

# BERNSTEIN-GAMMA FUNCTIONS AND EXPONENTIAL FUNCTIONALS OF LÉVY PROCESSES

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ABSTRACT. In this work we analyse the solution to the recurrence equation

$$\mathcal{M}_\Psi(z+1) = \frac{-z}{\Psi(-z)} \mathcal{M}_\Psi(z)$$

defined on a subset of the imaginary line and where  $\Psi$  runs through the set of all negative definite functions. Using the analytic Wiener-Hopf method we furnish the solution to this equation as a product of functions that extend the classical gamma function. These latter functions, being in bijection with the class of Bernstein functions, we call the Bernstein-gamma functions. Using their Weierstrass product representation we establish universal Stirling type asymptotic which is explicit in terms of the constituting Bernstein function. This allows the thorough understanding of the decay of  $|\mathcal{M}_\Psi(z)|$  along imaginary lines and an access to quantities important for many theoretical and applied studies in probability and analysis.

This functional equation appears as a central object in several recent studies ranging from analysis and spectral theory to probability theory. In this paper, as an application of the results above, we investigate from a global perspective the exponential functionals of Lévy processes whose Mellin transform satisfies the equation above. Although these variables have been intensively studied our new approach based on a combination of probabilistic and analytical techniques enables us to derive comprehensive properties and strengthen several results on the law of these random variables for some classes of Lévy processes that could be found in the literature. These encompass smoothness for its density, regularity and analytical properties, large and small asymptotic behaviour, including asymptotic expansions, bounds, and Mellin-Barnes representations of its successive derivatives. In some cases we also study the weak convergence of exponential functionals on a finite time horizon when the latter expands to infinity. As a result of new Wiener-Hopf and infinite product factorizations of the law of the exponential functional we deliver important intertwining relation between members of the class of positive self-similar semigroups. Some of the results presented in this paper have been announced in the note [59].

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## 1. INTRODUCTION

The main aim of this work is to develop an in-depth analysis of the solutions to the functional equation defined for any negative definite function  $\Psi$  by

$$(1.1) \quad \mathcal{M}_\Psi(z+1) = \frac{-z}{\Psi(-z)} \mathcal{M}_\Psi(z)$$

and valid (at least) on the domain  $i\mathbb{R} \setminus \mathcal{Z}_0(\Psi)$ , where we set  $\mathcal{Z}_0(\Psi) = \{z \in i\mathbb{R} : \Psi(-z) = 0\}$  and the negative definite functions are defined in (2.1).

The foremost motivation underlying this study is the methodology underpinning an approach developed by the authors for understanding the spectral decomposition of at least some non-self-adjoint Markov semigroups. This program has been carried out for a class of generalized Laguerre semigroups and thereby via a deterministic mapping for an equivalent class of positive self-similar semigroups, see [61], and this study has revealed that the solutions to the recurrence equations of type (1.1) play a central role in obtaining and quantitatively characterizing the spectral representation of the entire class of positive self-similar semigroups. A natural approach to derive and understand the solution to an equation defined on a subset of  $i\mathbb{R}$ , in this instance (1.1), stems from the classical Wiener-Hopf method. It is well-known that for any  $\Psi \in \overline{\mathcal{N}}$ , where  $\overline{\mathcal{N}}$  stands for the space of negative definite functions, we have the analytic Wiener-Hopf factorization

$$(1.2) \quad \Psi(z) = -\phi_+(-z)\phi_-(z), \quad z \in i\mathbb{R},$$

where  $\phi_+, \phi_- \in \mathcal{B}$ , that is  $\phi_\pm$  are Bernstein functions, see (2.2). Exploiting (1.2) the derivation and characterization of the solution of (1.1) can be reduced to considering an equation of the type

$$(1.3) \quad W_\phi(z+1) = \phi(z)W_\phi(z), \quad \phi \in \mathcal{B},$$

for  $z \in \mathbb{C}_{(0,\infty)} = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$ . In turn the solution to (1.3) can be represented on  $\mathbb{C}_{(0,\infty)}$  as an infinite Weierstrass product involving  $\phi \in \mathcal{B}$ , see [61, Chapter 6]. Here, in Theorem 3.2 we manage to characterize the main properties of  $W_\phi$ , as a meromorphic function on an identifiable complex strip via a couple of global parameters pertaining to all  $\phi \in \mathcal{B}$ . Also, new and informative asymptotic representations of  $W_\phi$  are offered and contained in Theorem 3.3. From them the asymptotic of  $W_\phi$  along  $a + i\mathbb{R}$  can be related to the geometry of the image

of  $\mathbb{C}_{(0,\infty)}$  via  $\phi \in \mathcal{B}$  and in many instances this asymptotic can be precisely computed or well-estimated as illustrated in Proposition 3.16. All results are reminiscent of the Stirling asymptotic for the gamma function which solves (1.3) with  $\phi(z) = z$ . For this purpose we call the functions  $W_\phi$  Bernstein-gamma functions. Due to their ubiquitous, albeit often unrecognised, presence in many theoretical studies they are an important class of special functions. If  $\mathbb{R}^+ = (0, \infty)$  the restriction of (1.3) on  $\mathbb{R}^+$  has been considered in a larger generality by [73] and for the class of Bernstein functions by [36]. More information on the literature can be found in Section 3.

These novel results on the general solution of (1.3), that is  $W_\phi$ , allow for an asymptotic representation and a complete characterization of the solution of (1.1), that is  $\mathcal{M}_\Psi$ , as a meromorphic function on an identifiable strip, in terms of four global parameters describing the analytical properties of  $\phi_+$ ,  $\phi_-$  and thereby of  $\Psi$  as stated by Theorem 2.1. In Theorem 2.5 we also conduct asymptotic analysis of  $|\mathcal{M}_\Psi(z)|$ . We wish to emphasize that (1.2) does not fully reduce the study of (1.1) to the decoupled investigation of (1.3) for  $\phi_+$  and  $\phi_-$ . In fact the usage of the interplay between  $\phi_+$  and  $\phi_-$  induced by (1.2) is the key to getting sharp and exhaustive results on the properties of  $\mathcal{M}_\Psi$  as illustrated by (2.16) of Theorem 2.5. The latter gives complete and quantifiable information as to the rate of polynomial decay of  $|\mathcal{M}_\Psi(z)|$  along complex lines of the type  $a + i\mathbb{R}$ , where  $a$  is fixed.

As a major application of our results on the solutions of functional equations of the type (1.1) we develop and present a general and unified study of the exponential functionals of Lévy processes. To facilitate the discussion of our main motivation, aims and achievements in light of the existing body of literature we recall that a possibly killed Lévy process  $\xi = (\xi_t)_{t \geq 0}$  is a.s. right-continuous, real-valued stochastic process which possesses stationary and independent increments that is killed at an independent of itself exponential random variable (time) of parameter  $q \geq 0$ , that is  $\mathbf{e}_q$  and  $\xi_t = \infty$  for any  $t \geq \mathbf{e}_q$ . Note that  $\mathbf{e}_0 = \infty$ . The law of a possibly killed Lévy process  $\xi$  is characterized via its characteristic exponent, i.e.  $\mathbb{E}[e^{z\xi_t}] = e^{\Psi(z)t}$ ,  $\Psi \in \overline{\mathcal{N}}$ , and there is a bijection between the class of possibly killed Lévy processes and  $\overline{\mathcal{N}}$ . Denote the exponential functional of the Lévy process  $\xi$  by

$$I_\Psi(t) = \int_0^t e^{-\xi_s} ds, \quad t \geq 0,$$

and its associated perpetuity by

$$(1.4) \quad I_\Psi = \int_0^\infty e^{-\xi_s} ds = \int_0^{\mathbf{e}_q} e^{-\xi_s} ds.$$

Its study has been initiated by Urbanik in [72] and proceeded by M. Yor with various co-authors [18, 36, 74]. There is also a number of subsequent and intermediate contributions to the study of these random variables, a small sample of which comprises of [4, 40, 52, 56, 58, 59, 61]. This is due to the fact that the exponential functionals appear and play a crucial role in various theoretical and applied contexts such as the spectral theory of some non-reversible Markov semigroups ([60, 61]), the study of random planar maps ([12]), limit theorems of Markov chains ([13]), positive self-similar Markov processes ([11, 16, 20, 55]), financial and insurance mathematics ([57, 35]), branching processes with immigration ([54]), fragmentation processes ([18]), random affine equations, perpetuities to name but a few. Starting from [46] it has become gradually evident that studying the Mellin transform of the exponential functional is the right tool in many contexts. For particular subclasses, that include and allow the study of the supremum of the stable process, this transform has been evaluated and sometimes via inversion the law of exponential functional has been obtained, see [8, 34, 39, 40, 41]. In this paper, as a consequence

of the detailed study of  $\mathcal{M}_\Psi$  and  $W_\phi$  above, and the fact that whenever  $I_\Psi < \infty$  the Mellin transform of  $I_\Psi$ , that is  $\mathcal{M}_{I_\Psi}$ , satisfies  $\mathcal{M}_{I_\Psi}(z) = \phi_-(0)\mathcal{M}_\Psi(z)$  at least on  $\text{Re}(z) \in (0, 1)$ , we obtain, refine and complement various results on the law of  $I_\Psi$ . Deriving complete and quantifiable information on the decay of  $|\mathcal{M}_\Psi(z)|$  along complex lines allows us to show that the law of  $I_\Psi$  is infinitely differentiable unless  $\xi$  is a compound Poisson process with strictly positive drift in which case (2.16) of Theorem 2.5 evaluates the minimum number of smooth derivatives the law of  $I_\Psi$  possesses. Under no restriction we provide a Mellin-Barnes representation for the law of  $I_\Psi$  and thereby bounds for the law of  $I_\Psi$  and its derivatives. In Theorem 2.7(4) and Corollary 2.12 we show that polynomial small asymptotic expansion is possible if and only if the Lévy process is killed, in which case we obtain explicit evaluation of the terms of this expansion. In Theorem 2.14 general results on the tail of the law are offered. These include the computation of the Pareto index for any exponential functional and under Cramer's condition, depending on the decay of  $|\mathcal{M}_{I_\Psi}(z)|$  and under minute additional requirements, the elucidation of the tail asymptotic and its extension to the level of the density and its derivatives. The latter for example immediately recovers the asymptotic behaviour of the density of the supremum of a stable Lévy process as investigated in [8, 25, 38, 56]. In Theorem 2.19 general results have also been derived for the law at zero. Finally, when  $\lim_{t \rightarrow \infty} I_\Psi(t) = \int_0^\infty e^{-\xi_s} ds = \infty$  and under the celebrated Spitzer's condition imposed on  $\xi$  we establish the weak convergence of  $\mathbb{P}(I_\Psi(t) \in dx)$  after proper rescaling in time and space. This result is particularly relevant in the world of random processes in random environments, where such information strengthens significantly the results of [44, 51]. We proceed by showing that the Wiener-Hopf type factorization of the law of  $I_\Psi$  which was proved in [52, 58] under various conditions, holds in fact in complete generality, see Theorem 2.27 which also contains additional interesting factorizations. By means of a classical relation between the entrance law of positive self-similar Markov processes and the law of the exponential functional of Lévy processes, we compute explicitly the Mellin transform of the former, see Theorem 2.29(1). Moreover, exploiting this relation and the Wiener-Hopf decomposition of the law of  $I_\Psi$  mentioned earlier, we derive some original intertwining relations between positive self-similar semigroups, see Theorem 2.29(2).

The outcome of this paper seems to reaffirm the power of complex analytical tools in probability theory. Even departing from a completely general perspective the Mellin transform is the key tool for understanding the exponential functional of a Lévy process. The reason for the latter is the possibility to represent the Mellin transform as a product combination of identifiable Bernstein-gamma functions and thus access quantifiable information about it as a meromorphic function and its asymptotic behaviour in a complex strip. However, as it can be most notably seen in the proofs of Theorem 2.5, Theorem 2.14 and Theorem 2.19, the most precise results depend on mixing analytical tools with probabilistic techniques and the properties of Lévy processes.

The paper is structured as follows: Section 2 is dedicated to the main results and their statements; Section 3 introduces and studies in detail the Bernstein-gamma functions; Section 4 considers the proofs of the results related to the functional equation (1.1); Section 5 furnishes the proofs for the results regarding the exponential functionals of Lévy processes; Section 6 deals with the factorizations of the law of the exponential functional and the intertwining between positive self-similar semigroups; the Appendix provides some additional information on Lévy processes and results on them that cannot be easily detected in the literature, e.g. the version of *équation amicale inversée* for killed Lévy processes.

## 2. MAIN RESULTS

**2.1. Wiener-Hopf factorization, Bernstein-Weierstrass representation and asymptotic analysis of the solution of (1.1).** We start by introducing some notation. We use  $\mathbb{N}$  for the set of non-negative integers and the standard notation  $\mathcal{C}^k(\mathbb{K})$  for the  $k$  times differentiable functions on some complex or real domain  $\mathbb{K}$ . The space  $\mathcal{C}_0^k(\mathbb{R}^+)$  stands for the  $k$  times differentiable functions which together with their  $k$  derivatives vanish at infinity, whereas  $\mathcal{C}_b^k(\mathbb{R}^+)$  requires only boundedness. For any  $z \in \mathbb{C}$  set  $z = |z|e^{i \arg z}$  with the branch of the argument function defined via the convention  $\arg : \mathbb{C} \mapsto (-\pi, \pi]$ . For any  $-\infty \leq a < b \leq \infty$ , we denote by  $\mathbb{C}_{(a,b)} = \{z \in \mathbb{C} : a < \operatorname{Re}(z) < b\}$  and for any  $a \in (-\infty, \infty)$  we set  $\mathbb{C}_a = \{z \in \mathbb{C} : \operatorname{Re}(z) = a\}$ . We use  $\mathbf{A}_{(a,b)}$  for the set of holomorphic functions on  $\mathbb{C}_{(a,b)}$ , whereas if  $-\infty < a$  then  $\mathbf{A}_{[a,b]}$  stands for the holomorphic functions on  $\mathbb{C}_{(a,b)}$  that can be extended continuously to  $\mathbb{C}_a$ . Similarly, we have the spaces  $\mathbf{A}_{[a,b]}$  and  $\mathbf{A}_{(a,b]}$ . Finally, we use  $\mathbf{M}_{(a,b)}$  for the set of meromorphic functions on  $\mathbb{C}_{(a,b)}$ . It is well-known that  $\Psi \in \overline{\mathcal{N}}$ , that is  $\Psi$  is a negative definite function, if and only if  $\Psi : i\mathbb{R} \rightarrow \mathbb{C}$  and it admits the following Lévy-Khintchine representation

$$(2.1) \quad \Psi(z) = \frac{\sigma^2}{2}z^2 + cz + \int_{-\infty}^{\infty} (e^{zr} - 1 - zr\mathbb{I}_{\{|r|<1\}}) \Pi(dr) - q,$$

where  $q \geq 0$ ,  $\sigma^2 \geq 0$ ,  $c \in \mathbb{R}$ , and, the sigma-finite measure  $\Pi$  satisfies the integrability condition  $\int_{-\infty}^{\infty} (1 \wedge r^2) \Pi(dr) < \infty$ . The class of Bernstein functions  $\mathcal{B}$  consists of all functions  $\phi \not\equiv 0$  that can be represented as follows

$$(2.2) \quad \phi(z) = \phi(0) + \mathbf{d}z + \int_0^{\infty} (1 - e^{-zy}) \mu(dy) = \phi(0) + \mathbf{d}z + z \int_0^{\infty} e^{-zy} \bar{\mu}(y) dy, \quad z \in \mathbb{C}_{[0,\infty)},$$

where  $\mathbb{C}_{[0,\infty)} = \{z \in \mathbb{C} : \operatorname{Re}(z) \geq 0\}$ ,  $\phi(0) \geq 0$ ,  $\mathbf{d} \geq 0$ ,  $\mu$  is a sigma-finite measure satisfying  $\int_0^{\infty} (1 \wedge y) \mu(dy) < \infty$  and  $\bar{\mu}(y) = \int_y^{\infty} \mu(dr)$ ,  $y \geq 0$ . With any function  $\phi \in \mathcal{B}$  since  $\phi \in \mathbf{A}_{[0,\infty)}$ , see (2.2), we associate the quantities

$$(2.3) \quad \mathbf{u}_\phi = \sup \{u \leq 0 : \phi(u) = 0\} \in [-\infty, 0]$$

$$(2.4) \quad \mathbf{a}_\phi = \inf \{u < 0 : \phi \in \mathbf{A}_{(u,\infty)}\} \in [-\infty, 0]$$

$$(2.5) \quad \bar{\mathbf{a}}_\phi = \max \{\mathbf{a}_\phi, \mathbf{u}_\phi\} = \sup \{u \leq 0 : \phi(u) = -\infty \text{ or } \phi(u) = 0\} \in [-\infty, 0],$$

which are well defined thanks to the form of  $\phi$ , see (2.2), and the convention  $\sup \emptyset = -\infty$  and  $\inf \emptyset = 0$ . Note that  $\phi$  is a non-zero constant if and only if (iff)  $\bar{\mathbf{a}}_\phi = -\infty$ . Indeed, otherwise, if  $\mathbf{a}_\phi = -\infty$  then necessarily from (2.2),  $\lim_{a \rightarrow \infty} \phi(-a) = -\infty$ . Hence,  $\mathbf{u}_\phi > -\infty$  and  $\bar{\mathbf{a}}_\phi \in (-\infty, 0]$ . For these quantities associated to  $\phi_+$ ,  $\phi_-$  in (1.2) for the sake of clarity we drop the subscript  $\phi$  and use  $\mathbf{u}_+$ ,  $\mathbf{u}_-$ ,  $\mathbf{a}_+$ ,  $\mathbf{a}_-$ ,  $\bar{\mathbf{a}}_+$ ,  $\bar{\mathbf{a}}_-$ . For  $\phi \in \mathcal{B}$ , we write the generalized Weierstrass product

$$W_\phi(z) = \frac{e^{-\gamma_\phi z}}{\phi(z)} \prod_{k=1}^{\infty} \frac{\phi(k)}{\phi(k+z)} e^{\frac{\phi'(k)}{\phi(k)} z}, \quad z \in \mathbb{C}_{(0,\infty)},$$

where

$$\gamma_\phi = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n \frac{\phi'(k)}{\phi(k)} - \log \phi(n) \right) \in \left[ -\ln \phi(1), \frac{\phi'(1)}{\phi(1)} - \ln \phi(1) \right].$$

Both  $W_\phi$ ,  $\gamma_\phi$  are known to exist and observe that if  $\phi(z) = z$ , then  $W_\phi$  corresponds to the Weierstrass product representation of the celebrated gamma function  $\Gamma$ , valid on  $\mathbb{C}/\{0, -1, -2, \dots\}$ , and  $\gamma_\phi$  is the Euler-Mascheroni constant, see e.g. [43], justifying both the terminology and notation. Note also that when  $z = n \in \mathbb{N}$  then  $W_\phi(n) = \prod_{k=1}^n \phi(k)$ . We are ready to state the

first of our central results which, for any  $\Psi \in \overline{\mathcal{N}}$ , provides an explicit representation of  $\mathcal{M}_\Psi$  in terms of generalized Bernstein-gamma functions.

**Theorem 2.1.** *Let  $\Psi \in \overline{\mathcal{N}}$  and recall that  $\Psi(z) = -\phi_+(-z)\phi_-(z)$ ,  $z \in i\mathbb{R}$ . Then, the mapping  $\mathcal{M}_\Psi$  defined by*

$$(2.6) \quad \mathcal{M}_\Psi(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-}(1-z)$$

satisfies the recurrence relation (1.1) on  $i\mathbb{R} \setminus \mathcal{Z}_0(\Psi)$  where  $\mathcal{Z}_0(\Psi) = \{z \in i\mathbb{R} : \Psi(-z) \neq 0\}$ . Setting  $\mathbf{a}_\Psi = \mathbf{a}_+ \mathbb{I}_{\{\bar{\mathbf{a}}_+ = 0\}} \leq 0$ , we have that

$$(2.7) \quad \mathcal{M}_\Psi \in \mathbf{A}_{(\mathbf{a}_\Psi, 1 - \bar{\mathbf{a}}_-)}.$$

If  $\mathbf{a}_\Psi = 0$ , then  $\mathcal{M}_\Psi$  extends continuously to  $i\mathbb{R} \setminus \{0\}$  if  $\phi'_+(0^+) = \infty$  or  $\bar{\mathbf{a}}_+ < 0$ , and otherwise  $\mathcal{M}_\Psi \in \mathbf{A}_{[0, 1 - \bar{\mathbf{a}}_-]}$ . In any case

$$(2.8) \quad \mathcal{M}_\Psi \in \mathbf{M}_{(\mathbf{a}_+, 1 - \mathbf{a}_-)}.$$

Let  $\mathbf{a}_+ \leq \bar{\mathbf{a}}_+ < 0$ . If  $\mathbf{u}_+ = -\infty$  or  $-\mathbf{u}_+ \notin \mathbb{N}$  then on  $\mathbb{C}_{(\mathbf{a}_+, 1 - \bar{\mathbf{a}}_-)}$ ,  $\mathcal{M}_\Psi$  has simple poles at all points  $-n$  such that  $-n > \mathbf{a}_+$ ,  $n \in \mathbb{N}$ . Otherwise, on  $\mathbb{C}_{(\mathbf{a}_+, 1 - \bar{\mathbf{a}}_-)}$ ,  $\mathcal{M}_\Psi$  has simple poles at all points  $-n$  such that  $n \in \mathbb{N} \setminus \{|\mathbf{u}_+|, |\mathbf{u}_+| + 1, \dots\}$ . In both cases the residues are of values  $\phi_+(0) \frac{\prod_{k=1}^n \Psi(k)}{n!}$  at each of those  $-n$  where we apply the convention  $\prod_{k=1}^0 = 1$ .

This theorem is proved in Section 4.1.

*Remark 2.2.* If  $\phi_+ \equiv \phi_-$ , then, for any  $z \in \mathbb{C}_{(\bar{\mathbf{a}}_- \vee \mathbf{a}_\Psi, (1 - \bar{\mathbf{a}}_-) \wedge (1 - \mathbf{a}_\Psi))}$

$$(2.9) \quad \mathcal{M}_\Psi(z) \mathcal{M}_\Psi(1-z) = \frac{\pi}{\sin \pi z},$$

and hence  $\mathcal{M}_\Psi(\frac{1}{2}) = \sqrt{\pi}$ . Thus, (2.9) is a generalized version of the reflection formula for the classical gamma function and offers further benefits for the understanding of the behaviour of  $\mathcal{M}_\Psi$ , see e.g. Theorem 2.5(2).

*Remark 2.3.* In view of the comprehensive asymptotic representation of  $|W_\phi(z)|$  for any  $\phi \in \mathcal{B}$ , see Theorem 3.3(3.19), (2.6) also provides asymptotic expansion for  $|\mathcal{M}_\Psi(z)|$ .

*Remark 2.4.* We note, from (1.2) and (2.3)-(2.5), that the quantities  $\mathbf{a}_+$ ,  $\mathbf{a}_-$ ,  $\mathbf{u}_+$ ,  $\mathbf{u}_-$ ,  $\bar{\mathbf{a}}_+$ ,  $\bar{\mathbf{a}}_-$  can be computed from the analytical properties of  $\Psi$ . For example, if  $\Psi \notin \mathbf{A}_{(u, 0)}$  for any  $u < 0$  then  $\mathbf{a}_- = 0$ . Similarly, if  $\overline{\lim}_{t \rightarrow \infty} \xi_t = -\underline{\lim}_{t \rightarrow \infty} \xi_t = \infty$  a.s. then clearly  $\bar{\mathbf{a}}_- = \bar{\mathbf{a}}_+ = 0$  as  $\phi_+(0) = \phi_-(0) = 0$ . The latter indicates that the ladder height processes of  $\xi$  are unkilld, see e.g. [9, Chapetr VI].

In view of the fact that, for any  $\phi \in \mathcal{B}$ ,  $W_\phi$  has a Stirling type asymptotic representation, see Theorem 3.3 below, and (2.6) holds, we proceed with a definition of two classes that will encapsulate different modes of decay of  $|\mathcal{M}_\Psi(z)|$  along complex lines. Put  $\mathbf{I}_\Psi = (0, 1 - \bar{\mathbf{a}}_-)$  and for any  $\beta \in [0, \infty]$ , we write

$$(2.10) \quad \begin{aligned} \overline{\mathcal{N}}_\beta &= \left\{ \Psi \in \overline{\mathcal{N}} : \lim_{|b| \rightarrow \infty} |b|^{\beta - \varepsilon} |\mathcal{M}_\Psi(a + ib)| = 0, \forall a \in \mathbf{I}_\Psi, \forall \varepsilon \in (0, \beta) \right\} \\ &\cap \left\{ \Psi \in \overline{\mathcal{N}} : \lim_{|b| \rightarrow \infty} |b|^{\beta + \varepsilon} |\mathcal{M}_\Psi(a + ib)| = \infty, \forall a \in \mathbf{I}_\Psi, \forall \varepsilon \in (0, \beta) \right\}, \end{aligned}$$

where if  $\beta = \infty$  we understand

$$(2.11) \quad \overline{\mathcal{N}}_\infty = \left\{ \Psi \in \overline{\mathcal{N}} : \lim_{|b| \rightarrow \infty} |b|^\beta |\mathcal{M}_\Psi(a + ib)| = 0, \forall a \in \mathbf{I}_\Psi, \forall \beta \geq 0 \right\}.$$

Moreover, for any  $\Theta > 0$  we set

$$(2.12) \quad \overline{\mathcal{N}}(\Theta) = \left\{ \Psi \in \overline{\mathcal{N}} : \overline{\lim}_{|b| \rightarrow \infty} \frac{\ln |\mathcal{M}_\Psi(a + ib)|}{|b|} \leq -\Theta, \forall a \in \mathbf{I}_\Psi \right\}.$$

Finally, we shall also need the set of regularly varying functions at 0. For this purpose we introduce some more notation. We use in the standard manner  $f \stackrel{a}{\sim} g$  (resp.  $f \stackrel{a}{=} O(g)$ ) for any  $a \in [-\infty, \infty]$ , to denote that  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = 1$  (resp.  $\overline{\lim}_{x \rightarrow a} \left| \frac{f(x)}{g(x)} \right| = C < \infty$ ). The notation  $o(\cdot)$  specifies that in the previous relations the constants are zero. We shall drop the superscripts if it is explicitly stated or clear that  $x \rightarrow a$ . We say that, for some  $\alpha \in [0, 1)$ ,  $f \in RV_\alpha \iff f(y) \stackrel{0}{\sim} y^\alpha \ell(y)$ , where  $\ell \in SV = RV_0$  is a slowly varying function, that is,  $\ell(cy) \stackrel{0}{\sim} \ell(y)$  for any  $c > 0$ . Furthermore,  $\ell \in SV$  is said to be quasi-monotone, if  $\ell$  is of bounded variation in a neighbourhood of zero and for any  $\gamma > 0$

$$(2.13) \quad \int_0^x y^\gamma |d\ell(y)| \stackrel{0}{=} O(x^\gamma \ell(x)).$$

With this notion we set

$$R_\alpha = \left\{ f \in RV_\alpha : y \mapsto \ell(y) = \frac{f(y)}{y^\alpha} \text{ is quasi-monotone} \right\}$$

and define, after recalling that  $\bar{\mu}(y) = \int_y^\infty \mu(dr)$ , see (2.2),

$$(2.14) \quad \mathcal{B}_{R_\alpha} = \{ \phi \in \mathcal{B} : \mathbf{d} = 0 \text{ and } \bar{\mu} \in R_\alpha \}.$$

Next, we define the class of Bernstein functions with a positive drift that is

$$(2.15) \quad \mathcal{B}_P = \{ \phi \in \mathcal{B} : \mathbf{d} > 0 \}.$$

Finally, we denote by  $\mu_+, \mu_-$  the measures associated to  $\phi_+, \phi_- \in \mathcal{B}$  stemming from (1.2) and  $\Pi_+(dy) = \Pi(dy)\mathbb{I}_{\{y>0\}}$ ,  $\Pi_-(dy) = \Pi(-dy)\mathbb{I}_{\{y>0\}}$  for the measure in (2.1). Finally,  $f(x^\pm)$  will stand throughout for the right, respectively left, limit at  $x$ . We provide an exhaustive claim concerning the decay of  $|\mathcal{M}_\Psi|$  along complex lines.

**Theorem 2.5.** *Let  $\Psi \in \overline{\mathcal{N}}$ .*

(1)  $\Psi \in \overline{\mathcal{N}}_{\mathbf{N}_\Psi}$  with

$$(2.16) \quad \mathbf{N}_\Psi = \begin{cases} \frac{v_-(0^+)}{\phi_-(0) + \bar{\mu}_-(0)} + \frac{\phi_+(0) + \bar{\mu}_+(0)}{\mathbf{d}_+} < \infty & \text{if } \mathbf{d}_+ > 0, \mathbf{d}_- = 0 \text{ and } \bar{\Pi}(0) = \int_{-\infty}^\infty \Pi(dy) < \infty, \\ \infty & \text{otherwise,} \end{cases}$$

where we have used implicitly the fact that if  $\mathbf{d}_+ > 0$  holds then  $\mu_-(dy) = v_-(y)dy\mathbb{I}_{\{y>0\}}$  with  $v_- \in C([0, \infty))$ .

(2) Moreover, if  $\phi_- \in \mathcal{B}_P$ , that is  $\mathbf{d}_- > 0$ , or  $\arg \phi_+ \equiv \arg \phi_-$  then  $\Psi \in \overline{\mathcal{N}}\left(\frac{\pi}{2}\right)$ . If  $\phi_- \in \mathcal{B}_{R_\alpha}$  or  $\phi_+ \in \mathcal{B}_{R_{1-\alpha}}$ , with  $\alpha \in (0, 1)$ , see (2.14) for the definition of regularly varying functions,

then  $\Psi \in \overline{\mathcal{N}}\left(\frac{\pi}{2}\alpha\right)$ . Finally, if  $\Theta_\pm = \frac{\pi}{2} + \Theta_{\phi_-} - \bar{\Theta}_{\phi_+} > 0$ , where  $\Theta_\phi = \lim_{b \rightarrow \infty} \frac{\int_0^{|b|} \arg \phi(1+iu)du}{|b|}$

and  $\bar{\Theta}_\phi = \overline{\lim}_{b \rightarrow \infty} \frac{\int_0^{|b|} \arg \phi(1+iu)du}{|b|}$ , then  $\Psi \in \overline{\mathcal{N}}(\Theta_\pm)$ .

This theorem is proved in Section 4.2.

*Remark 2.6.* It is known from [24, Chapter V, (5.3.11)] when  $\Psi(0) = 0$  and Proposition B.1 in generality that in (2.16)  $v_-(0^+) = \int_0^\infty u_+(y)\Pi_-(dy)$ , where  $u_+$  is the potential density discussed prior to Proposition 4.3. This formula also shows that in this case  $v_-(0^+) \in [0, \infty)$  and thus  $N_\Psi \in (0, \infty)$ .

**2.2. Exponential functional of Lévy processes.** We introduce the subclasses of  $\overline{\mathcal{N}}$

$$(2.17) \quad \mathcal{N} = \{\Psi \in \overline{\mathcal{N}} : \Psi(z) = -\phi_+(-z)\phi_-(z), z \in i\mathbb{R}, \text{ with } \phi_-(0) > 0\}$$

and

$$(2.18) \quad \mathcal{N}_\dagger = \{\Psi \in \overline{\mathcal{N}} : \Psi(0) = -\phi_+(0)\phi_-(0) = -q < 0\} \subseteq \mathcal{N}.$$

We note that

$$(2.19) \quad I_\Psi = \int_0^{e_q} e^{-\xi_s} ds < \infty \text{ a.s.} \iff \Psi \in \mathcal{N} \iff \phi_-(0) > 0,$$

which is evident when  $q = -\Psi(0) > 0$  and is due to the strong law of large numbers when  $q = 0$ . The latter includes the case  $\mathbb{E}[\xi_1 \mathbb{I}_{\{\xi_1 > 0\}}] = \mathbb{E}[-\xi_1 \mathbb{I}_{\{\xi_1 < 0\}}] = \infty$  but yet a.s.  $\lim_{t \rightarrow \infty} \xi_t = \infty$ . For an analytical criterion for the validity of the latter there is the celebrated Erickson's test, see [24, Section 6.7]. Let us write for any  $x \geq 0$ ,

$$F_\Psi(x) = \mathbb{P}(I_\Psi \leq x).$$

From [14], we know that the law of  $I_\Psi$  is absolute continuous with a density denoted by  $f_\Psi$ , i.e.  $F_\Psi^{(1)}(x) = f_\Psi(x)$  a.e.. Introduce the Mellin transform of the positive random variable  $I_\Psi$  denoted formally, for some  $z \in \mathbb{C}$ , as follows

$$(2.20) \quad \mathcal{M}_{I_\Psi}(z) = \mathbb{E}[I_\Psi^{z-1}] = \int_0^\infty x^{z-1} f_\Psi(x) dx.$$

We also use the ceiling function  $[\cdot] : [0, \infty) \mapsto \mathbb{N}$ , that is  $[x] = \min\{n \in \mathbb{N} : n > x\}$ .

**2.2.1. Regularity, analyticity and representations of the density and its successive derivatives.** We start our results on the exponential functional of Lévy processes by providing a result that can be regarded as a corollary to Theorem 2.5.

**Theorem 2.7.** *Let  $\Psi \in \mathcal{N}$ .*

(1) *We have*

$$(2.21) \quad \mathcal{M}_{I_\Psi}(z) = \phi_-(0)\mathcal{M}_\Psi(z) = \phi_-(0) \frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-}(1-z) \in \mathbf{A}_{(\mathfrak{a}_+, \mathbb{I}_{\{\bar{\mathfrak{a}}_+ = 0\}}, 1 - \bar{\mathfrak{a}}_-)} \cap \mathbf{M}_{(\mathfrak{a}_+, 1 - \mathfrak{a}_-)}$$

and  $\mathcal{M}_{I_\Psi}$  satisfies the recurrence relation (1.1) at least on  $i\mathbb{R} \setminus \mathcal{Z}_0(\Psi)$ .

(2) *If  $\phi_- \equiv 1$  then  $\text{Supp } I_\Psi = [0, \frac{1}{\mathfrak{d}_+}]$ , unless  $\phi_+(z) = \mathfrak{d}_+ z$ ,  $\mathfrak{d}_+ \in (0, \infty)$ , in which case  $\text{Supp } I_\Psi = \{\frac{1}{\mathfrak{d}_+}\}$  and if  $\phi_- \not\equiv 1$  and  $\phi_+(z) = z$  then  $\text{Supp } I_\Psi = [\frac{1}{\phi_-(\infty)}, \infty]$ , where we use the convention that  $\frac{1}{\infty} = 0$ . In all other cases  $\text{Supp } I_\Psi = [0, \infty]$ .*



(3)  $F_\Psi \in C_0^{[\mathbb{N}_\Psi]-1}(\mathbb{R}^+)$  and if  $\mathbb{N}_\Psi > 1$  (resp.  $\mathbb{N}_\Psi > \frac{1}{2}$ ) for any  $n = 0, \dots, [\mathbb{N}_\Psi] - 2$  and  $a \in (\mathbf{a}_+ \mathbb{I}_{\{\bar{\mathbf{a}}_+ = 0\}}, 1 - \bar{\mathbf{a}}_-)$ ,

$$(2.22) \quad f_\Psi^{(n)}(x) = (-1)^n \frac{\phi_-(0)}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^{-z-n} \frac{\Gamma(z+1)}{\Gamma(z+1-n)} \mathcal{M}_\Psi(z) dz,$$

where the integral is absolutely convergent for any  $x > 0$  (resp. is defined in the  $L^2$ -sense, as in the book of Titchmarsh [70]).

(4) Let  $\Psi \in \mathcal{N}_\dagger$ , i.e.  $\Psi(0) = -q < 0$  and let  $\mathbb{N}_+ = |\mathbf{u}_+| \mathbb{I}_{\{|\mathbf{u}_+| \in \mathbb{N}\}} + (|\mathbf{a}_+| + 1) \mathbb{I}_{\{|\mathbf{u}_+| \notin \mathbb{N}\}}$ . Then, we have, for any  $0 \leq n < \mathbb{N}_\Psi$ , any  $\mathbb{N} \ni M < \mathbb{N}_+$ ,  $a \in ((-M-1) \vee (\mathbf{a}_+ - 1), -M)$  and  $x > 0$ ,

$$(2.23) \quad F_\Psi^{(n)}(x) = q \sum_{k=1 \vee n}^M \frac{W_\Psi(k-1)}{(k-n)!} x^{k-n} + (-1)^{n+1} \frac{\phi_-(0)}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^{-z} \frac{\Gamma(z)}{\Gamma(z+1-n)} \mathcal{M}_\Psi(z+1) dz,$$

where, by analogy to the notation above, we have set  $W_\Psi(k-1) = \prod_{j=1}^{k-1} \Psi(j)$ , and by convention  $\prod_1^0 = 1$  and the sum vanishes if  $1 \vee n > M$ .

(5) If  $\Psi \in \overline{\mathcal{N}}(\Theta) \cap \mathcal{N}$ ,  $\Theta \in (0, \pi]$ , then  $f_\Psi$  is in fact even holomorphic in the sector  $\mathbb{C}(\Theta) = \{z \in \mathbb{C} : |\arg z| < \Theta\}$ .

This theorem is proved in Section 5.

*Remark 2.8.* Item (3) confirms the conjecture that  $f_\Psi \in C_0^\infty(\mathbb{R}^+)$  if there is an infinite activity in the underlying Lévy process, that is, when either  $\sigma^2 > 0$  and/or  $\int_{-\infty}^\infty \Pi(dy) = \infty$  in (2.1). Indeed, from Theorem 2.5(1), under each of these conditions in any case,  $\Psi \in \overline{\mathcal{N}}_\infty \cap \mathcal{N} = \mathcal{N}_\infty$ . The surprising fact is that  $\Psi \in \mathcal{N}_\infty$  and hence  $f_\Psi \in C_0^\infty(\mathbb{R}^+)$  even when the possibly killed underlying Lévy process is a pure compound Poisson process or it is a compound Poisson process with a negative drift, that is  $\phi_- \in \mathcal{B}_P$ , whereas if  $\phi_+ \in \mathcal{B}_P$  then we only know that  $\Psi \in \mathcal{N}_{\mathbb{N}_\Psi}$ ,  $\mathbb{N}_\Psi < \infty$ . Finally, if  $\phi_- \equiv 1$ ,  $\phi_+(z) = q + z$ ,  $q > 0$ , then  $\xi_t = t$  is killed at rate  $q$  and  $F_\Psi(x) = 1 - (1-x)^q$ ,  $x \in (0, 1)$ , which since  $\mathbb{N}_\Psi = q$ , see (2.16), confirms that the claim  $F_\Psi \in C_0^{[\mathbb{N}_\Psi]-1}(\mathbb{R}^+)$  is sharp unless  $q \in \mathbb{N}$ .

*Remark 2.9.* Let  $\phi_+(z) = z \in \mathcal{B}_P$  and  $\phi_-(0) > 0$  so that  $\Psi(z) = z\phi_-(z) \in \mathcal{N}$ . Then  $I_\Psi = \int_0^\infty e^{-\xi t} dt$  is a self-decomposable random variable, see [61, Chapter 5]. The rate of decay of the Fourier transform of  $I_\Psi$  has been computed as  $\lambda$  in the notation of [67]. One can check that  $\lambda = \mathbb{N}_\Psi$ . For all exponential functionals this work establishes and evaluates in terms of  $\mathbb{N}_\Psi$  the properties pertaining to the specific case of self-decomposable random variables  $I_\Psi$  related to  $\Psi(z) = z\phi_-(z) \in \mathcal{N}$  modulo to the discussion of whether and how precisely the smoothness of  $F_\Psi$  breaks down at the  $[\mathbb{N}_\Psi]$ -derivative.

*Remark 2.10.* Note that if  $\bar{\mathbf{a}}_+ = 0$ , (2.21) combined with  $\mathcal{M}_{I_\Psi}(z) = \mathbb{E}[I_\Psi^{z-1}]$ , see (2.20), and Theorem 3.2(4) evaluates all negative moments  $I_\Psi$  up to order  $-1 + \mathbf{a}_+$ . This recovers and extends the computation of [17, Proposition 2], which deals with the entire negative moments of  $I_\Psi$  when  $\mathbb{E}[\xi_1] = \Psi'(0^+) \in (0, \infty)$ .

*Remark 2.11.* If  $\bar{\mathbf{a}}_- < 0$  then it can be verified from (2.21) that  $\mathcal{M}_{I_\Psi}$  is the unique solution on the domain  $\mathbb{C}_{(0, -\bar{\mathbf{a}}_-)}$  of the functional equation (1.1), derived for the case  $\Psi \in \mathcal{N} \setminus \mathcal{N}_\dagger$ , that is  $q = 0$ , in [46] and when  $\Psi \in \mathcal{N}$  in [4]. If  $q = 0$  and  $\Psi'(0^+) \in (0, \infty)$  then according to Theorem 2.1,  $\mathcal{M}_{I_\Psi} \in \mathbf{A}_{[0,1]}$  and again (2.21) is a solution to (1.1) which in this case holds only on  $i\mathbb{R}$ .

Many theoretical papers on exponential functionals of Lévy processes depend on (1.1) for  $\mathcal{M}_{I_\Psi}$  to hold on a strip and on the availability of an explicit form of  $\Psi$ , e.g. [35, 41], to derive an expression for  $\mathcal{M}_{I_\Psi}$ . Here, (2.21) provides an immediate representation of  $\mathcal{M}_{I_\Psi}$  in terms of Bernstein-gamma functions.

Item (4) of Theorem 2.7 can be refined as follows.

**Corollary 2.12.** *Let  $\Psi \in \mathcal{N}_\dagger$ ,  $|\mathbf{a}_+| = \infty$  and  $-\mathbf{u}_+ \notin \mathbb{N}$ . Then*

$$F(x) \approx q \sum_{k=1}^{\infty} \frac{W_\Psi(k-1)}{k!} x^k$$

is the asymptotic expansion of  $F_\Psi$  at zero, that is, for any  $N \in \mathbb{N}$ ,  $F(x) - q \sum_{k=1}^N \frac{W_\Psi(k-1)}{k!} x^k \underset{0}{=} o(x^N)$ . The asymptotic expansion cannot be a convergent series for any  $x > 0$  unless  $\phi_+ \equiv 1$  or  $\phi_+(z) = \phi_+(0) + \mathbf{d}_+$  and  $\phi_-(\infty) < \infty$  and then in the first case it converges for  $x < \frac{1}{\mathbf{d}_+}$  and diverges for  $x > \frac{1}{\mathbf{d}_+}$ , and in the second it converges for  $x < \frac{1}{\mathbf{d}_+ \phi_-(\infty)}$  and diverges for  $x > \frac{1}{\mathbf{d}_+ \phi_-(\infty)}$ .

*Remark 2.13.* When  $\phi_+ \equiv 1$  it is well-known from [58, Corollary 1.3] and implicitly from [53] that the asymptotic expansion is convergent if and only if  $x < \frac{1}{\mathbf{d}_+}$ .

**2.2.2. Large asymptotic behaviour of the distribution and its successive derivatives.** We proceed by discussing the large asymptotic of the law and the density of  $I_\Psi$  in the general setting. For this purpose we introduce the well-known in the literature *non-lattice subclass* of  $\mathcal{B}$  and  $\mathcal{N}$ . Let, for some  $a \in \mathbb{R}$ ,

$$(2.24) \quad \mathcal{Z}_a(\Psi) = \{z \in \mathbb{C}_a : \Psi(z) = 0\} = \{z \in \mathbb{C}_a : \Psi(\bar{z}) = 0\},$$

where the latter identity follows easily from  $\overline{\Psi(z)} = \Psi(\bar{z})$ , see (2.1), and clearly for  $a = 0$   $\mathcal{Z}_0(\Psi) = \{z \in i\mathbb{R} : \Psi(z) = 0\} = \{z \in i\mathbb{R} : \Psi(-z) = 0\}$ . Then for any  $\Psi \in \mathcal{N}$ ,  $\phi \in \mathcal{B}$  set  $\Psi^\sharp(z) = \Psi(z) - \Psi(0) \in \mathcal{N} \setminus \mathcal{N}_\dagger$  and  $\phi^\sharp(z) = \phi(z) - \phi(0) \in \mathcal{B} \setminus \mathcal{B}_\dagger$ , where  $\mathcal{B}_\dagger = \{\phi \in \mathcal{B} : \phi(0) > 0\}$ . Then the *non-lattice subclass* is defined as follows

$$\Psi \in \mathcal{N}_{\mathcal{Z}} \iff \mathcal{Z}_0(\Psi^\sharp) = \{0\}$$

with identical definition and meaning for  $\mathcal{B}_{\mathcal{Z}}$ . We note that we use the terminology *non-lattice class* since the underlying Lévy processes does not live on a sublattice of  $\mathbb{R}$ , e.g. if  $\Psi \in \mathcal{N}_{\mathcal{Z}}$  then the support of the distribution of the underlying Lévy process  $\xi$  is either  $\mathbb{R}$  or  $\mathbb{R}^+$ . It is easily seen that  $\Psi \in \mathcal{N}_{\mathcal{Z}} \iff \phi_- \in \mathcal{B}_{\mathcal{Z}}$  and  $\phi_+ \in \mathcal{B}_{\mathcal{Z}}$ , see (1.2). Next, if  $\mathbf{u} \in (-\infty, 0)$ , see (2.3), we introduce the *weak non-lattice class* as follows

$$(2.25) \quad \begin{aligned} \Psi \in \mathcal{N}_{\mathcal{W}} &\iff \mathbf{u} \in (-\infty, 0) \text{ and } \exists k \in \mathbb{N} \text{ such that } \liminf_{|b| \rightarrow \infty} |b|^k |\Psi(\mathbf{u} + ib)| > 0 \\ &\iff \mathbf{u} \in (-\infty, 0) \text{ and } \exists k \in \mathbb{N} \text{ such that } \liminf_{|b| \rightarrow \infty} |b|^k |\phi_-(\mathbf{u} + ib)| > 0. \end{aligned}$$

We note that the *weak non-lattice class* seems not to have been introduced in the literature yet. Clearly,  $\mathcal{N} \setminus \mathcal{N}_{\mathcal{Z}} \subseteq \mathcal{N} \setminus \mathcal{N}_{\mathcal{W}}$  since  $\Psi^\sharp \in \mathcal{N} \setminus \mathcal{N}_{\mathcal{Z}}$  vanishes on  $\{k \in \mathbb{N} : \frac{2\pi i}{h}\}$ , where  $h > 0$  is the span of the lattice that supports the distribution of the unkilld Lévy process  $\xi^\sharp$ . We phrase our first main result which virtually encompasses all exponential functionals. We write throughout  $\overline{F}_\Psi(x) = 1 - F_\Psi(x)$  for the tail of  $I_\Psi$ .

**Theorem 2.14.** *Let  $\Psi \in \mathcal{N}$ .*

(1) If  $|\bar{\mathbf{a}}_-| < \infty$  (resp.  $|\bar{\mathbf{a}}_-| = \infty$ , that is  $-\Psi(-z) = \phi_+(z) \in \mathcal{B}$ ), then for any  $\underline{d} < |\bar{\mathbf{a}}_-| < \bar{d}$  (resp.  $\underline{d} < \infty$ ), we have that

$$(2.26) \quad \overline{\lim}_{x \rightarrow \infty} x^{\underline{d}} \bar{F}_\Psi(x) = 0,$$

$$(2.27) \quad \underline{\lim}_{x \rightarrow \infty} x^{\bar{d}} \bar{F}_\Psi(x) = \infty.$$

Therefore, in all cases,

$$(2.28) \quad \lim_{x \rightarrow \infty} \frac{\log \bar{F}_\Psi(x)}{\log x} = \bar{\mathbf{a}}_-.$$

(2) If in addition  $\Psi \in \mathcal{N}_{\mathcal{Z}}$ ,  $\bar{\mathbf{a}}_- = \mathbf{u}_- < 0$  and  $|\Psi'(u^+)| < \infty$  then

$$(2.29) \quad \bar{F}_\Psi(x) \asymp \frac{\phi_-(0)\Gamma(-\mathbf{u}_-)W_{\phi_-}(1+\mathbf{u}_-)}{\phi'_-(u^+)W_{\phi_+}(1-\mathbf{u}_-)}x^{\mathbf{u}_-}.$$

Moreover, if  $\Psi \in \mathcal{N}_\infty \cap \mathcal{N}_{\mathcal{W}}$  (resp.  $\Psi \in \mathcal{N}_{\mathbb{N}_\Psi}$ ,  $\mathbb{N}_\Psi < \infty$ ) then for every  $n \in \mathbb{N}$  (resp.  $n \leq \lceil \mathbb{N}_\Psi \rceil - 2$ )

$$(2.30) \quad f_\Psi^{(n)}(x) \asymp (-1)^n \frac{\phi_-(0)\Gamma(n+1-\mathbf{u}_-)W_{\phi_-}(1+\mathbf{u}_-)}{\phi'_-(u^+)W_{\phi_+}(1-\mathbf{u}_-)}x^{-n-1+\mathbf{u}_-}.$$

This theorem is proved in Section 5.5.

*Remark 2.15.* When  $\Psi \in \mathcal{N}_{\mathcal{Z}}$ ,  $\bar{\mathbf{a}}_- = \mathbf{u}_- < 0$  and  $|\Psi'(u^+)| < \infty$ , whose collective validity is referred to as the Cramer condition for the underlying Lévy process, it is well-known that  $\lim_{x \rightarrow \infty} x^{-\mathbf{u}_-} \bar{F}_\Psi(x) = C > 0$ , see [65, Lemma 4], that is (2.29) holds. Here, we evaluate explicitly  $C$  too. We emphasize that the ability to refine the tail result to densities at the expense of the minute requirement  $\Psi \in \mathcal{N}_{\mathcal{W}}$  comes from the representation (2.21), which measurably and almost invariably ensures fast decay of  $|\mathcal{M}_{I_\Psi}|$  along imaginary lines, see Theorem 2.5(2.16). This decay given by (2.16) can be extended to the line  $\mathbb{C}_{1-\mathbf{u}_-}$  only when  $\Psi \in \mathcal{N}_{\mathcal{W}}$ . We stress that the latter requirement is not needed for  $\bar{F}_\Psi(x)$  since it is non-increasing and one can apply the general and powerful Wiener-Ikehara theorem. Note that a good decay is never available for the quantity  $\mathbb{E}\left[e^{-z\xi_\infty}\right] = \phi_-(0)/\phi_-(-z)$ ,  $z \in \mathbb{C}_0$ , where  $\xi_\infty = \inf_{s \geq 0} \xi_s$ , and therefore for the tail  $\mathbb{P}(\xi_\infty > x)$ , since then  $|\phi_-(z)| \asymp \mathbf{d}|z| + o(|z|)$ , see Proposition 3.14(3).

*Remark 2.16.* The claim of item (1) is general. It is again a result of the decay  $|\mathcal{M}_{I_\Psi}|$  along complex lines and a monotone probabilistic approximation with Lévy processes that fall within the setting of item (2). Relation (2.28) is a strengthening of [4, Lemma 2] in that it quantifies precisely and estimates from below by  $-\infty$  the rate of the power decay of  $\bar{F}_\Psi(x)$  as  $x \rightarrow \infty$ . Since  $I_\Psi$  is also a perpetuity with thin tails, see [31], we provide very precise estimates for the tail behaviour of this class of perpetuities.

*Remark 2.17.* Note that in item (1) the case  $-\Psi(-z) = \phi_+(z) \in \mathcal{B}$ , see (1.2), corresponds to the Lévy process  $\xi$  behind  $\Psi$  being a possibly killed subordinator. This assumption is lacking in item (2) because the existence of  $-\mathbf{u}_- \in (0, \infty)$  precludes the case  $-\Psi(-z) = \phi_+(z)$ .

*Remark 2.18.* Since the supremum of a stable Lévy process can be related to a specific exponential functional for which (2.30) is valid then our result recovers the mains statements about the asymptotic of the density of the supremum of a Lévy process and its derivatives that appear in [25, 38].

