leq 0\}, \tag{1.1}$$

where $\mathbb{X} = (\mathbb{X}_t)_{t\geq 0}$ is the stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, for $t \geq 0$, by

$$\mathbb{X}_t = X_{\lambda_t}, \quad \text{where } \lambda_t = \inf\{s > 0;\ \chi_s > t\} \tag{1.2}$$

and $X = (X_t)_{t\geq 0}$ (resp. $\chi = (\chi_t)_{t\geq 0}$) is a self-similar of index $\alpha > 0$ (resp. of index $\beta > 0$ and a.s. increasing with infinite lifetime) Markov process issued from $x > 0$ (resp. issued from 0). We assume throughout that X and χ are independent.

Our investigation includes the first exit time to the positive half-line $\mathbb{T}$ of the Brownian motion, the Bessel processes and more generally of any non-degenerate stable Lévy processes, time-changed by the inverse of a β-stable subordinator with $0 < \beta < 1$. The recent years have witnessed the ubiquity of such non-Markovian dynamics in relation to the fractional Cauchy problem, see e.g. [32, 31, 23, 14], and, also due to their central role in diverse physical applications within the field of anomalous diffusion, see e.g. [20], and also for neuronal models for which their long range dependence feature is attractive, see e.g. [19]. There is a substantial literature devoted to the study and applications of the first passage times of non-Markovian dynamics such as Gaussian processes and semi-Markov (Markov processes time-changed with the inverse of a subordinator),

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see e.g. [2, 9, 10, 11, 12, 13, 15] including diverse applications in physics. However, unlike for Markov processes, this literature reveals that the lack of a general theory makes the analysis of such objects difficult and only very partial statistical information regarding these random variables has been obtained. For instance, for some Gaussian processes and for regularly varying semi-Markov processes, the persistence probabilities decay rate has been observed, meaning that the survival probabilities of the first passage time distribution has a power decay which is independent of the state variable, see the above references [12, 2] and the references therein. We shall also identify, among different fine properties, the persistence phenomena for the distribution of $\mathbb{T}$, see Theorem 1.1 3) below. We also mention that in the recent work [9], a Wiener-Hopf theory has been established for the first passage times over some stochastic boundaries of Lévy processes time-changed by the inverse of subordinators.

Denoting the law of the process by $\mathbb{P}_x$ when starting from $x > 0$, we recall that the stochastic process X is said to be self-similar of index $\alpha > 0$ (or α-self-similar) if the following identity

$$(X_{c^\alpha t}, \mathbb{P}_{cx})_{t\geq 0} \stackrel{(d)}{=} (cX_t, \mathbb{P}_x)_{t\geq 0} \tag{1.3}$$

holds in the sense of finite-dimensional distributions for any $c > 0$. Thus, since χ is a β-self-similar process with a.s. increasing paths, λ has clearly a.s. continuous and non-decreasing paths and is $\frac{1}{\beta}$-self-similar and non-Markovian. Since X and λ are independent, one easily gets that $\mathbb{X}$ is $\frac{\alpha}{\beta}$-self-similar and non-Markovian.

Note that every jump of the increasing self-similar Markov process χ corresponds to a plateau for its continuous inverse λ. In the physical literature, these periods are interpreted as trapping events in the dynamics of the particle $\mathbb{X}$ and thus slow down the dynamics of the original particle X. For this reason, in the framework of diffusion, the time-changed process $\mathbb{X}$ is often called a subdiffusion or an anomalous diffusion, see [20].

1.1. **The Lamperti mapping between self-similar Markov and Lévy processes.** Every self-similar Markov process issued from $x > 0$ and taken up to its first entrance in $(-\infty, 0]$ can be regarded as a positive self-similar Markov process with absorption at zero. Motivated by this observation, we recall that Lamperti [18] identifies a one-to-one mapping between the class of positive self-similar Markov processes and the class of Lévy processes. More specifically, one has, under $\mathbb{P}_x, x > 0$, that

$$X_t = x \exp\left(Y_{A_{tx^{-\alpha}}}\right), \quad 0 \leq t < \mathrm{T} = \inf\{s > 0;\ X_s = 0\}, \tag{1.4}$$

where $A_t = \inf\{s > 0; \int_0^s \exp(\alpha Y_u)du > t\}$. Here $Y = (Y_t)_{t\geq 0}$ as a Lévy process is a stochastic process with stationary and independent increments with càdlàg sample paths. Moreover, its law is fully characterized by its characteristic exponent $\Psi(z) = \log \mathbb{E}[e^{zY_1}], z \in i\mathbb{R}$, that takes the form

$$\Psi(z) = \Psi(0) + \frac{\sigma^2}{2}z^2 + \mathrm{a}z + \int_{\mathbb{R}} (e^{zy} - 1 - yz\mathbb{I}_{\{|y|<1\}})\Pi(dy), \tag{1.5}$$

in which $\sigma^2, -\Psi(0) \geq 0$ reflect the diffusion coefficient and the killing rate respectively, $\mathrm{a} \in \mathbb{R}$, is the coefficient of the linear part and Π is the Lévy measure that characterizes the jumps and satisfies the condition $\int_{\mathbb{R}}(1 \wedge |y|^2)\Pi(dy) < +\infty$ and $\Pi(\{0\}) = 0$. We shall also need the analytical Wiener-Hopf factorization of the Lévy-Khintchine exponent $\Psi_\alpha(z) = \Psi(\alpha z)$ of αY, which is given, for any $z \in i\mathbb{R}$, by

$$\Psi_\alpha(z) = -\phi_\alpha^-(z)\phi_\alpha^+(-z), \tag{1.6}$$

where $\phi_\alpha^\pm \in \mathbf{B}$, the set of Bernstein functions, that is

$$\phi_\alpha^\pm(0) \geq 0 \text{ and } \phi_\alpha^\pm(u) - \phi_\alpha^\pm(0) \text{ are of the form (1.8) below with .}$$

There is a direct probabilistic interpretation of $\phi_\alpha^\pm$ as the Laplace exponents of the so-called ladder height processes of αY, see [3, Chapter VI]. A classical example is the standard Brownian motion for which $\Psi_1(z) = z^2$ and $\phi_1^\pm(z) = z$, the latter corresponding to pure drift, that is the Brownian motion grows and decreases continuously.

In order to avoid the trivial situation when $\mathbb{T} = \infty$ $\mathbb{P}_x$-almost surely (a.s.), according to Lamperti, see also [28, Section 2.2], it suffices that

$$\mathrm{T} \stackrel{(d)}{=} x^\alpha \int_0^\infty \exp(\alpha Y_t)dt < \infty, \tag{1.7}$$

which in turn is equivalent to the assumption $\phi_\alpha^+(0) > 0$ in (1.6). For this reason, we consider the set

$$\mathcal{N} = \{\Psi \text{ of the form (1.5); } \Psi_\alpha(z) = -\phi_\alpha^-(z)\phi_\alpha^+(-z) \text{ with } \phi_\alpha^+(0) > 0\}.$$

Next, we denote by ϱ the subordinator associated to χ by the Lamperti mapping (1.4) (replacing α by β) and its law is characterized by the Bernstein function $\phi(z) = -\log \mathbb{E}[e^{-z\varrho_1}], \Re(z) \geq 0$, which is expressed as

$$\phi(z) = \mathrm{d}z + \int_0^\infty (1 - e^{-zy})\vartheta(dy), \tag{1.8}$$

where $\mathrm{d} \geq 0$ and ϑ is a Lévy measure such that $\int_0^\infty (1 \wedge y)\vartheta(dy) < +\infty$. Next, to ensure that the process χ can start from 0 which is then viewed as an entrance boundary, one needs in addition that $\int_0^\infty y\vartheta(dy) < +\infty$ which implies that

$$\mathbb{E}[\varrho_1] = \phi'(0^+) = \mathrm{d} + \int_0^\infty y\vartheta(dy) < +\infty, \tag{1.9}$$

see e.g. [8]. Moreover, it is easily seen that the Lamperti mapping yields that χ has a.s. increasing paths if and only if the ones of ϱ are also a.s. increasing. It is well known that the latter holds if $\phi(\infty) = \infty$ or equivalently either one of the following conditions

$$\mathrm{d} > 0 \text{ or/and } \vartheta(0,1) = \infty, \tag{1.10}$$

holds. Then, we write

$$\mathbf{B}_\varrho = \{\phi \in \mathbf{B};\ \phi(0) = 0, (1.9) \text{ and } (1.10) \text{ hold}\}$$

and we refer to the monograph [3] for a thorough account on Lévy processes. Next, for any $\phi \in \mathbf{B}$, we write

$$\mathfrak{a}_\phi = \sup\{u \geq 0;\ |\phi(-u)| < \infty\} \in [0,\infty] \text{ and } \mathfrak{a}_\phi^* = \sup\{u \geq 0;\ 0 \leq \phi(-u) < \infty\} \in [0,\infty) \tag{1.11}$$

and note that $\mathfrak{a}_\phi \geq \mathfrak{a}_\phi^*$ with $\mathfrak{a}_\phi^* = \infty$ if and only if ϕ is a constant, which is excluded for $\phi \in \mathbf{B}_\varrho$.

1.2. **Basic facts on the Bernstein-gamma functions.** We shall also need, for any $\phi \in \mathbf{B}$, the Bernstein-gamma function W_ϕ which is the unique positive-definite function, i.e. the Mellin transform of a positive measure, that solves the functional equation, for $\Re(z) > -\mathfrak{a}_\phi^*$,

$$W_\phi(z+1) = \phi(z)W_\phi(z), \quad W_\phi(1) = 1. \tag{1.12}$$

Its existence is established across the set $\mathbf{B}$ for example in [29, 28]. It is easily checked that for any integer n, $W_\phi(n+1) = \prod_{k=1}^n \phi(k)$ and for the convenience of the reader we recall some further facts regarding the Bernstein-gamma functions, which are thoroughly investigated in [28, Section 4]. For any $\phi \in \mathbf{B}$, it admits the following generalized Weierstrass product representation

$$W_\phi(z) = \frac{e^{-\gamma_\phi z}}{\phi(z)} \prod_{k=1}^\infty \frac{\phi(k)}{\phi(k+z)} e^{\frac{\phi'(k)}{\phi(k)}z}, \quad \Re(z) > -\mathfrak{a}_\phi^*,$$

with $\gamma_\phi = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{\phi'(k)}{\phi(k)} - \log\phi(n)\right)$, see [28, (4.7) and (4.8)]. Moreover, it is shown that W_ϕ is analytic on $\Re(z) > -\mathfrak{a}^*_\phi$ and meromorphic and zero-free on $\Re(z) > -\mathfrak{a}_\phi$, see [28, Theorem 4.1]. When $\phi(z) = z$ in (1.8), then W_ϕ boils down to the Euler gamma function and obviously in this case $\mathfrak{a}_\phi = \infty, \mathfrak{a}^*_\phi = 0$.

