

SPECTRAL PROJECTIONS CORRELATION STRUCTURE FOR SHORT-TO-LONG RANGE DEPENDENT PROCESSES

P. PATIE AND A. SRAPIONYAN

ABSTRACT. Let $\mathbf{X} = (\mathbf{X}_t)_{t \geq 0}$ be a stochastic process issued from $x \in \mathbb{R}$ that admits a marginal stationary measure ν , i.e. $\nu \mathbf{P}_t f = \nu f$ for all $t \geq 0$, where $\mathbf{P}_t f(x) = \mathbb{E}_x[f(\mathbf{X}_t)]$. In this paper we introduce the (resp. biorthogonal) spectral projections correlation functions which are expressed in terms of projections into the eigenspaces of \mathbf{P}_t (resp. and of its adjoint in the weighted Hilbert space $L^2(\nu)$). We obtain closed-form expressions involving eigenvalues, the condition number and/or the angle between the projections in the following different situations: when $\mathbf{X} = X$ with $X = (X_t)_{t \geq 0}$ being a Markov process, \mathbf{X} is the subordination of X in the sense of Bochner, and \mathbf{X} is a non-Markovian process which is obtained by time-changing X with an inverse of a subordinator. It turns out that these spectral projections correlation functions have different expressions with respect to these classes of processes which enables to identify substantial and deep properties about their dynamics. This interesting fact can be used to design original statistical tests to make inferences, for example, about the path properties of the process (presence of jumps), distance from symmetry (self-adjoint or non-self-adjoint) and short-to-long-range dependence. To reveal the usefulness of our results, we apply them to a class of non-self-adjoint Markov semigroups studied in [35], and then time-change by subordinators and their inverses.

1. INTRODUCTION

Stochastic processes play an important role in the investigation of random phenomena depending on time. When using a stochastic process for modeling or for statistical testing purposes, one should take into account its special features which indicate how well the process reflects the reality. Some of the most essential features include (but are not limited to) observing whether the process is Markovian or not, whether its trajectories are continuous or incorporate jumps, what type of range dependence it exhibits, and how far it is from symmetry (self-adjointness).

With the objective in mind, we introduce the concept of (biorthogonal) spectral projections correlation functions, see Definition 1.1 below. We proceed by computing explicitly these functions along with their large time asymptotic behavior for three classes of processes, namely Markov processes, Markov processes subordinated in the sense of Bochner and non-Markovian processes which are obtained by time-changing a Markov process with an inverse of a subordinator. These findings enable us to provide a unified and original framework for designing statistical tests that investigates critical properties of a stochastic process including the one described above. Indeed, in these three scenarios the (biorthogonal) spectral projections correlation functions have different expressions, involving some quantities characterizing the process, such as their eigenvalues with their associated condition number or the angle between the spectral projections.

We indicate that the recent years have witnessed the ubiquity of such non-Markovian dynamics in relation to the fractional Cauchy problem, see e.g. [19, 22, 26, 27, 32, 43], and, also due to their

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central role in diverse physical applications within the field of anomalous diffusion, see e.g. [30], as well as for neuronal models for which their long range dependence feature is attractive, see e.g. [28]. The solution of the fractional Cauchy problem, which replaces the integer order derivative by the Caputo derivative or even by a more general additive convolution operator, has a nice stochastic interpretation as it is related to the time-changed of the original Markov process by the inverse of a stable subordinator or a more general subordinator, see Toaldo [43] and, Baeumer and Meerschaert [4], Meerschaert et al. [29] among others. In their recent work, by observing that the fractional Caputo derivative of order $\alpha \in (0, 1)$ can also be expressed in terms of a multiplicative convolution operator, Patie and Srapionyan [37] introduced a class of such operators which also have the same self-similarity property as the Caputo derivative. Furthermore, they investigate the generalized Cauchy problems, defined with these operators, and provide a stochastic representation of its solution, expressed in terms of the right-inverse of increasing self-similar Markov processes. We also mention that Leonenko et al. [25] and Mijena and Nane [31] investigate the orthogonal spectral projections correlation structure in the framework of Pearson diffusions, i.e. diffusions with polynomial coefficients. More specifically, in [25], the authors discuss the case when a Pearson diffusion is time-changed by an inverse of an α -stable subordinator, $0 < \alpha < 1$. Whereas the authors of [31] consider a Pearson diffusion time-changed by an inverse of a linear combination of independent α - and β -stable subordinators, $0 < \alpha, \beta < 1$. In this work, we start with a general Markov process that admits an invariant measure with its associated semigroup not necessarily being self-adjoint and local, and then we perform a time-change with general subordinators and their inverses.

Finally, we emphasize that the notion of long-range dependence, also known as long memory, of stochastic processes has been and it is still a center of great interests in probability theory and its applications in the last decades. We refer for thorough and historical account of this concept to the recent monograph of Samorodnitsky [39]. The definitions of long-range dependence based on the second-order properties of a stationary stochastic process such as asymptotic behavior of covariances, spectral density, and variances of partial sums are among the most developed ones appearing in literature. These second-order properties are conceptually relatively simple and easy to estimate from the data. By far the most popular point of view on range dependence is through the rate of decay of covariance or correlation functions. Conceptually, short memory corresponds to a sufficiently fast rate of decay of the correlation (covariance) function as geometric decay, and long-range dependence corresponds to a sufficiently slow rate of decay of the correlation (covariance) function as power decay. It is also worth to mention that the relationship between long-range dependence and non-stationarity is a delicate one. The stationary long memory processes form a layer among the stationary processes that is "near the boundary" with non-stationary processes, or, alternatively, as the layer separating the non-stationary processes from the "usual" stationary processes. The processes in the "layer" resemble non-stationary models, and they are unusual stationary processes to such an extent that one can talk about a phase transition. Connecting the notion of long-range dependence to certain types of phase transitions fits well with the intuition of the term "long memory" describing a model that is out of the ordinary, see [39].

1.1. Preliminaries. Let $\mathbf{X} = (\mathbf{X}_t)_{t \geq 0}$ be a stochastic process defined on a sample filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and state space $E \subseteq \mathbb{R}$, endowed with a sigma-algebra \mathcal{E} . Let its associated family of linear operators $\mathbf{P} = (\mathbf{P}_t)_{t \geq 0}$ defined, for any $t \geq 0$ and $f \in \mathcal{B}_b(E)$, the space of bounded Borelian functions on E , by

$$\mathbf{P}_t f(x) = \mathbb{E}_x[f(\mathbf{X}_t)],$$

where \mathbb{E}_x stands for the expectation operator with respect to $\mathbb{P}_x(\mathbf{X}_0 = x) = 1$. Since $x \mapsto \mathbb{E}_x$ is \mathcal{E} -measurable, for any Radon measure ν , we use the notation

$$\nu \mathbf{P}_t f = \mathbb{E}_\nu[f(\mathbf{X}_t)] = \int_E \mathbb{E}_x[f(\mathbf{X}_t)] \nu(dx).$$

We say that a Radon measure ν on E is a *marginal stationary measure*, if for all $t \geq 0$,

$$(1.1) \quad \nu \mathbf{P}_t f = \nu f.$$

If \mathbf{X} is a Markov process and (1.1) holds, we say that ν is an *invariant measure*. Then, since ν is non-negative on E , we define the weighted Hilbert space

$$L^2(\nu) = \{f : E \rightarrow \mathbb{R} \text{ measurable; } \int_E f^2(x) \nu(dx) < \infty\},$$

endowed with the inner product $\langle \cdot, \cdot \rangle_\nu$, where $\langle f, g \rangle_\nu = \int_0^\infty f(x)g(x)\nu(dx)$, and norm $\|f\|_\nu = \sqrt{\langle f, f \rangle_\nu}$. Next, the operators \mathbf{P}_t , $t \geq 0$ being linear, positive and with total mass $\mathbf{P}_t \mathbb{1} = \mathbb{1}$ with $\mathbb{1}$ being the identity function on the appropriate space, we have, by Jensen's inequality, for any $f \in C_0(E) \subseteq \mathcal{B}_b(E)$ where $C_0(E)$ is the set of continuous functions on E vanishing at infinity,

$$\|\mathbf{P}_t f\|_\nu^2 = \int_E (\mathbf{P}_t f)^2(x) \nu(dx) \leq \int_E \mathbf{P}_t f^2(x) \nu(dx) = \nu f^2.$$

Thus, the Hahn-Banach theorem yields that we can extend \mathbf{P}_t as a contraction of $L^2(\nu)$. From now on, when there is no confusion, we denote by \mathbf{P}_t its extension to $L^2(\nu)$. Now, let $\mathbf{P}^* = (\mathbf{P}_t^*)_{t \geq 0}$ be the adjoint of \mathbf{P} in $L^2(\nu)$, i.e. for any $t \geq 0$ and $f, g \in L^2(\nu)$,

$$(1.2) \quad \langle \mathbf{P}_t f, g \rangle_\nu = \langle f, \mathbf{P}_t^* g \rangle_\nu.$$

We are now ready to state the following hypothesis.

Assumption 1. Let $\mathbb{N} \subseteq \mathbb{N}$ be a finite or a countable set, and for any $t \geq 0$, $(\mathcal{P}_n)_{n \in \mathbb{N}}$ (resp. $(\mathcal{V}_n)_{n \in \mathbb{N}}$) be a set of eigenfunctions of \mathbf{P}_t (resp. \mathbf{P}_t^*) in $L^2(\nu)$ in the sense that there exist distinct $(\lambda_n)_{n \in \mathbb{N}} \in \mathbb{R}_+$ such that for any $n \in \mathbb{N}$ and $t \geq 0$, we have

$$(1.3) \quad \mathbf{P}_t \mathcal{P}_n = e^{-\lambda_n t} \mathcal{P}_n,$$

$$(1.4) \quad \mathbf{P}_t^* \mathcal{V}_n = e^{-\lambda_n t} \mathcal{V}_n.$$

We may also find convenient to characterize the \mathcal{V}_n 's by duality using (1.2), i.e. $\langle \mathbf{P}_t f, \mathcal{V}_n \rangle_\nu = e^{-\lambda_n t} \langle f, \mathcal{V}_n \rangle_\nu$, for all $f \in L^2(\nu)$, which enables to identify the \mathcal{P}'_n and \mathcal{V}_n 's are the right and left-eigenfunctions respectively. Note that the assumption on $(\lambda_n)_{n \in \mathbb{N}}$ being of multiplicity 1 is in fact for sake of simplicity since we mean to consider only one of the eigenfunctions in the eigenspace associated to each eigenvalue. It is also worth pointing out that the definition of the spectral correlation requires the existence of left and/or right-eigenfunctions which imply that the point spectrum is not empty. Moreover, we emphasize that we suppose only that the spectrum has a component which is real and part of the discrete point spectrum but one could readily adjust the arguments to assume merely a complex point spectrum, continuous or discrete.

Next, without loss of generality, we assume that for any $n \in \mathbb{N}$,

$$\langle \mathcal{P}_n, \mathcal{V}_n \rangle_\nu = 1.$$

Indeed, if $\langle \mathcal{P}_n, \mathcal{V}_n \rangle_\nu = a_n \neq 0$ for $n \in \mathbb{N}$, then we could consider the sequences $\bar{\mathcal{P}}_n = \frac{\mathcal{P}_n}{\sqrt{|a_n|}}$ and $\bar{\mathcal{V}}_n = \frac{\mathcal{V}_n}{\sqrt{|a_n|}}$, for which, obviously, we have $\langle \bar{\mathcal{P}}_n, \bar{\mathcal{V}}_n \rangle_\nu = 1$ for $n \in \mathbb{N}$. We also note that the condition

$\langle \mathcal{P}_n, \mathcal{V}_n \rangle_\nu = 1$ does not constrain the norms of the sequences $(\mathcal{P}_n)_{n \in \mathbb{N}}$ and $(\mathcal{V}_n)_{n \in \mathbb{N}}$ to be 1, but it only follows from Cauchy-Schwarz inequality that, for any $n \in \mathbb{N}$,

$$1 = |\langle \mathcal{P}_n, \mathcal{V}_n \rangle_\nu| \leq \|\mathcal{P}_n\|_\nu \|\mathcal{V}_n\|_\nu.$$

In Lemma 4.1 below, we shall show that $(\mathcal{P}_n, \mathcal{V}_n)_{n \in \mathbb{N}}$ form a biorthogonal sequence in $L^2(\nu)$, i.e. $\langle \mathcal{P}_m, \mathcal{V}_n \rangle_\nu = \delta_{mn}$, where δ_{mn} is the Kronecker symbol defined in (2.1). In particular, if \mathbf{P} is self-adjoint in $L^2(\nu)$, i.e. for all $t \geq 0$, $\mathbf{P}_t = \mathbf{P}_t^*$, we have $\mathcal{P}_n = \mathcal{V}_n$ for $n \in \mathbb{N}$, and $(\mathcal{P}_n)_{n \in \mathbb{N}}$ form an orthonormal sequence in $L^2(\nu)$. Below we consider \mathbf{X} to belong to one of the following three families of stochastic processes.