Under specific conditions, see [56, 58] for the class of (possibly killed) spectrally negative Lévy processes and [38, 40] for some special instances, the density  $f_\Psi$  can be expanded into a converging series. This is achieved by a subtle pushing to infinity of the contour of the Mellin inversion when the analytic extension  $\mathcal{M}_{I_\Psi} \in \mathbf{M}_{(0,\infty)}$  is available. Our Theorem 2.5, which ensures a priori knowledge for the decay of  $|\mathcal{M}_{I_\Psi}(z)|$ , allows for various asymptotic expansions or evaluation of the speed of convergence of  $f_\Psi$  at infinity as long as  $\Psi \in \mathcal{N}_W$  and  $\mathcal{M}_{I_\Psi}$  extends analytically to the right of  $\mathbb{C}_{1-\mathbf{d}_+}$ . However, for sake of generality, we leave aside the study of some additional examples.

**2.2.3. Small asymptotic behaviour of the distribution and its successive derivatives.** We proceed with the small asymptotic behaviour.

**Theorem 2.19.** *Let  $\Psi \in \mathcal{N}$  then*

$$(2.31) \quad \lim_{x \rightarrow 0} \frac{F_\Psi(x)}{x} = -\Psi(0)$$

with  $f_\Psi(0^+) = \Psi(0)$ . Hence, if  $f_\Psi$  is continuous at zero or  $\Psi \in \mathcal{N}_{N_\Psi}$  with  $N_\Psi > 1$ , we have that

$$(2.32) \quad \lim_{x \rightarrow 0} f_\Psi(x) = f_\Psi(0) = -\Psi(0).$$

*This theorem is proved in Section 5.6.*

*Remark 2.20.* We stress that (2.31) when  $\Psi(0) < 0$  has appeared in [4, Theorem 7(i)] but its proof is essentially based on the non-trivial [53, Theorem 2.5]. Our proof is analytic and confirms our intuition that the function  $W_\phi$  on which all quantities are based is good enough to be thoroughly investigated from analytical perspective and yield results for the random variables whose Mellin transforms it represents. We note that (2.32) is a new result for general exponential functionals  $I_\Psi$ . For exponential functionals based on an increasing Lévy process it has already appeared in [53, Theorem 2.5] without any assumptions. This can be deduced from (2.31) if we additionally know that  $f_\Psi$  is continuous at zero. When  $N_\Psi \leq 1$  we know from Theorem 2.5(2.16) that  $\bar{\mu}_+(0) \leq \mathbf{d}_+ < \infty$  and then it is a trivial exercise to prove from the functional equation for  $f_\Psi$  in [53, Theorem 2.4], that  $f_\Psi$  is continuous at zero and henceforth (2.31) yields  $f_\Psi(0) = -\Psi(0)$ .

*Remark 2.21.* Denote by  $\mathcal{N}_- = \{\Psi \in \mathcal{N} : \phi_+(z) = \phi_+(0) + \mathbf{d}_+ z\}$  the class of the so-called spectrally negative Lévy processes, that is Lévy processes that do not jump upwards and assume that  $\phi_+(0) = 0$ . Then  $\Psi(z) = z\phi_-(z)$  and if  $\phi_-(\infty) = \infty$  the small-time asymptotic of  $f_\Psi$  and its derivatives, according to [61] reads off with  $\varphi_-$  such that  $\phi_-(\varphi_-(u)) = u$  as follows

$$(2.33) \quad f_\Psi^{(n)}(x) \underset{0}{\sim} \frac{C_{\phi_-} \phi_-(0)}{\sqrt{2\pi}} \frac{\varphi_-^n\left(\frac{1}{x}\right)}{x^n} \sqrt{\varphi_-'\left(\frac{1}{x}\right)} e^{-\int_{\phi_-(0)}^{\frac{1}{x}} \varphi_-(r) \frac{dr}{r}}.$$

If  $\phi_-(\infty) < \infty$  then according to Theorem 2.7(2) we have that  $\text{Supp } f_\Psi \in \left[\frac{1}{\phi_-(\infty)}, \infty\right)$ . Comparing the small asymptotic behaviour of  $f_\Psi$  in (2.33) with the one in (2.32) reveals that a simple killing of the underlying Lévy process leads to a dramatic change.

**2.2.4. Finiteness of negative moments and asymptotic behaviour for the exponential functionals on a finite time horizon.** For any  $\Psi \in \bar{\mathcal{N}}$  and  $t \geq 0$  let

$$I_\Psi(t) = \int_0^t e^{-\xi_s} ds.$$

We have the following claim which furnishes necessary and sufficient conditions for finiteness of negative moments of  $I_\Psi(t)$ .

**Theorem 2.22.** *Let  $\Psi \in \overline{\mathcal{N}} \setminus \mathcal{N}_\dagger$ . Then, for any  $t > 0$ ,*

$$(2.34) \quad \mathbb{E} [I_\Psi^{-a}(t)] < \infty \iff a \in (0, 1 - \mathbf{a}_+)$$

$$(2.35) \quad \mathbb{E} [I_\Psi^{-1+\mathbf{a}_+}(t)] < \infty \iff |\Psi(-\mathbf{a}_+)| < \infty \iff |\phi_+(\mathbf{a}_+)| < \infty$$

$$(2.36) \quad \mathbb{E} [I_\Psi^{-1}(t)] < \infty \iff |\Psi'(0^+)| < \infty$$

$$(2.37) \quad \mathbb{E} [I_\Psi^a(t)] = \infty \iff a > 1 - \mathbf{a}_+.$$

Finally, we have that for any  $a \in (0, 1 - \mathbf{a}_+)$

$$(2.38) \quad \lim_{t \rightarrow 0} t^a \mathbb{E} [I_\Psi^{-a}(t)] = 1.$$

*This theorem is proved in Section 5.7*

*Remark 2.23.* Some results as to the finiteness of  $\mathbb{E} [I_\Psi^{-a}(t)]$  appear in the recent preprint [51] but the authors limit their attention on the range  $a \in (0, -\mathbf{a}_+)$  which is substantially easier to prove via the relation (5.32).

Next, we consider the case

$$(2.39) \quad \mathcal{N}^c = \{\Psi \in \overline{\mathcal{N}} : \phi_-(0) = 0\} = \overline{\mathcal{N}} \setminus \mathcal{N},$$

that is all conservative Lévy processes such that  $\varliminf_{t \rightarrow \infty} \xi_t = -\infty$  a.s. and thus  $\mathcal{N}^c \cap \mathcal{N}_\dagger = \emptyset$ . In the setting of the next claim for any  $\Psi \in \mathcal{N}^c$  we use superscript  $r$  for  $\Psi^r(z) = \Psi(z) - r = -\phi_+^r(-z)\phi_-^r(z)$  and all related quantities. Recall that  $RV_\alpha$  stands for the class of regularly varying functions of index  $\alpha \in \mathbb{R}$  at zero. Then the following result elucidates the behaviour of the measures  $\mathbb{P}(I_\Psi(t) \in dx)$  as  $t \rightarrow \infty$  in quite a general framework.

**Theorem 2.24.** *Let  $\Psi \in \mathcal{N}^c$ .*

- (1) *Then for any  $a \in (\mathbf{a}_+ - 1, 0)$  such that  $-a \notin \mathbb{N}$  we get for any  $x > 0$ , any  $\mathbb{N} \ni n < 1 - \mathbf{a}_+$  and any  $a \in (n, \max\{n + 1, 1 - \mathbf{a}_+\})$*

$$(2.40) \quad \frac{e - 1}{e} \varliminf_{t \rightarrow \infty} \frac{\mathbb{P}(I_\Psi(t) \leq x)}{\kappa_-(\frac{1}{t})} \leq \phi_+(0) \sum_{k=1}^{n \wedge \mathbb{N}_+} \frac{|W_\Psi(k-1)|}{k!} x^k + \frac{x^a}{2\pi} \int_{-\infty}^{\infty} \frac{|\mathcal{M}_\Psi(-a + 1 + ib)|}{\sqrt{a^2 + b^2}} db,$$

where  $\kappa_-(\frac{1}{t}) = \phi_-^{\frac{1}{t}}(0)$ ,  $\mathbb{N}_+ = |\mathbf{u}_+| \mathbb{I}_{\{|\mathbf{u}_+| \in \mathbb{N}\}} + (|\mathbf{a}_+| + 1) \mathbb{I}_{\{|\mathbf{u}_+| \notin \mathbb{N}\}}$ ,  $W_\Psi(k-1) = \prod_{j=1}^{k-1} \Psi(j)$  and we use the convention  $\sum_1^0 = 0$ .

- (2) *Let now  $-\varliminf_{t \rightarrow \infty} \xi_t = \overline{\lim}_{t \rightarrow \infty} \xi_t = \infty$  a.s. or alternatively  $\phi_+(0) = \phi_-(0) = 0$  and assume also that  $\varliminf_{t \rightarrow \infty} \mathbb{P}(\xi_t < 0) = \rho \in [0, 1)$ , that is the celebrated Spitzer's condition holds. Then  $\kappa_-(r) = \phi_-^r(0) \in RV_\rho$  and for any  $a \in (0, 1 - \mathbf{a}_+)$  and any  $f \in \mathbf{C}_b(\mathbb{R}^+)$*

$$(2.41) \quad \lim_{t \rightarrow \infty} \frac{\mathbb{E} [I_\Psi^{-a}(t) f(I_\Psi(t))]}{\kappa_-(\frac{1}{t})} = \int_0^\infty f(x) \vartheta_a(dx),$$

where  $\vartheta_a$  is a finite positive measure on  $(0, \infty)$  such that for any  $c \in (a - 1 + \mathbf{a}_+, 0)$  and  $x > 0$

$$(2.42) \quad \vartheta_a(0, x) = -\frac{x^{-c}}{2\pi\Gamma(1-\rho)} \int_{-\infty}^{\infty} x^{-ib} \frac{\mathcal{M}_\Psi(c + 1 - a + ib)}{c + ib} db$$

and

$$(2.43) \quad \vartheta_a(\mathbb{R}^+) = \frac{1}{\Gamma(1-\rho)} \frac{\Gamma(1-a)}{W_{\phi_+}(1-a)} W_{\phi_-}(a).$$

This theorem is proved in Section 5.8.

*Remark 2.25.* Note that if  $\mathbb{E}[\xi_1] = 0$ ,  $\mathbb{E}[\xi_1^2] < \infty$  then we have that  $\kappa_-(r) \stackrel{0}{\sim} Cr^{\frac{1}{2}}$  and  $C$  can be elucidated to a degree from the Fristed's formula, [9, Chapter VI], which evaluates  $\kappa_-(r)$ . Therefore, if  $f(x) = x^{-a}f_1(x)$  with  $a \in (0, 1 - \mathbf{a}_+)$  and  $f_1 \in \mathcal{C}_b(\mathbb{R}^+)$  then  $\lim_{t \rightarrow \infty} C\sqrt{t}\mathbb{E}[f(I_t)] = \int_0^\infty f_1(x)\vartheta_a(dx)$ . This result has been at the core of two recent preprints which deal with the large temporal asymptotic behaviour of extinction and explosion probabilities of continuous state branching processes in Lévy random environment, see [44, 51]. However, there the authors need to impose some stringent restrictions such as  $f$  being ultimately non-increasing and of a specific form, and the Lévy process  $\xi$  to have some two-sided exponential moments. Thus, our Theorem 2.24 can significantly extend the aforementioned results of [44, 51] and furnish new ones when  $\mathbb{E}[\xi_1^2] = \infty$  and  $\lim_{t \rightarrow \infty} \mathbb{P}(\xi_t < 0) = \rho \in [0, 1)$ .

*Remark 2.26.* The main reason and necessity behind the imposition of additional conditions in [44, 51] seem to be due to the fact that the results are obtained through discretization of  $\xi$  and then either by reduction to more general results on specific random sums, see [51], or through reduction to a similar problem for random walks, see [44]. We refer to [1, 37] for the results in the random walk scenario.

**2.3. Intertwining relations of self-similar semigroups and factorization of laws.** By Mellin identification, we obtain as a straightforward consequence of the representation (2.21), the following probabilistic factorizations of the distribution of the exponential functional.

**Theorem 2.27.** *For any  $\Psi \in \mathcal{N}$ , the following multiplicative Wiener-Hopf factorizations of the exponential functional hold*

$$(2.44) \quad I_\Psi \stackrel{d}{=} I_{\phi_+} \times X_{\phi_-} \stackrel{d}{=} \bigotimes_{k=0}^{\infty} (C_k \mathfrak{B}_k X_\Psi \times \mathfrak{B}_{-k} Y_\Psi),$$

where  $\times$  stands for the product of independent random variables. The law of the positive variables  $X_\Psi, Y_\Psi$  are given by

$$(2.45) \quad \begin{aligned} \mathbb{P}(X_\Psi \in dx) &= \frac{1}{\phi_+(1)} (\bar{\mu}_+(-\ln x)dx + \phi_+(0)dx + \mathbf{d}_+\delta_1(dx)), \quad x \in (0, 1) \\ \mathbb{P}(Y_\Psi \in dx) &= \phi_-(0)\Upsilon_-(dx), \quad x > 1 \end{aligned}$$

where  $\Upsilon_-(dv) = U_-(d \ln(v))$ ,  $v > 1$  is the image of the potential measure  $U_-$  by the mapping  $y \mapsto \ln y$ ,

$$C_0 = e^{\gamma\phi_+ + \gamma\phi_- - \gamma + 1 - \frac{\phi'_+(1)}{\phi_+(1)}}, C_k = e^{\frac{1}{k+1} - \frac{\phi'_+(k+1)}{\phi_+(k+1)} - \frac{\phi'_-(k)}{\phi_-(k)}}, k = 1, 2, \dots,$$

where  $\gamma$  is the Euler-Mascheroni constant, and for integer  $k$ ,  $\mathfrak{B}_k X$  is the variable defined by

$$\mathbb{E}[f(\mathfrak{B}_k X)] = \frac{\mathbb{E}[X^k f(X)]}{\mathbb{E}[X^k]}.$$

*Remark 2.28.* Note that the first factorization in (2.44) is proved in [52, 58] under the assumptions  $\Pi(dx)\mathbb{1}_{\{x>0\}} = \pi_+(x)dx$  and  $\pi_+$  is non-decreasing on  $\mathbb{R}^+$  with the stronger relation  $X_{\phi_-} = I_\psi$  with  $\psi(z) = z\phi_-(z) \in \mathcal{N}_-$ . It has been announced in generality in [59] and building on it in [4]

the authors derive a new three term factorization of  $I_\Psi$ . The second factorization of (2.44) is new in such generality. When  $\Psi(z) = -\phi_+(-z) \in \mathcal{B}$  that is  $\xi$  is a subordinator then (2.44) is contained in [2, Theorem 3]. For the class of meromorphic Lévy processes,  $I_\Psi$  has been factorized in an infinite product of independent Beta random variables, see e.g. [35].

Next, we discuss some immediate results for the positive self-similar Markov process whose semigroups we call for brevity the positive self-similar semigroups and denote by  $K^\Psi = (K_t^\Psi)_{t \geq 0}$ . The dependence on  $\Psi \in \overline{\mathcal{N}}$  is due to the celebrated Lamperti transformation which identifies a bijection between the positive self-similar semigroups and  $\overline{\mathcal{N}}$ , see [42]. We recall that for some  $\alpha > 0$ , any  $f \in \mathcal{C}_0([0, \infty))$  and any  $x, c > 0$ ,  $K_t^\Psi f(cx) = K_{c^{-\alpha}t}^\Psi d_c f(x)$ , where  $d_c f(x) = f(cx)$  is the dilatation operator. Without loss of generality we consider  $\alpha = 1$  and we introduce the set  $\mathcal{N}_m = \{\Psi \in \mathcal{N} : q = 0, \phi'_+(0^+) < \infty, \mathcal{Z}_0(\Psi) = \{0\}\}$ . From (1.2), (2.17) and Theorem 3.2(2) it is clear that from probabilistic perspective the class  $\mathcal{N}_m$  stands for the conservative Lévy processes that do not live on a lattice which either drift to infinity and possess finite positive mean or oscillate but the ascending ladder height process has a finite mean, that is  $\phi'_+(0^+) < \infty$ . It is well-known from [15] that  $\Psi \in \mathcal{N}_m$  if and only if  $K^\Psi$  possesses an entrance law from zero. More specifically, there exists a family of probability measures  $\nu^\Psi = (\nu_t^\Psi)_{t > 0}$  such that for any  $f \in \mathcal{C}_0([0, \infty))$  and any  $t, s > 0$ ,  $\nu_{t+s}^\Psi f = \int_0^\infty f(x) \nu_{t+s}^\Psi(dx) = \int_0^\infty K_s^\Psi f(x) \nu_t^\Psi(dx) = \nu_t^\Psi K_s^\Psi f$ . We denote by  $V_\Psi$  the random variable whose law is  $\nu_1^\Psi$ .

**Theorem 2.29.** *Let  $\Psi \in \mathcal{N}_m$ .*

- (1) *Then the Mellin transform  $\mathcal{M}_{V_\Psi}$  of  $V_\Psi$  is the unique solution to the following functional equation with initial condition  $\mathcal{M}_{V_\Psi}(1) = 1$*

$$(2.46) \quad \mathcal{M}_{V_\Psi}(z+1) = \frac{\Psi(z)}{z} \mathcal{M}_{V_\Psi}(z), \quad z \in \mathbb{C}_{(\bar{a}, 1)},$$

*and admits the representation*

$$(2.47) \quad \mathcal{M}_{V_\Psi}(z) = \frac{1}{\phi'_+(0^+)} \frac{\Gamma(1-z)}{W_{\phi_+}(1-z)} W_{\phi_+}(z), \quad z \in \mathbb{C}_{(\bar{a}, 1)}.$$

- (2) *Let in addition  $\Pi(dx)\mathbb{I}_{\{x>0\}} = \pi_+(x)dx$ , with  $\pi_+$  non-increasing on  $\mathbb{R}^+$ , see (2.1). Then  $\Lambda_{\phi_+} f(x) = \mathbb{E}[f(xV_{\phi_+})]$  is a continuous linear operator from  $\mathcal{C}_0([0, \infty))$ , endowed with the uniform topology, into itself and we have the following intertwining identity on  $\mathcal{C}_0([0, \infty))$*

$$(2.48) \quad K_t^\psi \Lambda_{\phi_+} f = \Lambda_{\phi_+} K_t^\Psi f, \quad t \geq 0,$$

*where  $\psi(z) = z\phi_-(z) \in \mathcal{N}$ .*

This theorem is proved in Section 6.2.

*Remark 2.30.* The literature on intertwining of Markov semigroups is very rich and reveals that it is useful in a variety of contexts, see e.g. Diaconis and Fill [23] in relation with strong uniform times, by Carmona, Petit and Yor [21] in relation to the so-called selfsimilar saw tooth-processes, by Borodin and Corwin [19] in the context of Macdonald processes, by Pal and Shkolnikov [50] for linking diffusion operators, and, by Patie and Simon [63] to relate classical fractional operators. In this direction, it seems that the family of intertwining relations (2.48) are the first instances involving a Markov processes with two-sided jumps.

*Remark 2.31.* More recently, this type of commutation relations have proved to be a natural concept in some new developments of spectral theory. We refer to the work of Miclo [47] where it is shown that the notions of isospectrality and intertwining of some self-adjoint Markov operators are equivalent leading to an alternative view of the work of Bérard [5] on isospectral compact Riemannian manifolds, see also Arendt et al. [3] for similar developments that enable them to provide counterexamples to the famous Kac's problem. Intertwining is also the central idea in the recent works from the authors [60, 61] on the spectral analysis of classes of non-self-adjoint and non-local Markov semigroups. We also emphasize that the intertwining relation (2.48) and more generally the analytical properties of the solution of the recurrence equation (1.1) presented in this paper are critical in the spectral theory of the entire class of positive self-similar Markov semigroups developed in [62].

### 3. THE CLASS OF BERNSTEIN-GAMMA FUNCTIONS

Perhaps the most celebrated special function is the gamma function  $\Gamma$  introduced by Euler in [28]. Amongst its various properties is the fact that it satisfies the recurrence equation

$$(3.1) \quad \Gamma(z+1) = z\Gamma(z), \quad \Gamma(1) = 1,$$

valid on  $\mathbb{C} \setminus \mathbb{N}^-$ . For example, the relation (3.1) allows for the derivation of both the Weierstrass product representation of  $\Gamma(z)$  and the precise Stirling asymptotic expression for the behaviour of the gamma function as  $|z| \rightarrow \infty$  which are given respectively by

$$(3.2) \quad \begin{aligned} \Gamma(z) &= \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \frac{k}{k+z} e^{\frac{1}{k}z}, \\ \Gamma(z) &= \sqrt{2\pi} e^{z \log z - z - \frac{1}{2} \log z} \left( 1 + O\left(\frac{1}{z}\right) \right), \end{aligned}$$

where  $\gamma = \lim_{n \rightarrow \infty} (\sum_{k=1}^n \frac{1}{k} - \ln n)$  is the Euler-Mascheroni constant, see e.g. [43]. Recall from (2.2) that  $\phi \in \mathcal{B}$  if and only if  $\phi \not\equiv 0$  and, for  $z \in \mathbb{C}_{[0, \infty)}$ ,

$$(3.3) \quad \phi(z) = \phi(0) + \mathbf{d}z + \int_0^{\infty} (1 - e^{-zy}) \mu(dy) = \phi(0) + \mathbf{d}z + z \int_0^{\infty} e^{-zy} \bar{\mu}(y) dy,$$

$\phi(0) \geq 0$ ,  $\mathbf{d} \geq 0$  and  $\mu$  is a sigma-finite measure satisfying  $\int_0^{\infty} (1 \wedge y) \mu(dy) < \infty$ . Also here and hereafter we denote the tail of a measure  $\lambda$  by  $\bar{\lambda}(x) = \int_{|y| \geq x} \lambda(dy)$ , provided it exists. Due to its importance the class  $\mathcal{B}$  has been studied extensively in several monographs and papers, see e.g. [61, 68]. Here, we use it to introduce the class of Bernstein-gamma functions denoted by  $\mathcal{W}_{\mathcal{B}}$ , which appear in any main result above.

**Definition 3.1.** We say that  $W_{\phi} \in \mathcal{W}_{\mathcal{B}}$  if and only if for some  $\phi \in \mathcal{B}$

$$(3.4) \quad W_{\phi}(z+1) = \phi(z)W_{\phi}(z), \quad \operatorname{Re}(z) > 0; \quad W_{\phi}(1) = 1;$$

and there exists a positive random variable  $Y_{\phi}$  such that  $W_{\phi}(z+1) = \mathbb{E} \left[ Y_{\phi}^z \right]$ ,  $\operatorname{Re}(z) \geq 0$ .

Note that when, in (3.3),  $\phi(z) = z \in \mathcal{B}$  then  $W_{\phi}$  boils down to the gamma function with  $Y_{\phi}$  a standard exponential random variable. This yields to the well known integral representation of the gamma function  $\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx$  valid on  $\operatorname{Re}(z) > 0$ . The functions  $W_{\phi} \in \mathcal{W}_{\mathcal{B}}$  have already appeared explicitly, see [2, 36, 59, 61] or implicitly, [18, 46] in the literature. However, with the exception of [61, Chapter 6] we are not aware of other studies that focus on the



understanding of  $W_\phi$  as a holomorphic function on the complex half-plane  $\mathbb{C}_{(0,\infty)}$ . The latter is of significant importance at least for the following reasons. First, the class  $\mathcal{W}_\mathcal{B}$  arises in the spectral study of Markov semigroups and the quantification of its analytic properties in terms of  $\phi \in \mathcal{B}$  virtually opens the door to obtaining explicit information about most of the spectral objects and quantities of interest, see [61]. Then, the class  $\mathcal{W}_\mathcal{B}$  appears in the full explicit description of  $\mathcal{M}_\Psi$  and hence of  $\mathcal{M}_{I_\Psi}$ , see (2.6), and thus the understanding of its analytic properties yields detailed information about the law of those exponential functionals. Also, the class  $\mathcal{W}_\mathcal{B}$  contains some well-known special functions, e.g. the Barnes-gamma function and the q-gamma function related to the q-calculus, see [10, 22], [61, Remark 6.4], and the derivation of the analytic properties of  $\mathcal{W}_\mathcal{B}$  in general will render many special computations and efforts to direct application of the results concerning the functions comprising  $\mathcal{W}_\mathcal{B}$ . Equations of the type (3.4) have been considered on  $\mathbb{R}^+$  in greater generality. For example when  $\phi$  is merely a log-concave function on  $\mathbb{R}^+$ , Webster [73] has provided comprehensive results on the solution to (3.4), which we use readily throughout this work when possible since  $\phi \in \mathcal{B}$  is a log-concave function on  $\mathbb{R}^+$  itself.

In this Section, we start by stating the main results of our work concerning the class  $\mathcal{W}_\mathcal{B}$  and postpone their proofs to the subsections 3.1-3.7. In particular, we derive and state representations, asymptotic and analytical properties of  $W_\phi$ . To do so we introduce some notation. Similarly to (2.24) for  $\Psi$ , we write and have, for any  $a \geq \mathfrak{a}_\phi$ ,

$$(3.5) \quad \mathcal{Z}_a(\phi) = \{z \in \mathbb{C}_a : \phi(z) = 0\} = \{z \in \mathbb{C}_a : \phi(\bar{z}) = 0\}$$

and as in (2.3), (2.4) and (2.5),

$$(3.6) \quad \mathfrak{u}_\phi = \sup \{u \leq 0 : \phi(u) = 0\} \in [-\infty, 0]$$

$$(3.7) \quad \mathfrak{a}_\phi = \inf \{u < 0 : \phi \in \mathbf{A}_{(u,\infty)}\} \in [-\infty, 0]$$

$$(3.8) \quad \bar{\mathfrak{a}}_\phi = \max \{\mathfrak{a}_\phi, \mathfrak{u}_\phi\} = \sup \{u \leq 0 : \phi(u) = -\infty \text{ or } \phi(u) = 0\} \in [-\infty, 0].$$

The next theorem contains some easy but very useful results which stem from the existing literature. Before we state them we recall from [61, Chapter 6] that the class  $\mathcal{B}$  is in bijection with  $\mathcal{W}_\mathcal{B}$  via the absolutely convergent product on (at least)  $\mathbb{C}_{(0,\infty)}$

$$(3.9) \quad W_\phi(z) = \frac{e^{-\gamma_\phi z}}{\phi(z)} \prod_{k=1}^{\infty} \frac{\phi(k)}{\phi(k+z)} e^{\frac{\phi'(k)}{\phi(k)} z},$$

where

$$(3.10) \quad \gamma_\phi = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n \frac{\phi'(k)}{\phi(k)} - \ln \phi(n) \right) \in \left[ -\ln \phi(1), \frac{\phi'(1)}{\phi(1)} - \ln \phi(1) \right].$$

**Theorem 3.2.** *Let  $\phi \in \mathcal{B}$ .*

- (1)  $W_\phi \in \mathbf{A}_{(\bar{\mathfrak{a}}_\phi, \infty)} \cap \mathbf{M}_{(\mathfrak{a}_\phi, \infty)}$  and  $W_\phi$  is zero-free on  $\mathbb{C}_{(\mathfrak{a}_\phi, \infty)}$ . If  $\phi(0) > 0$  then  $W_\phi \in \mathbf{A}_{(0, \infty)}$  and  $W_\phi$  is zero-free on  $\mathbb{C}_{[0, \infty)}$ . If  $\phi(0) = 0$  (resp.  $\phi(0) = 0$  and  $\phi'(0^+) < \infty$ ) then  $W_\phi$  (resp.  $z \mapsto W_\phi(z) - \frac{1}{\phi'(0^+)z}$ ) extends continuously to  $i\mathbb{R} \setminus \mathcal{Z}_0(\phi)$  (resp.  $(i\mathbb{R} \setminus \mathcal{Z}_0(\phi)) \cup \{0\}$ ) and if  $\mathfrak{z} \in \mathcal{Z}_0(\phi)$  then  $\lim_{\substack{\operatorname{Re}(z) \geq 0 \\ z \rightarrow \mathfrak{z}}} \phi(z)W_\phi(z) = W_\phi(\mathfrak{z} + 1)$ .

- (2) There exists  $\mathfrak{z} \in \mathcal{Z}_0(\phi)$  with  $\mathfrak{z} \neq 0$  if and only if  $\phi(0) = \mathfrak{d} = 0$  and  $\mu = \sum_{n=0}^{\infty} c_n \delta_{\bar{h}k_n}$  with  $\sum_{n=1}^{\infty} c_n < \infty$ ,  $\bar{h} > 0$ ,  $k_n \in \mathbb{N}$  and  $c_n \geq 0$  for all  $n \in \mathbb{N}$ . In this case, the mappings  $z \mapsto e^{-z \ln \phi(\infty)} W_\phi(z)$  and  $z \mapsto |W_\phi(z)|$  are periodic with period  $\frac{2\pi i}{\bar{h}}$  on  $\mathbb{C}_{(0, \infty)}$ .

- (3) Assume that  $\mathbf{u}_\phi \in (-\infty, 0)$ ,  $\mathcal{Z}_{\mathbf{u}_\phi}(\phi) = \{\mathbf{u}_\phi\}$  and  $|\phi'(\mathbf{u}_\phi^+)| < \infty$ , the latter being always true if  $\mathbf{u}_\phi > \mathbf{a}_\phi$ . Then  $z \mapsto W_\phi(z) - \frac{W_\phi(1+\mathbf{u}_\phi)}{\phi'(\mathbf{u}_\phi^+)(z-\mathbf{u}_\phi)} \in \mathbf{A}_{[\mathbf{u}_\phi, \infty)}$ . In this setting  $\phi'(\mathbf{u}_\phi^+) = \mathbf{d} + \int_0^\infty ye^{-\mathbf{u}_\phi y} \mu(dy) \in (0, \infty]$ .
- (4) Assume that  $\mathbf{a}_\phi < \mathbf{u}_\phi \leq 0$  and put  $N_{\mathbf{a}_\phi} = \max\{n \in \mathbb{N} : \mathbf{u}_\phi - n > \mathbf{a}_\phi\} \in \mathbb{N} \cup \{\infty\}$ . Then there exists an open set  $\mathcal{O} \subset \mathbb{C}$  such that  $[\mathbf{u}_\phi - N_{\mathbf{a}_\phi}, 1] \subset \mathcal{O}$ , if  $N_{\mathbf{a}_\phi} < \infty$ , and  $(-\infty, 1] \subset \mathcal{O}$ , if  $N_{\mathbf{a}_\phi} = \infty$ , and  $W_\phi$  is meromorphic on  $\mathcal{O}$  with simple poles at  $\{\mathbf{u}_\phi - k\}_{0 \leq k < N_{\mathbf{a}_\phi} + 1}$  and residues  $\left\{ \mathfrak{R}_k = \frac{W_\phi(1+\mathbf{u}_\phi)}{\phi'(\mathbf{u}_\phi) \prod_{j=1}^k \phi(\mathbf{u}_\phi - j)} \right\}_{0 \leq k < N_{\mathbf{a}_\phi} + 1}$  with  $\prod_1^0 = 1$ .

Theorem 3.2 and especially the representation (3.9) allow for the understanding of the asymptotic behaviour of  $|W_\phi(z)|$ , as  $|z| \rightarrow \infty$ . We employ the floor and ceiling functions  $[u] = \max\{n \in \mathbb{N} : n \leq u\}$  and  $\lceil u \rceil = \min\{n \in \mathbb{N} : u > n\}$ . To be able to state the main asymptotic results we use the following notation. For any  $\phi \in \mathcal{B}$  we introduce several important functions that describe the asymptotic behaviour of  $|W_\phi(z)|$  in detail. Define formally, for any  $z = a + ib \in \mathbb{C}_{(0, \infty)}$ ,

$$(3.11) \quad A_\phi(z) = \int_0^{|b|} \arg \phi(a + iu) du.$$

The function  $A_\phi$  describes the asymptotics of  $|W_\phi(z)|$  along imaginary lines of the type  $\mathbb{C}_a$ ,  $a > 0$ . The next functions defined for  $a > 0$

$$(3.12) \quad G_\phi(a) = \int_1^{1+a} \ln \phi(u) du, \quad H_\phi(a) = \int_1^{1+a} \frac{u\phi'(u)}{\phi(u)} du \quad \text{and} \quad H_\phi^*(a) = a \left( \frac{\phi(a+1) - \phi(a)}{\phi(a)} \right)$$

appear in the asymptotic behaviour of  $W_\phi(x)$  for large  $x > 0$  which in turn can be seen as an extension of the Stirling formula for the classical gamma function. Finally, we introduce the functions that control the error coming from the approximations. For  $z = a + ib \in \mathbb{C}_{(0, \infty)}$ , writing  $P(u) = (u - [u])(1 - (u - [u]))$ , we set

$$(3.13) \quad E_\phi(z) = \frac{1}{2} \int_0^\infty P(u) \left( \ln \frac{|\phi(u+z)|}{\phi(u+a)} \right)'' du,$$

$$(3.14) \quad R_\phi(a) = \frac{1}{2} \int_1^\infty P(u) \left( \ln \frac{|\phi(u+a)|}{\phi(u)} \right)'' du,$$

and

$$(3.15) \quad T_\phi = \frac{1}{2} \int_1^\infty P(u) \left( \left( \frac{\phi'(u)}{\phi(u)} \right)^2 - \frac{\phi''(u)}{\phi(u)} \right) du.$$

Finally, we introduce subclasses of  $\mathcal{B}$  equivalent to  $\overline{\mathcal{N}}_\beta$  and  $\overline{\mathcal{N}}(\Theta)$ , see (2.10) and (2.12). For any  $\beta \in [0, \infty]$

$$(3.16) \quad \mathcal{B}_\beta = \left\{ \phi \in \mathcal{B} : \lim_{|b| \rightarrow \infty} |b|^{\beta-\varepsilon} |W_\phi(a+ib)| = 0, \forall a > \bar{\mathbf{a}}_\phi, \forall \varepsilon \in (0, \beta) \right\} \\ \cap \left\{ \phi \in \mathcal{B} : \lim_{|b| \rightarrow \infty} |b|^{\beta+\varepsilon} |W_\phi(a+ib)| = \infty, \forall a > \bar{\mathbf{a}}_\phi, \forall \varepsilon \in (0, \beta) \right\}$$

and any  $\theta \in (0, \frac{\pi}{2}]$

$$(3.17) \quad \mathcal{B}(\theta) = \left\{ \phi \in \mathcal{B} : \overline{\lim}_{|b| \rightarrow \infty} \frac{\ln |W_\phi(a+ib)|}{|b|} \leq -\theta, \forall a > \bar{a}_\phi \right\}.$$

We now state our second main result which can be thought of as the Stirling asymptotic for the Bernstein-gamma functions as recalled in (3.2).

**Theorem 3.3.** (1) For any  $a > 0$ , we have that

$$(3.18) \quad \sup_{\phi \in \mathcal{B}} \sup_{z \in \mathbb{C}(a, \infty)} |E_\phi(z)| < \infty \text{ and } \sup_{\phi \in \mathcal{B}} \sup_{c > a} |R_\phi(c)| < \infty.$$

Moreover, for any  $\phi \in \mathcal{B}$  and any  $z = a + ib \in \mathbb{C}(0, \infty)$ , we have that

$$(3.19) \quad |W_\phi(z)| = \frac{\sqrt{\phi(1)}}{\sqrt{\phi(a)\phi(1+a)|\phi(z)|}} e^{G_\phi(a) - A_\phi(z)} e^{-E_\phi(z) - R_\phi(a)}$$

with

$$(3.20) \quad \Theta_\phi(a+ib) = \frac{1}{|b|} A_\phi(a+ib) = \frac{1}{|b|} \int_a^\infty \ln \left( \frac{|\phi(u+i|b)|}{\phi(u)} \right) du \in \left[ 0, \frac{\pi}{2} \right].$$

As a result, for any  $b \in \mathbb{R}$ ,  $a \mapsto A_\phi(a+ib)$  is non-increasing on  $\mathbb{R}^+$  and if  $\phi^r(z) = r + dz + \int_0^\infty (1 - e^{-zy}) \mu(dy)$ ,  $r \geq 0$ , then  $r \mapsto A_{\phi^r}(z)$  is non-increasing for any  $z \in \mathbb{C}(0, \infty)$ .

(2) For any fixed  $b \in \mathbb{R}$  and large  $a$ , we have

$$(3.21) \quad |W_\phi(a+i|b)| = \frac{e^{-T_\phi}}{\sqrt{|\phi(z)|\phi(1)}} e^{a \ln \phi(a) - H_\phi(a) + H_\phi^*(a) - A_\phi(z)} \left( 1 + \mathcal{O}\left(\frac{1}{a}\right) \right),$$

where  $\lim_{a \rightarrow \infty} A_\phi(a+ib) = 0$ ,  $T_\phi = \lim_{a \rightarrow \infty} (E_\phi(a+ib) + R_\phi(a))$  is defined in (3.15) and

$$(3.22) \quad 0 \leq \underline{\lim}_{a \rightarrow \infty} \frac{H_\phi(a)}{a} \leq \overline{\lim}_{a \rightarrow \infty} \frac{H_\phi(a)}{a} \leq 1 \text{ and } 0 \leq \underline{\lim}_{a \rightarrow \infty} H_\phi^*(a) \leq \overline{\lim}_{a \rightarrow \infty} H_\phi^*(a) \leq 1.$$

(3) (a) If  $\phi \in \mathcal{B}_P$ , then, for any  $a > 0$  fixed and  $b \in \mathbb{R}$ ,

$$(3.23) \quad A_\phi(a+i|b|) = \frac{\pi}{2}|b| - \left( a + \frac{\phi(0)}{d} \right) \ln |b| - H(|b|),$$

where  $H(|b|) \cong o(|b|)$  and

$$\underline{\lim}_{b \rightarrow \infty} \frac{d H(b)}{\ln(b) \bar{\mu}\left(\frac{1}{b}\right)} \geq 1.$$

Thus  $\mathcal{B}_P \subseteq \mathcal{B}\left(\frac{\pi}{2}\right)$ .

(b) Next, let  $\phi \in \mathcal{B}_{R_\alpha}$ , see (2.14), with  $\alpha \in (0, 1)$ . Then, for any fixed  $a > 0$ ,

$$(3.24) \quad A_\phi(a+i|b|) \cong \frac{\pi}{2} \alpha |b| (1 + o(1))$$

and thus  $\mathcal{B}_{R_\alpha} \subseteq \mathcal{B}\left(\frac{\pi}{2}\alpha\right)$ .

(c) Let  $\phi \in \mathcal{B}_P^c$ , that is  $d = 0$ , such that  $\mu(dy) = v(y)dy$ . If  $v(0^+) < \infty$  exists and  $\|v\|_\infty = \sup_{x \geq 0} v(x) < \infty$ , then  $\phi \in \mathcal{B}_{N_\phi}$  with  $N_\phi = \frac{v(0^+)}{\phi(\infty)}$ . If  $v(0^+) = \infty$ ,  $v(y) = v_1(y) + v_2(y)$ ,  $v_1, v_2 \in L^1(\mathbb{R}^+)$ ,  $v_1 \geq 0$  is non-increasing in  $\mathbb{R}^+$ ,  $\int_0^\infty v_2(y)dy \geq 0$  and  $|v_2(y)| \leq \left( \int_y^\infty v_1(r)dr \right) \vee C$  for some  $C > 0$ , then  $\phi \in \mathcal{B}_\infty$ .

*Remark 3.4.* Note the beautiful dependence of (3.19) on the geometry of  $\phi(\mathbb{C}_{(0,\infty)}) \subseteq \mathbb{C}_{(0,\infty)}$ , see (3.33) for this inclusion. The more  $\phi$  shrinks  $\mathbb{C}_{(0,\infty)}$  the smaller the contribution of  $A_\phi$ . In fact  $\frac{1}{b}A_\phi(a+ib)$ , as  $b \rightarrow \infty$ , measures the fluctuations of the average angle along the contour  $\phi(\mathbb{C}_a)$ , which are necessarily of lesser order than those of  $\arg \phi(a+ib)$ ,  $b \rightarrow \infty$ , along  $\mathbb{C}_a$ .

*Remark 3.5.* When  $z = a$ , modulo to the specifications of the constants, the central result (3.19) has appeared for log-concave functions in [73, Theorem 6.3] and for Bernstein functions in [61, Theorem 5.1]. Here, we provide an explicit representation of the terms of the asymptotics of  $|W_\phi(a+ib)|$ , as  $a \rightarrow \infty$ , which depend on the real part of  $a$  solely.

*Remark 3.6.* Since  $W_\phi \in \mathbf{A}_{(\bar{\mathbf{a}}_\phi, \infty)} \cap \mathbf{M}_{(\mathbf{a}_\phi, \infty)}$  one can extend, via (3.4), the estimate (3.19), away from the poles residing in  $\mathbb{C}_{(\mathbf{a}_\phi, \bar{\mathbf{a}}_\phi]}$ , to  $z \in \mathbb{C}_{(\mathbf{a}_\phi, \infty)}$ . Indeed, setting, for any  $c \in \mathbb{R}$ ,  $c^\rightarrow = (\lfloor -c \rfloor + 1)\mathbb{I}_{\{c \leq 0\}}$ , then, for  $a > \mathbf{a}_\phi$  and  $z = a + ib$  not a pole, one has

$$(3.25) \quad W_\phi(a+ib) = W_\phi(a+a^\rightarrow+ib) \prod_{j=0}^{a^\rightarrow-1} \frac{1}{\phi(a+j+ib)}$$

with the convention that  $\prod_0^{-1} = 1$ .

*Remark 3.7.* With the help of additional notation and arguments the remainder term  $H(b)$  in (3.23) can be much better understood, see Proposition 3.16. When the convolutions of  $\bar{\mu}$  can be evaluated full asymptotic expansion of  $A_\phi$  can be achieved, see Remark 3.17 below.

*Remark 3.8.* The requirements for the case (3c) might seem stringent but in fact what they impose is that in a small positive neighbourhood of 0, the density can be decomposed as a non-increasing, integrable away from zero function and an oscillating error function  $v_2$  that is of smaller order than  $\int_y^\infty v_1(r)dr$ . This is obviously the case when the density  $v$  itself is non-increasing.

The next theorem contains alternative representations of  $W_\phi$ , which modulo to an easy extension to  $\mathbb{C}_{(0,\infty)}$  is due to [36] as well as a number of mappings that can be useful in a variety of contexts.

**Theorem 3.9.** *Let  $\phi, \underline{\phi} \in \mathcal{B}$ .*

(1)  $z \mapsto \log W_\phi(z+1) \in \mathcal{N}$  with

$$(3.26) \quad \log W_\phi(z+1) = (\ln \phi(1))z + \int_0^\infty (e^{-zy} - 1 - z(e^{-y} - 1)) \frac{\kappa(dy)}{y(e^y - 1)}$$

$\kappa(dy) = \int_0^y U(dy-r)(r\mu(dr) + \delta_{\mathbf{a}}(dr))$ , where  $U$  is the potential measure associated to  $\phi$ , see Proposition 3.14(5).

(2)  $z \mapsto \log(W_\phi(z+1)W_{\underline{\phi}}(1-z)) \in \mathcal{N}$

(3) If  $u \mapsto \frac{\phi}{\underline{\phi}}(u)$  is a non-zero completely monotone function then there exists a positive variable  $I$  which is moment determinate and such that, for all  $n \geq 0$ ,  $\mathbb{E}[I^n] = \frac{W_\phi(n+1)}{W_{\underline{\phi}}(n+1)}$ . Next, assume that  $\phi \in \mathcal{B}_{\mathbf{N}_\phi}$  and  $\underline{\phi} \in \mathcal{B}_{\mathbf{N}_{\underline{\phi}}}$ . If  $\mathbf{N} = \mathbf{N}_{\underline{\phi}} - \mathbf{N}_\phi > \frac{1}{2}$ , then the law of  $I$  is absolutely continuous with density  $f_I \in L^2(\mathbb{R}^+)$  and if  $\mathbf{N} > 1$  then  $f_I \in C_0^{[\mathbf{N}_\Psi]-2}(\mathbb{R}^+)$ .

- (4)  $u \mapsto \left(\frac{\phi'}{\phi} - \frac{\phi'}{\phi}\right)(u)$  is completely monotone if and only if  $z \mapsto \log \frac{W_\phi}{W_{\underline{\phi}}}(z+1) \in \mathcal{N}$ , that is, with the notation of the previous item,  $\log I$  is infinitely divisible on  $\mathbb{R}$ . An equivalent condition is that the measure  $U_\phi \star \mu_*$  is absolutely continuous with respect to the measure  $U_{\underline{\phi}} \star \mu_*$  with a density  $h \leq 1$ .

*Remark 3.10.* Note that in the trivial case  $\phi(z) = z$  then  $\kappa(dy) = dy$  and the representation (3.26) yields to the classical Malmstén formula for the gamma function, see [27]. With the recurrence equation (3.4), the Weierstrass product (3.9) and the Mellin transform of a positive random variable, see Definition 3.1, this integral provides a fourth representation that the set of functions  $W_\phi$  share with the classical gamma function, justifying our choice to name them the Bernstein-gamma functions.

*Remark 3.11.* We mention that when  $\phi(0) = 0$  the representation (3.26) for  $z \in \mathbb{R}^+$  appears in [36, Theorem 3.1] and for any  $\phi \in \mathcal{B}$  in [6, Theorem 2.2]. We also emphasize that with the aim of getting detailed information regarding bounds and asymptotic behaviours of  $|W_\phi(z)|$ , see Theorem 3.3, we found the Weierstrass product representation more informative to work with. However, as it is aptly illustrated by the authors of [36], the integral representation is useful for other purposes such as, for instance, for proving the multiplicative infinite divisibility property of some random variables.

*Remark 3.12.* The existence of Bernstein functions whose ratios are completely monotone, that is, the condition in item (3), has been observed by the authors in [61].

The final claim shows that the mapping  $\phi \in \mathcal{B} \mapsto W_\phi \in \mathcal{W}_{\mathcal{B}}$  is continuous with respect to the pointwise topology in  $\mathcal{B}$ . This handy result is widely used throughout.

**Lemma 3.13.** *Let  $(\phi_n)_{n \geq 0}$ ,  $\underline{\phi} \in \mathcal{B}$  and  $\lim_{n \rightarrow \infty} \phi_n(a) = \underline{\phi}(a)$  for all  $a > 0$ . Then  $\lim_{n \rightarrow \infty} W_{\phi_n}(z) = W_{\underline{\phi}}(z)$ ,  $z \in \mathbb{C}_{(0, \infty)}$ .*

Before providing the proofs of the previous claims, we collect some classical results concerning the set of Bernstein functions  $\mathcal{B}$  that will be useful also in several remaining parts of the paper. For thorough information on these functions, we refer to the excellent monograph [68]. Then we have the following claims which can be found in [61, Section 4].

**Proposition 3.14.** *Let  $\phi \in \mathcal{B}$ .*

- (1) For any  $z \in \mathbb{C}_{(0, \infty)}$ ,

$$(3.27) \quad \phi'(z) = \mathbf{d} + \int_0^\infty ye^{-zy} \mu(dy) = \mathbf{d} + \int_0^\infty e^{-zy} \bar{\mu}(y) dy - z \int_0^\infty e^{-zy} y \bar{\mu}(y) dy.$$

- (2) For any  $u \in \mathbb{R}^+$ ,

$$(3.28) \quad 0 \leq u\phi'(u) \leq \phi(u) \quad \text{and} \quad |\phi''(u)| \leq 2 \frac{\phi(u)}{u^2}.$$

- (3)  $\phi(u) \cong \mathbf{d}u + \mathfrak{o}(u)$  and  $\phi'(u) \cong \mathbf{d} + \mathfrak{o}(1)$ . Fix  $a > \mathfrak{a}_\phi$ , then  $|\phi(a+ib)| = |a+ib|(\mathbf{d} + \mathfrak{o}(1))$  as  $|b| \rightarrow \infty$ .

- (4) If  $\phi(\infty) < \infty$  and  $\mu$  is absolutely continuous then for any fixed  $a > \mathfrak{a}_\phi$ ,  $\lim_{|b| \rightarrow \infty} \phi(a+ib) = \phi(\infty)$ .

(5) The mapping  $u \mapsto \frac{1}{\phi(u)}$ ,  $u \in \mathbb{R}^+$ , is completely monotone, i.e. there exists a positive measure  $U$ , whose support is contained in  $[0, \infty)$ , called the potential measure, such that the Laplace transform of  $U$  is given via the identity

$$\frac{1}{\phi(u)} = \int_0^\infty e^{-uy} U(dy).$$

(6) In any case,

$$(3.29) \quad \lim_{u \rightarrow \infty} \frac{\phi(u \pm a)}{\phi(u)} = 1 \quad \text{uniformly for } a\text{-compact intervals on } \mathbb{R}^+.$$

**3.1. Proof of Theorem 3.2.** We start with item 1. The fact that  $W_\phi \in \mathbf{A}_{(\bar{a}_\phi, \infty)}$  is a consequence of [61, Theorem 6.1] and is essentially due to the recurrence equation (3.4) and the fact that  $\phi$  is zero-free on  $\mathbb{C}_{(\bar{a}_\phi, \infty)}$ . Also,  $W_\phi \in \mathbf{M}_{(a_\phi, \infty)}$  comes from the observation that  $0 \neq \phi \in \mathbf{A}_{(a_\phi, \infty)}$ , that is  $\phi$  can only have zeros of finite order, and (3.4) which allows a recurrent meromorphic extension to  $\mathbb{C}_{(a_\phi, \infty)}$ .  $W_\phi$  is zero-free on  $\mathbb{C}_{(\bar{a}_\phi, \infty)}$  follows from [61, Theorem 6.1 and Corollary 7.8] whereas the fact that  $W_\phi$  extends to zero-free on  $\mathbb{C}_{(a_\phi, \infty)}$  thanks to  $\phi \in \mathbf{A}_{(a_\phi, \infty)}$  and (3.4). If  $\phi(0) > 0$  then  $\mathcal{Z}_0(\phi) = \emptyset$  and hence the facts that  $W_\phi$  is zero-free on  $\mathbb{C}_{[0, \infty)}$  and  $W_\phi \in \mathbf{A}_{[0, \infty)}$  are immediate from  $W_\phi$  being zero-free on  $(0, \infty)$  and (3.4). However, when  $\phi(0) = 0$  relation (3.4) ensures that  $W_\phi$  extends continuously to  $i\mathbb{R} \setminus \mathcal{Z}_0(\phi)$  and clearly if  $\mathfrak{z} \in \mathcal{Z}_0(\phi)$  then  $\lim_{\operatorname{Re}(z) \geq 0, z \rightarrow \mathfrak{z}} \phi(z)W_\phi(z) = W_\phi(\mathfrak{z} + 1)$ . Finally, let us assume that  $\phi'(0^+) = \mathfrak{d} + \int_0^\infty y\mu(dy) < \infty$  and  $\{0\} \in \mathcal{Z}_0(\phi)$ , that is  $\phi(0) = 0$ . From the assumption  $\phi'(0^+) < \infty$  and the dominated convergence theorem, we get that  $\phi'$  extends to  $i\mathbb{R}$ , see (3.27). Therefore, from (3.4) and the assumption  $\phi(0) = 0$  we get, for any  $z \in \mathbb{C}_{(0, \infty)}$ , that

$$W_\phi(z + 1) = \phi(z)W_\phi(z) = (\phi(z) - \phi(0))W_\phi(z) = (\phi'(0^+)z + o(|z|))W_\phi(z).$$

Clearly, then the mapping  $z \mapsto W_\phi(z) - \frac{1}{\phi'(0^+)z}$  extends continuously to  $i\mathbb{R} \setminus (\mathcal{Z}_0(\phi) \setminus \{0\})$  provided  $\phi'(0^+) = \mathfrak{d} + \int_0^\infty y\mu(dy) > 0$ , which is apparently true. Let us deal with item (2). We note that

$$e^{-\phi(z)} = \mathbb{E} \left[ e^{-z\xi_1} \right], \quad z \in \mathbb{C}_{[0, \infty)},$$

where  $\xi = (\xi_t)_{t \geq 0}$  is a non-decreasing Lévy process (subordinator) as  $-\phi(-z) = \Psi(z) \in \bar{\mathcal{N}}$ , see (2.1). Thus, if  $\phi(z_0) = 0$  then  $\mathbb{E} \left[ e^{-z_0\xi_1} \right] = 1$ . If in addition,  $z_0 \in \mathbb{C}_{[0, \infty)} \setminus \{0\}$  then  $\phi(0) = 0$  and  $z_0 \in i\mathbb{R}$ . Next,  $\phi(z_0) = 0$  also triggers that  $\xi$  lives on a lattice of size, say  $\bar{h} > 0$ , which immediately gives that  $\mathfrak{d} = 0$  and  $\mu = \sum_{n=1}^\infty c_n \delta_{x_n}$  with  $\sum_{n=1}^\infty c_n < \infty$  and  $\forall n \in \mathbb{N}$  we have that  $x_n = \bar{h}k_n$ ,  $k_n \in \mathbb{N}$ ,  $c_n \geq 0$ . Finally,  $\bar{h}$  can be chosen to be the largest such that  $\xi$  lives on  $(\bar{h}n)_{n \in \mathbb{N}}$ . Thus,

$$\phi(z) = \sum_{n=1}^\infty c_n \left( 1 - e^{-z\bar{h}k_n} \right)$$

and we conclude that  $\phi$  is periodic with period  $\frac{2\pi i}{\bar{h}}$  on  $\mathbb{C}_{(0, \infty)}$ . Next, note that

$$(3.30) \quad \phi(\infty) = \lim_{u \rightarrow \infty} \phi(u) = \lim_{u \rightarrow \infty} \sum_{n=1}^\infty c_n \left( 1 - e^{-uhk_n} \right) = \sum_{n=1}^\infty c_n < \infty.$$

Then (3.10) implies that

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{\phi'(k)}{\phi(k)} = \sum_{k=1}^{\infty} \frac{\phi'(k)}{\phi(k)} = \gamma_{\phi} + \ln \phi(\infty).$$

Thus, from (3.9) we get that

$$W_{\phi}(z) = \frac{e^{-\gamma_{\phi} z}}{\phi(z)} \prod_{k=1}^{\infty} \frac{\phi(k)}{\phi(k+z)} e^{\frac{\phi'(k)}{\phi(k)} z} = \frac{e^{z \ln \phi(\infty)}}{\phi(z)} \prod_{k=1}^{\infty} \frac{\phi(k)}{\phi(k+z)}.$$

Hence, the claim for the  $\frac{2\pi i}{h}$  periodicity of the mappings  $z \mapsto e^{-z \ln \phi(\infty)} W_{\phi}(z)$  and  $z \mapsto |W_{\phi}(z)|$  follows immediately from the periodicity of  $\phi$ . Thus, item (2) is proved. Item (3) follows in the same manner as item (1) noting that when  $\mathbf{u}_{\phi} < 0$  then  $\bar{\mathbf{a}}_{\phi} = \mathbf{u}_{\phi}$ , see (3.6) and (3.8). The last item (4) is an immediate result from (3.4) and the fact that  $\phi' > 0$  on  $(\mathbf{a}_{\phi}, \infty)$ , see (3.27), that is  $\mathbf{u}_{\phi}$  is the unique zero of order one of  $\phi$  on  $(\mathbf{a}_{\phi}, \infty)$ . This ends the proof of Theorem 3.2.