The Bernstein-gamma functions are intimately linked to an important class of random variables namely the exponential functionals of Lévy processes, see e.g. [27, 28]. Exponential functionals are random variables represented by the integral in (1.7), where the horizon of integration is either infinity or an independent exponentially distributed random variable. Then, it is shown in [28, Theorem 2.4] that the Mellin transform of the exponential functionals can be evaluated explicitly in terms of the W_ϕ's as follows

$$\mathbb{E}\left[\left(\int_0^\infty e^{\alpha Y_t}dt\right)^{z-1}\right] = \phi^+_\alpha(0)\frac{\Gamma(z)}{W_{\phi^-_\alpha}(z)}W_{\phi^+_\alpha}(1-z), \quad \Re(z) \in (0,1), \tag{1.13}$$

where we used the notation introduced above. We emphasize that $\Re(z) \in (0,1)$ is a universal strip of analyticity across $\mathbf{B}$ which can be larger depending on the values of $\mathfrak{a}_\phi, \mathfrak{a}^*_\phi$ for $\phi = \phi^+_\alpha, \phi = \phi^-_\alpha$ and note the dependence on the Wiener-Hopf factors introduced in (1.6). We also point out that in [28] the exponential functional is defined with minus sign, that is $\int_0^\infty e^{-\alpha Y_t}dt$, which simply implies swapping of signs for $\phi^\pm_\alpha$ in all formulae. An in-depth study of these random variables is presented in [28]. The Bernstein-gamma functions also play a central role in the recent developments of the spectral theory of some classes of non-self-adjoint Markov operators, see e.g. [29] and [30].

1.3. **Main results.** To summarize, the process $\mathbb{X}$ is self-similar of index $\frac{\alpha}{\beta}$ starting from $x > 0$ and it is non-Markovian with possible upward and downward jumps depending on the support of the Lévy measure Π in (1.5). Indeed, the continuity of the paths of λ entails that the processes X and $\mathbb{X}$ have jumps of the same amplitude and direction. Moreover, from the Lamperti mapping it can be identified uniquely by the two Lévy-Khintchine exponents $\Psi_\alpha \in \mathcal{N}$ and $\phi_\beta \in \mathbf{B}_\varrho$, where we recall that Ψ_α is the Lévy-Khintchine exponent of αY with $\Psi_1 = \Psi$, see (1.4), and similarly for ϕ_β. To emphasize this connection we shall also use the notation

$$\mathbb{T}_{\Psi_\alpha}(\phi_\beta) = \mathbb{T} = \inf\{t > 0;\ \mathbb{X}_t \leq 0\} \tag{1.14}$$

and in the same spirit we may write $\mathrm{T}_{\Psi_\alpha} = \mathrm{T}$. As usual, we denote by $\mathtt{C}^\infty_0(\mathbb{R}^+)$ (resp. $\mathtt{C}^k_0(\mathbb{R}^+)$) the space of infinitely (resp. $k \in \mathbb{Z}^+$ times) continuously differentiable functions on $\mathbb{R}^+$ vanishing at ∞ along with its derivatives (resp. k derivatives). We are now ready to state our first main result.

Theorem 1.1. *Let $\Psi \in \mathcal{N}$ and $\phi \in \mathbf{B}_\varrho$ and $\alpha, \beta > 0$. Then, the following holds.*

1) For any $x > 0$,

$$\mathbb{E}_x\left[\mathbb{T}^z_{\Psi_\alpha}(\phi_\beta)\right] = x^{\frac{\alpha}{\beta}z}\frac{\phi^+_\alpha(0)}{\beta\phi'(0^+)}\frac{\Gamma(-\frac{z}{\beta})}{W_{\phi_\beta}(-\frac{z}{\beta})}\frac{\Gamma(\frac{z}{\beta}+1)W_{\phi^+_\alpha}(-\frac{z}{\beta})}{W_{\phi^-_\alpha}(\frac{z}{\beta}+1)},\ -\underline{\mathfrak{m}}_\mathbb{T} < \Re(z) < \overline{\mathfrak{m}}_\mathbb{T}, \tag{1.15}$$

where $\underline{\mathfrak{m}}_\mathbb{T} = \beta(\mathfrak{a}_{\phi^-_\alpha}\mathbb{I}_{\{\phi^-_\alpha(0)=0\}} + 1) \geq \beta > 0$ and $\overline{\mathfrak{m}}_\mathbb{T} = \beta(\mathfrak{a}_{\phi_\beta} \wedge \mathfrak{a}^*_{\phi^+_\alpha}) \geq 0$.

2) The law of $\mathbb{T}_{\Psi_\alpha}(\phi_\beta)$ is absolutely continuous with a density denoted by $f_{\mathbb{T}_{\Psi_\alpha}(\phi_\beta)}$ which has the following smoothness property

$$f_{\mathbb{T}_{\Psi_\alpha}(\phi_\beta)} \in \mathtt{C}^{\lceil \mathrm{N}\rceil - 2}_0\left(\mathbb{R}^+\right),$$

provided $\mathrm{N} = \mathrm{N}_{\phi_\beta} + \mathrm{N}_{\Psi_\alpha} \in (1,\infty]$, where with the obvious notation, see (1.8),

$$\mathrm{N}_{\phi_\beta} = \frac{\vartheta_\beta(0,\infty)}{\mathrm{d}_\beta}, \quad \mathrm{N}_{\Psi_\alpha} = \frac{\phi^-_\alpha(0) + \vartheta^-_\alpha(0,\infty)}{\mathrm{d}^-_\alpha} + \frac{\upsilon^+_\alpha(0^+)}{\phi^+_\alpha(0) + \vartheta^+_\alpha(0,\infty)} + \infty\mathbb{I}_{\{\mathrm{d}^+_\alpha > 0\}},$$

and, v_α^+ is the density of ϑ_α^+, whose existence is justified in the proof in the case where N_{Ψ_α} may be finite, namely $\mathrm{d}_\alpha^+ = 0 < \mathrm{d}_\alpha^-$.

3) Let us write simply $\mathfrak{c}_\alpha = \mathfrak{a}^*_{\phi_\alpha^+}$ and assume that $0 < \mathfrak{c}_\alpha < \mathfrak{a}_{\phi_\beta}$ with $\Psi_\alpha(-\mathfrak{c}_\alpha) = \phi_\alpha^+(-\mathfrak{c}_\alpha) = 0$, $|\Psi'_\alpha(-\mathfrak{c}_\alpha^+)| < \infty$ and $\{b \in \mathbb{R}; \Psi_\alpha(-\mathfrak{c}_\alpha + ib) = 0\} = \{0\}$ then

$$\lim_{t\to\infty} t^{\beta\mathfrak{c}_\alpha} \mathbb{P}_x(\mathbb{T}_{\Psi_\alpha}(\phi_\beta) > t) = \frac{\mathbb{E}_x\left[\mathbb{T}^{\beta\mathfrak{c}_\alpha}_{\Psi_\alpha}(\phi_\beta)\right]}{\mathfrak{c}_\alpha \phi_\alpha^{+\prime}(-\mathfrak{c}_\alpha^+)} \in (0,\infty).$$

Finally, if in addition $|\Psi''_\alpha(-\mathfrak{c}_\alpha^+)| < \infty$, $2 \leq \lceil \mathrm{N}_{\Psi_\alpha} \rceil < \infty$ (resp. $\lceil \mathrm{N}_{\Psi_\alpha} \rceil = \infty$ and there exists $k \in \mathbb{Z}^+$ such that $\liminf_{|b|\to\infty} |b|^k |\Psi_\alpha(-\mathfrak{c}_\alpha + ib)| > 0$) then for any $n \leq \lceil \mathrm{N}_{\Psi_\alpha} \rceil - 2 + (\lceil \mathrm{N}_{\phi_\beta} \rceil - 2)\mathbb{I}_{\{\lceil \mathrm{N}_{\phi_\beta} \rceil \geq 2\}}$ (resp. for any $n \in \mathbb{Z}^+$)

$$\lim_{t\to\infty} t^{\mathfrak{c}_\alpha + n + 1}\left(t^{\frac{1}{\beta}-1} f_{\mathbb{T}_{\Psi_\alpha}(\phi_\beta)}(t^{\frac{1}{\beta}})\right)^{(n)} = \beta(-1)^n C_{\mathfrak{c}_\alpha}(n) \frac{\mathbb{E}_x\left[\mathbb{T}^{\mathfrak{c}_\alpha}_{\Psi_\alpha}(\phi_\beta)\right]}{\mathfrak{c}_\alpha \phi_\alpha^{+\prime}(-\mathfrak{c}_\alpha^+)},$$

where $C_{\mathfrak{c}_\alpha}(n) = (1+\mathfrak{c}_\alpha)_{n-\overline{\mathrm{N}}_{\Psi_\alpha}} \sum_{k=0}^{\overline{\mathrm{N}}_{\Psi_\alpha}} \binom{\overline{\mathrm{N}}_{\Psi_\alpha}}{k} \frac{\Gamma(\overline{\mathrm{N}}_{\Psi_\alpha} - n + 1)}{\Gamma(k-n+1)} (-1)^k (1+\mathfrak{c}_\alpha)_k$, $\overline{\mathrm{N}}_{\Psi_\alpha} = \lceil \mathrm{N}_{\Psi_\alpha} \rceil - 2$ and $(1+\mathfrak{c}_\alpha)_k = \frac{\Gamma(1+\mathfrak{c}_\alpha + k)}{\Gamma(1+\mathfrak{c}_\alpha)}$.

Remark 1.2. *We note that a sufficient condition for $\mathbb{E}_x\left[\mathbb{T}_{\Psi_\alpha}(\phi_\beta)\right] < \infty$ is $\overline{\mathfrak{m}}_{\mathbb{T}} > 1$, which holds whenever $\beta(\mathfrak{a}_{\phi_\beta} \wedge \mathfrak{a}^*_{\phi_\alpha^+}) > 1$. However, inspecting (1.15) we observe that a necessary and sufficient condition for $\mathbb{T}_{\Psi_\alpha}(\phi_\beta)$ to have a finite mean is the continuous extension of the mapping $z \mapsto \frac{\Gamma(-\frac{z}{\beta})}{W_{\phi_\beta}(-\frac{z}{\beta})} W_{\phi_\alpha^+}(-\frac{z}{\beta})$ to the whole line $\Re(z) \in 1 + i\mathbb{R}$. The cases where it is possible can be inferred from* [28, Theorem 2.1] *for the ratio and* [28, Theorem 4.1] *for the second term.*

Remark 1.3. *It is worth pointing out that the density of $\mathrm{T} \stackrel{(d)}{=} x^\alpha \int_0^\infty \exp(\alpha Y_t)dt$ is in the set $\mathtt{C}_0^{\lceil \mathrm{N}_{\Psi_\alpha} \rceil - 2}(\mathbb{R}^+)$ provided that $\mathrm{N}_{\Psi_\alpha} > 1$, see the proof of Theorem 1.1 below or* [28, Theorem 2.4(3)]. *Therefore, when $\mathrm{N}_{\Psi_\alpha} < \infty$, $f_{\mathbb{T}_{\Psi_\alpha}(\phi_\beta)}$ is smoother by $\lceil \mathrm{N} \rceil - \lceil \mathrm{N}_{\Psi_\alpha} \rceil \geq 0$ number of derivatives than the aforementioned density. We thus conclude that the time-change smooths out the density of the absorption/extinction time.*

Remark 1.4. *Note that, in item 3), the condition of $-\mathfrak{c}_\alpha$ to be the unique zero of Ψ_α on the line $-\mathfrak{c}_\alpha + i\mathbb{R}$ is equivalent to the Lévy process Y being non-lattice, see the discussion prior to* [28, Theorem 2.11], *and hence of $\log \mathbb{X}$ being non-lattice. The requirement when $\lceil \mathrm{N}_{\Psi_\alpha} \rceil = \infty$ that there exists $k \in \mathbb{Z}^+$ such that $\liminf_{|b|\to\infty} |b|^k |\Psi_\alpha(-\mathfrak{c}_\alpha + ib)| > 0$ is equivalent to Y not being weak non-lattice, a new notion introduced in the aforementioned paper.*

In order to state our next main result, we define and provide some distributional and analytical properties of a family of random variables indexed by the set of Bernstein functions $\mathbf{B}$ that were introduced by the second author in [24] for a subset of $\mathbf{B}$ and generalize the Fréchet one. We recall that the latter is one of the three non-degenerate extreme value distribution functions arising as limits of properly renormalized running maxima of i.i.d. random variables. We shall need the notation, borrowed from [28, Theorem 2.3], $\overline{\Theta}_\phi = \limsup_{|b|\to\infty} \frac{\int_0^{|b|} \arg \phi(1+iu)du}{|b|} \in [0, \frac{\pi}{2}]$, $\phi \in \mathbf{B}$, and the argument function is defined from the main branch of the complex logarithm and we have $\arg : \mathbb{C} \mapsto (-\pi, \pi]$.