1.1.1. *Markov process.* First, let $\mathbf{X} = X$ with $X = (X_t)_{t \geq 0}$ a Markov process, and its associated semigroup be the family of linear operators $\mathbf{P} = P = (P_t)_{t \geq 0}$, defined, for any $t \geq 0$ and $f \in \mathcal{B}_b(E)$, by $P_t f(x) = \mathbb{E}_x[f(X_t)]$. Next, we assume that for $t \geq 0$ and $f \in \mathcal{B}_b(E)$ the mapping $t \mapsto P_t f$ is continuous (this is equivalent to the stochastic continuity property of the process X), and the semigroup P admits an invariant probability measure ν , i.e. $\nu P_t f = \nu f$. In such framework, a classical result states that the semigroup P can be extended to a strongly continuous contraction semigroup in $L^2(\nu)$, see e.g. Da Prato [13], and by an abuse of notation, we still denote its extension to $L^2(\nu)$ by P . Note that the adjoint of P in $L^2(\nu)$, P^* is the semigroup of a stochastic process which may not be necessarily a strong Markov one, but instead has the moderate Markov property, see e.g. Chung and Walsh [11, Chapter 13] for more details.

1.1.2. *Bochner subordination.* In Section 2 below, we also study the spectral projections correlation structure of subordinated Markov processes. *Bochner subordination* is a transformation of a Markov process to a new one through random time change by an independent subordinator, i.e. a real-valued Lévy process with non-decreasing sample paths, see e.g. [8, 9, 40]. From the operator semigroup perspective, Bochner subordination is a classical method for generating a new semigroup of linear operators on a Banach space from an existing one. More formally, using the notation of Section 1.1.1 above, for $P = (P_t)_{t \geq 0}$, a strongly continuous contraction semigroup in $L^2(\nu)$, and $(\mu_t)_{t \geq 0}$, a vaguely continuous convolution semigroup of probability measures on $[0, \infty)$, the subordination of P in the sense of Bochner is defined by

$$(1.5) \quad P_t^\varphi f(x) = \int_0^\infty P_s f(x) \mu_t(ds), \quad t \geq 0, f \in \mathcal{B}_b(E).$$

The superscript φ alludes to the Laplace exponent of $(\mu_t)_{t \geq 0}$, which is a Bernstein function with the following representation, for $\lambda \geq 0$,

$$(1.6) \quad \varphi(\lambda) = \varrho \lambda + \int_0^\infty (1 - e^{-\lambda y}) \vartheta(dy),$$

where $\varrho \geq 0$, and ϑ is a Lévy measure concentrated on \mathbb{R}_+ satisfying $\int_0^\infty (1 \wedge y) \vartheta(dy) < \infty$. Note that $(\mu_t)_{t \geq 0}$ gives rise to a Lévy subordinator $\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}$, which is assumed to be independent of X , and the law of \mathcal{T} is uniquely characterized by its Laplace exponent φ , that is, for $t, \lambda \geq 0$,

$$(1.7) \quad \mathbb{E} \left[e^{-\lambda \mathcal{T}_t} \right] = e^{-t\varphi(\lambda)}.$$

We write $\mathbf{X} = X_{\mathcal{T}} = (X_{\mathcal{T}_t})_{t \geq 0}$ for the Markov process associated with the semigroup $\mathbf{P}_t f(x) = P_t^\varphi f(x) = \mathbb{E}_x[f(X_{\mathcal{T}_t})]$. Moreover, one has that ν is also an invariant measure for the semigroup P^φ . Indeed, let $f \in \mathcal{B}_b(E)$ and assume, without loss of generality, f is non-negative, then, for $t \geq 0$, we have

$$\nu P_t^\varphi f = \langle P_t^\varphi f, \mathbb{1} \rangle_\nu = \int_0^\infty \langle P_s f, \mathbb{1} \rangle_\nu \mu_t(ds) = \int_0^\infty \nu f \mu_t(ds) = \nu f,$$

where we used Tonelli's theorem, the fact that ν is an invariant probability measure for P , and $\varphi(0) = 0$ in (1.6). Therefore, as above, P^φ can be extended to a contraction semigroup in $L^2(\nu)$. It is easy to note that the semigroup P^φ shares the same eigenspaces and co-eigenspaces (eigenspaces for the adjoint) as P , and, in particular, we have the following.

Proposition 1.1. *Let $(\mathcal{P}_n)_{n \in \mathbb{N}}$ and $(\mathcal{V}_n)_{n \in \mathbb{N}}$ be as defined in Assumption 1 with $\mathbf{P} = P$ of Section 1.1.1. Then, $(\mathcal{P}_n)_{n \in \mathbb{N}}$ and $(\mathcal{V}_n)_{n \in \mathbb{N}}$ are the eigenfunctions of the semigroup P^φ and its adjoint in $L^2(\nu)$, respectively, associated to the eigenvalues $(\varphi(\lambda_n))_{n \in \mathbb{N}}$.*

Proof. First, note that for $n \in \mathbb{N}$, $\mathcal{P}_n \in L^2(\nu)$, and for any $t \geq 0$, we have

$$P_t^\varphi \mathcal{P}_n = \int_0^\infty P_s \mathcal{P}_n \mu_t(ds) = \mathcal{P}_n \int_0^\infty e^{-\lambda_n s} \mu_t(ds) = e^{-t\varphi(\lambda_n)} \mathcal{P}_n,$$

where in the second equality we used (1.3), and the last step follows from (1.7). Next, for $f \in L^2(\nu)$, $n \in \mathbb{N}$ and $t \geq 0$, note that

$$\begin{aligned} \langle P_t^\varphi f, \mathcal{V}_n \rangle_\nu &= \int_E P_t^\varphi f(x) \mathcal{V}_n(x) \nu(dx) = \int_E \int_0^\infty P_s f(x) \mu_t(ds) \mathcal{V}_n(x) \nu(dx) \\ &= \int_0^\infty \int_E P_s f(x) \mathcal{V}_n(x) \nu(dx) \mu_t(ds) = \int_0^\infty \langle P_s f, \mathcal{V}_n \rangle_\nu \mu_t(ds) \\ &= \int_0^\infty \langle f, P_s^* \mathcal{V}_n \rangle_\nu \mu_t(ds) = \int_0^\infty e^{-\lambda_n s} \langle f, \mathcal{V}_n \rangle_\nu \mu_t(ds) = \langle f, \mathcal{V}_n \rangle_\nu e^{-t\varphi(\lambda_n)}, \end{aligned}$$

where in the last two steps we used (1.4) and (1.7), and we were allowed to change the order of integration using Fubini's theorem, since by Cauchy-Schwarz inequality, we have

$$\int_0^\infty |\langle P_s f, \mathcal{V}_n \rangle_\nu| \mu_t(ds) \leq \int_0^\infty \|P_s f\|_\nu \|\mathcal{V}_n\|_\nu \mu_t(ds) \leq \|f\|_\nu \|\mathcal{V}_n\|_\nu < \infty.$$

■

1.1.3. *Non-Markovian processes obtained by a time-change with an inverse of a subordinator.* Let \mathcal{T} denote the subordinator defined in (1.7), and define its right inverse, for $t > 0$, by

$$L_t = \inf\{s > 0; \mathcal{T}_s > t\}.$$

We point out that $t \mapsto \mathcal{T}_t$ is right-continuous and non-decreasing, and hence $t \mapsto L_t$ is also right-continuous and non-decreasing. In particular, when $t \mapsto \mathcal{T}_t$ is a.s. increasing, which is equivalent to $\varphi(\infty) = \infty$ in (1.6), then $t \mapsto L_t$ is continuous and $L_{\mathcal{T}_t} = t$ a.s., whereas $\mathcal{T}_{L_t} \geq t$ a.s. Next, let l_t denote the distribution of L_t , i.e. for any B Borelian set of \mathbb{R}_+ , $l_t(B) = \mathbb{P}(L_t \in B)$. Then, for any $\lambda \geq 0$ and $t \geq 0$, its Laplace transform is denoted by

$$(1.8) \quad \eta_t(\lambda) = \int_0^\infty e^{-\lambda s} l_t(ds).$$

For sake of simplicity, we assume that $\mathbb{P}(L_t < \infty) = \eta_t(0) = \int_0^\infty l_t(ds) = 1$ for all $t \geq 0$. However, all of the results presented below could be easily adapted to the case when $\int_0^\infty l_t(ds) < 1$ for some $t \geq 0$ (and hence, all $t \geq 0$). Let $\mathbf{P} = P^\eta = (P_t^\eta)_{t \geq 0}$ be the family of linear operators defined, for $f \in \mathcal{B}_b(E)$ and $t \geq 0$, by

$$P_t^\eta f(x) = \int_0^\infty P_s f(x) l_t(ds).$$

The corresponding time-changed process will be denoted by $\mathbf{X} = X_L = (X_{L_t})_{t \geq 0}$. As mentioned in the introduction above, this time-change with an inverse of a subordinator in specific situations was discussed in [25] and [31]. In the following we provide some basic properties of P^η .

Proposition 1.2. For any $f \in \mathcal{B}_b(E)$ and $t \geq 0$, $\nu P_t^\eta f = \nu f$, i.e. ν is a marginal stationary measure, and it is also a limiting distribution for P^η , i.e. $\lim_{t \rightarrow \infty} \nu P_t^\eta f = \nu f$. Moreover, for all $t \geq 0$, P_t^η can be extended to a contraction in $L^2(\nu)$.

Proof. Let $f \in \mathcal{B}_b(E)$ and non-negative, then, for any $t \geq 0$, we have, as above,

$$\nu P_t^\eta f = \langle P_t^\eta f, \mathbb{1} \rangle_\nu = \int_0^\infty \langle P_s f, \mathbb{1} \rangle_\nu l_t(ds) = \nu f,$$

where we used Tonelli's theorem, the fact that ν is an invariant measure for P and $\int_0^\infty l_t(ds) = 1$. Next, for a fixed $t \geq 0$ and any $f \in L^2(\nu)$, we note that

$$\begin{aligned} \|P_t^\eta f\|_\nu^2 &= \int_0^\infty (P_t^\eta f(x))^2 \nu(dx) = \int_0^\infty \left(\int_0^\infty P_s f(x) l_t(ds) \right)^2 \nu(dx) \\ &\leq \int_0^\infty \int_0^\infty (P_s f(x))^2 \nu(dx) l_t(ds) \\ &= \int_0^\infty \|P_s f\|_\nu^2 l_t(ds) \leq \|f\|_\nu^2, \end{aligned}$$

where we used Jensen's inequality and Tonelli's theorem, and in the last step we used the fact that P is a contraction semigroup in $L^2(\nu)$, and that the total mass of l_t is 1. \blacksquare

1.2. Spectral projections correlation functions. Notions of covariance and correlation functions have been intensively studied in the statistical literature. Let \mathbf{X} be a stochastic process, and ν be a Radon measure on the state space of \mathbf{X} . We define the *covariance and correlation functions under ν* in the following way. Let $s, t \geq 0$, then for any functions $f, g \in L^2(\nu)$,

$$(1.9) \quad C_\nu(f(\mathbf{X}_t), g(\mathbf{X}_s)) = \mathbb{E}_\nu[f(\mathbf{X}_t)g(\mathbf{X}_s)] - \mathbb{E}_\nu[f(\mathbf{X}_t)]\mathbb{E}_\nu[g(\mathbf{X}_s)],$$

$$(1.10) \quad \rho_\nu(f(\mathbf{X}_t), g(\mathbf{X}_s)) = \begin{cases} \frac{C_\nu(f(\mathbf{X}_t), g(\mathbf{X}_s))}{std_\nu(f(\mathbf{X}_t))std_\nu(g(\mathbf{X}_s))}, & \text{if } std_\nu(f(\mathbf{X}_t))std_\nu(g(\mathbf{X}_s)) > 0, \\ 0, & \text{if } std_\nu(f(\mathbf{X}_t))std_\nu(g(\mathbf{X}_s)) = 0, \end{cases}$$

where std_ν stands for the standard deviation defined by

$$std_\nu(f(\mathbf{X}_t)) = \sqrt{C_\nu(f(\mathbf{X}_t), f(\mathbf{X}_t))}.$$

Definition 1.1. When ν is a marginal stationary measure for \mathbf{X} and Assumption 1 holds, for $m, n \in \mathbb{N}$ and $t, s > 0$, we call $\rho_\nu(\mathcal{P}_m(\mathbf{X}_t), \mathcal{P}_n(\mathbf{X}_s))$ (resp. $\rho_\nu(\mathcal{P}_m(\mathbf{X}_t), \mathcal{V}_n(\mathbf{X}_s))$) (resp. biorthogonal spectral projections correlation functions).

The rest of the paper is organized as follows. In Section 2, we present the main results which include explicit expressions for the spectral projections correlation structure of non-reversible Markov processes, of their subordinated counterparts, as well as of non-Markovian processes, obtained by time-changing a Markov process with an inverse of a subordinator. In Section 3, we illustrate our results for the class of generalized Laguerre processes, which are associated with non-self-adjoint and non-local semigroups. The proofs of the main results are presented in Section 4.

2. MAIN RESULTS

Let us start with $\mathbf{X} = X$ a Markov process admitting an invariant probability measure ν , i.e. $\nu P_t f = \nu f$ for all $t \geq 0$ and $f \in L^2(\nu)$ where P is the $L^2(\nu)$ -semigroup. Recall from Assumption 1 that

$\mathbb{N} \subset \mathbb{N}$ is a finite or a countable set, and for any $t \geq 0$, $(\mathcal{P}_n)_{n \in \mathbb{N}}$ and $(\mathcal{V}_n)_{n \in \mathbb{N}}$ denote the sets of eigenfunctions of P_t and P_t^* , respectively. Next, for $m, n \in \mathbb{N}$, let δ_{mn} be the Kronecker symbol, i.e.

$$(2.1) \quad \delta_{mn} = \begin{cases} 0, & \text{if } m \neq n, \\ 1, & \text{if } m = n. \end{cases}$$

Then, we have the following characterization of the (biorthogonal) spectral projections correlation functions.

