

# SPECTRAL DECOMPOSITION OF FRACTIONAL OPERATORS AND A REFLECTED STABLE SEMIGROUP

P. PATIE AND Y. ZHAO

ABSTRACT. In this paper, we provide the spectral decomposition in Hilbert space of the  $C_0$ -semigroup  $P$  and its adjoint  $\widehat{P}$  having as generator, respectively, the Caputo and the right-sided Riemann-Liouville fractional derivatives of index  $1 < \alpha < 2$ . These linear operators, which are non-local and non-self-adjoint, appear in many recent studies in applied mathematics and also arise as the infinitesimal generators of some substantial processes such as the reflected spectrally negative  $\alpha$ -stable process. Our approach relies on intertwining relations that we establish between these semigroups and the semigroup of a Bessel type process whose generator is a self-adjoint second order differential operator. In particular, from this commutation relation, we characterize the positive real axis as the continuous point spectrum of  $P$  and provide a power series representation of the corresponding eigenfunctions. We also identify the positive real axis as the residual spectrum of the adjoint operator  $\widehat{P}$  and elucidates its role in the spectral decomposition of these operators. By resorting to the concept of continuous frames, we proceed by investigating the domain of the spectral operators and derive two representations for the heat kernels of these semigroups. As a by-product, we also obtain regularity properties for these latter and also for the solution of the associated Cauchy problem.

## 1. INTRODUCTION

Fractional calculus, in which derivatives and integrals of fractional order are defined and studied, is nearly as old as the classical calculus of integer orders. Ever since the first inquisition by L'Hopital and Leibniz in 1695, there has been an enormous amount of study on this topic for more than three centuries, with many mathematicians having suggested their own definitions that fit the concept of a non-integer order derivative. Among the most famous of these definitions are the Riemann-Liouville fractional derivative and the Caputo derivative, the latter being a reformulation of the former in order to use integer order initial conditions to solve fractional order differential equations. In this context, it is natural to consider the following Cauchy problem, for a smooth function  $f$  on  $x > 0$ ,

$$(1.1) \quad \begin{cases} \frac{d}{dt}u(t, x) = \mathbf{D}_\alpha u(t, x) \\ u(0, x) = f(x), \end{cases}$$

---

*Key words and phrases.* Fractional operators, continuous frames, non-self-adjoint integro-differential operators, Markov semigroups, Reflected stable processes, spectral theory  
*2010 Mathematical Subject Classification:* 35R11, 47G20, 60G52, 42C15, 47D07, 58C40.

This work was partially supported by ARC IAPAS, a fund of the Communauté française de Belgique. The authors are indebted to an anonymous referee who pointed out that one of the pathwise descriptions of the dual process was misleading in an earlier version of the manuscript and who has brought to our attention the reference [40].

where, for any  $1 < \alpha < 2$ , the linear operator  $\mathbf{D}_\alpha$  is either the Caputo  $\alpha$ -fractional derivative

$$(1.2) \quad \mathbf{D}_\alpha f(x) = {}^C D_+^\alpha f(x) = \int_0^x \frac{f^{([\alpha]+1)}(y)}{(x-y)^{\alpha-[\alpha]}} \frac{dy}{\Gamma([\alpha]+1-\alpha)},$$

with, for any  $k = 1, 2, \dots$ ,  $f^{(k)}(x) = \frac{d^k}{dx^k} f(x)$  stands for the  $k$ -th derivative of  $f$ , or, the right-sided Riemann-Liouville (RL) derivative

$$(1.3) \quad \mathbf{D}_\alpha f(x) = D_-^\alpha f(x) = \left( \frac{d}{dx} \right)^{[\alpha]+1} \int_x^\infty \frac{f(y)(y-x)^{[\alpha]-\alpha}}{\Gamma([\alpha]+1-\alpha)} dy,$$

with  $[\alpha]$  representing the integral part of  $\alpha$ . We point out that when  $\alpha = 2$ , in both cases,  $\mathbf{D}_2 f(x) = \frac{1}{2} f^{(2)}(x)$  is a second order differential operator.

In this paper, we aim at providing the spectral representation in  $L^2(\mathbb{R}_+)$  Hilbert space and regularities properties of the solution to the Cauchy problem (1.1).

The motivation underlying this study are several folds. On the one hand, the last three decades have witnessed the most intriguing leaps in engineering and scientific applications of such fractional operators, including but not limited to population dynamics, chemical technology, biotechnology and control of dynamical systems, and, we refer to the monographs of Kilbas et al. [20], Meerschaert and Sikorskii [26] and Sankaranarayanan [40] for excellent and recent accounts on fractional operators. On the other hand, some recent interesting studies have revealed that the linear operator  ${}^C D_+^\alpha$  is the infinitesimal generator of  $P = (P_t)_{t \geq 0}$  the Feller semigroup corresponding to the so-called spectrally negative reflected  $\alpha$ -stable process, see e.g. [2, 5, 36]. We will provide the formal definition of this process and semigroup in Section 2, and, we simply point out that the reflected Brownian motion is obtained in the limiting case  $\alpha = 2$ . The reflected  $\alpha$ -stable processes have been studied intensively in the stochastic processes literature. In particular, we mention that, in a recent paper, Baeumer et. al. [2] showed the interesting fact that the transition kernel of  $P$  allows to map the set of solutions of a Cauchy problem to its fractional (in time) analogue. Motivated by these findings, they provide a numerical method to approximate this transition kernel. In this perspective, in Theorem 6.2 below, we provide two analytical and simple expressions for this transition kernel.

Although the Cauchy problem for the fractional operators associated to reflected stable processes plays a central role in many fields of sciences, to the best of our knowledge, their spectral representation remain unclear. This seems to be attributed to the fact that there is not a unified theory for dealing with the spectral decomposition of non-local and non-self-adjoint operators, two properties satisfied, as we shall see in Proposition 2.1, by the fractional operators considered therein. For a nice account on classical and recent developments on this important topic, we refer to the two volume treatise of Dunford and Schwartz [16, 17] and the monograph of Davies [15], and the survey paper by Sjöstrand [43].

The purpose of this paper is to provide detailed information regarding the solution of the Cauchy problem (1.1) along with its elementary solution which corresponds to the transition probabilities of the Feller semigroups  $P$  and its dual  $\widehat{P}$ . More specifically, we provide a spectral representation of this solution in an integral form involving the absolutely continuous part of the spectral measure, the generalized Mittag-Leffler functions as eigenfunctions and a weak Fourier kernel, a terminology which is defined in [34] and recalled in Section 5. This kernel admits on a dense subset an integral representation which is given in terms of a function, having a simple

expression, that we name a residual function for the dual semigroup (or co-residual function for  $P$ ), as it is associated to elements in its residual spectrum. We refer to Section 5 for more precise definitions. As by-product of this spectral representation, we manage to derive regularity properties for the solution of (1.1) and also for the transition kernel. We already mention that we observe a cut-off phenomenon in the nature of the spectrum for the class of operators indexed by the parameter  $\alpha \in (1, 2]$ . Indeed, while the class of Bessel operators which include the limit case  $\mathbf{D}_2$ , i.e.  $\alpha = 2$ , has the positive axis  $(0, \infty)$  as continuous spectrum, we shall show that this axis corresponds, when  $\alpha \in (1, 2)$ , to the continuous point spectrum of the Caputo operator and the residual spectrum of the right-sided RL fractional operator.

Our approach relies on an in-depth analysis of an intertwining relation that we establish between the Caputo fractional operator and a second order differential operator of Bessel type, which the latter turns out to be the generator of a self-adjoint semigroup in  $L^2(\mathbb{R}_+)$ . This is combined with the theory of continuous frames that have been introduced recently in the mathematical physics literature, see [1]. This work complements nicely the recent works of Patie and Savov in [35] and [33] where such ideas are elaborated between linear operators having a common discrete point spectrum. We also mention that recently Kuznetsov and Kwasnicki [21] provide a representation of the transition kernel of  $\alpha$ -stable processes killed upon entering the negative real line, by inverting their resolvent density that they manage to compute explicitly. In this vein but in a more general context, Patie and Savov in the work in progress [34] explore further the idea developed in our paper to establish the spectral theory of the class of positive self-similar semigroups.

The rest of this paper is organized as follows. In Section 2, we introduce the reflected one-sided  $\alpha$ -stable processes and establish substantial analytical properties of the corresponding semigroups. In Section 3, we shall derive the intertwining relation between the spectrally negative reflected stable semigroup and the Bessel-type semigroup. From this link, we extract a set of eigenfunctions that are described in Section 4 which also includes some of their interesting properties such as the continuous upper frame property, completeness and large asymptotic behavior. In Section 5 we investigate the so-called co-residual functions. Finally, in Section 6 we gather all previous results to provide the spectral decomposition of the two semigroups  $P$  and  $\tilde{P}$  including two representations for their transition kernels. The regularity properties are also stated and proved in that Section.

**1.1. Notations.** Throughout, we denote by  $\mathbb{R}_+ = (0, \infty)$  the positive half-line. For any  $-\infty \leq \underline{a} < \bar{a} \leq \infty$ , we denote the strip  $\mathbb{C}_{(\underline{a}, \bar{a})} = \{z \in \mathbb{C}; \underline{a} < \Re(z) < \bar{a}\}$ , and write simply  $\mathbb{C}_+ = \mathbb{C}_{[0, \infty)}$ . We write  $\mathbb{C}_{(-\infty, 0)^c} = \{z \in \mathbb{C}; \arg(z) \neq \pi\}$  for the complex plane cut along the negative real axis. We also write  $L^2(\mathbb{R}_+)$  for the Hilbert space of square integrable Lebesgue measurable functions on  $\mathbb{R}_+$  endowed with the inner product  $\langle f, g \rangle = \int_0^\infty f(x)g(x)dx$  and the associated norm  $\|\cdot\|$ . For any weight function  $\nu$  defined on  $\mathbb{R}_+$ , i.e. a non-negative Lebesgue measurable function, we denote by  $L^2(\nu)$  the weighted Hilbert space endowed with the inner product  $\langle f, g \rangle_\nu = \int_0^\infty f(x)g(x)\nu(x)dx$  and its corresponding norm  $\|\cdot\|_\nu$ . We use  $C_0(\mathbb{R}_+)$  to denote the space of continuous real-valued functions on  $\mathbb{R}_+$  tending to 0 at infinity, which becomes a Banach space when endowed with the uniform topology  $\|\cdot\|_\infty$ . Additionally, we denote  $C_0^2(\mathbb{R}_+)$  to be the space of twice continuously differentiable functions on  $\mathbb{R}_+$ , which vanishes at both 0 and infinity, and  $C^\infty(\mathbb{R}_+)$  the space of functions with continuous derivatives on  $\mathbb{R}_+$  of all orders, and  $B_b(\mathbb{R}_+)$  the real-valued bounded Borel measurable functions on  $\mathbb{R}_+$ . For Banach spaces  $H_1, H_2$ ,

we define

$$\mathbf{B}(\mathbf{H}_1, \mathbf{H}_2) = \{L : \mathbf{H}_1 \rightarrow \mathbf{H}_2 \text{ linear and continuous mapping}\}.$$

In the case of one Banach space  $\mathbf{H}$ , the unital Banach algebra  $\mathbf{B}(\mathbf{H}, \mathbf{H})$  is simply denoted by  $\mathbf{B}(\mathbf{H})$ . Moreover, a semigroup  $P = (P_t)_{t \geq 0}$  where  $P_t \in \mathbf{B}(\mathbf{H})$  is called a positive  $\mathcal{C}_0$ -semigroup on  $\mathbf{H}$  if  $P_{t+s} = P_t \circ P_s$ ,  $P_t f \geq 0$  for  $f \geq 0$ , and for any functions  $f \in \mathbf{H}$ ,  $\|P_t f - f\|_{\mathbf{H}} \rightarrow 0$  as  $t \rightarrow 0$ . In the case when  $\mathbf{H} = \mathcal{C}_0(\mathbb{R}_+)$  endowed with the uniform topology, we say  $P$  is a Feller semigroup on  $\mathbb{R}_+$ . Furthermore, for an operator  $T \in \mathbf{B}(\mathbf{H}_1, \mathbf{H}_2)$ , we use the notation  $\text{Ran}(T)$  (resp.  $\text{Ker}(T)$ ) for the range (resp. the kernel) of  $T$  and  $\overline{\text{Ran}}(T)$  (resp.  $\overline{\text{Ker}}(T)$ ) for its closure. For any set of functions  $E \subseteq \mathbf{H}$ , we use  $\text{Span}(E)$  to denote the set of all linear combinations of functions in  $E$ , and  $\overline{\text{Span}}(E)$  for its closure. We now proceed to define a few further notations. For two functions  $f, g : \mathbb{R}_+ \rightarrow \mathbb{R}$ , we write  $f \stackrel{a}{=} O(g)$  (resp.  $f \stackrel{a}{=} o(g)$ ) if  $\limsup_{x \rightarrow a} \frac{f(x)}{g(x)} < \infty$  (resp.  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = 0$ ), and  $f \asymp g$  (resp.  $f \stackrel{a}{\sim} g$ ) if  $\exists c > 0$  such that  $c \leq \frac{f(x)}{g(x)} \leq c^{-1}$  for all  $x \in \mathbb{R}_+$  (resp. if  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = 1$  for some  $a \in \mathbb{R} \cup \{\pm\infty\}$ ). Finally, for any  $q \in \mathbb{R}_+$ , we write  $d_q f(x) = f(qx)$  and for any  $\alpha, \tau > 0$ , we set

$$(1.4) \quad \mathbf{e}_{\alpha, \tau}(x) = d_{\frac{1}{\tau}} \mathbf{e}_{\alpha}(x) = e^{-\tau x^{\alpha}}, \quad x > 0.$$

## 2. FRACTIONAL OPERATORS AND THE REFLECTED STABLE SEMIGROUP

Let  $Z = (Z_t)_{t \geq 0}$  be a spectrally negative  $\alpha$ -stable Lévy process with  $\alpha \in (1, 2)$ , defined on a filtered probability space  $(\Omega, \mathbf{F}, (\mathbf{F}_t)_{t \geq 0}, \mathbb{P} = (\mathbb{P}_x)_{x \in \mathbb{R}})$ . It means that  $Z$  is a process with stationary and independent increments, having no positive jumps, and its law is characterized, for  $t > 0$ , by

$$(2.1) \quad \log \mathbb{E}[e^{zZ_t}] = z^{\alpha} t, \quad z \in \mathbb{C}_+.$$

Here and below  $z^{\alpha}$  is the main branch of the complex analytic function in the complex half-plane  $\Re(z) \geq 0$ , so that  $1^{\alpha} = 1$ . Let  $X = (X_t)_{t \geq 0}$  be the process  $Z$  reflected at its infimum, that is, for any  $t \geq 0$ ,

$$X_t = \begin{cases} Z_t & \text{if } t < T_{(-\infty, 0]}^Z, \\ Z_t - \inf_{s \leq t} Z_s & \text{if } t \geq T_{(-\infty, 0]}^Z, \end{cases}$$

with  $T_{(-\infty, 0]}^Z = \inf\{t > 0; Z_t \leq 0\}$ , and we write, for any  $f \in \mathcal{B}_b(\mathbb{R}_+)$ ,  $t, x \geq 0$ ,

$$(2.2) \quad P_t f(x) = \mathbb{E}_x[f(X_t)],$$

where  $\mathbb{E}_x$  stands for the expectation operator associated to  $\mathbb{P}_x(Z_0 = x) = 1$ . Next, let  $\widehat{Z} = -Z$  be the dual process of  $Z$  (with respect to the Lebesgue measure), which is a spectrally positive  $\alpha$ -stable process, and, let  $\widehat{X} = (\widehat{X}_t)_{t \geq 0}$  be the process defined from  $\widehat{Z}$  by a random time-change as follows, for any  $t \geq 0$ ,

$$(2.3) \quad \widehat{X}_t = \widehat{Z}_{\widehat{\tau}_t},$$

where  $\widehat{\tau}_t = \inf\{u > 0; \widehat{A}_u > t\}$  and  $\widehat{A}_t = \int_0^t \mathbb{I}_{\{\widehat{Z}_s > 0\}} ds$ . We also write for any  $f \in \mathcal{B}_b(\mathbb{R}_+)$ ,  $t, x \geq 0$ ,

$$\widehat{P}_t f(x) = \widehat{\mathbb{E}}_x[f(\widehat{X}_t)],$$

where  $\widehat{\mathbb{E}}_x$  stands for the expectation operator associated to  $\widehat{\mathbb{P}}_x(\widehat{Z}_0 = x) = 1$ . We are now ready to state our first result.