Proposition 1.5. *1) For any $\phi \in \mathbf{B}$ and $\beta > 0$, there exists a positive random variable $\mathbb{F}_\beta(\phi)$ whose distribution is determined by*

$$\mathbb{E}\left[\mathbb{F}^z_\beta(\phi)\right] = \frac{\Gamma(1-\frac{z}{\beta})\Gamma(\frac{z}{\beta}+1)}{W_\phi(\frac{z}{\beta}+1)}, \ -\underline{\mathfrak{m}}_{\mathbb{F}} < \Re(z) < \overline{\mathfrak{m}}_{\mathbb{F}}, \tag{1.16}$$

where $\underline{\mathfrak{m}}_{\mathbb{F}} = \beta(\mathfrak{a}_\phi \mathbb{I}_{\{\phi(0)=0\}} + 1)$ *and* $\overline{\mathfrak{m}}_{\mathbb{F}} = \beta$.

2) Moreover, its law is absolutely continuous with a density $f_{\mathbb{F}_\beta(\phi)} \in \mathtt{C}_0^\infty(\mathbb{R}^+)$ *which admits an analytical extension to the sector* $S_\phi = \{z \in \mathbb{C};\, |\arg(z)| < \pi - \overline{\Theta}_\phi\}$ *given, for any* $c \in (-\frac{\underline{\mathfrak{m}}_{\mathbb{F}}-1}{\beta}, \frac{\overline{\mathfrak{m}}_{\mathbb{F}}+1}{\beta})$, *by the Mellin-Barnes integral*

$$f_{\mathbb{F}_\beta(\phi)}(t) = \frac{\beta}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-z\beta} \frac{\Gamma(-z+1+\frac{1}{\beta})\Gamma(z+1-\frac{1}{\beta})}{W_\phi(z+1-\frac{1}{\beta})} dz, \tag{1.17}$$

which expands, for $|t| > \phi^{-\frac{1}{\beta}}(\infty)$, *as*

$$f_{\mathbb{F}_\beta(\phi)}(t) = \beta t^{-\beta-1} \mathrm{I}_\phi(e^{i\pi} t^{-\beta}) \text{ where } \mathrm{I}_\phi(z) = \sum_{n=0}^{\infty} \frac{n+1}{\phi(n+1)} \frac{z^n}{W_\phi(n+1)}, |z| < \phi(\infty). \tag{1.18}$$

Remark 1.6. *From* [28, Theorem 4.7], *one can derive a Lévy-Khintchine type representation for the characteristic function of the real-valued variable* $\log \mathbb{F}_\beta(\phi), \phi \in \mathbf{B}$, *along with sufficient conditions on* ϕ *(or its characteristics) for this variable to be infinitely divisible, see also* [1] *for alternative conditions. This remark also applies to* $\log \mathbb{T}$ *defined in Theorem 1.1.*

We proceed by establishing some connections between this class of distributions and some distributions that have already appeared in the literature.

- *The Fréchet distribution*. When $\phi(u) = u$ above that is $W_\phi(n+1) = n!$, then $\mathbb{F}_\beta = \mathbb{F}_\beta(\phi)$ boils down to the classical Fréchet random variable of parameter $\beta > 0$, that is $f_{\mathbb{F}_\beta}(t) = \beta t^{-\beta-1} e^{-t^{-\beta}}, t > 0$.
- *The class of distributions introduced in* [24]. Let, for some fixed $\alpha > 0$, denote by $\psi(u) = (u-\alpha)\phi(u), u \geq 0$, with $\phi \in \mathbf{B}$, the Wiener-Hopf factorization of the Laplace exponent of a spectrally negative Lévy process which is either killed at an independent exponential time or with a negative mean. Then, it is shown in [24, Theorem 2.1], that s_ϕ is the density of a positive random variable, where

  $$s_\phi(t) = \alpha\phi(\alpha) t^{-2} \mathcal{I}_{\psi_{\triangleright\alpha}}(2; e^{i\pi} t^{-1}),\ t > 0,$$

  and, with the notation of the aforementioned paper, we used the fact that $\gamma = \alpha$, $\gamma_\alpha = 1$ and $C_\gamma = \psi'(\alpha) = \alpha\phi(\alpha)$, see [24, Proposition 2.4(2)], $\psi_{\triangleright\alpha}(u) = \psi(u+\alpha) = u\phi(u+\alpha)$ and

  $$\mathcal{I}_{\psi_{\triangleright\alpha}}(2; \alpha z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1)(\alpha z)^n}{\prod_{k=1}^{n} \psi(\alpha(k+1))} = \phi(\alpha) \sum_{n=0}^{\infty} \frac{(n+1) z^n}{\phi_\alpha(n+1) W_{\phi_\alpha}(n+1)} = \phi(\alpha) \mathrm{I}_{\phi_\alpha}(z).$$

  Since, it is well-known that $\psi(u) = (u-\alpha)\phi(u)$, as above, if and only if $\phi_{\triangleright\alpha}(u) = \phi(u+\alpha) \in \mathbf{B}_-$, where

  $$\mathbf{B}_- = \{\phi \in \mathbf{B};\ \text{ in (1.8) } \vartheta(dy) = v(y)dy \text{ with } v \text{ non-increasing on } \mathbb{R}_+\},$$

  we get that for all $\phi \in \mathbf{B}$ such that $\phi_{\triangleright\alpha} \in \mathbf{B}_-$, we have $s_\phi(t) = \frac{1}{\alpha} f_{\mathbb{F}_1(\phi_\alpha)}(\alpha t)$. Some illustrative examples are given in [24] and they include the reciprocal of the Gamma and Wright hypergeometric type random variables.

We now turn to the statement of the second main result for which we need the following. We recall, see e.g. [26], that the linear operator $\mathcal{S}_1 : f \mapsto \mathcal{S}_1 f(u) = \frac{u}{u+1} f(u+1)$ leaves invariant the set $\mathbf{B}$ of Bernstein functions. A slight extension of this transformation has been proposed in [29, Lemma 10.1.2.] and is defined as follows. Let us introduce the subset of Bernstein functions

$$\mathbf{B}_1 = \{\phi \in \mathbf{B};\ 0 \leq \phi(-u) < \infty \text{ for all } u \leq 1\} \tag{1.19}$$

and then for any $\phi \in \mathbf{B}_1$, we have

$$\mathcal{S}_\phi(u) = \frac{u}{u+1}\phi(u) \in \mathbf{B}. \tag{1.20}$$

To see that, we simply observe that on the one hand for any $\phi \in \mathbf{B}_1$, $\phi_1(u) = \phi(u-1) \in \mathbf{B}$ as it is well-defined, non-negative and clearly $\phi'(u-1)$ is completely monotone on $\mathbb{R}^+$ and on the other hand $\mathcal{S}_\phi = \mathcal{S}_1\phi_1$.

Theorem 1.7. *1) For any $\Psi \in \mathcal{N}_1 = \{\Psi \in \mathcal{N}$ with $\phi_\alpha^+ \in \mathbf{B}_1$ and (1.10) holds$\}$, we have the identity in law*

$$\mathbb{T}_{\Psi_\alpha}(\mathcal{S}_{\phi_\alpha^+}) \stackrel{(d)}{=} x^{\frac{\alpha}{\beta}}\mathbb{F}_\beta(\phi_\alpha^-). \tag{1.21}$$

2) In particular, for $\beta = 1$ and all $\Psi \in \mathcal{N}_1$ such that $\phi_\alpha^- \in \mathbf{B}_-$, we have

$$\mathbb{T}_{\Psi_\alpha}(\mathcal{S}_{\phi_\alpha^+}) \stackrel{(d)}{=} \mathrm{T}_\psi, \tag{1.22}$$

where $\mathrm{T}_\psi = \inf\{t > 0;\ \overline{X}_t \leq 0\}$ with $\overline{X} = (\overline{X}_t)_{t\geq 0}$ the spectrally negative α-self-similar positive Markov process associated, via the Lamperti mapping, to $\psi(z) = \frac{1}{\alpha}(z-\alpha)\phi^-(z) \in \mathcal{N}$.

3) Finally, when $\Psi \in \mathcal{N}_1^+ = \{\Psi \in \mathcal{N}_1$ with $\phi_\alpha^-(u) = \alpha u\}$, i.e. $\mathbb{X}$ is spectrally positive, then

$$\mathbb{T}_{\Psi_\alpha}(\mathcal{S}_{\phi_\alpha^+}) \stackrel{(d)}{=} (\alpha x^\alpha)^{\frac{1}{\beta}}\mathbb{F}_\beta. \tag{1.23}$$

Thus, for any $\Psi \in \mathcal{N}_1^+$, $\mathbb{T}_{\Psi_\alpha}(\mathcal{S}_{\phi_\alpha^+})$ is a positive self-decomposable variable.

Remark 1.8. *It is interesting to note in (1.21) that the passage times $\mathbb{T}_{\Psi_\alpha}(\mathcal{S}_{\phi_\alpha^+})$ have the same distribution for all $\mathbb{X}$ associated to the same descending ladder height exponent ϕ_α^-, independently of the ascending one and which reduces to a very simple law in the spectrally positive case.*

Remark 1.9. *In the same vein, it is also surprising to observe, from (1.22), that the passage time of the two processes have the same law whereas one process has two-sided jumps whereas the other one has only downward jumps. This leads to the interpretation that the specific time-change annihilates the role played by the ascending ladder height process of the underlying Lévy process in the dynamics of the upward jumps of $\mathbb{X}$. Although the identity (1.22) is in distribution, it would be very interesting to understand whether there is a pathwise explanation of this fact. We note also that $\beta = 1$ in the second item is required because the powers of x to the right-hand side of the expressions (1.7) and (1.21) must be the same. The first corresponds to the absorption time of $\overline{X}$ and the second to the extinction time of the respective $\mathbb{X}$.*

We postpone the proof of these results to the Section 3. We proceed instead with the description of an example that illustrates the previous results.