Theorem 2.1. *Let $m, n \in \mathbb{N}$. Then, for any $t \geq s > 0$, we have*

$$\rho_\nu(\mathcal{P}_m(X_t), \mathcal{V}_n(X_s)) = e^{-\lambda_m(t-s)} \kappa_\nu^{-1}(m) \delta_{mn},$$

and

$$\rho_\nu(\mathcal{P}_m(X_t), \mathcal{P}_n(X_s)) = e^{-\lambda_m(t-s)} c_\nu(n, m),$$

where $\kappa_\nu(m) = \|\mathcal{P}_m\|_\nu \|\mathcal{V}_m\|_\nu$ and $-1 \leq c_\nu(n, m) = \frac{\langle \mathcal{P}_n, \mathcal{P}_m \rangle_\nu}{\|\mathcal{P}_n\|_\nu \|\mathcal{P}_m\|_\nu} \leq 1$. Consequently, $c_\nu(n, n) = 1$ for any $n \in \mathbb{N}$.

The proof of Theorem 2.1 is presented in Section 4.1.

Remark 2.1. *We shall show in Lemma 4.1 below that $(\mathcal{P}_n, \mathcal{V}_n)_{n \in \mathbb{N}}$ form a biorthogonal sequence in $L^2(\nu)$ in the sense that $\langle \mathcal{P}_m, \mathcal{V}_n \rangle_\nu = \delta_{mn}$ for any $m, n \in \mathbb{N}$. Then, each (non-orthogonal) spectral projection is given by*

$$\mathcal{P}_m f = \langle f, \mathcal{P}_m \rangle_\nu \mathcal{V}_m, \quad \text{for } f \in L^2(\nu).$$

Moreover, in this context, the number

$$\kappa_\nu(m) = \|\mathcal{P}_m\|_\nu \|\mathcal{V}_m\|_\nu$$

is called the condition number of the eigenvalue λ_m and corresponds to the norm of the operator \mathcal{P}_m , see e.g. Davies [14]. The condition number measures how unstable the eigenvalues are under small perturbations of the operator P_t . We note that when $(\mathcal{P}_n, \mathcal{V}_n)_{n \in \mathbb{N}}$ form an orthonormal sequence, then $\kappa_\nu(m) = 1$.

Remark 2.2. *Recall that a biorthogonal system $(\mathcal{P}_n, \mathcal{V}_n)_{n \in \mathbb{N}}$ is called tame in $L^2(\nu)$ if $\mathbb{N} = \mathbb{N}$, it is complete (i.e. $\overline{\text{Span}(\mathcal{P}_n)_{n \in \mathbb{N}}} = L^2(\nu)$) and*

$$\kappa_\nu(m) = O(m^\beta),$$

for all $m \in \mathbb{N}$ and some β , i.e. there exists $b \in \mathbb{R}_+$ and $m_0 \in \mathbb{N}$ such that $|\kappa_\nu(m)| = \kappa_\nu(m) \leq bm^\beta$ for all $m \geq m_0$, see Davies [14]. Otherwise, we say that the system is wild. Note that if $(\mathcal{P}_n)_{n \in \mathbb{N}}$ is a basis in $L^2(\nu)$, then $\kappa_\nu(m)$ is uniformly bounded, so the system is tame with $\beta = 0$, see again [14].

Remark 2.3. *When $P = (P_t)_{t \geq 0}$ is a self-adjoint compact semigroup, then $\mathbb{N} = \mathbb{N}$ and $(\mathcal{P}_n)_{n \in \mathbb{N}} = (\mathcal{V}_n)_{n \in \mathbb{N}}$ form an orthonormal basis of $L^2(\nu)$. However, when P is non-self-adjoint, then $(\mathcal{P}_n, \mathcal{V}_n)_{n \in \mathbb{N}}$ do not form, in general, a basis of $L^2(\nu)$. A necessary condition for $(\mathcal{P}_n, \mathcal{V}_n)_{n \in \mathbb{N}}$ to form a basis is that the condition number $\kappa_\nu(m)$ is uniformly bounded. In this sense, the rate of growth of $\kappa_\nu(m)$ also can be seen as a measure of departure of these sequences from the basis property.*

Remark 2.4. *From the definition of the inner product, we note that $c_\nu(n, m) = \cos \angle(\mathcal{P}_n, \mathcal{P}_m)$ and $\arccos c_\nu(n, m)$ measures the angle between the polynomials \mathcal{P}_n and \mathcal{P}_m denoted by $\angle(\mathcal{P}_n, \mathcal{P}_m)$. In particular, the sequence $(\mathcal{P}_n)_{n \geq 0}$ is orthogonal if and only if $c_\nu(n, m) = 0$ for $n \neq m$.*

Lemma 2.1. For any $f, g \in L^2(\nu)$ and $t \geq 0$,

$$\rho_\nu(f(X_t), g(X_t)) = \frac{\langle f, g \rangle_\nu - \nu f \cdot \nu g}{\sqrt{\nu f^2 - (\nu f)^2} \cdot \sqrt{\nu g^2 - (\nu g)^2}}.$$

In particular, for any $m, n \in \mathbb{N}$ and $t \geq 0$,

$$\begin{aligned} \rho_\nu(\mathcal{P}_m(X_t), \mathcal{P}_n(X_t)) &= c_\nu(n, m), \\ \rho_\nu(\mathcal{P}_m(X_t), \mathcal{V}_n(X_t)) &= \kappa_\nu^{-1}(m) \delta_{mn}. \end{aligned}$$

The proof of Lemma 2.1 is presented in Section 4.2.

Remark 2.5. Note that if $f, g \in L^2(\nu)$ are such that $\nu f = \nu g = 0$, then for any $t \geq 0$

$$\rho_\nu(f(X_t), g(X_t)) = \frac{\langle f, g \rangle_\nu}{\|f\|_\nu \|g\|_\nu}.$$

We now proceed by studying the effect of the stochastic time-change in the analysis of the spectral projections correlation function. First, we start with Bochner subordination. To this end, recall that $\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}$ is a subordinator with Laplace exponent φ and transition kernel $\mu_t(ds)$, i.e.

$$\mathbb{E} \left[e^{-\lambda \mathcal{T}_t} \right] = \int_0^\infty e^{-\lambda s} \mu_t(ds) = e^{-t\varphi(\lambda)}, \quad \lambda > 0, t \geq 0,$$

where $\varphi(\lambda) = \varrho \lambda + \int_0^\infty (1 - e^{-\lambda y}) \vartheta(dy)$ with $\varrho \geq 0$, and ϑ being a Lévy measure satisfying $\int_0^\infty (1 \wedge y) \vartheta(dy) < \infty$. Denote the semigroup of the subordinated process by $P^\varphi = (P_t^\varphi)_{t \geq 0}$, i.e. for $f \in \mathcal{B}_b(E)$ and $t \geq 0$,

$$P_t^\varphi f(x) = \mathbb{E}_x[f(X_{\mathcal{T}_t})].$$

We recall from Section 1.1 that P^φ defines an $L^2(\nu)$ -Markov semigroup with ν as an invariant measure. By combining Proposition 1.1 and Theorem 2.1, we obtain the following characterization of the spectral projections correlation structure of the subordinated process.

Corollary 2.1. Moreover, for $m, n \in \mathbb{N}$ and $t \geq s > 0$, we have

$$(2.2) \quad \rho_\nu(\mathcal{P}_m(X_{\mathcal{T}_t}), \mathcal{V}_n(X_{\mathcal{T}_s})) = e^{-\varphi(\lambda_m)(t-s)} \kappa_\nu^{-1}(m) \delta_{mn},$$

and

$$(2.3) \quad \rho_\nu(\mathcal{P}_m(X_{\mathcal{T}_t}), \mathcal{P}_n(X_{\mathcal{T}_s})) = e^{-\varphi(\lambda_m)(t-s)} c_\nu(n, m).$$

Remark 2.6. Since $(\mathcal{P}_n, \mathcal{V}_n)_{n \in \mathbb{N}}$ form a biorthogonal sequence in $L^2(\nu)$, and are, respectively, the eigenfunctions of P^φ and its adjoint in $L^2(\nu)$, the correlation function $\rho_\nu(\mathcal{P}_m(X_{\mathcal{T}_t}), \mathcal{P}_n(X_{\mathcal{T}_s}))$ (resp. $\rho_\nu(\mathcal{P}_m(X_{\mathcal{T}_t}), \mathcal{V}_n(X_{\mathcal{T}_s}))$) is the (resp. biorthogonal) spectral projections correlation function of the process $(X_{\mathcal{T}_t})_{t \geq 0}$.

We continue with another stochastic time-change given by an inverse of a subordinator, which, as explained in Section 1.1, gives rise to a non-Markovian process. Recall that the inverse of the subordinator \mathcal{T} is defined for $t > 0$ by $L_t = \inf\{s > 0; \mathcal{T}_s > t\}$, its distribution is denoted by l_t , and its Laplace transform by η_t , that is for any $\lambda > 0$,

$$\eta_t(\lambda) = \int_0^\infty e^{-\lambda s} l_t(ds).$$

Also recall that we assume $\eta_t(0) = \int_0^\infty l_t(ds) = 1$ for all $t \geq 0$. Then, $P^\eta = (P_t^\eta)_{t \geq 0}$, defined, for $t \geq 0$ and $f \in L^2(\nu)$, by

$$P_t^\eta f(x) = \int_0^\infty P_s f(x) l_t(ds),$$

is a linear operator, and the corresponding time-changed process will be denoted by $X_L = (X_{L_t})_{t \geq 0}$. Note that Leonenko et al. [25], Mijena and Nane [31] and Aletti et al. [3] characterize the correlation structure of so-called Pearson diffusions when they are time-changed by an inverse of a linear combination of independent stable subordinators. We extend their methodology by first considering a general Markov process with biorthogonal spectral projections, and then time-changing it with an inverse of any independent subordinator. We also point out that by following a line of reasoning similar to the proof of Proposition 1.1, it can be shown that the biorthogonal sequence $(\mathcal{P}_n, \mathcal{V}_n)_{n \in \mathbb{N}}$ represents a set of eigenfunctions of the linear operator P_t^η , $t \geq 0$ and its adjoint in $L^2(\nu)$, respectively. Thus, $\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_s}))$ (resp. $\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{V}_n(X_{L_s}))$) is the (resp. biorthogonal) spectral projections correlation function of the process X_L . Finally, we set the following notation.

- (a) We write $f \stackrel{a}{\sim} g$ for $a \in [0, \infty]$ if $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = 1$. We may write $f(x) \stackrel{x \rightarrow a}{\sim} g(x)$ to emphasize dependency on the variable x .
 - (a1) f is called a long-tailed function if $\tau_y f(x) \stackrel{x \rightarrow \infty}{\sim} f(x)$ for any fixed $y > 0$, where $\tau_y f(x) = f(x + y)$ is the shift operator.
 - (a2) f is called slowly varying at 0 if $d_a f(x) \stackrel{x \rightarrow 0}{\sim} f(x)$ for any fixed $a > 0$, where $d_a f(x) = f(ax)$ is the dilation operator.
 - (a3) We say that f is strongly regularly varying at a with index $0 < \alpha < 1$ if $f \stackrel{a}{\sim} p_\alpha$, where $p_\alpha(x) = Cx^\alpha$ for some constant $C > 0$.
- (b) We write $f \stackrel{a}{\asymp} g$ if there exists a constant $C > 0$ such that $\frac{1}{C}g(x) \leq f(x) \leq Cg(x)$ for $x \geq a$.
- (c) We write $f = O(g)$ if $\overline{\lim}_{x \rightarrow \infty} \left| \frac{f(x)}{g(x)} \right| < \infty$.

We are now ready to state our last main result.

Theorem 2.2. *Let $m, n \in \mathbb{N}$. Then, for $t \geq s > 0$,*

$$(2.4) \quad \rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_s})) = c_\nu(n, m) \left(\lambda_m \int_0^s \eta_{t-r}(\lambda_m) U(dr) + \eta_t(\lambda_m) \right),$$

and

$$(2.5) \quad \rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{V}_n(X_{L_s})) = \kappa_\nu^{-1}(m) \delta_{mn} \left(\lambda_m \int_0^s \eta_{t-r}(\lambda_m) U(dr) + \eta_t(\lambda_m) \right),$$

where $U(dr) = \int_0^\infty \mathbb{P}(\mathcal{T}_t \in dr) dt$ is the renewal measure of the subordinator \mathcal{T} . Moreover, for any fixed $s > 0$,

$$c_\nu(n, m) \eta_t(\lambda_m) (\lambda_m \mathbb{E}[L_s] + 1) \leq \rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_s})) \leq c_\nu(n, m) \eta_{t-s}(\lambda_m) (\lambda_m \mathbb{E}[L_s] + 1),$$

$$\kappa_\nu^{-1}(m) \delta_{mn} \eta_t(\lambda_m) (\lambda_m \mathbb{E}[L_s] + 1) \leq \rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{V}_n(X_{L_s})) \leq \kappa_\nu^{-1}(m) \delta_{mn} \eta_{t-s}(\lambda_m) (\lambda_m \mathbb{E}[L_s] + 1).$$

Furthermore, if for a fixed $s > 0$, $\overline{\lim}_{t \rightarrow \infty} \frac{\eta_{t-s}(\lambda_m)}{\eta_t(\lambda_m)} = C$ for some constant $C = C(s, \lambda_m) (\lambda_m \mathbb{E}[L_s] + 1)$, then there exists $t_0 > 0$ such that

$$\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_s})) \stackrel{t_0}{\asymp} c_\nu(n, m) \eta_t(\lambda_m) (\lambda_m \mathbb{E}[L_s] + 1),$$

$$\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{V}_n(X_{L_s})) \stackrel{t_0}{\asymp} \kappa_\nu^{-1}(m) \delta_{mn} \eta_t(\lambda_m) (\lambda_m \mathbb{E}[L_s] + 1).$$

In particular, if $t \mapsto \eta_t(\lambda_m)$ is a long-tailed function, we have

$$(2.6) \quad \rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_s})) \stackrel{t \rightarrow \infty}{\asymp} c_\nu(n, m) \eta_t(\lambda_m) (\lambda_m \mathbb{E}[L_s] + 1),$$

$$(2.7) \quad \rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{V}_n(X_{L_s})) \stackrel{t \rightarrow \infty}{\asymp} \kappa_\nu^{-1}(m) \delta_{mn} \eta_t(\lambda_m) (\lambda_m \mathbb{E}[L_s] + 1).$$

The proof of this theorem is presented in Section 4.4. We complete this part with the following result which provides a sufficient condition for η_t to be a long-tailed function.