**Proposition 2.1.** (1)  $P$  is a positive contractive  $\mathcal{C}_0$ -semigroup on  $C_0(\mathbb{R}_+)$ , i.e. a Feller semigroup, whose infinitesimal generator is  $({}^C D_+^\alpha, \mathcal{D}_\alpha)$  where

$$\mathcal{D}_\alpha = \left\{ f \in C_0(\mathbb{R}_+); f(x) = \int_0^\infty \left( e^{-y} \mathcal{J}_\alpha(x) - \mathcal{J}'_\alpha(x-y) \mathbb{I}_{\{y < x\}} \right) g(y) dy, g \in C_0(\mathbb{R}_+) \right\},$$

with

$$(2.4) \quad \mathcal{J}_\alpha(z) = \frac{1}{\Gamma(1 + \frac{1}{\alpha})} \sum_{n=0}^{\infty} \frac{(e^{i\pi} z^\alpha)^n}{\Gamma(\alpha n + 1)}, \quad z \in \mathbb{C},$$

which is easily seen to define a function holomorphic on  $\mathbb{C}_{(-\infty, 0)^c}$ .

(2)  $P$  admits a unique extension as a contractive  $\mathcal{C}_0$ -semigroup on  $L^2(\mathbb{R}_+)$ , which is also denoted by  $P = (P_t)_{t \geq 0}$  when there is no confusion (otherwise we may denote  $P^F$  for the Feller semigroup). The domain of its infinitesimal generator  $L^X$  is given by

$$(2.5) \quad \mathcal{D}_\alpha(L^2(\mathbb{R}_+)) = \left\{ f \in L^2(\mathbb{R}_+); \int_{-\infty}^{\infty} |\mathcal{F}_f^+(\xi)|^2 |\xi|^{2\alpha} d\xi < \infty \right\}$$

where  $\mathcal{F}_f^+(\xi) = \int_0^\infty e^{i\xi x} f(x) dx$  is the one-sided Fourier transform of  $f$  taken in the  $L^2$  sense.

(3)  $\widehat{X}$  is the (weak) dual of  $X$  with respect to the Lebesgue measure. Moreover,  $\widehat{P}$  is a Feller semigroup which admits a unique extension as a contractive  $\mathcal{C}_0$ -semigroup on  $L^2(\mathbb{R}_+)$ , also denoted by  $\widehat{P}$ , which has  $(D_-^\alpha, \mathcal{D}_\alpha(L^2(\mathbb{R}_+)))$  as infinitesimal generator. Clearly as  $P \neq \widehat{P}$ , we get that  $P$  is non-self-adjoint in  $L^2(\mathbb{R}_+)$ .

**Remark 2.1.** We point out that when  $\alpha = 2$ ,  $P$  is the 1-dimensional Bessel semigroup, see [10, Appendix 1], which also belongs to the class of the so-called  $\alpha$ -Bessel semigroups, which are reviewed in more details in Appendix A. In this case,  $\widehat{P} = P$  and  $P$  is self-adjoint in  $L^2(\mathbb{R}_+)$ .

**Remark 2.2.** Note that the function  $\mathcal{J}_\alpha(e^{i\pi} z^{\frac{1}{\alpha}})$  is the (generalized) Mittag-Leffler function of parameters  $(\alpha, 1)$ , see e.g. [20] for a detailed account on this function.

In order to prove this Proposition, we first state and prove the following lemma, which generalizes [6, Lemma 2] and may be of independent interests.

**Lemma 2.1.** Let  $Y_t = Z_{\tau_t}, t \geq 0$ , where  $\tau_t = \inf\{u > 0; A_u > t\}$  and  $A_t = \int_0^t \mathbb{I}_{\{Z_s > 0\}} ds$ . Then  $(Y_t)_{t \geq 0}$  is a  $(\mathbf{F}_{\tau_t})_{t \geq 0}$  strong Markov process and for any  $f \in B_b(\mathbb{R}_+), t, x \geq 0$ , we have

$$(2.6) \quad P_t f(x) = \mathbb{E}_x[f(Y_t)].$$

Moreover,  $(Y_t)_{t \geq 0}$  and  $(\widehat{X}_t)_{t \geq 0}$  are dual processes with respect to the Lebesgue measure.

*Proof.* For any  $f \in B_b(\mathbb{R}_+), q > 0$ , let

$$U_q f(x) = \int_0^\infty e^{-qt} P_t f(x) dt, \quad U_q^\dagger f(x) = \int_0^\infty e^{-qt} \mathbb{E}_x \left[ f(X_t) \mathbb{I}_{\{t < T_0^X\}} \right] dt$$

be the resolvents of  $X$  and  $X^\dagger = (X_t^\dagger)_{t \geq 0}$ , the process  $X$  killed at time  $T_0^X = \inf\{t > 0; X_t = 0\}$ , respectively. It is easy to observe from the construction of  $X$  that  $T_0^X = T_{(-\infty, 0]}^Z$ . Moreover, by [39, Example 3],  $X$  can also be defined as the unique self-similar recurrent extension of  $X^\dagger$  and we get, from an application of the strong Markov property, that for all  $x \geq 0$ ,

$$(2.7) \quad U_q f(x) = U_q^\dagger f(x) + \mathbb{E}_x \left[ e^{-qT_0^X} \right] U_q f(0).$$

Next, since  $Z$  has paths of unbounded variation, by [22, Theorem 6.5], we have  $\mathbb{P}_x(T_{[0,\infty)}^Z = 0) = 1$  for  $x \geq 0$  and  $\mathbb{P}_x(T_{[0,\infty)}^Z > 0) = 1$  for any  $x < 0$ , where  $T_{[0,\infty)}^Z = \inf\{t > 0; Z_t \geq 0\}$ . Thus, the fine support of the additive functional  $(A_t)_{t \geq 0}$ , defined as the set  $\{x \in \mathbb{R}; \mathbb{P}_x(\tau_0 = 0) = 1\}$ , is plainly  $[0, \infty)$ . Moreover, as the Lévy process  $Z$  is a Feller process and therefore a Hunt process (see e.g. [12, Section 3.1]), we have from [19] that  $(Y_t)_{t \geq 0}$  is a  $(\mathbf{F}_{\tau_t})_{t \geq 0}$  strong Markov process, whose resolvent is defined, for  $f \in \mathbf{B}_b(\mathbb{R}_+)$ , by

$$V_q f(x) = \int_0^\infty e^{-qt} \mathbb{E}_x[f(Y_t)] dt.$$

Furthermore, it is easy to observe that  $A_t = t$  for any  $t \leq T_{(-\infty, 0]}^Z$  and thus  $\tau_t = t$  for any  $t < T_{(-\infty, 0]}^Z$ . On the other hand, since  $Z$  is a spectrally negative Lévy process with no Gaussian component,  $Z$  does not creep below, see e.g. [22, Exercise 7.4], and therefore  $T_0^Z = \inf\{t > 0; Z_t = 0\} > T_{(-\infty, 0]}^Z$  a.s., where a.s. throughout this proof, means  $\mathbb{P}_x$ -almost surely for all  $x > 0$ . Moreover, observe that a.s.

$$A_{T_0^Z} = \int_0^{T_{(-\infty, 0]}^Z} \mathbb{I}_{\{Z_s > 0\}} ds + \int_{T_{(-\infty, 0]}^Z}^{T_0^Z} \mathbb{I}_{\{Z_s > 0\}} ds = A_{T_{(-\infty, 0]}^Z} = T_{(-\infty, 0]}^Z.$$

Next, recalling that  $T_{(-\infty, 0]}^Z = T_0^X$ , we deduce from the previous identity that, with the obvious notation, a.s.

$$(2.8) \quad T_0^Y = A_{T_0^Z} = T_{(-\infty, 0]}^Z = T_0^X.$$

Since it is clear that  $Y_t = Z_{\tau_t} = Z_t = X_t$  for  $t < T_0^X$ , we have for any  $f \in \mathbf{B}_b(\mathbb{R}_+)$  and  $q > 0$ ,

$$V_q^\dagger f(x) = \int_0^\infty e^{-qt} \mathbb{E}_x \left[ f(Y_t) \mathbb{I}_{\{t < T_0^Y\}} \right] dt = \int_0^\infty e^{-qt} \mathbb{E}_x \left[ f(X_t) \mathbb{I}_{\{t < T_0^X\}} \right] dt = U_q^\dagger f(x).$$

Hence, the strong Markov property of  $(Y_t)_{t \geq 0}$  together with (2.8) yield that, for every  $x \geq 0$ ,

$$V_q f(x) = V_q^\dagger f(x) + \mathbb{E}_x \left[ e^{-qT_0^Y} \right] V_q f(0) = U_q^\dagger f(x) + \mathbb{E}_x \left[ e^{-qT_0^X} \right] V_q f(0).$$

Next, according to [6, Lemma 2] and after an obvious dual argument,  $(Y_t)_{t \geq 0}$  and  $(X_t)_{t \geq 0}$  have the same law under  $\mathbb{P}_0$  and therefore  $V_q f(0) = U_q f(0)$ . Hence

$$U_q f(x) = U_q^\dagger f(x) + \mathbb{E}_x \left[ e^{-qT_0^X} \right] U_q f(0) = U_q^\dagger f(x) + \mathbb{E}_x \left[ e^{-qT_0^X} \right] V_q f(0) = V_q f(x),$$

which proves the identity (2.6). Next, by [46, Proposition 4.4], we observe that  $(A_t)_{t \geq 0}$  and  $(\widehat{A}_t)_{t \geq 0}$  are dual additive functionals, both of which are finite for each  $t$  and continuous. Hence by [46, Theorem 4.5],  $(Y_t)_{t \geq 0}$  and  $(\widehat{X}_t)_{t \geq 0}$  are dual processes with respect to the Revuz measure associated to  $A$ , which, by [38], is the Lebesgue measure. This completes the proof of this lemma.  $\square$

*Proof of Proposition 2.1.* The Feller property of the semigroup  $P$  is given in [7, Proposition VI.1]. Moreover, the fact that the infinitesimal generator of  $P$  is  ${}^C D_+^\alpha$  has been proved in various papers, see e.g. [5] and [36], and the domain  $\mathcal{D}_\alpha$  is given in [36, Proposition 2.2], which completes the proof of the first item. Next, from [39, Lemma 3] and its proof, we know that, up to a multiplicative positive constant, the Lebesgue measure is the unique excessive measure for  $P$ , where with the notation of [39, Example 3],  $\gamma = 1 - \frac{1}{\alpha}$ . Thus, since  $X$  is stochastically continuous, see [23, Lemma 2.1], a classical result from the general theory of Markov semigroups, see e.g. [14,

Theorem 5.8], yields that the Feller semigroup  $P$  admits a unique extension as a contractive  $\mathcal{C}_0$ -semigroup on  $L^2(\mathbb{R}_+)$ . We now proceed to characterize the domain of the infinitesimal generator of the  $L^2(\mathbb{R}_+)$ -extension, denoted by  $\mathcal{D}^X$ . To this end, we first observe from [4, Theorem 12.16] that since  $Z$  is a Lévy process, its semigroup  $(\bar{P}_t)_{t \geq 0}$ , i.e.  $\bar{P}_t f(x) = \mathbb{E}_x[f(Z_t)]$ ,  $x \in \mathbb{R}$ , is a  $L^2(\mathbb{R})$ -Markov semigroup, and its infinitesimal generator, denoted by  $L^Z$ , has the following anisotropic Sobolev space as its domain

$$(2.9) \quad \mathcal{D}^Z = \left\{ \bar{f} \in L^2(\mathbb{R}); \int_{-\infty}^{\infty} |\mathcal{F}_{\bar{f}}(\xi)|^2 |\xi|^{2\alpha} d\xi < \infty \right\},$$

where  $\mathcal{F}_{\bar{f}}(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} \bar{f}(x) dx$  is the Fourier transform of  $\bar{f}$ . Now for a function  $f$  on  $\mathbb{R}^+$  we define its extension  $\bar{f} : \mathbb{R} \rightarrow \mathbb{R}$  as  $\bar{f}(x) = f(x)\mathbb{I}_{\{x > 0\}}$ . Then, for any  $f \in \overline{\mathcal{D}}_\alpha(L^2(\mathbb{R}_+)) = \{f \in \mathcal{D}_\alpha(L^2(\mathbb{R}_+)); \bar{f} \in C_0^2(\mathbb{R})\}$ , we have clearly  $\bar{f} \in \mathcal{D}^Z \cap C_0^2(\mathbb{R})$  and thus by combining [7, Section I.2] and [19, Theorem 2.1] we get, that for any  $x > 0$ ,

$$(2.10) \quad L^X f(x) = a(x)L^Z \bar{f}(x),$$

where  $a(x) = \mathbb{I}_{\{x > 0\}}$  from [19, (3.6)]. Therefore, since  $L^Z \bar{f} \in L^2(\mathbb{R})$ , it is obvious that  $L^X f \in L^2(\mathbb{R}_+)$ , which implies that  $f \in \mathcal{D}^X$ . Next, for any  $\tau > 0$ , let  $f_\tau(x) = \tau^3 x^3 e^{-\tau x}$ ,  $x > 0$ , then easy computations yield that for all  $\tau > 0$   $f_\tau \in \overline{\mathcal{D}}_\alpha(L^2(\mathbb{R}_+))$ , hence by the Wiener's theorem for Mellin transform  $\overline{\mathcal{D}}_\alpha(L^2(\mathbb{R}_+))$  is dense in  $L^2(\mathbb{R}_+)$  and therefore, for any  $f \in \mathcal{D}_\alpha(L^2(\mathbb{R}_+))$ , we can take  $(f_n)_{n \geq 0} \subset \overline{\mathcal{D}}_\alpha(L^2(\mathbb{R}_+)) \cap C_0^2(\mathbb{R}_+)$  such that  $f_n \rightarrow f$  in  $L^2(\mathbb{R}_+)$ . Writing  $\bar{f}_n$  and  $\bar{f}$  their corresponding extensions to  $L^2(\mathbb{R})$  as above, we still have  $\bar{f}_n \rightarrow \bar{f}$  in  $L^2(\mathbb{R})$  and  $\bar{f} \in \mathcal{D}^Z$ . Also note that for each  $\xi \in \mathbb{R}$ ,

$$(2.11) \quad \mathcal{F}_{L^X f_n}^+(\xi) = \mathcal{F}_{L^Z \bar{f}_n}(\xi) = (-i\xi)^\alpha \mathcal{F}_{\bar{f}_n}(\xi) \rightarrow (-i\xi)^\alpha \mathcal{F}_{\bar{f}}(\xi) = \mathcal{F}_{L^Z \bar{f}}(\xi),$$

where we used [4, Theorem 12.16] for the second and last identity. Therefore  $L^X f_n$  converges in  $L^2(\mathbb{R}_+)$  and  $f \in \mathcal{D}^X$  by the closedness of infinitesimal generator. This shows that  $\mathcal{D}_\alpha(L^2(\mathbb{R}_+)) \subseteq \mathcal{D}^X$ . On the other hand, take now  $f \in \mathcal{D}^X \cap C_0^2(\mathbb{R}_+)$  and let  $\bar{f}$  be constructed as above. Then by [19, Theorem 2.6] and recalling that the fine support of  $(A_t)_{t \geq 0}$  is  $\mathbb{R}_+$ , we have

$$(2.12) \quad L^Z \bar{f}(x) = \begin{cases} b(x)L^X f(x) & \text{for } x \geq 0, \\ 0 & \text{for } x < 0, \end{cases}$$

where, denoting  $\mathbb{I}_+(x) = \mathbb{I}_{\{x > 0\}}$ ,

$$b(x) = \lim_{t \rightarrow 0} \frac{\mathbb{E}_x[\int_0^t \mathbb{I}_{\{Z_s > 0\}} ds]}{t} = \lim_{t \rightarrow 0} \frac{\int_0^t \bar{P}_s \mathbb{I}_+(x) ds}{t} = \lim_{t \rightarrow 0} \bar{P}_t \mathbb{I}_+(x) = \mathbb{I}_{\{x > 0\}}$$

for each  $x \in \mathbb{R}$ . Therefore, we have

$$\int_{\mathbb{R}} (L^Z \bar{f}(x))^2 dx = \int_{\mathbb{R}} ((\mathbb{I}_{\{x < 0\}} + \mathbb{I}_{\{x \geq 0\}})L^Z \bar{f}(x))^2 dx = \int_0^\infty (L^Z \bar{f}(x))^2 dx = \int_0^\infty (L^X f(x))^2 dx,$$

which implies that  $\bar{f} \in \mathcal{D}^Z$  and  $f \in \mathcal{D}_\alpha(L^2(\mathbb{R}_+))$ . Next, since we have proved that  $\mathcal{D}_\alpha(L^2(\mathbb{R}_+)) \subseteq \mathcal{D}^X$  and  $\mathcal{D}_\alpha(L^2(\mathbb{R}_+)) \cap C_0^2(\mathbb{R}_+)$  is dense in  $L^2(\mathbb{R}_+)$ , we have that  $\mathcal{D}^X \cap C_0^2(\mathbb{R}_+)$  is also dense in  $L^2(\mathbb{R}_+)$ . Hence the same argument as above shows that (2.12) still holds for any  $f \in \mathcal{D}^X$ , which further proves that  $\mathcal{D}^X \subseteq \mathcal{D}_\alpha(L^2(\mathbb{R}_+))$  and completes the proof for the second argument. For the duality argument, we first observe from Lemma 2.1 that  $X$  and  $\hat{X}$  are dual processes with respect to the Lebesgue measure. Moreover, note that the minimal process  $X^\dagger$  belongs to the class of *positive  $\frac{1}{\alpha}$ -self-similar Markov processes* as introduced in [23], which also provides a

bijection between positive self-similar processes and Lévy processes stated as follows. Let us define, for any  $t \geq 0$ ,  $\vartheta_t = \inf\{u > 0; \int_0^u (X_s^\dagger)^{-\alpha} ds > t\}$ , then the process

$$(2.13) \quad \xi_t^\dagger = \log X_{\vartheta_t}^\dagger,$$

is a Lévy process killed at an independent exponential time. More specifically, by [36], the Laplace exponent of  $\xi^\dagger$  is

$$(2.14) \quad \psi^\dagger(u) = \frac{\Gamma(u+1)}{\Gamma(u-\alpha+1)}, \quad u > -1.$$