**3.2. Proof of Theorem 3.3(1).** First, the proof and claim of [61, Proposition 6.10(2)] show that, for any  $a > 0$  and any  $\phi \in \mathcal{B}$ ,  $\sup_{z \in \mathbb{C}(a, \infty)} |E_{\phi}(z)| \leq \frac{19}{8}a$  from where we get the first global bound in (3.18). Next, we know from the proof of [61, Proposition 6.10] and see in particular the expressions obtained for [61, (6.33) and (6.34)], that, for any  $z = a + ib \in \mathbb{C}(0, \infty)$ ,  $b > 0$ ,

$$\begin{aligned} |W_{\phi}(z)| &= W_{\phi}(a) \frac{\phi(a)}{|\phi(z)|} \sqrt{\left| \frac{\phi(z)}{\phi(a)} \right|} e^{-\int_0^{\infty} \ln \left| \frac{\phi(u+a+ib)}{\phi(a+bu)} \right| du} e^{-E_{\phi}(z)} \\ (3.31) \quad &= W_{\phi}(a) \sqrt{\left| \frac{\phi(a)}{\phi(z)} \right|} e^{-b\Theta_{\phi}(z)} e^{-E_{\phi}(z)}. \end{aligned}$$

We note that the term  $-E_{\phi}(z)$  is the limit in  $n$  of the error terms  $E_{\phi}^B(n, a) - E_{\phi}^B(n, a + ib)$  in the notation of the proof of [61, Proposition 6.10]. Thus, the last three terms of the second expression in (3.31) are in fact the quantity  $\frac{\phi(a)}{|\phi(z)|} Z_{\phi}(z)$  in the notation of [61, Proposition 6.10, (6.33)]. Let  $\log_0$  stand for the branch of the logarithm such that  $\arg z \in (-\pi, \pi]$ , that is, it coincides with our definition of the argument function. We note from (3.3) and (3.8) that, for any  $z = a + ib \in (\bar{\mathbf{a}}_{\phi}, \infty)$ ,

$$(3.32) \quad \operatorname{Re}(\phi(a + ib)) = \phi(0) + \mathbf{d}a + \int_0^{\infty} (1 - e^{-ay} \cos(by)) \mu(dy) \geq \phi(a) > 0,$$

and thus

$$(3.33) \quad \phi : \mathbb{C}(0, \infty) \rightarrow \mathbb{C}(0, \infty).$$

Therefore,  $\log_0 \phi \in \mathbf{A}(0, \infty)$ . By means of the integral expression in (3.20), an application of the Cauchy integral theorem to  $\log_0 \phi$  on the closed rectangular contour with vertices  $a + ib, u + ib, u$

and  $a$ , for any  $z = a + ib \in \mathbb{C}_{(0,\infty)}$ ,  $b > 0$ ,  $u > a$ , yields that

$$\begin{aligned}
b\Theta_\phi(a + ib) &= \int_a^\infty \ln \left( \frac{|\phi(y + ib)|}{\phi(y)} \right) dy = \lim_{u \rightarrow \infty} \int_a^u \ln \left( \frac{|\phi(y + ib)|}{\phi(y)} \right) dy \\
&= \lim_{u \rightarrow \infty} \operatorname{Re} \left( \int_a^u \log_0 \frac{\phi(y + ib)}{\phi(y)} dy \right) \\
&\stackrel{\text{Cauchy}}{=} \lim_{u \rightarrow \infty} \operatorname{Re} \left( \int_{u \rightarrow u+ib} \log_0 \phi(z) dz \right) - \operatorname{Re} \left( \int_{a \rightarrow a+ib} \log_0 \phi(z) dz \right) \\
&= \int_0^b \arg \phi(a + iy) dy - \lim_{u \rightarrow \infty} \int_0^b \arg \phi(u + iy) dy.
\end{aligned}$$

We investigate the last limit. Note that (3.3) gives that for  $z = a + ib \in \mathbb{C}_{(0,\infty)}$ ,  $b > 0$ ,

$$(3.34) \quad \operatorname{Im}(\phi(a + ib)) = db + \int_0^\infty e^{-ay} \sin(by) \mu(dy).$$

From (3.34) if  $\underline{b} \in (0, b)$  then by the dominated convergence theorem

$$\overline{\lim}_{a \rightarrow \infty} |\operatorname{Im}(\phi(a + i\underline{b}))| \leq \lim_{a \rightarrow \infty} \left( db + b \int_0^\infty e^{-ay} y \mu(dy) \right) = db.$$

Similarly, from (3.32), we have that

$$\underline{\lim}_{a \rightarrow \infty} \operatorname{Re}(\phi(a + i\underline{b})) = \infty \mathbb{I}_{\{a > 0\}} + (\phi(0) + \mu(0, \infty)) \mathbb{I}_{\{a = 0\}}.$$

From the last two relations we conclude that

$$\begin{aligned}
\overline{\lim}_{a \rightarrow \infty} \frac{|\operatorname{Im}(\phi(a + i\underline{b}))|}{\operatorname{Re}(\phi(a + i\underline{b}))} &\leq \frac{db}{\underline{\lim}_{a \rightarrow \infty} \operatorname{Re}(\phi(a + i\underline{b}))} \\
6 &= \lim_{a \rightarrow \infty} \frac{db}{2(\infty \mathbb{I}_{\{a > 0\}} + (\phi(0) + \mu(0, \infty)) \mathbb{I}_{\{a = 0\}})} = 0.
\end{aligned}$$

Therefore, since, from (3.32),  $\operatorname{Re}(\phi(a + i\underline{b})) > 0$ , we get that

$$(3.35) \quad \lim_{a \rightarrow \infty} \arg \phi(a + iy) = 0, \text{ uniformly on } y\text{-compact sets,}$$

and the second term on the right-hand side of the last relation of the equation above (3.34) vanishes. Since  $A_\phi(a + ib) = \int_0^b \arg \phi(a + iy) dy$  we conclude that  $b\Theta_\phi(a + ib) = A_\phi(a + ib)$ , for  $a + ib \in \mathbb{C}_{(0,\infty)}$ ,  $b > 0$ , and thus prove (3.20), as from [61, Proposition 6.10(1)] we have that  $\Theta(a + ib) \in [0, \frac{\pi}{2}]$ . Henceforth, we conclude the alternative expression to (3.31), for  $z = a + ib \in \mathbb{C}_{(0,\infty)}$ ,  $b > 0$ ,

$$(3.36) \quad |W_\phi(z)| = W_\phi(a) \sqrt{\left| \frac{\phi(a)}{\phi(z)} \right|} e^{-A_\phi(z)} e^{-E_\phi(z)}.$$

Since  $\left| \overline{W_\phi(z)} \right| = |W_\phi(\bar{z})|$  we conclude (3.36), for any  $z = a + ib \in \mathbb{C}_{(0,\infty)}$ ,  $b \neq 0$ . Since from (3.13),  $E_\phi(\operatorname{Re}(z)) = 0$ , we deduct that (3.36) holds for  $z = a \in \mathbb{R}^+$  too. Next, let us investigate



$W_\phi(a)$  in (3.36). Recall that, for  $a > 0$ , from (3.9) and (3.10) we get that

$$\begin{aligned}
 (3.37) \quad W_\phi(a) &= \frac{e^{-\gamma_\phi a}}{\phi(a)} \prod_{k=1}^{\infty} \frac{\phi(k)}{\phi(k+a)} e^{\frac{\phi'(k)}{\phi(k)} a} \\
 &= \frac{1}{\phi(a)} \lim_{n \rightarrow \infty} \prod_{k=1}^n \frac{\phi(k)}{\phi(k+a)} e^{a \ln \phi(n)} e^{a \left( \sum_{k=1}^n \frac{\phi'(k)}{\phi(k)} - \ln \phi(n) - \gamma_\phi \right)} \\
 &= \frac{1}{\phi(a)} \lim_{n \rightarrow \infty} \prod_{k=1}^n \frac{\phi(k)}{\phi(k+a)} e^{a \ln \phi(n)} = \frac{1}{\phi(a)} \lim_{n \rightarrow \infty} e^{-S_n(a) + a \ln \phi(n)},
 \end{aligned}$$

where  $S_n(a) = \sum_{k=1}^n \ln \frac{\phi(a+k)}{\phi(k)}$ . Then, we get, from [49, Section 8.2, (2.01), (2.03)] applied to the function  $\ln \frac{\phi(a+u)}{\phi(u)}$ ,  $u > 0$ , with  $m = 1$  in their notation, that

$$\begin{aligned}
 (3.38) \quad S_n(a) &= \int_1^n \ln \frac{\phi(a+u)}{\phi(u)} du + \frac{1}{2} \ln \frac{\phi(1+a)}{\phi(1)} + \frac{1}{2} \ln \frac{\phi(n+a)}{\phi(n)} + R_2(n, a) \\
 &= \frac{1}{2} \ln \frac{\phi(1+a)}{\phi(1)} - \int_1^{a+1} \ln \phi(u) du + \int_0^a \ln \phi(n+u) du + \frac{1}{2} \ln \frac{\phi(n+a)}{\phi(n)} + R_2(n, a),
 \end{aligned}$$

where, recalling that  $P(u) = (u - [u])(1 - (u - [u]))$ , for any  $a > 0$ ,

$$(3.39) \quad R_2(n, a) = \frac{1}{2} \int_1^n P(u) \left( \ln \frac{\phi(a+u)}{\phi(u)} \right)'' du.$$

Using Proposition 3.14(3.29) and (3.38), we get that

$$\begin{aligned}
 \lim_{n \rightarrow \infty} (S_n(a) - a \ln \phi(n)) &= \frac{1}{2} \ln \frac{\phi(1+a)}{\phi(1)} - \int_1^{a+1} \ln \phi(u) du + \lim_{n \rightarrow \infty} \left( \int_0^a \ln \frac{\phi(n+u)}{\phi(n)} du + R_2(n, a) \right) \\
 &= \frac{1}{2} \ln \frac{\phi(1+a)}{\phi(1)} - G_\phi(a) + \lim_{n \rightarrow \infty} R_2(n, a).
 \end{aligned}$$

Let us show that  $R_\phi(a) = \lim_{n \rightarrow \infty} R_2(n, a)$  exists. It follows from (3.39),  $\sup_{u>0} |P(u)| \leq \frac{1}{4}$  and the dominated convergence theorem since

$$(3.40) \quad \sup_{n \geq 1} \sup_{\phi \in \mathcal{B}} |R_2(n, a)| \leq \frac{1}{4} \sup_{\phi \in \mathcal{B}} \int_1^\infty \left( \left( \frac{\phi'(u)}{\phi(u)} \right)^2 + \left| \frac{\phi''(u)}{\phi(u)} \right| \right) du < 2,$$

where the finiteness follows from (3.28). Therefore, from (3.37), (3.39) and the existence of  $R_\phi(a)$ , we get that

$$W_\phi(a) = \frac{1}{\phi(a)} \sqrt{\frac{\phi(1)}{\phi(1+a)}} e^{G_\phi(a) - R_\phi(a)}.$$

Substituting this in (3.36) we prove (3.19). From (3.40) we also obtain the second global bound in (3.18). To conclude item (1) it remains to prove the two monotonicity properties. The monotonicity, for fixed  $b \in \mathbb{R}$ , of the mapping  $a \mapsto A_\phi(a + ib)$  on  $\mathbb{R}^+$  follows from (3.20) right away since for all  $u > 0, b \geq 0$ ,  $\ln \left( \frac{|\phi(u+ib)|}{\phi(u)} \right) \geq 0$ , see [61, Proposition 6.10, (6.32)]. Otherwise, set  $0 \leq q \leq q' < \infty$  and for any  $r \geq 0$ ,  $\phi^r(z) = r + \phi(0) + \phi^\sharp(z)$ . Then, the mapping

$$r \mapsto \ln \left( \frac{|\phi^r(u + i|b)|}{\phi^r(u)} \right) = \ln \left| 1 + \frac{\phi^\sharp(u + i|b) - \phi^\sharp(u)}{r + \phi(0) + \phi^\sharp(u)} \right|$$

is non-increasing on  $\mathbb{R}^+$  and (3.20) closes the proof of item (1).

We proceed by investigating in detail the principal functions that control the asymptotic behaviour of  $|W_\phi(z)|$ ,  $z = a + ib \in \mathbb{C}_{(0,\infty)}$  in (3.19). We start with the large asymptotic behaviour for large  $a$  and fixed  $b$ .

**3.3. Proof of Theorem 3.3(2).** Note that, for  $z = a + ib \in \mathbb{C}_{(0,\infty)}$ , some elementary algebra yields that

$$\begin{aligned} E_\phi(z) + R_\phi(a) &= \frac{1}{2} \int_0^1 \mathrm{P}(u) \left( \frac{\ln |\phi(u+z)|}{\ln \phi(u+a)} \right)'' du + \frac{1}{2} \int_1^\infty \mathrm{P}(u) \left( \frac{\ln |\phi(u+z)|}{\ln \phi(u)} \right)'' du \\ &:= \bar{E}_\phi(z) + \bar{R}_\phi(z). \end{aligned}$$

Then, noting that  $(\ln |\phi(u+z)|)'' \leq |(\log_0 \phi(u+z))''|$ , we easily obtain the estimate

$$|\bar{E}_\phi(z)| \leq \int_0^1 \left( \left| \frac{\phi'(u+z)}{\phi(u+z)} \right|^2 + \left( \frac{\phi'(u+a)}{\phi(u+a)} \right)^2 + \left| \frac{\phi''(u+z)}{\phi(u+z)} \right| + \left| \frac{\phi''(u+a)}{\phi(u+a)} \right| \right) du.$$

Clearly, from the proof of [61, Lemma 4.5(2)] combined with (3.28), we get that

$$(3.41) \quad \left| \frac{\phi'(a+ib)}{\phi(a+ib)} \right| \leq \sqrt{10} \frac{\phi'(a)}{\phi(a)} \leq \frac{\sqrt{10}}{a}$$

and

$$(3.42) \quad \left| \frac{\phi''(a+ib)}{\phi(a+ib)} \right| \leq \sqrt{10} \frac{\phi''(a)}{\phi(a)} \leq \frac{2\sqrt{10}}{a^2},$$

from where we deduce that  $\lim_{a \rightarrow \infty} |\bar{E}_\phi(z)| = 0$ . Next, we look into  $\bar{R}_\phi(z)$ . Note that

$$\begin{aligned} \bar{R}_\phi(z) &= \frac{1}{2} \int_1^\infty \mathrm{P}(u) (\ln |\phi(u+z)|)'' du - \frac{1}{2} \int_1^\infty \mathrm{P}(u) (\ln \phi(u))'' du \\ &= \bar{R}_1(z) + \bar{R}_2(z). \end{aligned}$$

Clearly,  $\bar{R}_2$  is the right-hand side of (3.15) and we proceed to show that  $\lim_{a \rightarrow \infty} |\bar{R}_1(z)| = 0$ . To do so we simply repeat the work done for  $\bar{E}_\phi(z)$  to conclude that

$$|\bar{R}_1(z)| \leq \frac{1}{8} \int_1^\infty \left| \frac{\phi'(u+z)}{\phi(u+z)} \right|^2 + \left| \frac{\phi''(u+z)}{\phi(u+z)} \right| du.$$

Since each integrand converges to zero as  $a \rightarrow \infty$ , see (3.41) and (3.42), we invoke again (3.41) and (3.42) to ensure that the dominated convergence theorem is applicable. Therefore, for any fixed  $b \in \mathbb{R}$ ,

$$\lim_{a \rightarrow \infty} e^{-E_\phi(a+ib) - R_\phi(a)} = e^{-T_\phi}$$

and the very first term in the product on the right-hand side of (3.21) is established. For any  $\phi \in \mathcal{B}$ , we deduce by performing an integration by parts in the expression of  $G_\phi$  in (3.12) that, for any  $a > 0$ ,

$$(3.43) \quad G_\phi(a) = (a+1) \ln \phi(a+1) - \ln \phi(1) - H_\phi(a).$$

Next, note that, for large  $a$ ,

$$a \ln \frac{\phi(a+1)}{\phi(a)} = a \ln \left( 1 + \frac{\phi(a+1) - \phi(a)}{\phi(a)} \right) = H_\phi^*(a) + a \mathrm{O} \left( \left( \frac{\phi(a+1) - \phi(a)}{\phi(a)} \right)^2 \right),$$

where the last asymptotic relation follows from  $\lim_{a \rightarrow \infty} \frac{\phi(a+1)}{\phi(a)} = 1$ , see (3.29). However, from (3.28) we get that  $a \frac{\phi'(a)}{\phi(a)} \leq 1$ ,  $a > 0$ , and, since  $\phi'$  is non-increasing, we conclude that

$$a \ln \frac{\phi(1+a)}{\phi(a)} = H_\phi^*(a) + O\left(\frac{1}{a}\right).$$

Thus, putting pieces together we deduce from (3.43) that

$$(3.44) \quad G_\phi(a) = a \ln \phi(a) - H_\phi(a) + H_\phi^*(a) + \ln \phi(a) - \ln \phi(1) + O\left(\frac{1}{a}\right).$$

This combined with (3.19) and (3.15) leads to the remaining terms in the product in (3.21) which completes the proof of Theorem 3.2(2).

**3.4. Proof of Theorem 3.3(3c).** The proof is based on the observations of Remark 4.4 which simply crystallize the main ingredients of the lengthy proof of Proposition 4.3 stated below.

Finally, we discuss the term  $A_\phi(z)$  in (3.19) which governs the asymptotics along complex lines  $\mathbb{C}_a$ ,  $a > \bar{a}_\phi$ , see Remark 3.4. It can be simplified for two general subclasses of  $\mathcal{B}$ , namely the case when  $\mathfrak{d} > 0$ , that is  $\phi \in \mathcal{B}_P$ , and the class  $\mathcal{B}_{R_\alpha}$ , see (2.14). We start with the former case.

**3.5. Proof of Theorem 3.3(3a).** We start the proof by introducing some notation and quantities. For a measure (or a function)  $\mu$  on  $\mathbb{R}$ ,  $\mathcal{F}_\mu(-ib) = \int_{-\infty}^{\infty} e^{-iby} \mu(dy)$  stands for its Fourier transform. Also, we use  $\lambda * \gamma$  to denote the convolution between measures and/or functions. Next, for any measure  $\lambda$  on  $\mathbb{R}$  such that  $\|\lambda\|_{TV} := \int_{-\infty}^{\infty} |\lambda(dy)| < 1$  we define

$$(3.45) \quad \mathfrak{L}_\lambda(dy) = \sum_{n=1}^{\infty} \frac{\lambda^{*n}(dy)}{n} \quad \text{and} \quad \underline{\mathfrak{L}}_\lambda(dy) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\lambda^{*n}(dy)}{n}.$$

Clearly,  $\|\mathfrak{L}_\lambda\|_{TV} < \infty$  and  $\|\underline{\mathfrak{L}}_\lambda\|_{TV} < \infty$ . Finally, we use for a measure (resp. function)  $\lambda_a(dy) = e^{-ay} \lambda(dy)$  (resp.  $\lambda_a(y) = e^{-ay} \lambda(y)$ ) with  $a \in \mathbb{R}$ . Let from now on  $\lambda$  be a measure on  $\mathbb{R}^+$ . If  $\|\lambda_a\|_{TV} < 1$  for some  $a > 0$ , then by virtue of the fact that  $\lambda_a^{*n}(dy) = e^{-any} \lambda^{*n}(dy)$ ,  $y \in (0, \infty)$ , (3.45) holds locally on  $(0, \infty)$  for  $\lambda_0 = \lambda$ . Let next  $\lambda(dy) = \lambda(y)dy$ ,  $y \in (0, \infty)$ , and  $\lambda \in (\mathbb{L}^1(\mathbb{R}^+), *)$ , i.e. the  $C^*$  algebra of the integrable functions on  $\mathbb{R}^+$ , that is  $\mathbb{L}^1(\mathbb{R}^+)$  considered as a subalgebra of  $(\mathbb{L}^1(\mathbb{R}), *)$  which is endowed with the convolution operation as a multiplication. Note that formally

$$-\log_0(1 - \mathcal{F}_\lambda) = \mathcal{F}_{\mathfrak{L}_\lambda} = \sum_{n=1}^{\infty} \frac{\mathcal{F}_\lambda^n}{n} \quad \text{and} \quad \log_0(1 + \mathcal{F}_\lambda) = \mathcal{F}_{\underline{\mathfrak{L}}_\lambda} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\mathcal{F}_\lambda^n}{n}.$$

Then, the Wiener-Lévy theorem for normed algebras will be shown below to yield that

$$(3.46) \quad \exists \mathfrak{L}_\lambda(dy) = \underline{\mathfrak{L}}_\lambda(y)dy, \quad y \in (0, \infty) \quad \text{s.t.} \quad \mathfrak{L}_\lambda \in (\mathbb{L}^1(\mathbb{R}^+), *) \iff \text{Supp } \mathcal{F}_\lambda \cap (-\infty, -1] = \emptyset,$$

$$(3.47) \quad \exists \mathfrak{L}_\lambda(dy) = \mathfrak{L}_\lambda(y)dy, \quad y \in (0, \infty) \quad \text{s.t.} \quad \mathfrak{L}_\lambda \in (\mathbb{L}^1(\mathbb{R}^+), *) \iff \text{Supp } \mathcal{F}_\lambda \cap [1, \infty) = \emptyset,$$

that is  $\underline{\mathfrak{L}}_\lambda$  (resp.  $\mathfrak{L}_\lambda$ ) is an element of  $(\mathbb{L}^1(\mathbb{R}^+), *)$  if the support of  $\mathcal{F}_\lambda$  is a strict subset of the domain of analyticity of the function  $\log_0(1+z)$  (resp.  $\log_0(1-z)$ ). We then have the following claim.

**Proposition 3.15.** *Let  $\lambda$  be a real-valued measure on  $(0, \infty)$ . If  $\|\lambda\|_{TV} < 1$  then both  $\|\underline{\mathfrak{L}}_\lambda\|_{TV} < \infty$  and  $\|\underline{\mathfrak{L}}_\lambda\|_{TV} < \infty$ . Furthermore, let  $\lambda \in (\mathbb{L}^1(\mathbb{R}^+), *)$ . If  $\text{Supp } \mathcal{F}_\lambda \cap (-\infty, -1] = \emptyset$  (resp.  $\text{Supp } \mathcal{F}_\lambda \cap [1, \infty) = \emptyset$ ) then  $\underline{\mathfrak{L}}_\lambda \in (\mathbb{L}^1(\mathbb{R}^+), *)$  (resp.  $\underline{\mathfrak{L}}_\lambda \in (\mathbb{L}^1(\mathbb{R}^+), *)$ ) and*

$$(3.48) \quad \log_0(1 + \mathcal{F}_\lambda(-ib)) = \mathcal{F}_{\underline{\mathfrak{L}}_\lambda}(-ib) \text{ if } \underline{\mathfrak{L}}_\lambda \in (\mathbb{L}^1(\mathbb{R}^+), *)$$

$$(3.49) \quad -\log_0(1 - \mathcal{F}_\lambda(-ib)) = \mathcal{F}_{\underline{\mathfrak{L}}_\lambda}(-ib) \text{ if } \underline{\mathfrak{L}}_\lambda \in (\mathbb{L}^1(\mathbb{R}^+), *).$$

*Proof.* Let  $\|\lambda\|_{TV} < 1$ . Since, for all  $n \in \mathbb{N}$ ,  $\|\lambda^{*n}\|_{TV} \leq \|\lambda\|_{TV}^n$  and  $\sup_{b \in \mathbb{R}} |\mathcal{F}_\lambda^n(ib)| \leq \|\lambda\|_{TV}^n$ , then (3.45), (3.46), (3.47) and the Taylor expansion about zero of  $\log_0(1 \pm z)$  yield all results in the case  $\|\lambda\|_{TV} < 1$ . Let the support of  $\mathcal{F}_\lambda$  does not intersect with  $[1, \infty)$  (resp.  $(-\infty, -1]$ ), which is the region where  $\log_0(1 - z)$  (resp.  $\log_0(1 + z)$ ) is not holomorphic. Then we get from the Wiener-Lévy theorem, see [48, Section 7.7] that  $\underline{\mathfrak{L}}_\lambda = c_1 \delta_0 + \lambda^+$  (resp.  $\underline{\mathfrak{L}}_\lambda = c_2 \delta_0 + \lambda^-$ ), where the Dirac mass  $\delta_0$  completes  $(\mathbb{L}^1(\mathbb{R}), *)$  to a semistable Banach algebra with unity,  $c_1, c_2 \in \mathbb{R}$  and  $\lambda^\pm \in \mathbb{L}^1(\mathbb{R})$ . Since the Taylor expansion of  $\log_0(1 \pm z)$  is approximated with monomials of order greater or equal to 1 and  $\mathcal{F}_{\delta_0} \equiv 1$  then  $c_1 = c_2 = 0$ . Also, considering  $a > 0$  big enough such that  $\|\lambda_a\|_{TV} < 1$  then (3.45) confirms that  $\underline{\mathfrak{L}}_{\lambda_a}, \underline{\mathfrak{L}}_{\lambda_a} \in \mathbb{L}^1(\mathbb{R}^+)$ . Since  $\underline{\mathfrak{L}}_{\lambda_a}(dy) = e^{-ay} \underline{\mathfrak{L}}_\lambda(dy)$  and  $\underline{\mathfrak{L}}_{\lambda_a}(dy) = e^{-ay} \underline{\mathfrak{L}}_\lambda(dy)$  we conclude that  $\lambda^\pm \in \mathbb{L}^1(\mathbb{R}^+)$  and hence all the remaining claims.  $\square$

Let us assume that  $d > 0$ . Then, for each  $z = a + ib, a > 0$ , we have from the second expression in (3.3) that

$$(3.50) \quad \phi(z) = dz \left( 1 + \frac{\phi(0)}{dz} + \mathcal{F}_{\bar{\mu}_{a,d}}(-ib) \right),$$

where we have set  $\bar{\mu}_{a,d}(y) = \frac{1}{d} e^{-ay} \bar{\mu}(y), y \in (0, \infty)$ . With the preceding notation we can state the next result which with the help of Lemma 4.1 and the substitution

$$H(b) := \int_0^\infty \frac{1 - \cos(by)}{y} \underline{\mathfrak{L}}_{\bar{\mu}_{a,d}}(dy)$$

concludes the proof of Theorem 3.2(3a).

**Proposition 3.16.** *Assume that  $\phi \in \mathcal{B}_P$  and  $z = a + ib \in \mathbb{C}_a$ , with  $a > 0$  fixed. We have*

(3.51)

$$\begin{aligned} A_\phi(z) &= |b| \arctan\left(\frac{|b|}{a}\right) - \frac{\left(a + \frac{\phi(0)}{d}\right)}{2} \ln\left(1 + \frac{b^2}{a^2}\right) - \int_0^\infty \frac{1 - \cos(by)}{y} \underline{\mathfrak{L}}_{\bar{\mu}_{a,d}}(dy) + \bar{A}_\phi(a) \\ &\approx \frac{\pi}{2} |b| - \left(a + \frac{\phi(0)}{d}\right) \ln |b| - o(|b|), \end{aligned}$$

where  $\underline{\mathfrak{L}}_{\bar{\mu}_{a,d}}$  is related to  $\bar{\mu}_{a,d}(y) = d^{-1} e^{-ay} \bar{\mu}(y)$  via (3.46) and  $|\bar{A}_\phi(a)| < \infty$  for all  $a > 0$ . For all  $a > 0$  big enough,  $\|\bar{\mu}_{a,d}\|_{TV} < 1$  and (3.45) relates  $\underline{\mathfrak{L}}_{\bar{\mu}_{a,d}}(dy)$  to  $\bar{\mu}_{a,d}(y) dy, y \in (0, \infty)$ . Also, for those  $a > 0$  such that  $\|\bar{\mu}_{a,d}\|_{TV} < 1$ , we have that

$$(3.52) \quad \int_0^\infty \frac{1 - \cos(by)}{y} \underline{\mathfrak{L}}_{\bar{\mu}_{a,d}}(dy) = \int_0^{|b|} \arctan\left(\frac{\text{Im}(\mathcal{F}_{\bar{\mu}_{a,d}}(iu))}{1 + \text{Re}(\mathcal{F}_{\bar{\mu}_{a,d}}(iu))}\right) du$$

with, as  $b \rightarrow \infty$ ,

$$(3.53) \quad \arctan\left(\frac{\text{Im}(\mathcal{F}_{\bar{\mu}_{a,d}}(ib))}{1 + \text{Re}(\mathcal{F}_{\bar{\mu}_{a,d}}(ib))}\right) = \text{Im}(\mathcal{F}_{\bar{\mu}_{a,d}}(ib)) \left(1 + O\left(\left(\text{Im}(\mathcal{F}_{\bar{\mu}_{a,d}}(ib))\right)^2\right)\right)$$

and

$$(3.54) \quad \lim_{b \rightarrow \infty} \frac{\mathbf{d}}{\ln(b)\bar{\mu}\left(\frac{1}{b}\right)} \int_0^\infty \frac{1 - \cos(by)}{y} \underline{\mathfrak{L}}_{\bar{\mu}_{a,\mathbf{d}}}(dy) \geq 1.$$

*Remark 3.17.* We note that (3.52) coupled with (3.53) give a more tractable way to compute  $A_\phi$  in (3.23). When  $\|\bar{\mu}_{a,\mathbf{d}}\|_{TV} < 1$  thanks to (3.45) we have that

$$\int_0^\infty \frac{1 - \cos(by)}{y} \underline{\mathfrak{L}}_{\bar{\mu}_{a,\mathbf{d}}}(dy) = \int_0^\infty \frac{1 - \cos(by)}{y} \sum_{n=1}^\infty (-1)^{n-1} \frac{\bar{\mu}_{a,\mathbf{d}}^{*n}(dy)}{n}$$

and thus via (3.51) we have precise information for the asymptotic expansion of  $A_\phi$  when the convolutions of  $\bar{\mu}_{a,\mathbf{d}}(y) = \mathbf{d}^{-1}e^{-ay}\bar{\mu}(y)$  or equivalently of  $\bar{\mu}$  are accessible. For example, if  $\bar{\mu}_{a,\mathbf{d}}(y) \stackrel{0}{\sim} \mathbf{d}^{-1}y^{-\alpha}$ ,  $\alpha \in (0, 1)$ , then with  $B(a, b)$ ,  $a, b > 0$ , standing for the classical Beta function and  $C_n = \mathbf{d}^{-n} \prod_{j=1}^{n-1} B(j - j\alpha, 1 - \alpha)$ ,  $\bar{\mu}_{a,\mathbf{d}}^{*n}(y) \stackrel{0}{\sim} C_n y^{n-1-n\alpha}$  and, with the obvious notation for asymptotic behaviour of densities of measures,

$$\underline{\mathfrak{L}}_{\bar{\mu}_{a,\mathbf{d}}}(dy) \stackrel{0}{\sim} \left( \sum_{\frac{1}{1-\alpha} > n \geq 1} (-1)^{n-1} C_n y^{n-1-n\alpha} + o(1) \right) dy.$$

A substitution in (3.23) and elementary calculations yield that

$$\int_0^\infty \frac{1 - \cos(by)}{y} \underline{\mathfrak{L}}_{\bar{\mu}_{a,\mathbf{d}}}(dy) \approx \sum_{\frac{1}{1-\alpha} > n \geq 1} (-1)^{n-1} \tilde{C}_n b^{1-n+n\alpha} + O(1)$$

with  $\tilde{C}_n = C_n \int_0^\infty \frac{1 - \cos(v)}{v^{2-n(1-\alpha)}} dv$ ,  $n \in \left[1, \frac{1}{1-\alpha}\right)$ , and the asymptotic expansion of  $A_\phi$  follows.

*Proof.* Let for the course of the proof  $z = a + ib \in \mathbb{C}_a$ ,  $a > 0$ , and since  $|W_\phi(a + ib)| = |W_\phi(a - ib)|$  without loss of generality we assume throughout that  $b > 0$ . From (3.50) we have modulo to  $(-\pi, \pi]$

$$(3.55) \quad \arg \phi(z) = \arg z + \arg \left( 1 + \frac{\phi(0)}{\mathbf{d}z} + \mathcal{F}_{\bar{\mu}_{a,\mathbf{d}}}(-ib) \right).$$

However, an application of the Riemann-Lebesgue lemma to the function  $\bar{\mu}_{a,\mathbf{d}} \in (L^1(\mathbb{R}^+), *)$  yields, as  $b \rightarrow \infty$ , that

$$(3.56) \quad |\mathcal{F}_{\bar{\mu}_{a,\mathbf{d}}}(-ib)| = o(1).$$

Therefore, for all  $b$  big enough,

$$(3.57) \quad \arg \left( 1 + \frac{\phi(0)}{\mathbf{d}(a + ib)} + \mathcal{F}_{\bar{\mu}_{a,\mathbf{d}}}(-ib) \right) = \arg \left( 1 + \mathcal{F}_{\bar{\mu}_{a,\mathbf{d}}}(-ib) \right) - \frac{\phi(0)b}{\mathbf{d}(a^2 + b^2)} + O\left(\frac{1}{b^2}\right).$$

Also

$$(3.58) \quad \begin{aligned} \operatorname{Im}(\mathcal{F}_{\bar{\mu}_{a,\mathbf{d}}}(-ib)) &= - \int_0^\infty \sin(by) \bar{\mu}_{a,\mathbf{d}}(y) dy \\ &= - \frac{1}{b} \int_0^\infty (1 - \cos(by)) \mu_{a,\mathbf{d}}(dy) < 0, \end{aligned}$$

since for  $a > 0$ ,  $y \mapsto \bar{\mu}_{a,\mathbf{d}}(y) = \mathbf{d}^{-1}e^{-ay}\bar{\mu}(y)$  is strictly decreasing on  $\mathbb{R}^+$  and  $\mu_{a,\mathbf{d}}(dy) = d\bar{\mu}_{a,\mathbf{d}}(y)$ ,  $y \in (0, \infty)$ , is not supported on a lattice. Therefore, from (3.56), (3.58) and the

fact that  $b \mapsto \mathcal{F}_{\bar{\mu}_{a,d}}(-ib)$  is continuous we deduce that  $\text{Supp } \mathcal{F}_{\bar{\mu}_{a,d}} \cap (-\infty, -1] = \emptyset$ . Therefore, Proposition 3.15 gives that  $\underline{\mathcal{L}}_{\bar{\mu}_{a,d}} \in (\mathbb{L}^1(\mathbb{R}^+), *)$  and from (3.48), for all  $b > 0$ ,

$$(3.59) \quad \begin{aligned} \arg(1 + \mathcal{F}_{\bar{\mu}_{a,d}}(-ib)) &= \text{Im}(\log_0(1 + \mathcal{F}_{\bar{\mu}_{a,d}}(-ib))) \\ &= \text{Im}(\mathcal{F}_{\underline{\mathcal{L}}_{\bar{\mu}_{a,d}}}(-ib)) = - \int_0^\infty \sin(by) \underline{\mathcal{L}}_{\bar{\mu}_{a,d}}(dy). \end{aligned}$$

Then, from (3.11) due to (3.55), (3.57) and (3.59) we deduce that, for any  $a, b > 0$ ,

$$\begin{aligned} A_\phi(a+ib) &= \int_0^b \arg(a+iu) du + \int_0^b \arg\left(1 + \frac{\phi(0)}{\mathfrak{d}(a+iu)} + \mathcal{F}_{\bar{\mu}_{a,d}}(-iu)\right) du + \tilde{A}_\phi(a) \\ &= \int_0^b \arctan\left(\frac{u}{a}\right) du - \frac{\phi(0)}{2\mathfrak{d}} \ln\left(1 + \frac{b^2}{a^2}\right) - \int_0^\infty \frac{1 - \cos(by)}{y} \underline{\mathcal{L}}_{\bar{\mu}_{a,d}}(dy) + \bar{A}_\phi(a), \end{aligned}$$

where  $\tilde{A}_\phi(a)$  is the error induced when  $\arg \phi(z) \neq \arg z + \arg\left(1 + \frac{\phi(0)}{\mathfrak{d}z} + \mathcal{F}_{\bar{\mu}_{a,d}}(-ib)\right) \notin (-\pi, \pi]$  in (3.55) which can happen only for small  $u$  thanks to (3.56) and (3.57), and  $\bar{A}_\phi(a)$  is the sum of  $\tilde{A}_\phi(a)$  and the integral of the error term in (3.57) and clearly  $|\tilde{A}_\phi(a)| < \infty$ . Then the first relation in (3.51) follows by a simple integration by parts. The asymptotic relation in (3.51) comes from  $\underline{\mathcal{L}}_{\bar{\mu}_{a,d}} \in (\mathbb{L}^1(\mathbb{R}^+), *)$  and the auxiliary claim that for any  $h \in (\mathbb{L}^1(\mathbb{R}^+), *)$

$$(3.60) \quad \left| \int_0^\infty \frac{1 - \cos(by)}{y} h(y) dy \right| = o(|b|),$$

which follows from the Riemann-Lebesgue lemma invoked in the middle term of

$$\left| \int_0^\infty \frac{1 - \cos(by)}{y} h(y) dy \right| = \left| \int_0^b \int_0^\infty \sin(uy) h(y) dy du \right| \leq o(1) \left| \int_0^b du \right|.$$

Finally, since  $\lim_{a \rightarrow \infty} \int_0^\infty e^{-ay} \bar{\mu}(y) dy = 0$  then  $\lim_{a \rightarrow \infty} \|\bar{\mu}_{a,d}\|_{TV} = 0$  and thus eventually, for some  $a$  large enough,  $\|\bar{\mu}_{a,d}\|_{TV} < 1$  and  $\sup_{b \in \mathbb{R}} |\mathcal{F}_{\bar{\mu}_{a,d}}(ib)| < 1$ . Choose such  $a > 0$ . Then, from (3.56) with  $z \in \mathbb{C}_a$ ,  $b > 0$ ,

(3.61)

$$\arg(1 + \mathcal{F}_{\bar{\mu}_{a,d}}(-ib)) = \arctan\left(\frac{\text{Im}(\mathcal{F}_{\bar{\mu}_{a,d}}(-ib))}{1 + \text{Re}(\mathcal{F}_{\bar{\mu}_{a,d}}(-ib))}\right) = -\arctan\left(\frac{\text{Im}(\mathcal{F}_{\bar{\mu}_{a,d}}(ib))}{1 + \text{Re}(\mathcal{F}_{\bar{\mu}_{a,d}}(ib))}\right),$$

and using the latter in (3.59) then (3.52) follows upon simple integration of (3.59). The asymptotic relation (3.53) follows from (3.56), i.e.  $|\mathcal{F}_{\bar{\mu}_{a,d}}(-ib)| = o(1)$  combined with the Taylor expansion of  $\arctan x$ . Next, consider the bound (3.54). Since from (3.58)  $\text{Im}(\mathcal{F}_{\bar{\mu}_{a,d}}(ib)) > 0$

and, as  $b \rightarrow \infty$ ,  $|\operatorname{Im}(\mathcal{F}_{\bar{\mu}_{a,d}}(ib))| = o(1)$  we get from (3.52), (3.53), (3.61) and (3.59) that

$$\begin{aligned} \int_0^\infty \frac{1 - \cos(by)}{y} \underline{\mathfrak{L}}_{\bar{\mu}_{a,d}}(dy) &= \int_0^{|b|} \arctan\left(\frac{\operatorname{Im}(\mathcal{F}_{\bar{\mu}_{a,d}}(iu))}{1 + \operatorname{Re}(\mathcal{F}_{\bar{\mu}_{a,d}}(iu))}\right) du \\ &\approx (1 + o(1)) \int_0^b \operatorname{Im}(\mathcal{F}_{\bar{\mu}_{a,d}}(iu)) du \\ &\stackrel{3.58}{=} (1 + o(1)) \int_0^\infty \frac{1 - \cos(by)}{y} \bar{\mu}_{a,d}(y) dy \\ &\geq (1 + o(1)) \int_0^1 \frac{1 - \cos(by)}{y} \bar{\mu}_{a,d}(y) dy \\ &\geq (1 + o(1)) \bar{\mu}_{a,d}\left(\frac{1}{b}\right) \int_1^b \frac{1 - \cos(y)}{y} dy \approx \bar{\mu}_{a,d}\left(\frac{1}{b}\right) \ln(b). \end{aligned}$$

This proves (3.54) since  $\bar{\mu}_{a,d}(b^{-1}) = e^{-ab^{-1}} \mathbf{d}^{-1} \bar{\mu}(b^{-1}) \approx \mathbf{d}^{-1} \bar{\mu}(b^{-1})$ .  $\square$

**3.6. Proof of Theorem 3.3(3b).** Let  $\phi \in \mathcal{B}_{R_\alpha}$  with  $\alpha \in (0, 1)$  and let  $z = a + ib \in \mathbb{C}_a$ ,  $a > 0$ . Then, there exists  $\alpha \in (0, 1)$  such that  $\bar{\mu}(y) = y^{-\alpha} \ell(y)$  and  $\ell$  is quasi-monotone, see (2.13) and (2.14). Recall the second relation of (3.3) which, since in this setting  $\mathbf{d} = 0$ , takes the form

$$(3.62) \quad \phi(z) = \phi(0) + z \int_0^\infty e^{-iby} e^{-ay} \bar{\mu}(y) dy = \phi(0) + z \int_0^\infty e^{-iby} y^{-\alpha} e^{-ay} \ell(y) dy.$$

Since the mapping  $y \mapsto \ell(y) e^{-ay}$  is clearly quasi-monotone we conclude from [69, Theorem 1.39] that, for fixed  $a > 0$  and  $b \rightarrow \infty$ ,

$$\int_0^\infty e^{-iby} y^{-\alpha} e^{-ay} \ell(y) dy \approx \Gamma(1 - \alpha) \left( b e^{\frac{i\pi}{2}} \right)^{\alpha-1} \ell\left(\frac{1}{b}\right).$$

Therefore, from (3.62) and the last relation we obtain, as  $b \rightarrow \infty$ , that

$$\arg \phi(z) = \arg z + \arg \left( \int_0^\infty e^{-iby} y^{-\alpha} e^{-ay} \ell(y) dy + \frac{\phi(0)}{z} \right) \approx \arg z + \frac{\pi(\alpha - 1)}{2} \approx \frac{\pi}{2} \alpha,$$

which proves (3.24) by using the definition of  $A_\phi$  in (3.11). This together with Lemma 4.1 establishes the claim.

**3.7. Proof of Lemma 3.13.** Let  $Y_{\phi_n}, Y_{\underline{\phi}}, n \in \mathbb{N}$ , be the random variables associated to  $W_{\phi_n}, W_{\underline{\phi}}, n \in \mathbb{N}$ , see Definition 3.1. Clearly, since for any  $\phi \in \mathcal{B}$

$$\mathbb{E}[e^{tY_\phi}] = \sum_{k=0}^\infty t^k \frac{\mathbb{E}[Y_\phi^k]}{k!} = \sum_{k=0}^\infty t^k \frac{W_\phi(k+1)}{k!} = \sum_{k=0}^\infty t^k \frac{\prod_{j=1}^k \phi(j)}{k!}$$

and Proposition 3.14(3) holds, we conclude that  $\mathbb{E}[e^{tY_\phi}]$  is well defined for  $t < \frac{1}{\mathbf{d}} \in (0, \infty]$ . However,  $\lim_{n \rightarrow \infty} \phi_n(a) = \underline{\phi}(a)$  implies that  $\mathbf{d}^* = \sup_{n \geq 0} \mathbf{d}_n < \infty$ , where  $\mathbf{d}_n$  are the linear terms in (3.3). Therefore,  $\mathbb{E}[e^{zY_{\underline{\phi}}}]$ ,  $\mathbb{E}[e^{zY_{\phi_n}}]$ ,  $n \in \mathbb{N}$ , are analytic in  $\mathbb{C}_{(-\infty, \min\{\frac{1}{\mathbf{d}}, \frac{1}{\mathbf{d}^*}\})} \supsetneq \mathbb{C}_{(-\infty, 0]}$ .

Moreover, for any  $k \in \mathbb{N}$ ,

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E} \left[ Y_{\phi_n}^k \right] &= \lim_{n \rightarrow \infty} W_{\phi_n}(k+1) \\ &= \lim_{n \rightarrow \infty} \prod_{j=1}^k \phi_n(j) = W_{\underline{\phi}}(k+1) = \mathbb{E} \left[ Y_{\underline{\phi}}^k \right]. \end{aligned}$$

The last two observations trigger that  $\lim_{n \rightarrow \infty} Y_{\phi_n} \stackrel{d}{=} Y_{\underline{\phi}}$ , see [29, p.269, Example (b)]. Therefore,  $\lim_{n \rightarrow \infty} W_{\phi_n}(z) = W_{\underline{\phi}}(z)$ ,  $z \in \mathbb{C}_{(0, \infty)}$ , which concludes the proof.

**3.8. Proof of Theorem 3.9.** The first item is proved, for any  $\phi \in \mathcal{B}$ , by Berg in [6, Theorem 2.2]. The second item follows readily from the first one after recalling that if  $\Psi \in \mathcal{N}$  then  $-\Psi(-z) \in \mathcal{N}$ . For the item (3), since  $\frac{\phi}{\underline{\phi}}$  is completely monotone, and we recall that for any  $n \in \mathbb{N}$ ,  $W_{\phi}(n+1) = \prod_{k=1}^n \phi(k)$ , we first get from [7, Theorem 1.3] that the sequence  $\left( f_n = \frac{W_{\phi}(n+1)}{W_{\phi}(n)} \right)_{n \geq 0}$  is the moment sequence of a positive variable  $I$ . Next, from the recurrence equation (3.4) combined with the estimates stated in (3), we deduce that  $\frac{f_{n+1}}{f_n} = \frac{\phi(n)}{\phi(n)} \stackrel{\infty}{=} O(n)$ . Thus, there exists  $A > 0$  such that for any  $a < A$ ,

$$\mathbb{E} [e^{aI}] = \sum_{n=0}^{\infty} \frac{f_n}{n!} a^n < \infty,$$

implying that  $I$  is moment determinate. Next, with the notation of the statement, if  $\mathbb{N} = \mathbb{N}_{\underline{\phi}} - \mathbb{N}_{\phi} > \frac{1}{2}$  then, as  $|\mathcal{M}_I(i|b| - \frac{1}{2})| = \left| \frac{W_{\phi}(i|b| + \frac{1}{2})}{W_{\underline{\phi}}(i|b| + \frac{1}{2})} \right| \stackrel{\infty}{=} O(|b|^{\mathbb{N}})$  and, from Theorem 3.2, for any  $\phi \in \mathcal{B}$ ,  $W_{\phi} \in \mathbf{A}_{(0, \infty)}$  and is zero-free on  $\mathbb{C}_{(0, \infty)}$ , we obtain that  $b \mapsto \mathcal{M}_I(ib - \frac{1}{2}) \in L^2(\mathbb{R})$  and hence by the Parseval identity for Mellin transform we conclude that  $f_I \in L^2(\mathbb{R}^+)$ . Finally if  $\mathbb{N} > 1$  then the result follows from a similar estimate for the Mellin transform which allows to use a Mellin inversion technique to prove the claim in this case. For the last item, we first observe, from (3.27) and Proposition 3.14(5), that for any  $\phi \in \mathcal{B}$  and  $u > 0$ ,

$$(3.63) \quad \frac{\phi'(u)}{\phi(u)} = \int_0^{\infty} e^{-uy} \kappa(dy)$$

where we recall that  $\kappa(dy) = \int_0^y U(dy - r)(r\mu(dr) + \delta_a(dr))$ . Thus,

$$(3.64) \quad \underline{\Psi}(z) = \ln \frac{W_{\phi}(z+1)}{W_{\underline{\phi}}(z+1)} = \ln \frac{\phi(1)}{\underline{\phi}(1)} z + \int_0^{\infty} (e^{-zy} - 1 - z(e^{-y} - 1)) \frac{\kappa(dy) - \underline{\kappa}(dy)}{y(e^y - 1)}$$

where, as in (3.63), we have set  $\frac{\phi'(u)}{\phi(u)} = \int_0^{\infty} e^{-uy} \underline{\kappa}(dy)$ . Next, since plainly  $\frac{\phi'(1)}{\phi(1)} - \frac{\phi'(1)}{\underline{\phi}(1)} < \infty$ , we have that the measure  $e^{-y} K(dy) = e^{-y} (\kappa(dy) - \underline{\kappa}(dy))$  is finite on  $\mathbb{R}^+$ . Thus, by the Lévy-Khintchine formula, see [9],  $\underline{\Psi} \in \mathcal{N}$  if and only if  $K$  is a positive measure, which by Bernstein theorem, see e.g. [29], is equivalent to the mapping  $u \mapsto \left( \frac{\phi'}{\phi} - \frac{\phi'}{\underline{\phi}} \right)(u)$  to be completely monotone. The last equivalent condition being immediate from the definition of  $K$ , the proof of the theorem is completed.