2. Illustrative examples

Our running example will be based on stable Lévy processes and positive powers theoreof. Let us assume that $X = Z^{\frac{1}{b}}$ where Z is an $\mathfrak{a}$-stable Lévy process, $0 < \mathfrak{a} \leq 2$, with positivity parameter $\rho = \mathbb{P}(Z_1 > 0)$, killed upon entering into the negative half-line and $0 < b$. This is easily seen to be a positive self-similar Markov process of index $0 < \alpha = \mathfrak{a}b$ and it is associated, via the Lamperti mapping, to a Lévy process with Lévy-Khintchine exponent expressed in terms of its Wiener-Hopf factors as follows

$$\Psi_\alpha(z) = -\frac{\Gamma(1+\alpha z)}{\Gamma(1-\mathfrak{a}(1-\rho)+\alpha z)}\frac{\Gamma(\mathfrak{a}-\alpha z)}{\Gamma(\mathfrak{a}(1-\rho)-\alpha z)} = -\phi_\alpha^-(z)\phi_\alpha^+(-z), \tag{2.1}$$

see [17, Section 2.1]. The special case where $Z=\chi$ is a β-stable subordinator with $\beta\in(0,1)$ is of separate interest. Indeed, χ is clearly an increasing positive self-similar Markov process of index β and it is associated via the Lamperti mapping to

$$\phi_\beta(z)=\frac{\Gamma(\beta+\beta z)}{\Gamma(\beta z)}, \tag{2.2}$$

where we have simply specialized (2.1) for $b=\rho=1,\alpha=\beta$.

2.1. **Example related to Theorem 1.1.** We take X to be a standard one-dimensional Brownian motion and χ to be an independent β-stable subordinator. From (2.2) we have that $\phi_\beta(z)=\frac{\Gamma(\beta z+\beta)}{\Gamma(\beta z)}$ from which it is immediate to check that for any $c>0$, $f(z)=c\frac{\Gamma(\beta z+1)}{z}$ solves $f(z+1)=\phi_\beta(z)f(z),\Re(z)>0$. Then, $c=\frac{1}{\phi'_\beta(0^+)}=\frac{1}{\beta\phi'(0^+)}=\frac{1}{\Gamma(1+\beta)}$ normalizes f so that

$$f(z)=W_{\phi_\beta}(z)=\frac{1}{\Gamma(1+\beta)}\frac{\Gamma(\beta u+1)}{z}=\frac{\Gamma(\beta z)}{\Gamma(\beta)} \text{ and } W_{\phi_\beta}(1)=1. \tag{2.3}$$

The standard Brownian motion is self-similar of index $\mathfrak{a}=2$ and applying (2.1) with $b=1,\mathfrak{a}=\alpha=2,\rho=\frac{1}{2}$ we get that the associated Lévy-Khintchine exponent via the Lamperti transformation is $\Psi_2(z)=4z^2-2z$ with $\phi_2^+(z)=2z+1$ and $\phi_2^-(z)=2z$. Therefore Ψ_2 is simply the Lévy-Khintchine exponent of a Brownian motion with variance 8 and negative drift of value -2. We then substitute in (1.15) to get

$$\begin{aligned}\mathbb{E}_x\left[\mathbb{T}^z_{\Psi_\alpha}(\phi_\beta)\right] &= x^{\frac{2}{\beta}z}\frac{1}{\Gamma(1+\beta)}\frac{\Gamma(-\frac{z}{\beta})\Gamma(\beta)}{\Gamma(-z)}\frac{\Gamma(\frac{z}{\beta}+1)\Gamma(-2\frac{z}{\beta}+1)}{2^{\frac{z}{\beta}}\Gamma(\frac{z}{\beta}+1)},\\ &= x^{\frac{2}{\beta}z}\frac{1}{2^{\frac{z}{\beta}}\beta}\frac{\Gamma(-\frac{z}{\beta})}{\Gamma(-z)}\Gamma(-2\frac{z}{\beta}+1),\quad -\infty<\Re(z)<\frac{\beta}{2},\end{aligned}$$

where we have checked that $W_{\phi_\alpha^-}(z)=2^{z-1}\Gamma(z),W_{\phi_\alpha^+}(z)=\Gamma(2z+1),\mathfrak{a}^*_{\phi_\alpha^+}=\frac{1}{2},\mathfrak{a}_{\phi_\alpha^-}=\infty,\mathfrak{a}_{\phi_\beta}=1$, see (1.11) and (1.12), and we have used the recurrent property of the gamma function. We highlight that the result above can be inferred from the factorization (3.6) below since χ_1 is simply the stable subordinator at time 1, whose moments are known and recalled in Remark 3.2, and the respective absorption time for the Brownian motion, T, is known to be distributed as the inverse Gamma distribution.

2.2. **Example related to Theorem 1.7.** Assume that in (2.1), $0<b\le 1-\rho$ and hence $0<\alpha=\mathfrak{a}b\le\mathfrak{a}(1-\rho)$. Note that $\Psi_\alpha(0)>0$ unless $\rho=1$ (resp. $\rho=0$), which we exclude as in this case Z is a positive (resp. a negative) Lévy process and does not hit the negative (resp. positive) half-line. Let us assume for sake of simplicity that $\beta=1$ and write $\phi=\phi_1$, then, as $0<\rho<1$, one easily checks that $\phi_\alpha^+\in\mathbf{B}_1$ and gets

$$\phi(u)=\mathcal{S}_{\phi_\alpha^+}(u)=\frac{u}{u+1}\frac{\Gamma(\mathfrak{a}+\alpha u)}{\Gamma(\mathfrak{a}(1-\rho)+\alpha u)}\in\mathbf{B}.$$

Next, we have that $W_{\phi_\alpha^-}$ solves, for $u>0$, the equation

$$W_{\phi_\alpha^-}(u+1)=\frac{\Gamma(1+\alpha u)}{\Gamma(1-\mathfrak{a}(1-\rho)+\alpha u)}W_{\phi_\alpha^-}(u),\quad W_{\phi_\alpha^-}(1)=1.$$

Recalling that the *Barnes gamma function* G satisfies the functional equation, for $u,\tau>0$,

$$G(u+1;\tau)=\Gamma\left(\frac{u}{\tau}\right)G(u;\tau)$$

see e.g. [17, (24)], we get

$$W_{\phi_\alpha^-}(u+1)=\frac{G(1+\frac{1-\mathfrak{a}(1-\rho)}{\alpha};\frac{1}{\alpha})}{G(\frac{1}{\alpha}+1;\frac{1}{\alpha})}\frac{G(u+\frac{1}{\alpha}+1;\frac{1}{\alpha})}{G(u+1+\frac{1-\mathfrak{a}(1-\rho)}{\alpha};\frac{1}{\alpha})}.$$

Easy algebra yields that

$$f_{\mathbb{T}_{\Psi_\alpha}(\mathcal{S}_{\phi_\alpha^+})}(t)=x^{\alpha}t^{-2}\mathrm{I}_G(e^{i\pi}(x^{-\alpha}t)^{-1}),\ t>0,$$

where we have set $\mathrm{I}_G=\mathrm{I}_{\phi_\alpha^-}$ with, for any $z\in\mathbb{C}$,

$$\mathrm{I}_G(z)=\frac{G(\frac{1}{\alpha}+1;\frac{1}{\alpha})}{\alpha G(1+\frac{1-\mathfrak{a}(1-\rho)}{\alpha};\frac{1}{\alpha})}\sum_{n=0}^{\infty}\frac{\Gamma(1-\mathfrak{a}(1-\rho-b)+\alpha n)}{\Gamma(\alpha+\alpha n)}\frac{G(n+2+\frac{1-\mathfrak{a}(1-\rho)}{\alpha};\frac{1}{\alpha})}{G(n+\frac{1}{\alpha}+2;\frac{1}{\alpha})}z^n.$$

Finally, setting $\rho=1-\frac{1}{\mathfrak{a}}$ with $1<\mathfrak{a}\leq 2$, that is Z and hence $\mathbb{X}$ is spectrally positive, we obtain indeed that $\phi_\alpha^-(z)=\frac{\Gamma(1+\alpha z)}{\Gamma(1-\mathfrak{a}(1-\rho)+\alpha z)}=\alpha z$ and, from (1.23), we get that when $\mathbb{X}$ starts from $\alpha^{-\frac{1}{\alpha}}$ then $\mathbb{T}_{\Psi_\alpha}(\mathcal{S}_{\phi_\alpha^+})$ has the Fréchet distribution of parameter 1.

3. Proofs

Throughout, for a non-negative random variable X we use the notation

$$\mathcal{M}_X(z)=\mathbb{E}[X^z]$$

for at least any $z\in i\mathbb{R}$, the imaginary line, meaning that $\mathcal{M}_X(z-1)$ is its Mellin transform.

3.1. **Proof of Theorem 1.1.** We start by recalling that χ is the increasing self-similar Markov process of index $\beta>0$ starting from 0 and associated via the Lamperti mapping to the Bernstein function $\phi\in\mathbf{B}_\varrho$. We denote by $\lambda=(\lambda_t)_{t\geq 0}$ its continuous right-inverse, see (1.2).

Lemma 3.1. *For any $t>0$ and $\Re(z)>0$,*

$$\mathcal{M}_{\lambda_t}(z)=\frac{t^{z\beta}}{\beta\phi'(0^+)}\frac{\Gamma(z)}{W_{\phi_\beta}(z)},\tag{3.1}$$

where we recall that $\phi_\beta(u)=\phi(\beta u)\in\mathbf{B}$. The law of λ_t is absolutely continuous for all $t>0$. Moreover, for any $q\in\mathbb{C}$,

$$\mathbb{E}\left[e^{q\lambda_t}\right]=\frac{1}{\beta\phi'(0^+)}\bar{\mathrm{I}}_{\phi_\beta}(qt^{\beta}),\tag{3.2}$$

where $\bar{\mathrm{I}}_{\phi_\beta}(q)=\sum_{n=0}^{\infty}\frac{q^n}{nW_{\phi_\beta}(n)}$. Consequently the law of λ_t is, for all $t>0$, moment determinate.