Note that by writing  $\theta$  for the largest non-negative root of the convex function  $\psi^\dagger$ , it is easy to check that  $\theta = \alpha - 1 \in (0, 1)$ . Hence by [39, Section 5], there exists a dual process of  $X^\dagger$ , denoted by  $\widehat{X}^\dagger$ , with the Lebesgue measure serving as the reference measure. Moreover,  $\widehat{X}^\dagger$  is also a positive  $\frac{1}{\alpha}$ -self-similar process with its corresponding Lévy process denoted by  $\widehat{\xi}^\dagger$ , which is the dual of the Lévy process obtained from  $\xi^\dagger$  by means of Doob  $h$ -transform via the invariant function  $h(x) = e^{\theta x}$ ,  $x \in \mathbb{R}$ . Therefore, the Laplace exponent of  $\widehat{\xi}^\dagger$  takes the form, for  $u < 0$ ,

$$\widehat{\psi}(u) = \psi^\dagger(-u + \theta) = \psi^\dagger(-u + \alpha - 1) = \frac{\Gamma(\alpha - u)}{\Gamma(-u)}.$$

Note that  $\widehat{\xi}^\dagger$  drifts to  $-\infty$  a.s. and thus  $\widehat{X}^\dagger$  has a a.s. finite lifetime  $T_0^{\widehat{X}^\dagger} = \inf\{t > 0; \widehat{X}_t^\dagger \leq 0\}$ . Hence by recalling that  $X$  can be viewed as the recurrent extension of  $X^\dagger$  that leaves 0 continuously a.s., we deduce from [39, Lemma 6] that  $\widehat{X}$  can also be viewed as the recurrent extension of  $\widehat{X}^\dagger$  which leaves 0 by a jump according to the jump-in measure  $Cx^{-\alpha}$ ,  $x, C > 0$ . The Feller property of the semigroup of such recurrent extension has been shown in [9, Proposition 3.1], while the existence of the  $L^2(\mathbb{R}_+)$ -extension follows by the same argument than the one we developed for  $P$ . Moreover, from [4, Theorem 12.16], we deduce easily that  $\mathcal{D}^{\widehat{Z}} = \mathcal{D}^Z$ , hence using the same method as above, we get that  $\mathcal{D}^{\widehat{X}} = \mathcal{D}^X = \mathcal{D}_\alpha(L^2(\mathbb{R}_+))$ . Finally, using the same arguments as in (2.11), we see that for any  $f \in \mathcal{D}_\alpha(L^2(\mathbb{R}_+))$ ,

$$\mathcal{F}_{L^{\widehat{X}}f}^+(\xi) = \mathcal{F}_{L^{\widehat{Z}}f}(\xi) = (i\xi)^\alpha \mathcal{F}_f(\xi).$$

Comparing this identity with [18, Lemma 2.1 and Theorem 2.3], we conclude that  $L^{\widehat{X}}f = D_-^\alpha f$  on  $\mathcal{D}_\alpha(L^2(\mathbb{R}_+))$ . This completes the proof.  $\square$

### 3. INTERTWINING RELATIONSHIP

We say that a linear operator  $\Lambda$  is a *multiplicative operator* if it admits the following representation, for any  $f \in \mathcal{B}_b(\mathbb{R}_+)$ ,

$$\Lambda f(x) = \int_0^\infty f(xy)\lambda(y)dy,$$

for some integrable function  $\lambda$ . When in addition  $\lambda$  is the density of the law of a random variable  $\mathbf{X}$ , i.e.  $\lambda(y) \geq 0$  and  $\langle 1, \lambda \rangle = 1$ , we say that  $\Lambda$  is a *Markov multiplicative operator*. Moreover,  $\mathcal{M}_\lambda = \mathcal{M}_\Lambda = \mathcal{M}_{\mathbf{X}}$  is called a Markov multiplier where for at least  $\Re(s) = 1$ ,

$$\mathcal{M}_\Lambda(s) = \int_0^\infty y^{s-1}\lambda(y)dy,$$



is the Mellin transform of  $\lambda$ . By adapting the developments in [45, 2.1.9] based on the Fourier transform, we also have that if  $\int_0^\infty y^{-\frac{1}{2}}\lambda(y)dy < \infty$  then  $\Lambda \in \mathbf{B}(L^2(\mathbb{R}_+))$  with, for any  $f \in L^2(\mathbb{R}_+)$ ,

$$(3.1) \quad \mathcal{M}_{\Lambda f}(s) = \mathcal{M}_\Lambda(1-s)\mathcal{M}_f(s).$$

Note that this latter provides that  $\Lambda$  is one-to-one in  $L^2(\mathbb{R}_+)$  if  $\mathcal{M}_\Lambda(1-s) \neq 0$ . We also recall from [31] that if  $s \mapsto \mathcal{M}_\lambda(s)$  is defined, absolutely integrable and uniformly decays to zero along the lines of the strip  $s \in \mathbb{C}_{(\underline{a}, \bar{a})}$  for some  $\underline{a} < \bar{a}$ , then the Mellin inversion theorem applies to yield, for any  $x > 0$ ,

$$(3.2) \quad \lambda(x) = \frac{1}{2\pi i} \int_{\underline{a}-i\infty}^{\bar{a}+i\infty} x^{-s} \mathcal{M}_\lambda(s) ds, \quad \underline{a} < a < \bar{a}.$$

Now we are ready to state the following.

**Theorem 3.1.** *Let us write, for any  $\alpha \in (1, 2)$ ,*

$$(3.3) \quad \mathcal{M}_{\Lambda_\alpha}(s) = \frac{\Gamma(\frac{s-1}{\alpha} + 1)\Gamma(\frac{s}{\alpha})}{\Gamma(\frac{1}{\alpha})\Gamma(s)}, \quad s \in \mathbb{C}_+.$$

*Then, the following holds.*

- (1)  $\mathcal{M}_{\Lambda_\alpha}$  is a Markov multiplier and  $\Lambda_\alpha \in \mathbf{B}(L^2(\mathbb{R}_+)) \cap \mathbf{B}(C_0(\mathbb{R}_+))$ . Moreover, it is one-to-one on  $C_0(\mathbb{R}_+)$ , and, in  $L^2(\mathbb{R}_+)$ ,  $\overline{\text{Ran}}(\Lambda_\alpha) = L^2(\mathbb{R}_+)$ .
- (2) Moreover, for any  $t \geq 0$  and  $f \in L^2(\mathbb{R}_+)$ , the following intertwining relation holds

$$(3.4) \quad P_t \Lambda_\alpha f = \Lambda_\alpha Q_t f,$$

where  $Q = (Q_t)_{t \geq 0}$  is the  $L^2(\mathbb{R}_+)$ -extension of the  $\alpha$ -Bessel self-adjoint semigroup as defined in Appendix A.

- (3) Consequently, we have, for any  $f \in \mathcal{D}_\mathbf{L}(L^2(\mathbb{R}_+))$ ,

$$(3.5) \quad {}^C D_+^\alpha \Lambda_\alpha f = \Lambda_\alpha \mathbf{L} f,$$

where the fractional operator  ${}^C D_+^\alpha$  was defined in (1.2), while the second order differential operator  $\mathbf{L}$  and its  $L^2(\mathbb{R}_+)$ -domain  $\mathcal{D}_\mathbf{L}(L^2(\mathbb{R}_+))$  are defined in (A.1) and (A.7), respectively.

The proof of this Theorem is split into three steps. First, we show that (3.3) is indeed a Markov multiplier. Then, we establish the identity (3.4) in the space  $C_0(\mathbb{R}_+)$ . Finally, by remarking that  $C_0(\mathbb{R}_+)$  is dense in  $L^2(\mathbb{R}_+)$ , we can extend the intertwining identity to  $L^2(\mathbb{R}_+)$  by a continuity argument.

**3.1. The Markov multiplicative operator  $\Lambda_\alpha$ .** In order to prove Theorem 3.1(1), which provides some substantial properties of  $\Lambda_\alpha$ , we shall need the following claims.

**Lemma 3.1.** *Let us define*

$$(3.6) \quad g_\alpha(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{\alpha})\Gamma(\alpha n + 1)}{\Gamma(n + \frac{1}{\alpha})(n!)^2} (e^{i\pi} z^\alpha)^n,$$

then  $g_\alpha$  is holomorphic on  $\mathbb{C}_{(-\infty, 0)^c}$ . Moreover,  $g_\alpha \in L^2(\mathbb{R}_+)$  with  $\Lambda_\alpha g_\alpha = \mathbf{e}_\alpha$  where  $\mathbf{e}_\alpha$  is defined in (1.4).

*Proof.* First, from the Stirling approximation

$$(3.7) \quad \Gamma(a) \approx \sqrt{2\pi} a^{a-\frac{1}{2}} e^{-a},$$

see [24, (1.4.25)], we get that  $\frac{\Gamma(\alpha n + \alpha + 1)\Gamma(n + \frac{1}{\alpha})}{\Gamma(\alpha n + 1)\Gamma(n + 1 + \frac{1}{\alpha})(n + 1)^2} \approx O(n^{\alpha-3})$ , hence, as  $\alpha \in (1, 2)$ ,  $g_\alpha$  is holomorphic on  $\mathbb{C}_{(-\infty, 0)^c}$ . We now proceed to show that  $g_\alpha \in L^2(\mathbb{R}_+)$ . To this end, let us define, for  $0 < \Re(s) < 1$ ,

$$\mathcal{M}_\alpha(s) = \frac{\Gamma(1 + \frac{1}{\alpha})\Gamma(\frac{s}{\alpha})\Gamma(1 - s)}{\Gamma(1 - \frac{s}{\alpha})\Gamma(\frac{1-s}{\alpha})}$$

and we first aim at proving that  $\mathcal{M}_\alpha = \mathcal{M}_{g_\alpha}$  the Mellin transform of  $g_\alpha$ . For this purpose, observe that  $s \mapsto \mathcal{M}_\alpha(s)$  is holomorphic on  $\mathbb{C}_{(0,1)}$  and then consider the contour integral  $I_{N,B} = \frac{1}{2\pi i} \int_{C_{N,B}} z^{-s} \mathcal{M}_\alpha(s) ds$  where  $C_{N,B}$  is the rectangle with vertices at  $\frac{1}{2} \pm iB$  and  $-\alpha N - \frac{\alpha}{2} \pm iB$  for some large  $N \in \mathbb{N}$  and  $B > 0$ . Then we can obviously split  $I_{N,B}$  into four parts, namely  $I_{N,B} = I_1 + I_2 + I_3 + I_4$  where

$$I_1 = \frac{1}{2\pi i} \int_{\frac{1}{2} + iB}^{-\alpha N - \frac{\alpha}{2} + iB} z^{-s} \mathcal{M}_\alpha(s) ds, \quad I_2 = \frac{1}{2\pi i} \int_{-\alpha N - \frac{\alpha}{2} + iB}^{-\alpha N - \frac{\alpha}{2} - iB} z^{-s} \mathcal{M}_\alpha(s) ds,$$

$$I_3 = \frac{1}{2\pi i} \int_{-\alpha N - \frac{\alpha}{2} - iB}^{\frac{1}{2} - iB} z^{-s} \mathcal{M}_\alpha(s) ds, \quad I_4 = \frac{1}{2\pi i} \int_{\frac{1}{2} - iB}^{\frac{1}{2} + iB} z^{-s} \mathcal{M}_\alpha(s) ds.$$

Next, observing from the Stirling approximation, see e.g. [31, (2.1.8)], that for fixed  $a \in \mathbb{R}$ ,

$$(3.8) \quad |\Gamma(a + ib)| \approx C |b|^{a-\frac{1}{2}} e^{-\frac{\pi}{2}|b|},$$

with  $C = C(a) > 0$ , we deduce, for some  $C_\alpha > 0$ , that

$$(3.9) \quad |\mathcal{M}_\alpha(a + ib)| \approx C_\alpha |b|^{\frac{3}{\alpha}a - a - \frac{1}{\alpha}} e^{-\frac{\pi}{2}(1 - \frac{1}{\alpha})|b|},$$

and, hence

$$(3.10) \quad \left| z^{-(a+ib)} \mathcal{M}_\alpha(a + ib) \right| \approx C_\alpha |z|^{-a} |b|^{\frac{3}{\alpha}a - a - \frac{1}{\alpha}} e^{-\frac{\pi}{2}(1 - \frac{1}{\alpha})|b| + \arg(z)b}.$$

Therefore, if  $|\arg(z)| < \frac{\pi}{2}(1 - \frac{1}{\alpha})$  and  $N$  is kept fixed, we have both

$$(3.11) \quad \lim_{B \rightarrow \infty} |I_1| = \lim_{B \rightarrow \infty} |I_3| = 0.$$

For the integral  $I_2$ , we have

$$|I_2| \leq \frac{1}{2\pi} |z|^{\alpha N + \frac{\alpha}{2}} \int_{-\infty}^{\infty} e^{\arg(z)b} \left| \frac{\Gamma(1 + \frac{1}{\alpha})\Gamma(-N - \frac{1}{2} + i\frac{b}{\alpha})\Gamma(1 + \alpha N + \frac{\alpha}{2} + ib)}{\Gamma(N + \frac{3}{2} - i\frac{b}{\alpha})\Gamma(N + \frac{1}{2} + \frac{1}{\alpha} - i\frac{b}{\alpha})} \right| db$$

$$= \frac{1}{2} |z|^{\alpha N + \frac{\alpha}{2}} \int_{-\infty}^{\infty} e^{\arg(z)b} \left| \frac{\Gamma(1 + \frac{1}{\alpha})\Gamma(1 + \alpha N + \frac{\alpha}{2} + ib)}{\Gamma(N + \frac{3}{2} - i\frac{b}{\alpha})^2 \Gamma(N + \frac{1}{2} + \frac{1}{\alpha} - i\frac{b}{\alpha}) \cosh(\frac{\pi b}{\alpha})} \right| db$$

where we have used the reflection formula for the gamma function. Using the Stirling approximation again, it is easy to derive, for large  $N$ , the upper bound

$$\left| \frac{\Gamma(1 + \alpha N + \frac{\alpha}{2} + ib)}{\Gamma(N + \frac{3}{2} - i\frac{b}{\alpha})^2} \right| \leq C e^{N(\alpha \log \alpha - \alpha - 2)} N^{(\alpha-2)N + \frac{\alpha-1}{2}}$$

which is uniform in  $b \in \mathbb{R}$  and where  $C > 0$ . Moreover, recalling, from [31, (5.1.3)], that, for  $N \geq 1$  and  $b \in \mathbb{R}$ ,  $|\Gamma(N + \frac{1}{2} + \frac{1}{\alpha} - i\frac{b}{\alpha})| \geq \frac{\Gamma(N + \frac{1}{2} + \frac{1}{\alpha})}{\cosh^{\frac{1}{2}}(\frac{\pi b}{\alpha})}$ , we find

$$|I_2| \leq C \frac{e^{N(\alpha \log \alpha - \alpha - 2)} N^{(\alpha-2)N + \frac{\alpha-1}{2}}}{\Gamma(N + \frac{1}{2} + \frac{1}{\alpha})} \int_{-\infty}^{\infty} \frac{e^{\arg(z)b}}{\cosh^{\frac{1}{2}}(\frac{\pi b}{\alpha})} db$$

where the last integral converges absolutely whenever  $|\arg(z)| < \frac{\pi}{2\alpha}$ . For such  $z$ , since  $1 < \alpha < 2$ , we get that  $\lim_{N \rightarrow \infty} |I_2| = 0$ . Therefore, combining this with (3.11), we have, for  $|\arg(z)| < \frac{\pi}{2\alpha} \wedge \frac{\pi}{2}(1 - \frac{1}{\alpha}) = \frac{\pi}{2\alpha}$ ,

$$\lim_{N, B \rightarrow \infty} I_{N, B} = \lim_{B \rightarrow \infty} I_4 = \frac{1}{2\pi i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} z^{-s} \mathcal{M}_\alpha(s) ds.$$