## 4. THE FUNCTIONAL EQUATION (1.1)

**4.1. Proof of Theorem 2.1.** Recall that by definition  $\mathcal{M}_\Psi(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-}(1-z)$ , see (2.6). From (1.2) and (1.3) it is clear that formally for  $z \in i\mathbb{R}$

$$(4.1) \quad \mathcal{M}_\Psi(z+1) = \frac{\Gamma(z+1)}{W_{\phi_+}(z+1)} W_{\phi_-}(-z) = \frac{z\Gamma(z)}{\phi_+(z)W_{\phi_+}(z)} \frac{W_{\phi_-}(1-z)}{\phi_-(-z)} = \frac{-z}{\Psi(-z)} \mathcal{M}_\Psi(z).$$

However, from Theorem 3.2(1) it is clear that  $W_{\phi_+}$  (resp.  $W_{\phi_-}$ ) extend continuously to  $i\mathbb{R} \setminus \mathcal{Z}_0(\phi_+)$  (resp.  $i\mathbb{R} \setminus \mathcal{Z}_0(\phi_-)$ ). Since from (1.2) we have that  $\mathcal{Z}_0(\Psi) = \mathcal{Z}_0(\phi_+) \cup \mathcal{Z}_0(\phi_-)$ , see (2.24) and (3.5) for the definition of the sets of zeros, we conclude that  $\mathcal{M}_\Psi$  satisfies (4.1) on  $i\mathbb{R} \setminus \mathcal{Z}_0(\Psi)$ . The fact that  $\mathcal{M}_\Psi \in \mathbf{A}_{(0, 1-\bar{\alpha}_-)} \cap \mathbf{M}_{(\mathfrak{a}_+, 1-\mathfrak{a}_-)}$  then follows from the facts that  $W_\phi \in \mathbf{A}_{(\bar{\alpha}_\phi, \infty)} \cap \mathbf{M}_{(\mathfrak{a}_\phi, \infty)}$  and  $W_\phi$  is zero-free on  $\mathbb{C}_{(\mathfrak{a}_\phi, \infty)}$  for any  $\phi \in \mathcal{B}$ , see Theorem 3.2(1), which lead to

$$(4.2) \quad \frac{1}{W_{\phi_+}} \in \mathbf{A}_{(\mathfrak{a}_+, \infty)} \text{ and } W_{\phi_-}(1-\cdot) \in \mathbf{A}_{(-\infty, 1-\bar{\alpha}_-)}.$$

Thus, (2.8), that is  $\mathcal{M}_\Psi \in \mathbf{M}_{(\mathfrak{a}_+, 1-\mathfrak{a}_-)}$ , in general, and (2.7), that is  $\mathcal{M}_\Psi \in \mathbf{A}_{(\mathfrak{a}_\Psi, 1-\bar{\alpha}_-)}$ , when  $\mathfrak{a}_\Psi = \mathfrak{a}_+ \mathbb{I}_{\{\bar{\alpha}_+ = 0\}} = 0$  follow. Note that when  $0 = \bar{\alpha}_+ > \mathfrak{a}_+$ , i.e.  $\mathfrak{a}_\Psi = \mathfrak{a}_+ < 0$ , then necessarily  $\phi'_+(a^+) = \mathfrak{d} + \int_0^\infty ye^{ay} \mu_+(dy) \in (0, \infty)$  for any  $a > \mathfrak{a}_+$ , see (3.27), and  $\bar{\alpha}_+ = 0$ , see (3.8), is the only zero of  $\phi_+$  on  $(\mathfrak{a}_+, \infty)$ . Therefore, Theorem 3.2(4) applies and yields that at  $z = 0$ ,  $W_{\phi_+}$  has a simple pole, which through (3.4) and  $\phi_+ < 0$  on  $(\mathfrak{a}_+, 0)$  is propagated to all  $n \in \mathbb{N}$  such that  $-n > \mathfrak{a}_+$ . These simple poles however are simple zeros for  $\frac{1}{W_{\phi_+}} \in \mathbf{A}_{(\mathfrak{a}_+, \infty)}$  which cancel the poles of  $\Gamma$ . Thus,  $\mathcal{M}_\Psi \in \mathbf{A}_{(\mathfrak{a}_\Psi, 1-\bar{\alpha}_-)}$  and (2.7) is established. For  $z \in \mathbb{C}_{(0, \infty)}$  we have that

$$(4.3) \quad \mathcal{M}_\Psi(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-}(1-z) = \frac{\phi_+(z)}{z} \frac{\Gamma(z+1)}{W_{\phi_+}(z+1)} W_{\phi_-}(1-z)$$

From Theorem 3.2(1) if  $\phi_+(0) > 0$ , that is  $\mathfrak{a}_+ < 0$ , then  $W_{\phi_+} \in \mathbf{A}_{[0, \infty)}$  and  $W_{\phi_+}$  is zero-free on  $\mathbb{C}_{[0, \infty)}$ , and the pole of  $\Gamma$  at zero is uncontested, see (4.3). Therefore,  $\mathcal{M}_\Psi$  extends continuously to  $i\mathbb{R} \setminus \{0\}$  in this case. The same follows from (4.3) when  $\phi'_+(0^+) = \infty$ . Let next  $\phi_+(0) = 0 = \mathfrak{a}_+$  and  $\phi'_+(0^+) < \infty$ . Then (4.3) shows that the claim  $\mathcal{M}_\Psi \in \mathbf{A}_{[0, 1-\mathfrak{a}_-)}$  clearly follows. The fact that  $\mathcal{M}_\Psi \in \mathbf{M}_{(\mathfrak{a}_+, 1-\mathfrak{a}_-)}$ , that is (2.8) is apparent from (4.2). We proceed with the final assertions. Let  $\mathfrak{a}_+ \leq \bar{\alpha}_+ < 0$ . If  $-\mathfrak{u}_+ \notin \mathbb{N}$  then  $\mathfrak{u}_+$  is the only zero of  $\phi_+$  on  $(\mathfrak{a}_+, \infty)$  and if  $\mathfrak{u}_+ = -\infty$  since  $\bar{\alpha}_+ < 0$  then  $\phi_+$  has no zeros on  $(\mathfrak{a}_+, \infty)$  at all. Henceforth, from (3.4) we see that  $W_{\phi_+}$  does not possess poles at the negative integers. Thus, the poles of the function  $\Gamma$  are uncontested. If  $-\mathfrak{u}_+ \in \mathbb{N}$  and  $\mathfrak{u}_+ = \mathfrak{a}_+$  there is nothing to prove, whereas if  $\mathfrak{u}_+ > \mathfrak{a}_+$  then Theorem 3.2(3) shows that  $W_{\phi_+}$  has a simple pole at  $\mathfrak{u}_+$ . Thus  $\frac{1}{W_{\phi_+}}$  has a simple zero at  $\mathfrak{u}_+$ . Then, (3.4) propagates the zeros to all  $\mathfrak{a}_+ < -n \leq \mathfrak{u}_+$  cancelling the poles of  $\Gamma$  at those locations. The values of the residues are easily computed via the recurrent equation (1.3) for  $W_{\phi_+}, W_{\phi_-}$ , the Wiener-Hopf factorization (1.2), the form of  $\mathcal{M}_\Psi$ , see (2.6), and the residues of the gamma function which are of value  $\frac{(-1)^n}{n!}$  at  $-n$ .

**4.2. Proof of Theorem 2.5.** Before we commence the proof we introduce some more notation. We use  $f \asymp g$  to denote the existence of two positive constants  $0 < C_1 < C_2 < \infty$  such that  $C_1 \leq \underline{\lim}_{x \rightarrow a} \left| \frac{f(x)}{g(x)} \right| \leq \overline{\lim}_{x \rightarrow a} \left| \frac{f(x)}{g(x)} \right| \leq C_2$ , where  $a$  is usually 0 or  $\infty$ . The relation  $f \lesssim g$ , that will be employed from now on, requires only that  $\overline{\lim}_{x \rightarrow a} \left| \frac{f(x)}{g(x)} \right| \leq C_2 < \infty$ .

We recall from (2.15) that  $\mathcal{B}_P = \{\phi \in \mathcal{B} : \mathfrak{d} > 0\}$  and  $\mathcal{B}_P^c$  is its complement. Appealing to various auxiliary results below we consider Theorem 2.5(1) first. Throughout the proof we use (2.6), that is

$$(4.4) \quad \mathcal{M}_\Psi(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-}(1-z) \in \mathbf{A}_{(0,1-\bar{a})}.$$

We note from Definition 3.1 and (2.44) of Theorem 2.27 that  $W_{\phi_-}(z)$  and  $\frac{\Gamma(z)}{W_{\phi_+}(z)}$  are Mellin transforms of positive random variables. Therefore the bounds

$$(4.5) \quad |\mathcal{M}_\Psi(z)| \leq \frac{\Gamma(a)}{W_{\phi_+}(a)} |W_{\phi_-}(1-z)|$$

$$(4.6) \quad |\mathcal{M}_\Psi(z)| \leq \frac{|\Gamma(z)|}{|W_{\phi_+}(z)|} |W_{\phi_-}(1-a)|$$

hold for  $z \in \mathbb{C}_a$ ,  $a \in (0, 1 - \bar{a})$ . From (4.5) and (4.6) we have that  $\Psi \in \overline{\mathcal{N}}_\infty$ , see (2.11), if and only if either  $\phi_- \in \mathcal{B}_\infty$  and/or  $\left| \frac{\Gamma(z)}{W_{\phi_+}(z)} \right|$  decays faster than any polynomial along  $a + i\mathbb{R}$ ,  $\forall a \in (0, 1 - \bar{a})$ . The former certainly holds if  $\phi_- \in \mathcal{B}_P$ , that is  $\mathfrak{d} > 0$ , since from Proposition 4.3(1) we have the even stronger  $\phi_- \in \mathcal{B}(\frac{\pi}{2}) \subset \mathcal{B}_\infty$ , and the latter if  $\phi_+ \in \mathcal{B}_P^c$ , see Proposition 4.2(4.11). Also, if  $\phi_+ \in \mathcal{B}_P$  then Proposition 4.2(4.11) holds for all  $u > 0$  iff  $\bar{\mu}_+(0) = \infty$  and hence from (4.6) we deduce that  $\Psi \in \overline{\mathcal{N}}_\infty$ . Next, from Proposition 4.3(2) if  $\phi_+ \in \mathcal{B}_P$ ,  $\bar{\mu}_+(0) < \infty$  then

$$\phi_- \in \mathcal{B}_\infty \iff \Psi \in \overline{\mathcal{N}}_\infty \iff \bar{\Pi}(0) = \infty.$$

Therefore to confirm the second line of (2.16) we ought to check only that if  $\Psi \in \overline{\mathcal{N}}$ ,  $\phi_+ \in \mathcal{B}_P$  and  $\phi_- \in \mathcal{B}_P^c$  then

$$(4.7) \quad \bar{\Pi}(0) = \infty \iff \bar{\Pi}_-(0) = \infty \text{ or } \bar{\mu}_+(0) = \infty.$$

However, if  $\bar{\Pi}_-(0) < \infty$  and  $\bar{\mu}_+(0) < \infty$  then since  $\phi_+ \in \mathcal{B}_P$  the Lévy process is a positive linear drift plus compound Poisson process which proves the backward direction of (4.7). The forward part is identical. In fact the expression for  $\mathbb{N}_\Psi$  in the first line of (2.16) is derived as the sum of the rate of polynomial decay of  $\left| \frac{\Gamma(z)}{W_{\phi_+}(z)} \right|$  in Proposition 4.2 and of  $|W_{\phi_-}(z)|$  in Proposition 4.3(3) coupled with (4.4). The assertions  $\phi_- \in \mathcal{B}_P \implies \Psi \in \overline{\mathcal{N}}(\frac{\pi}{2})$  and  $\phi_- \in \mathcal{B}_{R_\alpha}, \phi_+ \in \mathcal{B}_{R_{1-\alpha}} \implies \Psi \in \overline{\mathcal{N}}(\frac{\pi}{2}\alpha)$  of item (2) follow from (4.5) and (4.6) with the help of items (3a) and (3b) of Theorem 3.3. Let  $\arg \phi_+ = \arg \phi_-$  hold and choose  $a = \frac{1}{2}$ . Then from (3.19) and (4.4) we see that modulo two constants, as  $b \rightarrow \infty$ ,

$$\left| \mathcal{M}_\Psi \left( \frac{1}{2} + ib \right) \right| \asymp \frac{\sqrt{|\phi_+(\frac{1}{2} + ib)|}}{\sqrt{|\phi_-(\frac{1}{2} - ib)|}} \left| \Gamma \left( \frac{1}{2} + ib \right) \right|.$$

From (3.32), that is  $\operatorname{Re}(\phi_-(\frac{1}{2} - ib)) > \phi_-(\frac{1}{2}) > 0$ , from Proposition 3.14(3) and the standard asymptotic for the gamma function

$$(4.8) \quad |\Gamma(a + ib)| = \sqrt{2\pi} |b|^{a-\frac{1}{2}} e^{-\frac{\pi}{2}|b|} (1 + o(1)),$$

as  $b \rightarrow \infty$  and  $a$  fixed, see [32, 8.328.1], we see that  $\Psi \in \overline{\mathcal{N}}(\frac{\pi}{2})$ . Finally, the last claim of  $\Psi \in \overline{\mathcal{N}}(\Theta_+)$  follows readily from (4.4) and (3.19). This ends the proof.  $\square$

The next sequence of results are used in the proof above. Recall the classes  $\mathcal{B}_\infty$  and  $\mathcal{B}(\theta)$ , see (3.16) and (3.17). We start with the following useful lemma.

**Lemma 4.1.** *Let  $\phi \in \mathcal{B}$  and assume that there exists  $\widehat{a} > \bar{a}_\phi$  such that for all  $n \in \mathbb{N}$ ,  $\overline{\lim}_{|b| \rightarrow \infty} |b|^n |W_\phi(\widehat{a} + ib)| = 0$  (resp. there exists  $\theta \in (0, \frac{\pi}{2}]$  such that  $\overline{\lim}_{|b| \rightarrow \infty} \frac{\ln |W_\phi(\widehat{a} + ib)|}{|b|} \leq -\theta$ ) then  $\phi \in \mathcal{B}_\infty$  (resp.  $\phi \in \mathcal{B}(\theta)$ ).*

*Proof.* To prove the claims we rely on the Lindelöf's theorem combined with the functional equation (3.4) in Definition 3.1. First, assume that  $\exists \widehat{a} > \bar{a}_\phi$  such that  $\overline{\lim}_{|b| \rightarrow \infty} |b|^n |W_\phi(\widehat{a} + ib)| = 0$ ,  $\forall n \in \mathbb{N}$  and since  $|W_\phi(\widehat{a} + ib)| = |W_\phi(\widehat{a} - ib)|$  consider  $b > 0$  only. The recurrent equation (3.4) and  $|\phi(\widehat{a} + ib)| = db + o(|b|)$ , as  $b \rightarrow \infty$  and  $\widehat{a} > \bar{a}_\phi$  fixed, see Proposition 3.14(3), yield that  $\overline{\lim}_{|b| \rightarrow \infty} |b|^n |W_\phi(1 + \widehat{a} + ib)| = 0$ ,  $\forall n \in \mathbb{N}$ . Then, we apply the Lindelöf's theorem to the strip  $\mathbb{C}_{[\widehat{a}, \widehat{a}+1]}^+ = \mathbb{C}_{[\widehat{a}, \widehat{a}+1]} \cap \{b \geq 0\}$  and to the functions  $f_n(z) = z^n W_\phi(z)$ ,  $n \in \mathbb{N}$ , which are holomorphic on  $\mathbb{C}_{[\widehat{a}, \widehat{a}+1]}^+$ . Indeed, from our assumptions and the observation above, we have that, for every  $n \in \mathbb{N}$  and some finite constants  $C_n > 0$ ,

$$\sup_{z \in \partial \mathbb{C}_{[\widehat{a}, \widehat{a}+1]}^+} |f_n(z)| \leq C_n$$

and clearly from Definition 3.1

$$\sup_{z \in \mathbb{C}_{[\widehat{a}, \widehat{a}+1]}^+} \left| \frac{f_n(z)}{z^n} \right| = \sup_{z \in \mathbb{C}_{[\widehat{a}, \widehat{a}+1]}^+} |W_\phi(z)| = \sup_{v \in [\widehat{a}, \widehat{a}+1]} W_\phi(v) < \infty.$$

Thus, we conclude from the Lindelöf's strip theorem that

$$\sup_{z \in \mathbb{C}_{[\widehat{a}, \widehat{a}+1]}^+} |f_n(z)| = \sup_{z \in \mathbb{C}_{[\widehat{a}, \widehat{a}+1]}^+} |z^n W_\phi(z)| \leq C_n,$$

see [30, Theorem 1.0.1], which is a discussion of the celebrated paper by Phragmén and Lindelöf, that is [64]. Finally, (3.4) and  $|\phi(a + ib)| = db + o(b)$ , as  $b \rightarrow \infty$ , allows us to deduce that

$$\overline{\lim}_{|b| \rightarrow \infty} |b|^n |W_\phi(a + ib)| = 0, \forall n \in \mathbb{N}, \forall a \geq \widehat{a}.$$

To conclude the claim for  $a \in (\bar{a}_\phi, \widehat{a})$  we use (3.4) in the opposite direction and (3.32), that is  $\operatorname{Re}(\phi(a + ib)) \geq \phi(a) > 0$ , for any  $a > \bar{a}_\phi$ , to get that

$$(4.9) \quad \frac{1}{\phi(a)} |W_\phi(1 + a + ib)| \geq \frac{1}{|\phi(a + ib)|} |W_\phi(1 + a + ib)| = |W_\phi(a + ib)|.$$

Thus,  $\phi \in \mathcal{B}_\infty$ . Next, assume that  $\exists \widehat{a} > \bar{a}_\phi$ ,  $\theta \in (0, \frac{\pi}{2}]$  such that  $\overline{\lim}_{|b| \rightarrow \infty} \frac{\ln |W_\phi(\widehat{a} + ib)|}{|b|} \leq -\theta$ .

Then, arguing as above, we conclude that this relation holds for  $\widehat{a} + 1$  too. Then, on  $\mathbb{C}_{[\widehat{a}, \widehat{a}+1]}^+$ , for the function  $f_\varepsilon(z) = W_\phi(z) e^{-i(\theta - \varepsilon)z}$ ,  $\varepsilon \in (0, \theta)$ , we have from Definition 3.1 with some  $C > 0$ ,  $D = D(\widehat{a}) > 0$ , that

$$\sup_{z \in \partial \mathbb{C}_{[\widehat{a}, \widehat{a}+1]}^+} |f_\varepsilon(z)| \leq C \quad \text{and} \quad \sup_{b \geq 0} \sup_{v \in [\widehat{a}, \widehat{a}+1]} |f_\varepsilon(v + ib)| \leq D e^{(\theta - \varepsilon)b} \leq D e^{\frac{\pi}{2}b}.$$

This suffices to apply [30, Theorem 1.0.1] with  $\tilde{f}_\varepsilon(z) = f_\varepsilon(e^{i\frac{\pi}{2}}z)$ . Therefore, we conclude that  $|f_\varepsilon(z)| \leq C$  on  $\mathbb{C}_{[\widehat{a}, \widehat{a}+1]}^+$  and thus, for all  $v \in [\widehat{a}, \widehat{a} + 1]$ ,

$$\overline{\lim}_{b \rightarrow \infty} \frac{\ln |W_\phi(v + ib)|}{b} \leq -\theta + \varepsilon.$$

Sending  $\varepsilon \rightarrow 0$  we conclude that, for all  $v \in [\widehat{a}, \widehat{a} + 1]$ ,

$$(4.10) \quad \overline{\lim}_{b \rightarrow \infty} \frac{\ln |W_\phi(v + ib)|}{b} \leq -\theta.$$

Therefore, from the identities  $|W_\phi(a + ib)| = |W_\phi(a - ib)|$  and (3.4), the relations  $|\phi(a + ib)| = db + o(b)$ , as  $b \rightarrow \infty$  and  $a > \bar{a}_\phi$  fixed, see Proposition 3.14(3), and (4.9) we deduce that (4.10) holds for  $a > \bar{a}_\phi$ . Thus, we deduce that  $\phi \in \mathcal{B}(\theta)$  and conclude the entire proof.  $\square$

The proof of Theorem 2.5 via (4.5) and (4.6) hinges upon the assertions of Proposition 4.2 and Proposition 4.3. Let us examine  $\left| \frac{\Gamma(z)}{W_{\phi^*}(z)} \right|$ , that is (4.6), first.

**Proposition 4.2.** *Let  $\phi \in \mathcal{B}_P^c$  then for any  $u \geq 0$  and  $a > 0$  fixed*

$$(4.11) \quad \lim_{|b| \rightarrow \infty} |b|^u \left| \frac{\Gamma(a + ib)}{W_\phi(a + ib)} \right| = 0.$$

*If  $\phi \in \mathcal{B}_P$  then (4.11) holds for any  $u < \frac{1}{d}(\phi(0) + \bar{\mu}(0)) \in (0, \infty]$ . In fact, if  $\bar{\mu}(0) < \infty$  the limit in (4.11) is infinity for all  $u > \frac{1}{d}(\phi(0) + \bar{\mu}(0))$ . Finally, regardless of the value of  $\bar{\mu}(0)$ , we have, as  $b \rightarrow \infty$  and for any  $a > 0$  such that  $\mathbf{d}^{-1} \int_0^\infty e^{-ay} \bar{\mu}(y) dy < 1$ ,*

$$(4.12) \quad \left| \frac{\Gamma(a + ib)}{W_\phi(a + ib)} \right| \lesssim e^{-\frac{\bar{\mu}(\frac{1}{b}) + \phi(0)}{d} \ln b}.$$

*Proof.* Let  $\phi \in \mathcal{B}$ . Fix  $a > 0$  and without loss of generality assume that  $b > 0$ . Applying (4.8) to  $|\Gamma(a + ib)|$  and (3.19) to  $|W_\phi(a + ib)|$  we get, as  $b \rightarrow \infty$ , that

$$(4.13) \quad \left| \frac{\Gamma(a + ib)}{W_\phi(a + ib)} \right| \asymp b^{a - \frac{1}{2}} \sqrt{|\phi(a + ib)|} e^{A_\phi(a + ib) - \frac{\pi}{2}b}.$$

It therefore remains to estimate  $A_\phi(a + ib)$  in the different scenarios stated. Let us start with  $d = 0$  or equivalently  $\phi \in \mathcal{B}_P^c$ . Then from (3.3) we get that

$$\begin{aligned} \phi(a + ib) &= \phi(0) + \int_0^\infty (1 - e^{-ay} \cos(by)) \mu(dy) + i \int_0^\infty \sin(by) e^{-ay} \mu(dy) \\ &= \operatorname{Re}(\phi(a + ib)) + i \operatorname{Im}(\phi(a + ib)). \end{aligned}$$

Clearly,  $\operatorname{Re}(\phi(a + ib)) \geq \phi(a) > 0$ , see (3.32). Next,

$$(4.14) \quad \lim_{b \rightarrow \infty} \frac{|\operatorname{Im}(\phi(a + ib))|}{b} = 0,$$

which follows from Proposition 3.14(3). These facts allow us to deduce that, for any  $M > 0$  and any  $u > u(M) > 0$ ,

$$\begin{aligned} |\arg(\phi(a + iu))| &= \left| \arctan \left( \frac{\operatorname{Im}(\phi(a + iu))}{\operatorname{Re}(\phi(a + iu))} \right) \right| \\ &\leq \arctan \left( \frac{|\operatorname{Im}(\phi(a + iu))|}{\phi(a)} \right) = \frac{\pi}{2} - \arctan \left( \frac{\phi(a)u}{u |\operatorname{Im}(\phi(a + iu))|} \right) \\ &\leq \frac{\pi}{2} - \arctan \left( \frac{M\phi(a)}{u} \right), \end{aligned}$$

where the last inequality follows from (4.14). Therefore, from the definition of  $A_\phi$ , see (3.11), we get that for any  $b > u(M)$ ,

$$|A_\phi(a + ib)| \leq \int_0^b |\arg \phi(a + iu)| du \leq \frac{\pi}{2}b - \int_{u(M)}^b \arctan\left(\frac{M\phi(a)}{u}\right) du.$$

However, since  $\arctan x \stackrel{0}{\sim} x$ , we see that  $\exists u'$  big enough such that for any  $b > u'$

$$|A_\phi(a + ib)| \leq \frac{\pi}{2}b - \frac{M\phi(a)}{2}(\ln b - \ln u').$$

Plugging this in (4.13) and using the fact that, for a fixed  $a > 0$ ,  $|\phi(a + ib)| \stackrel{\infty}{\asymp} o(|a + ib|)$ , when  $\mathfrak{d} = 0$ , see Proposition 3.14(3), we easily get that, as  $b \rightarrow \infty$ ,

$$\left| \frac{\Gamma(a + ib)}{W_\phi(a + ib)} \right| \lesssim b^a e^{-\frac{M\phi(a)}{2} \ln b} = b^{a - \frac{M\phi(a)}{2}}.$$

Since  $M$  is arbitrary we conclude (4.11) when  $\phi \in \mathcal{B}_P^c$ . Assume next that  $\phi \in \mathcal{B}_P$  and without loss of generality that  $b > 0$ . Then from (3.51) of Proposition 3.16 we get for the exponent of (4.13) that, as  $b \rightarrow \infty$  and any fixed  $a > 0$ ,

$$\begin{aligned} A_\phi(a + ib) - \frac{\pi}{2}b &= -b \arctan\left(\frac{a}{b}\right) - \left(a + \frac{\phi(0)}{\mathfrak{d}}\right) \ln b \\ &\quad - \frac{\left(a + \frac{\phi(0)}{\mathfrak{d}}\right)}{2} \left(\ln\left(1 + \frac{b^2}{a^2}\right) - \ln b^2\right) - \int_0^\infty \frac{1 - \cos(by)}{y} \mathfrak{L}_{\bar{\mu}_{a,\mathfrak{d}}}(dy) + \bar{A}_\phi(a) \\ &= -\left(a + \frac{\phi(0)}{\mathfrak{d}}\right) \ln b - \frac{\left(a + \frac{\phi(0)}{\mathfrak{d}}\right)}{2} \ln\left(1 + \frac{b^2}{a^2}\right) + O(1), \end{aligned}$$

where we have used implicitly that  $\arctan \frac{a}{b} + \arctan \frac{b}{a} = \frac{\pi}{2}$ . Therefore, as  $b \rightarrow \infty$ , (4.13) is simplified to

$$\left| \frac{\Gamma(a + ib)}{W_\phi(a + ib)} \right| \asymp b^{a - \frac{1}{2}} \sqrt{|\phi(a + ib)|} e^{-\left(a + \frac{\phi(0)}{\mathfrak{d}}\right) \ln b - \int_0^\infty \frac{1 - \cos(by)}{y} \mathfrak{L}_{\bar{\mu}_{a,\mathfrak{d}}}(dy)}.$$

We recall that  $\mathfrak{L}_{\bar{\mu}_{a,\mathfrak{d}}}(dy)$  is the measure associated to the measure  $\bar{\mu}_{a,\mathfrak{d}}(y)dy = \mathfrak{d}^{-1}e^{-ay}\bar{\mu}(y)dy$  as defined in (3.46). When  $\phi \in \mathcal{B}_P$  we have that  $|\phi(a + ib)| \sim \mathfrak{d}b$ , as  $b \rightarrow \infty$  and  $a > 0$  fixed, see Proposition 3.14(3), and thus (4.13) is simplified further to

$$(4.15) \quad \left| \frac{\Gamma(a + ib)}{W_\phi(a + ib)} \right| \asymp e^{-\frac{\phi(0)}{\mathfrak{d}} \ln b - \int_0^\infty \frac{1 - \cos(by)}{y} \mathfrak{L}_{\bar{\mu}_{a,\mathfrak{d}}}(dy)}.$$

Next, choose  $a > a_0 > 0$  so as to have  $\|\bar{\mu}_{a,\mathfrak{d}}\|_{TV} = \mathfrak{d}^{-1} \int_0^\infty e^{-ay}\bar{\mu}(y)dy < 1$ . Then (3.54) of Proposition 3.16 applies and proves (4.12) regardless of the value of  $\bar{\mu}(0)$ . Moreover, (4.12) also settles (4.11) for those  $a > a_0$  and any  $u \geq 0$  whenever  $\bar{\mu}(0) = \infty$ . However, since

$$\left| \frac{\Gamma(1 + a + ib)}{W_\phi(1 + a + ib)} \right| = \frac{|a + ib|}{|\phi(a + ib)|} \left| \frac{\Gamma(a + ib)}{W_\phi(a + ib)} \right|$$

and  $\lim_{b \rightarrow \infty} \frac{|a + ib|}{|\phi(a + ib)|} = \mathfrak{d}^{-1}$ , see Proposition 3.14(3), we trivially conclude (4.11), when  $\bar{\mu}(0) = \infty$ , for any  $a > 0$ ,  $u \geq 0$ . Next, let  $\bar{\mu}(0) < \infty$  and choose again  $a > a_0$  so that  $\|\bar{\mu}_{a,\mathfrak{d}}\|_{TV} < 1$ . Then

(3.45) holds and thanks to Proposition 4.10 we have that  $\bar{\mu}_{a,d}^{*n}(y) = e^{-ay} \frac{\bar{\mu}^{*n}(y)}{d^n}$ . Therefore, on  $(0, \infty)$ ,

$$\begin{aligned} \mathfrak{L}_{\bar{\mu}_{a,d}}(dy) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\bar{\mu}_{a,d}^{*n}(y)}{n} dy \\ &= e^{-ay} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\bar{\mu}^{*n}(y)}{d^n n} dy = \frac{1}{d} e^{-ay} \bar{\mu}(y) dy + h(y) dy. \end{aligned}$$

Now since  $\bar{\mu}(0) < \infty$  and therefore  $\bar{\mu} \in L^\infty(\mathbb{R}^+)$  we have from Proposition 4.10(4.74) applied with  $a = a' = 0$  that

$$|h(y)| \leq e^{-ay} \sum_{n=2}^{\infty} \frac{\bar{\mu}^{*n}(y)}{d^n n} dy \leq y e^{-ay} \sum_{n=2}^{\infty} \frac{\bar{\mu}^n(0) y^{n-2}}{d^n n!}.$$

Thus

$$\left| \int_0^\infty \frac{1 - \cos(by)}{y} h(y) dy \right| \leq 2 \sum_{n=2}^{\infty} \frac{\bar{\mu}^n(0)}{n(n-1) d^n a^{n-1}} < \infty \iff a \geq \frac{\bar{\mu}(0)}{d}.$$

Therefore, if we choose  $a > a_0 \vee \frac{\bar{\mu}(0)}{d}$  we see that the term above does not contribute to the asymptotic in (4.15). We are then left with the relation

$$(4.16) \quad \left| \frac{\Gamma(a+ib)}{W_\phi(a+ib)} \right| \asymp e^{-\frac{\phi(0)}{d} \ln b - \frac{1}{d} \int_0^\infty \frac{1 - \cos(by)}{y} e^{-ay} \bar{\mu}(y) dy}.$$

First, since  $y \mapsto e^{-ay} \frac{\bar{\mu}(y)}{y}$  is integrable on  $(1, \infty)$ , the Riemann-Lebesgue lemma yields that

$$(4.17) \quad \lim_{b \rightarrow \infty} \int_1^\infty \frac{1 - \cos(by)}{y} e^{-ay} \bar{\mu}(y) dy = \int_1^\infty e^{-ay} \bar{\mu}(y) \frac{dy}{y}.$$

Next, recall that  $\bar{\mu}(0) < \infty$ . Henceforth, from the dominated convergence theorem

$$(4.18) \quad \lim_{b \rightarrow \infty} \int_0^{\frac{1}{b}} \frac{1 - \cos(by)}{y} e^{-ay} \bar{\mu}(y) dy = \lim_{b \rightarrow \infty} \int_0^1 \frac{1 - \cos y}{y} e^{-a \frac{y}{b}} \bar{\mu}\left(\frac{y}{b}\right) dy = \bar{\mu}(0) \int_0^1 \frac{1 - \cos y}{y} dy.$$

Recall that  $y \mapsto \bar{\mu}_a(y) = e^{-ay} \bar{\mu}(y)$  is decreasing on  $\mathbb{R}^+$ , thus defining a measure  $\bar{\mu}_a(dv)$  on  $(0, \infty)$ . Therefore, the remaining portion of the integral in the exponent of (4.16) is written as

$$(4.19) \quad \begin{aligned} \int_{\frac{1}{b}}^1 \frac{1 - \cos(by)}{y} e^{-ay} \bar{\mu}(y) dy &= \int_{\frac{1}{b}}^1 \frac{1 - \cos(by)}{y} \left( \bar{\mu}_a(y) - \bar{\mu}_a\left(\frac{1}{b}\right) + \bar{\mu}_a\left(\frac{1}{b}\right) \right) dy \\ &= \bar{\mu}_a\left(\frac{1}{b}\right) \left( \ln b - \int_1^b \frac{\cos y}{y} dy \right) \\ &\quad - \int_{\frac{1}{b}}^1 \frac{1}{y} \left( \bar{\mu}_a\left(\frac{1}{b}\right) - \bar{\mu}_a(y) \right) dy + \int_{\frac{1}{b}}^1 \frac{\cos(by)}{y} \int_{\frac{1}{b}}^y \bar{\mu}_a(dv) dy. \end{aligned}$$

Clearly, then

$$\overline{\lim}_{b \rightarrow \infty} \left| \int_{\frac{1}{b}}^1 \frac{\cos(by)}{y} \int_{\frac{1}{b}}^y \bar{\mu}_a(dv) dy \right| \leq \overline{\lim}_{b \rightarrow \infty} \int_{\frac{1}{b}}^1 \left| \int_{bv}^b \frac{\cos y}{y} dy \right| \bar{\mu}_a(dv) = 0,$$

since  $\overline{\lim}_{b \rightarrow \infty} \left| \int_{bv}^b \frac{\cos y}{y} dy \right| = 0$ , for all  $v \in (0, 1)$ , and the validity of the dominated convergence theorem which is due to  $\sup_{x \geq 1} \left| \int_x^\infty \frac{\cos y}{y} dy \right| < \infty$  and  $\int_0^1 |\bar{\mu}_a(dv)| = \bar{\mu}(0) - e^{-1}\bar{\mu}(1) < \infty$ . Henceforth, from (4.17), (4.18) and (4.19), we obtain, as  $b \rightarrow \infty$ , that

$$\left| \int_0^\infty \frac{1 - \cos(by)}{y} e^{-ay} \bar{\mu}(y) dy - \bar{\mu}(0) \ln b \right| \leq \left( \bar{\mu}(0) - \bar{\mu}_a \left( \frac{1}{b} \right) \right) \ln b + \int_{\frac{1}{b}}^1 (\bar{\mu}(0) - \bar{\mu}_a(y)) \frac{dy}{y} + C + o(1)$$

where  $C > 0$ . However, it can be seen easily that the first and the second term on the right-hand side are of order  $o(\ln b)$  and therefore, as  $b \rightarrow \infty$ ,

$$\int_0^\infty \frac{1 - \cos(by)}{y} e^{-ay} \bar{\mu}(y) dy = \bar{\mu}(0) \ln b + o(\ln b).$$

This fed in (4.16) yields

$$(4.20) \quad \left| \frac{\Gamma(a + ib)}{W_\phi(a + ib)} \right| \asymp e^{-\frac{\phi(0)}{a} \ln b - \frac{\bar{\mu}(0)}{a} \ln b + o(\ln b)},$$

which proves (4.11) for  $u < \frac{1}{a}(\phi(0) + \bar{\mu}(0))$  and shows that the limit in (4.11) is infinity for  $u > \frac{1}{a}(\phi(0) + \bar{\mu}(0))$ . This concludes the proof of Proposition 4.2.  $\square$

Proposition 4.2 essentially deals exhaustively with the proof of Theorem 2.5 via the term  $\left| \frac{\Gamma(z)}{W_{\phi_+}(z)} \right|$  in (4.6). Since  $\left| \frac{\Gamma(z)}{W_{\phi_+}(z)} \right|$  decays faster than any polynomial except when  $\phi_+ \in \mathcal{B}_P$  and  $\bar{\mu}_+(0) < \infty$ , it remains to discuss this scenario. Before stating and proving Proposition 4.3 we introduce some more notation needed throughout below. We recall that with any  $\phi \in \mathcal{B}$  we have an associated non-decreasing (possibly killed at independent exponential rate of parameter  $\phi(0) > 0$ ) Lévy process  $\xi$ . Then the potential measure of  $\xi$  and therefore of  $\phi$  is defined as

$$(4.21) \quad U(dy) = \int_0^\infty e^{-\phi(0)t} \mathbb{P}(\xi_t \in dy) dt, \quad y > 0,$$

and from Proposition 3.14(5) we get that, for  $z \in \mathbb{C}_{(0, \infty)}$ ,

$$(4.22) \quad \int_0^\infty e^{-zy} U(dy) = \frac{1}{\phi(z)}.$$

The renewal or potential function  $U(y) = \int_0^y U(dx)$ ,  $y > 0$ , is subadditive on  $(0, \infty)$ . Recall that if  $\Psi \in \bar{\mathcal{N}}$  then  $\Psi(z) = -\phi_+(-z)\phi_-(z)$  and we have two potential measures  $U_\pm$  related to  $\phi_\pm$  respectively. If in addition  $\phi \in \mathcal{B}_P$  then it is well known from [9, Chapter III] that the potential density  $u(y) = \frac{U(dy)}{dy}$  exists. Moreover, it is continuous, strictly positive and bounded on  $[0, \infty)$ , that is  $\|u\|_\infty < \infty$ . Furthermore, [26, Proposition 1] establishes that in this case

$$(4.23) \quad u(y) = \sum_{j=0}^{\infty} \frac{(-1)^j}{\mathfrak{d}^{j+1}} (\mathbf{1} * (\phi(0) + \bar{\mu})^{*j})(y) = \frac{1}{\mathfrak{d}} + \tilde{u}(y), \quad y \geq 0,$$

where  $\mathbf{1}(y) = \mathbb{I}_{\{y > 0\}}$  stands for the Heavyside function and  $f * g(x) = \int_0^x f(x-v)g(v)dv$  represents the convolution of two functions. We keep the last notation for convolutions of measures too. Then we have the result.

**Proposition 4.3.** *Let  $\phi_+ \in \mathcal{B}_P$  and  $\bar{\mu}_+(0) < \infty$ . Then we have the following scenarios.*

- (1) *If  $\phi_- \in \mathcal{B}_P$  then  $\phi_- \in \mathcal{B}(\frac{\pi}{2})$ .*

(2) If  $\phi_- \in \mathcal{B}_P^c$  then  $\phi_- \in \mathcal{B}_\infty \iff \bar{\Pi}_-(0) = \infty$ .

(3) If  $\phi_- \in \mathcal{B}_P^c$  and  $\bar{\Pi}_-(0) < \infty$  then for any  $u < \frac{\int_0^\infty u_+(y)\Pi_-(dy)}{\phi_-(0)+\bar{\mu}_-(0)}$  and any  $a > 0$  we have that  $\lim_{|b| \rightarrow \infty} |b|^u |W_{\phi_-}(a+ib)| = 0$ . If  $u > \frac{\int_0^\infty u_+(y)\Pi_-(dy)}{\phi_-(0)+\bar{\mu}_-(0)}$  then for any  $a > 0$  the following  $\lim_{|b| \rightarrow \infty} |b|^u |W_{\phi_-}(a+ib)| = \infty$  is valid.

*Remark 4.4.* We stress that the validity of item (2), that is  $\phi_- \in \mathcal{B}_\infty$ , can be established for general  $\phi \in \mathcal{B}$  as long as the following three conditions are satisfied. First, the Lévy measure of  $\phi$  is absolutely continuous, that is  $\mu(dy) = v(y)dy$ ,  $y > 0$ , and  $v(0) = \infty$ . Secondly,  $v(y) = v_1(y) + v_2(y)$  such that  $v_1, v_2 \in L^1(\mathbb{R}^+)$  and  $v_1 \geq 0$  is non-increasing on  $\mathbb{R}^+$ . Finally,  $\int_0^\infty v_2(y)dy \geq 0$  and  $|v_2(x)| \leq (\int_x^\infty v_1(y)dy) \vee C$  for some  $C > 0$  on  $\mathbb{R}^+$ . Imposing  $\phi_+ \in \mathcal{B}_P$  and  $\bar{\mu}_+(0) < \infty$  ensures precisely those conditions for  $\phi_-$ . In fact Lemma 4.5, Proposition 4.6 and Lemma 4.7 modulo to (4.58),(4.59) serve only the purpose to check the validity of those conditions. For item (3) for general  $\phi \in \mathcal{B}$  it suffices to assume that  $\mu(dy) = v(y)dy$ ,  $y > 0$ ,  $v(0^+) = \lim_{y \rightarrow 0} v(y) < \infty$  and  $v \in L^\infty(\mathbb{R}^+)$ . Also note that  $v_-(0^+) = \int_0^\infty u_+(y)\Pi_-(dy)$ .

*Proof.* When  $\phi_- \in \mathcal{B}_P$  it follows immediately from (3.51) of Proposition 3.16 that  $\phi_- \in \mathcal{B}(\frac{\pi}{2})$  and item (1) is proved. Let us proceed with item (2). Since  $\phi_+ \in \mathcal{B}_P$  from [24, Chapter V, (5.3.11)] or Proposition B.1 we can obtain the differentiated version of (B.1), that is

$$(4.24) \quad \mu_-(dy) = v_-(y)dy = \int_0^\infty u_+(v)\Pi_-(y+dv)dy.$$

Note that

$$(4.25) \quad \bar{\mu}_-(0) = \int_0^\infty v_-(y)dy = \infty \iff \int_0^1 \bar{\Pi}_-(y)dy = \infty.$$

Indeed, from (4.24), for any  $y < 1$ ,

$$\left( \inf_{0 \leq v \leq 1} u_+(v) \right) (\bar{\Pi}_-(y) - \bar{\Pi}_-(1)) \leq v_-(y) \leq \|u_+\|_\infty \bar{\Pi}_-(y).$$

Then  $\inf_{0 \leq v \leq 1} u_+(v) > 0$  since  $u_+(0) = \frac{1}{\mathfrak{d}_+} > 0$ ,  $u_+ \in \mathfrak{C}([0, \infty))$  and  $u_+$  never touches 0 whenever  $\mathfrak{d}_+ > 0$ , see [9, Chapter III], and (4.25) is established. Next  $\phi_+ \in \mathcal{B}_P$  and  $\bar{\mu}_+(0) < \infty$  trigger the simultaneous validity of Lemma 4.5, Lemma 4.7 and Lemma 4.8 provided  $\bar{\mu}_-(0) = \infty$  or equivalently  $\int_0^1 \bar{\Pi}_-(y)dy = \infty$ , see (4.25) above. Assume the latter and note that  $\bar{\mu}_-(0) = \infty$  is only needed for Lemma 4.8 so that  $\phi_-^c(\infty) = \infty$  is valid when  $\mathfrak{d}_- = 0$ , as it is the case in item (2) here. Then, we can always choose  $c \in \mathbb{R}^+$  such that (4.63) holds for any  $a \geq a_c > 0$ , namely, for all  $b$  large enough

$$\arg(\phi_-(a+ib)) = \arg(\phi_-^c(a+ib))(1 + o(1)) + \arg(1 - \mathcal{F}_{\Xi_a^c}(-ib))$$

with  $\Xi_a^c$  as in Lemma 4.8. Thus, from the definition of  $A_\phi$ , see (3.11), we get that, as  $b \rightarrow \infty$ ,

$$A_\phi(a+ib) = A_{\phi_-^c}(a+ib)(1 + o(1)) + \int_0^b \arg(1 - \mathcal{F}_{\Xi_a^c}(-iu))du.$$

However, since in (3.19) of Theorem 3.3,  $G_\phi$  does not depend on  $b$ , whereas  $E_\phi$  and  $R_\phi$  are uniformly bounded on  $\mathbb{C}_a$  and  $\phi_- \in \mathcal{B}$  and  $A_\phi \geq 0$ , we conclude that, for every  $a > a_c$  fixed and



any  $\eta \in (0, \frac{1}{2})$ , as  $b \rightarrow \infty$ ,

$$(4.26) \quad \begin{aligned} |W_{\phi_-}(a+ib)| &\lesssim \frac{|\phi_-^c(a+ib)|^{\frac{1}{2}+\eta}}{\sqrt{|\phi_-(a+ib)|}} e^{-\int_0^b \arg(1-\mathcal{F}_{\Xi_a^c}(-iu))du} |W_{\phi_-^c}(a+ib)|^{1+2\eta} \\ |W_{\phi_-}(a+ib)| &\gtrsim \frac{|\phi_-^c(a+ib)|^{\frac{1}{2}-\eta}}{\sqrt{|\phi_-(a+ib)|}} e^{-\int_0^b \arg(1-\mathcal{F}_{\Xi_a^c}(-iu))du} |W_{\phi_-^c}(a+ib)|^{1-2\eta}. \end{aligned}$$

However, the Lévy measure of  $\phi_-^c$  is  $\mu_-^c(dy) = \mathbb{I}_{\{y < c\}} \mathbf{d}_+^{-1} \bar{\Pi}_-^c(y) dy$ , see (4.56), and then since  $\bar{\Pi}_-^c(y)$  is non-increasing on  $(0, \infty)$  we deduce via integration by parts of (4.56) that

$$\Psi^c(z) = z\phi_-^c(z) = \phi_-(0)z + \frac{1}{\mathbf{d}_+} \int_{-c}^0 (e^{zr} - 1 - zr) \Pi^c(-dr) \in \bar{\mathcal{N}}.$$

However, [61, Theorem 5.1 (5.3)] shows that  $\phi_-^c \in \mathcal{B}_\infty \iff \bar{\Pi}_-^c(0) = \infty$  (the latter being equivalent to  $\mathbf{N}_\tau = \infty$  in the notation of [61] and  $W_{\phi_-^c} = \mathcal{M}_{V_\psi}$  therein). Moreover, since  $\bar{\Pi}_-^c(y) = (\bar{\Pi}_-(y) - \bar{\Pi}_-(c)) \mathbb{I}_{\{y \leq c\}}$ , see Lemma 4.5, we obtain that  $\phi_-^c \in \mathcal{B}_\infty \iff \bar{\Pi}_-(0) = \infty$ . It remains henceforth to understand the terms to the right-hand side of (4.26) and show that they cannot disrupt the subexponential decay brought in by  $|W_{\phi_-^c}(a+ib)|^{1+2\eta}$ . With the notation and the claim of Lemma 4.8 we have that  $\lim_{a \rightarrow \infty} \|\Xi_a^c\|_{TV} = 0$  and thus there exists  $a_0 > a_c$  such that for all  $a > a_0$ ,  $\|\Xi_a^c\|_{TV} < 1$ . Therefore, from (3.49) of Proposition 3.15 we get that for all  $u \in \mathbb{R}$ ,  $\log_0(1 - \mathcal{F}_{\Xi_a^c}(-iu)) = -\mathcal{F}_{\Xi_a^c}(-iu)$ . Moreover,  $\|\Xi_a^c\|_{TV} < 1$  implies that the first expression in (3.45) holds. Henceforth,

$$(4.27) \quad \begin{aligned} \arg(1 - \mathcal{F}_{\Xi_a^c}(-iu)) &= \text{Im}(\log_0(1 - \mathcal{F}_{\Xi_a^c}(-iu))) \\ &= -\text{Im}(\mathcal{F}_{\Xi_a^c}(-iu)) - \sum_{n=2}^{\infty} \frac{\text{Im}(\mathcal{F}_{\Xi_a^c}^n(-iu))}{n} \\ &= -\text{Im}(\mathcal{F}_{\Xi_a^c}(-iu)) - \sum_{n=2}^{\infty} \frac{\text{Im}(\mathcal{F}_{(\Xi_a^c)^{*n}}(-iu))}{n}. \end{aligned}$$

We work with the remainder term of (4.26), that is  $\int_0^b \arg(1 - \mathcal{F}_{\Xi_a^c}(-iu)) du$  and  $b > 0$ . We start with the infinite sum in (4.27) clarifying that, for each  $n \geq 1$ ,

$$(4.28) \quad \text{Im}(\mathcal{F}_{(\Xi_a^c)^{*n}}(-iu)) = -\int_0^\infty \sin(uy) (\chi_a^c)^{*n}(y) dy,$$

where from Lemma 4.8  $\chi_a^c(y) dy = e^{-ay} \chi^c(y) dy$  is the density of  $\Xi_a^c(dy)$  and for any  $a \geq a_c$ ,  $\|\chi_a^c\|_\infty < \infty$ . By definition  $a_0 = \max\{a_0, a_c\}$ . We work from now on with  $0 < a_0 < a$ . Next, note that, for each  $b > 0$ , from Proposition 4.10(4.74), we have that

$$(4.29) \quad \begin{aligned} \int_0^b \int_0^\infty \left| \sin(uy) \sum_{n=2}^{\infty} \frac{(\chi_a^c)^{*n}(y)}{n} \right| dy du &\leq b \int_0^\infty \sum_{n=2}^{\infty} \|\chi_{a_0}^c\|_\infty^n \frac{y^{n-1} e^{-(a-a_0)y}}{n!} dy \\ &= b \sum_{n=2}^{\infty} \frac{\|\chi_{a_0}^c\|_\infty^n}{n(a-a_0)^n} < \infty \iff a - a_0 > \|\chi_{a_0}^c\|_\infty. \end{aligned}$$

So since (4.26) is valid for any  $a > a_0 > a_c$  we, from now on, fix  $a > 2\|\chi_{a_0}^c\|_\infty + a_0$ . Then (4.29) allows via integration by parts in (4.30) below to conclude using (4.27), (4.28) and Proposition

4.10(4.74) that, for any  $b > 0$ ,

$$\begin{aligned}
(4.30) \quad & \left| \int_0^b (\arg(1 - \mathcal{F}_{\Xi_a^c}(-iu)) + \operatorname{Im}(\mathcal{F}_{\Xi_a^c}(-iu))) du \right| = \left| \int_0^b \sum_{n=2}^{\infty} \frac{\operatorname{Im}(\mathcal{F}_{(\Xi_a^c)^{*n}}(-iu))}{n} du \right| \\
& = \left| \int_0^b \sum_{n=2}^{\infty} \frac{\int_0^{\infty} \sin(uy) (\chi_a^c)^{*n}(y) dy}{n} du \right| = \left| \int_0^{\infty} \frac{1 - \cos(by)}{y} \sum_{n=2}^{\infty} \frac{(\chi_a^c)^{*n}(y)}{n} dy \right| \\
& \leq 2 \int_0^{\infty} \sum_{n=2}^{\infty} \frac{\|\chi_{a_0}^c\|_{\infty}^n y^{n-2} e^{-(a-a_0)y}}{n!} dy = 2 \sum_{n=2}^{\infty} \frac{\|\chi_{a_0}^c\|_{\infty}^n}{n(n-1)(a-a_0)^{n-1}} < \infty.
\end{aligned}$$

Since the right-hand side is independent of  $b > 0$  we deduct that for  $a > 2\|\chi_{a_0}^c\|_{\infty} + a_0$ , (4.26) is simplified to

$$\begin{aligned}
(4.31) \quad & |W_{\phi_{-}}(a+ib)| \lesssim \frac{|\phi_{-}^c(a+ib)|^{\frac{1}{2}+\eta}}{\sqrt{|\phi_{-}(a+ib)|}} e^{\int_0^b \operatorname{Im}(\mathcal{F}_{\Xi_a^c}(-iu)) du} |W_{\phi^c}(a+ib)|^{1+2\eta} \\
& |W_{\phi_{-}}(a+ib)| \gtrsim \frac{|\phi_{-}^c(a+ib)|^{\frac{1}{2}-\eta}}{\sqrt{|\phi_{-}(a+ib)|}} e^{\int_0^b \operatorname{Im}(\mathcal{F}_{\Xi_a^c}(-iu)) du} |W_{\phi^c}(a+ib)|^{1-2\eta}
\end{aligned}$$

Next from (4.28)

$$\begin{aligned}
(4.32) \quad & \left| \int_0^b \operatorname{Im}(\mathcal{F}_{\Xi_a^c}(-iu)) du \right| = \left| \int_0^{\infty} \frac{1 - \cos(by)}{y} \chi_a^c(y) dy \right| \\
& \leq \left| \int_0^1 \frac{1 - \cos(by)}{y} \chi_a^c(y) dy \right| + \left| \int_1^{\infty} \frac{1 - \cos(by)}{y} \chi_a^c(y) dy \right|.
\end{aligned}$$

From the Riemann-Lebesgue lemma applied to the absolutely integrable function  $\chi_a^c(y)y^{-1}\mathbb{I}_{\{y>1\}}$  we get that

$$\lim_{b \rightarrow \infty} \left| \int_1^{\infty} \frac{1 - \cos(by)}{y} \chi_a^c(y) dy \right| = \left| \int_1^{\infty} \frac{\chi_a^c(y)}{y} dy \right| =: D_a.$$