Proposition 2.1. *For any $\lambda > 0$, there exists a positive random variable X_λ such that $\eta_t(\lambda)$ is the tail of its distribution, i.e. $\eta_t(\lambda) = \mathbb{P}(X_\lambda > t)$, $t > 0$. Moreover, $\eta_t(\lambda)$ is a long-tailed distribution if φ is strongly regularly varying at 0.*

The proof of this proposition is presented in Section 4.3.

2.1. Interpretation of the (biorthogonal) spectral projections correlation functions for statistical properties. The results presented above regarding the (biorthogonal) spectral projections correlation functions and their asymptotic behavior provide an interesting approach for designing statistical tests in order to identify substantial properties of a stochastic process. More formally, we start by assuming that the sample $\widehat{\mathbf{X}} = (\widehat{\mathbf{X}}_1, \dots, \widehat{\mathbf{X}}_T)$, with $T \in \mathbb{N}$ large, is coming from a stochastic process \mathbf{X} which belongs to some family with a marginal stationary measure $(\nu_i)_{i \in I}$ and associated biorthogonal sequence $\left((\mathcal{P}_n^{(i)}, \mathcal{V}_n^{(i)})_{n \in \mathbb{N}} \right)_{i \in I}$ as defined in Assumption 1, where I is the index set of the family. For example, in the case when \mathbf{X} belongs to the family of generalized Laguerre processes presented in Section 3.1 below, we can consider one element from each of the following sub-families: a pure diffusion, a diffusion component and jumps with finite activity, a diffusion component and jumps with infinite activity, and a pure jump process. Now, based on the (biorthogonal) spectral projections correlation structure, one can identify

- (a) how far from symmetry (self-adjointness) the process is,
- (b) what type of range dependence (short-to-long) it displays, and
- (c) the path properties of the process (cádlág or continuous paths).

For designing statistical tests, one can rely on the estimates of $\kappa_{\nu_i}(m)$ and/or $c_{\nu_i}(n, m)$, $i \in I$, $n, m \in \mathbb{N}$. Since $(\kappa_{\nu_i}(m))_{m \in \mathbb{N}}$ contain information about both of the sequences $(\mathcal{P}_n^{(i)})_{n \in \mathbb{N}}$ and $(\mathcal{V}_n^{(i)})_{n \in \mathbb{N}}$, below we describe some statistical tests involving the condition number. However, the estimates of $c_{\nu_i}(n, m)$ can be useful to further refine the search of the process. More precisely, based on the main results presented in Section 2, one can make the following implications.

- (a) To study the possible *departure from symmetry* of $\widehat{\mathbf{X}}$, see Remark 2.3, following the results provided by Lemma 2.1, we first take $t = s = k$ for some $k \in \{1, \dots, T\}$ and $m = n \in \mathbb{N}$. Then, since the marginal stationary measure guarantees that the statistical properties of the process do not change over time, for each $i \in I$, we compute the empirical estimates of the condition number $\kappa_{\nu_i}(m)$ for some $m \in \mathbb{N}$, by

$$(2.8) \quad \begin{aligned} \widehat{\kappa}_{\nu_i}^{-1}(m) &= \widehat{\rho}_{\nu_i}(\mathcal{P}_m^{(i)}(\widehat{\mathbf{X}}_k), \mathcal{V}_m^{(i)}(\widehat{\mathbf{X}}_k)) \\ &= \frac{\sum_{j=1}^T \left(\mathcal{P}_m^{(i)}(\widehat{\mathbf{X}}_j) - \overline{\mathcal{P}}_m^{(i)}(\widehat{\mathbf{X}}) \right) \left(\mathcal{V}_m^{(i)}(\widehat{\mathbf{X}}_j) - \overline{\mathcal{V}}_m^{(i)}(\widehat{\mathbf{X}}) \right)}{\sqrt{\sum_{j=1}^T \left(\mathcal{P}_m^{(i)}(\widehat{\mathbf{X}}_j) - \overline{\mathcal{P}}_m^{(i)}(\widehat{\mathbf{X}}) \right)^2} \sqrt{\sum_{j=1}^T \left(\mathcal{V}_m^{(i)}(\widehat{\mathbf{X}}_j) - \overline{\mathcal{V}}_m^{(i)}(\widehat{\mathbf{X}}) \right)^2}}, \end{aligned}$$

where $\overline{\mathcal{P}}_m^{(i)}(\widehat{\mathbf{X}}) = \frac{1}{T} \sum_{j=1}^T \mathcal{P}_m^{(i)}(\widehat{\mathbf{X}}_j)$ and $\overline{\mathcal{V}}_m^{(i)}(\widehat{\mathbf{X}}) = \frac{1}{T} \sum_{j=1}^T \mathcal{V}_m^{(i)}(\widehat{\mathbf{X}}_j)$ are the sample means. Next, we compute the theoretical condition number by

$$\kappa_{\nu_i}(m) = \|\mathcal{P}_m^{(i)}\|_{\nu_i} \|\mathcal{V}_m^{(i)}\|_{\nu_i}.$$

Finally, to identify the couple $(\mathcal{P}_n^{(i)}, \mathcal{V}_n^{(i)})_{n \in \mathbb{N}}$, we choose $\epsilon_S > 0$, and check if

$$(2.9) \quad |\kappa_{\nu_i}(m) - \widehat{\kappa}_{\nu_i}(m)| < \epsilon_S.$$

For the next step, for sake of simplicity, we suppose that there is only one $\bar{i} \in I$ such that the condition (2.9) is satisfied.

- (b) To assess the *range dependence* of the sample, we study the asymptotic behavior of the empirical correlation $\widehat{\rho}_{\nu_i}(\mathcal{P}_m^{(\bar{i})}(\widehat{\mathbf{X}}_k), \mathcal{V}_m^{(\bar{i})}(\widehat{\mathbf{X}}_j))$, $k, j \in \{1, \dots, T\}$, $k > j$, $m \in \mathbb{N}$. More formally, we first compute $\widehat{\kappa}_{\nu_i}(m)$ by (2.8), fix some $j \in \{1, \dots, T\}$ (one can simply set $j = 1$ or $j = 2$), and we proceed by studying

$$(2.10) \quad g_{\lambda_m}(k) = \widehat{\kappa}_{\nu_i}(m) \cdot \widehat{\rho}_{\nu_i}(\mathcal{P}_m^{(\bar{i})}(\widehat{\mathbf{X}}_k), \mathcal{V}_m^{(\bar{i})}(\widehat{\mathbf{X}}_j)).$$

Now, if $k \mapsto g_{\lambda_m}(k)$, $j < k \in \{1, \dots, T\}$ exhibits exponential decay with respect to λ_m , then we have short-range dependence. In contrast, if it exhibits a polynomial decay, then the process has long-range dependence, and, in particular, it is not a (subordinated) Markov process. We remark that although these two cases are the most popular ones discussed in the literature, depending on the rate of decay of the correlation function, the process can exhibit short-to-long-range dependence. The concept of long-range dependence has been repeatedly used to describe properties of financial time series such as stock prices, foreign exchange rates, market indices and commodity prices. In this context, based on the behavior of (biorthogonal) spectral projections correlation functions, in their working paper [20], the authors provide a more detailed empirical study to detect the (short-to-long-) range dependence in volatility in financial markets.

- (c) Finally, to study the *path properties* of the process, i.e. the presence of jumps and their activity, we study the behavior of $\widehat{\kappa}_{\nu_i}(m)$ for large m . To illustrate this with a specific example, let us consider the class of generalized Laguerre processes introduced in Section 3.1. Note that this class encompasses a range of symmetries and jumps. Then, one can identify the following cases.
- (i) If $\widehat{\kappa}_{\nu_i}(m) = 1$, $m \in \mathbb{N}$, then the process is a pure diffusion, see Section 3.2.
 - (ii) If $\widehat{\kappa}_{\nu_i}(m) = O(m^\beta)$ for some β , then the process has both a diffusion component and a jump component with finite activity, see Section 3.3.
 - (iii) If $\widehat{\kappa}_{\nu_i}(m) = O(e^{\epsilon m})$ for any $\epsilon > 0$, then, similarly, the process has both diffusion and jump components while in this case jumps have infinite activity, see Section 3.4.
 - (iv) If $\widehat{\kappa}_{\nu_i}(m) = O(e^{m^\beta})$ for some β , then the process is a pure jump process.

The problem of deciding whether the continuous-time process which models an economic or financial time series has continuous paths or exhibits jumps is an important issue. For example, Ait-Sahalia and Jacod [1] design a test to identify the presence of jumps in a discretely observed semimartingale, based on power variations sampled at different frequencies. Furthermore, in this setting, the authors of [2] propose statistical tests to discriminate between the finite and infinite activity of jumps in a semimartingale. We emphasize that our approach allows one to design a statistical test in order to identify both the presence and the types of jumps.

3. EXAMPLES

In this section, we illustrate the results of Section 2 for the class of generalized Laguerre semigroups which have been studied in depth by Patie and Savov in [35]. To investigate the behavior of (biorthogonal) spectral projections correlation structure in various scenarios, we first discuss two important examples of subordinators and their inverses.

Example 1. Let \mathcal{T} be an α -stable subordinator, i.e. in (1.7), $\varphi(\lambda) = \lambda^\alpha$, $0 < \alpha < 1$. We recall from [25] that for any $\lambda > 0$ and $t \geq 0$, the Laplace transform of its inverse is given by

$$\eta_t(\lambda) = E_\alpha(-\lambda t^\alpha),$$

where E_α is the Mittag-Leffler function defined by $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$ for $z \in \mathbb{C}$. On the other hand, since $U(ds) = \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds$, $s > 0$, we have that $\mathbb{E}[L_s] = U(0, s) = \int_0^s U(dr) = \frac{s^\alpha}{\Gamma(1+\alpha)}$. Now, Corollary 2.1 yields that for any $m, n \in \mathbb{N}$ and $t \geq s > 0$,

$$\begin{aligned} \rho_\nu(\mathcal{P}_m(X_{\mathcal{T}_t}), \mathcal{V}_n(X_{\mathcal{T}_s})) &= e^{-\lambda_m^\alpha(t-s)} \kappa_\nu^{-1}(m) \delta_{mn}, \\ \rho_\nu(\mathcal{P}_m(X_{\mathcal{T}_t}), \mathcal{P}_n(X_{\mathcal{T}_s})) &= e^{-\lambda_m^\alpha(t-s)} c_\nu(n, m). \end{aligned}$$

Next, it follows from Theorem 2.2 that

$$\begin{aligned} \rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_s})) &= c_\nu(n, m) \left(\lambda_m \int_0^s \eta_{t-r}(\lambda_m) U(dr) + \eta_t(\lambda_m) \right) \\ &= c_\nu(n, m) \left(\lambda_m \int_0^s E_\alpha(-\lambda_m(t-r)^\alpha) \frac{r^{\alpha-1}}{\Gamma(\alpha)} dr + E_\alpha(-\lambda_m t^\alpha) \right) \\ &= \frac{c_\nu(n, m) \lambda_m t^\alpha}{\Gamma(\alpha)} \int_0^{s/t} \frac{E_\alpha(-\lambda_m t^\alpha (1-z)^\alpha)}{z^{1-\alpha}} dz + c_\nu(n, m) E_\alpha(-\lambda_m t^\alpha). \end{aligned}$$

Note that since φ is strongly regularly varying at 0, we have, by Proposition 2.1, that $t \mapsto \eta_t$ is a long-tailed function. Furthermore, it is well known, see e.g. [25], that when $t \rightarrow \infty$,

$$\eta_t(\lambda) = E_\alpha(-\lambda t^\alpha) \sim \frac{1}{\Gamma(1-\alpha)\lambda t^\alpha}.$$

Hence, from Theorem 2.2, we deduce that for a fixed $s > 0$, when $t \rightarrow \infty$, we have

$$(3.1) \quad \rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_s})) \sim \frac{c_\nu(n, m)}{\Gamma(1-\alpha)t^\alpha} \left(\frac{1}{\lambda_m} + \frac{s^\alpha}{\Gamma(1+\alpha)} \right),$$

and, similarly,

$$\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{V}_n(X_{L_s})) \sim \frac{\kappa_\nu^{-1}(m) \delta_{mn}}{\Gamma(1-\alpha)t^\alpha} \left(\frac{1}{\lambda_m} + \frac{s^\alpha}{\Gamma(1+\alpha)} \right).$$

When X is a Pearson diffusion, we note that (3.1) boils down to the case discussed in [25]. Finally, since the correlation functions decay in a polynomial rate of $\alpha \in (0, 1)$, here the process X_L exhibits long-range dependence.