Remark 3.2. *Note that when χ is a β-stable subordinator, $0<\beta<1$, we recover the well-know fact that in* (3.2)*, $\bar{\mathrm{I}}_{\phi_\beta}$ is the Mittag-Leffler function of index β. Indeed, from* (2.3) *the Bernstein-gamma function related to ϕ_β is in this case $W_{\phi_\beta}(z)=\frac{1}{\Gamma(1+\beta)}\frac{\Gamma(\beta z+1)}{z}$ with $\beta\phi'(0^+)=\Gamma(1+\beta)$ and thus*

$$\mathbb{E}\left[e^{q\lambda_t}\right]=\frac{1}{\beta\phi'(0^+)}\sum_{n=0}^{\infty}\frac{1}{n}\frac{(t^\beta q)^n}{W_{\phi_\beta}(n)}=\sum_{n=0}^{\infty}\frac{(t^\beta q)^n}{\Gamma(\beta n+1)}.$$

Proof. For any bounded Borel function f and $t>0$, we have that

$$\begin{aligned}\mathbb{E}[f(\lambda_t)] &= \mathbb{E}[f(t^\beta\lambda_1)]=\int_0^\infty f(t^\beta s)\mathbb{P}(\lambda_1\in ds)=\frac{1}{\beta}\int_0^\infty s^{-\frac{1}{\beta}-1}f(t^\beta s)\mathbb{P}(\chi_1\in ds^{-\frac{1}{\beta}})\\ &= \int_0^\infty f((t/u)^\beta)\mathbb{P}(\chi_1\in du)=\mathbb{E}\left[f\left(t^\beta\chi_1^{-\beta}\right)\right],\end{aligned}\tag{3.3}$$

where, recalling that χ is increasing, we have used the identities $\mathbb{P}(\lambda_1 \leq s) = \mathbb{P}(\chi_s \geq 1) = \mathbb{P}(\chi_1 \geq s^{-\frac{1}{\beta}})$. Then, according to [28, Theorem 2.24], we deduce that for any $\Re(z) > 0$,

$$\mathcal{M}_{\lambda_t}(z) \quad = \quad t^{z\beta}\mathcal{M}_{\chi_1^\beta}(-z) = t^{z\beta}\frac{1}{\beta\phi'(0^+)}\frac{\Gamma(z)}{W_{\phi_\beta}(z)}. \tag{3.4}$$

Note that to derive the last identity, we used the fact that the process $\chi^\beta = (\chi_t^\beta)_{t\geq 0}$ is a 1-self-similar increasing Markov process associated to the subordinator $\beta\varrho$ whose Laplace exponent is ϕ_β combined with [28, Theorem 2.24] which provides the Mellin transform of χ_1^β when starting from zero. Next, since $\phi \in \mathbf{B}_\varrho$, one can apply [5, Theorem 1(iii)] to get that, for any bounded Borel function f,

$$\mathbb{E}[f(\chi_1^\beta)] = \frac{1}{\beta\phi'(0^+)}\mathbb{E}\left[\frac{1}{I}f\left(\frac{1}{I}\right)\right]$$

where $I = \int_0^\infty e^{-\beta\varrho_t}dt$. Since from [4] the distribution of I is known to be absolutely continuous, we deduce, using also (3.3), the same property for the law of λ_t for any $t > 0$. Finally, by an expansion of the exponential function combined with an application of a standard Fubini argument, the previous identity and the recurrence relation for the gamma function, one gets

$$\mathbb{E}\left[e^{q\lambda_t}\right] = \sum_{n=0}^{\infty}\mathbb{E}[\lambda_t^n]\frac{q^n}{n!} = \frac{1}{\beta\phi'(0^+)}\sum_{n=0}^{\infty}\frac{1}{n}\frac{(t^\beta q)^n}{W_{\phi_\beta}(n)},$$

where, by using the functional equation (1.12), the series is easily checked to be absolutely convergent on $|q|t^\beta < \phi(\infty)$. Since $\phi \in \mathbf{B}_\varrho$ then $\phi(\infty) = \infty$ and hence $\bar{\mathrm{I}}_{\phi_\beta}$ defines an entire function. The last claim is then immediate. □

Proposition 3.3. *Let $\Psi \in \mathcal{N}$ and $\phi \in \mathbf{B}_\varrho$. For any $x > 0$, we have $\mathbb{P}_x$ a.s.*

$$\mathbb{T}_{\Psi_\alpha}(\phi_\beta) \stackrel{(d)}{=} \chi_{\mathrm{T}_{\Psi_\alpha}}, \tag{3.5}$$

where we recall that $\mathrm{T}_{\Psi_\alpha} = \inf\{t > 0;\ X_t \leq 0\}$ where X is an α-self-similar positive Markov process associated to Ψ via the Lamperti mapping. Consequently

$$\mathbb{T}_{\Psi_\alpha}(\phi_\beta) \stackrel{(d)}{=} \chi_1 \times \mathrm{T}_{\Psi_\alpha}^{\frac{1}{\beta}}, \tag{3.6}$$

where $\times$ stands for the product of two independent random variables.

Proof. First, recall that $t \mapsto \lambda_t$ is a.s. continuous and thus for any $x > 0$, we have $\mathbb{P}_x$ a.s.

$$\mathbb{T}_{\Psi_\alpha}(\phi_\beta) = \inf\{t > 0;\ \mathbb{X}_t \leq 0\} = \inf\{t > 0;\ \lambda_t \in \{s > 0;\ X_s \leq 0\}\} = \inf\{t > 0;\ \lambda_t = \mathrm{T}_{\Psi_\alpha}\}.$$

Thus, from the independence of χ and $\mathbb{X}$ which entails that T_{Ψ_α} is not a jump time of χ, and since the right-inverse of λ is χ which follows as χ is a right-continuous function, we get that

$$\mathbb{T}_{\Psi_\alpha}(\phi_\beta) = \inf\{t > 0;\ \lambda_t > \mathrm{T}_{\Psi_\alpha}\} = \chi_{\mathrm{T}_{\Psi_\alpha}},$$

which provides (3.5). Finally, (3.6) follows immediately by using again the independence of χ and T_{Ψ_α}, and the fact that χ is a self-similar process of index β. □

3.1.1. *End of the proof of Theorem 1.1.* Let $\Psi \in \mathcal{N}$ and $\phi \in \mathbf{B}_\varrho$ and write simply here $\mathbb{T}$ for $\mathbb{T}_{\Psi_\alpha}(\phi_\beta)$ and T for T_{Ψ_α}. By independence of the variables T and χ_1, and recalling that χ_1^β is a 1-self-similar increasing Markov process associated to the subordinator $\beta\varrho$ whose Laplace exponent is $\phi_\beta(\cdot) = \phi(\beta\cdot)$, we get, for any $-1 < \Re(z) < 0$, that

$$\mathcal{M}_{\mathbb{T}}(\beta z) = \mathcal{M}_{\chi_1^\beta}(z)\mathcal{M}_{\mathrm{T}}(z) = \frac{1}{\beta\phi'(0^+)}\frac{\Gamma(-z)}{W_{\phi_\beta}(-z)}x^{\alpha z}\phi_\alpha^+(0)\frac{\Gamma(z+1)}{W_{\phi_\alpha^-}(z+1)}W_{\phi_\alpha^+}(-z),$$

where for the second identity we have used (3.4), the identity (1.7), that is $\mathrm{T} \stackrel{(d)}{=} x^{\alpha} \int_0^{\infty} \exp(\alpha Y_t)dt$ under $\mathbb{P}_x, x > 0$, and the expression of the Mellin transform of the so-called exponential functional which is found in [28, Theorem 2.4]. The expression (1.15) follows then readily and another change of variable yields that the Mellin transform of $\mathbb{T}$ (under $\mathbb{P}_x, x > 0$) is given by

$$\mathcal{M}_{\mathbb{T}}(z-1) = x^{\frac{\alpha}{\beta}(z-1)} \frac{\phi_\alpha^+(0)}{\beta \phi'(0^+)} \frac{\Gamma(-\frac{z}{\beta}+\frac{1}{\beta})}{W_{\phi_\beta}(-\frac{z}{\beta}+\frac{1}{\beta})} \frac{\Gamma(\frac{z}{\beta}+1-\frac{1}{\beta}) W_{\phi_\alpha^+}(-\frac{z}{\beta}+\frac{1}{\beta})}{W_{\phi_\alpha^-}(\frac{z}{\beta}+1-\frac{1}{\beta})}. \tag{3.7}$$

Then, [28, Theorem 2.3(2.13)] yields that the mappings $z \mapsto \frac{\Gamma(-\frac{z}{\beta}+\frac{1}{\beta})}{W_{\phi_\beta}(-\frac{z}{\beta}+\frac{1}{\beta})}$, $z \mapsto \frac{\Gamma(\frac{z}{\beta}+1-\frac{1}{\beta})}{W_{\phi_\alpha^-}(\frac{z}{\beta}+1-\frac{1}{\beta})}$, and $z \mapsto W_{\phi_\alpha^+}(-\frac{z}{\beta}+\frac{1}{\beta})$ are analytical on $\Re(z) < \beta \mathfrak{a}_{\phi_\beta}+1$ (recall that $\phi_\beta(0)=0$), $\Re(z) > -\beta(\mathfrak{a}_{\phi_\alpha^-}\mathbb{I}_{\{\phi_\alpha^-(0)=0\}}+1)+1$ and $\Re(z) < \beta \mathfrak{a}^*_{\phi_\alpha^+}+1$ respectively. Putting pieces together and changing variables gives that $\mathcal{M}_{\mathbb{T}}(z)$ is analytical on the strip $\{z \in \mathbb{C};\ -\beta(\mathfrak{a}_{\phi_\alpha^-}\mathbb{I}_{\{\phi_\alpha^-(0)=0\}}+1) < \Re(z) < \beta(\mathfrak{a}_{\phi_\beta} \wedge \mathfrak{a}^*_{\phi_\alpha^+})\}$. Next, the fact that the law of $\mathbb{T}$ is absolutely continuous with density $f_{\mathbb{T}}$ follows from the absolute continuity of the law of the random variable $\int_0^\infty \exp(\alpha Y_t)dt$, see [4], combined with the identities (3.6) and (1.7).

To understand the smoothness of $f_{\mathbb{T}}$ we investigate the decay along complex lines of the terms of (1.15). From [28, Theorem 2.4(3)] for any $-\frac{1}{\beta} < a < 0$ and for any $p < \mathrm{N}_{\phi_\beta} = \frac{\vartheta_\beta(0,\infty)}{\mathrm{d}_\beta}$, with $\mathrm{N}_{\phi_\beta} = \infty$ provided $\vartheta_\beta(0,\infty) = \infty$ or $\mathrm{d}_\beta = 0$, we have

$$\lim_{|b|\to\infty} |b|^p \left| \frac{\Gamma(-\frac{a}{\beta} - i\frac{b}{\beta})}{W_{\phi_\beta}(-\frac{a}{\beta} - i\frac{b}{\beta})} \right| = 0, \tag{3.8}$$

whereas for any $p > \mathrm{N}_{\phi_\beta}$

$$\lim_{|b|\to\infty} |b|^p \left| \frac{\Gamma(-\frac{a}{\beta} - i\frac{b}{\beta})}{W_{\phi_\beta}(-\frac{a}{\beta} - i\frac{b}{\beta})} \right| = \infty, \tag{3.9}$$

which is possible if and only if $\mathrm{N}_{\phi_\beta} < \infty$. Also, from [28, Theorem 2.3], for any

$$p < \mathrm{N}_\Psi \text{ where } \mathrm{N}_\Psi = \frac{\phi_\alpha^-(0) + \vartheta_\alpha^-(0,\infty)}{d_\alpha^-} + \frac{v_\alpha^+(0^+)}{\phi_\alpha^+(0) + \vartheta_\alpha^+(0,\infty)}$$

we have that, for any $-\frac{1}{\beta} < a < 0$,

$$\lim_{|b|\to\infty} |b|^p \left| \mathcal{M}_{\mathrm{T}}\left(\frac{a+ib}{\beta}\right) \right| = \lim_{|b|\to\infty} |b|^p \left| \frac{\Gamma\left(1+\frac{a}{\beta}+i\frac{b}{\beta}\right)}{W_{\phi_\alpha^-}\left(1+\frac{a}{\beta}+i\frac{b}{\beta}\right)} W_{\phi_\alpha^+}\left(\frac{-a-ib}{\beta}\right) \right| = 0 \tag{3.10}$$

with $\mathrm{N}_\Psi = \infty$ unless $\Psi(z) - \mathrm{a}z$ is bounded with $\mathrm{a} < 0$, that is Y is a compound Poisson processes with a strictly negative drift a, in which case, for any $p > \mathrm{N}_\Psi \in [0,\infty)$,

$$\lim_{|b|\to\infty} |b|^p \left| \mathcal{M}_{\mathrm{T}}\left(\frac{a+ib}{\beta}\right) \right| = \lim_{|b|\to\infty} |b|^p \left| \frac{\Gamma\left(1+\frac{a}{\beta}+i\frac{b}{\beta}\right)}{W_{\phi_\alpha^-}\left(1+\frac{a}{\beta}+i\frac{b}{\beta}\right)} W_{\phi_\alpha^+}\left(\frac{-a-ib}{\beta}\right) \right| = \infty. \tag{3.11}$$