Therefore, using the fact that  $\|\chi_a^c\|_{\infty} < \infty$ , see Lemma 4.8, we conclude, for all  $b > 1$  big enough, that

$$\begin{aligned}
(4.33) \quad & \left| \int_0^b \operatorname{Im}(\mathcal{F}_{\Xi_a^c}(-iu)) du \right| \leq \int_0^b \frac{1 - \cos y}{y} \left| \chi_a^c\left(\frac{y}{b}\right) \right| dy + 2D_a \\
& \leq \|\chi_a^c\|_{\infty} \left( \int_0^1 \frac{1 - \cos y}{y} dy + \int_1^b \frac{1 - \cos y}{y} dy \right) + 2D_a \\
& \leq \|\chi_a^c\|_{\infty} \int_0^1 \frac{1 - \cos y}{y} dy + \|\chi_a^c\|_{\infty} \ln b + \tilde{D}_a,
\end{aligned}$$

where  $\tilde{D}_a = 2D_a + \sup_{b>1} \left| \int_1^b \frac{\cos y}{y} dy \right| < \infty$ . This allows us to conclude in (4.31), as  $b \rightarrow \infty$ ,

$$\begin{aligned}
(4.34) \quad & |W_{\phi_{-}}(a+ib)| \lesssim b^{\|\chi_a^c\|_{\infty} + \frac{1}{2} + \eta} |W_{\phi^c}(a+ib)|^{1+2\eta} \\
& |W_{\phi_{-}}(a+ib)| \gtrsim b^{-\|\chi_a^c\|_{\infty} - \frac{1}{2}} |W_{\phi^c}(a+ib)|^{1-2\eta},
\end{aligned}$$

where we have also used the standard relation  $|\phi(a+ib)| \approx db + o(b)$ , see Proposition 3.14(3), and for any  $a > 0$  and any  $\phi \in \mathcal{B}$ ,  $|\phi(a+ib)| \geq \operatorname{Re}(\phi(a+ib)) \geq \phi(a) > 0$ , see (3.32). Hence, as mentioned below (4.26) from [61, Theorem 5.1 (5.3)] we have that  $\phi^c \in \mathcal{B}_{\infty} \iff \bar{\Pi}_{-}^c(0) = \infty$

and since  $\bar{\Pi}_-^c(y) = (\bar{\Pi}_-(y) - \bar{\Pi}_-(c)) \mathbb{I}_{\{y \leq c\}}$ , see Lemma 4.5, we conclude that  $\phi_-^c \in \mathcal{B}_\infty \iff \bar{\Pi}_-(0) = \infty$ . This together with (4.34) and Lemma 4.1 shows that

$$\int_0^1 \bar{\Pi}_-(y) dy = \infty \implies \bar{\Pi}_-(0) = \infty \implies \phi_- \in \mathcal{B}_\infty.$$

Let next  $\int_0^1 \bar{\Pi}_-(y) dy < \infty$  but  $\bar{\Pi}_-(0) = \infty$ . Unfortunately, we cannot easily use similar comparison as above despite that  $\phi_-^c \in \mathcal{B}_\infty \iff \bar{\Pi}_-^c(0) = \infty$ . In fact Lemma 4.8 fails to give a good and quick approximation of  $\arg \phi_-$  with  $\arg \phi_-^c$ . We choose a different route. An easy computation involving (4.24),  $u_+(0) = \frac{1}{\mathbf{d}_+} \in (0, \infty)$ , see (4.23), and  $u_+ \in \mathcal{C}([0, \infty))$ , gives that

$$(4.35) \quad \lim_{y \rightarrow 0} v_-(y) = \infty,$$

since for any  $\varepsilon > 0$ ,  $v_-(y) \geq (\inf_{v \in [0, \varepsilon]} u_+(v)) (\bar{\Pi}_-(y) - \bar{\Pi}_-(y + \varepsilon))$ . Since we aim to show that  $\phi_- \in \mathcal{B}_\infty$  from Lemma 4.1 we can work again with a single  $a$  which we will choose later. From  $|W_{\phi_-}(a + ib)| = |W_{\phi_-}(a - ib)|$  we can focus on  $b > 0$  only as well. From the alternative expression for  $A_{\phi_-}$ , see (3.20) in the claim of Theorem 3.2(1), we get that

$$(4.36) \quad \begin{aligned} A_{\phi_-}(a + ib) &= b\Theta_{\phi_-}(a + ib) = \int_a^\infty \ln \left( \left| \frac{\phi_-(u + ib)}{\phi_-(u)} \right| \right) du \\ &\geq \int_a^\infty \ln \left( \left| 1 + \frac{\operatorname{Re}(\phi_-(u + ib) - \phi_-(u))}{\phi_-(u)} \right| \right) du. \end{aligned}$$

Next, we note that since  $\mathbf{d}_- = 0$ ,

$$\frac{\operatorname{Re}(\phi_-(u + ib) - \phi_-(u))}{\phi_-(u)} = \frac{\int_0^\infty (1 - \cos(by)) e^{-uy} v_-(y) dy}{\phi_-(u)} \geq 0.$$

Moreover, since  $\bar{\mu}_-(0) < \infty$ ,

$$\lim_{u \rightarrow \infty} \sup_{b \in \mathbb{R}} \int_0^\infty (1 - \cos(by)) e^{-uy} v_-(y) dy \leq \lim_{u \rightarrow \infty} 2 \int_0^\infty e^{-uy} v_-(y) dy = 0$$

and  $\lim_{u \rightarrow \infty} \phi_-(u) = \phi_-(\infty) < \infty$ , see (3.3). We choose  $\hat{a} > 0$  large enough so that  $\phi_-(u) > \frac{\phi_-(\infty)}{2}$  and  $\sup_{b \in \mathbb{R}} \int_0^\infty (1 - \cos(by)) e^{-uy} v_-(y) dy \leq \frac{\phi_-(\infty)}{4}$ , for all  $u \geq \hat{a}$ . Therefore from (4.36) and  $\ln(1 + x) \geq Cx$ ,  $\forall x < \frac{1}{2}$  with some  $C > 0$ , we deduce that for any  $\varepsilon > 0$  and  $b > \frac{1}{\varepsilon}$

$$\begin{aligned} A_{\phi_-}(\hat{a} + ib) &\geq C \int_{\hat{a}}^\infty \frac{\int_0^\infty (1 - \cos(by)) e^{-uy} v_-(y) dy}{\phi_-(u)} du \\ &\geq \frac{C}{\phi_-(\infty)} \int_{\hat{a}}^\infty \int_0^\infty (1 - \cos(by)) e^{-uy} v_-(y) dy du \\ &= \frac{C}{\phi_-(\infty)} \int_0^\infty (1 - \cos(by)) e^{-\hat{a}y} v_-(y) \frac{dy}{y} \geq \frac{C}{\phi_-(\infty)} \int_{\frac{1}{b}}^\varepsilon (1 - \cos(by)) e^{-\hat{a}y} v_-(y) \frac{dy}{y} \\ &\geq \frac{C}{\phi_-(\infty)} e^{-\hat{a}\varepsilon} \int_1^{b\varepsilon} (1 - \cos y) v_-\left(\frac{y}{b}\right) \frac{dy}{y} \\ &\geq \frac{C}{\phi_-(\infty)} e^{-\hat{a}\varepsilon} \left( \inf_{v \in (0, \varepsilon)} v_-(v) \right) \int_1^{b\varepsilon} (1 - \cos y) \frac{dy}{y}. \end{aligned}$$

However, since  $\int_1^\infty \frac{\cos y}{y} dy < \infty$  we conclude that for any  $\varepsilon > 0$ , as  $b \rightarrow \infty$ ,

$$(4.37) \quad \lim_{b \rightarrow \infty} \frac{A_{\phi_-}(\hat{a} + ib)}{\ln b} \geq \frac{C}{\phi_-(\infty)} e^{-\hat{a}\varepsilon} \inf_{v \in (0, \varepsilon)} v_-(v).$$

Now, (4.35) and (4.37) prove the claim  $\phi_- \in \mathcal{B}_\infty$  since for fixed  $\widehat{a}$  and as  $b \rightarrow \infty$

$$W_{\phi_-}(a+ib) \asymp \frac{1}{\sqrt{|\phi_-(\widehat{a}+ib)|}} e^{-A_{\phi_-}(\widehat{a}+ib)} \lesssim \frac{1}{\sqrt{|\phi_-(\widehat{a}+ib)|}} e^{-\frac{C}{\phi_-(\infty)} e^{-\widehat{a}\varepsilon} (\inf_{v \in (0, \varepsilon)} v_-(v)) \ln(b)},$$

see (3.19), and  $|\phi_-(\widehat{a}+ib)| \geq \phi_-(\widehat{a}) > 0$ , see (3.32). We conclude item (2). We proceed with the proof of item (3). Assume then that  $\overline{\Pi}_-(0) < \infty$ . In this case we study directly  $A_{\phi_-}$ . Since (4.24) holds in any situation when  $\phi_+ \in \mathcal{B}_P$  then  $\mu_-(dy) = v_-(y) dy$ ,  $y \in (0, \infty)$ , and

$$\|v_-\|_\infty = \sup_{y \geq 0} v_-(y) \leq \|u_+\|_\infty \sup_{y > 0} \overline{\Pi}_-(y) = \|u_+\|_\infty \overline{\Pi}_-(0) = A < \infty.$$

Note that  $\phi_-(\infty) = \phi_-(0) + \bar{\mu}_-(0)$  and put  $v_a^*(y) = \frac{e^{-ay}}{\phi_-(\infty)} v_-(y)$ ,  $y > 0$ . Then, clearly from the first expression in (3.3), for  $z = a + ib \in \mathbb{C}_{(0, \infty)}$ ,  $a > 0$ ,

$$(4.38) \quad \phi_-(z) = \phi_-(0) + \bar{\mu}_-(0) - \int_0^\infty e^{-iby} e^{-ay} v_-(y) dy = \phi_-(\infty) (1 - \mathcal{F}_{v_a^*}(-ib)).$$

From  $\|v_-\|_\infty \leq A$  then for all  $a$  big enough we have that  $\|v_a^*\|_{TV} < 1$ . Fix such  $a$ . Then  $\forall b \in \mathbb{R}$  we deduce from (3.49) of Proposition 3.15 and (3.45) that

$$(4.39) \quad \begin{aligned} \arg(\phi_-(a+ib)) &= \operatorname{Im}(\log_0(1 - \mathcal{F}_{v_a^*}(-ib))) \\ &= -\operatorname{Im}(\mathcal{F}_{v_a^*}(-ib)) - \sum_{n=2}^\infty \frac{\operatorname{Im}(\mathcal{F}_{v_a^*}(-ib))^n}{n} = -\operatorname{Im}(\mathcal{F}_{v_a^*}(-ib)) + g_a(b). \end{aligned}$$

Since  $\|v^*\|_\infty \leq \frac{A}{\phi_-(\infty)} < \infty$  with  $v^* = v_0^*$  we can show repeating without modification (4.29) and (4.30) above and in the process estimating the convolutions  $(v_a^*)^{*n}$ ,  $n \geq 2$ , using Proposition 4.10(4.74) with  $a' = 0$ , that  $\sup_{b>0} \left| \int_0^b g_a(v) dv \right| < \infty$  and thus it does not contribute more than a constant to  $A_{\phi_-}$  at least for this fixed  $a$  big enough. Without loss of generality work with  $b > 0$ . Then, from the definition of  $A_{\phi_-}$ , see (3.11), (4.39) and the preceding discussion, we get that

$$\begin{aligned} A_{\phi_-}(a+ib) &= \int_0^b \arg(\phi_-(a+iu)) du = \int_0^b g_a(u) du - \int_0^b \operatorname{Im}(\mathcal{F}_{v_a^*}(-iu)) du \\ &= \int_0^b g_a(u) du + \int_0^\infty \frac{1 - \cos(by)}{y} v_a^*(y) dy, \end{aligned}$$

where  $\sup_{b>0} \left| \int_0^b g_a(u) du \right| < \infty$ . Estimating precisely as in (4.32) and (4.33), since  $y \mapsto v_a^*(y) y^{-1} \mathbb{1}_{\{y>1\}} \in L^1(\mathbb{R}^+)$  and  $\|v^*\|_\infty \leq \frac{A}{\phi_-(\infty)} < \infty$  one gets that for some positive constant  $C'_a$

$$(4.40) \quad \left| A_{\phi_-}(a+ib) - \int_1^b \frac{1 - \cos y}{y} v_a^*\left(\frac{y}{b}\right) dy \right| \leq C'_a.$$

We investigate the contribution of the integral as  $b \rightarrow \infty$ . Fix  $\rho \in (0, 1)$ . Then, clearly,

$$(4.41) \quad \sup_{b>1} \int_{b\rho}^b \frac{1 - \cos y}{y} v_a^*\left(\frac{y}{b}\right) dy \leq 2 \|v_a^*\|_\infty |\ln \rho|.$$

Next put

$$v_a^*(0) = \lim_{y \rightarrow 0} \frac{\int_0^\infty u_+(v) \overline{\Pi}_-(y+dv)}{\phi_-(\infty)} = \frac{\int_0^\infty u_+(v) \overline{\Pi}_-(dv)}{\phi_-(0) + \bar{\mu}_-(0)} < \infty,$$

which follows from (4.24) and the continuity of  $u_+$ . Thus,  $v_a^*$  is continuous at zero. Set  $\bar{v}_a(y) = v_a^*(y) - v_a^*(0)$ , for  $y \in (0, \infty)$ . Then

$$(4.42) \quad \int_1^{b\rho} \frac{1 - \cos y}{y} v_a^* \left( \frac{y}{b} \right) dy = \int_1^{b\rho} \frac{1 - \cos y}{y} \bar{v}_a \left( \frac{y}{b} \right) dy + v_a^*(0) \left( \ln b - \int_1^b \frac{\cos y}{y} dy \right).$$

However, since  $v_a^*$  is continuous at zero we are able to immediately conclude that

$$\limsup_{\rho \rightarrow 0} \sup_{y \leq b\rho} \left| \bar{v}_a \left( \frac{y}{b} \right) \right| = \lim_{\rho \rightarrow 0} \bar{v}(\rho) = 0,$$

where  $\bar{v}(\rho) = \sup_{v \leq \rho} |\bar{v}_a(v)|$ , and thus

$$(4.43) \quad \left| \int_1^{b\rho} \frac{1 - \cos y}{y} \bar{v}_a \left( \frac{y}{b} \right) dy \right| \leq 2 \left( \sup_{y \leq \rho} |\bar{v}_a(y)| \right) \ln b = 2\bar{v}(\rho) \ln b.$$

We then combine (4.41), (4.42) and (4.43) in (4.40) to get for any  $\rho \in (0, 1)$  and some constant  $C_{a,\rho} > 0$  that

$$(4.44) \quad |A_{\phi_-}(a + ib) - v_a^*(0) \ln b| \leq C_{a,\rho} + 2\bar{v}(\rho) \ln b.$$

Thus, for all  $a$  big enough and all  $\rho \in (0, 1)$  we have from (3.19) of Theorem 3.3 that

$$(4.45) \quad |W_{\phi_-}(a + ib)| \asymp \frac{1}{\sqrt{|\phi_-(a + ib)|}} e^{-v_a^*(0) \ln b - 2\bar{v}(\rho) \ln b - C_{a,\rho}}.$$

Since  $\lim_{\rho \rightarrow 0} \bar{v}(\rho) = 0$  and  $v_a^*(0) = v^*(0) = \frac{v_-(0)}{\phi_-(\infty)}$  this settles the proof for item (3) at least for all  $a$  big enough. However, since  $\mu_-$  is absolutely continuous, for any  $a > 0$  fixed, we have from Proposition 3.14(4) that  $\lim_{|b| \rightarrow \infty} \phi_-(a + ib) = \phi_-(\infty)$ . From  $W_{\phi_-}(1 + a + ib) = \phi_-(a + ib) W_{\phi_-}(a + ib)$ , see (3.4), we then get that (4.45) holds for any  $a > 0$  up to a multiplicative constant. This concludes the proof of item (3) and therefore of Proposition 4.3.  $\square$

The first auxiliary result uses (4.23) and the celebrated *équation amicale inversée*, see [24, Chapter V, (5.3.11)], extended easily to killed Lévy processes in Proposition B.1, to decompose and relate the Lévy measure of  $\phi_+$  to the Lévy measure associated to the Lévy process underlying  $\Psi \in \bar{\mathcal{N}}$ . This decomposition is used to relate  $\phi_-$  and hence  $W_{\phi_-}$  to  $\phi_-^c \in \mathcal{B}$  and  $W_{\phi_-^c}$  as in the proof of Proposition 4.3(2).

**Lemma 4.5.** *Let  $\Psi \in \bar{\mathcal{N}}$  such that  $\phi_+ \in \mathcal{B}_P$  and  $\bar{\mu}_+(0) < \infty$ . Then with the decomposition of the potential density  $u_+(x) = \frac{1}{d_+} + \tilde{u}_+(x)$ ,  $x \geq 0$ , see (4.23),  $\exists c_0 = c_0(\Psi) \in (0, \infty)$  such that for any  $c \in (0, c_0)$  the following identity of measures holds on  $(0, c)$*

$$(4.46) \quad \begin{aligned} \mu_-(dy) &= \frac{1}{d_+} (\bar{\Pi}_-^c(y) + (\bar{\Pi}_-(c) - \bar{\Pi}_-(y + c))) dy \\ &+ \int_c^\infty u_+(v) \Pi_-(y + dv) dy + \int_0^c \tilde{u}_+(v) \Pi_-(y + dv) dy \\ &= \frac{1}{d_+} \bar{\Pi}_-^c(y) dy + \tau_1^c(y) dy + \tau_2^c(y) dy, \end{aligned}$$

where  $\bar{\Pi}_-^c(y) = (\bar{\Pi}_-(y) - \bar{\Pi}_-(c)) \mathbb{I}_{\{y \leq c\}}$ ,

$$(4.47) \quad \tau_1^c(y) = \mathbb{I}_{\{y \leq \frac{c}{2}\}} \int_0^{\frac{c}{2}} \tilde{u}_+(v) \Pi_-(y + dv)$$

and

$$(4.48) \quad \begin{aligned} \tau_2^c(y) = & \left( \int_0^{\frac{c}{2}} \tilde{u}_+(v) \Pi_-(y+dv) \mathbb{I}_{\{y \in (\frac{c}{2}, c)\}} + \int_{\frac{c}{2}}^c \tilde{u}_+(v) \Pi_-(y+dv) \mathbb{I}_{\{y < c\}} \right) dy + \\ & + (\bar{\Pi}_-(c) - \bar{\Pi}_-(y+c)) \mathbb{I}_{\{y < c\}} dy + \left( \int_c^\infty u_+(v) \Pi_-(y+dv) \right) \mathbb{I}_{\{y < c\}} dy. \end{aligned}$$

Moreover,  $\sup_{y \in (0, c)} |\tau_2^c(y)| < \infty$  and for some  $C^* > 0$

$$(4.49) \quad |\tau_1^c(y)| \leq C^* \left( \int_0^{\frac{c}{2}} \bar{\Pi}_-(\rho+y) - \bar{\Pi}_-\left(\frac{c}{2}+y\right) d\rho \right) \mathbb{I}_{\{y \leq \frac{c}{2}\}}.$$

Finally, both  $\tau_1^c$  and  $\tau_2^c$  are absolutely integrable on  $(0, c)$ .

*Proof.* Let  $c > 0$ . Recall (4.24) and decompose as follows

$$(4.50) \quad \begin{aligned} \mu_-(dy) = v_-(y) dy &= \int_0^\infty u_+(v) \Pi_-(y+dv) dy \\ &= \int_0^c u_+(v) \Pi_-(y+dv) dy + \int_c^\infty u_+(v) \Pi_-(y+dv) dy. \end{aligned}$$

Recall also that the potential measure  $U_+$  and its density  $u_+$  have been discussed prior to the statement of Proposition 4.3. Most notably (4.23) gives that

$$(4.51) \quad u_+(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{\mathbf{d}_+^{j+1}} \left( \mathbf{1} * (\phi_+(0) + \bar{\mu}_+)^{*j} \right) (x) =: \frac{1}{\mathbf{d}_+} + \tilde{u}_+(x).$$

For any  $c > 0$ , plugging the right-hand side of (4.51) in the first term of the last identity of (4.50) we get upon trivial rearrangement the first identity in (4.46). The expressions for  $\tau_1^c$ ,  $\tau_2^c$ , see (4.47) and (4.48), are up to a mere choice. Trivially, for the second term in (4.48), we get that  $(\bar{\Pi}_-(c) - \bar{\Pi}_-(y+c)) \mathbb{I}_{\{y < c\}} \leq \bar{\Pi}_-(c) \mathbb{I}_{\{y \leq c\}}$  and it is integrable on  $(0, c)$ . Also since  $\|u_+\|_\infty < \infty$ , which is thanks to  $\phi_+ \in \mathcal{B}_P$ , we deduce that

$$\left( \int_c^\infty u_+(v) \Pi_-(y+dv) \right) \mathbb{I}_{\{y < c\}} \leq \|u_+\|_\infty \bar{\Pi}_-(y+c) \mathbb{I}_{\{y < c\}} \leq \|u_+\|_\infty \bar{\Pi}_-(c) \mathbb{I}_{\{y < c\}}.$$

Clearly, the upper bound is finite and integrable on  $(0, c)$ . Thus, the last term in (4.48) has been dealt with. Finally, using in an evident manner (4.51), we study the first term

$$(4.52) \quad \begin{aligned} & \left| \int_0^c \tilde{u}_+(v) \Pi_-(y+dv) \right| \mathbb{I}_{\{y \in (\frac{c}{2}, c)\}} + \left| \int_{\frac{c}{2}}^c \tilde{u}_+(v) \Pi_-(y+dv) \right| \mathbb{I}_{\{y < c\}} \\ & \leq \left( \frac{1}{\mathbf{d}_+} + \|u_+\|_\infty \right) \left( \bar{\Pi}_-(y) \mathbb{I}_{\{c > y > \frac{c}{2}\}} + \bar{\Pi}_-\left(\frac{c}{2}\right) \mathbb{I}_{\{y < c\}} \right). \end{aligned}$$

However, the upper bound in (4.52) is clearly both bounded and integrable on  $(0, c)$ . Thus, we have proved that  $\sup_{y \in (0, c)} |\tau_2^c(y)| < \infty$  and  $\tau_2^c$  is absolutely integrable on  $(0, c)$ . It remains to investigate  $\tau_1^c$ . Note that the term defining (4.51) has, for any  $x \geq 0$ , the form

$$\mathbf{1} * (\phi_+(0) + \bar{\mu}_+) (x) = \int_0^x (\phi_+(0) + \bar{\mu}_+(y)) dy = \phi_+(0)x + \int_0^x \bar{\mu}_+(y) dy.$$

Since  $\bar{\mu}_+(0) < \infty$  we conclude that

$$\mathbf{1} * (\phi_+(0) + \bar{\mu}_+) (x) \stackrel{0}{\sim} (\phi_+(0) + \bar{\mu}_+(0)) x.$$

Then, since  $1 * (\phi_+(0) + \bar{\mu}_+)$  is non-decreasing on  $\mathbb{R}^+$ , [26, (4.2)] gives, for any  $j \in \mathbb{N}$ , that

$$\left( \mathbf{1} * (\phi_+(0) + \bar{\mu}_+)^{*j} \right) (x) \leq \left( \mathbf{1} * (\phi_+(0) + \bar{\mu}_+(x)) \right)^j$$

and we conclude that for some  $h > 0$  and all  $x \in (0, h)$ ,

$$\left( \mathbf{1} * (\phi_+(0) + \bar{\mu}_+)^{*j} \right) (x) \leq 2^j (\phi_+(0) + \bar{\mu}_+(0))^j x^j.$$

Therefore from (4.51), for  $x < \frac{1}{4(\phi_+(0) + \bar{\mu}_+(0))} \wedge h$ ,

$$(4.53) \quad |\tilde{u}_+(x)| \leq C^* x,$$

where  $C^* > 0$  is some positive constant. Hence, from now on, we choose an arbitrary  $c < c_0 = \frac{1}{4(\phi_+(0) + \bar{\mu}_+(0))} \wedge h$ . Using (4.53) in (4.47) we get that

$$|\tau_1^c(y)| \leq C^* \left( \int_0^{\frac{c}{2}} v \Pi_-(y + dv) \right) \mathbb{I}_{\{y \leq \frac{c}{2}\}}$$

and (4.49) follows by integration by parts. However, from (4.49) we get for  $y \in (0, \frac{c}{2})$  that

$$(4.54) \quad \begin{aligned} \frac{\tau_1^c(y)}{C^*} &\leq \int_0^{\frac{c}{2}} \left( \bar{\Pi}_-(\rho + y) - \bar{\Pi}_-\left(\frac{c}{2} + y\right) \right) d\rho \\ &\leq \bar{\Pi}_-(y) := \int_y^c \bar{\Pi}_-(v) dv \end{aligned}$$

and since

$$\int_0^c \bar{\Pi}_-(y) dy = \int_0^c \int_y^c \bar{\Pi}_-(v) dv dy \leq \int_0^c \int_0^c \bar{\Pi}_-(v) dv dy = \int_0^c v \bar{\Pi}_-(v) dv < \infty,$$

we conclude that  $|\tau_1^c|$  is integrable on  $(0, \frac{c}{2})$ .  $\square$

We keep the notation of Lemma 4.5 and introduce the function

$$(4.55) \quad \tau^c(y) = \tau_1^c(y) + \tau_2^c(y) + v_-(y) \mathbb{I}_{\{y \geq c\}}.$$

The next result is technical.

**Proposition 4.6.** *Let  $\Psi \in \bar{\mathcal{N}}$  such that  $\phi_+ \in \mathcal{B}_P$  and  $\bar{\mu}_+(0) < \infty$ . Then for any  $d \in [0, \bar{\mu}_-(0))$  there exists  $c_1 = c_1(d, \Psi) \in (0, c_0)$  such that  $\int_0^\infty \tau^c(y) dy \in (d, \bar{\mu}_-(0))$  for all  $c \in (0, c_1)$ .*

*Proof.* Note that from (4.55) and Lemma 4.5 for all  $c \in (0, c_0)$

$$\int_c^\infty \tau^c(y) dy = \int_c^\infty v_-(y) dy = \bar{\mu}_-(c).$$

By simple inspection of (4.47) and (4.48) we note that the only potential negative contribution to  $\tau^c$  comes from the terms whose integrands are  $\tilde{u}_+$ . Since (4.53), that is  $|\tilde{u}_+(x)| \leq C^* x$ , holds for all  $x \in (0, c_0)$  clearly an upper bound of those terms is the expression

$$\bar{\tau}(y) = C^* \int_0^c v \Pi_-(y + dv) \mathbb{I}_{\{y < c\}}.$$

Therefore, integrating by parts, we get that

$$\begin{aligned} \int_0^c \bar{\tau}(y) dy &= C^{r^*} \int_0^c \int_0^c (\bar{\Pi}_-(y+w) - \bar{\Pi}_-(y+c)) dw dy \\ &\leq C^{r^*} \int_0^c \int_0^c \bar{\Pi}_-(y+w) dw dy \leq C^{r^*} \int_0^c \bar{\bar{\Pi}}_-(y) dy. \end{aligned}$$

As the upper bound tends to 0 as  $c \rightarrow 0$  we conclude the claim as the negative contribution of  $\tau_1^c, \tau_2^c$  cannot exceed in absolute value this quantity, whereas the positive contribution exceeds  $\int_c^\infty \tau^c(y) dy = \bar{\mu}_-(c)$  which converges to  $\bar{\mu}_-(0)$ .  $\square$

Lemma 4.5 allows us to prove the following result which transforms the decomposition of  $\mu_-$  on  $(0, c)$  to a decomposition of  $\phi_-$ . We stress that although one of the terms in the aforementioned decomposition is a Bernstein function with better understood properties, the second term need not belong to  $\mathcal{B}$ .

**Lemma 4.7.** *Let  $\Psi \in \bar{\mathcal{N}}$  such that  $\phi_+ \in \mathcal{B}_P$  and  $\bar{\mu}_+(0) < \infty$ . Let  $c \in (0, c_0)$  so that Lemma 4.5 is valid. Then, the function*

$$(4.56) \quad \phi_-^c(z) = \phi_-(0) + \mathbf{d}_- z + \int_0^c (1 - e^{-zy}) \frac{\bar{\Pi}_-^c(y)}{\mathbf{d}_+} dy \in \mathcal{B}$$

and with the definition of  $\tau^c$ , see (4.55),

$$(4.57) \quad \phi_-(z) = \phi_-^c(z) + \int_0^\infty (1 - e^{-zy}) \tau^c(y) dy = \phi_-^c(z) + \tilde{\phi}_-^c(z).$$

For any such choice and  $a > 0$  fixed

$$(4.58) \quad \lim_{|b| \rightarrow \infty} \operatorname{Im} \left( \tilde{\phi}_-^c(a + ib) \right) = 0 \quad \text{and} \quad \lim_{|b| \rightarrow \infty} \operatorname{Re} \left( \tilde{\phi}_-^c(a + ib) \right) = \tilde{\phi}_-^c(\infty) = \int_0^\infty \tau^c(y) dy,$$

whereas

$$(4.59) \quad \lim_{|b| \rightarrow \infty} \operatorname{Re} (\phi_-^c(a + ib)) = \phi_-^c(\infty) \quad \text{and} \quad \operatorname{Im} (\phi_-^c(a + ib)) \geq 0.$$

Finally, there exist  $d \in [0, \bar{\mu}_-(0))$ ,  $c_2 = c_2(d, \Psi) \in (0, c_0)$  such that for any  $c \in (0, c_2)$  we have that  $\tilde{\phi}_-^c(\infty) = \int_0^\infty \tau^c(y) dy > 0$ .

*Proof.* Note that (4.56) can be defined for any  $c > 0$  but we fix  $c \in (0, c_0)$  ensuring the validity of Lemma 4.5. The validity of (4.57) is a simple rearrangement. Next, (4.58) follows from the application of the Riemann-Lebesgue lemma to the function  $\tau^c \in L^1(\mathbb{R}^+)$ . The latter is a consequence from the absolute integrability of its constituting components  $\tau_1^c, \tau_2^c$ , see Lemma 4.5, and the fact that  $\bar{\mu}_-(c) = \int_c^\infty v_-(y) dy < \infty$ , see (3.3). Next, recall that (4.25) states

$$(4.60) \quad \bar{\mu}_-(0) = \int_0^\infty v_-(y) dy = \infty \iff \int_0^1 \bar{\Pi}_-(y) dy = \infty.$$

Thus, if  $\bar{\mu}_-(0) = \infty$  (4.60) shows that the Lévy measure of  $\phi_-^c$ , that is the quantity  $\bar{\Pi}_-^c(y) dy = (\bar{\Pi}_-(y) - \bar{\Pi}_-(c)) \mathbb{I}_{\{y < c\}} dy$ , assigns infinite mass on  $(0, c)$  and is absolutely continuous therein. However, the latter facts trigger the validity of [66, Theorem 27.7] and thus the distribution of the non-increasing Lévy process underlying  $\phi_-^c$  is absolutely continuous. This, in turn, thanks to the Riemann-Lebesgue lemma yields to

$$(4.61) \quad \lim_{|b| \rightarrow \infty} e^{-\phi_-^c(a+ib)} = \lim_{|b| \rightarrow \infty} \mathbb{E} \left[ e^{-(a+ib)\xi_1^c} \right] = \lim_{|b| \rightarrow \infty} \int_0^\infty e^{-ax-ibx} h_c(x) dx = 0,$$



where  $\xi_1^c$  is the non-decreasing Lévy process associated to  $\phi^c$  taken at time 1 and on  $(0, \infty)$

$$h_c(x)dx = \mathbb{P}(\xi_1^c \in dx).$$

Thus, the first assertion of (4.59) is valid. It is clearly valid if  $\mathfrak{d}_- > 0$  as well. In both cases  $\phi^c(\infty) = \infty$ . It remains to settle the first statement of (4.59) when  $\bar{\mu}_-(0) < \infty$  and  $\mathfrak{d}_- = 0$ . It follows from the Riemann-Lebesgue lemma and (4.56) wherein by assumption  $\mathfrak{d}_- = 0$  and

$$\lim_{|b| \rightarrow \infty} \int_0^c e^{-ay - i|b|y} \frac{\bar{\Pi}^c(y)}{\mathfrak{d}_+} dy = 0$$

which in turn comes from (4.60) which implies that  $\int_0^c \bar{\Pi}^c(y) dy < \infty$ . Regardless of  $\bar{\mu}_-(0)$  being finite or not the second claim of (4.59) follows by integration by parts of  $\bar{\Pi}^c(y) = \int_y^c \Pi_-(dr)$ ,  $y \in (0, c)$ , in (4.56) or the proof of [61, Lemma 4.6] since  $\bar{\Pi}^c$  is non-increasing on  $\mathbb{R}^+$ . The final claim of the Lemma follows easily from the assertion of Proposition 4.6.  $\square$

The next result is the first step to the understanding of the quantity  $A_{\phi_-}$  via studying the integrand  $\arg \phi_-$ . We always fix  $c$  such that  $\tilde{\phi}^c(\infty) > 0$  in Lemma 4.7 and all claims of Lemma 4.5 hold, which from the final assertion of Lemma 4.7 is always possible as long as  $\bar{\mu}_-(0) > 0$ . We then decompose  $\arg \phi_-$  as a sum of  $\arg \phi_-^c$  and an error term and we simplify the latter. For this purpose we introduce some further notation. Let in the sequel (4.57) hold. Then we denote by  $U^c(dy)$ ,  $y > 0$ , the potential measure of the subordinator associated to  $\phi^c$  and by  $U_a^c(dy) = e^{-ay}U^c(dy)$ . Recall that  $\tau_a^c(y) = e^{-ay}\tau^c(y)$ ,  $y > 0$ , where  $\tau^c$  is defined in (4.55). Then, the following claim holds.

**Lemma 4.8.** *Let  $\Psi \in \bar{\mathcal{N}}$  such that  $\phi_+ \in \mathcal{BP}$  and  $\bar{\mu}_+(0) < \infty$ . Assume furthermore that  $\bar{\mu}_-(0) = \infty$  or equivalently  $\int_0^1 \bar{\Pi}_-(y) dy = \infty$ , see (4.60). Fix  $a > 0$  and  $c \in (0, c_0)$  so that Lemma 4.5 is valid. Then, modulo to  $(-\pi, \pi]$  for all  $b > 0$  and directly for all  $b$  large enough*

$$(4.62) \quad \begin{aligned} \arg(\phi_-(a+ib)) &= \arg(\phi_-^c(a+ib)) + \arg\left(1 + \frac{\tilde{\phi}^c(a+ib)}{\phi^c(a+ib)}\right) \\ &= \arg(\phi_-^c(a+ib)) + \arg\left(1 + \tilde{\phi}^c(\infty)\mathcal{F}_{U_a^c}(-ib) - \mathcal{F}_{U_a^c * \tau_a^c}(-ib)\right), \end{aligned}$$

where  $\phi_-^c, \tilde{\phi}^c$  are as in the decomposition (4.57). For any  $c \in (0, c_0)$  there exists  $a_c > 0$  such that for any  $a \geq a_c$  and as  $b \rightarrow \infty$

$$(4.63) \quad \arg(\phi_-(a+ib)) = \arg(\phi_-^c(a+ib)) (1 + o(1)) + \arg(1 - \mathcal{F}_{\Xi_a^c}(-ib)),$$

where  $\Xi_a^c$  is an absolutely continuous finite measure on  $\mathbb{R}^+$ . Moreover, its density  $\chi_a^c$  is such that  $\chi_a^c \in L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$  and  $\lim_{a \rightarrow \infty} \|\Xi_a^c\|_{TV} = 0$ .

*Proof.* Since the assumptions of Lemma 4.5 and Lemma 4.7 are satisfied we conclude that  $\phi_-(z) = \phi_-^c(z) + \tilde{\phi}^c(z)$ , see (4.57). Then, modulo to  $(-\pi, \pi]$ , the first identity of (4.62) is immediate whereas the second one follows from the fact that

$$\frac{\tilde{\phi}^c(a+ib)}{\phi^c(a+ib)} = \frac{1}{\phi^c(a+ib)} \left( \int_0^\infty \tau^c(y) dy - \mathcal{F}_{\tau_a^c}(-ib) \right) = \frac{\tilde{\phi}^c(\infty)}{\phi^c(a+ib)} - \frac{\mathcal{F}_{\tau_a^c}(-ib)}{\phi^c(a+ib)},$$

see (4.57), and (4.22). Note that (4.59) implies that  $\arg(\phi^c(a+ib)) \in [0, \frac{\pi}{2}]$  at least for  $b$  large enough. Moreover, since  $\bar{\mu}_-(0) = \infty$  we note that

$$(4.64) \quad \lim_{|b| \rightarrow \infty} \operatorname{Re}(\phi^c(a+ib)) = \phi^c(\infty) = \infty,$$

see (4.59). Also, (4.64) together with (4.58) yields that  $\arg\left(1 + \frac{\tilde{\phi}_-^c(a+ib)}{\phi_-^c(a+ib)}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$  at least for  $b$  large enough. Henceforth, (4.62) holds directly for such  $b$ . From (4.22), (4.64) and  $\tilde{\phi}_-^c(\infty) > 0$  we get that for all  $b$  large enough

$$(4.65) \quad 0 < \operatorname{Re}\left(\tilde{\phi}_-^c(\infty) \mathcal{F}_{U_a^c}(-ib)\right) = \tilde{\phi}_-^c(\infty) \operatorname{Re}\left(\frac{1}{\phi_-^c(a+ib)}\right) = \tilde{\phi}_-^c(\infty) \frac{\operatorname{Re}(\phi_-^c(a-ib))}{|\phi_-^c(a+ib)|^2}.$$

Therefore, from (4.65) we conclude that for all  $b$  large enough

$$(4.66) \quad \begin{aligned} & \arg\left(1 + \tilde{\phi}_-^c(\infty) \mathcal{F}_{U_a^c}(-ib) - \mathcal{F}_{U_a^c * \tau_a^c}(-ib)\right) \\ &= \arg\left(1 + \tilde{\phi}_-^c(\infty) \mathcal{F}_{U_a^c}(-ib)\right) + \arg\left(1 - \frac{\mathcal{F}_{U_a^c * \tau_a^c}(-ib)}{1 + \tilde{\phi}_-^c(\infty) \mathcal{F}_{U_a^c}(-ib)}\right). \end{aligned}$$

From (4.22), as  $b \rightarrow \infty$ , we get with the help of the second fact in (4.59), that is  $\operatorname{Im}(\phi_-^c(a+ib)) \geq 0$ , and (4.64) that

$$(4.67) \quad \begin{aligned} H(b) &= |\operatorname{Im}(\mathcal{F}_{U_a^c}(-ib))| = \frac{|\operatorname{Im}(\overline{\phi_-^c(a+ib)})|}{|\phi_-^c(a+ib)|^2} \\ &= \frac{\operatorname{Im}(\phi_-^c(a+ib))}{(\operatorname{Re}(\phi_-^c(a+ib)))^2 + (\operatorname{Im}(\phi_-^c(a+ib)))^2} = o(1). \end{aligned}$$

From (4.59) and (4.64) we have again that  $\arg(\phi_-^c(a+ib)) = \arctan\left(\frac{\operatorname{Im}(\phi_-^c(a+ib))}{\operatorname{Re}(\phi_-^c(a+ib))}\right)$  and we aim to show, as  $b \rightarrow \infty$ , that

$$(4.68) \quad H(b) = |\operatorname{Im}(\mathcal{F}_{U_a^c}(-ib))| = o(\arg(\phi_-^c(a+ib))).$$

Fix  $n \in \mathbb{N}$  and note from (4.67) that  $\lim_{b \rightarrow \infty} nH(b) = 0$ . Therefore, from the second fact in (4.59) and (4.64), for all  $b$  large enough,

$$\tan(nH(b)) \leq 2nH(b) \leq \frac{2n}{\operatorname{Re}(\phi_-^c(a+ib))} \frac{\operatorname{Im}(\phi_-^c(a+ib))}{\operatorname{Re}(\phi_-^c(a+ib))} = o(1) \frac{\operatorname{Im}(\phi_-^c(a+ib))}{\operatorname{Re}(\phi_-^c(a+ib))}.$$

Therefore, since from (4.59),  $\operatorname{Im}(\phi_-^c(a+ib)) \geq 0$ , as  $b \rightarrow \infty$ ,

$$\begin{aligned} n \overline{\lim}_{b \rightarrow \infty} \frac{H(b)}{\arg(\phi_-^c(a+ib))} &\leq \overline{\lim}_{b \rightarrow \infty} \frac{\arctan\left(o(1) \frac{\operatorname{Im}(\phi_-^c(a+ib))}{\operatorname{Re}(\phi_-^c(a+ib))}\right)}{\arg(\phi_-^c(a+ib))} \\ &\leq \overline{\lim}_{b \rightarrow \infty} \frac{\arctan\left(\frac{\operatorname{Im}(\phi_-^c(a+ib))}{\operatorname{Re}(\phi_-^c(a+ib))}\right)}{\arg(\phi_-^c(a+ib))} = 1. \end{aligned}$$

Hence, since  $n \in \mathbb{N}$  is arbitrary we conclude (4.68). However, from (4.65), that is the inequality  $\operatorname{Re}\left(\tilde{\phi}_-^c(\infty) \mathcal{F}_{U_a^c}(-ib)\right) > 0$ , elementary geometry in the complex plane and (4.67) as  $b \rightarrow \infty$

$$\begin{aligned} \left|\arg\left(1 + \tilde{\phi}_-^c(\infty) \mathcal{F}_{U_a^c}(-ib)\right)\right| &= \left|\arg\left(1 + \tilde{\phi}_-^c(\infty) \operatorname{Re}(\mathcal{F}_{U_a^c}(-ib)) + i \tilde{\phi}_-^c(\infty) \operatorname{Im}(\mathcal{F}_{U_a^c}(-ib))\right)\right| \\ &\leq \left|\arg\left(1 + i \tilde{\phi}_-^c(\infty) \operatorname{Im}(\mathcal{F}_{U_a^c}(-ib))\right)\right| = \left|\tan\left(\tilde{\phi}_-^c(\infty) \operatorname{Im}(\mathcal{F}_{U_a^c}(-ib))\right)\right| \\ &\lesssim \tilde{\phi}_-^c(\infty) |\operatorname{Im}(\mathcal{F}_{U_a^c}(-ib))| \stackrel{4.68}{=} o(\arg(\phi_-^c(a+ib))) \end{aligned}$$

and therefore we deduce easily for the right-hand side of (4.66) that as  $b \rightarrow \infty$

$$(4.69) \quad \begin{aligned} & \arg \left( 1 + \tilde{\phi}_-^c(\infty) \mathcal{F}_{U_a^c}(-ib) - \mathcal{F}_{U_a^c * \tau_a^c}(-ib) \right) \\ &= \arg \left( 1 - \frac{\mathcal{F}_{U_a^c * \tau_a^c}(-ib)}{1 + \tilde{\phi}_-^c(\infty) \mathcal{F}_{U_a^c}(-ib)} \right) + o(\arg(\phi_-^c(a + ib))). \end{aligned}$$

To confirm (4.63) we need to study the first in the second line of (4.69). From Lemma 4.9 below we know that for any  $a > 0$ ,  $G_a^c(dy) = U_a^c * \tau_a^c(dy) = g_a^c(y)dy$  with  $g_a^c \in L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$ . Also, for fixed  $c \in (0, c_0)$  from (4.22) we get that

$$U_a^c(\mathbb{R}^+) = \int_0^\infty e^{-ay} U^c(dy) = \frac{1}{\phi_-^c(a)}.$$

However, since  $\int_0^1 \bar{\Pi}_-(y)dy = \infty$  then we obtain from  $\bar{\Pi}_-^c(y) = (\bar{\Pi}_-(y) - \bar{\Pi}_-(c)) \mathbb{I}_{\{y < c\}}$  that  $\int_0^1 \bar{\Pi}_-^c(y)dy = \infty$  and thus from (4.56) we conclude that  $\lim_{a \rightarrow \infty} \phi_-^c(a) = \infty$  and hence  $\lim_{a \rightarrow \infty} U_a^c(\mathbb{R}^+) = 0$ . Thus, we choose  $a_c > 0$  such that  $U_a^c(\mathbb{R}^+) < \frac{1}{4\tilde{\phi}_-^c(\infty)}$  for all  $a \geq a_c$  and work with arbitrary such  $a$ . Therefore  $\sup_{b \in \mathbb{R}} \left| \tilde{\phi}_-^c(\infty) \mathcal{F}_{U_a^c}(-ib) \right| < \frac{1}{4}$  and then we can deduct that

$$\frac{\mathcal{F}_{G_a^c}(-ib)}{1 + \tilde{\phi}_-^c(\infty) \mathcal{F}_{U_a^c}(-ib)} = \mathcal{F}_{G_a^c}(-ib) \sum_{n=0}^{\infty} (-1)^n \left( \tilde{\phi}_-^c(\infty) \right)^n \mathcal{F}_{U_a^c}^n(-ib).$$

Since  $G_a^c(dy) = g_a^c(y)dy$ , formally, the right-hand side is the Fourier transform of a measure  $\Xi_a^c$  supported on  $\mathbb{R}^+$  with density

$$(4.70) \quad \chi_a^c(y) = g_a^c(y) + \sum_{n=1}^{\infty} (-1)^n \left( \tilde{\phi}_-^c(\infty) \right)^n \int_0^y g_a^c(y-v) (U_a^c)^{*n}(dv).$$

However, it is immediate with the assumptions and observations above that

$$\|\chi_a^c\|_\infty \leq \|g_a^c\|_\infty + \|g_a^c\|_\infty \sum_{n=1}^{\infty} \left( \tilde{\phi}_-^c(\infty) U_a^c(\mathbb{R}^+) \right)^n < \|g_a^c\|_\infty \sum_{n=0}^{\infty} \frac{1}{4^n} < \infty$$

and

$$(4.71) \quad \|\Xi_a^c\|_{TV} = \int_0^\infty |\chi_a^c(y)| dy \leq \int_0^\infty |g_a^c(y)| dy \left( 1 + \sum_{n=1}^{\infty} \left( \tilde{\phi}_-^c(\infty) U_a^c(\mathbb{R}^+) \right)^n \right) \leq 2\|G_a^c\|_{TV} < \infty.$$

Therefore,  $\Xi_a^c$  is a well-defined finite measure with density  $\chi_a^c \in L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$  and from (4.69) we get for all  $a \geq a_c$  and any  $b \in \mathbb{R}$  that

$$\arg \left( 1 - \frac{\mathcal{F}_{U_a^c * \tau_a^c}(-ib)}{1 + \tilde{\phi}_-^c(\infty) \mathcal{F}_{U_a^c}(-ib)} \right) = \arg(1 - \mathcal{F}_{\Xi_a^c}(-ib)).$$

Combining this, (4.69) and (4.62) we conclude (4.63) for any  $a \geq a_c$  and as  $b \rightarrow \infty$ . The final claims are also immediate from the discussion above. We just note from (4.71) that  $\|\Xi_a^c\|_{TV} \leq 2\|G_a^c\|_{TV}$  and Lemma 4.9 shows that  $\lim_{a \rightarrow \infty} \|\Xi_a^c\|_{TV} = 0$ .  $\square$

In the next result we discuss the properties of the measure  $U_a^c * \tau_a^c$  used in the proof above.

**Lemma 4.9.** Fix  $a > 0$ . The measure  $G_a^c(dy) = U_a^c * \tau_a^c(dx)$  has a bounded on  $(0, \infty)$  density

$$g_a^c(x) = e^{-ax} \int_0^x \tau^c(x-v) U^c(dv) = e^{-ax} g^c(x)$$

and  $g_a^c \in L^1(\mathbb{R}^+)$  with  $\lim_{a \rightarrow \infty} \|G_a^c\|_{TV} = 0$ .

*Proof.* The existence and the form of  $\chi_a$  is immediate from the definition of convolution and the fact that  $\tau_a^c(y) dy = e^{-ay} \tau^c(y) dy$ ,  $y > 0$ . Recall from (4.55) that

$$\tau^c(y) dy = (\tau_1^c(y) + \tau_2^c(y) + \mathbb{I}_{\{y>c\}} v_-(y)) dy.$$

Then,  $A_1 = \|\tau_2^c\|_\infty + \|v_- \mathbb{I}_{\{y>c\}}\|_\infty < \infty$  follows from Lemma 4.5 and (4.50). Therefore, we have with some constant  $A_3 > 0$

$$(4.72) \quad \sup_{x>0} e^{-ax} \int_0^x |\tau_2^c(x-y) + v_-(x-y) \mathbb{I}_{\{x-y>c\}}| U^c(dy) \leq \sup_{x>0} A_1 e^{-ax} U^c(x) \\ \leq A_2 \sup_{x>0} x e^{-ax} \leq A_3,$$

where we have used the fact that  $U^c : \mathbb{R}^+ \mapsto \mathbb{R}^+$  is subadditive, see [9, p.74]. It remains to study the portion coming from  $\tau_1^c(y)$ , which according to Lemma 4.5(4.47) is supported on  $(0, \frac{c}{2})$  and is bounded by the expression in (4.49). Thus, recalling that  $\overline{\Pi}_-^c(y) = (\overline{\Pi}_-(y) - \overline{\Pi}_-(c)) \mathbb{I}_{\{y<c\}}$ , we get that

$$(4.73) \quad \sup_{x>0} e^{-ax} \int_0^x |\tau_1^c(x-y)| U^c(dy) = \sup_{x>0} e^{-ax} \int_{\max\{0, x-\frac{c}{2}\}}^x |\tau_1^c(x-y)| U^c(dy) \\ \leq C^* \sup_{x>0} e^{-ax} \int_{\max\{0, x-\frac{c}{2}\}}^x \left( \int_0^{\frac{c}{2}} \overline{\Pi}_-^c(\rho + x-y) - \overline{\Pi}_-^c\left(\frac{c}{2} + x-y\right) d\rho \right) U^c(dy) \\ \leq C^* \sup_{x>0} e^{-ax} \int_{\max\{0, x-\frac{c}{2}\}}^x \left( \int_0^{\frac{c}{2}} \overline{\Pi}_-^c(\rho + x-y) d\rho \right) U^c(dy) \\ \leq C^* \sup_{x>0} e^{-ax} \int_0^x \overline{\Pi}_-^c(x-y) U^c(dy) \\ \leq C^* \sup_{x>0} e^{-ax} = C^*,$$

where for the first inequality we have used the bound (4.49) and for the very last one that, see [9, Chapter III, Proposition 2],

$$\int_0^x \overline{\Pi}_-^c(x-y) U^c(dy) = \mathbb{P}\left(\mathbf{e}_{\phi^c(0)} > T_{(x, \infty)}^\sharp\right) \leq 1,$$

where  $\mathbf{e}_{\phi^c(0)}$  is an exponential random variable with parameter  $\phi^c(0) = \phi_-(0) \geq 0$ , see (4.56),  $T_{(x, \infty)}^\sharp$  is the first passage time above  $x > 0$  of the unkilld subordinator related to  $(\phi^c)^\sharp(z) = \phi^c(z) - \phi^c(0) \in \mathcal{B}$  and, from (4.56),  $\overline{\Pi}_-^c(y) = \int_y^\infty \overline{\Pi}_-^c(v) dv$  is the tail of the Lévy measure associated to  $(\phi^c)^\sharp$ . Summing (4.72) and (4.73) yields that  $\|g_a^c\|_\infty \leq A_3 + C^* < \infty$ . Finally,  $g_a^c \in L^1(\mathbb{R}^+)$  follows immediately from the estimate before the last estimate in (4.72) and (4.73). Clearly, from them we also get  $\lim_{a \rightarrow \infty} \|G_a^c\|_{TV} = 0$  which ends the proof.  $\square$

Recall that for any  $a \geq 0$  and any function  $f : \mathbb{R}^+ \mapsto \mathbb{R}$ ,  $f_a(y) = e^{-ay} f(y)$ ,  $y > 0$ . The next proposition is trivial.