Example 2. Let \mathcal{T} be a Poisson subordinator with mean $\frac{1}{\theta}$, i.e. in (1.7), $\varphi(\lambda) = \theta(1 - e^{-\lambda})$. Then, for the inverse Poisson subordinator, we have that L_t follows $\text{Gam}([t+1], 1/\theta)$, see Leonenko et al. [24]. Using the moment generating function of a Gamma random variable, we get

$$(3.2) \quad \eta_t(\lambda) = \int_0^\infty e^{-\lambda s} l_t(ds) = \left(1 + \frac{\lambda}{\theta} \right)^{-[t+1]},$$

and thus $U(0, s) = \mathbb{E}[L_s] = \frac{[s+1]}{\theta}$. Then, it follows from Corollary 2.1 that for any $m, n \in \mathbb{N}$ and $t \geq s > 0$,

$$\begin{aligned} \rho_\nu(\mathcal{P}_m(X_{\mathcal{T}_t}), \mathcal{V}_n(X_{\mathcal{T}_s})) &= e^{-\theta(1-e^{-\lambda_m})(t-s)} \kappa_\nu^{-1}(m) \delta_{mn}, \\ \rho_\nu(\mathcal{P}_m(X_{\mathcal{T}_t}), \mathcal{P}_n(X_{\mathcal{T}_s})) &= e^{-\theta(1-e^{-\lambda_m})(t-s)} c_\nu(n, m). \end{aligned}$$

Now, when $t \rightarrow \infty$, note that $\eta_t(\lambda) \sim (1 + \frac{\lambda}{\theta})^{-t}$. Consequently $\lim_{t \rightarrow \infty} \frac{\eta_{t-s}(\lambda_m)}{\eta_t(\lambda_m)} \neq 1$, and therefore (2.6) and (2.7) do not hold. However, we are able to compute the exact formulas for the (biorthogonal) spectral projections correlation functions of X_L as follows. First, noting that for any $k \in \mathbb{N}$ such that $k < t$, $-[t - k + 1] = k - [t + 1]$, we have

$$\begin{aligned} \lambda_m \int_0^s \eta_{t-r}(\lambda_m) U(dr) + \eta_t(\lambda_m) &= \frac{\lambda_m}{\theta} \sum_{k=0}^{[s]} \left(1 + \frac{\lambda_m}{\theta}\right)^{-[t-k+1]} + \left(1 + \frac{\lambda_m}{\theta}\right)^{-[t+1]} \\ (3.3) \qquad \qquad \qquad &= \left(1 + \frac{\lambda_m}{\theta}\right)^{-[t+1]} \left(2 - \left(1 + \frac{\lambda_m}{\theta}\right)^{[s+1]}\right). \end{aligned}$$

Thus, it follows from Theorem 2.2 that

$$\begin{aligned} \rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_s})) &= c_\nu(n, m) \left(1 + \frac{\lambda_m}{\theta}\right)^{-[t+1]} \left(2 - \left(1 + \frac{\lambda_m}{\theta}\right)^{[s+1]}\right), \\ \rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{V}_n(X_{L_s})) &= \kappa_\nu^{-1}(m) \delta_{mn} \left(1 + \frac{\lambda_m}{\theta}\right)^{-[t+1]} \left(2 - \left(1 + \frac{\lambda_m}{\theta}\right)^{[s+1]}\right). \end{aligned}$$

Since the spectral projections correlation functions decay in an exponential rate, the time-changed process X_L exhibits short-range dependence although it is non-Markovian.

3.1. A short review of the generalized Laguerre semigroups. In this section, we provide a short description of the so-called generalized Laguerre processes introduced and studied by Patie and Savov [35], see also Patie et al. [36]. We point out that these processes have been recently used to model asset price dynamics in Jarrow et al. [21]. To this end, let $\tilde{\mathbf{A}}$ be the infinitesimal generator of classical Laguerre process which in financial literature was introduced in 1985 by John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross [12], and is known as a Cox-Ingersoll-Ross (for short CIR) process, i.e. for at least $f \in C_0^2(\mathbb{R}_+)$, we have

$$(3.4) \qquad \qquad \qquad \tilde{\mathbf{A}}f(x) = \sigma^2 x f''(x) + (\beta + \sigma^2 - x) f'(x),$$

where $\beta, \sigma \geq 0$. We say that a semigroup $P = (P_t)_{t \geq 0}$ is a *generalized Laguerre (gL) semigroup* if its infinitesimal generator is given, for a smooth function f on $x > 0$, by

$$(3.5) \qquad \qquad \mathbf{A}f(x) = \tilde{\mathbf{A}}f(x) + \int_0^\infty (f(e^{-y}x) - f(x) + yx f'(x)) \Pi(x, dy),$$

where $\Pi(x, dy) = \frac{\Pi(dy)}{x}$, with Π being a Lévy measure concentrated on $(0, \infty)$ and satisfying the integrability condition $\int_0^\infty (y^2 \wedge y) \Pi(dy) < \infty$. We call the corresponding process $X = (X_t)_{t \geq 0}$ a *generalized Laguerre process*. Note that when $\Pi(0, \infty) = 0$, then P boils down to the semigroup of a classical Laguerre process. Moreover, from [35, Theorem 1.6] we have that the semigroup P admits a unique invariant measure, which in this case is absolutely continuous with a density that we denote by ν , and write the Hilbert space $L^2(\nu)$ as in Section 1. Recall that P can be extended to a contraction semigroup in $L^2(\nu)$, and by an abuse of notation, we still denote it by P . Now, [35, Theorem 1.11] yields that if $\bar{\Pi}(y) = \int_y^\infty \Pi(dr)$ is strongly regularly varying at 0 with some index $\alpha \in (0, 1)$, then, for any $f \in L^2(\nu)$ and $t > T_\Pi$ for some explicit T_Π (with $T_\Pi = 0$ when $\sigma^2 > 0$), we have the following spectral expansion,

$$P_t f = \sum_{n=0}^{\infty} e^{-nt} \langle f, \mathcal{V}_n \rangle_\nu \mathcal{P}_n \quad \text{in } L^2(\nu),$$

where $(\mathcal{P}_n, \mathcal{V}_n)_{n \geq 0}$ form a biorthogonal sequence of $L^2(\nu)$, and are expressed as follows:

$$\mathcal{P}_n(x) = \sum_{k=0}^n (-1)^k \frac{\binom{n}{k}}{W_\phi(k+1)} x^k \in L^2(\nu),$$

and,

$$\mathcal{V}_n(x) = \frac{\mathcal{R}^{(n)}\nu(x)}{\nu(x)} = \frac{(x^n \nu(x))^{(n)}}{n! \nu(x)} \in L^2(\nu),$$

with the last equation serving as a definition of the Rodrigues operator. Here, $W_\phi(1) = 1$ and, for $n \in \mathbb{N}$, $W_\phi(n+1) = \prod_{k=1}^n \phi(k)$, where ϕ is the Bernstein function, see (1.6), which takes the form

$$\phi(\lambda) = \beta + \sigma^2 \lambda + \int_0^\infty (1 - e^{-\lambda y}) \bar{\Pi}(y) dy,$$

with Π, β, σ^2 as in (3.5). Furthermore, by [35, Theorem 7.3 and Proposition 8.4] we have, that for any $n \geq 0$ and $t \geq 0$, \mathcal{P}_n (resp. \mathcal{V}_n) is an eigenfunction for P_t (resp. P_t^*) associated to the eigenvalue e^{-nt} , i.e. $\mathcal{P}_n, \mathcal{V}_n \in L^2(\nu)$ and

$$P_t \mathcal{P}_n(x) = e^{-nt} \mathcal{P}_n(x) \quad \text{and} \quad P_t^* \mathcal{V}_n(x) = e^{-nt} \mathcal{V}_n(x),$$

with $(P_t^*)_{t \geq 0}$ being the adjoint of $(P_t)_{t \geq 0}$ in $L^2(\nu)$. Therefore, in this case, we have $\lambda_n = n$, $n \in \mathbb{N}$.

Next, we describe the eigenvalue expansions of specific instances of the generalized Laguerre semi-groups which illustrate the different situations that are ranging from the self-adjoint case to perturbation of a self-adjoint differential operator through non-local operators without diffusion component. We study their spectral projections correlation structure, and discuss some of their important properties as are range dependence and symmetry (self-adjointness), among others.

3.2. The self-adjoint diffusion case. For any $\beta > 0$, the infinitesimal generator of the classical Laguerre process takes the form

$$\mathbf{A}_\beta f(x) = x f''(x) + (\beta + 1 - x) f'(x).$$

Note that this is the infinitesimal generator of a one-dimensional diffusion often referred in the literature as the CIR process. The eigenfunctions are given by

$$\bar{\mathcal{L}}_n^{(\beta)}(x) = \sqrt{\mathbf{c}_n(\beta)} \mathcal{L}_n^{(\beta)}(x),$$

where $\mathbf{c}_n(\beta) = \frac{\Gamma(n+1)\Gamma(\beta+1)}{\Gamma(n+\beta+1)}$ and $\mathcal{L}_n^{(\beta)}(x) = \sum_{k=0}^n (-1)^k \binom{n+\beta}{n-k} \frac{x^k}{k!}$ is the associated Laguerre polynomial of order β . Denote by

$$(3.6) \quad \gamma_\beta(dx) = \frac{x^\beta e^{-x}}{\Gamma(\beta+1)} dx, \quad x > 0,$$

the law of a Gamma random variable with parameter $(\beta + 1)$. Then, the semigroup is self-adjoint in $L^2(\gamma_\beta)$ and the sequence $(\bar{\mathcal{L}}_n^{(\beta)})_{n \geq 0}$ forms an orthonormal basis of $L^2(\gamma_\beta)$. In particular, this means that, for $n, m \in \mathbb{N}$, we have that

$$\kappa_{\gamma_\beta}(m) = 1 \quad \text{and} \quad c_{\gamma_\beta}(n, m) = \delta_{nm}.$$

Hence, it follows from Theorem 2.1, that for $m, n \in \mathbb{N}$ and $t \geq s > 0$,

$$\rho_{\gamma_\beta}(\bar{\mathcal{L}}_m^{(\beta)}(X_t), \bar{\mathcal{L}}_n^{(\beta)}(X_s)) = e^{-m(t-s)} \delta_{nm}.$$

Now, from Corollary 2.1, Theorem 2.2 and Examples 1 and 2 we have the following additional results.

- Let \mathcal{T} be an α -**stable subordinator**, i.e. $\varphi(\lambda) = \lambda^\alpha$, $0 < \alpha < 1$, see Example 1. Then, for any $t \geq s > 0$,

$$\rho_{\gamma_\beta}(\overline{\mathcal{L}}_m^{(\beta)}(X_{\mathcal{T}_t}), \overline{\mathcal{L}}_n^{(\beta)}(X_{\mathcal{T}_s})) = e^{-m^\alpha(t-s)} \delta_{nm}.$$

- Let \mathcal{T} be a **Poisson subordinator** with parameter θ , i.e. $\varphi(\lambda) = \theta(1 - e^{-\lambda})$, see Example 2. Then, for any $t \geq s > 0$,

$$\rho_{\gamma_\beta}(\overline{\mathcal{L}}_m^{(\beta)}(X_{\mathcal{T}_t}), \overline{\mathcal{L}}_n^{(\beta)}(X_{\mathcal{T}_s})) = e^{-\theta(1 - e^{-m})(t-s)} \delta_{nm}.$$

- Let L be the **inverse of an α -stable subordinator**, see Example 1. Then,

$$\rho_{\gamma_\beta}(\overline{\mathcal{L}}_m^{(\beta)}(X_{L_t}), \overline{\mathcal{L}}_n^{(\beta)}(X_{L_s})) = \frac{\delta_{nm} m t^\alpha}{\Gamma(\alpha)} \int_0^{s/t} \frac{E_\alpha(-m t^\alpha (1-z)^\alpha)}{z^{1-\alpha}} dz + \delta_{nm} E_\alpha(-m t^\alpha).$$

Furthermore, for a fixed $s > 0$, when $t \rightarrow \infty$,

$$\rho_{\gamma_\beta}(\overline{\mathcal{L}}_m^{(\beta)}(X_{L_t}), \overline{\mathcal{L}}_n^{(\beta)}(X_{L_s})) \sim \frac{\delta_{nm}}{\Gamma(1-\alpha)t^\alpha} \left(\frac{1}{m} + \frac{s^\alpha}{\Gamma(1+\alpha)} \right).$$

- Let L be the **inverse of a Poisson subordinator** with parameter θ , see Example 2. Then, for any $t \geq s > 0$,

$$\rho_{\gamma_\beta}(\overline{\mathcal{L}}_m^{(\beta)}(X_{L_t}), \overline{\mathcal{L}}_n^{(\beta)}(X_{L_s})) = \delta_{nm} \left(1 + \frac{m}{\theta}\right)^{-[t+1]} \left(2 - \left(1 + \frac{m}{\theta}\right)^{[s+1]}\right).$$