Collecting the decay in (3.8) and (3.10) we therefore get that, for any $a \in (-\beta(\mathfrak{a}_{\phi_\alpha^-}\mathbb{I}_{\{\phi_\alpha^-(0)=0\}}+1), \beta(\mathfrak{a}_{\phi_\beta} \wedge \mathfrak{a}^*_{\phi_\alpha^+}))$ and any $p < \mathrm{N}$ (resp. $p > \mathrm{N}$) where we recall that $\mathrm{N} = \mathrm{N}_{\phi_\beta} + \mathrm{N}_\Psi \in [0,\infty]$ we have that

$$\lim_{|b|\to\infty} |b|^p |\mathcal{M}_{\mathbb{T}}(a+ib)| = 0 \quad (\text{resp. } \lim_{|b|\to\infty} |b|^p |\mathcal{M}_{\mathbb{T}}(a+ib)| = \infty).$$

If $\mathrm{N}>1$ by Mellin inversion we deduce that $f_{\mathbb{T}} \in \mathtt{C}_0^{\lceil \mathrm{N}\rceil-2}(\mathbb{R}^+)$, see [22] or [28, (7.10)]. The sufficient conditions for $\mathrm{N}=\infty$ are easily derived. We see that they preclude $d_\alpha^+ = 0 < d_\alpha^-$. When the latter holds the fact that $\vartheta_\alpha^+(dy) = v_\alpha^+(y)dy, y>0$, with $v_\alpha^+(0^+) \in (0,\infty)$ follows from [28, Proposition B.2]. Next, from (3.6), we get that for all $t>0$,

$$\mathbb{P}_x(\mathbb{T}>t) = \int_0^\infty \mathbb{P}_x(\mathrm{T}^{\frac{1}{\beta}} > t/r) f_{\chi_1}(r)dr,$$

where $f_{\chi_1}(t)dt = \mathbb{P}(\chi_1 \in dt), t>0$, whose existence is justified in the proof of Proposition 3.1. Recall, from (3.4), that

$$\mathcal{M}_{\chi_1}(z-1) = \frac{1}{\beta\phi'(0^+)} \frac{\Gamma(-\frac{z}{\beta}+\frac{1}{\beta})}{W_{\phi_\beta}(-\frac{z}{\beta}+\frac{1}{\beta})},\ \Re(z) < \overline{\mathfrak{m}}_\chi = \beta \mathfrak{a}_{\phi_\beta}+1,$$

and, from [28, Theorem 2.11(2)], under the conditions of the claim, one gets that

$$\lim_{t\to\infty} t^{\beta \mathfrak{c}_\alpha} \mathbb{P}_x(\mathrm{T}^{\frac{1}{\beta}} > t) = \frac{\mathbb{E}_x[\mathrm{T}^{\mathfrak{c}_\alpha}]}{\mathfrak{c}_\alpha \phi_\alpha^{+\prime}(-\mathfrak{c}_\alpha^+)}, \tag{3.12}$$

where we have used that $\mathbb{E}_x[\mathrm{T}^{\mathfrak{c}_\alpha}] = x^\alpha \mathbb{E}\left[\left(\int_0^\infty \exp(\alpha Y_t)dt\right)^{\mathfrak{c}_\alpha}\right]$ and the expression of the moments of the latter functional in [28, Theorem 2.4]. Plainly

$$t \mapsto t^{\beta \mathfrak{c}_\alpha + 1} \mathbb{P}_x(\mathrm{T}^{\frac{1}{\beta}} > t) \text{ is bounded on } (0,a] \text{ for any } a>0. \tag{3.13}$$

Hence, one has all the conditions of [6, Theorem 4.1.6] to conclude utilizing (3.6) that

$$\lim_{t\to\infty} t^{\beta \mathfrak{c}_\alpha} \mathbb{P}_x(\mathbb{T}>t) = \lim_{t\to\infty} t^{\beta \mathfrak{c}_\alpha} \mathbb{P}_x(\mathrm{T}^{\frac{1}{\beta}} \times \chi_1 > t) = \mathcal{M}_{\chi_1}(\beta \mathfrak{c}_\alpha) \frac{\mathbb{E}_x[\mathrm{T}^{\mathfrak{c}_\alpha}]}{\mathfrak{c}_\alpha \phi_\alpha^{+\prime}(-\mathfrak{c}_\alpha^+)} = \frac{\mathbb{E}_x[\mathbb{T}^{\mathfrak{c}_\alpha}]}{\mathfrak{c}_\alpha \phi_\alpha^{+\prime}(-\mathfrak{c}_\alpha^+)},$$

which is the first claim of item 3). Then, (3.6) raised to the power β yields that

$$f_{\mathbb{T}^\beta}(t) = \int_0^\infty f_{\mathrm{T}}(t/r) f_{\chi_1^\beta}(r)\frac{dr}{r},$$

where we have set $f_{\mathrm{T}}(t)dt = \mathbb{P}_x(\mathrm{T}\in dt)$ and $f_{\mathbb{T}^\beta}(t) = \frac{t^{\frac{1}{\beta}-1}}{\beta} f_{\mathbb{T}}(t^{\frac{1}{\beta}})$. Next, for any $n \le \lceil \mathrm{N}\rceil - 2$, from the general theory of Mellin transform, see [22, 11.7], one obtains that the Mellin transform of $f_{\mathbb{T}^\beta}^{(n)} \in \mathtt{C}_0^{\lceil \mathrm{N}\rceil-2-n}(\mathbb{R}^+)$, is given, a priori in the sense of distribution, for any z in the strip $S_{\mathbb{T}^\beta,n} = \{z\in\mathbb{C};\ -\mathfrak{a}_{\phi_\alpha^-}\mathbb{I}_{\{\phi_\alpha^-(0)=0\}} + n < \Re(z) < 1+n+\mathfrak{a}_{\phi_\beta} \wedge \mathfrak{a}^*_{\phi_\alpha^+}\}$, by

$$\begin{aligned} \mathcal{M}_{f_{\mathbb{T}^\beta}^{(n)}}(z-1) &= \frac{\Gamma(z)}{\Gamma(z-n)} \mathcal{M}_{\mathbb{T}^\beta}(z-1-n) \\ &= \frac{\Gamma(z)}{\Gamma(z-n)} \mathcal{M}_{\mathrm{T}}(z-1-n)\mathcal{M}_{\chi_1^\beta}(z-1-n) \end{aligned} \tag{3.14}$$

Since, the Stirling formula, see [28, (6.10)], yields that, for any $z=a+ib$, and large b,

$$\left|\frac{\Gamma(z)}{\Gamma(z-n)}\right| \le C|b|^n \tag{3.15}$$

for some $C>0$, we get that, for all $\epsilon>0$ and $z=a+ib\in S_{\mathbb{T}^\beta,n}$,

$$\left|\mathcal{M}_{f_{\mathbb{T}^\beta}^{(n)}}(z-1)\right| \le C|b|^{n-\mathrm{N}-\epsilon}. \tag{3.16}$$

Thus, as $n\le\lceil \mathrm{N}\rceil - 2, z\mapsto \mathcal{M}_{f_{\mathbb{T}^\beta}^{(n)}}(z-1)$ is integrable along imaginary lines and hence it is the Mellin transform in the classical sense of $f_{\mathbb{T}^\beta}^{(n)}$. Now, set $\overline{\mathrm{N}}_{\Psi_\alpha} = \lceil \mathrm{N}_{\Psi_\alpha}\rceil - 2$ and $\overline{\mathrm{N}}_{\phi_\beta} = \lceil \mathrm{N}_{\phi_\beta}\rceil - 2$, and we shall consider the two cases $n\le \overline{\mathrm{N}}_{\Psi_\alpha}$ and $\overline{\mathrm{N}}_{\Psi_\alpha} < n \le \overline{\mathrm{N}}_{\Psi_\alpha} + \overline{\mathrm{N}}_{\phi_\beta}\mathbb{I}_{\{\overline{\mathrm{N}}_{\phi_\beta}>0\}}$. Let us assume first that $n \leq \overline{\mathrm{N}}_{\Psi_\alpha}$ and proceeding as above, combining (3.11) and (3.15), we get that, for all $\epsilon > 0$ and $z = a + ib \in S_{\mathrm{T},n} = \{z \in \mathbb{C};\ -\mathfrak{a}_{\phi_\alpha^-}\mathbb{I}_{\{\phi_\alpha^-(0)=0\}} + n < \Re(z) < 1 + n + \mathfrak{c}_\alpha\}$,

$$\left|\mathcal{M}_{f_{\mathrm{T}}^{(n)}}(z-1)\right| = \left|\frac{\Gamma(z)}{\Gamma(z-n)}\mathcal{M}_{\mathrm{T}}(z-1-n)\right| \leq C|b|^{n-\overline{\mathrm{N}}_{\Psi_\alpha}-\epsilon} \tag{3.17}$$

and deduce that $f_{\mathrm{T}}^{(n)} \in \mathtt{C}_0^{\overline{\mathrm{N}}_{\Psi_\alpha}-n}(\mathbb{R}^+)$. Moreover, by the Mellin inversion formula, we have, that for any $c \in (-\mathfrak{a}_{\phi_\alpha^-}\mathbb{I}_{\{\phi_\alpha^-(0)=0\}} - 2, \mathfrak{c}_\alpha - 1)$ and all $t > 0$,

$$\begin{aligned}|f_{\mathrm{T}}^{(n)}(t)| &= \left|\frac{(-1)^n}{2\pi i}\int_{c+n-i\infty}^{c+n+i\infty} t^{-z}\mathcal{M}_{f_{\mathrm{T}}^{(n)}}(z-1)dz\right| \\ &\leq Ct^{-c-n}\int_{c+n-i\infty}^{c+n+i\infty}\left|\mathcal{M}_{f_{\mathrm{T}}^{(n)}}(z-1)\right|dz \\ &\leq \bar{C}t^{-c-n}, \end{aligned} \tag{3.18}$$

for some $C, \bar{C} > 0$. Appealing again to [28, Theorem 2.11(2)], under the conditions of the claim, we get that, for any $n \leq \overline{\mathrm{N}}_{\Psi_\alpha}$,

$$\lim_{t\to\infty} t^{\mathfrak{c}_\alpha+n+1} f_{\mathrm{T}}^{(n)}(t) = (-1)^n(1+\mathfrak{c}_\alpha)_n \frac{\mathbb{E}_x\left[\mathrm{T}^{\mathfrak{c}_\alpha}\right]}{\mathfrak{c}_\alpha \phi_\alpha^{+\prime}(-\mathfrak{c}_\alpha^+)}. \tag{3.19}$$