Hence an application of Cauchy's integral theorem yields

$$(3.12) \quad \frac{1}{2\pi i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} z^{-s} \mathcal{M}_\alpha(s) ds = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{\alpha})\Gamma(\alpha n + 1)}{\Gamma(n + \frac{1}{\alpha})(n!)^2} (-1)^n z^{\alpha n} = g_\alpha(z)$$

where we sum over the poles  $s = -\alpha n, n = 0, 1, \dots$  of  $\Gamma(\frac{s}{\alpha})$  with residues  $\frac{\alpha(-1)^n}{n!}$ . This shows that  $\mathcal{M}_{g_\alpha} = \mathcal{M}_\alpha$ . Since  $\alpha \in (1, 2)$ , we have, from (3.9), that  $b \mapsto \mathcal{M}_\alpha(\frac{1}{2} + ib) \in L^2(\mathbb{R})$  and by the Parseval identity for the Mellin transform we conclude that  $g_\alpha \in L^2(\mathbb{R}_+)$ . Finally, by means of a standard application of Fubini theorem, see e.g. [44, Section 1.77]), one shows that, for any  $x > 0$ ,

$$\Lambda_\alpha g_\alpha(x) = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{\alpha})\Gamma(\alpha n + 1)}{\Gamma(n + \frac{1}{\alpha})(n!)^2} \mathcal{M}_{\Lambda_\alpha}(\alpha n + 1) (-1)^n x^{\alpha n} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{\alpha n}}{n!} = \mathbf{e}_\alpha(x),$$

where we used the expression (3.3). This completes the proof of the lemma.  $\square$

Next, let us show that  $\mathcal{M}_{\Lambda_\alpha}$  is the Mellin transform of a random variable that we denote by  $I_\alpha$ . To this end, we write, for any  $u > 0$ ,

$$\phi_\alpha(u) = \frac{\Gamma(\alpha u + 1)}{\Gamma(\alpha u + 1 - \alpha)} \frac{1}{u - 1 + \frac{1}{\alpha}} = \frac{\alpha}{\Gamma(2 - \alpha)} + \int_0^\infty (1 - e^{-uy}) \frac{\alpha(\alpha - 1)}{\Gamma(2 - \alpha)} \frac{e^{-\frac{y}{\alpha}}}{(1 - e^{-\frac{y}{\alpha}})^\alpha} dy,$$

where the second identity follows after some standard computation, see e.g. [32, (4.2)]. As plainly  $\int_0^\infty (y \wedge 1) \frac{e^{-\frac{y}{\alpha}}}{(1 - e^{-\frac{y}{\alpha}})^\alpha} dy < \infty$ , we get, from [41, Theorem 3.2], that  $\phi_\alpha$  is a Bernstein function, whose definition is given in [41, Definition 3.2]. Moreover, by [41, Section 5],  $\phi_\alpha$  is the Laplace exponent of a subordinator, that is an increasing process with stationary and independent increments, which we denote by  $(\xi_t)_{t \geq 0}$ . Next, observing that for any  $n \in \mathbb{N}$ ,

$$(3.13) \quad \mathcal{M}_{\Lambda_\alpha}(\alpha n + 1) = \frac{n! \Gamma(n + \frac{1}{\alpha})}{\Gamma(\frac{1}{\alpha}) \Gamma(\alpha n + 1)} = \frac{n!}{\prod_{k=1}^n \phi_\alpha(k)},$$

we deduce, from [11, Proposition 3.3], that  $(\mathcal{M}_{\Lambda_\alpha}(\alpha n + 1))_{n \geq 0}$  is the Stieltjes moment sequence of the random variable  $\int_0^\infty e^{-\xi t} dt$ . Moreover, observe from its definition (3.3) and applications of the recurrence relation of the gamma function that  $\mathcal{M}_{\Lambda_\alpha}$  satisfies the functional equation, on  $s \in \mathbb{C}_+$ ,

$$\mathcal{M}_{\Lambda_\alpha}(\alpha s + 1) = \frac{s}{\phi_\alpha(s)} \mathcal{M}_{\Lambda_\alpha}(\alpha(s - 1) + 1), \quad \mathcal{M}_{\Lambda_\alpha}(1) = 1,$$

hence, by a uniqueness argument developed in [33, Section 7], we have

$$(3.14) \quad \mathcal{M}_{\Lambda_\alpha}(s+1) = \mathbb{E} \left[ \left( \int_0^\infty e^{-\xi t} dt \right)^{\frac{s}{\alpha}} \right] = \mathbb{E} [I_\alpha^s].$$

Consequently  $\mathcal{M}_{\Lambda_\alpha}(s)$  is the Mellin transform of the variable  $I_\alpha = \left( \int_0^\infty e^{-\xi s} ds \right)^{\frac{1}{\alpha}}$ . Finally, since the law of  $\int_0^\infty e^{-\xi t} dt$  is known to be absolutely continuous, see e.g. [33, Proposition 7.7], so is the one of  $I_\alpha$ , therefore we conclude that  $\mathcal{M}_{\Lambda_\alpha}$  is indeed a Markov multiplier, which provides the first claim of Theorem 3.1(1). Next, the one-to-one property of  $\Lambda_\alpha$  follows from the fact that the mapping  $s \mapsto \mathcal{M}_{\Lambda_\alpha}(s)$  is clearly zero-free on the line  $1 + i\mathbb{R}$ . Moreover, writing  $\lambda_\alpha$  the density of  $I_\alpha$ , we have by dominated convergence that for any  $f \in C_0(\mathbb{R}_+)$ ,  $\Lambda_\alpha f \in C_0(\mathbb{R}_+)$  with  $\|\Lambda_\alpha f\|_\infty \leq \|f\|_\infty$ , that is,  $\Lambda_\alpha \in \mathbf{B}(C_0(\mathbb{R}_+))$ . On the other hand, for  $f \in L^2(\mathbb{R}_+)$ , Jensen's inequality and a change of variable yield

$$\|\Lambda_\alpha f\|^2 = \int_0^\infty \mathbb{E} [f(xI_\alpha)]^2 dx \leq \int_0^\infty \mathbb{E} [f^2(xI_\alpha)] dx = \mathbb{E} [I_\alpha^{-1}] \|f\|^2$$

where  $\mathbb{E} [I_\alpha^{-1}] = \mathcal{M}_{\Lambda_\alpha}(0) = \frac{\Gamma(1-\frac{1}{\alpha})}{\Gamma(1+\frac{1}{\alpha})} < \infty$ . Hence  $\Lambda_\alpha \in \mathbf{B}(L^2(\mathbb{R}_+))$ . Moreover, from Lemma 3.1, it is easy to conclude that  $\Lambda_\alpha d_q g_\alpha = d_q e_\alpha$  for all  $q > 0$ , where  $d_q g_\alpha \in L^2(\mathbb{R}_+)$  since  $q d_q$  is a unitary operator in  $L^2(\mathbb{R}_+)$ . Hence, by the well-known result that  $\overline{\text{Span}(d_q e_\alpha)_{q>0}} = L^2(\mathbb{R}_+)$ , we have that  $\Lambda_\alpha$  has a dense range in  $L^2(\mathbb{R}_+)$ , which completes the proof of Theorem 3.1(1).

**3.2. Proofs of Theorem 3.1(2) and (3).** We recall that a collection of  $\sigma$ -finite measures  $(\eta_t)_{t>0}$  is called an *entrance law* for the semigroup  $P$  if for any  $t, s > 0$  and any  $f \in C_0(\mathbb{R}_+)$ ,  $\eta_t P_s f = \eta_{t+s} f$  where  $\eta_t f = \int_0^\infty f(x) \eta_t(dx)$ . We also recall from Appendix A that  $G_\alpha$  is the  $\frac{1}{\alpha}$  power of a gamma variable with parameter  $\frac{1}{\alpha} > 0$ , that is  $\mathbb{P}(G_\alpha \in dy) = \frac{e^{-y} y^{\frac{1}{\alpha}-1}}{\Gamma(\frac{1}{\alpha})} dy, y > 0$ . Now we are ready to state the following Lemma.

**Lemma 3.2.**  *$P$  admits an entrance law  $(\eta_t)_{t>0}$  defined for any  $t > 0$  by  $\eta_t f = \eta_1 d_t f = \int_0^\infty f(ty) \eta_1(dy)$  where  $\eta_1(dy) = \lambda_{\mathbf{X}_\alpha}(y) dy$ , with  $\lambda_{\mathbf{X}_\alpha} \in L^2(\mathbb{R}_+)$ , is the probability measure of a variable  $\mathbf{X}_\alpha$ . Its Mellin transform takes the form*

$$(3.15) \quad \mathcal{M}_{\mathbf{X}_\alpha}(s) = \frac{\Gamma(s)}{\Gamma(\frac{s}{\alpha} + 1 - \frac{1}{\alpha})}, \quad s \in \mathbb{C}_+.$$

Moreover, we have the following factorization of the variable  $G_\alpha$

$$G_\alpha \stackrel{d}{=} \mathbf{X}_\alpha \times I_\alpha,$$

where  $\stackrel{d}{=}$  stands for the identity in distribution and  $\mathbf{X}_\alpha$  is considered independent of  $I_\alpha$ , which we recall was characterized in (3.14).

*Proof.* First, let us observe from (3.15) that for any  $n \geq 0$ ,

$$(3.16) \quad \mathcal{M}_{\mathbf{X}_\alpha}(\alpha n + 1) = \frac{\Gamma(\alpha n + 1)}{n!} = \frac{\prod_{k=1}^n \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha(k-1) + 1)}}{n!} = \prod_{k=1}^n \frac{\psi^\dagger(\alpha k)}{k},$$

where we recall from the proof of Proposition 2.1 that  $\psi^\dagger(u) = \frac{\Gamma(u+1)}{\Gamma(u-\alpha+1)}, u > \alpha - 1$ , is the Laplace exponent of the killed Lévy process  $\xi^\dagger$  defined in (2.13). Then by [3, Theorem 1], we deduce that  $(\mathcal{M}_{\mathbf{X}_\alpha}(\alpha n + 1))_{n \geq 0}$  is the moment sequence of the variable  $X_1^\alpha$  under  $\mathbb{P}_0$ , for

which we used the fact that since  $X$  is a  $\frac{1}{\alpha}$ -self-similar process,  $X^\alpha$  is a 1-self-similar process whose minimal process is associated, through the Lamperti mapping, to a Lévy process with Laplace exponent  $\psi_\alpha(u) = \psi^\dagger(\alpha u)$ . Moreover, note from (3.15) that  $\mathcal{M}_{\mathbf{X}_\alpha}$  satisfies the functional equation  $\mathcal{M}_{\mathbf{X}_\alpha}(\alpha s + 1) = \frac{\psi_\alpha(s)}{s} \mathcal{M}_{\mathbf{X}_\alpha}(\alpha(s-1) + 1)$  with  $\mathcal{M}_{\mathbf{X}_\alpha}(1) = 1$ , hence by a uniqueness argument, see again [33, Section 7], we conclude that  $\mathcal{M}_{\mathbf{X}_\alpha}(s+1) = \mathbb{E}_0[X_1^s]$  is indeed the Mellin transform of  $X_1$  under  $\mathbb{P}_0$ . Using again the Stirling approximation (3.8), we see that  $|\mathcal{M}_{\mathbf{X}_\alpha}(\frac{1}{2} + ib)| \stackrel{\pm\infty}{\asymp} O\left(|b|^{\frac{1}{2\alpha} - \frac{1}{2}} e^{-\frac{\pi}{2}(1 - \frac{1}{\alpha})|b|}\right)$ , and thus  $b \mapsto \mathcal{M}_{\mathbf{X}_\alpha}(\frac{1}{2} + ib) \in L^2(\mathbb{R})$ . Hence by Mellin inversion and Parseval identity, we get that the law of  $\mathbf{X}_\alpha$  is absolutely continuous with a density  $\lambda_{\mathbf{X}_\alpha} \in L^2(\mathbb{R}_+)$ . Now, recalling that  $\eta_1(dy) = \lambda_{\mathbf{X}_\alpha}(y)dy$  and for any  $t > 0$ ,  $\eta_t f = \eta_1 d_t f$ , we get, from (3.16) augmented by a moment identification that  $(\eta_t)_{t>0}$  is an entrance law for the semigroup  $P$ . Finally, from the expression of  $\mathcal{M}_{\Lambda_\alpha}$  in (3.3), we conclude that for  $s \in \mathbb{C}_+$ ,

$$\mathcal{M}_{\mathbf{X}_\alpha}(s)\mathcal{M}_{\Lambda_\alpha}(s) = \frac{\Gamma(s)}{\Gamma(\frac{s}{\alpha} + 1 - \frac{1}{\alpha})} \frac{\Gamma(\frac{s-1}{\alpha} + 1)\Gamma(\frac{s}{\alpha})}{\Gamma(\frac{1}{\alpha})\Gamma(s)} = \frac{\Gamma(\frac{s}{\alpha})}{\Gamma(\frac{1}{\alpha})} = \mathcal{M}_{G_\alpha}(s)$$

where we used for the last identity the expression (A.3). We complete the proof by invoking the injectivity of the Mellin transform.  $\square$

We are now ready to prove the intertwining relation stated in Theorem 3.1(2). First, since  $s \mapsto \mathcal{M}_{\mathbf{X}_\alpha}(s)$  is zero-free on the line  $1 + i\mathbb{R}$ , we again conclude that the Markov operator  $\Lambda_{\mathbf{X}_\alpha}$  associated to the positive variable  $\mathbf{X}_\alpha$ , i.e.  $\Lambda_{\mathbf{X}_\alpha} f(x) = \int_0^\infty f(xy)\lambda_{\mathbf{X}_\alpha}(y)dy$ , is injective on  $C_0(\mathbb{R}_+)$ . This combined with the fact that the law of  $G_\alpha$  is the entrance law at time 1 of the semigroup  $Q$  and with the factorization of this latter stated in Lemma 3.2 provide all conditions for the application of [11, Proposition 3.2], which gives that for any  $t \geq 0$  and  $f \in C_0(\mathbb{R}_+)$ , the following intertwining relationship between the Feller semigroups  $(P_t^F)_{t \geq 0}$  and  $(Q_t^F)_{t \geq 0}$ ,

$$(3.17) \quad P_t^F \Lambda_\alpha f = \Lambda_\alpha Q_t^F f.$$

in  $C_0(\mathbb{R}_+)$ . Furthermore, since  $C_0(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$  is dense in  $L^2(\mathbb{R}_+)$ , we can extend the intertwining identity into  $L^2(\mathbb{R}_+)$  by continuity of the involved operators and complete the proof of Theorem 3.1(2). Finally, Theorem 3.1(3) follows directly from (3.4) by recalling that  ${}^C D_+^\alpha$  and  $\mathbf{L}$  are the infinitesimal generators of  $P$  and  $Q$ , respectively, where the  $L^2(\mathbb{R}_+)$ -domain of  $\mathbf{L}$  is given in (A.7). This concludes the proof of Theorem 3.1.

#### 4. EIGENFUNCTIONS AND UPPER FRAMES

We start by recalling a few definitions concerning the spectrum of linear operators and we refer to [17, XV.8] for a thorough account on these objects. Let  $P \in \mathbf{B}(L^2(\mathbb{R}_+))$ . We say that a complex number  $\mathfrak{z} \in S(P)$ , the spectrum of  $P$ , if  $P - \mathfrak{z}I$  does not have an inverse in  $L^2(\mathbb{R}_+)$  with the following three distinctions:

- $\mathfrak{z} \in S_p(P)$ , the point spectrum, if  $\text{Ker}(P - \mathfrak{z}I) \neq \{0\}$ . In this case, we say a function  $f_\mathfrak{z}$  is an eigenfunction for  $P$ , associated to the eigenvalue  $\mathfrak{z}$ , if  $f_\mathfrak{z} \in \text{Ker}(P - \mathfrak{z}I)$ .
- $\mathfrak{z} \in S_c(P)$ , the continuous spectrum, if  $\text{Ker}(P - \mathfrak{z}I) = \{0\}$  and  $\overline{\text{Ran}(P - \mathfrak{z}I)} = L^2(\mathbb{R}_+)$  but  $\text{Ran}(P - \mathfrak{z}I) \subsetneq L^2(\mathbb{R}_+)$ .
- $\mathfrak{z} \in S_r(P)$ , the residual spectrum, if  $\text{Ker}(P - \mathfrak{z}I) = \{0\}$  and  $\overline{\text{Ran}(P - \mathfrak{z}I)} \subsetneq L^2(\mathbb{R}_+)$ .

Moreover, we also recall from [1] that a collection of functions  $(g_q)_{q>0}$  is a *frame* for  $L^2(\mathbb{R}_+)$  if for all  $q > 0$   $g_q \in L^2(\mathbb{R}_+)$  and there exists constants  $A, B > 0$ , called the *frame bounds*, such that, for all  $f \in L^2(\mathbb{R}_+)$ ,

$$A\|f\|^2 \leq \int_0^\infty \langle f, g_q \rangle^2 dq \leq B\|f\|^2.$$

Moreover, we say  $(g_q)_{q>0}$  is *upper frame* if it only satisfies the second inequality. Finally, recalling that  $\mathcal{J}_\alpha$  was defined in (2.4), we are ready to state the following claims which include the expression along with substantial properties of the set of eigenfunctions of  $P_t$ .