3.3. Small perturbation of the Laguerre semigroup. Let $b \geq 1$, and take $\sigma^2 = 1$, $\beta = \frac{b^2-1}{b}$ and $\overline{\Pi}(y) = e^{-by}$, $y \geq 0$ in (3.5), i.e. we consider, for f smooth,

$$\mathbf{A}^{(b)} f(x) = x f''(x) + \left(\frac{b^2-1}{b} + 1 - x \right) f'(x) + \frac{b}{x} \int_0^\infty (f(e^{-y}x) - f(x) + y x f'(x)) e^{-by} dy.$$

The associated semigroup is ergodic with a unique invariant measure ν_b ,

$$\nu_b(dx) = \frac{(1+x)}{b+1} \gamma_{b-1}(dx), \quad x > 0,$$

with $\gamma_{b-1}(dx)$ as in (3.6). Then, the eigenfunctions and co-eigenfunctions $(\mathcal{P}_n^{(b)}, \mathcal{V}_n^{(b)})_{n \geq 0}$ are expressed in terms of Laguerre polynomials $(\mathcal{L}_n^{(b)})_{n \geq 0}$ as follows, $n \geq 0$,

$$\begin{aligned} \mathcal{P}_n^{(b)}(x) &= \mathbf{c}_n(b+1) \mathcal{L}_n^{(b+1)}(x) - \frac{\mathbf{c}_n(b+1)}{b} x \mathcal{L}_{n-1}^{(b+2)}(x), \\ \mathcal{V}_n^{(b)}(x) &= \frac{1}{x+1} \mathcal{L}_n^{(b-1)}(x) + \frac{x}{x+1} \mathcal{L}_n^{(b)}(x), \end{aligned}$$

where $\mathbf{c}_n(\cdot)$ and $(\mathcal{L}_n^{(b)})_{n \geq 0}$ are defined as in Section 3.2, see [35, Example 3.2]. Next, it follows from [35, Theorem 2.2 and (3.9)] that

$$\|\mathcal{P}_n^{(b)}\|_{\nu_b} = O(1), \quad \|\mathcal{V}_n^{(b)}\|_{\nu_b} = O(n^{(b+1)/2}).$$

Then, for any $m \in \mathbb{N}$, we have

$$\kappa_{\nu_b}(m) = O(m^{(b+1)/2}),$$

and therefore, the biorthogonal sequence $(\mathcal{P}_n^{(b)}, \mathcal{V}_n^{(b)})_{n \in \mathbb{N}}$ is tame, see Remark 2.2 for definition. Thus, it follows from Theorem 2.1 that

$$\rho_{\nu_b}(\mathcal{P}_m^{(b)}(X_t), \mathcal{V}_n^{(b)}(X_s)) = e^{-m(t-s)} \kappa_{\nu_b}^{-1}(m) \delta_{nm}.$$

Now, from Corollary 2.1, Theorem 2.2 and Examples 1 and 2 we have the following results.

- Let \mathcal{T} be an α -stable subordinator, i.e. $\varphi(\lambda) = \lambda^\alpha$, $0 < \alpha < 1$, see Example 1. Then, for any $t \geq s > 0$,

$$\rho_{\nu_b}(\mathcal{P}_m^{(b)}(X_{\mathcal{T}_t}), \mathcal{V}_n^{(b)}(X_{\mathcal{T}_s})) = e^{-m^\alpha(t-s)} \kappa_{\nu_b}^{-1}(m) \delta_{nm}.$$

- Let \mathcal{T} be a Poisson subordinator with parameter θ , i.e. $\varphi(\lambda) = \theta(1 - e^{-\lambda})$, see Example 2. Then, for any $t \geq s > 0$

$$\rho_{\nu_b}(\mathcal{P}_m^{(b)}(X_{\mathcal{T}_t}), \mathcal{V}_n^{(b)}(X_{\mathcal{T}_s})) = e^{-\theta(1 - e^{-m})(t-s)} \kappa_{\nu_b}^{-1}(m) \delta_{nm}.$$

- Let L be the inverse of an α -stable subordinator, see Example 1. Then, for a fixed $s > 0$, when $t \rightarrow \infty$,

$$\rho_{\nu_b}(\mathcal{P}_m^{(b)}(X_{L_t}), \mathcal{V}_n^{(b)}(X_{L_s})) \sim \frac{\kappa_{\nu_b}^{-1}(m) \delta_{mn}}{\Gamma(1 - \alpha) t^\alpha} \left(\frac{1}{m} + \frac{s^\alpha}{\Gamma(1 + \alpha)} \right).$$

- Let L be the inverse of a Poisson subordinator with parameter θ , see Example 2. Then, for any $t \geq s > 0$,

$$\rho_{\nu_b}(\mathcal{P}_m^{(b)}(X_{L_t}), \mathcal{V}_n^{(b)}(X_{L_s})) = \kappa_{\nu_b}^{-1}(m) \delta_{mn} \left(1 + \frac{m}{\theta} \right)^{-[t+1]} \left(2 - \left(1 + \frac{m}{\theta} \right)^{[s+1]} \right).$$

3.4. The Gauss-Laguerre semigroup. We next consider the Gauss-Laguerre semigroup $P^{\alpha,b} = (P_t^{\alpha,b})_{t \geq 0}$ which has been introduced and extensively studied in [34], and which is an instance of the generalized Laguerre semigroups, see [35, Example 3.3]. In particular, its infinitesimal generator, for any $\alpha \in (0, 1)$ and $b \in [1 - \frac{1}{\alpha}, \infty)$, and for any given smooth function f , takes the form

$$\mathbf{A}^{(\alpha,b)} f(x) = (b_\alpha - x) f'(x) + \frac{\sin(\alpha\pi)}{\pi} x \int_0^1 f''(xy) g_{\alpha,b}(y) dy, \quad x > 0,$$

where $b_\alpha = \frac{\Gamma(\alpha b + \alpha + 1)}{\Gamma(\alpha b + 1)}$ and

$$g_{\alpha,b}(y) = \frac{\Gamma(\alpha)}{b + \frac{1}{\alpha} + 1} y^{b + \frac{1}{\alpha} + 1} {}_2F_1(\alpha(b + 1) + 1, \alpha + 1; \alpha(b + 1) + 2; y^{\frac{1}{\alpha}}),$$

with ${}_2F_1$ the Gauss hypergeometric function. The associated semigroup $P^{\alpha,b} = (P_t^{\alpha,b})_{t \geq 0}$ is a non-self-adjoint contraction in $L^2(\mathbf{e}_{\alpha,b})$, where

$$\mathbf{e}_{\alpha,b}(dx) = \frac{x^{b + \frac{1}{\alpha} - 1} e^{-x^{\frac{1}{\alpha}}}}{\Gamma(\alpha b + 1)} dx, \quad x > 0,$$

is its unique invariant measure. For any $x \geq 0$, we set $\mathcal{P}_0^{(\alpha,b)}(x) = 1$ and for any $n \geq 1$,

$$\begin{aligned} \mathcal{P}_n^{(\alpha,b)}(x) &= \Gamma(\alpha b + 1) \sum_{k=0}^n (-1)^k \frac{\binom{n}{k}}{\Gamma(\alpha k + \alpha b + 1)} x^k, \\ \mathcal{V}_n^{(\alpha,b)}(x) &= \frac{(-1)^n}{n! \mathbf{e}_{\alpha,b}(x)} (x^n \mathbf{e}_{\alpha,b}(x))^{(n)}, \end{aligned}$$

which are the eigenfunctions and co-eigenfunctions of $P^{\alpha,b}$. It is worth mentioning that in [34, Proposition 3.3] the authors show that the $\mathcal{V}_n^{(\alpha,b)}$'s can be expressed in terms of sequences of polynomials as well. Then, it follows from [34, Proposition 2.3] and [35, Theorem 2.2] that

$$\|\mathcal{P}_n^{(\alpha,b)}\|_{\mathbf{e}_{\alpha,b}} = O(1), \quad \|\mathcal{V}_n^{(\alpha,b)}\|_{\mathbf{e}_{\alpha,b}} = O(e^{T_\alpha n}),$$

where $T_\alpha = -\ln(2^\alpha - 1)$. Then, we have that for any $m \in \mathbb{N}$,

$$\kappa_{\mathbf{e}_{\alpha,b}}(m) = O(e^{T_\alpha m}),$$

hence, in this case, the biorthogonal sequence $(\mathcal{P}_n^{(\alpha,b)}, \mathcal{V}_n^{(\alpha,b)})_{n \in \mathbb{N}}$ is a wild system in $L^2(\mathbf{e}_{\alpha,b})$. Now, it follows from Theorem 2.1 that

$$\rho_{\mathbf{e}_{\alpha,b}}(\mathcal{V}_n^{(\alpha,b)}(X_s), \mathcal{P}_m^{(\alpha,b)}(X_t)) = e^{-m(t-s)} \kappa_{\mathbf{e}_{\alpha,b}}^{-1}(m) \delta_{nm}.$$

Next, from Corollary 2.1, Theorem 2.2 and Examples 1 and 2 we have the following additional results.

- Let \mathcal{T} be an **α -stable subordinator**, i.e. $\varphi(\lambda) = \lambda^\alpha$, $0 < \alpha < 1$, see Example 1. Then, for any $t \geq s > 0$,

$$\rho_{\mathbf{e}_{\alpha,b}}(\mathcal{P}_m^{(\alpha,b)}(X_{\mathcal{T}_t}), \mathcal{V}_n^{(\alpha,b)}(X_{\mathcal{T}_s})) = e^{-m^\alpha(t-s)} \kappa_{\mathbf{e}_{\alpha,b}}^{-1}(m) \delta_{nm}.$$

- Let \mathcal{T} be a **Poisson subordinator** with parameter θ , i.e. $\varphi(\lambda) = \theta(1 - e^{-\lambda})$, see Example 2. Then, for any $t \geq s > 0$

$$\rho_{\mathbf{e}_{\alpha,b}}(\mathcal{P}_m^{(\alpha,b)}(X_{\mathcal{T}_t}), \mathcal{V}_n^{(\alpha,b)}(X_{\mathcal{T}_s})) = e^{-\theta(1-e^{-m})(t-s)} \kappa_{\mathbf{e}_{\alpha,b}}^{-1}(m) \delta_{nm}.$$

- Let L be the **inverse of an α -stable subordinator**, see Example 1. Then, for a fixed $s > 0$, when $t \rightarrow \infty$,

$$\rho_{\mathbf{e}_{\alpha,b}}(\mathcal{P}_m^{(\alpha,b)}(X_{L_t}), \mathcal{V}_n^{(\alpha,b)}(X_{L_s})) \sim \frac{\kappa_{\mathbf{e}_{\alpha,b}}^{-1}(m) \delta_{mn}}{\Gamma(1-\alpha)t^\alpha} \left(\frac{1}{m} + \frac{s^\alpha}{\Gamma(1+\alpha)} \right).$$

- Let L be the **inverse of a Poisson subordinator** with parameter θ , see Example 2. Then, for any $t \geq s > 0$,

$$\rho_{\mathbf{e}_{\alpha,b}}(\mathcal{P}_m^{(\alpha,b)}(X_{L_t}), \mathcal{V}_n^{(\alpha,b)}(X_{L_s})) = \kappa_{\mathbf{e}_{\alpha,b}}^{-1}(m) \delta_{mn} \left(1 + \frac{m}{\theta}\right)^{-[t+1]} \left(2 - \left(1 + \frac{m}{\theta}\right)^{[s+1]}\right).$$

4. PROOFS OF THE MAIN RESULTS

4.1. **Proof of Theorem 2.1.** We split the proof of Theorem 2.1 into several intermediary lemmas.