**Proposition 4.10.** *For any function  $f$ ,  $f_a^{*n}(y) = e^{-ay} f^{*n}(y)$ ,  $y \in (0, \infty)$ . If  $f_{a'} \in L^\infty(\mathbb{R}^+)$  for some  $a' \geq 0$  then for any  $a \geq a'$ , all  $n \geq 1$  and  $y > 0$*

$$(4.74) \quad |f_a^{*n}(y)| \leq \|f_{a'}\|_\infty^n \frac{y^{n-1} e^{-(a-a')y}}{(n-1)!}.$$

*Proof.* First  $f_a^{*n}(y) = e^{-ay} f^{*n}(y)$ ,  $y \in (0, \infty)$ , is a triviality. Then, (4.74) is proved by elementary inductive hypothesis based on the immediate observation

$$|f_a^{*2}(y)| = e^{-ay} \left| \int_0^y f(y-v)f(v)dv \right| \leq \|f_{a'}\|_\infty^2 y e^{-(a-a')y}, \quad y > 0.$$

□

## 5. PROOFS FOR EXPONENTIAL FUNCTIONALS OF LÉVY PROCESSES

**5.1. Regularity, analyticity and representations of the density: Proof of Theorem 2.7(1).** Recall that for any  $\Psi \in \overline{\mathcal{N}}$  and we have that  $\mathcal{M}_\Psi$  satisfies (1.1) that is

$$(5.1) \quad \mathcal{M}_\Psi(z+1) = \frac{-z}{\Psi(-z)} \mathcal{M}_\Psi(z)$$

at least for  $z \in \mathbb{C}_{(0, -\bar{a})}$ , see Theorem 2.1, and recall the quantity  $\bar{a}$ , see (2.5). Therefore, for any  $\Psi \in \mathcal{N} \subset \overline{\mathcal{N}}$  such that  $\bar{a} < 0$  from Remark 2.11,  $\mathcal{M}_{I_\Psi}$  solves (5.1) at least for  $z \in \mathbb{C}_{(0, -\bar{a})}$ . Since by definition if  $\Psi \in \mathcal{N} \iff \phi_-(0) > 0$ , see (2.17), we proceed to show that  $\mathcal{M}_{I_\Psi}(z) = \phi_-(0) \mathcal{M}_\Psi(z)$  or that the identities in (2.21) hold. First, thanks to [61, Proposition 6.8] we have that

$$\mathbb{E} \left[ I_{\phi_*}^{z-1} \right] = \frac{\Gamma(z)}{W_{\phi_*}(z)}, \quad z \in \mathbb{C}_{(0, \infty)}.$$

Furthermore, it is immediate to verify that  $\phi_-(0) W_{\phi_-}(1-z)$ ,  $z \in \mathbb{C}_{(-\infty, 1-\bar{a})}$ , is the Mellin transform of the random variable  $X_{\phi_-}$  defined via the identity

$$(5.2) \quad \mathbb{E} [g(X_{\phi_-})] = \phi_-(0) \mathbb{E} \left[ \frac{1}{Y_{\phi_*}} g \left( \frac{1}{Y_{\phi_*}} \right) \right],$$

where from Definition 3.1  $Y_{\phi_*}$  is the random variable associated to  $W_{\phi_*} \in \mathcal{W}_B$  and  $g \in \mathcal{C}_b(\mathbb{R}^+)$ . Therefore

$$(5.3) \quad \begin{aligned} \mathcal{M}_{I_\Psi}^\circ(z) &= \mathbb{E} \left[ I_{\phi_*}^{z-1} \right] \mathbb{E} \left[ X_{\phi_*}^{z-1} \right] = \phi_-(0) \mathcal{M}_\Psi(z) \\ &= \phi_-(0) W_{\phi_*}(1-z) \frac{\Gamma(z)}{W_{\phi_*}(z)}, \quad z \in \mathbb{C}_{(0, 1-\bar{a})}, \end{aligned}$$

is the Mellin transform of  $I_{\phi_*} \times X_{\phi_*}$  and solves (5.1) with  $\mathcal{M}_{I_\Psi}^\circ(1) = 1$  on  $\mathbb{C}_{(0, -\bar{a})}$ . Therefore, both  $\mathcal{M}_{I_\Psi}^\circ, \mathcal{M}_{I_\Psi}$  solve (5.1) on  $\mathbb{C}_{(0, -\bar{a})}$  with  $\mathcal{M}_{I_\Psi}^\circ(1) = \mathcal{M}_{I_\Psi}(1) = 1$  and are holomorphic on  $\mathbb{C}_{(0, 1-\bar{a})}$ . However, note that  $\mathcal{M}_{I_\Psi}^\circ$  is zero-free on  $(0, 1-\bar{a})$  since  $\Gamma$  is zero-free and according to Theorem 3.2  $W_{\phi_*}$  is zero-free on  $(\bar{a}, \infty)$ . Thus, we conclude that  $f(z) \mathcal{M}_{I_\Psi}^\circ(z) = \mathcal{M}_{I_\Psi}(z)$  with  $f$  some entire holomorphic periodic function of period one, that is  $f(z+1) = f(z)$ ,  $z \in \mathbb{C}_{(0, 1]}$ ,

and  $f(1) = 1$ . Next, considering  $z = a + ib \in \mathbb{C}_{(0,1-\bar{a}_-)}$ ,  $a$  fixed and  $|b| \rightarrow \infty$ , we get that

$$\begin{aligned} |f(z)| &= \left| \frac{\mathcal{M}_{I_\Psi}(z)}{\mathcal{M}_{I_\Psi}^\circ(z)} \right| \leq \frac{\mathbb{E}[I_\Psi^{a-1}]}{\left| \mathcal{M}_{I_\Psi}^\circ(z) \right|} \\ &= \frac{\mathbb{E}[I_\Psi^{a-1}]}{\phi_-(0)} \frac{|W_{\phi_+}(z)|}{|\Gamma(z)W_{\phi_-}(1-z)|} = O(1) \frac{|W_{\phi_+}(z)|}{|\Gamma(z)W_{\phi_-}(1-z)|}. \end{aligned}$$

Since  $a > 0$  is fixed, we apply (3.19) to  $|W_{\phi_+}(z)|$  and (3.25) and (3.19) to  $|W_{\phi_-}(1-z)|$  to obtain the inequality

$$(5.4) \quad |f(z)| \leq O(1) \frac{e^{-A_{\phi_+}(a+ib)+A_{\phi_-}(a+(1-a)^{\rightarrow} -ib)}}{\sqrt{|\phi_+(z)|} |\Gamma(z)|} \times \prod_{j=0}^{(1-a)^{\rightarrow}-1} |\phi_-(z+j)|,$$

where we recall that  $c^{\rightarrow} = (\lfloor -c \rfloor + 1) \mathbb{I}_{\{c \leq 0\}}$ . Also, we have regarded any term in (3.19) and (3.25) depending on  $a$  solely as a constant included in  $O(1)$ . From Proposition 3.14(3) we have, for  $a > 0$  fixed, that  $|\phi(a + i|b|)| \approx |b|(\mathfrak{d} + o(1))$ . We recall, from (3.32), that, for any  $a > 0$ ,

$$|\phi(a + ib)| \geq \operatorname{Re}(\phi(a + ib)) \geq \phi(0) + \mathfrak{d}a + \int_0^\infty (1 - e^{-ay}) \mu(dy) = \phi(a) > 0.$$

Applying these observations to  $\phi_+$ ,  $\phi_-$ , in (5.4) and invoking (3.20), that is  $A_\phi \geq 0$  and  $A_\phi(a + ib) \leq \frac{\pi}{2}|b|$ , we get, as  $|b| \rightarrow \infty$ ,

$$(5.5) \quad |f(z)| \leq |b|^{(1-a)^{\rightarrow}} \frac{e^{|b|\frac{\pi}{2}}}{|\Gamma(z)|} O(1).$$

However, from the well-known Stirling asymptotic for the gamma functions, see (4.8), (5.5) is further simplified, as  $|b| \rightarrow \infty$ , to

$$|f(z)| \leq |b|^{(1-a)^{\rightarrow} - a + \frac{1}{2}} e^{|b|\pi} O(1) = o\left(e^{2\pi|b|}\right).$$

However, the fact that  $f$  is entire periodic with period 1 and  $|f(z)| = o\left(e^{2\pi|b|}\right)$ ,  $|b| \rightarrow \infty$ , for  $z \in \mathbb{C}_{(0,1-\bar{a}_-)}$  implies by a celebrated criterion for the maximal growth of periodic entire functions, see [45, p.96, (36)], that  $f(z) = f(1) = 1$ . Hence,  $\mathcal{M}_{I_\Psi}(z) = \mathcal{M}_{I_\Psi}^\circ(z) = \phi_-(0)\mathcal{M}_\Psi(z)$ , which concludes the proof of Theorem 2.1 verifying (2.21) whenever  $\bar{a}_- < 0$ . Recall that  $\Psi \in \bar{\mathcal{N}}$  is

$$(5.6) \quad \Psi(z) = \frac{\sigma^2}{2}z^2 + cz + \int_{-\infty}^\infty (e^{zr} - 1 - zr\mathbb{I}_{\{|r|<1\}}) \Pi(dr) - q,$$

see (2.1). Next, assume that  $\bar{a}_- = 0$ , see (2.5), and that either  $-\Psi(0) = q > 0$  or  $\Psi'(0^+) = \mathbb{E}[\xi_1] \in (0, \infty]$  with  $q = 0$  hold in the definition of  $\Psi \in \mathcal{N}$ , i.e. (2.17). For any  $\eta > 0$  modify the Lévy measure  $\Pi$  in (5.6) as follows

$$(5.7) \quad \Pi_\eta(dr) = \mathbb{I}_{\{r>0\}}\Pi(dr) + \mathbb{I}_{\{r<0\}}e^{\eta r}\Pi(dr).$$

Denote then by  $\Psi_\eta$  the Lévy -Khintchine exponent based on  $c, \sigma^2$  taken from  $\Psi$  and Lévy measure  $\Pi_\eta$ . Then set  $\Psi_\eta(z) = -\phi_+^\eta(-z)\phi_-^\eta(z)$ , see (1.2). However, (5.7) and (5.6) imply that  $\Psi_\eta$  (resp.  $\phi_-^\eta$ ) extends holomorphically at least to  $\mathbb{C}_{(-\eta,0)}$  (resp.  $\mathbb{C}_{(-\eta,\infty)}$ ), which immediately triggers that  $\bar{a}_{\phi_-^\eta} < 0$  if  $\Psi(0) = -\Psi_\eta(0) = q > 0$  in (5.6). However, when  $\Psi'(0^+) = \mathbb{E}[\xi_1] \in (0, \infty]$  and  $q = 0$  are valid then we get that  $\Psi_\eta'(0^+) = \mathbb{E}[\xi_1^\eta] \geq \Psi'(0^+) = \mathbb{E}[\xi_1] > 0$  since relation (5.7) shows that  $\xi^\eta$  is derived from  $\xi$  via a removal of negative jumps only. Henceforth, a.s.  $\lim_{t \rightarrow \infty} \xi_t^\eta = \infty$

which shows that the downgoing ladder height process associated to  $\phi^\eta$  is killed, that is  $\phi^\eta(0) > 0$ . This combined with  $\phi^\eta \in \mathbf{A}_{(-\eta, \infty)}$  gives that  $\bar{\mathbf{a}}_{\phi^\eta} < 0$ , see (2.5). Therefore, we have that (2.21) is valid for  $I_{\Psi_\eta}$  and the probabilistic interpretation of  $\mathcal{M}_{I_{\Psi_\eta}}$  above gives that

$$(5.8) \quad I_{\Psi_\eta} \stackrel{d}{=} X_{\phi^\eta} \times I_{\phi^\eta}.$$

However, since (5.7) corresponds to the thinning of the negative jumps of  $\xi$  we conclude that  $I_{\Psi_\eta} \leq I_\Psi$  and clearly a.s.  $\lim_{\eta \rightarrow 0} I_{\Psi_\eta} = I_\Psi$ . Moreover, from [52, Lemma 4.9] we have that  $\lim_{\eta \rightarrow 0} \phi_\pm^\eta(u) = \phi_\pm(u)$ ,  $u \geq 0$ . Therefore, with the help of Lemma 3.13 we deduce that  $\lim_{\eta \rightarrow 0} W_{\phi_\pm^\eta}(z) = W_{\phi_\pm}(z)$ ,  $z \in \mathbb{C}_{(0, \infty)}$ , and establish (2.21) via

$$\begin{aligned} \mathcal{M}_{I_\Psi}(z) &= \lim_{\eta \rightarrow 0} \mathcal{M}_{I_{\Psi_\eta}}(z) = \lim_{\eta \rightarrow 0} \phi_-^\eta(0) \frac{\Gamma(z)}{W_{\phi_+^\eta}(z)} W_{\phi_-^\eta}(1-z) \\ &= \phi_-(0) \frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-}(1-z), \quad z \in \mathbb{C}_{(0,1)}. \end{aligned}$$

It remains to consider the case  $\Psi \in \mathcal{N}$ ,  $q = 0$ ,  $\mathbb{E}[\xi_1 \mathbb{I}_{\{\xi_1 > 0\}}] = \mathbb{E}[-\xi_1 \mathbb{I}_{\{\xi_1 < 0\}}] = \infty$  and  $\lim_{t \rightarrow \infty} \xi_t = \infty$  hold. We do so by killing the Lévy process  $\xi$  at rate  $r > 0$ . Therefore, with the obvious notation,  $\Psi^r(z) = -\phi_+^r(-z)\phi_-^r(z)$  and  $\lim_{r \rightarrow 0} \phi_\pm^r(z) = \phi_\pm(z)$ ,  $z \in \mathbb{C}_{(0, \infty)}$ , since  $\phi_\pm^r$  are the exponents of the bivariate ladder height processes  $(\tau^\pm, H^\pm)$  as introduced in Section A and Proposition C.1 holds. Also a.s.  $\lim_{r \rightarrow 0} I_{\Psi^r} = I_\Psi$ . However, since (2.21) holds whenever  $r > 0$  we conclude (2.21) in this case by virtue of Lemma 3.13.  $\square$

**5.2. Proof of Theorem 2.7(2).** We use the identity (2.44) that is

$$(5.9) \quad I_\Psi \stackrel{d}{=} I_{\phi_*} \times X_{\phi_*},$$

where  $I_{\phi_*}$  is the exponential functional of the possibly killed negative subordinator associated to  $\phi_* \in \mathcal{B}$ . It is well known from [33, Lemma 2.1] that  $\text{Supp } I_{\phi_*} = \left[0, \frac{1}{\mathbf{d}_*}\right]$  unless  $\phi_*(z) = \mathbf{d}_* z$  in which case  $\text{Supp } I_{\phi_*} = \left\{\frac{1}{\mathbf{d}_*}\right\}$ . When  $\mathbf{d}_* = 0$  then  $\text{Supp } I_\Psi = \text{Supp } I_{\phi_*} = [0, \infty]$  in any case. Assume that  $\mathbf{d}_* > 0$  and note from (3.21) that

$$\ln \mathbb{E}[Y_{\phi_*}^n] = \ln W_{\phi_*}(n+1) \stackrel{\infty}{\approx} n \ln \phi_-(n+1),$$

which clearly shows that  $\text{Supp } Y_{\phi_*} \subseteq [0, \ln(\phi_-(\infty))]$  and  $\text{Supp } Y_{\phi_*} \not\subseteq [0, \ln(\phi_-(\infty)) - \varepsilon]$ , for all  $\varepsilon > 0$ , where  $Y_{\phi_*}$  is the random variable associated to  $W_{\phi_*}$ , see Definition 3.1. However,  $Y_{\phi_*}$  is multiplicative infinitely divisible, see  $\mathcal{R} \stackrel{d}{=} Y_{\phi_*}$  in the notation of [36, Section 3.2], or [6, Theorem 2.2]. Then  $\ln Y_{\phi_*}$  is infinitely divisible and again according to [36, Section 3.2] its Lévy measure,  $\Theta$  in their notation, is carried by  $\mathbb{R}^-$  and  $\Theta(-dx)$  is equivalent to  $k(dx) = \int_0^x U_-(dx-y)y\mu_-(dy)$ . However, when  $\mathbf{d}_* > 0$  from (4.24) we have

$$\mu_-(dy) = v_-(y)dy = \left( \int_0^\infty u_+(v)\Pi_-(y+dv) \right) dy$$

and as  $u_+ > 0$  on  $\mathbb{R}^+$ , as mentioned in the proof of Lemma 4.7, then  $v_- > 0$  on  $\mathbb{R}^+$ . Also, in this case  $v_-(y) = \int_0^\infty u_+(v)\Pi_-(y+dv)$  is at least a càdlàg function, see [24, Chapter 5, Theorem 16]

or in more generality the differentiated version of (B.1) below. Thus, for any  $c > 0$ ,

$$\int_0^c k(dx) = \int_0^c \int_y^c (x-y)v_-(x-y)dxU_-(dy) > 0$$

and a celebrated criterion in [71] shows that  $\text{Supp } \ln Y_{\phi_-} = [-\infty, \ln \phi_-(\infty)]$ . Finally, from (5.2) we deduce that  $\text{Supp } X_{\phi_-} = \left[\frac{1}{\phi_-(\infty)}, \infty\right]$ . From (5.9) this concludes the proof.  $\square$

**5.3. Proof of items (3) and (5) of Theorem 2.7.** The smoothness and the analyticity in each of the cases follow by a simple Mellin inversion

$$(5.10) \quad f_{\Psi}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x^{-a-ib} \mathcal{M}_{I_{\Psi}}(a+ib) db$$

due to the implied polynomial  $\mathbf{N}_{\Psi} \in (0, \infty]$  or exponential  $\Theta \in [0, \frac{\pi}{2}]$  decay of  $|\mathcal{M}_{I_{\Psi}}|$ , taking derivatives and utilizing the dominated convergence theorem. Thus, (2.22) follows in ordinary sense. To prove its validity in the  $L^2$ -sense it is sufficient according to [70] to have that  $b: \mathbb{R} \mapsto \mathcal{M}_{I_{\Psi}}(a+ib) \in L^2(\mathbb{R})$ , which is the case when  $\mathbf{N}_{\Psi} > 1/2$ .  $\square$

**5.4. Proof of Theorem 2.7(4).** We start with an auxiliary result. It shows that the decay of  $|\mathcal{M}_{\Psi}|$  can be extended to the left provided  $\mathbf{a}_+ < 0$ .

**Proposition 5.1.** *Let  $\Psi \in \overline{\mathcal{N}}$ . Then, the decay of  $|\mathcal{M}_{\Psi}(z)|$  which is always of value  $\mathbf{N}_{\Psi} > 0$ , see (2.16), is preserved along  $\mathbb{C}_a$  for any  $a \in (\mathbf{a}_+, 0]$ .*

*Proof.* Let  $\mathbf{a}_+ < 0$  and take any  $a \in (\max\{\mathbf{a}_+, -1\}, 0)$ . Then we can meromorphically extend via (1.1) and (1.2) to derive that

$$(5.11) \quad \mathcal{M}_{\Psi}(a+ib) = \frac{\Psi(-a-ib)}{-a-ib} \mathcal{M}_{\Psi}(a+1+ib) = \frac{\phi_+(a+ib)\phi_-(-a-ib)}{a+ib} \mathcal{M}_{\Psi}(a+1+ib).$$

Then, Proposition 3.14(3) gives that  $|\Psi(-a-ib)| = O(b^2)$  and the result when  $\Psi \in \overline{\mathcal{N}}_{\infty}$  follows. If  $\Psi \in \overline{\mathcal{N}}_{\mathbf{N}_{\Psi}}$ ,  $\mathbf{N}_{\Psi} < \infty$ , then according to Theorem 2.5 we have that  $\phi_+ \in \mathcal{B}_P$ ,  $\phi_- \in \mathcal{B}_P^c$ ,  $\overline{\Pi}(0) < \infty$  and  $v_-(y) = \mu_-(dy)/dy$ . Therefore, again from Proposition 3.14(3),  $\lim_{|b| \rightarrow \infty} \left| \frac{\phi_+(a+ib)}{a+ib} \right| = \mathbf{d}_+ > 0$  and from Proposition 3.14(4),  $\lim_{|b| \rightarrow \infty} \phi_-(-a-ib) = \phi_-(\infty)$ . With these observations,  $\mathbf{N}_{\Psi}$  is preserved in the decay through (5.11). We recur this argument for any  $a \in (\mathbf{a}_+, -1)$ , if  $\mathbf{a}_+ < -1$ . For the case  $a = 0$  taking  $b \neq 0$  and then using the recurrent equation (3.4) applied to (2.6) we observe that

$$\mathcal{M}_{\Psi}(ib) = \frac{\phi_+(ib)}{ib} \frac{\Gamma(1+ib)}{W_{\phi_+}(1+ib)} W_{\phi_-}(1-ib).$$

However, if  $\Psi \in \overline{\mathcal{N}}_{\infty}$  then either  $\phi_- \in \mathcal{B}_{\infty}$  and/or  $\left| \frac{\Gamma(1+ib)}{W_{\phi_+}(1+ib)} \right|$  decays subexponentially. Then Proposition 3.14(3) shows the same subexponential decay for  $|\mathcal{M}_{\Psi}(ib)|$ . If  $\Psi \in \overline{\mathcal{N}}_{\mathbf{N}_{\Psi}}$ ,  $\mathbf{N}_{\Psi} < \infty$ , then as above  $\phi_+ \in \mathcal{B}_P$  and thus  $\lim_{|b| \rightarrow \infty} \left| \frac{\phi_+(ib)}{ib} \right| = \mathbf{d}_+ > 0$  and we conclude the proof.  $\square$

We are ready to start the proof of Theorem 2.7(4). A standard relation of Mellin transforms gives that the restriction of

$$(5.12) \quad \mathcal{M}_{I_{\Psi}}^*(z) = -\frac{1}{z} \mathcal{M}_{I_{\Psi}}(z+1) = -\frac{\phi_-(0)}{z} \mathcal{M}_{\Psi}(z+1) \in \mathbf{A}_{(-1,0)} \cap \mathbf{M}_{(\mathbf{a}_+-1, -\mathbf{a}_+)},$$



on  $\mathbb{C}_{(-1,0)}$  is the Mellin transform in the distributional sense of  $F_\Psi(x) = \mathbb{P}(I_\Psi \leq x)$ , where we recall (2.21) for the form and the analytical properties of  $\mathcal{M}_{I_\Psi}$ . Note next that in this assertion we only consider  $\Psi \in \mathcal{N}_+$ , that is  $q = -\Psi(0) > 0$ . From Theorem 2.1 and (5.12) we get that if  $\mathbf{u}_+ = -\infty$  or  $-\mathbf{u}_+ \notin \mathbb{N}$  then  $\mathcal{M}_{I_\Psi}^*$  has simple poles at all points in the set  $\{-1, \dots, -\lceil 1 - \mathbf{a}_+ \rceil + 1\}$  and otherwise if  $-\mathbf{u}_+ \in \mathbb{N}$  simple poles at all points of  $\{-1, \dots, \mathbf{u}_+\}$ . The decay of  $|\mathcal{M}_{I_\Psi}^*(z)|$  along  $\mathbb{C}_a$ ,  $a \in (-1, 0)$  is  $\mathbf{N}_\Psi + 1$  since the decay of  $|\mathcal{M}_{I_\Psi}(z)|$  along  $\mathbb{C}_a$ ,  $a \in (0, 1)$ , is of order  $\mathbf{N}_\Psi$ , see Theorem 2.5(1). However, thanks to Proposition 5.1 the decay of  $|\mathcal{M}_{I_\Psi}^*(z)|$  is of order  $\mathbf{N}_\Psi + 1 > 1$  along  $\mathbb{C}_a$ ,  $a \in (\mathbf{a}_+ - 1, 0)$ . Therefore, (5.12) is the Mellin transform of  $F_\Psi(x)$  in ordinary sense. Moreover, with  $\mathbf{N}_+ = |\mathbf{u}_+| \mathbb{I}_{\{|\mathbf{u}_+| \in \mathbb{N}\}} + (\lceil |\mathbf{a}_+| + 1 \rceil) \mathbb{I}_{\{|\mathbf{u}_+| \notin \mathbb{N}\}}$ ,  $\mathbb{N} \ni M < \mathbf{N}_+$  and  $a \in ((-M - 1) \vee (\mathbf{a}_+ - 1), -M)$  so that  $a + M \in (-1, 0)$  we apply the Cauchy theorem in the Mellin inversion for  $F_\Psi(x)$  to get that

$$(5.13) \quad \begin{aligned} F_\Psi(x) &= -\frac{\phi_-(0)}{2\pi i} \int_{a+M-i\infty}^{a+M+i\infty} x^{-z} \frac{\mathcal{M}_\Psi(z+1)}{z} dz \\ &= q \sum_{k=1}^M \frac{\prod_{j=1}^{k-1} \Psi(j)}{k!} x^k - \frac{\phi_-(0)}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^{-z} \frac{\mathcal{M}_\Psi(z+1)}{z} dz, \end{aligned}$$

since the residues are of values  $\frac{\phi_-(0)}{k} \phi_+(0) \frac{\prod_{j=1}^{k-1} \Psi(j)}{(k-1)!} = q \frac{\prod_{j=1}^{k-1} \Psi(j)}{k!}$  at each of those poles at  $-k$  and are computed as the residues of  $\mathcal{M}_\Psi$ , see Theorem 2.1, and the contribution of the other terms of (5.12), that is  $-\phi_-(0)/z$ . We recall that by convention  $\sum_{j=1}^0 = 0$ . Thus, we prove (2.23) for  $n = 0$ . The derivative of order  $n$  is easily established via differentiating (5.13) as long as  $0 \leq n < \mathbf{N}_\Psi$ .

*Proof of Corollary 2.12.* We sketch the proof of Corollary 2.12. If  $|\mathbf{a}_+| = \infty$  and  $-\mathbf{u}_+ \notin \mathbb{N}$  then the fact the sum in (2.23) taken to infinity, that is

$$F(x) = q \sum_{k=1}^{\infty} \frac{\prod_{j=1}^{k-1} \Psi(j)}{k!} x^k$$

is an asymptotic expansion is immediate. From the identity  $\Psi(j) = -\phi_+(-j)\phi_-(j)$  and (3.3) we see that

$$\phi_+(-j) = \phi_-(0) - \mathbf{d}_+ j - \int_0^\infty (e^{jy} - 1) \mu_+(dy)$$

and clearly  $|\phi_+(-j)|$  has an exponential growth in  $j$  if  $\mu_+$  is not identically zero. In the latter case the asymptotic series cannot be a convergent series for any  $x > 0$ . If  $\phi_+(z) = \phi_+(0) + \mathbf{d}_+ z$  then  $\Psi(j) \stackrel{\infty}{\sim} \mathbf{d}_+ j \phi_-(j)$  and hence

$$\frac{\prod_{j=1}^{k-1} |\Psi(j)|}{k!} \stackrel{\infty}{\sim} \frac{\mathbf{d}_+^{k-1}}{k} \prod_{j=1}^{k-1} \phi_-(j).$$

Therefore,  $F$  is absolutely convergent if  $x < \frac{1}{\mathbf{d}_+ \phi_-(\infty)}$  and divergent if  $x > \frac{1}{\mathbf{d}_+ \phi_-(\infty)}$ . Finally, if  $\phi_+ \equiv 1$  then  $\frac{\prod_{j=1}^{k-1} |\Psi(j)|}{k!} \stackrel{\infty}{\sim} \frac{\prod_{j=1}^{k-1} \phi_-(j)}{k!}$  and since from Proposition 3.14(3),  $\phi_-(j) \stackrel{\infty}{\sim} j(\mathbf{d}_+ + o(1))$ , we deduct that  $F$  is absolutely convergent for  $x < \frac{1}{\mathbf{d}_+}$  and divergent for  $x > \frac{1}{\mathbf{d}_+}$ .  $\square$   $\square$

**5.5. Proof of Theorem 2.14.** Recall the definitions of the *lattice class* and the *weak non-lattice class*, see around (2.25). We start with a result which discusses when the decay of  $|\mathcal{M}_\Psi(z)|$  can be extended to the line  $\mathbb{C}_{1-\mathbf{u}_-}$ .

**Proposition 5.2.** *Let  $\Psi \in \mathcal{N}$ . If  $\Psi \in \mathcal{N}_\infty \cap \mathcal{N}_\mathcal{W}$  then the subexponential decay of  $|\mathcal{M}_{I_\Psi}(z)|$  along  $\mathbb{C}_{1-\mathbf{u}_-}$  is preserved. Otherwise, if  $\Psi \in \mathcal{N}_{\mathbb{N}_\Psi}$ ,  $\mathbb{N}_\Psi < \infty$ , then the same polynomial decay is valid for  $|\mathcal{M}_{I_\Psi}|$  along  $\mathbb{C}_{1-\mathbf{u}_-}$ .*

*Proof.* Let  $-\mathbf{u}_- = -\bar{\mathbf{a}}_- \in (0, \infty)$ . Using (1.2) we write for  $b \neq 0$

$$(5.14) \quad \mathcal{M}_\Psi(1 - \mathbf{u}_- + ib) = \frac{\mathbf{u}_- - ib}{\Psi(\mathbf{u}_- - ib)} \mathcal{M}_\Psi(-\mathbf{u}_- + ib).$$

Assume first that  $\Psi \in \mathcal{N}_\infty$  then we choose  $k_0 \in \mathbb{N}$ , whose existence is guaranteed since  $\Psi \in \mathcal{N}_\mathcal{W}$ , such that  $\lim_{|b| \rightarrow \infty} |b|^{k_0} |\Psi(\mathbf{u}_- + ib)| > 0$ . Premultiplying (5.14) with  $|b|^{-k_0}$  and taking absolute values we conclude that  $|\mathcal{M}_\Psi(1 - \mathbf{u}_- + ib)|$  has subexponential decay. Recall from Theorem 2.5(1) that  $\Psi \in \mathcal{N}_{\mathbb{N}_\Psi}$  with  $\mathbb{N}_\Psi < \infty \iff \phi_+ \in \mathcal{B}_P$ ,  $\phi_- \in \mathcal{B}_P^c$ ,  $\bar{\Pi}(0) < \infty$  and from (4.50) that the Lévy measure behind  $\phi_-$  is absolutely continuous. Then, we conclude from Proposition 3.14(3) that  $\lim_{|b| \rightarrow \infty} \frac{|\phi_+(-\mathbf{u}_- + ib)|}{|\mathbf{u}_- + ib|} = \mathbf{d}_+ > 0$  and from an easy extension of Proposition 3.14(4) that  $\lim_{|b| \rightarrow \infty} \phi_-(\mathbf{u}_- + ib) = \phi_-(\infty) > 0$ . Therefore, we conclude that in (5.14),

$$\lim_{|b| \rightarrow \infty} \frac{|\mathbf{u}_- - ib|}{|\Psi(\mathbf{u}_- - ib)|} = \frac{1}{\mathbf{d}_+ \phi_-(\infty)} > 0.$$

This shows that the speed of decay of  $|\mathcal{M}_\Psi(-\mathbf{u}_- + ib)|$  and therefore via (2.21) the speed of decay of  $|\mathcal{M}_{I_\Psi}(z)|$  are preserved.  $\square$

We start with the proof of (2.29) of item (2). Note that an easy computation related to the Mellin transforms shows that the restriction of

$$(5.15) \quad \overline{\mathcal{M}_{I_\Psi}^*}(z) = \frac{1}{z} \mathcal{M}_{I_\Psi}(z + 1) \in \mathbf{A}_{(-1,0)} \cap \mathbf{M}_{(\mathbf{a}_+, -1, -\mathbf{a}_-)},$$

to  $\mathbb{C}_{(0, -\bar{\mathbf{a}}_-)}$ , if  $-\bar{\mathbf{a}}_- > 0$ , is the Mellin transform of  $\bar{F}_\Psi(x) = \mathbb{P}(I_\Psi \geq x)$  in the distributional sense. Next, note from Theorem 2.5(1) that since  $\mathbb{N}_\Psi > 0$  for every  $\Psi \in \mathcal{N}$  then along  $\mathbb{C}_a$ ,  $a \in (0, -\bar{\mathbf{a}}_-)$ ,  $|\overline{\mathcal{M}_{I_\Psi}^*}(z)|$  decays subexponentially if  $\Psi \in \mathcal{N}_\infty$  or by speed of  $\mathbb{N}_\Psi + 1 > 1$  otherwise. Therefore,  $\overline{\mathcal{M}_{I_\Psi}^*}$  is the Mellin transform of  $\bar{F}_\Psi(x)$  in the ordinary sense. When  $\Psi \in \mathcal{N}_\mathcal{Z}$ , (2.29) is known modulo to an unknown constant, see [65, Lemma 4]. The value of this constant, that is  $\frac{\phi_-(0)\Gamma(-\mathbf{u}_-)W_{\phi_-}(1+\mathbf{u}_-)}{\phi'_-(\mathbf{u}_-^+)W_{\phi_+}(1-\mathbf{u}_-)}$ , can be immediately computed as in (5.22) below. We proceed to establish (2.30). For this purpose we assume that either  $\Psi \in \mathcal{N}_\infty \cap \mathcal{N}_\mathcal{W}$  or  $\Psi \in \mathcal{N}_{\mathbb{N}_\Psi}$ ,  $\mathbb{N}_\Psi < \infty$ . In any case, whenever  $\mathbb{N}_\Psi > 1$ , the Mellin inversion theorem applies and yields that, for any  $z \in \mathbb{C}_a$ ,  $a \in (0, 1 - \bar{\mathbf{a}}_-)$ ,

$$(5.16) \quad f_\Psi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x^{-a-ib} \mathcal{M}_{I_\Psi}(a + ib) db.$$

However, the assumptions  $\bar{\mathbf{a}}_- = \mathbf{u}_- < 0$ ,  $\Psi(\mathbf{u}_-) = -\phi_+(-\mathbf{u}_-) \phi_-(\mathbf{u}_-) = 0$  and  $|\Psi'(\mathbf{u}_-^+)| < \infty$  of item (2) together with (1.2) lead to

$$\Psi'(\mathbf{u}_-^+) = \phi'_-(\mathbf{u}_-^+) \phi_+(-\mathbf{u}_-)$$

and hence  $|\Psi'(\mathbf{u}^+)| < \infty$  implies that  $|\phi'_-(\mathbf{u}^+)| < \infty$ . This observation, the form of  $\mathcal{M}_\Psi$ , see (4.4) and the fact that  $\Psi \in \mathcal{N}_Z \subset \mathcal{N}_W$  permit us to write

$$(5.17) \quad \begin{aligned} \mathcal{M}_{I_\Psi}(z) &= \phi_-(0) \frac{\Gamma(z)}{W_{\phi_*}(z)} W_{\phi_-}(1-z) \\ &= \phi_-(0) \frac{\Gamma(z)}{W_{\phi_*}(z)} P(1-z) + \frac{\phi_-(0) W_{\phi_-}(1+\mathbf{u}_-)}{\phi'_-(\mathbf{u}^+)} \frac{1}{1-\mathbf{u}_- - z} \frac{\Gamma(z)}{W_{\phi_*}(z)}, \end{aligned}$$

with  $P$  from Theorem 3.2(3) having the form and the property that

$$P(z) = W_{\phi_-}(z) - \frac{W_{\phi_-}(1+\mathbf{u}_-)}{\phi'_-(\mathbf{u}^+)(z-\mathbf{u}_-)} \in \mathbf{A}_{[\mathbf{u}_-, \infty)}.$$

Therefore, (5.17) shows that  $\mathcal{M}_{I_\Psi} \in \mathbf{A}_{(0, 1-\mathbf{u}_-)}$  extends continuously to  $\mathbb{C}_{1-\mathbf{u}_-} \setminus \{1-\mathbf{u}_-\}$ . Next, we show that the contour in (5.16) can be partly moved to the line  $\mathbb{C}_{1-\mathbf{u}_-}$  at least for  $|b| > c > 0$  for any  $c > 0$ . For this purpose we observe from Proposition 5.2 that whenever  $\Psi \in \mathcal{N}_W \cap \mathcal{N}_\infty$  (resp.  $\Psi \in \mathcal{N}_{N_\Psi} \cap \mathcal{N}$ ,  $N_\Psi < \infty$ ) the decay of  $|\mathcal{M}_{I_\Psi}(z)|$  extends with the same subexponential (resp. polynomial) speed to the complex line  $\mathbb{C}_{1-\mathbf{u}_-}$ . Then, for any  $c > 0$ ,  $a \in (0, 1-\mathbf{u}_-)$  and  $x > 0$ , thanks to the Cauchy integral theorem valid because  $N_\Psi > 1$ ,

$$(5.18) \quad \begin{aligned} f_\Psi(x) &= \frac{1}{2\pi i} \int_{z=a+ib} x^{-z} \mathcal{M}_{I_\Psi}(z) dz \\ &= x^{\mathbf{u}_- - 1} \frac{1}{2\pi} \int_{|b|>c} x^{-ib} \mathcal{M}_{I_\Psi}(1-\mathbf{u}_- + ib) db + \frac{1}{2\pi i} \int_{B^\downarrow(1-\mathbf{u}_-, c)} x^{-z} \mathcal{M}_{I_\Psi}(z) dz, \\ &= f_\Psi^*(x, c) + f_\Psi^*(x, c), \end{aligned}$$

where  $B^\downarrow(1-\mathbf{u}_-, c) = \{z \in \mathbb{C} : |z-1+\mathbf{u}_-| = c \text{ and } \operatorname{Re}(z-1+\mathbf{u}_-) \leq 0\}$  that is a semi-circle centered at  $1-\mathbf{u}_-$ . We note that the Riemann-Lebesgue lemma applied to the absolutely integrable function  $g(b) = \mathcal{M}_{I_\Psi}(1-\mathbf{u}_- + ib) \mathbb{I}_{\{|b|>c\}}$  yields that

$$(5.19) \quad \lim_{x \rightarrow \infty} x^{1-\mathbf{u}_-} f_\Psi^*(x, c) = \lim_{x \rightarrow \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ib \ln x} \mathcal{M}_{I_\Psi}(1-\mathbf{u}_- + ib) \mathbb{I}_{\{|b|>c\}} db = 0.$$

Using (5.17) we write that

$$(5.20) \quad \begin{aligned} f_\Psi^*(x, c) &= \frac{\phi_-(0)}{2\pi i} \int_{B^\downarrow(1-\mathbf{u}_-, c)} x^{-z} \frac{\Gamma(z)}{W_{\phi_*}(z)} P(1-z) dz \\ &+ \frac{1}{2\pi i} \frac{\phi_-(0) W_{\phi_-}(1+\mathbf{u}_-)}{\phi'_-(\mathbf{u}^+)} \int_{B^\downarrow(1-\mathbf{u}_-, c)} x^{-z} \frac{1}{1-\mathbf{u}_- - z} \frac{\Gamma(z)}{W_{\phi_*}(z)} dz \\ &= f_\Psi^{**}(x, c) + f_\Psi^{***}(x, c). \end{aligned}$$

Since  $\frac{\Gamma(z)}{W_{\phi_*}(z)} P(1-z) \in \mathbf{A}_{(0, 1-\mathbf{u}_-]}$  we can use the Cauchy integral theorem to show that, for every  $c > 0$  fixed,

$$(5.21) \quad \begin{aligned} |f_\Psi^{**}(x, c)| &= \left| \frac{\phi_-(0)}{2\pi} \int_{-c}^c x^{-1+\mathbf{u}_- - ib} \frac{\Gamma(1-\mathbf{u}_- + ib)}{W_{\phi_*}(1-\mathbf{u}_- + ib)} P(\mathbf{u}_- - ib) db \right| \\ &\leq x^{\mathbf{u}_- - 1} \frac{c \phi_-(0)}{\pi} \sup_{z=1-\mathbf{u}_- + ib; b \in (-c, c)} \left| \frac{\Gamma(z)}{W_{\phi_*}(z)} P(1-z) \right|. \end{aligned}$$

Next, we consider  $f_{\Psi}^{***}(x, c)$ . Since  $\frac{\Gamma(z)}{W_{\phi_+}(z)} \in \mathbf{A}_{(0, \infty)}$ , choosing  $c$  small enough we have by the Cauchy's residual theorem that

$$\begin{aligned} f_{\Psi}^{***}(x, c) &= \frac{\phi_-(0)W_{\phi_-}(1 + \mathbf{u}_-)\Gamma(1 - \mathbf{u}_-)}{\phi'_-(\mathbf{u}_+^+)W_{\phi_+}(1 - \mathbf{u}_-)}x^{\mathbf{u}_- - 1} \\ &\quad + \frac{1}{2\pi i} \frac{\phi_-(0)W_{\phi_-}(1 + \mathbf{u}_-)}{\phi'_-(\mathbf{u}_+^+)} \int_{B^\dagger(1 - \mathbf{u}_-, c)} x^{-z} \frac{1}{1 - \mathbf{u}_- - z} \frac{\Gamma(z)}{W_{\phi_+}(z)} dz, \end{aligned}$$

where  $B^\dagger(1 - \mathbf{u}_-, c) = \{z \in \mathbb{C} : |z - 1 + \mathbf{u}_-| = c \text{ and } \operatorname{Re}(z - 1 + \mathbf{u}_-) \geq 0\}$ . However, on  $B^\dagger(1 - \mathbf{u}_-, c)$ , we have that  $z = 1 - \mathbf{u}_- + ce^{i\theta}$ ,  $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ , and thus for any such  $\theta$

$$\lim_{x \rightarrow \infty} |x^{1 - \mathbf{u}_-} x^{-z}| = \lim_{x \rightarrow \infty} x^{-c \cos \theta} = 0.$$

Therefore, since

$$\sup_{z \in B^\dagger(1 - \mathbf{u}_-, c)} \left| \frac{1}{1 - \mathbf{u}_- - z} \frac{\Gamma(z)}{W_{\phi_+}(z)} \right| \leq \frac{1}{c} \sup_{z \in B^\dagger(1 - \mathbf{u}_-, c)} \left| \frac{\Gamma(z)}{W_{\phi_+}(z)} \right| < \infty,$$

we can apply the dominated convergence theorem to the integral term in the representation of  $f_{\Psi}^{***}$  above to conclude that for all  $c > 0$  small enough

$$(5.22) \quad \lim_{x \rightarrow \infty} x^{1 - \mathbf{u}_-} f_{\Psi}^{***}(x, c) = \frac{\phi_-(0)W_{\phi_-}(1 + \mathbf{u}_-)\Gamma(1 - \mathbf{u}_-)}{\phi'_-(\mathbf{u}_+^+)W_{\phi_+}(1 - \mathbf{u}_-)}.$$

Combining (5.22) and (5.21), we get from (5.20) that

$$\lim_{c \rightarrow 0} \lim_{x \rightarrow \infty} x^{1 - \mathbf{u}_-} f_{\Psi}^*(x, c) = \lim_{c \rightarrow 0} \lim_{x \rightarrow \infty} x^{1 - \mathbf{u}_-} f_{\Psi}^{***}(x, c) = \frac{\phi_-(0)W_{\phi_-}(1 + \mathbf{u}_-)\Gamma(1 - \mathbf{u}_-)}{\phi'_-(\mathbf{u}_+^+)W_{\phi_+}(1 - \mathbf{u}_-)}.$$

However, since (5.19) holds too, we get from (5.18) that

$$\lim_{x \rightarrow \infty} x^{1 - \mathbf{u}_-} f_{\Psi}(x) = \frac{\phi_-(0)W_{\phi_-}(1 + \mathbf{u}_-)\Gamma(1 - \mathbf{u}_-)}{\phi'_-(\mathbf{u}_+^+)W_{\phi_+}(1 - \mathbf{u}_-)}.$$

This completes the proof of (2.30) of item (2) for  $n = 0$ . Its claim for any  $n \in \mathbb{N}$ ,  $n \leq \lceil \mathbf{N}_{\Psi} \rceil - 2$ , follows by the same techniques as above after differentiating (5.18) and observing that we have thanks to Proposition 5.1 that for every  $n \in \mathbb{N}$ ,  $n \leq \lceil \mathbf{N}_{\Psi} \rceil - 2$ ,

$$\lim_{b \rightarrow \infty} |b|^n |\mathcal{M}_{I_{\Psi}}(1 - \mathbf{u}_- + ib)| = 0$$

and  $|b|^n |\mathcal{M}_{I_{\Psi}}(1 - \mathbf{u}_- + ib)|$  is integrable.

Let us proceed with the proof of item (1). We note that in any case the Mellin transform of  $\overline{F}_{\Psi}(x)$ , that is  $\overline{\mathcal{M}}_{I_{\Psi}}^*$  in (5.15), has a decay of value  $\mathbf{N}_{\Psi} + 1 > 1$  along  $\mathbb{C}_a$ ,  $a \in (0, -\overline{\mathbf{a}}_-)$ , if  $-\overline{\mathbf{a}}_- > 0$ . Consider first (2.26), that is  $\overline{\lim}_{x \rightarrow \infty} x^{\underline{d}} \overline{F}_{\Psi}(x) = 0$  for  $\underline{d} < |\overline{\mathbf{a}}_-|$ . Let  $-\overline{\mathbf{a}}_- > 0$ , as otherwise there is nothing to prove in (2.26), and choose  $\underline{d} \in (0, -\overline{\mathbf{a}}_-)$ . By a Mellin inversion as in (5.18) and with  $a = \underline{d} + \varepsilon < -\overline{\mathbf{a}}_-$ ,  $\varepsilon > 0$ , we get that on  $\mathbb{R}^+$

$$\overline{F}_{\Psi}(x) = x^{-\underline{d} - \varepsilon} \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} x^{-ib} \overline{\mathcal{M}}_{I_{\Psi}}^*(\underline{d} + \varepsilon + ib) db \right| \leq C_{\underline{d}} x^{-\underline{d} - \varepsilon}$$

and (2.26) follows. This also settles the claim (2.27) when  $-\Psi(-z) = \phi_+(z) \in \mathcal{B}$  since then  $\overline{\mathcal{M}}_{I_{\Psi}}^*(z) = \frac{1}{z} \frac{\Gamma(z)}{W_{\phi_+}(z)} \in \mathbf{A}_{(0, \infty)}$ ,  $|\overline{\mathbf{a}}_-| = \infty$  and we can choose  $\underline{d}$  as big as we wish. It remains therefore to prove (2.27) assuming that  $-\Psi(-z) \not\equiv \phi_+(z)$ . If  $\Pi(dy) \equiv 0 dy$  on  $(-\infty, 0)$  then necessarily  $\phi_-(z) = \phi_-(0) + \mathbf{d}_- z$ ,  $\mathbf{d}_- > 0$ , and even the stronger item (2) is applicable since it can

be immediately shown that  $\Psi \in \mathcal{N}_{\mathcal{W}}$  since  $\lim_{|b| \rightarrow \infty} \left| \phi_- \left( -\frac{\phi_-(0)}{a} + ib \right) \right| = \infty$ , see (2.25). We can assume from now that  $\Pi(dy) \not\equiv 0$  on  $(-\infty, 0)$  and  $\Psi \in \mathcal{N}_{\mathcal{Z}}$ . If the conditions of item (2) are violated we proceed by approximation to furnish them. Recall (5.6) and with the given  $\Psi$  define  $\forall \eta > 0$ ,

$$(5.23) \quad \Psi^\eta(z) = \frac{\sigma^2}{2} z^2 + cz + \int_{\mathbb{R}} (e^{zr} - 1 - zr \mathbb{I}_{\{|r| < 1\}}) e^{-\eta r^2 \mathbb{I}_{\{r < -1\}}} \Pi(dr) - q.$$

Put  $\Pi^\eta(dr) = e^{-\eta r^2 \mathbb{I}_{\{r < -1\}}} \Pi(dr)$ . Since  $\Pi^\eta$  is absolutely continuous with respect to  $\Pi$  it is clear that  $\forall \eta > 0$ ,  $\Psi \in \mathcal{N}_{\mathcal{Z}} \implies \Psi^\eta \in \mathcal{N}_{\mathcal{Z}}$  and set from (1.2),  $\Psi^\eta(z) = -\phi_+^\eta(-z) \phi_-^\eta(z)$ . Let  $\xi^\eta$  be the Lévy process underlying  $\Psi^\eta$  and note that the transformation  $\Psi \mapsto \Psi^\eta$  leaves  $q$  invariant and has the sole effect of truncating at the level of paths some of the negative jumps smaller than  $-1$  of the underlying Lévy process  $\xi$ . Therefore, pathwise

$$(5.24) \quad I_\Psi = \int_0^{e_q} e^{-\xi_s} ds \geq \int_0^{e_q} e^{-\xi_s^\eta} ds = I_{\Psi^\eta}.$$

Next, it is clear from (5.23) that  $\Psi^\eta \in \mathbf{A}_{(-\infty, 0)}$  and then it is immediate that  $(\Psi^\eta)'' > 0$  on  $\mathbb{R}^-$ , and thus  $\Psi^\eta$  is convex on  $\mathbb{R}^-$ . Moreover, clearly

$$\lim_{u \rightarrow -\infty} \frac{1}{|u|} \int_{-\infty}^0 (e^{ur} - 1 - ru \mathbb{I}_{\{|r| < 1\}}) \Pi^\eta(dr) = \infty$$

since  $\Pi(dy)$  is not identically the zero measure on  $\mathbb{R}^-$ . Therefore,  $\lim_{u \rightarrow -\infty} \Psi^\eta(u) = \infty$ . Thus, immediately we conclude that  $-\mathbf{u}_{\phi_-^\eta} \in (0, \infty)$  if  $\Psi(0) = -q < 0$ . Also, if  $q = 0$ , then necessarily, regardless of whether  $\mathbb{E}[\xi_1] \in (0, \infty]$  or  $\mathbb{E}[\xi_1 \mathbb{I}_{\{\xi_1 > 0\}}] = \mathbb{E}[-\xi_1 \mathbb{I}_{\{\xi_1 < 0\}}] = \infty$  with  $\lim_{t \rightarrow \infty} \xi_t = \infty$  hold for  $\xi$ , then  $\mathbb{E}[\xi_1^\eta] \in (0, \infty]$ , see (5.23), and thus  $\phi_-^\eta(0) > 0$  leads to  $-\mathbf{u}_{\phi_-^\eta} \in (0, \infty)$ . However, from (5.24) we get that

$$\mathbb{P}(I_\Psi > x) \geq \mathbb{P}(I_{\Psi^\eta} > x)$$

and from (2.29) of item (2) we obtain, for any  $\epsilon > 0$  and fixed  $\eta > 0$ , that with some  $C > 0$

$$\overline{\lim}_{x \rightarrow \infty} x^{-\mathbf{u}_{\phi_-^\eta} + \epsilon} \mathbb{P}(I_\Psi > x) \geq \overline{\lim}_{x \rightarrow \infty} x^{-\mathbf{u}_{\phi_-^\eta} + \epsilon} \mathbb{P}(I_{\Psi^\eta} > x) = C \overline{\lim}_{x \rightarrow \infty} x^\epsilon = \infty.$$

Thus, it remains to show that  $\lim_{\eta \rightarrow 0} \mathbf{u}_{\phi_-^\eta} = \bar{\mathbf{a}}_- \in (-\infty, 0]$ , where  $\bar{\mathbf{a}}_- > -\infty$  is evident since  $z \mapsto -\Psi(-z) \notin \mathcal{B}$ . It is clear from the identity

$$\begin{aligned} \Psi^\eta(u) &= \frac{\sigma^2}{2} u^2 + cu + \int_0^\infty (e^{ur} - 1 - ur \mathbb{I}_{\{r < 1\}}) \Pi(dr) \\ &\quad + \int_{-\infty}^0 (e^{ur} - 1 - ur \mathbb{I}_{\{r > -1\}}) e^{-\eta r^2 \mathbb{I}_{\{r < -1\}}} \Pi(dr), \end{aligned}$$

that for any  $u \leq 0$ ,  $\Psi^\eta(u)$  is increasing as  $\eta \downarrow 0$  and  $\lim_{\eta \rightarrow 0} \Psi^\eta(u) = \Psi(u)$ . Fix  $\eta^* > 0$  small enough.