**Theorem 4.1.** (1) For any  $q, t > 0$ ,  $d_q \mathcal{J}_\alpha$  is an eigenfunction for  $P_t$  associated to the eigenvalue  $e^{-q^\alpha t}$ . Consequently, we have  $(e^{-q^\alpha t})_{q>0} \subseteq S_p(P_t)$ .

(2) Let the linear operator  $\mathcal{H}_\alpha$  be defined for any  $f \in L^2(\mathbb{R}_+)$  by

$$(4.1) \quad \mathcal{H}_\alpha f(q) = \int_0^\infty f(x) \mathcal{J}_\alpha(qx) dx, \quad q > 0,$$

then  $\mathcal{H}_\alpha \in \mathbf{B}(L^2(\mathbb{R}_+))$  with  $\|\mathcal{H}_\alpha\| = \sup_{\|f\|=1} \|\mathcal{H}_\alpha f\| \leq \frac{\Gamma(1-\frac{1}{\alpha})}{\Gamma(1+\frac{1}{\alpha})}$ . Consequently, the collection of functions  $(d_q \mathcal{J}_\alpha)_{q \geq 0}$  is a dense upper frame for  $L^2(\mathbb{R}_+)$ , with upper frame bound  $\frac{\Gamma(1-\frac{1}{\alpha})}{\Gamma(1+\frac{1}{\alpha})}$ .

(3) For any  $k \in \mathbb{N}$ ,  $\mathcal{J}_\alpha^{(k)}$  admits the following asymptotic expansion for large  $x > 0$ ,

$$(4.2) \quad \mathcal{J}_\alpha^{(k)}(x) \approx \frac{x^{-k-\alpha}}{\pi \Gamma(1 + \frac{1}{\alpha})} \sum_{n=0}^{\infty} a_{n,k} x^{-\alpha n}$$

where  $a_{n,k} = (-1)^{n+k} \Gamma(\alpha n + \alpha + k) \sin(\pi \alpha (n+1))$  and  $\approx$  means that for any  $N \in \mathbb{N}$ ,  $\mathcal{J}_\alpha^{(k)}(x) - \frac{x^{-k-\alpha}}{\pi \Gamma(1 + \frac{1}{\alpha})} \sum_{n=0}^N a_{n,k} x^{-\alpha n} \in \mathcal{O}(x^{-k-\alpha-\alpha(N+1)})$ .

**Remark 4.1.** Note that there is a cut-off in the nature of the spectrum when one considers the family of operators  $P$  indexed by  $\alpha \in (1, 2)$ . Indeed, when  $\alpha = 2$ , then  $\mathcal{J}_2 = J_2 \notin L^2(\mathbb{R}_+)$  (see (A.4) for the definition of the Bessel-type function  $J_2$ ) and hence, for all  $q, t > 0$ ,  $e^{-q^2 t} \notin S_p(P_t)$  but instead  $e^{-q^2 t} \in S_c(P_t)$ .

*Proof.* First, we recall that  $J_\alpha$ , the Bessel-type function, is defined in (A.4) as an holomorphic function on  $\mathbb{C}_{(-\infty, 0)^c}$ . As  $\alpha \in (1, 2)$  and  $J_\alpha(x) \in \mathcal{O}(x^{\frac{\alpha-2}{4}})$ , see e.g. [42], we get that  $J_\alpha \in C_0(\mathbb{R}_+)$ . Hence, as above, applying Fubini's theorem, we obtain, for  $x > 0$ , that

$$(4.3) \quad \Lambda_\alpha J_\alpha(x) = \alpha \sum_{n=0}^{\infty} \frac{(e^{i\pi} x^\alpha)^n}{n! \Gamma(n + \frac{1}{\alpha})} \mathcal{M}_{\Lambda_\alpha}(\alpha n + 1) = \frac{1}{\Gamma(1 + \frac{1}{\alpha})} \sum_{n=0}^{\infty} \frac{(e^{i\pi} x^\alpha)^n}{\Gamma(\alpha n + 1)} = \mathcal{J}_\alpha(x),$$

which shows, since  $\Lambda_\alpha \in \mathbf{B}(C_0(\mathbb{R}_+))$ , that both  $\mathcal{J}_\alpha \in C_0(\mathbb{R}_+)$  and  $d_q \mathcal{J}_\alpha \in C_0(\mathbb{R}_+)$  for all  $q > 0$ . Thus, we can use the relation (3.17) to get, for all  $q > 0$  and  $x \geq 0$ ,

$$(4.4) \quad P_t^F d_q \mathcal{J}_\alpha(x) = P_t^F \Lambda_\alpha d_q J_\alpha(x) = \Lambda_\alpha Q_t^F d_q J_\alpha(x) = e^{-q^\alpha t} \Lambda_\alpha d_q J_\alpha(x) = e^{-q^\alpha t} d_q \mathcal{J}_\alpha(x).$$

Next, proceeding as in the proof of Lemma 3.1, we get, for  $|\arg(z)| < (\frac{1}{\alpha} - \frac{1}{2}) \pi$ , that

$$\mathcal{J}_\alpha(z) = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} z^{-s} \mathcal{M}_{\mathcal{J}_\alpha}(s) ds,$$

where, for  $0 < \Re(s) < \alpha$ ,

$$(4.5) \quad \mathcal{M}_{\mathcal{J}_\alpha}(s) = \frac{\Gamma(1 - \frac{s}{\alpha})\Gamma(\frac{s}{\alpha})}{\Gamma(1 - s)\Gamma(\frac{1}{\alpha})}.$$

Since from (3.8),  $|\mathcal{M}_{\mathcal{J}_\alpha}(\frac{1}{2} + ib)| \stackrel{\pm\infty}{=} O(|b|^{\alpha - \frac{\alpha}{2}} e^{-\frac{\pi}{2}(\frac{2}{\alpha} - 1)|b|})$ , we get, by the Parseval identity, that  $\mathcal{J}_\alpha \in L^2(\mathbb{R}_+)$ . Moreover, since  $P^F$  coincides with its extension  $P$  on  $C_0(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ , we conclude from (4.4) that, for all  $q, t > 0$ ,  $d_q \mathcal{J}_\alpha$  is an eigenfunction of  $P_t$  with eigenvalue  $e^{-q^\alpha t}$ . This completes the proof of the first item. Next, with  $\widehat{\Lambda}_\alpha \in \mathbf{B}(L^2(\mathbb{R}_+))$  as the  $L^2(\mathbb{R}_+)$ -adjoint of  $\Lambda_\alpha \in \mathbf{B}(L^2(\mathbb{R}_+))$ , we have, for any  $f \in L^2(\mathbb{R}_+)$ ,

$$\|\mathcal{H}_\alpha f\|^2 = \int_0^\infty \langle f, \Lambda_\alpha d_q J_\alpha \rangle^2 dq = \|H_\alpha \widehat{\Lambda}_\alpha f\|^2 = \|\widehat{\Lambda}_\alpha f\|^2 \leq \frac{\Gamma^2(1 - \frac{1}{\alpha})}{\Gamma^2(1 + \frac{1}{\alpha})} \|f\|^2$$

where we used successively the identity (4.3), the definition as well as the unitary property of  $H_\alpha$ , see Proposition A.1, and the identity  $\|\widehat{\Lambda}_\alpha\| = \|\Lambda_\alpha\| \leq \frac{\Gamma(1 - \frac{1}{\alpha})}{\Gamma(1 + \frac{1}{\alpha})}$ , giving the second claim. Furthermore, for any  $k \in \mathbb{N}$ , by a classical argument since the first series below is easily checked to be uniformly convergent in  $z \in \mathbb{C}_{(-\infty, 0)^c}$ , we get

$$z^k \Gamma(1 + \frac{1}{\alpha}) \mathcal{J}_\alpha^{(k)}(z) = \sum_{n=0}^\infty \frac{(e^{i\pi} z^\alpha)^n}{\Gamma(\alpha n - k + 1)} = {}_1\Psi_1(e^{i\pi} z^\alpha) \approx z^{-\alpha} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{\Gamma(n+1) z^{-\alpha n}}{\Gamma(-\alpha n - \alpha - k + 1)},$$

where, for  $|\arg(z)| < \frac{\pi}{2}(2 - \alpha)$ , we used [30, Theorem 1], with the notation therein, that is,  ${}_1\Psi_1$  stands for the Wright function and we made the choice of parameters  $p = q = 1, \alpha_1 = 1, a_1 = 1, \beta_1 = \alpha, b_1 = -k + 1, \kappa = 1 + \beta_1 - \alpha_1 = \alpha \in (1, 2)$ . The proof is completed by an application of the reflection formula for the gamma function.  $\square$

## 5. CO-RESIDUAL FUNCTIONS

In this section, we focus on characterizing the spectrum of the  $L^2(\mathbb{R}_+)$ -adjoint operator  $\widehat{P} = (\widehat{P}_t)_{t \geq 0}$ . We point out that the non-self-adjointness of  $P_t$  does not ensure the existence of eigenfunctions for  $\widehat{P}_t$ . In fact, we shall show in the following Lemma that the point spectrum of  $\widehat{P}_t$  is empty and  $S_p(P_t) = S_r(\widehat{P}_t)$ , the residual spectrum of  $\widehat{P}_t$ .

**Lemma 5.1.** *For each  $t \geq 0$ ,  $S_p(\widehat{P}_t) = \emptyset$  and  $(e^{-q^\alpha t})_{q > 0} \subseteq S_r(\widehat{P}_t) = S_p(P_t)$ .*

*Proof.* Assume that there exists  $\mathfrak{z} \in S_p(\widehat{P}_t)$ , then there exists a non-zero function  $f_\mathfrak{z} \in L^2(\mathbb{R}_+)$  such that  $\widehat{P}_t f_\mathfrak{z} = \mathfrak{z} f_\mathfrak{z}$ . Moreover, since  $\Lambda_\alpha$  has a dense range in  $L^2(\mathbb{R}_+)$ , we see that  $\text{Ker}(\widehat{\Lambda}_\alpha) = \{0\}$  and therefore  $g_\mathfrak{z} = \widehat{\Lambda}_\alpha f_\mathfrak{z} \neq 0$  with  $g_\mathfrak{z} \in L^2(\mathbb{R}_+)$  as  $\widehat{\Lambda}_\alpha \in \mathbf{B}(L^2(\mathbb{R}_+))$ . Now by the adjoint intertwining relation of (3.4), we have

$$Q_t g_\mathfrak{z} = Q_t \widehat{\Lambda}_\alpha f_\mathfrak{z} = \widehat{\Lambda}_\alpha \widehat{P}_t f_\mathfrak{z} = \mathfrak{z} \widehat{\Lambda}_\alpha f_\mathfrak{z} = \mathfrak{z} g_\mathfrak{z},$$

which implies that  $\mathfrak{z} \in S_p(Q_t)$ , a contradiction to the fact that  $S_p(Q_t) = \emptyset$ . Therefore we have  $S_p(\widehat{P}_t) = \emptyset$  and moreover, from the known fact that  $S_r(\widehat{P}_t) \cup S_p(\widehat{P}_t) = S_p(P_t)$ , we conclude that  $(e^{-q^\alpha t})_{q > 0} \subseteq S_r(\widehat{P}_t)$ .  $\square$

Next, we will characterize a sequence of the so-called residual functions associated to  $S_r(\widehat{P}_t)$ , by means of (weak) Fourier kernels. To this end, we first recall from [34] that a linear operator  $\widehat{\mathcal{H}}$  is called a weak Fourier kernel if there exists a linear space  $\mathcal{D}(\widehat{\mathcal{H}})$  dense in  $L^2(\mathbb{R}_+)$  and  $\mathcal{M}_{\widehat{\mathcal{H}}} : \frac{1}{2} + i\mathbb{R} \rightarrow \mathbb{C}$  such that, for any  $f \in \mathcal{D}(\widehat{\mathcal{H}})$ ,

$$(5.1) \quad b \mapsto \mathcal{M}_{\widehat{\mathcal{H}}_f} \left( \frac{1}{2} + ib \right) = \mathcal{M}_{\widehat{\mathcal{H}}} \left( \frac{1}{2} + ib \right) \mathcal{M}_f \left( \frac{1}{2} - ib \right) \in L^2(\mathbb{R}).$$

**Theorem 5.1.** *Let us write, for  $s \in \mathbb{C}_{(0,1)}$ ,*

$$(5.2) \quad \mathcal{M}_{\widehat{\mathcal{H}}_\alpha}(s) = \frac{\Gamma\left(\frac{1}{\alpha}\right)\Gamma(s)}{\Gamma\left(\frac{1-s}{\alpha}\right)\Gamma\left(1 - \frac{1}{\alpha} + \frac{s}{\alpha}\right)}.$$

*Then the following statements hold.*

(1)  $\widehat{\mathcal{H}}_\alpha$  is a weak Fourier kernel and  $\mathcal{D} \subseteq \mathcal{D}(\widehat{\mathcal{H}}_\alpha)$ , where the linear space  $\mathcal{D}$  is defined by

$$(5.3) \quad \mathcal{D} = \left\{ f \in L^2(\mathbb{R}_+); \left| \mathcal{M}_f \left( \frac{1}{2} + ib \right) \right| \stackrel{\pm\infty}{\asymp} O\left(|b|^{-\frac{1}{2}-\epsilon} e^{-\frac{(2-\alpha)\pi}{2\alpha}|b|}\right) \text{ for some } \epsilon > 0 \right\}.$$

Moreover, we have  $\widehat{\mathcal{H}}_\alpha \mathcal{H}_\alpha f = f$  on  $L^2(\mathbb{R}_+)$  and  $\mathcal{H}_\alpha \widehat{\mathcal{H}}_\alpha g = g$  on  $\mathcal{D}(\widehat{\mathcal{H}}_\alpha)$ . Consequently,  $\widehat{\mathcal{H}}_\alpha$  is a self-adjoint operator on  $\mathcal{D}(\widehat{\mathcal{H}}_\alpha)$ .

(2) We have  $\text{Ran}(\Lambda_\alpha) \subseteq \mathcal{D}(\widehat{\mathcal{H}}_\alpha)$ , and, on  $L^2(\mathbb{R}_+)$ ,  $\widehat{\mathcal{H}}_\alpha \Lambda_\alpha = H_\alpha$  and  $\mathcal{H}_\alpha = \Lambda_\alpha H_\alpha$ .

(3) For any  $f \in L^2(\mathbb{R}_+)$  and  $t, q > 0$ , we have  $\widehat{\mathcal{H}}_\alpha P_t \Lambda_\alpha f(q) = e^{-q^\alpha t} \widehat{\mathcal{H}}_\alpha \Lambda_\alpha f(q)$ . Moreover, for any  $1 < \kappa < \frac{\alpha}{2-\alpha}$ ,  $\mathcal{E}_\kappa = \text{Span}(\mathbf{e}_{\kappa, \tau})_{\tau > 0}$  is a dense subset of  $L^2(\mathbb{R}_+)$ , and, for all  $f \in \mathcal{E}_\kappa$ , we have the integral representation, for almost every (a.e.)  $q > 0$ ,

$$(5.4) \quad \widehat{\mathcal{H}}_\alpha f(q) = \int_0^\infty f(x) \widehat{\mathcal{J}}_\alpha(qx) dx$$

where, for  $|\arg(z)| < \pi$ , we set, with  $\pi_\alpha = \frac{\pi}{\alpha}$ ,

$$(5.5) \quad \widehat{\mathcal{J}}_\alpha(z) = \frac{\Gamma\left(\frac{1}{\alpha}\right)}{\pi} \sin(\pi_\alpha - z \sin(\pi_\alpha)) e^{-z \cos(\pi_\alpha)}.$$

We say that  $d_q \widehat{\mathcal{J}}_\alpha$  is a residual function for  $\widehat{P}_t$  (or co-residual function for  $P_t$ ) associated to residual spectrum value  $e^{-q^\alpha t}$ .