On the other hand, we deduce, from (3.21), that the mapping $t \mapsto t^{\mathfrak{c}_\alpha+n+3}f_{\mathrm{T}}^{(n)}(t)$ is bounded on any interval $(0, a], a > 0$. Then, the mapping $z \mapsto \mathcal{M}_{\chi_1^\beta}(z-1-n)$, being analytical on the half-plane $\Re(z) < \mathfrak{a}_{\phi_\beta} + \frac{1}{\beta} - n$, is the Mellin transform of the measurable function $t^{-n}f_{\chi_1^\beta}(t)$. Thus, we observe, from (3.20), the Mellin convolution which translates, for any $t > 0$, as

$$f_{\mathbb{T}^\beta}^{(n)}(t) = \int_0^\infty f_{\mathrm{T}}^{(n)}(t/r) f_{\chi_1^\beta}(r)\frac{dr}{r^{n+1}}.$$

Hence, we can use [6, Theorem 4.1.6] to get

$$\lim_{t\to\infty} t^{\mathfrak{c}_\alpha+n+1} f_{\mathbb{T}^\beta}^{(n)}(t) = (-1)^n(1+\mathfrak{c}_\alpha)_n \frac{\mathbb{E}_x\left[\mathrm{T}^{\mathfrak{c}_\alpha}\right]}{\mathfrak{c}_\alpha \phi_\alpha^{+\prime}(-\mathfrak{c}_\alpha^+)}\mathcal{M}_{\chi_1^\beta}(\mathfrak{c}_\alpha) = (-1)^n(1+\mathfrak{c}_\alpha)_n \frac{\mathbb{E}_x\left[\mathbb{T}_{\Psi_\alpha}^{\beta\mathfrak{c}_\alpha}(\phi_\beta)\right]}{\mathfrak{c}_\alpha \phi_\alpha^{+\prime}(-\mathfrak{c}_\alpha^+)},$$

which provides the statement for $n \leq \overline{\mathrm{N}}_{\Psi_\alpha}$. Finally, assume that $\overline{\mathrm{N}}_{\Psi_\alpha} < n \leq \overline{\mathrm{N}}_{\Psi_\alpha} + \overline{\mathrm{N}}_{\phi_\beta}\mathbb{I}_{\{\overline{\mathrm{N}}_{\phi_\beta}>0\}}$ and note from (3.20) that, for any $z = a + ib \in S_{\mathbb{T}^\beta,n}$,

$$\begin{aligned}\mathcal{M}_{f_{\mathbb{T}^\beta}^{(n)}}(z-1) &= \frac{\Gamma(z)}{\Gamma(z-n)}\mathcal{M}_{\mathrm{T}}(z-1-n)\mathcal{M}_{\chi_1^\beta}(z-1-n) \\ &= \frac{\Gamma(z)}{\Gamma(z-\overline{\mathrm{N}}_{\Psi_\alpha})}\mathcal{M}_{\mathrm{T}}\left(z-1-n\right)\frac{\Gamma(z-\overline{\mathrm{N}}_{\Psi_\alpha})}{\Gamma(z-n)}\mathcal{M}_{\chi_1^\beta}(z-1-n).\end{aligned} \tag{3.20}$$

Then, we recall that (the first identity serves as setting up a notation)

$$\mathcal{M}_{\overline{f}_{(\mathrm{T},n)}^{(\overline{\mathrm{N}}_{\Psi_\alpha})}}(z-1) = \frac{\Gamma(z)}{\Gamma(z-\overline{\mathrm{N}}_{\Psi_\alpha})}\mathcal{M}_{\mathrm{T}}\left(z-1-n\right) = \frac{\Gamma(z)}{\Gamma(z-\overline{\mathrm{N}}_{\Psi_\alpha})}\mathcal{M}_{\mathrm{T}}\left(z+\overline{\mathrm{N}}_{\Psi_\alpha}-n-1-\overline{\mathrm{N}}_{\Psi_\alpha}\right)$$

is the Mellin transform of the function $\overline{f}_{(\mathrm{T},n)}^{(\overline{\mathrm{N}}_{\Psi_\alpha})}(t) = (t^{\overline{\mathrm{N}}_{\Psi_\alpha}-n}f_{\mathrm{T}}(t))^{(\overline{\mathrm{N}}_{\Psi_\alpha})} \in \mathtt{C}_0(\mathbb{R}^+)$. We also identify, by uniqueness of the Mellin transform and combining the estimates (3.9) and (3.15),

$$\mathcal{M}_R(z) = \frac{\Gamma(z-\overline{\mathrm{N}}_{\Psi_\alpha})}{\Gamma(z-n)}\mathcal{M}_{\chi_1^\beta}(z-1-n) = \frac{\Gamma(z-\overline{\mathrm{N}}_{\Psi_\alpha})}{\Gamma(z-\overline{\mathrm{N}}_{\Psi_\alpha}-(n-\overline{\mathrm{N}}_{\Psi_\alpha}))}\mathcal{M}_{\chi_1^\beta}(z-\overline{\mathrm{N}}_{\Psi_\alpha}-1-(n-\overline{\mathrm{N}}_{\Psi_\alpha}))$$

as the Mellin transform of the continuous function $t^{-\overline{\mathrm{N}}_{\Psi_\alpha}} f^{(n-\overline{\mathrm{N}}_{\Psi_\alpha})}_{\chi_1^\beta}(t)$, as by assumption $1 \leq n - \overline{\mathrm{N}}_{\Psi_\alpha} \leq \overline{\mathrm{N}}_{\phi_\beta}$. As the gamma function has simple poles at $-n, n = 0, 1, \ldots$, we have by analytical continuation that $z \mapsto \mathcal{M}_R(z)$ is analytical on the half-plane $\Re(z) < \mathfrak{a}_{\phi_\beta} + 1 + n$. We also deduce, by Mellin convolution, that for any $t > 0$,

$$f^{(n)}_{\mathbb{T}^\beta}(t) = \int_0^\infty \overline{f}^{(\overline{\mathrm{N}}_{\Psi_\alpha})}_{\mathrm{T}}(t/r) r^{-\overline{\mathrm{N}}_{\Psi_\alpha}-1} f^{(n-\overline{\mathrm{N}}_{\Psi_\alpha})}_{\chi_1^\beta}(r)dr.$$

Next combining the identity

$$\overline{f}^{(\overline{\mathrm{N}}_{\Psi_\alpha})}_{\mathrm{T}}(t) = \sum_{k=0}^{\overline{\mathrm{N}}_{\Psi_\alpha}} \binom{\overline{\mathrm{N}}_{\Psi_\alpha}}{k} \frac{\Gamma(\overline{\mathrm{N}}_{\Psi_\alpha} - n + 1)}{\Gamma(k - n + 1)} t^{k-n} f^{(k)}_{\mathrm{T}}(t)$$

with the estimate (3.19) yields that

$$\begin{aligned} \lim_{t\to\infty} t^{\mathfrak{c}_\alpha+n+1} \overline{f}^{(\overline{\mathrm{N}}_{\Psi_\alpha})}_{(\mathrm{T},n)}(t) &= \sum_{k=0}^{\overline{\mathrm{N}}_{\Psi_\alpha}} \binom{\overline{\mathrm{N}}_{\Psi_\alpha}}{k} \frac{\Gamma(\overline{\mathrm{N}}_{\Psi_\alpha} - n + 1)}{\Gamma(k - n + 1)} \lim_{t\to\infty} t^{\mathfrak{c}_\alpha+1+k} f^{(k)}_{\mathrm{T}}(t) \\ &= \frac{\mathbb{E}_x\left[\mathrm{T}^{\mathfrak{c}_\alpha}\right]}{\mathfrak{c}_\alpha \phi_\alpha^{+\prime}(-\mathfrak{c}_\alpha^+)} \sum_{k=0}^{\overline{\mathrm{N}}_{\Psi_\alpha}} \binom{\overline{\mathrm{N}}_{\Psi_\alpha}}{k} \frac{\Gamma(\overline{\mathrm{N}}_{\Psi_\alpha} - n + 1)}{\Gamma(k - n + 1)} (-1)^k (1 + \mathfrak{c}_\alpha)_k. \end{aligned}$$

Moreover, by the Mellin inversion formula, we have, that for any $c \in (n + \mathfrak{a}_{\phi_\alpha^-} \mathbb{I}_{\{\phi_\alpha^-(0)=0\}} - 2, n + \mathfrak{c}_\alpha - 1)$ and all $t > 0$,

$$|\overline{f}^{(\overline{\mathrm{N}}_{\Psi_\alpha})}_{(\mathrm{T},n)}(t)| = \left| \frac{(-1)^n}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-z} \mathcal{M}_{\overline{f}^{(\overline{\mathrm{N}}_{\Psi_\alpha})}_{(\mathrm{T},n)}}(z-1)dz \right| \leq \bar{C} t^{-c},$$

for some $\bar{C} > 0$. Since $t \mapsto t^{\mathfrak{c}_\alpha+n+3} \overline{f}^{(\overline{\mathrm{N}}_{\Psi_\alpha})}_{(\mathrm{T},n)}(t)$ is bounded on any interval $(0, a], a > 0$, one can use again [6, Theorem 4.1.6] to get

$$\begin{aligned} \lim_{t\to\infty} t^{\mathfrak{c}_\alpha+n+1} f^{(n)}_{\mathbb{T}^\beta}(t) &= \mathcal{M}_R(\mathfrak{c}_\alpha + n + 1) \frac{\mathbb{E}_x\left[\mathrm{T}^{\mathfrak{c}_\alpha}\right]}{\mathfrak{c}_\alpha \phi_\alpha^{+\prime}(-\mathfrak{c}_\alpha^+)} \sum_{k=0}^{\overline{\mathrm{N}}_{\Psi_\alpha}} \binom{\overline{\mathrm{N}}_{\Psi_\alpha}}{k} \frac{\Gamma(\overline{\mathrm{N}}_{\Psi_\alpha} - n + 1)}{\Gamma(k - n + 1)} (-1)^k (1 + \mathfrak{c}_\alpha)_k \\ &= \frac{\Gamma(\mathfrak{c}_\alpha + n + 1 - \overline{\mathrm{N}}_{\Psi_\alpha})}{\Gamma(\mathfrak{c}_\alpha + 1)} \frac{\mathbb{E}_x\left[\mathrm{T}^{\mathfrak{c}_\alpha}\right]}{\mathfrak{c}_\alpha \phi_\alpha^{+\prime}(-\mathfrak{c}_\alpha^+)} \sum_{k=0}^{\overline{\mathrm{N}}_{\Psi_\alpha}} \binom{\overline{\mathrm{N}}_{\Psi_\alpha}}{k} \frac{\Gamma(\overline{\mathrm{N}}_{\Psi_\alpha} - n + 1)}{\Gamma(k - n + 1)} (-1)^k (1 + \mathfrak{c}_\alpha)_k \end{aligned}$$

which after rearranging the terms completes the proof.