Then, clearly,

$$\Psi(\mathbf{u}_{\phi_-^{\eta^*}}) = \lim_{\eta \rightarrow 0} \Psi^\eta(\mathbf{u}_{\phi_-^{\eta^*}}) \geq \Psi^{\eta^*}(\mathbf{u}_{\phi_-^{\eta^*}}) = 0.$$

If  $\Psi(\mathbf{u}_{\phi_-^{\eta^*}}) = \infty$  then from (1.2) we get that  $\phi_-(\mathbf{u}_{\phi_-^{\eta^*}}) = -\infty$  and thus  $\mathbf{u}_{\phi_-^{\eta^*}} \leq \bar{\mathbf{a}}_-$ . Similarly, if  $\Psi(\mathbf{u}_{\phi_-^{\eta^*}}) \in [0, \infty)$  then  $\phi_-(\mathbf{u}_{\phi_-^{\eta^*}}) \leq 0$  and again  $\mathbf{u}_{\phi_-^{\eta^*}} \leq \bar{\mathbf{a}}_-$ . Therefore,

$$\underline{\mathbf{u}} = \overline{\lim}_{\eta \rightarrow 0} \mathbf{u}_{\phi_-^\eta} \leq \bar{\mathbf{a}}_-.$$

Assume that  $\underline{u} < \bar{a}_-$  and choose  $u \in (\underline{u}, \bar{a}_-)$ . Then  $\Psi^\eta(u) < 0$  and thus  $\Psi(u) = \lim_{\eta \rightarrow 0} \Psi^\eta(u) \leq 0$ . However,  $\Psi(u) \in (0, \infty]$ , for  $u < \bar{a}_-$ , which triggers  $\underline{u} = \bar{a}_-$ . Moreover, the monotonicity of  $\Psi^\eta$  when  $\eta \downarrow 0$  shows that in fact

$$\lim_{\eta \rightarrow 0} \underline{u}_{\phi^\eta} = \underline{u} = \bar{a}_-$$

and we conclude the statement (2.27) when  $\Psi \in \mathcal{N}_{\mathcal{Z}}$ . Next, if  $\Psi \notin \mathcal{N}_{\mathcal{Z}}$  then as in Theorem 3.2(2) one can check that  $\Pi(dx) = \sum_{n=-\infty}^{\infty} c_n \delta_{\bar{h}n}(dx)$ ,  $\sum_{n=-\infty}^{\infty} c_n < \infty$ ,  $c_n \geq 0$ ,  $\bar{h} > 0$ , and  $\sigma^2 = c = 0$  in (5.6). The underlying Lévy process,  $\xi$ , is a possibly killed compound Poisson process living on the lattice  $\{\bar{h}n\}_{n=-\infty}^{\infty}$ . We proceed by approximation. Set  $\Psi_d(z) = \Psi(z) + dz$ ,  $d > 0$ , with underlying Lévy process  $\xi^d$ . Clearly,  $\xi_t^d = \xi_t + dt$ ,  $t \geq 0$ , and hence  $I_\Psi \geq I_{\Psi_d}$  and, for any  $x > 0$ ,  $\mathbb{P}(I_\Psi > x) \geq \mathbb{P}(I_{\Psi_d} > x)$ . However,  $\Psi_d \in \mathcal{N}_{\mathcal{Z}}$  and therefore from (2.27) for any  $\bar{d} > |\bar{a}^d|$ , where from  $\Psi_d(z) = -\phi_+^d(-z)\phi_-^d(z)$ ,  $\bar{a}^d = \bar{a}_{\phi^d}$ , see (3.8), we deduce that

$$\liminf_{x \rightarrow \infty} x^{\bar{d}} \mathbb{P}(I_\Psi > x) \geq \liminf_{x \rightarrow \infty} x^{\bar{d}} \mathbb{P}(I_{\Psi_d} > x) = \infty.$$

To establish (2.27) for  $\Psi$  it remains to confirm that  $\lim_{d \rightarrow 0} \bar{a}^d = \bar{a}_-$ . Note that adding  $dz$  to  $\Psi(z)$  does not alter its range of analyticity and hence with the obvious notation  $\underline{a}_- = \underline{a}_{\phi^d}$ . Set

$$T = \inf \{t > 0 : \xi_t < 0\} \in (0, \infty] \text{ and } T^d = \inf \{t > 0 : \xi_t^d < 0\} \in (0, \infty].$$

Clearly, from  $\xi_t^d = \xi_t + dt$  we get that a.s.  $\lim_{d \rightarrow 0} (T^d, \xi_{T^d}) = (T, \xi_T)$ . However,  $(T, \xi_T)$  and  $(T^d, \xi_{T^d})$  define the distribution of the bivariate descending ladder time and height processes of  $\Psi$  and  $\Psi_d$ , see Section A. Therefore, since from Lemma A.1 we can choose  $\phi_-, \phi_-^d$  to represent the descending ladder height process, that is  $\phi_- = k_-$ ,  $\phi_-^d = k_-^d$  in the notation therein, we conclude that  $\lim_{d \rightarrow 0} \phi_-^d(z) = \phi_-(z)$  for any  $z \in \mathbb{C}_{(a_-, \infty)}$  and hence  $\lim_{d \rightarrow 0} \underline{u}_{\phi^d} = \underline{u}_-$ . Thus  $\lim_{d \rightarrow 0} \bar{a}^d = \bar{a}_-$ . This, concludes Theorem 2.14.  $\square$

**5.6. Proof of Theorem 2.19.** Recall from (5.12) that  $\mathcal{M}_{I_\Psi}^*(z)$  is the Mellin transform of  $F_\Psi(x)$  at least on  $\mathbb{C}_{(-1,0)}$ . We record and re-express it with the help of (2.6) and (3.4) as

$$\begin{aligned} \mathcal{M}_{I_\Psi}^*(z) &= -\frac{1}{z} \mathcal{M}_{I_\Psi}(z+1) = -\frac{\phi_-(0)}{z} \mathcal{M}_\Psi(z+1) \\ (5.25) \quad &= -\frac{\phi_-(0)}{z} \frac{\Gamma(z+1)}{W_{\phi_+}(z+1)} W_{\phi_-}(-z), \quad z \in \mathbb{C}_{(-1,0)}. \end{aligned}$$

From Theorem 2.1 we deduce that  $\mathcal{M}_{I_\Psi} \in \mathbf{A}_{(0, 1-a_-)}$  and that it extends continuously to  $\mathbb{C}_0 \setminus \{0\}$ . Moreover, Proposition 5.1 applied to  $\mathcal{M}_{I_\Psi}(z)$  for  $z \in i\mathbb{R}$  shows that the decay of  $|\mathcal{M}_{I_\Psi}(z)|$  is preserved along  $\mathbb{C}_0$ . Therefore, either  $|\mathcal{M}_{I_\Psi}^*(z)|$  decays subexponentially along  $\mathbb{C}_a$ ,  $a \in [-1, 0)$ , or it decays polynomially with a speed of  $N_\Psi + 1 > 1$ . Via a Mellin inversion, choosing a contour, based on the line  $\mathbb{C}_{-1}$  and a semi-circle, as in the proof of Theorem 2.14, see (5.18) and (5.19), we get that, for any  $c \in (0, \frac{1}{2})$ , as  $x \rightarrow 0$ ,

$$(5.26) \quad F_\Psi(x) = \frac{1}{2\pi i} \int_{B^{\uparrow}(-1, c)} x^{-z} \mathcal{M}_{I_\Psi}^*(z) dz + o(x),$$

where only the contour is changed to  $B^{\uparrow}(-1, c) = \{z \in \mathbb{C} : |z+1| = c \text{ and } \operatorname{Re}(z+1) \geq 0\}$ . Apply (3.4) to write from (5.25)

$$(5.27) \quad \mathcal{M}_{I_\Psi}^*(z) = -\frac{\phi_-(0)}{z} \frac{\phi_+(z+1)}{z+1} \frac{\Gamma(z+2)}{W_{\phi_+}(z+2)} W_{\phi_-}(-z) = \frac{\phi_+(z+1)}{z+1} \mathcal{M}_{I_\Psi}^{**}(z).$$

Clearly,  $\mathcal{M}_{I_\Psi}^{**} \in \mathbf{A}_{(-2,0)}$ . Recall that  $\phi_+^\sharp(z) = \phi_+(z) - \phi_+(0) \in \mathcal{B}$ . Then, we have, noting  $\mathcal{M}_{I_\Psi}^{**}(-1) = \phi_-(0)$ , that

$$\begin{aligned} \mathcal{M}_{I_\Psi}^*(z) &= \frac{\phi_+(0)\phi_-(0)}{z+1} + \frac{\phi_+^\sharp(z+1)}{z+1} \mathcal{M}_{I_\Psi}^{**}(-1) + \phi_+(z+1) \frac{\mathcal{M}_{I_\Psi}^{**}(z) - \mathcal{M}_{I_\Psi}^{**}(-1)}{z+1} \\ &= \mathcal{M}_1^*(z) + \mathcal{M}_2^*(z) + \mathcal{M}_3^*(z). \end{aligned}$$

Plugging this in (5.26) we get and set

$$(5.28) \quad F_\Psi(x) = \frac{1}{2\pi i} \int_{B^\dagger(-1,c)} x^{-z} \sum_{j=1}^3 \mathcal{M}_j^*(z) dz + o(x) = \sum_{j=1}^3 F_j(x) + o(x).$$

Since  $z \mapsto \frac{\mathcal{M}_3^*(z)}{\phi_+(z+1)} \in \mathbf{A}_{(-2,0)}$  and  $\phi_+(0) \in [0, \infty)$  then precisely as in (5.21) we get that

$$(5.29) \quad \lim_{c \rightarrow 0} \lim_{x \rightarrow 0} x^{-1} F_3(x) = 0.$$

Next, from (3.3) and  $\mathcal{M}_{I_\Psi}^{**}(-1) = \phi_-(0)$  we get that

$$\mathcal{M}_2^*(z) = \frac{\phi_+^\sharp(z+1)}{z+1} \mathcal{M}_{I_\Psi}^{**}(-1) = \phi_-(0) \int_0^\infty e^{-(z+1)y} \bar{\mu}_+(y) dy + \phi_-(0) \mathbf{d}_+.$$

Clearly, if  $(\phi_+^\sharp)'(0) = \phi_+'(0) = \int_0^\infty \bar{\mu}_+(y) dy < \infty$ , then the same arguments used above to prove (5.29) yield that

$$(5.30) \quad \lim_{c \rightarrow 0} \lim_{x \rightarrow 0} x^{-1} F_2(x) = \lim_{c \rightarrow 0} \lim_{x \rightarrow 0} \frac{1}{2\pi i} \int_{B^\dagger(-1,c)} x^{-z} \mathcal{M}_2^*(z) dz = 0.$$

Also, similarly, we deduce that in any case the term  $\phi_-(0) \mathbf{d}_+$  does not contribute to (5.30) and therefore assume that  $\mathbf{d}_+ = 0$  in the sequel. However, if

$$(\phi_+^\sharp)'(0^+) = \phi_+'(0^+) = \infty$$

we could not apply this argument. We then split  $\bar{\mu}_+(y) = \bar{\mu}_+(y) \mathbb{I}_{\{y>1\}} + \bar{\mu}_+(y) \mathbb{I}_{\{y \leq 1\}}$  and write accordingly

$$\mathcal{M}_2^*(z) = \phi_-(0) \int_1^\infty e^{-(z+1)y} \bar{\mu}_+(y) dy + \phi_-(0) \int_0^1 e^{-(z+1)y} \bar{\mu}_+(y) dy = \mathcal{M}_{2,1}^*(z) + \mathcal{M}_{2,2}^*(z).$$

However,  $|\mathcal{M}_{2,2}^*(-1)| = \phi_-(0) \int_0^1 \bar{\mu}_+(y) dy < \infty$  and the portion of  $\mathcal{M}_{2,2}^*$  in  $F_2$  is negligible in the sense of (5.30). Then, we need discuss solely the contribution of  $\mathcal{M}_{2,1}^*$  to  $F_2$  that is

$$\begin{aligned} F_2^*(x) &= \frac{\phi_-(0)}{2\pi i} \int_{B^\dagger(-1,c)} x^{-z} \int_1^\infty e^{-(z+1)y} \bar{\mu}_+(y) dy dz \\ &= \frac{\phi_-(0)}{2\pi i} \int_1^\infty \int_{B^\dagger(-1,c)} x^{-z} e^{-(z+1)y} dz \bar{\mu}_+(y) dy. \end{aligned}$$

The interchange is possible since evidently on  $B^\dagger(-1,c)$  we have that

$$\sup_{z \in B^\dagger(-1,c)} |\mathcal{M}_{2,1}^*(z)| < \infty.$$

The latter in turn follows from

$$\sup_{z \in B^\dagger(-1,c)} |\mathcal{M}_{2,1}^*(z)| = \sup_{z \in B^\dagger(-1,c)} \frac{|\phi_+(z+1)|}{|z+1|} = \sup_{z \in B^\dagger(-1,c)} \frac{|\phi_+(z+1)|}{c} < \infty.$$

However, for any  $x, y > 0$ ,  $z \mapsto x^{-z}e^{-(z+1)y}$  is an entire function and an application of the Cauchy theorem to the closed contour  $B^{\downarrow}(-1, c) \cup \{-1 + i\beta, \beta \in [-c, c]\}$  implies that

$$\int_{B^{\downarrow}(-1, c)} x^{-z}e^{-(z+1)y} dz = ix \int_{-c}^c e^{-i\beta(\ln x + y)} d\beta = 2x \frac{\sin(c(\ln x + y))}{\ln x + y} i.$$

Therefore, integrating by parts after representing  $\bar{\mu}_+(y) = \int_y^\infty \mu_+(dv)$ , we get that

$$\begin{aligned} F_2^*(x) &= x \frac{\phi_-(0)}{\pi} \lim_{A \rightarrow \infty} \int_1^A \frac{\sin(c(\ln x + y))}{\ln x + y} \bar{\mu}_+(y) dy \\ &= x \frac{\phi_-(0)}{\pi} \lim_{A \rightarrow \infty} \int_1^A \int_{c(1+\ln x)}^{c(v+\ln x)} \frac{\sin(y)}{y} dy \mu_+(dv) \\ &\quad + x \frac{\phi_-(0)}{\pi} \lim_{A \rightarrow \infty} \bar{\mu}_+(A) \int_{c(\ln x+1)}^{c(A+\ln x)} \frac{\sin(y)}{y} dy \\ &= x \frac{\phi_-(0)}{\pi} \int_1^\infty \int_{c(1+\ln x)}^{c(v+\ln x)} \frac{\sin(y)}{y} dy \mu_+(dv) \end{aligned}$$

since

$$\sup_{a < b; a, b \in \mathbb{R}} \left| \int_a^b \frac{\sin(y)}{y} dy \right| < \infty,$$

the mass of  $\mu_+(dv)$  on  $(1, \infty)$  is simply  $\bar{\mu}_+(1) < \infty$  and  $\lim_{A \rightarrow \infty} \bar{\mu}_+(A) = 0$ . Also, this allows via the dominated convergence theorem to deduce that

$$\lim_{x \rightarrow 0} x^{-1} F_2^*(x) = \frac{\phi_-(0)}{\pi} \lim_{x \rightarrow 0} \int_1^\infty \int_{c(1+\ln x)}^{c(v+\ln x)} \frac{\sin(y)}{y} dy \mu_+(dv) = 0$$

and thus with the reasoning above to verify that (5.30) holds, that is  $\lim_{c \rightarrow 0} \lim_{x \rightarrow 0} x^{-1} F_2(x) = 0$ . The latter together with (5.29) applied in (5.28) allows us to understand the asymptotic of  $x^{-1}F(x)$ , as  $x \rightarrow 0$ , in terms of  $F_1$ . However, from its very definition, (5.28) and  $\Psi(0) = -\phi_-(0)\phi_+(0)$ , see (1.2),

$$F_1(x) = -\frac{\Psi(0)}{2\pi i} \int_{B^{\downarrow}(-1, c)} \frac{x^{-z}}{z+1} dz = -\Psi(0)x + \frac{\Psi(0)}{2\pi i} \int_{B^{\downarrow}(-1, c)} \frac{x^{-z}}{z+1} dz,$$

where we recall that

$$B^{\downarrow}(-1, c) = \{z \in \mathbb{C} : |z+1| = c \text{ and } \operatorname{Re}(z+1) \leq 0\}.$$

Clearly, if  $z \in B^{\downarrow}(-1, c) \setminus \{-1 \pm ic\}$  then  $\lim_{x \rightarrow 0} |x^{-1}x^{-z}| = 0$ . Therefore, for any  $c \in (0, \frac{1}{2})$ ,

$$(5.31) \quad \lim_{x \rightarrow 0} x^{-1} F_\Psi(x) = \lim_{x \rightarrow 0} x^{-1} F_1(x) = -\Psi(0) \in [0, \infty).$$

Thus (2.31) holds true. When  $\Psi \in \mathcal{N}_{\mathbb{N}_\Psi}$ , for some  $\mathbb{N}_\Psi > 1$ , all arguments above applied to  $\mathcal{M}_{I_\Psi}^*$  can be carried over directly to  $\mathcal{M}_{I_\Psi}$  but at  $z = 0$ . When  $p_\Psi$  is continuous at zero then the result is immediate from (2.31). Thus, we obtain (2.32).  $\square$

**5.7. Proof of Theorem 2.22.** Let  $\Psi \in \bar{\mathcal{N}} \setminus \mathcal{N}_\dagger$  that is  $\Psi(0) = 0$ , see (2.18). Recall that  $I_\Psi(t) = \int_0^t e^{-\xi s} ds$ . Then if  $\mathfrak{a}_+ < 0$  we get that for any  $a \in (0, -\mathfrak{a}_+)$

$$(5.32) \quad \mathbb{E} [I_\Psi^{-a}(t)] \leq t^{-a} \mathbb{E} [e^{a \sup_{v \leq t} \xi v}] < \infty,$$



where the finiteness of the exponential moments of  $\sup_{v \leq t} \xi_v$  of order less than  $-\mathbf{a}_+$  follows from the definition of  $\mathbf{a}_+$ , see (2.4), that is  $\phi_+ \in \mathbf{A}_{(a, \infty)}$  for  $a \in (0, -\mathbf{a}_+)$ , see e.g. [9, Chapter VI] or [24, Chapter 4]. This of course settles (2.34), that is  $a \in (0, 1 - \mathbf{a}_+) \implies \mathbb{E} [I_{\Psi}^{-a}(t)] < \infty$ , when  $\mathbf{a}_+ = -\infty$ . We rewrite (5.6) as follows

$$(5.33) \quad \begin{aligned} \Psi(z) &= \tilde{\Psi}(z) + \Psi^*(z) \\ &= \left( \frac{\sigma^2}{2} z^2 + cz + \int_{-\infty}^1 (e^{zr} - 1 - zr \mathbb{I}_{\{|r| < 1\}}) \Pi(dr) \right) + \int_1^{\infty} (e^{zr} - 1) \Pi(dr), \end{aligned}$$

where  $\tilde{\Psi}, \Psi^* \in \overline{\mathcal{N}} \setminus \mathcal{N}_{\dagger}$  and from (5.33)  $(\xi_s)_{s \geq 0} = (\tilde{\xi}_s + \xi_s^*)_{s \geq 0}$  where  $\tilde{\xi}, \xi^*$  are independent Lévy processes with Lévy-Khintchine exponents  $\tilde{\Psi}, \Psi^*$  respectively. Set as usual  $\tilde{\Psi}(z) = -\tilde{\phi}_+(-z)\tilde{\phi}_-(z)$  and note from (5.33) that  $\tilde{\Psi} \in \mathbf{A}_{(0, \infty)}$  and hence  $\tilde{\phi}_+ \in \mathbf{A}_{(-\infty, 0)}$  or equivalently  $\mathbf{a}_{\tilde{\phi}_+} = -\infty$ , see (2.4). Similarly to (5.32) we get that, for any  $a \in (0, 1 - \mathbf{a}_+)$ ,

$$(5.34) \quad \mathbb{E} [I_{\Psi}^{-a}(t)] \leq \mathbb{E} [e^{a \sup_{v \leq t} \tilde{\xi}_v}] \mathbb{E} [I_{\Psi^*}^{-a}(t)].$$

However, since  $\mathbf{a}_{\tilde{\phi}_+} = -\infty$  we conclude from (5.32) that in fact  $\mathbb{E} [e^{a \sup_{v \leq t} \tilde{\xi}_v}] < \infty$  for any  $a > 0$ . If  $\Psi^* \equiv 0$  there is nothing to prove. So let  $h = \Pi\{(1, \infty)\} > 0$  and write  $\xi_s^* = \sum_{j=1}^{N_s} X_j$  where  $(N_s)_{s \geq 0}$  is a Poisson counting process with  $N_s \sim \text{Poisson}(hs)$  and  $(X_j)_{j \geq 1}$  are i.i.d. random variables with law

$$\mathbb{P}(X_1 \in dx) = \mathbb{I}_{\{x > 1\}} \frac{\Pi(dx)}{h}.$$

It is a well-known fact that

$$\mathbb{E} [e^{\lambda X_1}] < \infty \iff \lambda < -\mathbf{a}_+.$$

Set  $S_n = \sum_{j=1}^n X_j$  with  $S_0 = 0$ . Then

$$I_{\Psi^*}(t) = t \mathbb{I}_{\{N_t=0\}} + \mathbb{I}_{\{N_t>0\}} \left( \sum_{j=1}^{N_t} e_j e^{-S_{j-1}} + \left( t - \sum_{j=1}^{N_t} e_j \right) e^{-S_{N_t}} \right)$$

with  $(e_j)_{j \geq 1}$  a sequence of independent identically distributed random variables with exponential law of parameter  $h$ . Clearly then

$$(5.35) \quad \begin{aligned} \mathbb{E} [I_{\Psi}(t)^{-a}] &= t^{-a} \mathbb{P}(N_t = 0) + \sum_{n=1}^{\infty} \mathbb{E} \left[ \frac{1}{\left( \sum_{j=1}^n e_j e^{-S_{j-1}} + \left( t - \sum_{j=1}^n e_j \right) e^{-S_n} \right)^a}; N_t = n \right] \\ &= t^{-a} \mathbb{P}(N_t = 0) + \sum_{n=1}^{\infty} A_n. \end{aligned}$$

Note that

$$\{N_t = n\} = \left\{ \sum_{j=1}^n e_j \leq \frac{t}{2}; \sum_{j=1}^{n+1} e_j > t \right\} \cup \left\{ \sum_{j=1}^n e_j \in \left( \frac{t}{2}, t \right); \sum_{j=1}^{n+1} e_j > t \right\}.$$

We observe that with some  $C(t) < \infty$ , for any  $t > 0$ ,

$$(5.36) \quad \sup_{s \in (0, t]} \mathbb{P}(N_s \geq n - 1) \leq C(t) \frac{t^{n-1} h^{n-1}}{(n-1)!},$$

which is an elementary consequence of  $N_s \sim \text{Poisson}(sh)$ . We split the quantity  $A_n = A_n^{(1)} + A_n^{(2)}$  by considering the two possible mutually exclusive cases for the event  $\{N_t = n\}$  discussed above. In the first scenario we have the following sequence of relations

$$\begin{aligned}
(5.37) \quad A_n^{(1)} &= \mathbb{E} \left[ \frac{1}{\left( \sum_{j=1}^n e_j e^{-S_{j-1}} + \left( t - \sum_{j=1}^n e_j \right) e^{-S_n} \right)^a}; \sum_{j=1}^n e_j \leq \frac{t}{2}; \sum_{j=1}^{n+1} e_j > t \right] \\
&\leq \mathbb{E} \left[ \frac{1}{\left( e_1 + \frac{t}{2} e^{-S_n} \right)^a}; e_1 \leq \frac{t}{2} \right] \mathbb{P}(N_t \geq n-1) \\
&\stackrel{5.36}{\leq} C(t) \frac{h^n t^{n-1}}{(n-1)!} \mathbb{E} \left[ \int_0^{\frac{t}{2}} \frac{1}{\left( x + \frac{t}{2} e^{-S_n} \right)^a} e^{-hx} dx \right] \leq C(t) \frac{h^n t^{n-1}}{(n-1)!} \mathbb{E} \left[ \int_0^{\frac{t}{2}} \frac{1}{\left( x + \frac{t}{2} e^{-S_n} \right)^a} dx \right] \\
&\leq C(t) \frac{h^n t^{n-1}}{(n-1)!} \left( \frac{2^{a-1}(a-1)}{t^{a-1}} \mathbb{E} \left[ e^{(a-1)S_n} \right] \mathbb{I}_{\{a>1\}} + (\mathbb{E}[S_n] + \ln(4)) \mathbb{I}_{\{a=1\}} + \frac{t^{1-a}}{1-a} \mathbb{I}_{\{a \in (0,1)\}} \right) \\
&= C(t) \frac{h^n t^{n-1}}{(n-1)!} \left( \frac{2^{a-1}}{(a-1)t^{a-1}} \left( \mathbb{E} \left[ e^{(a-1)X_1} \right] \right)^n \mathbb{I}_{\{a>1\}} + (n\mathbb{E}[X_1] + \ln(4)) \mathbb{I}_{\{a=1\}} \right) \\
&\quad + C(t) \frac{h^n t^{n-1}}{(n-1)!} \frac{t^{1-a}}{1-a} \mathbb{I}_{\{a \in (0,1)\}},
\end{aligned}$$

where for the terms containing  $\mathbb{I}_{\{a=1\}}$  and  $\mathbb{I}_{\{a \in (0,1)\}}$  in the derivation of the last inequality we have used that  $S_n \geq n > 0$  since  $X_1 \geq 1$ . In the second scenario we observe that the following inclusion holds

$$(5.38) \quad \left\{ \sum_{j=1}^n e_j \in \left( \frac{t}{2}, t \right); \sum_{j=1}^{n+1} e_j > t \right\} \subseteq \bigcup_{j=1}^n \left\{ e_j \geq \frac{t}{2n} \right\} \cap \left\{ \sum_{1 \leq i \leq n; i \neq j} e_i < t \right\}.$$

Clearly then for any  $j$ , the events  $\{e_j \geq \frac{t}{2n}\}$  and  $\{\sum_{1 \leq i \leq n; i \neq j} e_i < t\}$  are independent and moreover

$$\mathbb{P} \left( \sum_{1 \leq i \leq n; i \neq j} e_i < t \right) = \mathbb{P} \left( \sum_{1 \leq i \leq n-1} e_i < t \right) = \mathbb{P}(N_t \geq n-1).$$

We are therefore able to estimate  $A_n^{(2)}$  using the relation between events in (5.38) in the following manner

$$\begin{aligned}
A_n^{(2)} &= \mathbb{E} \left[ \frac{1}{\left( \sum_{j=1}^n e_j e^{-S_{j-1}} + \left( t - \sum_{j=1}^n e_j \right) e^{-S_n} \right)^a}; \sum_{j=1}^n e_j \in \left( \frac{t}{2}, t \right); \sum_{j=1}^{n+1} e_j > t \right] \\
&\leq \left( \left| \frac{1}{1-a} \right| (2n)^a t^{1-a} \mathbb{I}_{\{a \neq 1\}} + \ln(2n) \mathbb{I}_{\{a=1\}} \right) \mathbb{P}(N_t \geq n-1) \\
&\quad + \mathbb{I}_{\{n>1\}} \left( \sum_{j=2}^n h \mathbb{E} \left[ \int_0^t \frac{1}{\left( x + \frac{t}{2n} e^{-S_{j-1}} \right)^a} dx \right] \right) \mathbb{P}(N_t \geq n-1),
\end{aligned}$$

where the terms in the second line correspond to the case  $\{e_1 \geq \frac{t}{2n}\}$  whereas the term in the third line corresponds to other scenarios, that is  $\bigcup_{j=2}^n \{e_j \geq \frac{t}{2n}\}$ , and a disintegration of  $e_1$ .

However, performing the integration and estimating precisely as in (5.37) we get with the help of (5.36) that

$$\begin{aligned}
(5.39) \quad A_n^{(2)} &\leq \left( \left| \frac{1}{1-a} \right| (2n)^a t^{1-a} \mathbb{I}_{\{a \neq 1\}} + \ln(2n) \mathbb{I}_{\{a=1\}} \right) \mathbb{P}(N_t \geq n-1) \\
&+ \mathbb{I}_{\{n>1\}} \mathbb{I}_{\{a>1\}} C(t) \frac{t^{n-1} h^n}{(n-1)!} \sum_{j=2}^n \frac{(2j)^{a-1}}{(a-1)t^{a-1}} \left( \mathbb{E} \left[ e^{(a-1)X_1} \right] \right)^{j-1} \\
&+ \mathbb{I}_{\{n>1\}} \mathbb{I}_{\{a=1\}} C(t) \frac{t^{n-1} h^n}{(n-1)!} \sum_{j=2}^n (j \mathbb{E}[X_1] + \ln(4j)) \\
&+ \mathbb{I}_{\{n>1\}} \mathbb{I}_{\{a \in (0,1)\}} C(t) \frac{t^{n-1} h^n}{(n-1)!} \sum_{j=2}^n \frac{t^{1-a}}{1-a} \mathbb{I}_{\{a \in (0,1)\}} \\
&\leq \left( \left| \frac{1}{1-a} \right| (2n)^a t^{1-a} \mathbb{I}_{\{a \neq 1\}} + \ln(2n) \mathbb{I}_{\{a=1\}} \right) \mathbb{P}(N_t \geq n-1) \\
&+ C(t) \frac{t^{n-a} h^n}{(n-1)!} \frac{(4n)^a}{a-1} \left( \mathbb{E} \left[ e^{(a-1)X_1} \right] \right)^{n-1} \mathbb{I}_{\{a>1\}} + C(t) \frac{t^{n-1} h^n}{(n-1)!} (n^2 + \ln(4n)) \mathbb{E}[X_1] \mathbb{I}_{\{a=1\}} \\
&+ C(t) \frac{t^{n-a} h^n}{(n-1)!} \frac{h}{1-a} n \mathbb{I}_{\{a \in (0,1)\}}.
\end{aligned}$$

Therefore from  $A_n = A_n^{(1)} + A_n^{(2)}$ , (5.37) and (5.39) applied in (5.35) we easily get that  $\mathbb{E}[I_\Psi(t)^{-a}] < \infty$  whenever:  $a \in (0, 1)$ ;  $a = 1$  and  $\mathbb{E}[X_1] < \infty$  and  $a > 1$  and  $\mathbb{E}[e^{(a-1)X_1}] < \infty$ . However, it is very well-known fact that

$$\mathbb{E}[X_1] < \infty \iff \mathbb{E}[\max\{\xi_1, 0\}] < \infty \iff |\Psi'(0^+)| < \infty$$

and if  $\mathbf{a}_+ < 0$  then

$$\mathbb{E} \left[ e^{(a-1)X_1} \right] < \infty \iff a \in (0, 1 - \mathbf{a}_+)$$

and

$$\mathbb{E} \left[ e^{\mathbf{a}_+ X_1} \right] < \infty \iff |\Psi(-\mathbf{a}_+)| < \infty \iff |\phi_+(\mathbf{a}_+)| < \infty.$$

Hence via (5.34) the relation (2.34) and the backward directions of (2.35) and (2.36) follow. Let us provide a lower bound for  $\mathbb{E}[I_\Psi^{-a}(t)]$  whenever  $a \geq 1$ . We again utilize the processes  $\tilde{\xi}, \xi^*$  in the manner

$$\begin{aligned}
\mathbb{E}[I_\Psi(t)^{-a}] &\geq \mathbb{E} \left[ I_\Psi(t)^{-a}; \sup_{v \leq t} |\tilde{\xi}_s| \leq 1; N_t = 1 \right] \\
&\geq e^{-a} \mathbb{P} \left( \sup_{v \leq t} |\tilde{\xi}_s| \leq 1 \right) \mathbb{E} [I_{\Psi^*}^{-a}(t); N_t = 1] \\
&\geq e^{-at} \mathbb{P} \left( \sup_{v \leq t} |\tilde{\xi}_s| \leq 1 \right) \mathbb{P}(e_2 > t) h e^{-ht} \mathbb{E} \left[ \int_0^t \frac{1}{(x + t e^{-X_1})^a} dx \right] \\
&= e^{-at} \mathbb{P} \left( \sup_{v \leq t} |\tilde{\xi}_s| \leq 1 \right) \mathbb{P}(e_2 > t) h e^{-ht} \\
&\quad \times \left( \left( \frac{t^{1-a}}{a-1} \mathbb{E} \left[ e^{(a-1)X_1} \right] - 1 \right) \mathbb{I}_{\{a>1\}} + \mathbb{E}[X_1] \mathbb{I}_{\{a=1\}} \right).
\end{aligned}$$

The very last term is infinity iff  $a - 1 > \mathbf{a}_+$ ,  $\mathbb{E}[X_1] = \infty$  and  $a = 1$ , or  $a - 1 = \mathbf{a}_+$  and  $\mathbb{E}[e^{\mathbf{a}_+ X_1}] = \infty$ . This shows the forward directions of (2.35) and (2.36) and relation (2.35) and (2.37). It remains to show (2.38). From  $A_n = A_n^{(1)} + A_n^{(2)}$  and (5.37) and (5.39) we get that  $t^a A_n = O(t)$  as  $t$  goes to zero and therefore from (5.35) we get that

$$\lim_{t \rightarrow 0} t^a \mathbb{E}[I_\Psi^{-a}(t)] = \lim_{t \rightarrow 0} \mathbb{P}(N_t = 0) = 1.$$

This establishes the validity of (2.38).

**5.8. Proof of Theorem 2.24.** Recall that  $\Psi \in \mathcal{N}^c = \overline{\mathcal{N}} \setminus \mathcal{N} = \{\Psi \in \overline{\mathcal{N}} : \phi_-(0) = 0\}$  and as usual set

$$\Psi_r(z) = \Psi(z) - r = -\phi_+^r(-z)\phi_-^r(z) \in \mathcal{N}, r \geq 0.$$

We also repeat that  $I_\Psi(t) = \int_0^t e^{-\xi s} ds$ ,  $t \geq 0$  and  $I_\Psi(t)$  is non-decreasing in  $t$ . The relation (2.40) then follows from the immediate bound

$$\mathbb{P}(I_{\Psi_r} \leq x) = r \int_0^\infty e^{-rt} \mathbb{P}(I_\Psi(t) \leq x) dt \geq (1 - e^{-1}) \mathbb{P}\left(I_\Psi\left(\frac{1}{r}\right) \leq x\right),$$

combined with the representation (2.23) with  $n = 0$  for  $\mathbb{P}(I_{\Psi_r} \leq x)$ , the identity  $\kappa_-(r)\kappa_+(r) = \phi_-^r(0)\phi_+^r(0) = -\Psi(0) = r$  which is valid since

$$\Psi \in \mathcal{N}^c \implies \Psi_r(0) = \Psi(0) - r = -r$$

and the fact that (A.4) holds. Thus, Theorem 2.24(1) is settled and we proceed with Theorem 2.24(2). Denote by  $\mathcal{M}_r^{*,a}$ ,  $a \in (0, 1)$ , the Mellin transform of the cumulative distribution function of the measure  $y^{-a} \mathbb{P}(I_{\Psi_r} \in dy) \mathbb{I}_{\{y>0\}}$ , say  $V_a^r$ . Following (5.25) we conclude that

$$(5.40) \quad \mathcal{M}_r^{*,a}(z) = -\frac{1}{z} \mathcal{M}_{I_{\Psi_r}}(z + 1 - a) = -\frac{\kappa_-(r)}{z} \mathcal{M}_{\Psi_r}(z + 1 - a), z \in \mathbb{C}_{(a-1,0)}$$

and at least  $\mathcal{M}_r^{*,a} \in \mathbf{A}_{(a-1,0)}$  since  $\mathcal{M}_{\Psi_r} \in \mathbf{A}_{(0,1)}$ , see Theorem 2.1(2.7). Similarly, as for any  $\Psi \in \overline{\mathcal{N}}$  we have that  $\mathbf{N}_\Psi > 0$ , we deduct that  $\lim_{|b| \rightarrow \infty} |b|^\beta |\mathcal{M}_r^{*,a}(c + ib)| = 0$  for some  $\beta \in (1, 1 + \mathbf{N}_\Psi)$  and any  $c \in (a - 1, 0)$ . Therefore, by Mellin inversion, for any  $x > 0$ ,

$$(5.41) \quad \begin{aligned} V_a^r(x) &= \int_0^x y^{-a} \mathbb{P}(I_{\Psi_r} \in dy) = \frac{x^{-c}}{2\pi} \int_{-\infty}^\infty x^{-ib} \mathcal{M}_r^{*,a}(c + ib) db \\ &= -\kappa_-(r) \frac{x^{-c}}{2\pi} \int_{-\infty}^\infty x^{-ib} \frac{\mathcal{M}_{\Psi_r}(c + 1 - a + ib)}{c + ib} db. \end{aligned}$$

However, since

$$(5.42) \quad V_a^r(x) = r \int_0^\infty e^{-rt} \int_0^x y^{-a} \mathbb{P}(I_\Psi(t) \in dy) dt,$$

we have that

$$\lim_{r \rightarrow 0} \int_0^\infty e^{-rt} \int_0^x y^{-a} \mathbb{P}(I_\Psi(t) \in dy) dt = \lim_{r \rightarrow 0} \left( -\frac{\kappa_-(r)}{r} \frac{x^{-c}}{2\pi} \int_{-\infty}^\infty x^{-ib} \frac{\mathcal{M}_{\Psi_r}(c + 1 - a + ib)}{c + ib} db \right).$$

From (5.55), (5.56) and (5.57) of Lemma 5.3 with  $\beta \in (0, \mathbf{N}_\Psi)$  we conclude that the dominated convergence theorem applies and yields that

$$\lim_{r \rightarrow 0} \frac{r}{\kappa_-(r)} \int_0^\infty e^{-rt} \int_0^x y^{-a} \mathbb{P}(I_\Psi(t) \in dy) dt = -\frac{x^{-c}}{2\pi} \int_{-\infty}^\infty x^{-ib} \frac{1}{c + ib} \mathcal{M}_\Psi(c + 1 - a + ib) db.$$

Let  $-\liminf_{t \rightarrow \infty} \xi_t = \overline{\lim}_{t \rightarrow \infty} \xi_t = \infty$  a.s. or alternatively  $\kappa_+(0) = \phi_+(0) = \phi_-(0) = \kappa_-(0) = 0$ . Assume also that  $\lim_{t \rightarrow \infty} \mathbb{P}(\xi_t < 0) = \rho \in [0, 1)$ . Then from the discussion succeeding [24, Chapter 7, (7.2.3)] (a chapter dedicated to the Spitzer's condition) we have that  $\kappa_- \in RV_\rho$ . Then, since  $\frac{\kappa_-(r)}{r} = \frac{1}{\kappa_+(r)}$ ,  $\kappa_+ \in RV_{1-\rho}$  with  $1 - \rho > 0$ , and

$$(5.43) \quad \int_0^x y^{-a} \mathbb{P}(I_\Psi(t) \in dy) = \mathbb{E} [I_\Psi^{-a}(t) \mathbb{I}_{\{I_\Psi(t) \leq x\}}]$$

is non-increasing in  $t$  for any fixed  $x > 0$  and  $a \in (0, 1)$ , from a classical Tauberian theorem and the monotone density theorem, see [9, Section 0.7], we conclude from (5.41) that, for any  $x > 0$  and any  $c \in (a - 1, 0)$ ,

$$(5.44) \quad \lim_{t \rightarrow \infty} t \kappa_+ \left( \frac{1}{t} \right) \int_0^x y^{-a} \mathbb{P}(I_\Psi(t) \in dy) = -\frac{x^{-c}}{2\pi\Gamma(1-\rho)} \int_{-\infty}^{\infty} x^{-ib} \frac{1}{c+ib} \mathcal{M}_\Psi(c+1-a+ib) db.$$

With  $t \kappa_+ \left( \frac{1}{t} \right) = \frac{1}{\kappa_-\left(\frac{1}{t}\right)}$  we deduce that

$$\frac{y^{-a} \mathbb{P}(I_\Psi(t) \in dy)}{\kappa_-\left(\frac{1}{t}\right)} \text{ converges vaguely to } \vartheta_a,$$

whose distribution function is simply the integral to the right-hand side of (5.44). To show that it converges weakly, and thus prove (2.41) and (2.42), we need only show that

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E} [I_\Psi^{-a}(t)]}{\kappa_-\left(\frac{1}{t}\right)} < \infty.$$

However, this is immediate from the fact that  $\mathcal{M}_{I_{\Psi_r}}(z)$ ,  $r > 0$ , is always well defined on  $\mathbb{C}_{(0,1)}$ , see (2.21) of Theorem 2.1,  $\lim_{r \rightarrow 0} \mathcal{M}_{\Psi_r}(1-a) = \mathcal{M}_\Psi(1-a)$ ,  $a \in (0, 1)$ , justified in (5.55) below, and utilizing again a Tauberian theorem and the monotone density theorem to

$$(5.45) \quad \mathcal{M}_{I_{\Psi_r}}(1-a) = \kappa_-(r) \mathcal{M}_{\Psi_r}(1-a) = r \int_0^\infty e^{-rt} \mathbb{E} [I_\Psi^{-a}(t)] dt.$$

By putting  $f \equiv 1$  in (2.41) we also prove (2.43) for any  $a \in (0, 1)$ . Next, assume that  $\mathbf{a}_+ < 0$  and fix any  $r > 0$ . For any  $a \in (0, 1 - \mathbf{a}_+)$  set

$$(5.46) \quad H(r, a) = \int_1^\infty e^{-rt} \mathbb{E} [I_\Psi^{-a}(t)] dt.$$

From Theorem 2.22 we have that  $\mathbb{E} [I_\Psi^{-a}(t)] < \infty$  for all  $t \geq 0$  and any  $a \in (0, 1 - \mathbf{a}_+)$ . Therefore from  $\mathbb{E} [I_\Psi^{-a}(1)] \geq \mathbb{E} [I_\Psi^{-a}(t)]$  for  $t \geq 1$  we conclude that

$$H(a, r) \leq \mathbb{E} [I_\Psi^{-a}(1)] \int_1^\infty e^{-rt} dt < \infty$$

and thus  $H(r, z)$  can be extended analytically so that  $H(r, \cdot) \in \mathbf{A}_{[0, 1 - \mathbf{a}_+]}$ . From (5.43) we immediately see that, for any  $x > 0$ ,

$$H(a, r) \geq \int_1^\infty e^{-rt} \int_0^x y^{-a} \mathbb{P}(I_\Psi(t) \in dy) dt$$

and we conclude, re-expressing (5.42) as

$$(5.47) \quad \begin{aligned} \frac{1}{r} V_a^r(x) &= \int_0^1 e^{-rt} \int_0^x y^{-a} \mathbb{P}(I_\Psi(t) \in dy) dt + \int_1^\infty e^{-rt} \int_0^x y^{-a} \mathbb{P}(I_\Psi(t) \in dy) dt \\ &= \int_0^1 e^{-rt} \int_0^x y^{-a} \mathbb{P}(I_\Psi(t) \in dy) dt + W_x(r, -a), \end{aligned}$$

that  $W_x(r, \cdot) \in \mathbf{A}_{(\mathbf{a}_+ - 1, 0]}$ . Next, since the analyticity of  $\Psi_r$  obviously coincides with that of  $\Psi$  we conclude that  $\mathbf{a}_+ = a_{\phi_+^r}$  and  $\mathbf{a}_- = a_{\phi_-^r}$  for any  $r \geq 0$ . Moreover, from  $\phi_+(0) = 0$  then  $\phi_+ < 0$  on  $(\mathbf{a}_+, 0)$  and  $\mathbf{u}_+ = 0$ , and since  $\lim_{r \rightarrow 0} \phi_+^r(a) = \phi_+(a)$  for any  $a > \mathbf{a}_+$ , see (C.2), we easily deduct that

$$\lim_{r \rightarrow 0} \mathbf{u}_{\phi_+^r} = \mathbf{u}_+ = 0$$

and for all  $r \leq r_0$ ,  $\mathbf{u}_{\phi_+^r} \in (\max\{-1, \mathbf{a}_+\}, 0)$ . Therefore since  $-\mathbf{u}_{\phi_+^r}$  is not an integer from Theorem 2.1 we conclude that for all  $r \leq r_0$ ,  $\mathcal{M}_{\Psi_r} \in \mathbf{M}_{(\mathbf{a}_+, 1 - \mathbf{a}_-)}$  and it has simple poles with residues  $\kappa_-(r) \frac{\prod_{k=1}^n \Psi_r(k)}{n!}$ , with  $\prod_{k=1}^0 = 1$ , at all non-positive integers  $-n$  such that  $-n > \mathbf{a}_+$ . Henceforth, for any  $-n_0 > \mathbf{a}_+$ ,  $n_0 \in \mathbb{N} \cup \{0\}$ , we have that

$$(5.48) \quad \mathcal{M}_{\Psi_r}(z) = \kappa_+(r) \sum_{n=0}^{n_0} \frac{\prod_{k=1}^n \Psi_r(k)}{n!} \frac{1}{z+n} + \mathcal{M}_{\Psi_r}^{(n_0)}(z)$$

with  $\mathcal{M}_{\Psi_r}^{(n_0)} \in \mathbf{A}_{(\max\{-n_0 - 1, \mathbf{a}_+\}, 1 - \mathbf{a}_-)}$ . Also as for any  $\Psi \in \overline{\mathcal{N}}$  we have that  $N_\Psi > 0$ , see Theorem 2.5, we conclude from (5.48) that at least for any  $\beta \in (0, \min\{N_{\Psi_r}, 1\})$  and any  $c \in (\max\{-n_0 - 1, \mathbf{a}_+\}, 1 - \mathbf{a}_-)$

$$(5.49) \quad \lim_{|b| \rightarrow \infty} |b|^\beta \left| \mathcal{M}_{\Psi_r}^{(n_0)}(c + ib) \right| = 0.$$

For  $a \in (0, 1)$ ,  $c \in (a - 1, 0)$  and  $-n_0 > \mathbf{a}_+$ ,  $n_0 \in \mathbb{N} \cup \{0\}$ , (5.41) together with (5.47), (5.48), (5.49) and  $\kappa_+(r)\kappa_-(r) = r$ , allow us to re-express (5.41) as follows

$$(5.50) \quad \begin{aligned} \frac{1}{r} V_a^r(x) &\stackrel{5.47}{=} \int_0^1 e^{-rt} \int_0^x y^{-a} \mathbb{P}(I_\Psi(t) \in dy) dt + W_x(r, -a) \\ &\stackrel{5.41}{=} -\frac{\kappa_-(r)}{r} \frac{1}{2\pi i} \int_{z \in \mathbb{C}_c} x^{-z} \frac{\mathcal{M}_{\Psi_r}(z + 1 - a)}{z} dz \\ &\stackrel{5.48}{=} \sum_{n=0}^{n_0} \frac{\prod_{k=1}^n \Psi_r(k)}{n!} \frac{1}{1 - a + n} (x^{1-a+n} \mathbb{I}_{\{x \leq 1\}} - \mathbb{I}_{\{x > 1\}}) \\ &\quad - \frac{\kappa_-(r)}{r} \frac{1}{2\pi i} \int_{z \in \mathbb{C}_c} x^{-z} \frac{\mathcal{M}_{\Psi_r}^{(n_0)}(z + 1 - a)}{z} dz, \end{aligned}$$

where the first term in the very last identity stems from the fact that for  $a \in (0, 1)$  the function  $-\frac{1}{z(z+1-a+n)}$  is the Mellin transform of the function  $\frac{1}{1-a+n} (x^{1-a+n} \mathbb{I}_{\{x \leq 1\}} - \mathbb{I}_{\{x > 1\}})$ . However, from  $W_x(r, \cdot) \in \mathbf{A}_{(\mathbf{a}_+ - 1, 0]}$  as noted beneath (5.47), the fact that  $\mathcal{M}_{\Psi_r}^{(n_0)} \in \mathbf{A}_{(\max\{-n_0 - 1, \mathbf{a}_+\}, 1 - \mathbf{a}_-)}$  and since we can choose  $c < 0$  as close to zero as we wish, we deduct upon substitution  $-a \mapsto \zeta$  in

(5.50) and equating the second and forth terms in (5.50) that as a function of  $\zeta$   
 (5.51)

$$\begin{aligned}
 W_x(r, \zeta) &+ \frac{\kappa_-(r)}{r} \frac{1}{2\pi i} \int_{z \in \mathbb{C}_c} x^{-z} \frac{\mathcal{M}_{\Psi_r}^{(n_0)}(z+1+\zeta)}{z} dz \\
 &= \sum_{n=0}^{n_0} \frac{\prod_{k=1}^n \Psi_r(k)}{n!} \frac{1}{1+\zeta+n} \left( x^{1+\zeta+n} \mathbb{I}_{\{x \leq 1\}} - \mathbb{I}_{\{x > 1\}} \right) - \int_0^1 e^{-rt} \int_0^x y^\zeta \mathbb{P}(I_\Psi(t) \in dy) dt \\
 &:= \sum_{n=0}^{n_0} \frac{\prod_{k=1}^n \Psi_r(k)}{n!} \frac{1}{1+\zeta+n} \left( x^{1+\zeta+n} \mathbb{I}_{\{x \leq 1\}} - \mathbb{I}_{\{x > 1\}} \right) - G_x(r, \zeta) \\
 &= \tilde{G}_x(r, \zeta) \in \mathbf{A}_{(\max\{-n_0-2, \mathfrak{a}_+-1\}, 0]},
 \end{aligned}$$

where

$$G_x(r, \zeta) = \int_0^1 e^{-rt} \int_0^x y^\zeta \mathbb{P}(I_\Psi(t) \in dy) dt$$

and  $\tilde{G}_x(r, \cdot) \in \mathbf{A}_{(\max\{-n_0-2, \mathfrak{a}_+-1\}, 0]}$  is a consequence of  $W_x(r, \zeta) \in \mathbf{A}_{(\mathfrak{a}_+-1, 0]}$  and the fact that  $\mathcal{M}_{\Psi_r}^{(n_0)} \in \mathbf{A}_{(\max\{-n_0-1, \mathfrak{a}_+\}, 1-\mathfrak{a}_+)}$ . Note that for any  $k \in \mathbb{N}$  using the Taylor formula for  $e^{-x}$

$$\begin{aligned}
 G_x(r, \zeta) &= \sum_{j=0}^k (-1)^j \frac{r^j}{j!} \int_0^1 t^j \int_0^x y^\zeta \mathbb{P}(I_\Psi(t) \in dy) dt + \int_0^1 f_{k+1}(t) \int_0^x y^\zeta \mathbb{P}(I_\Psi(t) \in dy) dt \\
 (5.52) \quad &:= \sum_{j=0}^k (-1)^j \frac{r^j}{j!} H_j(x, \zeta) + \tilde{H}_{k+1}(x, \zeta).
 \end{aligned}$$

However, as  $|\int_0^x y^\zeta \mathbb{P}(I_\Psi(t) \in dy)| \leq \mathbb{E}[I_\Psi(t)^{\operatorname{Re}(\zeta)}]$ ,  $\overline{\lim}_{t \rightarrow 0} t^{-k-1} f_{k+1}(t) < \infty$  and

$$\lim_{t \rightarrow 0} t^{-\operatorname{Re}(\zeta)} \mathbb{E}[I_\Psi(t)^{\operatorname{Re}(\zeta)}] = 1,$$

for any  $\operatorname{Re}(\zeta) \in (\mathfrak{a}_+ - 1, 0)$ , see Theorem 2.22(2.38), we conclude that

$$H_j(x, \cdot) \in \mathbf{A}_{(\max\{-1-j, -1+\mathfrak{a}_+\}, 0)} \text{ and } \tilde{H}_{k+1}(x, \cdot) \in \mathbf{A}_{(\max\{-2-k, -1+\mathfrak{a}_+\}, 0)}.$$

Set  $n'$  the largest integer smaller than  $1 - \mathfrak{a}_+$ . Then, if  $k < n'$ , from (5.51) applied with  $n_0 = n'$ , we deduce that  $H_k$  is meromorphic on  $(\mathfrak{a}_+ - 1, 0)$  with simple poles at  $\{-n', \dots, -k-1\}$ . Set in these instances

$$H_k(x, \zeta) = \sum_{j=k+1}^{n'} \frac{b(k, x)}{\zeta + j} + H'_k(x, \zeta)$$

with  $H'_k(x, \cdot) \in \mathbf{A}_{(\mathfrak{a}_+-1, 0)}$ . Then

$$G_x(r, \zeta) = \sum_{j=1}^{n'} \frac{P_{j,k}(x, r)}{\zeta + j} + Q_{n'}(x, r) H_{n'}^*(x, \zeta)$$

with  $P_{j,k}, j = 1, \dots, k$  and  $Q_k$  polynomials in  $r$  and  $H_{n'}^*(x, \cdot) \in \mathbf{A}_{(\mathfrak{a}_+-1, 0)}$ . However, since  $\Psi_r(z) = \Psi(z) - r$  we deduce that  $\prod_{k=1}^n \Psi_r(k)$  are polynomials in  $r$  and since from the definition of  $n'$ , (5.51) defines analytic function, that is  $\tilde{G}_x(r, \cdot) \in \mathbf{A}_{(\mathfrak{a}_+-1, 0]}$ , we conclude that

$$\sum_{n=0}^{n'} \frac{\prod_{k=1}^n \Psi_r(k)}{n!} \frac{1}{1+\zeta+n} \left( x^{1+\zeta+n} \mathbb{I}_{\{x \leq 1\}} - \mathbb{I}_{\{x > 1\}} \right) - G_x(r, \zeta) = -Q_{n'}(x, r) H_{n'}^*(x, \zeta).$$

Clearly then

$$\lim_{r \rightarrow 0} Q_{n'}(x, r) H_{n'}^*(x, \zeta) = Q_{n'}(x, 0) H_{n'}^*(x, \zeta)$$

and from (5.51) we conclude that, for any  $-a \in (\mathfrak{a}_+ - 1, 0)$ ,

$$(5.53) \quad \lim_{r \rightarrow 0} \left( W_x(r, -a) + \frac{\kappa_-(r)}{r} \frac{1}{2\pi i} \int_{z \in \mathbb{C}_c} x^{-z} \frac{\mathcal{M}_{\Psi_r}^{(n')}(z+1-a)}{z} dz \right) = -Q_{n'}(x, 0) H_{n'}^*(x, -a).$$

Since  $\kappa_+(r)\kappa_-(r) = r$  then from (5.48)

$$\frac{\kappa_-}{r} \left| \mathcal{M}_{\Psi_r}(z) - \mathcal{M}_{\Psi_r}^{(n')}(z) \right| \leq \kappa_+(r) \frac{\kappa_-(r)}{r} \sum_{n=0}^{n'} \left| \frac{\prod_{k=1}^n \Psi_r(k)}{n!} \frac{1}{z+n} \right| = O(1) \sum_{n=0}^{n'} \left| \frac{\prod_{k=1}^n \Psi(k)}{n!} \frac{1}{z+n} \right|$$

and therefore by our freedom for fixed  $-a \in (\mathfrak{a}_+ - 1, 0)$  to choose  $|c+1-a|$  to be non-integer and  $c \in (a-1+\mathfrak{a}_+, 0)$  we get that

$$(5.54) \quad \begin{aligned} & \frac{\kappa_-(r)}{r} \left| \int_{z \in \mathbb{C}_c} x^{-z} \frac{\mathcal{M}_{\Psi_r}(z+1-a)}{z} dz - \int_{z \in \mathbb{C}_c} x^{-z} \frac{\mathcal{M}_{\Psi_r}^{(n')}(z+1-a)}{z} dz \right| \\ & \leq x^{-c} O(1) \int_{-\infty}^{\infty} \sum_{n=0}^{n'} \left| \frac{\prod_{k=1}^n \Psi(k)}{n!} \frac{1}{1-a+c+ib+n} \right| \frac{db}{|c+ib|} = O(1), \end{aligned}$$

where in the last step we have invoked the dominated convergence theorem. Finally, from these observations (5.55), (5.56) and (5.57) of Lemma 5.3 we deduct from (5.53) and (5.54) that as  $r \rightarrow 0$

$$\begin{aligned} W_x(r, -a) &= \int_1^{\infty} e^{-rt} \int_0^x y^{-a} \mathbb{P}(I_{\Psi}(t) \in dx) dt \\ &= -\frac{\kappa_-(r)}{r} \frac{1}{2\pi i} \int_{z \in \mathbb{C}_c} x^{-z} \frac{\mathcal{M}_{\Psi_r}^{(n')}(z+1-a)}{z} dz - Q_{n'}(x, r) H_{n'}^*(x, -a) \\ &= -\frac{\kappa_-(r)}{r} \frac{1}{2\pi i} \int_{z \in \mathbb{C}_c} x^{-z} \frac{\mathcal{M}_{\Psi_r}^{(n')}(z+1-a)}{z} dz + \frac{\kappa_-(r)}{r} \frac{1}{2\pi i} \int_{z \in \mathbb{C}_c} x^{-z} \frac{\mathcal{M}_{\Psi_r}(z+1-a)}{z} dz \\ &\quad - Q_{n'}(x, r) H_{n'}^*(x, -a) - \frac{\kappa_-(r)}{r} \frac{1}{2\pi i} \int_{z \in \mathbb{C}_c} x^{-z} \frac{\mathcal{M}_{\Psi_r}(z+1-a)}{z} dz \\ &\stackrel{\circ}{\sim} -\frac{\kappa_-(r)}{r} \frac{1}{2\pi i} \int_{z \in \mathbb{C}_c} x^{-z} \frac{\mathcal{M}_{\Psi}(z+1-a)}{z} dz \end{aligned}$$

and we conclude (2.41) and (2.42) as in the case  $a \in (0, 1)$ . We show, for  $a \in (0, 1 - \mathfrak{a}_+)$  that

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[I_{\Psi}^{-a}(t)]}{\kappa_-(\frac{1}{t})} = \vartheta_a(\mathbb{R}^+) < \infty$$

by decomposing (5.45) precisely as  $V_a^T(x)$  in (5.50) and proceeding as there. This also verifies the expression for  $\vartheta_a(\mathbb{R}^+)$  in (2.43).