*Proof.* First, since from (3.8) and  $0 < a < 1$  fixed,  $\left| \mathcal{M}_{\widehat{\mathcal{H}}_\alpha}(a + ib) \right| \stackrel{\pm\infty}{\asymp} O\left(|b|^{a-\frac{1}{2}} e^{-\frac{(2-\alpha)\pi}{2\alpha}|b|}\right)$ , we deduce from (5.3) and the fact that, for all  $b \in \mathbb{R}$ ,  $\left| \mathcal{M}_f\left(\frac{1}{2} - ib\right) \right| = \left| \mathcal{M}_f\left(\frac{1}{2} + ib\right) \right|$ , that

$$b \mapsto \left| \mathcal{M}_{\widehat{\mathcal{H}}_\alpha} \left( \frac{1}{2} + ib \right) \mathcal{M}_f \left( \frac{1}{2} - ib \right) \right| \stackrel{\pm\infty}{\asymp} O\left(|b|^{-\frac{1}{2}-\epsilon}\right) \in L^2(\mathbb{R}).$$

Therefore  $\mathcal{D} \subseteq \mathcal{D}(\widehat{\mathcal{H}}_\alpha)$ . Next, observing that for any  $1 < \kappa < \frac{\alpha}{2-\alpha}$ ,  $\tau > 0$  and  $s \in \mathbb{C}_{(0,\infty)}$ ,

$$(5.6) \quad \mathcal{M}_{\mathbf{e}_{\kappa, \tau}}(s) = \int_0^\infty x^{s-1} e^{-\tau x^\kappa} dx = \tau^{-\frac{s}{\kappa}} \kappa^{-1} \Gamma\left(\frac{s}{\kappa}\right),$$

we get  $\left| \mathcal{M}_{\mathbf{e}_{\kappa, \tau}}\left(\frac{1}{2} - ib\right) \right| \stackrel{\pm\infty}{\asymp} O\left(|b|^{\frac{1}{2\kappa}-\frac{1}{2}} e^{-\frac{\pi}{2\kappa}|b|}\right)$ , and thus, for any  $1 < \kappa < \frac{\alpha}{2-\alpha}$ ,  $(\mathbf{e}_{\kappa, \tau})_{\tau > 0} \subseteq \mathcal{D}$ . Moreover, since  $(\mathbf{e}_{\kappa, \tau})_{\tau > 0}$  is dense in  $L^2(\mathbb{R}_+)$ , we obtain that  $\mathcal{D}(\widehat{\mathcal{H}}_\alpha)$  is dense in  $L^2(\mathbb{R}_+)$  and



therefore  $\widehat{\mathcal{H}}_\alpha$  is a weak Fourier kernel. Next, using the definition of  $\mathcal{H}_\alpha f$  in (4.1), we get, by performing a change of variable in (3.1), that for any  $f \in L^2(\mathbb{R}_+)$  and  $s \in \frac{1}{2} + i\mathbb{R}$ ,

$$(5.7) \quad \mathcal{M}_{\mathcal{H}_\alpha f}(s) = \mathcal{M}_{\mathcal{J}_\alpha}(s) \mathcal{M}_f(1-s) = \frac{\Gamma(1-\frac{s}{\alpha}) \Gamma(\frac{s}{\alpha})}{\Gamma(\frac{1}{\alpha}) \Gamma(1-s)} \mathcal{M}_f(1-s).$$

Thus, for such  $s$ , we have

$$\mathcal{M}_{\widehat{\mathcal{H}}_\alpha}(s) \mathcal{M}_{\mathcal{H}_\alpha f}(1-s) = \frac{\Gamma(\frac{1}{\alpha}) \Gamma(s)}{\Gamma(\frac{1-s}{\alpha}) \Gamma(1-\frac{1}{\alpha}+\frac{s}{\alpha})} \mathcal{M}_{\mathcal{H}_\alpha f}(1-s) = \mathcal{M}_f(s).$$

Therefore, an application of the Parseval identity yields that  $\mathcal{H}_\alpha f \in \mathcal{D}(\widehat{\mathcal{H}}_\alpha)$  and  $\widehat{\mathcal{H}}_\alpha \mathcal{H}_\alpha f = f$  for all  $f \in L^2(\mathbb{R}_+)$ . Similarly, one gets that  $\mathcal{H}_\alpha \widehat{\mathcal{H}}_\alpha g = g$  for all  $g \in \mathcal{D}(\widehat{\mathcal{H}}_\alpha)$ . Next, from (4.1) one gets readily that  $\mathcal{H}_\alpha$  is self-adjoint in  $L^2(\mathbb{R}_+)$ , hence  $\widehat{\mathcal{H}}_\alpha$  is also self-adjoint as the inverse operator of  $\mathcal{H}_\alpha$ , which concludes the proof of Theorem 5.1(1). Next, from (3.1), we have, for any  $f \in L^2(\mathbb{R}_+)$ ,  $\mathcal{M}_{\Lambda_\alpha f}(s) = \mathcal{M}_f(s) \mathcal{M}_{\Lambda_\alpha}(1-s)$ , therefore for at least  $s \in \frac{1}{2} + i\mathbb{R}$ ,

$$\begin{aligned} \mathcal{M}_{\widehat{\mathcal{H}}_\alpha}(s) \mathcal{M}_{\Lambda_\alpha f}(1-s) &= \frac{\Gamma(\frac{1}{\alpha}) \Gamma(s)}{\Gamma(\frac{1-s}{\alpha}) \Gamma(1-\frac{1}{\alpha}+\frac{s}{\alpha})} \mathcal{M}_{\Lambda_\alpha}(s) \mathcal{M}_f(1-s) \\ &= \frac{\Gamma(\frac{s}{\alpha})}{\Gamma(\frac{1-s}{\alpha})} \mathcal{M}_f(1-s) = \mathcal{M}_{H_\alpha f}(s) \end{aligned}$$

where the Mellin transform of  $H_\alpha f$  is given in (A.6). Since from Proposition A.1,  $H_\alpha f \in L^2(\mathbb{R}_+)$ , we get, by the Parseval identity, that  $\Lambda_\alpha f \in \mathcal{D}(\widehat{\mathcal{H}}_\alpha)$  and  $\widehat{\mathcal{H}}_\alpha \Lambda_\alpha f = H_\alpha f$  for any  $f \in L^2(\mathbb{R}_+)$ . Combine this relation with the self-inverse property of  $H_\alpha$  from Proposition A.1, we have, for any  $f \in L^2(\mathbb{R}_+)$ , that  $\widehat{\mathcal{H}}_\alpha \Lambda_\alpha H_\alpha f = H_\alpha H_\alpha f = f$ , which implies  $\mathcal{H}_\alpha = \Lambda_\alpha H_\alpha$  and finishes the proof of Theorem 5.1(2). Next, combining the intertwining relation (3.4) and the spectral expansion (A.8) for  $Q_t$ , we get that, for any  $f \in L^2(\mathbb{R}_+)$ ,  $t > 0$ ,

$$(5.8) \quad P_t \Lambda_\alpha f = \Lambda_\alpha Q_t f = \Lambda_\alpha H_\alpha \mathbf{e}_{\alpha,t} H_\alpha f = \mathcal{H}_\alpha \mathbf{e}_{\alpha,t} H_\alpha f.$$

Hence, by observing that  $P_t \Lambda_\alpha f \in \text{Ran}(\mathcal{H}_\alpha)$  and by means of Theorem 5.1(1), we get

$$\widehat{\mathcal{H}}_\alpha P_t \Lambda_\alpha f = \widehat{\mathcal{H}}_\alpha \mathcal{H}_\alpha \mathbf{e}_{\alpha,t} H_\alpha f = \mathbf{e}_{\alpha,t} H_\alpha f = \mathbf{e}_{\alpha,t} \widehat{\mathcal{H}}_\alpha \Lambda_\alpha f.$$

Finally, since, from above, we have, for any  $1 < \kappa < \frac{\alpha}{2-\alpha}$ ,  $(\mathbf{e}_{\kappa,\tau})_{\tau>0} \subseteq \mathcal{D}$ , we get that  $\widehat{\mathcal{H}}_\alpha \mathbf{e}_{\kappa,\tau} \in L^2(\mathbb{R}_+)$  and by combining (5.2) with (5.6), that, for at least  $s \in \frac{1}{2} + i\mathbb{R}$ ,

$$\mathcal{M}_{\widehat{\mathcal{H}}_\alpha \mathbf{e}_{\kappa,\tau}}(s) = \frac{\tau^{\frac{s-1}{\kappa}} \Gamma(\frac{1}{\alpha}) \Gamma(s) \Gamma(\frac{1-s}{\kappa})}{\kappa \Gamma(1-\frac{1}{\alpha}+\frac{s}{\alpha}) \Gamma(\frac{1-s}{\alpha})}.$$

By following a line of reasoning similar to the one used in the proof of Lemma 3.1, we obtain

$$\widehat{\mathcal{H}}_\alpha \mathbf{e}_{\kappa,\tau}(q) = \frac{\Gamma(\frac{1}{\alpha})}{\kappa} \sum_{n=0}^{\infty} \frac{(-1)^n \tau^{-\frac{n+1}{\kappa}} \Gamma(\frac{n+1}{\kappa})}{n! \Gamma(1-\frac{1}{\alpha}-\frac{n}{\alpha}) \Gamma(\frac{n+1}{\alpha})} q^n = \frac{\Gamma(\frac{1}{\alpha})}{\kappa \pi} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\frac{n+1}{\kappa}) \sin((n+1)\pi\alpha)}{\tau^{\frac{n+1}{\kappa}} n!} q^n,$$

which defines an entire function since, by the Stirling approximation (3.7),  $\frac{\Gamma(\frac{n+1}{\kappa})}{n! \Gamma(\frac{n}{\kappa})} \cong O\left(n^{\frac{1}{\kappa}-1}\right)$  and  $\kappa > 1$ . On the other hand, since for any  $x > 0$ ,

$$\frac{\pi}{\Gamma(\frac{1}{\alpha})} \widehat{\mathcal{J}}_\alpha(x) = \cos(\pi\alpha) \Im\left(e^{-xe^{i\pi\alpha}}\right) + \sin(\pi\alpha) \Re\left(e^{-xe^{i\pi\alpha}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sin((n+1)\pi\alpha) x^n,$$

a standard application of Fubini's theorem, see again [44, Section 1.77], yields

$$\begin{aligned} \int_0^\infty \mathbf{e}_{\kappa,\tau}(x) \widehat{\mathcal{J}}_\alpha(qx) dx &= \frac{\Gamma\left(\frac{1}{\alpha}\right)}{\pi} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \sin((n+1)\pi_\alpha) q^n \int_0^\infty e^{-\tau x^\kappa} x^n dx \\ &= \frac{\Gamma\left(\frac{1}{\alpha}\right)}{\kappa\pi} \sum_{n=0}^\infty \frac{(-1)^n \Gamma\left(\frac{n+1}{\kappa}\right) \sin((n+1)\pi_\alpha)}{\tau^{\frac{n+1}{\kappa}} n!} q^n = \widehat{\mathcal{H}}_\alpha \mathbf{e}_{\kappa,\tau}(q), \end{aligned}$$

from which we conclude that  $\widehat{\mathcal{H}}_\alpha f(q) = \int_0^\infty f(x) \widehat{\mathcal{J}}_\alpha(qx) dx$  for all  $f \in \mathcal{E}_\kappa$ . This completes the proof.  $\square$

## 6. SPECTRAL REPRESENTATION, HEAT KERNEL AND SMOOTHNESS PROPERTIES

We have now all the ingredients for stating and proving the spectral representation of the semigroups  $P$  and  $\widehat{P}$  along with the representation of the heat kernel.

### 6.1. Spectral expansions of $P$ and $\widehat{P}$ in Hilbert spaces and the heat kernel.

**Theorem 6.1.** (1) For any  $g \in L^2(\mathbb{R}_+)$  and  $t > 0$ , we have in  $L^2(\mathbb{R}_+)$

$$(6.1) \quad \widehat{P}_t g = \widehat{\mathcal{H}}_\alpha \mathbf{e}_{\alpha,t} \mathcal{H}_\alpha g.$$

(2) The heat kernel of  $P$  admits the representation

$$(6.2) \quad P_t(x, y) = \int_0^\infty e^{-q^\alpha t} \mathcal{J}_\alpha(qx) \widehat{\mathcal{J}}_\alpha(qy) dq,$$

where the integral is locally uniformly convergent in  $(t, x, y) \in \mathbb{R}_+^3$ .

(3) For any  $t > T_\alpha$ , we have in  $L^2(\mathbb{R}_+)$ ,

$$(6.3) \quad P_t f = \mathcal{H}_\alpha \mathbf{e}_{\alpha,t} \widehat{\mathcal{H}}_\alpha f$$

where

(a) if  $f \in \mathcal{D}(\widehat{\mathcal{H}}_\alpha)$  then  $T_\alpha = 0$ ,

(b) otherwise if  $f \in L^2(\bar{\mathbf{e}}_{\kappa,\eta})$  for some  $\kappa \geq \frac{\alpha}{\alpha-1}$  and  $\eta > 0$ , where we set  $\bar{\mathbf{e}}_{\kappa,\eta}(x) = e^{\eta x^\kappa}$ ,  $x > 0$ , then  $T_\alpha = \frac{\eta}{\alpha-1} \left(2 \frac{\alpha-1}{\alpha\eta} \cos((\alpha+1)\pi_\alpha)\right)^\alpha \mathbb{I}_{\{\kappa(\alpha-1)=\alpha\}}$  and  $\widehat{\mathcal{H}}_\alpha f(q) = \int_0^\infty f(y) \widehat{\mathcal{J}}_\alpha(qy) dy$ .

**Remark 6.1.** We mention that  $\mathcal{D}(\widehat{\mathcal{H}}_\alpha) \setminus L^2(\bar{\mathbf{e}}_{\kappa,\eta}) \neq \emptyset$  meaning that the two conditions (3a) and (3b) are applicable under different situations. For instance, for  $0 < \beta < \min\left(\frac{\alpha}{2-\alpha}, \frac{\alpha}{\alpha-1}\right)$ ,  $\mathbf{e}_\beta \in \text{Ran}(\Lambda_\alpha) \setminus L^2(\bar{\mathbf{e}}_{\kappa,\eta})$ , as one can show that  $\Lambda_\alpha B_\beta = \mathbf{e}_\beta$  with  $x \mapsto B_\beta(x) = \sum_{n=0}^\infty \frac{(-1)^n}{\mathcal{M}_{\Lambda_\alpha}(\beta n+1)n!} x^{\beta n} \in L^2(\mathbb{R}_+)$ .

*Proof.* First, since for any  $f \in L^2(\mathbb{R}_+)$ ,  $\widehat{P}_t f \in L^2(\mathbb{R}_+)$ , we get, for all  $q > 0$ ,

$$\mathcal{H}_\alpha \widehat{P}_t f(q) = \left\langle \widehat{P}_t f, d_q \mathcal{J}_\alpha \right\rangle = \langle f, P_t d_q \mathcal{J}_\alpha \rangle = e^{-q^\alpha t} \langle f, d_q \mathcal{J}_\alpha \rangle = e^{-q^\alpha t} \mathcal{H}_\alpha f(q)$$

where we used Theorem 4.1(1) for  $d_q \mathcal{J}_\alpha \in L^2(\mathbb{R}_+)$  and for the third identity. Therefore, we can apply Theorem 5.1(1) to get that for any  $g \in L^2(\mathbb{R}_+)$ ,

$$\widehat{P}_t g = \widehat{\mathcal{H}}_\alpha \mathcal{H}_\alpha \widehat{P}_t g = \widehat{\mathcal{H}}_\alpha \mathbf{e}_{\alpha,t} \mathcal{H}_\alpha g,$$

which proves (6.1). On the other hand, for any  $f \in \mathcal{D}(\widehat{\mathcal{H}}_\alpha), g \in L^2(\mathbb{R}_+)$ , we have, using the self-adjoint property of  $\widehat{\mathcal{H}}_\alpha$  and  $\mathcal{H}_\alpha$ , see Theorem 5.1(1),

$$\langle P_t f, g \rangle = \langle f, \widehat{P}_t g \rangle = \langle f, \widehat{\mathcal{H}}_\alpha \mathbf{e}_{\alpha,t} \mathcal{H}_\alpha g \rangle = \langle \mathcal{H}_\alpha \mathbf{e}_{\alpha,t} \widehat{\mathcal{H}}_\alpha f, g \rangle,$$

which proves (6.3) for  $f \in \mathcal{D}(\widehat{\mathcal{H}}_\alpha)$  and  $T_\alpha = 0$ , that is, the claim (3a). Next, let us consider the density function  $\lambda_{\mathbf{X}_\alpha} \in L^2(\mathbb{R}_+)$  of the random variable  $\mathbf{X}_\alpha$ , which we recall was studied in Lemma 3.2. Then using (5.7) again, it is easy to deduce that  $\mathcal{M}_{\mathcal{H}_\alpha \lambda_{\mathbf{X}_\alpha}}(s) = \frac{\Gamma(\frac{s}{\alpha})}{\Gamma(\frac{1}{\alpha})}$ , which coincides with the Mellin transform of  $G_\alpha$  (see (A.3)). Hence we have, for all  $q > 0$ , that  $\mathcal{H}_\alpha \lambda_{\mathbf{X}_\alpha}(q) = \lambda_{G_\alpha}(q) = \frac{e^{-q^\alpha}}{\Gamma(1+\frac{1}{\alpha})}$ . Therefore, we see that for any  $\tau > 0, q \mapsto \mathbf{e}_{\alpha,t} \mathcal{H}_\alpha d_\tau \lambda_{\mathbf{X}_\alpha}(q) = \frac{e^{-(\tau^{-\alpha}+t)q^\alpha}}{\tau \Gamma(1+\frac{1}{\alpha})} \in \mathcal{E}_\alpha$ . Hence, using Theorem 5.1(3), we can write

$$(6.4) \quad \widehat{P}_t d_\tau \lambda_{\mathbf{X}_\alpha}(y) = \widehat{\mathcal{H}}_\alpha \mathbf{e}_{\alpha,t} \mathcal{H}_\alpha d_\tau \lambda_{\mathbf{X}_\alpha}(y) = \int_0^\infty e^{-q^{\alpha t}} \widehat{\mathcal{J}}_\alpha(qy) \int_0^\infty \lambda_{\mathbf{X}_\alpha}(\tau x) \mathcal{J}_\alpha(qx) dx dq.$$