3.2. **Proof of Proposition 1.5.** Let $\phi \in \mathbf{B}$, denote by ϱ its associated subordinator and write

$$\mathrm{I}_\phi = \int_0^\infty e^{-\varrho_t} dt.$$

Then, for any $\Re(z) > 0$, we have

$$\mathcal{M}_{\mathrm{I}_\phi}(z) = \frac{\Gamma(z+1)}{W_\phi(z+1)},$$

see e.g. [29] and recalling that $\mathbb{F}_\beta$ stands for the Fréchet random variable of parameter $\beta > 0$, we have, for any $\Re(z) < \beta$,

$$\mathcal{M}_{\mathbb{F}_\beta}(z) = \Gamma\left(-\frac{z}{\beta} + 1\right).$$

Hence, by (shifted) Mellin transform identification, we deduce, from (1.15), that for any $\phi \in \mathbf{B}$ and $\beta > 0$

$$\mathbb{F}_\beta(\phi) \stackrel{(d)}{=} \mathbb{F}_\beta \times \mathrm{I}_\phi^{\frac{1}{\beta}}.$$

Next, performing a change of variables yields that its Mellin transform takes the form

$$\mathcal{M}_{\mathbb{F}_\beta(\phi)}(z-1) = \frac{\Gamma(-\frac{z}{\beta}+1+\frac{1}{\beta})\Gamma(\frac{z}{\beta}+1-\frac{1}{\beta})}{W_\phi(\frac{z}{\beta}+1-\frac{1}{\beta})}. \tag{3.21}$$

The proof of Theorem 1.1 combined with the analyticity of the gamma function to the right-half plane entails that the mapping $z \mapsto \mathcal{M}_{\mathbb{F}_\beta(\phi)}(z-1)$ is analytical on the strip $S_\beta = \{z \in \mathbb{C};\ 1-\beta(\mathfrak{a}_\phi \mathbb{I}_{\{\phi(0)=0\}}+1) < \Re(z) < 1+\beta\}$ and for any $\epsilon > 0$ and $z = a+ib \in S_\beta$,

$$\left|\mathcal{M}_{\mathbb{F}_\beta(\phi)}(z-1)\right| \leq e^{-|b|\frac{\Theta_\epsilon(\phi)}{\beta}} \tag{3.22}$$

where $\Theta_\epsilon(\phi) = \pi - \overline{\Theta}_\phi - \epsilon \geq \frac{\pi}{2} - \epsilon$ and recall that $\overline{\Theta}_\phi = \limsup_{b\to\infty} \frac{\int_0^{|b|} \arg\phi(1+iu)du}{|b|}$. Hence according to the theory of Mellin transforms, the law of $\mathbb{F}_\beta(\phi)$ is absolutely continuous with a density $f_{\mathbb{F}_\beta(\phi)} \in \mathtt{C}_0^\infty(\mathbb{R}^+)$ and which is analytical on the sector $S_\phi = \{z \in \mathbb{C};\ |\arg(z)| < \pi - \overline{\Theta}_\phi\}$ and admits the Mellin Barnes representation

$$\begin{aligned} f_{\mathbb{F}_\beta(\phi)}(t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-z}\frac{\Gamma(-\frac{z}{\beta}+1+\frac{1}{\beta})\Gamma(\frac{z}{\beta}+1-\frac{1}{\beta})}{W_\phi(\frac{z}{\beta}+1-\frac{1}{\beta})}dz \\ &= \frac{\beta}{2\pi i}\int_{\frac{c}{\beta}-i\infty}^{\frac{c}{\beta}+i\infty} t^{-\beta z}\frac{\Gamma(-z+1+\frac{1}{\beta})\Gamma(z+1-\frac{1}{\beta})}{W_\phi(z+1-\frac{1}{\beta})}dz, \end{aligned}$$

which is absolutely integrable on S_ϕ for any $c \in S_\beta$. An application of Cauchy Theorem, see [29, Proof of Lemma 8.16] for the details of similar arguments, gives that

$$f_{\mathbb{F}_\beta(\phi)}(t) = \beta t^{-\beta-1}\sum_{n=0}^{\infty}\frac{\Gamma(n+2)}{W_\phi(n+2)}\frac{(-t^{-\beta})^n}{n!} = \beta t^{-\beta-1}\sum_{n=0}^{\infty}\frac{n+1}{\phi(n+1)}\frac{(-t^{-\beta})^n}{W_\phi(n+1)},$$

where, for the last identity, we used the recurrence relation $W_\phi(n+2) = \phi(n+1)W_\phi(n+1)$ and the one of the gamma function to get that the series is convergent for $|t|^{-\beta} < \phi(\infty)$.

3.3. **Proof of Theorem 1.7.** First, recall that since $\Psi \in \mathcal{N}_1$ then $\phi_\alpha^+ \in \mathbf{B}_1$ and thus $\phi_\beta(u) = \phi(\beta u) = \mathcal{S}_{\phi_\alpha^+}(u) = \frac{u}{u+1}\phi_\alpha^+(u) \in \mathbf{B}$. Moreover, since (1.10) holds for ϕ_α^+, we have that either $\mathrm{d}_\alpha^+ > 0$ or $\vartheta_\alpha^+(0,1) = \infty$. Thus, as from e.g. [29, Proposition 4.1(3)], with the obvious notation,

$$\mathrm{d}_\beta = \lim_{u\to\infty}\frac{\phi_\beta(u)}{u} = \lim_{u\to\infty}\frac{u}{u+1}\frac{\phi_\alpha^+(u)}{u} = \mathrm{d}_\alpha^+$$

and if $\vartheta_\alpha^+(0,1) = \infty$ with hence $\mathrm{d}_\beta = \mathrm{d}_\alpha^+ = 0$, then

$$\vartheta_\alpha^+(0,1) = \lim_{u\to\infty}\phi_\alpha^+(u) = \lim_{u\to\infty}\frac{u}{u+1}\phi_\alpha^+(u) = \lim_{u\to\infty}\phi_\beta(u) = \vartheta_\beta(0,1).$$

Consequently $\phi_\beta = \mathcal{S}_{\phi_\alpha^+}$ satisfies the condition (1.10). On the other hand we have

$$\phi_\beta'(0^+) = \beta\phi'(0^+) = \phi_\alpha^+(0) \tag{3.23}$$

which follows easily from the definition of ϕ_β and from [29, Proposition 4.1(4.4)] that gives that for all $u \geq 0$, $0 \leq u(\phi_\alpha^+)'(u) \leq \phi_\alpha^+(u) - \phi_\alpha^+(0)$ and hence $\lim_{u\to 0^+} u(\phi_\alpha^+)'(u) = 0$. Putting pieces together we deduce that $\phi_\beta = \mathcal{S}_{\phi_\alpha^+} \in \mathbf{B}_\varrho$. Next, from the definition of $W_{\phi_\alpha^+}$ in (1.12), as $\phi_\alpha^+ \in \mathbf{B}_1 \subset \mathbf{B}$, and writing $\overline{W}(u) = \frac{1}{u}W_{\phi_\alpha^+}(u), u > 0$, we have that $\overline{W}(1) = 1$ and

$$\overline{W}(u+1) = \frac{\phi_\alpha^+(u)}{u+1}W_{\phi_\alpha^+}(u) = \phi_\beta(u)\frac{W_{\phi_\alpha^+}(u)}{u} = \phi_\beta(u)\overline{W}(u).$$

Thus invoking the uniqueness argument given in [33] yields that $\overline{W}(u) = W_{\phi_\beta}(u)$, that is

$$W_{\phi_\beta}(u) = \frac{1}{u}W_{\phi_\alpha^+}(u).$$

Hence, we deduce from (1.15) and an application of the recurrence relation of the gamma function that, for any $-\beta < \Re(z) < 0$,

$$\begin{aligned}\mathcal{M}_{\mathbb{T}_{\Psi_\alpha}(\mathcal{S}_{\phi_\alpha^+})}(z) &= x^{\frac{\alpha}{\beta}z}\frac{\phi_\alpha^+(0)}{\beta\phi'(0^+)}\frac{\Gamma(-\frac{z}{\beta})}{W_{\phi_\beta}(-\frac{z}{\beta})}\frac{\Gamma(\frac{z}{\beta}+1)W_{\phi_\alpha^+}(-\frac{z}{\beta})}{W_{\phi_\alpha^-}(\frac{z}{\beta}+1)}\\ &= x^{\frac{\alpha}{\beta}z}\frac{\Gamma(1-\frac{z}{\beta})\Gamma(\frac{z}{\beta}+1)}{W_{\phi_\alpha^-}(\frac{z}{\beta}+1)},\end{aligned} \tag{3.24}$$

where we used the identity (3.23). Comparing this expression with (1.16) yields, by uniqueness of the Mellin transform, the first identity in law (1.21). Next, set $\beta = 1$, from [29, Theorem 2.4(1)], we get that $\mathbb{T}_{\Psi_\alpha}(\mathcal{S}_{\phi_\alpha^+}) \overset{(d)}{=} x^\alpha \int_0^\infty e^{\alpha\overline{Y}_t}dt$ with $\overline{Y} = (\overline{Y}_t)_{t\geq 0}$ a spectrally negative Lévy process with characteristic exponent $\psi(z) = \frac{1}{\alpha}(z-\alpha)\phi^-(z)$ where we used the fact that $\phi_\alpha^+(0)W_{\phi_\alpha^+}(z) = \Gamma(1+z)$, that is $\phi_\alpha^+(z) = z+1$. The second claim follows since, from (1.7), we also have $\mathrm{T}_\psi \overset{(d)}{=} x^\alpha \int_0^\infty e^{\alpha\overline{Y}_t}dt$. Finally if now $\Psi \in \mathcal{N}_1^+ = \{\Psi \in \mathcal{N}_1 \text{ with } \phi_\alpha^-(u) = \alpha u\}$, we have $W_{\phi_\alpha^-}(\frac{z}{\beta}+1) = \alpha^{\frac{z}{\beta}}\Gamma(\frac{z}{\beta}+1)$ and thus (3.24) entails that in this case

$$\mathcal{M}_{\mathbb{T}_{\Psi_\alpha}(\mathcal{S}_{\phi_\alpha^+})}(z) = (\alpha x^\alpha)^{\frac{z}{\beta}}\Gamma\left(1-\frac{z}{\beta}\right),$$

which completes the proof after mentioning that the self-decomposability property of the Fréchet distribution is found, for $0 < \beta \leq 1$, in [7] and for $\beta > 1$, as an instance of a general result in [25, Section 3(1)] and also in [16, Lemma 1] where in these papers the authors use the notation $\beta = \frac{1}{\alpha}, 0 < \alpha < 1$.

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