Lemma 4.1. *The sequence $(\mathcal{P}_n, \mathcal{V}_n)_{n \in \mathbb{N}}$, defined in Assumption 1, forms a biorthogonal sequence in $L^2(\nu)$, i.e. for any $n, m \in \mathbb{N}$,*

$$(4.1) \quad \langle \mathcal{P}_n, \mathcal{V}_m \rangle_\nu = \delta_{nm}.$$

Proof. First, recall that, given Assumption 1, in Section 1.1 we assumed, without loss of generality, that for any $n \in \mathbb{N}$, $\langle \mathcal{P}_n, \mathcal{V}_n \rangle_\nu = 1$. Therefore, we need to show that $\langle \mathcal{P}_n, \mathcal{V}_m \rangle_\nu = 0$ when $n \neq m$. Then, note that for all $t \geq 0$ and $m, n \in \mathbb{N}$,

$$\langle \mathcal{P}_n, \mathcal{V}_m \rangle_\nu = e^{\lambda_n t} \langle P_t \mathcal{P}_n, \mathcal{V}_m \rangle_\nu = e^{\lambda_n t} \langle \mathcal{P}_n, P_t^* \mathcal{V}_m \rangle_\nu = e^{(\lambda_n - \lambda_m)t} \langle \mathcal{P}_n, \mathcal{V}_m \rangle_\nu$$

where in the first and last equality we used (1.3) and (1.4) respectively. Therefore,

$$\left(1 - e^{(\lambda_n - \lambda_m)t}\right) \langle \mathcal{P}_n, \mathcal{V}_m \rangle_\nu = 0.$$

Hence, since we assumed that the eigenvalues are of multiplicity 1, $\lambda_n \neq \lambda_m$ if $n \neq m$. Thus, $\langle \mathcal{P}_n, \mathcal{V}_m \rangle_\nu = 0$, which concludes the proof. \blacksquare

Lemma 4.2. *Let $f \in L^2(\nu)$. Then, for any $t \geq 0$,*

$$(4.2) \quad \text{std}_\nu(f(X_t)) = \sqrt{\nu f^2 - (\nu f)^2}.$$

In particular, if f is such that $\nu f = 0$, then

$$(4.3) \quad \text{std}_\nu(f(X_t)) = \sqrt{\nu f^2} = \|f\|_\nu.$$

Proof. The first claim immediately follows, for any $t \geq 0$, from the sequence of equalities

$$std_\nu(f(X_t)) = \sqrt{\nu P_t f^2 - (\nu P_t f)^2} = \sqrt{\nu f^2 - (\nu f)^2}.$$

Finally, if $\nu f = 0$, then we have

$$(4.4) \quad std_\nu(f(X_t)) = \sqrt{\nu f^2} = \|f\|_\nu. \quad \blacksquare$$

Lemma 4.3. *Let $f \in L^2(\nu)$. Then, for any $m \in \mathbb{N}$ and $t \geq s > 0$,*

$$C_\nu(\mathcal{P}_m(X_t), f(X_s)) = e^{-\lambda_m(t-s)} \langle \mathcal{P}_m, f \rangle_\nu.$$

Proof. First, Lemma 4.1 yields that $\langle \mathcal{P}_m, \mathcal{V}_n \rangle_\nu = \delta_{mn}$, $m, n \in \mathbb{N}$. In particular, since ν is invariant, for any $m \in \mathbb{N}$, $\nu P_t \mathcal{P}_m = \nu \mathcal{P}_m = \langle \mathcal{P}_m, \mathbb{1} \rangle_\nu = \delta_{0m} = 0$, where we used the fact that the constant function $\mathbb{1}$ is an eigenfunction for P_t since $P_t \mathbb{1} = \mathbb{1}$ for all $t \geq 0$. Similarly, since $P_0 = P_0^* = \mathbb{1}$, then $\nu \mathcal{P}_m = \nu \mathcal{V}_m = \delta_{0m} = 0$. Then, from the definition of the covariance function given in (1.9), we obtain, for any $t \geq s > 0$,

$$\begin{aligned} C_\nu(\mathcal{P}_m(X_t), f(X_s)) &= \mathbb{E}_\nu[\mathcal{P}_m(X_t) f(X_s)] - \mathbb{E}_\nu[\mathcal{P}_m(X_t)] \mathbb{E}_\nu[f(X_s)] \\ &= \mathbb{E}_\nu[\mathcal{P}_m(X_t) f(X_s)] - \nu P_t \mathcal{P}_m \nu P_s f \\ &= \mathbb{E}_\nu[\mathcal{P}_m(X_t) f(X_s)]. \end{aligned}$$

Next, using the Markov property and (1.3), we get

$$\begin{aligned} \mathbb{E}_\nu[\mathcal{P}_m(X_t) f(X_s)] &= \mathbb{E}_\nu[\mathbb{E}_{X_s}[\mathcal{P}_m(X_{t-s})] f(X_s)] \\ &= \mathbb{E}_\nu[P_{t-s} \mathcal{P}_m(X_s) f(X_s)] \\ &= e^{-\lambda_m(t-s)} \mathbb{E}_\nu[\mathcal{P}_m(X_s) f(X_s)] \\ &= e^{-\lambda_m(t-s)} \nu P_s \mathcal{P}_m f = e^{-\lambda_m(t-s)} \nu \mathcal{P}_m f \\ &= e^{-\lambda_m(t-s)} \langle \mathcal{P}_m, f \rangle_\nu, \end{aligned}$$

where in the second last equality we used the fact that ν is an invariant measure for P . \blacksquare

We are now ready to prove Theorem 2.1. First, recall from the proof of Lemma 4.3 that for any $m \in \mathbb{N}$, $\nu \mathcal{P}_m = \nu \mathcal{V}_m = 0$. Next, it follows from Lemma 4.2 that for any $t \geq 0$ and $m \in \mathbb{N}$,

$$std_\nu(\mathcal{P}_m(X_t)) = \|\mathcal{P}_m\|_\nu \quad \text{and} \quad std_\nu(\mathcal{V}_m(X_t)) = \|\mathcal{V}_m\|_\nu.$$

Then, using Lemma 4.3 with $f = \mathcal{V}_n$ and $f = \mathcal{P}_m$, respectively, we get, for any $t \geq s > 0$ and $n, m \in \mathbb{N}$, that

$$\begin{aligned} \rho_\nu(\mathcal{P}_m(X_t), \mathcal{P}_n(X_s)) &= \frac{e^{-\lambda_m(t-s)} \langle \mathcal{P}_n, \mathcal{P}_m \rangle_\nu}{\|\mathcal{P}_n\|_\nu \|\mathcal{P}_m\|_\nu} = e^{-\lambda_m(t-s)} c_\nu(n, m), \\ \rho_\nu(\mathcal{P}_m(X_t), \mathcal{V}_n(X_s)) &= \frac{e^{-\lambda_m(t-s)} \delta_{mn}}{\|\mathcal{P}_m\|_\nu \|\mathcal{V}_n\|_\nu} = e^{-\lambda_m(t-s)} \kappa_\nu^{-1}(m) \delta_{mn}, \end{aligned}$$

where we recall that for $m, n \in \mathbb{N}$, $\kappa_\nu(m) = \|\mathcal{P}_m\|_\nu \|\mathcal{V}_m\|_\nu$ and $c_\nu(n, m) = \frac{\langle \mathcal{P}_n, \mathcal{P}_m \rangle_\nu}{\|\mathcal{P}_n\|_\nu \|\mathcal{P}_m\|_\nu}$. Then, by symmetry, it is easy to note that for any $t, s > 0$, we have

$$\rho_\nu(\mathcal{P}_m(X_t), \mathcal{P}_n(X_s)) = e^{-\lambda_m(t-s) - \lambda_n(s-t)} c_\nu(n, m).$$

Finally, the Cauchy-Schwarz inequality entails that $|\langle \mathcal{P}_n, \mathcal{P}_m \rangle_\nu| \leq \|\mathcal{P}_n\|_\nu \|\mathcal{P}_m\|_\nu$ and hence $-1 \leq c_\nu(n, m) \leq 1$. Moreover, when $n = m$, we have that for any $n \in \mathbb{N}$,

$$c_\nu(n, n) = \frac{\langle \mathcal{P}_n, \mathcal{P}_n \rangle_\nu}{\|\mathcal{P}_n\|_\nu \|\mathcal{P}_n\|_\nu} = \frac{\|\mathcal{P}_n\|_\nu^2}{\|\mathcal{P}_n\|_\nu^2} = 1,$$

and this concludes the proof of Theorem 2.1. ■

4.2. Proof of Lemma 2.1. The definitions of the covariance and correlation functions in (1.9) and (1.10) give that for any $t \geq 0$,

$$\begin{aligned}
\rho_\nu(f(X_t), g(X_t)) &= \frac{\mathbb{E}_\nu[f(X_t)g(X_t)] - \mathbb{E}_\nu[f(X_t)]\mathbb{E}_\nu[g(X_t)]}{std_\nu(f(X_t))std_\nu(g(X_t))} \\
&= \frac{\nu P_t f g - \nu P_t f \cdot \nu P_t g}{\sqrt{\nu f^2 - (\nu f)^2} \cdot \sqrt{\nu g^2 - (\nu g)^2}} \\
&= \frac{\nu f g - \nu f \cdot \nu g}{\sqrt{\nu f^2 - (\nu f)^2} \cdot \sqrt{\nu g^2 - (\nu g)^2}} \\
(4.5) \qquad &= \frac{\langle f, g \rangle_\nu - \nu f \cdot \nu g}{\sqrt{\nu f^2 - (\nu f)^2} \cdot \sqrt{\nu g^2 - (\nu g)^2}}
\end{aligned}$$

where in the third equality we used the fact that ν is an invariant measure for P_t . Next, for $n, m \in \mathbb{N}$, taking $f = \mathcal{P}_m$ with $g = \mathcal{P}_n$ and $g = \mathcal{V}_n$ in (4.5), and using Lemma 4.1 and Lemma 4.2, we get, for any $t \geq 0$,

$$\begin{aligned}
\rho_\nu(\mathcal{P}_m(X_t), \mathcal{P}_n(X_t)) &= \frac{\langle \mathcal{P}_m, \mathcal{P}_n \rangle_\nu}{\|\mathcal{P}_m\|_\nu \cdot \|\mathcal{P}_n\|_\nu} = c_\nu(n, m), \\
\rho_\nu(\mathcal{P}_m(X_t), \mathcal{V}_n(X_t)) &= \frac{\langle \mathcal{P}_m, \mathcal{V}_n \rangle_\nu}{\|\mathcal{P}_m\|_\nu \cdot \|\mathcal{V}_n\|_\nu} = \kappa_\nu^{-1}(m)\delta_{mn}
\end{aligned}$$

where we recall that for any $n \in \mathbb{N}$, $\nu \mathcal{P}_n = \nu \mathcal{V}_n = 0$. ■

4.3. Proof of Proposition 2.1. For a function f , we write $\mathcal{L}_f(q) = \int_0^\infty e^{-qz} f(z) dz$ and we use the same notation for the Laplace transform of a measure. Then, for any $\lambda > 0$, denoting

$$(4.6) \qquad U_\lambda(dw) = \int_0^\infty \mathbb{P}(\mathcal{T}_z \in dw) e^{-\lambda z} dz, \quad w \geq 0,$$

the λ -potential measure of \mathcal{T} , we have, for any $q > 0$,

$$\begin{aligned}
\mathcal{L}_{U_\lambda}(q) &= \int_0^\infty e^{-qw} \int_0^\infty \mathbb{P}(\mathcal{T}_z \in dw) e^{-\lambda z} dz \\
&= \int_0^\infty e^{-\lambda z} \int_0^\infty e^{-qw} \mathbb{P}(\mathcal{T}_z \in dw) dz \\
(4.7) \qquad &= \int_0^\infty e^{-\lambda z} e^{-z\varphi(q)} dz = \frac{1}{\lambda + \varphi(q)}.
\end{aligned}$$

Next, as $\varphi(0) = 0$, see (1.6), $\int_0^\infty U_\lambda(dw) = \frac{1}{\lambda}$. Thus, writing, for any $t \geq 0$, $\bar{U}_\lambda(t) = \lambda \int_t^\infty U_\lambda(dw)$ and changing the order of integration justified by an application of Tonelli's theorem, we get that for any $q > 0$,

$$(4.8) \qquad \mathcal{L}_{\bar{U}_\lambda}(q) = \frac{1}{q} - \lambda \frac{1}{q(\lambda + \varphi(q))} = \frac{\varphi(q)}{q(\lambda + \varphi(q))}.$$

On the other hand, it is well known that the Laplace transform of $t \mapsto \eta_t(\lambda)$, where we recall that $\eta_t(\lambda) = \int_0^\infty e^{-\lambda s} l_t(ds)$, takes the form, for any $q > 0$,

$$(4.9) \qquad \mathcal{L}_{\eta_t(\lambda)}(q) = \frac{\varphi(q)}{q(\lambda + \varphi(q))},$$

see e.g. Mijena and Nane [31]. Therefore, the injectivity of the Laplace transform implies that for any $t \geq 0$,

$$(4.10) \quad \eta_t(\lambda) = \lambda \int_t^\infty U_\lambda(dw).$$

Here, writing $\tilde{U}_\lambda(dw) = \lambda U_\lambda(dw)$, $w > 0$, we have

$$(4.11) \quad \eta_t(\lambda) = \int_t^\infty \tilde{U}_\lambda(dr).$$

Since $\eta_t(\lambda)$ is decreasing in t , $\eta_0(\lambda) = 1$ and $\lim_{t \rightarrow \infty} \eta_t(\lambda) = 0$, we deduce that $\eta_t(\lambda)$ is a tail of a probability measure, i.e. there exists a random variable X_λ such that $\eta_t(\lambda) = \mathbb{P}(X_\lambda > t) = \int_t^\infty \tilde{U}_\lambda(dr)$, $t > 0$. Next, assuming that φ is strongly regularly varying at 0, i.e. $\varphi(q) \stackrel{0}{\sim} Cq^\alpha$, $0 < \alpha < 1$ for some constant $C > 0$, using (4.7), we obtain

$$1 - \mathcal{L}_{\tilde{U}_\lambda}(q) = 1 - \int_0^\infty e^{-qr} \tilde{U}_\lambda(dr) = \frac{\varphi(q)}{\lambda + \varphi(q)} \stackrel{0}{\sim} \frac{C}{\lambda} q^\alpha.$$

Then, it follows from a Tauberian theorem, see e.g. [7, Corollary 8.1.7], that equivalently we have

$$\eta_t(\lambda) \stackrel{t \rightarrow \infty}{\sim} \frac{C}{\lambda} \frac{t^{-\alpha}}{\Gamma(1 - \alpha)}.$$

Thus, for any $a > 0$, we get that

$$\lim_{t \rightarrow \infty} \frac{\eta_{\log(at)}(\lambda)}{\eta_{\log t}(\lambda)} = \lim_{t \rightarrow \infty} \frac{(\log a + \log t)^{-\alpha}}{(\log t)^{-\alpha}} = 1.$$