Next, from (4.2) we deduce that  $|\mathcal{J}_\alpha(x)| \stackrel{0}{=} O(1)$  and  $|\mathcal{J}_\alpha(x)| \stackrel{\infty}{=} O(x^{-\alpha})$  and thus, since  $\lambda_{\mathbf{X}_\alpha}$  is a probability density function,  $\int_0^\infty \lambda_{\mathbf{X}_\alpha}(\tau x) |\mathcal{J}_\alpha(qx)| dx \leq C(1 + q^{-\alpha})$  for some  $C = C(\tau) > 0$ . On the other hand, from (5.5), we get that there exists  $\widehat{C} > 0$  such that for all  $y > 0, |\widehat{\mathcal{J}}_\alpha(y)| \leq \widehat{C} e^y$ , which justifies an application of Fubini theorem to obtain

$$(6.5) \quad \int_0^\infty e^{-q^{\alpha t}} \widehat{\mathcal{J}}_\alpha(qy) \int_0^\infty \lambda_{\mathbf{X}_\alpha}(\tau x) \mathcal{J}_\alpha(qx) dx dq = \int_0^\infty \lambda_{\mathbf{X}_\alpha}(\tau x) \int_0^\infty e^{-q^{\alpha t}} \mathcal{J}_\alpha(qx) \widehat{\mathcal{J}}_\alpha(qy) dq dx.$$

Now let us define the Mellin convolution operator  $\mathcal{X}$  by  $\mathcal{X}f(\tau) = \int_0^\infty f(y) \lambda_{\mathbf{X}_\alpha}(\tau y) dy$  and, since  $\lambda_{\mathbf{X}_\alpha} \in L^2(\mathbb{R}_+)$ ,  $\mathcal{X} \in \mathbf{B}(L^2(\mathbb{R}_+))$  and by performing a change of variable in (3.1) we get, from (3.15),  $\mathcal{M}_{\mathcal{X}}(s) = \mathcal{M}_{\lambda_{\mathbf{X}_\alpha}}(s) = \frac{\Gamma(s)}{\Gamma(\frac{s}{\alpha} + 1 - \frac{1}{\alpha})}$  which is clearly zero-free on  $\Re(s) = 1$  entailing that  $\mathcal{X}$  is one-to-one in  $L^2(\mathbb{R}_+)$ . Moreover, by means of the same upper bounds used above, we deduce that for any  $y$  fixed,

$$x \mapsto \int_0^\infty e^{-q^{\alpha t}} \mathcal{J}_\alpha(qx) \widehat{\mathcal{J}}_\alpha(qy) dq \in L^2(\mathbb{R}_+)$$

and thus the right-hand side of (6.5) is in  $L^2(\mathbb{R}_+)$  and hence from (6.4), we get that, for any  $\tau > 0$ ,

$$\widehat{P}_t d_\tau \lambda_{\mathbf{X}_\alpha}(y) = \int_0^\infty \lambda_{\mathbf{X}_\alpha}(\tau x) \int_0^\infty e^{-q^{\alpha t}} \mathcal{J}_\alpha(qx) \widehat{\mathcal{J}}_\alpha(qy) dq dx.$$

The one-to-one property of  $\mathcal{X}$  implies that the transition kernel of  $\widehat{P}_t$ , denoted by  $\widehat{P}_t(y, x)$ , can be represented, for a.e.  $y > 0$ , as  $\widehat{P}_t(y, x) = \int_0^\infty e^{-q^{\alpha t}} \mathcal{J}_\alpha(qx) \widehat{\mathcal{J}}_\alpha(qy) dq$ . Since the last integral is also locally uniformly convergent for any  $(t, x, y) \in \mathbb{R}_+^3$ , and  $\widehat{\mathcal{J}}_\alpha$  is continuous, the identity holds everywhere. This last fact combined with the duality stated in Proposition 2.1(3) yield the expression (6.2) by recalling, from Proposition 2.1(3), that since the Lebesgue measure serves as reference measure we get that  $P_t(x, y) = \widehat{P}_t(y, x), t, x, y > 0$ . While (3a) has been proved above,

we now proceed to the justification of (3b). First, by the Cauchy-Schwarz inequality, observe that for any  $f \in L^2(\bar{\mathbf{e}}_{\kappa,\eta})$ , writing  $\widehat{\mathcal{J}}_{\bar{\mathbf{e}}}(qy) = \frac{\widehat{\mathcal{J}}_{\alpha}(qy)}{\bar{\mathbf{e}}_{\kappa,\eta}(y)}$ , we have

$$\int_0^\infty f(y)\widehat{\mathcal{J}}_{\alpha}(qy)dy = \int_0^\infty f(y)\widehat{\mathcal{J}}_{\bar{\mathbf{e}}}(qy)\bar{\mathbf{e}}_{\kappa,\eta}(y)dy \leq \|f\|_{\bar{\mathbf{e}}_{\kappa,\eta}} \left\| d_q \widehat{\mathcal{J}}_{\bar{\mathbf{e}}} \right\|_{\bar{\mathbf{e}}_{\kappa,\eta}}.$$

Moreover, since for all  $y > 0$ ,  $\left| \widehat{\mathcal{J}}_{\alpha}(y) \right| \leq C e^{-y \cos(\pi\alpha)}$ ,  $C > 0$ , we have by an application of the Laplace method, see e.g. [29, Ex.7.3 p.84], that for large  $q$ ,

$$\left\| d_q \widehat{\mathcal{J}}_{\bar{\mathbf{e}}} \right\|_{\bar{\mathbf{e}}_{\kappa,\eta}}^2 \leq C^2 \int_0^\infty e^{-\eta y^\kappa} e^{-2q \cos(\pi\alpha)y} dy \cong O\left(q^a e^{c_\kappa q^{\frac{\kappa}{\kappa-1}}}\right),$$

where  $a > 0$  and we set  $c_\kappa = (\kappa - 1)\eta^{\frac{1}{1-\kappa}} \left(\frac{2 \cos((\alpha+1)\pi\alpha)}{\kappa}\right)^{\frac{\kappa}{\kappa-1}} > 0$  since  $\kappa > \alpha > 1$ . Note that  $c_{\frac{\alpha}{\alpha-1}} = T_\alpha$  and for any  $t > T_\alpha$ , since  $\alpha \geq \frac{\kappa}{\kappa-1}$ ,  $q \mapsto F_\kappa(q) = \mathbf{e}_{\alpha,t}(q) \left(C + q^a e^{c_\kappa q^{\frac{\kappa}{\kappa-1}}}\right)$  is integrable on  $\mathbb{R}_+$ . This justifies an application of Fubini Theorem which gives that, for such  $f, t$  and  $x > 0$ ,

$$(6.6) \quad P_t f(x) = \int_0^\infty f(y) \int_0^\infty e^{-q^\alpha y} \mathcal{J}_\alpha(qx) \widehat{\mathcal{J}}_\alpha(qy) dq dy.$$

Finally, as  $F_\kappa \in L^2(\mathbb{R}_+)$  and from Theorem 6.1, the sequence  $(d_q \mathcal{J}_\alpha)_{q>0}$  is an upper frame, we obtain that in fact  $P_t f \in L^2(\mathbb{R}_+)$  and, in  $L^2(\mathbb{R}_+)$ ,  $P_t f = \mathcal{H}_\alpha \mathbf{e}_{\alpha,t} \widehat{\mathcal{H}}_\alpha f$  with  $\widehat{\mathcal{H}}_\alpha f(q) = \int_0^\infty f(y) \widehat{\mathcal{J}}_\alpha(qy) dy$ . This completes the proof.  $\square$

**6.2. Regularity properties.** Finally, we extract from the spectral decomposition stated in Theorem 6.1 the following regularity properties as well as an alternative representation of the heat kernel.

**Theorem 6.2.** (1) For any  $f \in L^2(\bar{\mathbf{e}}_{\kappa,\eta}) \cup \mathcal{D}(\widehat{\mathcal{H}}_\alpha)$ ,  $(t, x) \mapsto P_t f(x) \in C^\infty((T_\alpha, \infty) \times \mathbb{R}_+)$  and  $T_\alpha$  was defined in Theorem 6.1.

(2) We have  $(t, x, y) \mapsto P_t(x, y) \in C^\infty(\mathbb{R}_+^3)$  and, for any non-negative integers  $k, \mathbf{p}, \mathbf{q}$ ,

$$(6.7) \quad \frac{d^k}{dt^k} P_t^{(\mathbf{p}, \mathbf{q})}(x, y) = (-1)^k \int_0^\infty q^{\alpha k} e^{-q^\alpha t} (d_q \mathcal{J}_\alpha)^{(\mathbf{p})}(x) \left(d_q \widehat{\mathcal{J}}_\alpha\right)^{(\mathbf{q})}(y) dq$$

where the integral is locally uniformly convergent in  $(t, x, y) \in \mathbb{R}_+^3$ .

(3) Moreover, the heat kernel can be written in a series form as

$$(6.8) \quad P_t(x, y) = \sum_{n=0}^\infty (1+t)^{-n-\frac{1}{\alpha}} \mathcal{P}_n(x^\alpha) \mathcal{V}_n\left(y(1+t)^{-\frac{1}{\alpha}}\right),$$

where  $\mathcal{P}_n(x) = \frac{1}{\Gamma(1+\frac{1}{\alpha})} \sum_{k=0}^n (-1)^k \frac{\binom{n}{k} k!}{\Gamma(\alpha k + 1)} x^k$  and  $\mathcal{V}_n(y) = \frac{1}{n!} \int_0^\infty q^{\alpha n} e^{-q^\alpha} \widehat{\mathcal{J}}_\alpha(qy) dq$  and the series is locally uniformly convergent in  $(t, x, y) \in \mathbb{R}_+^3$ .

*Proof.* We actually prove only the item (2) as the first item follows by developing similar arguments. First, from Theorem 4.1 and Theorem 5.1, we have that  $\mathcal{J}_\alpha, \widehat{\mathcal{J}}_\alpha \in C^\infty(\mathbb{R}_+)$  and for any  $x, y > 0$  fixed and non-negative integers  $k, \mathbf{p}, \mathbf{q}$ ,

$$\left| \frac{d^k}{dt^k} e^{-q^\alpha t} (d_q \mathcal{J}_\alpha)^{(\mathbf{p})}(x) (d_q \widehat{\mathcal{J}}_\alpha)^{(\mathbf{q})}(y) \right| \cong O\left(q^{\alpha(k-1)+\mathbf{q}} e^{-q^\alpha t + qy}\right)$$

and

$$\left| \frac{d^k}{dt^k} e^{-q^\alpha t} (d_q \mathcal{J}_\alpha)^{(p)}(x) (d_q \widehat{\mathcal{J}}_\alpha)^{(q)}(y) \right| \stackrel{0}{=} O\left(q^{\alpha k + p + q}\right).$$

Hence (6.2) yields

$$\frac{d^k}{dt^k} P_t^{(p,q)}(x, y) = (-1)^k \int_0^\infty q^{\alpha k} e^{-q^\alpha t} (d_q \mathcal{J}_\alpha)^{(p)}(x) (d_q \widehat{\mathcal{J}}_\alpha)^{(q)}(y) dq$$

where the integral is locally uniformly convergent in  $(t, x, y) \in \mathbb{R}_+^3$ . Hence  $(t, x, y) \mapsto P_t(x, y) \in C^\infty(\mathbb{R}_+^3)$ . To prove (6.8), we first observe, from [13, Proposition 2.1(ii)], that for any  $x, q \in \mathbb{R}_+$ ,

$$(6.9) \quad e^{q^\alpha} \mathcal{J}_\alpha(qx) = \sum_{n=0}^\infty \mathcal{P}_n(x^\alpha) \frac{q^{\alpha n}}{n!},$$

which by substitution in (6.2) gives, assuming, for a moment, that one may interchange the sum and integral,

$$P_t(x, y) = \int_0^\infty e^{-q^\alpha(t+1)} \widehat{\mathcal{J}}_\alpha(qy) \sum_{n=0}^\infty \frac{\mathcal{P}_n(x^\alpha)}{n!} q^{\alpha n} dq = \sum_{n=0}^\infty (1+t)^{-n-\frac{1}{\alpha}} \mathcal{P}_n(x^\alpha) \mathcal{V}_n\left(y(1+t)^{-\frac{1}{\alpha}}\right).$$

In order to justify the interchange we provide some uniform bounds for large  $n$  of  $\mathcal{P}_n$  and  $\mathcal{V}_n$ . First, since  $z \mapsto \mathcal{J}_\alpha(z^{\frac{1}{\alpha}})$  is an entire function of order  $\overline{\lim}_{n \rightarrow \infty} \frac{n \ln n}{\Gamma(\alpha n + 1)} = \frac{1}{\alpha}$  and type 1, by following a line of reasoning similar to the proof of [33, Theorem 8.4(5)], we obtain the following sequence of inequalities, valid for all  $x > 0$  and  $n$  large,

$$\begin{aligned} |\mathcal{P}_n(x)| \leq \mathcal{P}_n(-x) &= \frac{n!}{2\pi i} x^n \oint_{nx} e^{\frac{z}{x}} \mathcal{J}_\alpha\left(-z^{\frac{1}{\alpha}}\right) \frac{dz}{z^{n+1}} \\ &\leq e^{n^{\frac{1}{\alpha}} x^{\frac{1}{\alpha}}} \frac{n! e^{-n \ln n}}{2\pi} \int_0^{2\pi} e^{n \cos \theta} d\theta = O\left(n^{\frac{1}{2}} e^{(xn)^{\frac{1}{\alpha}}}\right) \end{aligned}$$

where the contour is a circle centered at 0 with radius  $nx > 0$  and for the last inequality we used the bound  $n! \leq e^{1-n} n^{n-\frac{1}{2}}$ . Hence, we have, for all fixed  $x > 0$  and  $n$  large,

$$(6.10) \quad |\mathcal{P}_n(x^\alpha)| = O\left(n^{\frac{1}{2}} e^{xn^{\frac{1}{\alpha}}}\right).$$

Next, since for any  $q > 0$ ,  $|\widehat{\mathcal{J}}_\alpha(q)| \leq \widehat{\mathcal{J}}_\alpha(-q) \leq C e^q$ , for some constant  $C = C(\alpha) > 0$ , we get, for all  $y > 0$  and  $n \in \mathbb{N}$ ,

$$|\mathcal{V}_n(y)| \leq \frac{C}{n!} \int_0^\infty e^{-q^\alpha} q^{\alpha n} \sum_{k=0}^\infty \frac{(yq)^k}{k!} dq = C \sum_{k=0}^\infty \frac{\Gamma(n+1 + \frac{k}{\alpha})}{n! k!} y^k$$

where for the equality we use the integral representation of the gamma function. Now, by performing the same computations that in the proof of [35, Proposition 2.2], we get, for all  $y > 0$  and  $n$  large,

$$(6.11) \quad |\mathcal{V}_n(y)| = O\left(n e^{\bar{c}_\alpha y n^{\frac{1}{\alpha}}}\right)$$

where  $\bar{c}_\alpha = \frac{\alpha}{\alpha-1}$ . Hence combining the bounds (6.10) and (6.11), we obtain, for any fixed  $x, y, t > 0$  and large  $n$ ,

$$(6.12) \quad (1+t)^{-n-\frac{1}{\alpha}} \left| \mathcal{P}_n(x^\alpha) \mathcal{V}_n\left(y(1+t)^{-\frac{1}{\alpha}}\right) \right| = O\left(n^{\frac{3}{2}} e^{(\bar{c}_\alpha y (1+t)^{-\frac{1}{\alpha}} + x) n^{\frac{1}{\alpha}} - \ln(1+t)n}\right),$$

which justifies the interchange and completes the proof.  $\square$

APPENDIX A. THE  $\alpha$ -BESSEL SEMIGROUP AND THE OPERATOR  $H_\alpha$

We say  $Q = (Q_t)_{t \geq 0}$  is an  $\alpha$ -Bessel semigroup with index  $1 < \alpha < 2$  if it is a Feller semigroup whose infinitesimal generator is given by

$$(A.1) \quad \mathbf{L}f(x) = \frac{2}{\alpha^2} x^{2-\alpha} f^{(2)}(x) + \frac{2}{\alpha} \left( \frac{2}{\alpha} - 1 \right) x^{1-\alpha} f^{(1)}(x), \quad x > 0,$$

where  $f \in \mathcal{D}_{\mathbf{L}} = \{f \in C_0(\mathbb{R}_+); \mathbf{L}f \in C_0(\mathbb{R}_+), f^+(0) = 0\}$ , the domain of  $\mathbf{L}$ , with  $f^+(x) = \lim_{h \downarrow 0} \frac{f(x+h) - f(x)}{s(x+h) - s(x)}$  is the right-derivative of  $f$  with respect to the scale function  $s(x) = \frac{x^{\alpha-1}}{\alpha-1}$ . We point out that

$$(A.2) \quad Q_t f(x) = K_t \mathbf{p}_{\frac{1}{\alpha}} f(x^\alpha), \quad x > 0,$$

where  $K = (K_t)_{t \geq 0}$  is the semigroup of a squared Bessel process of dimension  $\frac{2}{\alpha}$ , or equivalently of order  $\frac{1}{\alpha} - 1$  and  $\mathbf{p}_{\frac{1}{\alpha}} f(x) = f(x^{\frac{1}{\alpha}})$ . We refer in this part to [10, Appendix 1] for concise information on squared Bessel processes that can be easily transferred to  $Q$  by means of the identity (A.2). Furthermore, writing  $(\vartheta_t)_{t > 0}$  for the entrance law of  $Q$ , we have  $\vartheta_t f = \int_0^\infty f(ty) \lambda_{G_\alpha}(y) dy$  where  $\lambda_{G_\alpha}(y) = \frac{e^{-y^\alpha}}{\Gamma(1 + \frac{1}{\alpha})}$ ,  $y > 0$ , is the density of the variable  $G_\alpha$ . Note that  $G_\alpha^\alpha$  is simply a gamma variable of parameter  $\frac{1}{\alpha}$ , the law of this latter being the entrance law at time 1 of  $K$ . The Mellin transform of  $G_\alpha$  is given by

$$(A.3) \quad \mathcal{M}_{G_\alpha}(s) = \frac{\Gamma(\frac{s}{\alpha})}{\Gamma(\frac{1}{\alpha})}, \quad \Re(s) > 0.$$

Next, defining the function  $J_\alpha$ , for  $z \in \mathbb{C}_{(-\infty, 0)^c}$ , by

$$(A.4) \quad J_\alpha(z) = \alpha \sum_{n=0}^{\infty} \frac{(e^{i\pi} z^\alpha)^n}{n! \Gamma(n + \frac{1}{\alpha})},$$

we can deduce from [10, Appendix 1] that for any  $q, t, x \geq 0$ ,

$$Q_t d_q J_\alpha(x) = e^{-q^\alpha t} d_q J_\alpha(x).$$

Next, we introduce the linear operator defined, for a smooth function  $f$  on  $q > 0$ , by

$$(A.5) \quad H_\alpha f(q) = \int_0^\infty J_\alpha(qx) f(x) dx.$$

Then,  $H_\alpha$  has the following properties reminiscent of the classical Hankel transform.