The proof above is relies on the ensuing claims.

**Lemma 5.3.** *Let  $\Psi \in \overline{\mathcal{N}}$  and for any  $r \geq 0$ ,  $\Psi_r(z) = \Psi(z) - r$ . Fix  $a \in (\mathfrak{a}_+, 1)$  such that  $-a \notin \mathbb{N}$ . Then for any  $z \in \mathbb{C}_a$  we have that*

$$(5.55) \quad \lim_{r \rightarrow 0} \mathcal{M}_{\Psi_r}(z) = \mathcal{M}_{\Psi}(z).$$



Moreover, for any  $\widehat{b} > 0$  and  $\mathfrak{r} < \infty$ ,

$$(5.56) \quad \sup_{0 \leq r \leq \mathfrak{r}} \sup_{|b| \leq \widehat{b}} |\mathcal{M}_{\Psi_r}(a + ib)| < \infty.$$

Finally, for any  $\beta < \mathbf{N}_{\Psi}$ , we have that

$$(5.57) \quad \overline{\lim}_{|b| \rightarrow \infty} |b|^\beta \sup_{0 \leq r \leq \mathfrak{r}} |\mathcal{M}_{\Psi_r}(a + ib) - \mathcal{M}_{\Psi}(a + ib)| = 0.$$

*Proof.* Let  $r \geq 0$ . Set  $\Psi_r(z) = \Psi(z) - r = -\phi_+^r(-z)\phi_-^r(z)$ . Since the analyticity of  $\Psi_r$  obviously coincides with that of  $\Psi$  we conclude that  $\mathfrak{a}_+ = a_{\phi_+^r}$  and  $\mathfrak{a}_- = a_{\phi_-^r}$  for any  $r \geq 0$ . We start with some preparatory work by noting that for  $a \in (\mathfrak{a}_+, 1)$  and any  $r \geq 0$ ,

$$(5.58) \quad \sup_{b \in \mathbb{R}} |W_{\phi_-^r}(1 - a - ib)| \leq W_{\phi_-^r}(1 - a),$$

since, from Definition 3.1,  $W_{\phi_-^r}$  is the moment transform of the random variable  $Y_{\phi_-^r}$ . Also, for any non-integer  $a \in (\mathfrak{a}_+, 1)$ ,  $z = a + ib \in \mathbb{C}_a$  and any  $r \geq 0$  we get from (3.25) that

$$(5.59) \quad \frac{\Gamma(a + ib)}{W_{\phi_+^r}(a + ib)} = \left( \prod_{j=0}^{a^- - 1} \frac{\phi_+^r(a + j + ib)}{a + j + ib} \right) \frac{\Gamma(a + a^- + ib)}{W_{\phi_+^r}(a + a^- + ib)},$$

where we recall that  $c^- = (\lfloor -c \rfloor + 1) \mathbb{I}_{\{c \leq 0\}}$ . This leads to

$$(5.60) \quad \sup_{b \in \mathbb{R}} \left| \frac{\Gamma(a + ib)}{W_{\phi_+^r}(a + ib)} \right| \leq \sup_{b \in \mathbb{R}} \left( \prod_{j=0}^{a^- - 1} \frac{|\phi_+^r(a + j + ib)|}{|a + j + ib|} \right) \frac{\Gamma(a + a^-)}{W_{\phi_+^r}(a + a^-)}$$

since for  $a > 0$ ,  $\frac{\Gamma(a + ib)}{W_{\phi_+^r}(a + ib)}$  is the Mellin transform of  $I_{\phi_+^r}$ , see the proof of Theorem 2.27 in section 5.1. Next, observe from (3.19) that for any  $0 \leq r \leq \mathfrak{r}$ ,  $z = a + ib$  and fixed  $a > 0$

$$(5.61) \quad \sup_{0 \leq r \leq \mathfrak{r}} |W_{\phi_+^r}(z)| = \sup_{0 \leq r \leq \mathfrak{r}} \left( \frac{\sqrt{\phi_+^r(1)}}{\sqrt{\phi_+^r(a)\phi_+^r(1+a)|\phi_+^r(z)|}} e^{G_{\phi_+^r}(a) - A_{\phi_+^r}(z)} e^{-E_{\phi_+^r}(z) - R_{\phi_+^r}(a)} \right).$$

First, the error term, namely the product term above, is uniformly bounded over the whole class of Bernstein functions, see (3.18). Second, from Proposition C.1, we have that

$$\lim_{r \rightarrow 0} \phi_+^r(1) = \phi_+(1); \quad \lim_{r \rightarrow 0} \phi_+^r(a) = \phi_+(a) \quad \text{and} \quad \lim_{r \rightarrow 0} \phi_+^r(1 + a) = \phi_+(1 + a).$$

Similarly, from (3.12)  $\lim_{r \rightarrow 0} G_{\phi_+^r}(a) = G_{\phi_+}(a)$ . Therefore, (5.61) is simplified to

$$(5.62) \quad \sup_{0 \leq r \leq \mathfrak{r}} |W_{\phi_+^r}(z)| \asymp \sup_{0 \leq r \leq \mathfrak{r}} \frac{1}{|\phi_+^r(z)|} e^{-A_{\phi_+^r}(z)}.$$

However, according to Theorem 3.3(1),  $A_{\phi_+^r}(z)$  are non-increasing in  $r$ . Henceforth, (5.62) yields that, for any  $z = a + ib$ ,  $a > 0$ ,

$$(5.63) \quad C'_a \inf_{0 \leq r \leq \mathfrak{r}} \left( \frac{\sqrt{|\phi_+(z)|}}{\sqrt{|\phi_+^r(z)|}} \right) |W_{\phi_+}(z)| \leq \sup_{0 \leq r \leq \mathfrak{r}} |W_{\phi_+^r}(z)| \leq C_a \sup_{0 \leq r \leq \mathfrak{r}} \left( \frac{\sqrt{|\phi_+^r(z)|}}{\sqrt{|\phi_+^r(z)|}} \right) |W_{\phi_+^r}(z)|,$$

where  $C_a, C'_a$  are two absolute constants. Next, from Proposition C.1(C.2) we have, for any  $z \in \mathbb{C}_{(0,\infty)}$ , that  $\lim_{r \rightarrow 0} \phi_{\pm}^r(z) = \phi_{\pm}(z)$  and hence from Lemma 3.13 one obtains, for any  $z \in \mathbb{C}_{(0,\infty)}$ ,

$$\lim_{r \rightarrow 0} W_{\phi_{\pm}^r}(z) = W_{\phi_{\pm}}(z).$$

Also from Proposition C.1(C.2) and (5.59) we get that for any non-integer  $a \in (\mathbf{a}_+, 1)$  and fixed  $z \in \mathbb{C}_a$

$$\begin{aligned} \lim_{r \rightarrow 0} \frac{\Gamma(z)}{W_{\phi_{\pm}^r}(z)} &= \lim_{r \rightarrow 0} \left( \prod_{j=0}^{a^{\rightarrow}-1} \frac{\phi_{\pm}^r(z+j)}{|z+j|} \frac{\Gamma(z+a^{\rightarrow})}{W_{\phi_{\pm}^r}(z+a^{\rightarrow})} \right) \\ &= \prod_{j=0}^{a^{\rightarrow}-1} \frac{\phi_{\pm}(z+j)}{|z+j|} \frac{\Gamma(z+a^{\rightarrow})}{W_{\phi_{\pm}}(z+a^{\rightarrow})} = \frac{\Gamma(z)}{W_{\phi_{\pm}}(z)}. \end{aligned}$$

Recall (2.6)

$$\mathcal{M}_{\Psi}(z) = \phi_{-}(0) \frac{\Gamma(z)}{W_{\phi_{+}}(z)} W_{\phi_{-}}(1-z).$$

Therefore, have that

$$\begin{aligned} \lim_{r \rightarrow 0} \mathcal{M}_{\Psi_r}(z) &= \lim_{r \rightarrow 0} \phi_{-}^r(0) \frac{\Gamma(z)}{W_{\phi_{\pm}^r}(z)} W_{\phi_{\pm}^r}(1-z) \\ &= \phi_{-}(0) \frac{\Gamma(z)}{W_{\phi_{+}}(z)} W_{\phi_{-}}(1-z) \\ &= \mathcal{M}_{\Psi}(z), \quad z \in \mathbb{C}_a, \end{aligned}$$

and (5.55) follows. Next, (5.59) and (5.63) give with the help of (2.6) that, for any  $\mathfrak{r} \in \mathbb{R}^+$ ,

(5.64)

$$\begin{aligned} \sup_{0 \leq r \leq \mathfrak{r}} |\mathcal{M}_{\Psi_r}(a+ib)| &= \sup_{0 \leq r \leq \mathfrak{r}} \prod_{j=0}^{a^{\rightarrow}-1} \frac{|\phi_{\pm}^r(a+j+ib)|}{|a+j+ib|} \frac{|\Gamma(a+a^{\rightarrow}+ib)|}{|W_{\phi_{\pm}^r}(a+a^{\rightarrow}+ib)|} |W_{\phi_{\pm}^r}(1-a-ib)| \\ &\leq C_a \left[ \sup_{0 \leq r \leq \mathfrak{r}} \frac{\sqrt{|\phi_{\pm}^r(a+a^{\rightarrow}+ib)|}}{\sqrt{|\phi_{\pm}(a+a^{\rightarrow}+ib)|}} \sup_{0 \leq r \leq \mathfrak{r}} \frac{\sqrt{|\phi_{\pm}^r(1-a-ib)|}}{\sqrt{|\phi_{\pm}^r(1-a-ib)|}} \right. \\ &\quad \times \sup_{0 \leq r \leq \mathfrak{r}} \prod_{j=0}^{a^{\rightarrow}-1} \frac{|\phi_{\pm}^r(a+j+ib)|}{|a+j+ib|} \\ &\quad \left. \times \frac{|\Gamma(a+a^{\rightarrow}+ib)|}{|W_{\phi_{\pm}(a+a^{\rightarrow}+ib)|}} |W_{\phi_{\pm}^r}(1-a-ib)| \right] \\ &= C_a (J_1(b) \times J_2(b) \times J_3(b)). \end{aligned}$$

However, Proposition C.1(C.2) triggers that

$$\begin{aligned} \sup_{|b| \leq \widehat{b}} J_1(b) &= \sup_{|b| \leq \widehat{b}} \left( \sup_{0 \leq r \leq \mathfrak{r}} \frac{\sqrt{|\phi_{\pm}^r(a+a^{\rightarrow}+ib)|}}{\sqrt{|\phi_{\pm}(a+a^{\rightarrow}+ib)|}} \sup_{0 \leq r \leq \mathfrak{r}} \frac{\sqrt{|\phi_{\pm}^r(1-a-ib)|}}{\sqrt{|\phi_{\pm}^r(1-a-ib)|}} \right) \\ &\leq \sup_{|b| \leq \widehat{b}} \left( \sup_{0 \leq r \leq \mathfrak{r}} \frac{\sqrt{|\phi_{\pm}^r(a+a^{\rightarrow}+ib)|}}{\sqrt{|\phi_{\pm}(a+a^{\rightarrow})|}} \sup_{0 \leq r \leq \mathfrak{r}} \frac{\sqrt{|\phi_{\pm}^r(1-a-ib)|}}{\sqrt{|\phi_{\pm}^r(1-a)|}} \right) \end{aligned}$$

since  $1 - a > 0$ ,  $a + a^\rightarrow > 0$  and (3.32) holds, that is  $\operatorname{Re}(\phi(a + ib)) \geq \phi(a) > 0$ . The same is valid for  $\sup_{|b| \leq b} J_3(b)$  (resp.  $\sup_{|b| \leq b} J_2(b)$ ) thanks to (5.58), (5.60) and  $\lim_{r \rightarrow 0} W_{\phi_+^r}(z) = W_{\phi_+}(z)$  (resp. (C.2)). Henceforth, (5.56) follows. It remains to show (5.57). Let  $\mathbf{N}_\Psi = \infty$  first. We note that for any  $r > 0$ ,

$$\Psi \in \mathcal{N}_\infty \iff \Psi_r \in \mathcal{N}_\infty, \text{ or equivalently, } \mathbf{N}_\Psi = \infty \iff \mathbf{N}_{\Psi_r} = \infty.$$

This is due to the fact that the decay of  $|\mathcal{M}_\Psi|$  along complex lines is only determined by the Lévy triplet  $(c, \sigma^2, \Pi)$ , see (5.6), which is unaffected in this case. Henceforth, we are ready to consider the terms in (5.64) and observe that for any  $\beta > 0$ ,

$$(5.65) \quad \overline{\lim}_{|b| \rightarrow \infty} |b|^\beta J_3(b) = \overline{\lim}_{|b| \rightarrow \infty} |b|^\beta \frac{|\Gamma(a + a^\rightarrow + ib)|}{|W_{\phi_+}(a + a^\rightarrow + ib)|} |W_{\phi_+^\varepsilon}(1 - a - ib)| = 0,$$

because if

$$\overline{\lim}_{|b| \rightarrow \infty} |b|^\beta \frac{|\Gamma(a + a^\rightarrow + ib)|}{|W_{\phi_+}(a + a^\rightarrow + ib)|} > 0$$

then from Theorem 2.5(1) and Proposition C.1  $\phi_+ \in \mathcal{B}_P$ ,  $\bar{\mu}_+(0) < \infty$ , which implies that  $\phi_+^\varepsilon \in \mathcal{B}_P$ ,  $\bar{\mu}_+^\varepsilon(0) < \infty$  and hence

$$\lim_{|b| \rightarrow \infty} |b|^\beta |W_{\phi_+^\varepsilon}(1 - a - ib)| = 0.$$

Next from (C.3) and Proposition 3.14(3) we deduct that

$$(5.66) \quad \begin{aligned} \overline{\lim}_{|b| \rightarrow \infty} J_2(b) &= \overline{\lim}_{|b| \rightarrow \infty} \sup_{0 \leq r \leq \tau} \left( \prod_{j=0}^{a^\rightarrow - 1} \frac{|\phi_+^r(a + j + ib)|}{|a + j + ib|} \right) \\ &\leq \overline{\lim}_{|b| \rightarrow \infty} \left( \prod_{j=0}^{a^\rightarrow - 1} \frac{\sup_{0 \leq r \leq \tau} (|\phi_+^r(a + j + ib) - \phi_+(a + j + ib)|) + |\phi_+(a + j + ib)|}{|b|} \right) < \infty. \end{aligned}$$

Finally, from (3.32), (C.2), (C.3) and Proposition 3.14(3) we conclude that

$$(5.67) \quad \begin{aligned} \overline{\lim}_{|b| \rightarrow \infty} \frac{J_1(b)}{|b|^2} &= \overline{\lim}_{|b| \rightarrow \infty} \frac{1}{|b|^2} \sup_{0 \leq r \leq \tau} \frac{\sqrt{|\phi_+^r(a + a^\rightarrow + ib)|}}{\sqrt{|\phi_+(a + a^\rightarrow + ib)|}} \sup_{0 \leq r < \tau} \frac{\sqrt{|\phi_+^\varepsilon(1 - a - ib)|}}{\sqrt{|\phi_+^\varepsilon(1 - a - ib)|}} \\ &\leq C_a \sup_{0 \leq r \leq \tau} \frac{1}{\sqrt{|\phi_+(a + a^\rightarrow)|}} \frac{1}{\sqrt{|\phi_+^\varepsilon(a + a^\rightarrow)|}} < \infty. \end{aligned}$$

Collecting the estimates (5.65), (5.66) and (5.67) we prove (5.57) when  $\mathbf{N}_\Psi = \infty$  since from (5.64)

$$\lim_{|b| \rightarrow \infty} |b|^\beta \sup_{0 \leq r \leq \tau} |\mathcal{M}_{\Psi_r}(a + ib)| \leq C_a \lim_{|b| \rightarrow \infty} \frac{J_1(b)}{|b|^2} J_2(b) |b|^{\beta+2} J_3(b) = 0.$$

Assume next that  $\mathbf{N}_\Psi < \infty$  which triggers from Theorem 2.5(1) and Proposition C.1 that  $\phi_+$ ,  $\phi_+^\varepsilon \in \mathcal{B}_P$  with  $\mathbf{d}_+ = \mathbf{d}_+^\varepsilon, \forall r > 0$ ,  $\phi_-$ ,  $\phi_-^\varepsilon \in \mathcal{B}_P^c$  and  $\bar{\Pi}(0) < \infty$ . The latter implies that  $\bar{\mu}_\pm^\varepsilon(0) < \infty, r \geq 0$ . Thus, from (C.1) and (C.2) of Proposition C.1, we conclude, from (2.16), that

$$(5.68) \quad \begin{aligned} \lim_{r \rightarrow 0} \mathbf{N}_{\Psi_r} &= \lim_{r \rightarrow 0} \left( \frac{v_-^r(0^+)}{\phi_-^r(0) + \bar{\mu}_-^r(0)} \right) + \lim_{r \rightarrow 0} \frac{\phi_+^r(0) + \bar{\mu}_+^r(0)}{\mathbf{d}_+} \\ &= \left( \frac{v_-(0^+)}{\phi_-(0) + \bar{\mu}_-(0)} \right) + \frac{\phi_+(0) + \bar{\mu}_+(0)}{\mathbf{d}_+} = \mathbf{N}_\Psi, \end{aligned}$$

wherein it has not only been checked that  $\lim_{r \rightarrow 0} v^r(0^+) = v_-(0^+)$  yet. However, from Remark 2.6 we have that  $v^r(0^+) = \int_0^\infty u^r(y) \Pi_-(dy)$ , since  $\Pi^r = \Pi$ , whenever  $N_{\Psi^r} < \infty$ . Also, in this case, from (4.23), we have since  $\mathbf{d}_+ = \mathbf{d}_+^r$  that

$$u_+^r(y) = \frac{1}{\mathbf{d}_+} + \sum_{j=1}^{\infty} \frac{(-1)^j}{\mathbf{d}_+^{j+1}} (\mathbf{1} * (\phi_+^r(0) + \bar{\mu}_+^r)^{*j})(y) = \frac{1}{\mathbf{d}_+} + \tilde{u}_+^r(y), \quad y \geq 0.$$

The infinite sum above is locally uniformly convergent, see the proof of [26, Proposition 1], and therefore we can show using  $\lim_{r \rightarrow 0} \phi_+^r(0) = \phi_+(0)$ ,  $\lim_{r \rightarrow 0} \bar{\mu}_+^r = \bar{\mu}_+$  and (C.1) and (C.2) of Proposition C.1, that  $\lim_{r \rightarrow 0} v^r(0^+) = v_-(0^+)$ . Thus, (5.68) holds true. Note that since  $\mathbf{d}_+ > 0$  the Lévy process underlying  $\Psi^r$  is not a compound Poisson process and hence from Lemma A.1 we have that that  $\mu_{\pm}^r(dy) = \int_0^\infty e^{-rt - \Psi(0)t} \mu_{\pm}^\sharp(dt, dy)$ , where  $\mu^\sharp$  stands for the Lévy measure of the conservative Lévy process underlying  $\Psi^\sharp(z) = \Psi(z) - \Psi(0) = \Psi^r(z)$ . Therefore, in the sense of measures on  $(0, \infty)$ ,  $\mu_{\pm}^r(dy) \leq \mu_{\pm}^\sharp(dy)$ , for all  $r \geq 0$ . Since  $\bar{\mu}_+^r(0) < \infty$ ,  $r \geq 0$ , and  $\phi_-, \phi_-^r \in \mathcal{B}_P^c$  we conclude from Proposition C.1(C.2) that for any  $a > \mathbf{a}_+$  and  $\tau > 0$

$$(5.69) \quad \sup_{b \in \mathbb{R}} \sup_{0 \leq r \leq \tau} |\phi_-^r(a + ib)| \leq \sup_{0 \leq r \leq \tau} (\phi_-^r(0)) + \int_0^\infty (e^{ay} + 1) \mu_-^\sharp(dy) < \infty.$$

Also from  $\mathbf{d}_+^r = \mathbf{d}_+ > 0$  and  $\bar{\mu}_+^r(0) < \infty$ ,  $r \geq 0$ , we obtain, for fixed  $a > \mathbf{a}_+$ ,  $-a \notin \mathbb{N}$ , that

$$(5.70) \quad \sup_{b \in \mathbb{R}} \sup_{0 \leq r \leq \tau} \frac{|\phi_+^r(a + ib)|}{|a + ib|} \leq \sup_{0 \leq r \leq \tau} \frac{\phi_+^r(0)}{|a|} + \mathbf{d}_+ + \int_0^\infty (e^{ay} + 1) \mu_+^\sharp(dy) < \infty.$$

Therefore from (5.70)

$$(5.71) \quad \overline{\lim}_{|b| \rightarrow \infty} J_2(b) = \overline{\lim}_{|b| \rightarrow \infty} \sup_{0 \leq r \leq \tau} \left( \prod_{j=0}^{a^- - 1} \frac{|\phi_+^r(a + j + ib)|}{|a + j + ib|} \right) < \infty$$

and from (5.69) and (3.32)

$$(5.72) \quad \begin{aligned} \overline{\lim}_{|b| \rightarrow \infty} J_1(b) &= \overline{\lim}_{|b| \rightarrow \infty} \sup_{0 \leq r \leq \tau} \frac{\sqrt{|\phi_+^r(a + a^- + ib)|}}{\sqrt{|\phi_+(a + a^- + ib)|}} \sup_{0 \leq r < \tau} \frac{\sqrt{|\phi_-^r(1 - a - ib)|}}{\sqrt{|\phi_-^r(1 - a - ib)|}} \\ &\leq C_a \sup_{0 \leq r \leq \tau} \frac{1}{\sqrt{|\phi_+(a + a^-)|}} \frac{1}{\sqrt{|\phi_-^r(a + a^-)|}} < \infty. \end{aligned}$$

Relations (5.71) and (5.72) allow the usage of (5.64) to the effect that

$$(5.73) \quad \sup_{0 \leq r \leq \tau} \sup_{|b| \leq \hat{b}} |\mathcal{M}_{\Psi^r}(a + ib)| \leq C_a \frac{|\Gamma(a + a^- + ib)|}{|W_{\phi_+}(a + a^- + ib)|} |W_{\phi_-^r}(1 - a - ib)|.$$

Then (5.57) follows from (5.73) and (5.68) as  $\tau$  can be chosen as small as we wish and thus  $N_{\Psi^\tau}$  as close as we need to  $N_\Psi$ . This concludes the proof of this lemma.  $\square$

## 6. INTERTWINING BETWEEN SELF-SIMILAR SEMIGROUPS AND FACTORIZATION OF LAWS

**6.1. Proof of Theorem 2.27.** As in the case  $\bar{\mathbf{a}}_- < 0$ , see section 5.1, we recognize  $\frac{\Gamma(z)}{W_{\phi_+}(z)}$  as the Mellin transform of the random variable  $I_{\phi_+}$  and  $\phi_-(0)W_{\phi_-}(1 - z)$  as the Mellin transform of  $X_{\phi_-}$  as defined in (5.2). This leads to the first factorization (2.44) of Theorem 2.27. Next, we proceed with the proof of the second identity in law of Theorem 2.27. We simply express in (2.21)

the product representations of the functions  $W_{\phi_+}, W_{\phi_-}, \Gamma$ , see (3.9), to obtain, for  $z \in \mathbb{C}_{(-1,0)}$ , that

$$(6.1) \quad \begin{aligned} \mathcal{M}_{I_\Psi}(z+1) &= \mathbb{E}[I_\Psi^z] = \phi_-(0) \frac{\Gamma(z+1)}{W_{\phi_+}(z+1)} W_{\phi_-}(-z) \\ &= \frac{e^{z\left(\gamma_{\phi_+} + \gamma_{\phi_-} - \gamma + 1 - \frac{\phi'_+(1)}{\phi_+(1)}\right)}}{\phi_-(-z)} \frac{\phi_-(0)}{\phi_+(1)(1+z)} \prod_{k=2}^{\infty} \frac{\phi_-(k-1)}{\phi_-(k-1-z)} \frac{k\phi_+(k+z)}{\phi_+(k)(k+z)} C_\Psi^z(k), \end{aligned}$$

where

$$C_\Psi(k) = e^{\left(\frac{\phi'_-(k-1)}{\phi_-(k-1)} + \frac{1}{k} - \frac{\phi'_+(k)}{\phi_+(k)}\right)}.$$

Performing a change of variable in Proposition 3.14(5) (resp. in the expression (3.3)), we get, recalling that  $\Upsilon_-(dv) = U_-(\ln(v))$ ,  $v > 1$ , is the image of  $U_-$  via the mapping  $y \mapsto \ln y$  that

$$\begin{aligned} \int_1^\infty y^z \Upsilon_-(dy) &= \frac{1}{\phi_-(-z)} \\ \int_0^1 y^z (\bar{\mu}_+(-\ln y) dy + \phi_+(0) dy + \mathbf{d}_+ \delta_1(dy)) &= \frac{\phi_+(1+z)}{(1+z)}. \end{aligned}$$

Note that the last identities prove (2.45), that is

$$\begin{aligned} \mathbb{P}(X_\Psi \in dx) &= \frac{1}{\phi_+(1)} (\bar{\mu}_+(-\ln x) dx + \phi_+(0) dx + \mathbf{d}_+ \delta_1(dx)), \quad x \in (0, 1) \\ \mathbb{P}(Y_\Psi \in dx) &= \phi_-(0) \Upsilon_-(dx), \quad x > 1 \end{aligned}$$

are probability measures on  $\mathbb{R}^+$ . The in it is trivial that, for any  $z \in i\mathbb{R}$ ,

$$\int_0^\infty x^z \mathbb{P}(X_\Psi \times Y_\Psi \in dx) = \frac{\phi_-(0)}{\phi_+(1)} \frac{\phi_+(1+z)}{\phi_-(-z)(1+z)}.$$

Then it is clear that for  $k = 1, \dots$ , we have that

$$\mathbb{E}[f(\mathfrak{B}_k X_\Psi)] \mathbb{E}[f(\mathfrak{B}_{-k} Y_\Psi)] = \frac{\phi_-(k)}{\phi_-(k-z)} \frac{(k+1)}{\phi_+(k+1)} \frac{\phi_+(k+1+z)}{(k+1+z)},$$

where we recall that

$$\mathbb{E}[f(\mathfrak{B}_k X)] = \frac{\mathbb{E}[X^k f(X)]}{\mathbb{E}[X^k]}$$

and evidently  $\mathbb{E}[X_\Psi^k] = \phi_+(1) \frac{\phi_+(k+1)}{k+1}$  and  $\mathbb{E}[Y_\Psi^{-k}] = \frac{\phi_-(0)}{\phi_-(k)}$ . This concludes the proof.  $\square$

**6.2. Proof of Theorem 2.29.** Let  $\Psi \in \mathcal{N}_m$ . If  $\Psi(0^+) \in (0, \infty)$ , that is, the underlying Lévy process drifts to infinity, (2.47) and hence (2.46) can be verified directly from [16] wherein it is shown that  $\mathbb{E}[f(V_\Psi)] = \frac{1}{\mathbb{E}[I_\Psi^{-1}]} \mathbb{E}\left[\frac{1}{I_\Psi} f\left(\frac{1}{I_\Psi}\right)\right]$  for any  $f \in \mathcal{C}_0([0, \infty))$  and (2.21). Indeed from the latter we easily get that  $\mathbb{E}[I_\Psi^{-1}] = \phi_-(0) \phi'_+(0^+)$ . Then a substitution yields the result. If  $\Psi(0^+) = 0$  that is the underlying process oscillates we proceed by approximation. Set  $\Psi_\tau(z) = \Psi(z) + \tau z$  and note that  $\Psi'_\tau(0^+) = \tau > 0$ . Then (2.47) and hence (2.46) are valid for  $\Psi_\tau(z) = -\phi_\tau^\tau(-z) \phi_\tau^\tau(z)$ . From the celebrated Fristedt's formula, see (A.2) below, Lemma A.1 and fact that the underlying process is conservative, that is  $\Psi(0) = 0$ , we get that

$$\phi_\tau^\tau(z) = h^\tau(0) e^{\int_0^\infty \int_0^\infty (e^{-t} - e^{-zx}) \frac{\mathbb{P}(\xi_t + \tau t \in dx)}{t} dt}, \quad z \in \mathbb{C}_{[0, \infty)},$$

where  $h^\tau(0) = 1$  since the Lévy process  $\xi^\tau$  corresponding to  $\Psi_\tau$  is not a compound Poisson process, see (A.3). Then as  $\lim_{\tau \rightarrow 0} \mathbb{P}(\xi_t + \tau t \in \pm dx) = \mathbb{P}(\xi_t \in \pm dx)$  weakly on  $[0, \infty)$  we conclude that  $\lim_{\tau \rightarrow 0} \phi_\pm^\tau(z) = \phi_\pm(z)$ ,  $z \in \mathbb{C}_{[0, \infty)}$ . This together with the obvious  $\lim_{\tau \rightarrow 0} \Psi_\tau(z) = \Psi(z)$  gives that  $\lim_{\tau \rightarrow 0} \phi_\pm^\tau(z) = \phi_\pm(z)$ ,  $z \in \mathbb{C}_{[0, \infty)}$ . Thus, from Lemma 3.13 we get that  $\lim_{\tau \rightarrow 0} W_{\phi_\pm^\tau}(z) = W_{\phi_\pm}(z)$  on  $\mathbb{C}_{(0, \infty)}$ . Therefore, (2.47), that is

$$\lim_{\tau \rightarrow 0} \mathcal{M}_{V_{\Psi_\tau}}(z) = \lim_{\tau \rightarrow 0} \frac{1}{(\phi_\pm^\tau(0^+))'} \frac{\Gamma(1-z)}{W_{\phi_\pm^\tau}(1-z)} W_{\phi_\pm^\tau}(z) = \frac{1}{\phi_\pm'(0^+)} \frac{\Gamma(1-z)}{W_{\phi_\pm}(1-z)} W_{\phi_\pm}(z), \quad z \in \mathbb{C}_{(\bar{\alpha}, 1)},$$

holds provided that  $\lim_{\tau \rightarrow 0} (\phi_\pm^\tau(0^+))' = \phi_\pm'(0^+)$ . However, from  $\lim_{\tau \rightarrow 0} \phi_\pm^\tau(z) = \phi_\pm(z)$  we deduce from the second expression in (3.3) with  $\phi_\pm(0) = \phi_\pm^\tau(0) = 0$  that on  $\mathbb{C}_{(0, \infty)}$

$$\lim_{\tau \rightarrow 0} \left( \mathfrak{d}_\pm^\tau + \int_0^\infty e^{-zy} \bar{\mu}_\pm^{\mathfrak{d}_\pm^\tau}(y) dy \right) = \mathfrak{d}_\pm + \int_0^\infty e^{-zy} \bar{\mu}_\pm(y) dy.$$

Since by assumption  $\phi_\pm'(0^+) < \infty$  and hence  $(\phi_\pm^\tau)'(0^+) < \infty$ . Then, in an obvious manner from (3.27) we can get that

$$\lim_{\tau \rightarrow 0} (\phi_\pm^\tau)'(0^+) = \lim_{\tau \rightarrow 0} \left( \mathfrak{d}_\pm^\tau + \int_0^\infty \bar{\mu}_\pm^{\mathfrak{d}_\pm^\tau}(y) dy \right) = \mathfrak{d}_\pm + \int_0^\infty \bar{\mu}_\pm(y) dy = \phi_\pm'(0^+).$$

Thus, item (1) is settled. All the claims of item (2) follow from the following sequence of arguments. First that  $\Pi(dx) = \pi_+(x)dx$ ,  $x > 0$ ,  $\pi_+$  non-increasing on  $\mathbb{R}^+$  and [52] imply that (2.44) is precised to  $I_\Psi \stackrel{d}{=} I_{\phi_\pm} \times I_\psi$ , where  $\psi(z) = z\phi_\pm(z) \in \mathcal{N}_m$ . Secondly, this factorization is transferred to  $V_\Psi \stackrel{d}{=} V_{\phi_\pm} \times V_\psi$  via (2.47). Finally the arguments in the proof of [61, Theorem 7.1] depend on the latter factorization of the entrance laws and the zero-free property of  $\mathcal{M}_{V_\Psi}(z)$  for  $z \in \mathbb{C}_{(0, 1)}$  which via (2.47) is a consequence of Theorem 3.2 which yields that  $W_\phi(z)$  is zero free on  $\mathbb{C}_{(0, \infty)}$  for any  $\phi \in \mathcal{B}$ .

## APPENDIX A. SOME FLUCTUATION DETAILS ON LÉVY PROCESSES AND THEIR EXPONENTIAL FUNCTIONAL

Recall that a Lévy process  $\xi = (\xi_t)_{t \geq 0}$  is a real-valued stochastic process which possesses stationary and independent increments with a.s. right-continuous paths. We allow killing of the Lévy process by means of the following procedure. For  $q \geq 0$  pick an exponential variable  $\mathbf{e}_q$ , of parameter  $q \geq 0$ , independent of  $\xi$ , such that  $\xi_t = \infty$  for any  $t \geq \mathbf{e}_q$ . Note that  $\mathbf{e}_0 = \infty$  a.s. and in this case the Lévy process is conservative that is unkilld. The law of a possibly killed Lévy process  $\xi$  is characterized via its characteristic exponent, i.e.  $\log \mathbb{E} [e^{z\xi_t}] = \Psi(z)t$ , where  $\Psi : i\mathbb{R} \rightarrow \mathbb{C}$  admits the following Lévy-Khintchine representation

$$(A.1) \quad \Psi(z) = \frac{\sigma^2}{2}z^2 + cz + \int_{-\infty}^{\infty} (e^{zr} - 1 - zr\mathbb{I}_{\{|r| < 1\}}) \Pi(dr) - q,$$

where  $q \geq 0$  is the killing rate,  $\sigma^2 \geq 0$ ,  $c \in \mathbb{R}$ , and, the Lévy measure  $\Pi$  satisfies the integrability condition  $\int_{-\infty}^{\infty} (1 \wedge r^2) \Pi(dr) < +\infty$ . With each Lévy process, say  $\xi$ , there are the bivariate ascending and descending ladder height and time processes  $(\tau^\pm, H^\pm) = (\tau_t^\pm, H_t^\pm)_{t \geq 0}$  associated to  $\xi$  via  $(H_t^\pm)_{t \geq 0} = (\xi_{\tau_t^\pm})_{t \geq 0}$  and we refer to [9, Chapter VI] for more information on these processes. Since these processes are bivariate subordinators we denote by  $k_\pm$  their Laplace exponents. The celebrated Fristedt's formula, see [9, Chapter VI, Corollary 10], then evaluates those on  $z \in \mathbb{C}_{[0, \infty)}$ ,  $q \geq 0$ , as

$$(A.2) \quad k_\pm(q, z) = e^{-\int_0^\infty \int_0^\infty (e^{-t} - e^{-zx - qt}) \mathbb{P}(\xi_t^\# \in dx) \frac{dt}{t}},$$

where  $\xi^\#$  is a conservative Lévy process constructed from  $\xi$  by letting it to evolve on an infinite time horizon. Set

$$(A.3) \quad h(q) = e^{-\int_0^\infty (e^{-t} - e^{-qt}) \mathbb{P}(\xi_t^\# = 0) \frac{dt}{t}}$$

and note that  $h : [0, \infty) \mapsto \mathbb{R}^+$  is an increasing, positive function. Then, the analytical form of the Wiener-Hopf factorization of  $\Psi \in \overline{\mathcal{N}}$  is given by the expressions

$$(A.4) \quad \Psi(z) = -\phi_+(-z)\phi_-(z) = -h(q)k_+(q, -z)k_-(q, z), \quad z \in i\mathbb{R},$$

where  $\phi_\pm \in \mathcal{B}$  with  $\phi_\pm(0) \geq 0$  and characteristics of  $\phi_\pm$ , that is  $(\phi_\pm(0), \mathbf{d}_\pm, \mu_\pm)$ , depend on  $q = -\Psi(0) \geq 0$ . Then we have the result.

**Lemma A.1.** *For any  $\Psi \in \overline{\mathcal{N}}$  it is possible to choose  $\phi_+ = h(q)k_+$  and  $\phi_- = k_-$ . The function  $h : [0, \infty) \mapsto \mathbb{R}^+$  is not identical to 1 if and only if  $\overline{\Pi}(0) < \infty$ ,  $\sigma^2 = c = 0$ , see (A.1), that is  $\xi$  is a compound Poisson process. Then, on  $\mathbb{R}^+$ ,  $\mu_-(dy) = \int_0^\infty e^{-qy_1} \mu_-^\#(dy_1, dy)$  and  $\mu_+(dy) = h(q) \int_0^\infty e^{-qy_1} \mu_+^\#(dy_1, dy)$ , where  $\mu_\pm^\#(dy_1, dy)$  are the Lévy measures of the bivariate ascending and descending ladder height and time processes associated to the conservative Lévy process  $\xi^\#$ .*

*Proof.* The proof is straightforward from [24, p.27] and the fact that for fixed  $q \geq 0$ ,  $k_\pm \in \mathcal{B}$ .  $\square$

We refer to the excellent monographs [9] and [66] for background on the probabilistic and path properties of general Lévy processes and their associated Lévy-Khintchine exponent  $\Psi \in \overline{\mathcal{N}}$ .

## APPENDIX B. A SIMPLE EXTENSION OF THE CELEBRATED ÉQUATION AMICALE INVERSÉE

When  $\Psi \in \overline{\mathcal{N}}$  with  $\Psi(0) = 0$  the Vigon's celebrated *équation amicale inversée*, see [24, 5.3.4] states that

$$(B.1) \quad \bar{\mu}_-(y) = \int_0^\infty \bar{\Pi}_-(y+v)U_+(dv), \quad y > 0,$$

where  $\mu_-$  is the Lévy measure of the descending ladder height process and  $U_+$  is the potential measure associated to the ascending ladder height process, see Section A and relation (4.22). We now extend (B.1) to all  $\Psi \in \overline{\mathcal{N}}$ .

**Proposition B.1.** *Let  $\Psi \in \overline{\mathcal{N}}$  and recall that  $\Psi(z) = -\phi_+(-z)\phi_-(z)$ ,  $z \in i\mathbb{R}$ . Then (B.1) holds.*

*Proof.* Recall that  $\Psi^\sharp(z) = \Psi(z) - \Psi(0) \in \overline{\mathcal{N}}$  and  $\Psi^\sharp$  corresponds to a conservative Lévy process. From Lemma A.1 we have that  $\mu_-(dy) = \int_0^\infty e^{-qt} \mu_-^\sharp(dt, dy)$ ,  $y > 0$ . However, from [24, Corollary 6, Chapter 5] we have that

$$\mu_-^\sharp(dt, dy) = \int_0^\infty U_+^\sharp(dt, dv)\Pi_-(v+dy),$$

where  $U_-^\sharp$  is the bivariate potential measure associated to  $((\tau^-)^\sharp, (H^-)^\sharp)$ , see [24, Chapter 5] for more details. Therefore,

$$\bar{\mu}_-(y) = \int_0^\infty \int_0^\infty e^{-qt} U_+^\sharp(dt, dv) \bar{\Pi}_-(v+y).$$

Assume first that the underlying Lévy process is not a compound Poisson process. Then from [24, p. 50] and Lemma A.1, we have, for any  $\eta > 0$ ,

$$\frac{1}{\phi_+(\eta)} = \frac{1}{k_+(q, \eta)} = \int_0^\infty e^{-\eta v} \int_0^\infty e^{-qt} U_+^\sharp(dt, dv)$$

and from Proposition 3.14(5) we conclude that  $U_+(dv) = \int_0^\infty e^{-qt} U_+^\sharp(dt, dv)$  since

$$\frac{1}{\phi_+(\eta)} = \int_0^\infty e^{-\eta v} U_+(dv).$$

Thus (B.1) is established for any  $\Psi \in \overline{\mathcal{N}}$  such that the underlying Lévy process is not a compound Poisson process. In the case of compound Poisson process the claim follows by a modification of the proof in [24, p. 50] accounting for the function  $h(q)$  appearing in (A.4) which is missed therein since in this case  $h(q)k_+(q, 0)k_-(q, 0) = q$ .  $\square$

## APPENDIX C. SOME REMARKS ON KILLED LÉVY PROCESSES

The next claim is also a general fact that seems not to have been recorded in the literature at least in such a condensed form.

**Proposition C.1.** *Let  $\Psi \in \overline{\mathcal{N}}$  and for any  $r > 0$ ,  $\Psi_r(z) = \Psi(z) - r = \phi_+^r(-z)\phi_-^r(z)$ ,  $z \in i\mathbb{R}$ , with the notation  $(\phi_\pm^r(0), \mathbf{d}_\pm^r, \mu_\pm^r)$  for the triplets defining the Bernstein functions  $\phi_\pm^r$ . Then, for any  $r > 0$ ,  $\mathbf{d}_\pm^r = \mathbf{d}_\pm$  and*

$$\bar{\mu}_+^r(0) = \infty \iff \bar{\mu}_+(0) = \infty \text{ and } \bar{\mu}_-^r(0) = \infty \iff \bar{\mu}_-(0) = \infty.$$



Moreover, with  $\Psi^\sharp(z) = \Psi(z) - \Psi(0) = \Psi_r^\sharp(z) \in \overline{\mathcal{N}}$  we get that weakly on  $(0, \infty)$

$$(C.1) \quad \lim_{r \rightarrow 0} \mu_{\pm}^r(dx) = \mu_{\pm}(dx)$$

and therefore for any  $a > \mathbf{a}_+$  and  $[b_1, b_2] \subset \mathbb{R}$  with  $-\infty < b_1 < 0 < b_2 < \infty$

$$(C.2) \quad \overline{\lim}_{\tau \rightarrow 0} \sup_{b \in [b_1, b_2]} \sup_{0 \leq r \leq \tau} |\phi_+^r(a + ib) - \phi_+(a + ib)| = 0$$

and

$$(C.3) \quad \overline{\lim}_{\tau \rightarrow 0} \sup_{b \in \mathbb{R} \setminus [b_1, b_2]} \sup_{0 \leq r \leq \tau} \frac{|\phi_+^r(a + ib) - \phi_+(a + ib)|}{|b|} = 0.$$

Relations (C.2) and (C.3) also hold with  $\phi_-, \phi_-^r$  for any fixed  $a > \mathbf{a}_-$ .

*Proof.* The Lévy process  $\xi^r$  underlying  $\Psi_r$  is killed at rate  $\Psi(0) + r$  but otherwise possesses the same Lévy triplet  $(c, \sigma^2, \Pi)$  as  $\xi$ . Therefore, for any  $r > 0$ ,  $\mathbf{d}_\pm^r = \mathbf{d}_\pm$ ,

$$\bar{\mu}_+^r(0) = \infty \iff \bar{\mu}_+(0) = \infty \text{ and } \bar{\mu}_-^r(0) = \infty \iff \bar{\mu}_-(0) = \infty$$

since those are local properties unaffected by the additional killing rate. Moreover, even  $\mathbf{a}_{\phi_\pm^r} = \mathbf{a}_{\phi_\pm}$ , see (3.7), since the analyticity of  $\Psi$  and hence of  $\phi_\pm$  is unaltered. Next, (C.1) follows immediately from Lemma A.1 as it represents  $\mu_\pm^r$  in terms of the Lévy measure of the ladder height processes of the conservative process underlying  $\Psi^\sharp$ , that is

$$\mu_-(dy) = \int_0^\infty e^{-(r+\Psi(0))y_1} \mu_-^\sharp(dy_1, dy),$$

and  $\lim_{r \rightarrow 0} h(r + \Psi(0)) = h(\Psi(0))$ . It remains to prove (C.2) and (C.3). Fix  $a > \mathbf{a}_+$  and  $[b_1, b_2]$  as in the statement. Then from the second expression of (3.3)

$$(C.4) \quad \sup_{b \in [b_1, b_2]} |\phi_+^r(a + ib) - \phi_+(a + ib)| \leq |\phi_+^r(0) - \phi_+(0)| \\ + 2 \max\{|b_1| + |a|, b_2 + |a|\} \int_0^\infty e^{-ay} |\bar{\mu}_+(y) - \bar{\mu}_+^r(y)| dy.$$

Clearly, from the celebrated Fristedt's formula, see (A.2), Lemma A.1 and the monotone convergence theorem

$$\lim_{r \rightarrow 0} \phi_+^r(0) = \lim_{r \rightarrow 0} h(\Psi(0) + r) k_+(r + \Psi(0), 0) = \lim_{r \rightarrow 0} h(\Psi(0) + r) e^{\int_0^\infty \int_0^\infty (e^{-t} - e^{-(\Psi(0)+r)t}) \frac{\mathbb{P}(\xi_t^\sharp \in dx)}{t} dt} \\ = h(\Psi(0)) e^{\int_0^\infty \int_0^\infty (e^{-t} - e^{-\Psi(0)t}) \frac{\mathbb{P}(\xi_t^\sharp \in dx)}{t} dt} = h(\Psi(0)) k_+(\Psi(0), 0) = \phi_+(0),$$

where  $\xi^\sharp$  is the conservative Lévy process underlying  $\Psi^\sharp(z) = \Psi(z) - \Psi(0)$ . Next, from Lemma A.1 it follows that for any  $y > 0$  and any  $\tau > 0$

$$(C.5) \quad \sup_{0 \leq r \leq \tau} e^{-ay} \bar{\mu}_+^r(y) \leq h(\tau) e^{-ay} \bar{\mu}_+^\sharp(y)$$

with the latter being integrable on  $(0, \infty)$  since  $a > \mathbf{a}_+$ . Moreover, again from Lemma A.1 we get that for any  $y > 0$

$$\sup_{0 \leq r \leq \tau} |\bar{\mu}_+(y) - \bar{\mu}_+^r(y)| = \sup_{0 \leq r \leq \tau} \left| \int_0^\infty (1 - e^{-rt}) e^{-\Psi(0)t} \mu_+^\sharp(dt, (y, \infty)) \right| \\ \leq \tau \int_0^1 t \mu_+^\sharp(dt, (y, \infty)) + \int_1^\infty (1 - e^{-\tau t}) \mu_+^\sharp(dt, (y, \infty))$$

provided  $\xi$  underlying  $\Psi$  is not a compound Poisson process and

$$\begin{aligned} & \sup_{0 \leq r \leq \tau} |\bar{\mu}_+(y) - \bar{\mu}_+^r(y)| \\ & \leq h(\Psi(0)) \left( \tau \int_0^1 t e^{-\Psi(0)t} \mu_+^\sharp(dt, (y, \infty)) + \int_1^\infty (1 - e^{-\tau t}) e^{-\Psi(0)t} \mu_+^\sharp(dt, (y, \infty)) \right) \\ & \quad + (h(\Psi(0) + \tau) - h(\Psi(0))) \int_0^\infty e^{-\Psi(0)t} \mu_+^\sharp(dt, \mathbb{R}^+) \end{aligned}$$

otherwise. Evidently, in both cases, the right-hand side goes to zero as  $\tau \rightarrow 0$  for any  $y > 0$  and this together with (C.5) and the dominated convergence theorem show from (C.4) that (C.2) holds true. In fact (C.3) follows in the same manner from (C.4) by first dividing by  $2 \max\{|b| + |a|\}$  for  $b \in \mathbb{R} \setminus [b_1, b_2]$ .  $\square$

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