**Proposition A.1.**  *$H_\alpha$  is a unitary and self-inverse operator on  $L^2(\mathbb{R}_+)$ , i.e.  $\|H_\alpha f\| = \|f\|$  and  $H_\alpha H_\alpha f = f$  for all  $f \in L^2(\mathbb{R}_+)$ . Moreover, for any  $f \in L^2(\mathbb{R}_+)$ , the Mellin transform of  $H_\alpha f$  is given by*

$$(A.6) \quad \mathcal{M}_{H_\alpha f}(s) = \mathcal{M}_{J_\alpha}(s) \mathcal{M}_f(1-s) = \frac{\Gamma(\frac{s}{\alpha})}{\Gamma(\frac{1-s}{\alpha})} \mathcal{M}_f(1-s), \quad s \in \mathbb{C}_{(0,1)}.$$

*Proof.* First, note that  $J_\alpha(x) = \alpha x^{\frac{\alpha-1}{2}} J_{\frac{1}{\alpha}-1} \left( 2x^{\frac{\alpha}{2}} \right)$  where  $J_{\frac{1}{\alpha}-1}$  denotes the standard Bessel function of the first kind of order  $\frac{1}{\alpha} - 1$ , see e.g. [24, Section 5.3]. Then recall that the standard Hankel transform is defined, for any  $g \in L^2(m)$  where  $m(dx) = x dx$ , as

$$H_\alpha g(r) = \int_0^\infty J_{\frac{1}{\alpha}-1}(rx) g(x) x dx, \quad r > 0.$$

Then by [37, Chapter 9],  $H_\alpha$  is unitary and self-inverse on  $L^2(m)$ , i.e. for any  $g \in L^2(m)$ , we have  $\|H_\alpha g\|_m = \|g\|_m$  and  $H_\alpha H_\alpha g = g$ . Now for any  $f \in L^2(\mathbb{R}_+)$ , we set  $g(x) = x^{\frac{1}{\alpha}-1} f\left(\left(\frac{x}{2}\right)^{\frac{2}{\alpha}}\right)$ . Then it can be easily checked, through a standard change of variable, that  $g \in L^2(m)$  and  $\|g\|_m^2 = \alpha 2^{\frac{2}{\alpha}-1} \|f\|^2$ . Therefore, by applying a change of variable, one gets

$$\begin{aligned} \|H_\alpha f\|^2 &= \int_0^\infty \left| \int_0^\infty f(x) J_\alpha(qx) dx \right|^2 dq = \frac{1}{2^{\frac{2}{\alpha}}} \int_0^\infty \left| \int_0^\infty J_{\frac{1}{\alpha}-1}(q^{\frac{\alpha}{2}} y) g(y) y dy \right|^2 \frac{dq}{q^{1-\alpha}} \\ &= \frac{1}{2^{\frac{2}{\alpha}}} \int_0^\infty q^{\alpha-1} \left| H_\alpha g(q^{\frac{\alpha}{2}}) \right|^2 dq = \frac{1}{\alpha 2^{\frac{2}{\alpha}-1}} \|H_\alpha g\|_m^2 = \frac{1}{\alpha 2^{\frac{2}{\alpha}-1}} \|g\|_m^2 = \|f\|^2. \end{aligned}$$

This proves that  $H_\alpha$  is a unitary operator. Next, for any  $f \in L^2(\mathbb{R}_+)$ , again by change of variable, we have

$$H_\alpha H_\alpha f(y) = \int_0^\infty J_\alpha(qy) \int_0^\infty f(x) J_\alpha(qx) dx dq = \frac{y^{\frac{\alpha-1}{2}}}{2^{\frac{1}{\alpha}-1}} H_\alpha H_\alpha g(2y^{\frac{\alpha}{2}}) = \frac{y^{\frac{\alpha-1}{2}}}{2^{\frac{1}{\alpha}-1}} g(2y^{\frac{\alpha}{2}}) = f(y),$$

which proves that  $H_\alpha$  is self-inverse. Next, using again a change of variable in (3.1), we have  $\mathcal{M}_{H_\alpha f}(s) = \mathcal{M}_{J_\alpha}(s) \mathcal{M}_f(1-s)$ , where for  $0 < \Re(s) < 1$ ,  $\mathcal{M}_{J_\alpha}(s) = \frac{\Gamma(\frac{s}{\alpha})}{\Gamma(\frac{1-s}{\alpha})}$  can be proved by the Mellin-Barnes integral representation of Bessel functions, see e.g. [31, Section 3.4.3], which gives that

$$\frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} z^{-s} \frac{\Gamma(\frac{s}{\alpha})}{\Gamma(\frac{1-s}{\alpha})} ds = \alpha \sum_{n=0}^{\infty} \frac{(e^{i\pi} z^\alpha)^n}{n! \Gamma(n + \frac{1}{\alpha})} = J_\alpha(z).$$

This concludes the proof of the Proposition.  $\square$

Next, by referring to [10, Chapter II], we see that the speed measure of  $Q$  is (up to a multiplicative positive constant) the Lebesgue measure, hence  $Q$  extends uniquely to a self-adjoint contractive  $\mathcal{C}_0$ -semigroup on  $L^2(\mathbb{R}_+)$ , also denoted by  $Q$  when there is no confusion (otherwise, we may denote  $Q^F$  for the Feller semigroup). The infinitesimal generator  $\mathbf{L}$  of this  $L^2(\mathbb{R}_+)$ -extension is an unbounded self-adjoint operator on  $L^2(\mathbb{R}_+)$ , and, by [25, Remark 3.1], its  $L^2(\mathbb{R}_+)$ -domain, denoted by  $\mathcal{D}_{\mathbf{L}}(L^2(\mathbb{R}_+))$ , is given by

$$(A.7) \quad \mathcal{D}_{\mathbf{L}}(L^2(\mathbb{R}_+)) = \{f \in L^2(\mathbb{R}_+); \mathbf{L}f \in L^2(\mathbb{R}_+), f^+(0) = 0\}.$$

Moreover, for any  $t \geq 0$ ,  $Q_t \in \mathbf{B}(L^2(\mathbb{R}_+))$  with  $S(Q_t) = S_c(Q_t) = (e^{-q^\alpha t})_{q \geq 0}$  and  $S_p(Q_t) = S_r(Q_t) = \emptyset$ . Finally, using the spectral expansion of the self-adjoint squared Bessel operator  $K_t$ , see e.g. [28, Section 6] and [27], one can deduce that for any  $t > 0$  and  $f \in L^2(\mathbb{R}_+)$ ,  $Q_t f$  has the following spectral expansion in  $L^2(\mathbb{R}_+)$ ,

$$(A.8) \quad Q_t f = H_\alpha \mathbf{e}_{\alpha,t} H_\alpha f.$$

## REFERENCES

- [1] J.-P. Antoine and P. Balazs. Frames and semi-frames. *J. Phys. A*, 44(47):479501, 2, 2011.
- [2] B. Baeumer, M. Kovács, M. M. Meerschaert, R. L. Schilling, and P. Straka. Reflected spectrally negative stable processes and their governing equations. *Trans. Amer. Math. Soc.*, 368(1):227–248, 2016.
- [3] M. Barczy and L. Döring. On entire moments of self-similar Markov processes. *Stoch. Anal. Appl.*, 31(2):191–198, 2013.
- [4] C. Berg and G. Forst, Potential theory on locally compact abelian groups. *Springer-Verlag*, 1975

- [5] V. Bernyk, R. C. Dalang, and G. Peskir. Predicting the ultimate supremum of a stable Lévy process with no negative jumps. *Ann. Probab.*, 39(6):2385–2423, 2011.
- [6] J. Bertoin. An extension of Pitman’s theorem for spectrally positive lévy processes. *Ann. Probab.*, 20(3):1464–1483, 1992.
- [7] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.
- [8] J. Bertoin and M. Yor. *The entrance laws of self-similar Markov processes and exponential functionals of Lévy processes*. *Potential Analysis*, 17: 389400, 2002
- [9] R. M. Blumenthal. *On construction of Markov processes*, volume 63 of *Z. Wahrsch. Verw. Gebiete*. 63(4):433–444, 1983.
- [10] A. N. Borodin and P. Salminen. *Handbook of Brownian Motion - Facts and Formulae*. Probability and its Applications. Birkhäuser Verlag, Basel, 2<sup>nd</sup> edition, 2002.
- [11] Ph. Carmona, F. Petit, and M. Yor. On the distribution and asymptotic results for exponential functionals of Lévy processes. In M. Yor (ed.) *Exponential functionals and principal values related to Brownian motion*. *Biblioteca de la Rev. Mat. Iberoamericana*, pages 73–121, 1997.
- [12] K. L. Chung and J. B. Walsh. *Markov processes, Brownian motion, and time symmetry*, volume 249 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, New York, second edition, 2005.
- [13] T. Craven and G. Csordas. Jensen polynomials and the Turán and Laguerre inequalities. *Pacific J. Math.*, 136(2):241–260, 1989.
- [14] G. Da Prato. *An introduction to infinite-dimensional analysis*. Universitext. Springer-Verlag, Berlin, 2006. Revised and extended from the 2001 original by Da Prato.
- [15] E. B. Davies. *Linear operators and their spectra*, volume 106 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2007.
- [16] N. Dunford and J. T. Schwartz. *Linear operators. Part II: Spectral theory. Self adjoint operators in Hilbert space*. With the assistance of William G. Bade and Robert G. Bartle. Interscience Publishers John Wiley & Sons New York-London, 1963.
- [17] N. Dunford and J. T. Schwartz. *Linear operators. Part III: Spectral operators*. Interscience Publishers [John Wiley & Sons, Inc.], New York-London-Sydney, 1971. With the assistance of William G. Bade and Robert G. Bartle, Pure and Applied Mathematics, Vol. VII.
- [18] V. J. Ervin and J.P. Roop. Variational formulation for the stationary fractional advection dispersion equation. *Numer. Methods Partial Differential Equations*, (22):558–576, 2006.
- [19] H. Gzyl. Infinitesimal generators of time changed processes. *Ann. Probab.*, 8(4):716–726, 1980.
- [20] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and applications of fractional differential equations*, volume 204 of *North-Holland Mathematics Studies*. Elsevier Science B.V., Amsterdam, 2006.
- [21] A. Kuznetsov and M. Kwasnicki. Spectral analysis of stable processes on the positive half-line. *available at <http://arxiv.org/abs/1509.06435>*, 2015.
- [22] A. E. Kyprianou. *Introductory lectures on fluctuations of Lévy processes with applications*. Universitext. Springer-Verlag, Berlin, 2006.
- [23] J. Lamperti. Semi-stable Markov processes. I. *Z. Wahrsch. Verw. Geb.*, 22:205–225, 1972.
- [24] N. N. Lebedev. *Special functions and their applications*. Dover Publications Inc., New York, 1972. Revised edition, translated from the Russian and edited by Richard A. Silverman, Unabridged and corrected republication.
- [25] H. P. McKean, Jr. Elementary solutions for certain parabolic partial differential equations. *Trans. Amer. Math. Soc.*, 82:519–548, 1956.
- [26] M. M. Meerschaert and A. Sikorskii. *Stochastic models for fractional calculus*, volume 43 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 2012.
- [27] B. Muckenhoupt and E. M. Stein. Classical expansions and their relation to conjugate harmonic functions. *Trans. Amer. Math. Soc.*, 118:17–92, 1965.
- [28] A. Nowak and K. Stempak. On  $L^p$ -contractivity of Laguerre semigroups. *Illinois J. Math.*, 56(2):433–452, 2012.
- [29] F. W. J. Olver. *Introduction to Asymptotics and Special Functions*. Academic Press, 1974.
- [30] R. B. Paris. Exponentially small expansions in the asymptotics of the Wright function. *J. Comput. Appl. Math.*, 234(2):488–504, 2010.
- [31] R. B. Paris and D. Kaminski. *Asymptotics and Mellin-Barnes integrals*, volume 85 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2001.



- [32] P. Patie. Infinite divisibility of solutions to some self-similar integro-differential equations and exponential functionals of Lévy processes. *Ann. Inst. Henri Poincaré Probab. Stat.*, 45(3):667–684, 2009.
- [33] P. Patie and M. Savov. Spectral expansions of non-self-adjoint generalized Laguerre semigroups. *Submitted*, 162 p., 2015.
- [34] P. Patie and M. Savov. Spectral theory of positive self-similar semigroups. *Working paper*, 2015.
- [35] P. Patie and M. Savov. Cauchy problem of the non-self-adjoint Gauss-Laguerre semigroups and uniform bounds of generalized Laguerre polynomials. *J. Spectr. Theory*, to appear, 2016.
- [36] P. Patie and T. Simon. Intertwining certain fractional derivatives. *Potential Anal.*, 36(4):569–587, 2012.
- [37] A. D. Poularikas, editor. *The transforms and applications handbook*. The Electrical Engineering Handbook Series. CRC Press, Boca Raton, FL; IEEE Press, New York, 1996.
- [38] D. Revuz. Mesures associées aux fonctionnelles additives de Markov. I. *Trans. Amer. Math. Soc.*, 148:501–531, 1970.
- [39] V. Rivero. Recurrent extensions of self-similar Markov processes and Cramér’s condition. *Bernoulli*, 11(3):471–509, 2005.
- [40] H. Sankaranarayanan. *Grünwald-type approximations and boundary conditions for one-sided fractional derivative operators*. PhD thesis, University of Otago, 2014.
- [41] R. L. Schilling, R. Song, and Z. Vondraček. *Bernstein functions*, volume 37 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 2010. Theory and applications.
- [42] N. Sekeljic. Asymptotic expansion of bessel functions: Applications to electromagnetics. *Dynamics at the Horsetooth*, 2A, 2010.
- [43] J. Sjöstrand. Some results on nonselfadjoint operators: a survey. In *Further progress in analysis*, pages 45–74. World Sci. Publ., Hackensack, NJ, 2009.
- [44] E. C. Titchmarsh. *The theory of functions*. Oxford University Press, Oxford, 1939.
- [45] E. C. Titchmarsh. *Introduction to the theory of Fourier integrals*. Chelsea Publishing Co., New York, third edition, 1986.
- [46] J. B. Walsh. Markov processes and their functionals in duality. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 24:229–246, 1972.

SCHOOL OF OPERATIONS RESEARCH AND INFORMATION ENGINEERING, CORNELL UNIVERSITY, ITHACA, NY 14853.

*E-mail address:* pp396@orie.cornell.edu

SCHOOL OF OPERATIONS RESEARCH AND INFORMATION ENGINEERING, CORNELL UNIVERSITY, ITHACA, NY 14853.

*E-mail address:* yz645@cornell.edu