Therefore, $t \mapsto \eta_{\log(t)}(\lambda)$ is slowly varying at infinity, and thus $t \mapsto \eta_t(\lambda)$ is long-tailed, see e.g. [16, Lemma 2.15], which completes the proof of the proposition. ■

4.4. Proof of Theorem 2.2. Writing for $t, s \geq 0$, $H_{t,s}(u, v) = \mathbb{P}(L_t \leq u, L_s \leq v)$, we have that the independence of X and L entails that for any $m, n \in \mathbb{N}$,

$$\begin{aligned} \rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_s})) &= \int_0^\infty \int_0^\infty \rho_\nu(\mathcal{P}_m(X_u), \mathcal{P}_n(X_v)) H_{t,s}(du, dv), \\ \rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{V}_n(X_{L_s})) &= \int_0^\infty \int_0^\infty \rho_\nu(\mathcal{P}_m(X_u), \mathcal{V}_n(X_v)) H_{t,s}(du, dv). \end{aligned}$$

Next, recalling that c_ν is symmetric, i.e. $c_\nu(n, m) = c_\nu(m, n)$ for any $m, n \in \mathbb{N}$, Theorem 2.1 gives that for any $u, v \geq 0$,

$$\rho_\nu(\mathcal{P}_m(X_u), \mathcal{P}_n(X_v)) = c_\nu(n, m) \left(e^{-\lambda_m(u-v)} \mathbf{1}_{\{u > v\}} + e^{-\lambda_n(v-u)} \mathbf{1}_{\{u \leq v\}} \right)$$

and hence

$$\begin{aligned} \rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_s})) &= \int_0^\infty \int_0^\infty c_\nu(n, m) \left(e^{-\lambda_m(u-v)} \mathbf{1}_{\{u > v\}} + e^{-\lambda_n(v-u)} \mathbf{1}_{\{u \leq v\}} \right) H_{t,s}(du, dv) \\ (4.12) \quad &= c_\nu(n, m) \int_0^\infty \int_0^\infty \left(e^{-\lambda_m(u-v)} \mathbf{1}_{\{u > v\}} + e^{-\lambda_n(v-u)} \mathbf{1}_{\{u \leq v\}} \right) H_{t,s}(du, dv). \end{aligned}$$

Now, let $(u, v) \mapsto F(u, v)$ be a function of bounded variation such that $u \mapsto F(u, v)$ and $v \mapsto F(u, v)$ are also of bounded variation. Then, writing $F(du, v)$, $F(u, dv)$ and $F(du, dv)$, we mean the one dimensional measures generated by the sections $u \mapsto F(u, v)$, $v \mapsto F(u, v)$ and the two dimensional

measure generated by $(u, v) \mapsto F(u, v)$ respectively. For such a function F , recall the bivariate integration by parts formula

$$\begin{aligned}
\int_0^\infty \int_0^\infty F(u, v) H_{t,s}(du, dv) &= \int_0^\infty \int_0^\infty H_{t,s}([u, \infty] \times [v, \infty]) F(du, dv) \\
&+ \int_0^\infty H_{t,s}([u, \infty] \times (0, \infty]) F(du, 0) \\
&+ \int_0^\infty H_{t,s}((0, \infty] \times [v, \infty]) F(0, dv) \\
&+ F(0, 0) H_{t,s}((0, \infty] \times (0, \infty]),
\end{aligned}
\tag{4.13}$$

see e.g. Gill et al. [18, Lemma 2.2]. Let us apply this formula to

$$F(u, v) = e^{-\lambda_m(u-v)} \mathbf{1}_{\{u>v\}} + e^{-\lambda_n(v-u)} \mathbf{1}_{\{u\leq v\}}, \quad (u, v) \in \mathbb{R}_+^2,$$

which is clearly of bounded variation. Then, writing $\bar{H}_{t,s}(u, v) = \mathbb{P}(L_t \geq u, L_s \geq v)$ and $\bar{H}_t(u) = \mathbb{P}(L_t \geq u)$,

$$\begin{aligned}
\int_0^\infty \int_0^\infty F(u, v) H_{t,s}(du, dv) &= \int_0^\infty \int_0^\infty \bar{H}_{t,s}(u, v) F(du, dv) + \int_0^\infty \bar{H}_t(u) F(du, 0) \\
&+ \int_0^\infty \bar{H}_s(v) F(0, dv) + 1,
\end{aligned}
\tag{4.14}$$

where we used that as $\mathbb{P}(L_t = 0) = 0$, $\mathbb{P}(L_t > 0) = 1$ for all $t > 0$, $F(0, 0) = 1$ and H is a distribution function. Note that $F(du, v) = (-\lambda_m e^{-\lambda_m(u-v)} \mathbf{1}_{\{u>v\}} + \lambda_n e^{-\lambda_n(v-u)} \mathbf{1}_{\{u\leq v\}}) du$ for all $v \geq 0$. Thus, an integration by parts yields that

$$\begin{aligned}
\int_0^\infty \bar{H}_t(u) F(du, 0) &= \int_0^\infty (1 - \mathbb{P}(L_t < u)) (-\lambda_m e^{-\lambda_m u}) du \\
&= e^{-\lambda_m u} \bar{H}_t(u) \Big|_0^\infty + \int_0^\infty e^{-\lambda_m u} l_t(u) du = \eta_t(\lambda_m) - 1,
\end{aligned}$$

and similarly,

$$\int_0^\infty \bar{H}_s(v) F(0, dv) = \eta_s(\lambda_n) - 1.$$

Hence, (4.12) reduces to

$$c_\nu(n, m) \int_0^\infty \int_0^\infty F(u, v) H_{t,s}(du, dv) = c_\nu(n, m) (I(t, s) + \eta_t(\lambda_m) + \eta_s(\lambda_n) - 1),$$

where we have set

$$I(t, s) = \int_0^\infty \int_0^\infty \bar{H}_{t,s}(u, v) F(du, dv).$$

Then, observing that $I(s, t) = I(t, s)$, we assume, without loss of generality, that $s \leq t$ and we write $I(t, s) = I_1(t, s) + I_2(t, s) + I_3(t, s)$, where

$$\begin{aligned}
I_1(t, s) &= \int_0^\infty \int_0^v \bar{H}_{t,s}(u, v) F(du, dv), \quad I_2(t, s) = \int_0^\infty \int_{u=v}^\infty \bar{H}_{t,s}(u, v) F(du, dv), \\
I_3(t, s) &= \int_0^\infty \int_v^\infty \bar{H}_{t,s}(u, v) F(du, dv).
\end{aligned}$$

Then, as the inverse of the subordinator \mathcal{T} is non-decreasing, $\overline{H}_{t,s}(u, v) = \mathbb{P}(L_t \geq u, L_s \geq v) = \mathbb{P}(L_s \geq v) = \overline{H}_s(v)$ for $u \leq v$ and $F(du, dv) = -\lambda_n^2 e^{-\lambda_n(v-u)} dudv$ for $u < v$. Thus,

$$\begin{aligned}
I_1(t, s) &= \int_0^\infty \int_0^v \overline{H}_s(v) F(du, dv) \\
&= -\lambda_n^2 \int_0^\infty \int_0^v \overline{H}_s(v) e^{\lambda_n(u-v)} dudv \\
&= -\lambda_n \int_0^\infty \overline{H}_s(v) (1 - e^{-\lambda_n v}) dv \\
&= -\lambda_n \int_0^\infty \overline{H}_s(v) dv + \lambda_n \int_0^\infty e^{-\lambda_n v} \overline{H}_s(v) dv \\
&= -\lambda_n \mathbb{E}[L_s] + \lambda_n \int_0^\infty e^{-\lambda_n v} \mathbb{P}(L_s \geq v) dv \\
&= -\lambda_n \mathbb{E}[L_s] - \eta_s(\lambda_n) + 1,
\end{aligned}$$

where in the last identity we have performed an integration by parts. We also note that

$$(4.17) \quad \mathbb{E}[L_s] = \int_0^\infty \overline{H}_s(v) dv = \int_0^\infty \mathbb{P}(L_s \geq v) dv = U(0, s).$$

Next, writing simply $f_v(u)du = F(du, v) = (-\lambda_m e^{-\lambda_m(u-v)} 1_{\{u>v\}} + \lambda_n e^{-\lambda_n(v-u)} 1_{\{u \leq v\}}) du$, we remark that the mapping $u \mapsto f_v(u)$ has a jump of size $(\lambda_m + \lambda_n)$ at the point $u = v$. Then,

$$I_2(t, s) = \int_0^\infty \int_{u=v} \overline{H}_s(v) F(du, dv) = (\lambda_m + \lambda_n) \int_0^\infty \overline{H}_s(v) dv = (\lambda_m + \lambda_n) \mathbb{E}[L_s].$$

Finally, as $F(du, dv) = -\lambda_m^2 e^{-\lambda_m(u-v)} dudv$ for $u > v$, we deduce that

$$I_3(t, s) = -\lambda_m^2 \int_0^\infty \overline{H}_{t,s}(u, v) \int_v^\infty e^{-\lambda_m(u-v)} dudv,$$

and we proceed by computing the joint tail distribution of the pair (L_t, L_s) , that is $\overline{H}_{t,s}(u, v) = \mathbb{P}(L_t \geq u, L_s \geq v)$. Note that since L is the inverse of \mathcal{T} , then $\{L_t \geq u\} = \{\mathcal{T}_u \leq t\}$, and thus $\mathbb{P}(L_t \geq u, L_s \geq v) = \mathbb{P}(\mathcal{T}_u \leq t, \mathcal{T}_v \leq s)$. Now, since as a Lévy process \mathcal{T} has stationary and independent increments, it follows, recalling that $s \leq t$,

$$\begin{aligned}
\overline{H}_{t,s}(u, v) &= \mathbb{P}(\mathcal{T}_u \leq t, \mathcal{T}_v \leq s) = \mathbb{P}((\mathcal{T}_u - \mathcal{T}_v) + \mathcal{T}_v \leq t, \mathcal{T}_v \leq s) \\
&= \int_0^s \mathbb{P}(\mathcal{T}_v \in dr) \int_0^{t-r} \mathbb{P}(\mathcal{T}_{u-v} \in dw).
\end{aligned}$$

Using Fubini's theorem and performing the change of variable $z = u - v$, we get

$$\begin{aligned}
I_3(t, s) &= -\lambda_m^2 \int_0^\infty \int_0^s \mathbb{P}(\mathcal{T}_v \in dr) \int_0^{t-r} \mathbb{P}(\mathcal{T}_{u-v} \in dw) \int_v^\infty e^{-\lambda_m(u-v)} dudv \\
&= -\lambda_m^2 \int_0^s \int_0^{t-r} \int_0^\infty e^{-\lambda_m z} \mathbb{P}(\mathcal{T}_z \in dw) dz \int_0^\infty \mathbb{P}(\mathcal{T}_v \in dr) dv \\
(4.18) \quad &= -\lambda_m^2 \int_0^s \int_0^{t-r} \int_0^\infty e^{-\lambda_m z} \mathbb{P}(\mathcal{T}_z \in dw) dz U(dr)
\end{aligned}$$

where in the last step, from the definition of the renewal measure, we have used that $\int_0^\infty \mathbb{P}(\mathcal{T}_v \in dr) dv = U(dr)$. Now, taking $\lambda = \lambda_m$ in (4.6) and (4.10) in the proof of Proposition 2.1, we have $U_{\lambda_m}(dw) = \int_0^\infty \mathbb{P}(\mathcal{T}_z \in dw) e^{-\lambda_m z} dz$, $w > 0$ and $\eta_t(\lambda_m) = \lambda_m \int_t^\infty U_{\lambda_m}(dw)$. Hence, using (4.17),

the expression of $I_3(t, s)$ in (4.18) reduces to

$$\begin{aligned}
I_3(t, s) &= -\lambda_m^2 \int_0^s \int_0^{t-r} U_{\lambda_m}(dw)U(dr) \\
&= \lambda_m \int_0^s \eta_{t-r}(\lambda_m)U(dr) - \lambda_m \int_0^s U(dr) \\
&= \lambda_m \int_0^s \eta_{t-r}(\lambda_m)U(dr) - \lambda_m \mathbb{E}[L_s].
\end{aligned}$$

Finally, putting all pieces together, we obtain that for $t \geq s > 0$,

$$\begin{aligned}
\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_s})) &= c_\nu(n, m)(I_1(t, s) + I_2(t, s) + I_3(t, s) + \eta_t(\lambda_m) + \eta_s(\lambda_n) - 1) \\
&= c_\nu(n, m) \left(-\lambda_n \mathbb{E}[L_s] - \eta_s(\lambda_n) + 1 + (\lambda_m + \lambda_n) \mathbb{E}[L_s] \right. \\
&\quad \left. + \lambda_m \int_0^s \eta_{t-r}(\lambda_m)U(dr) - \lambda_m \mathbb{E}[L_s] + \eta_t(\lambda_m) + \eta_s(\lambda_n) - 1 \right) \\
(4.19) \qquad \qquad \qquad &= c_\nu(n, m) \left(\lambda_m \int_0^s \eta_{t-r}(\lambda_m)U(dr) + \eta_t(\lambda_m) \right)
\end{aligned}$$

which provides the claim (2.4). We proceed by studying the spectral projections correlation structure of $\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{V}_n(X_{L_s}))$ for $t \geq s > 0$ and $m, n \in \mathbb{N}$. Note that, for any $u, v \geq 0$,

$$(4.20) \qquad \qquad \rho_\nu(\mathcal{P}_m(X_u), \mathcal{V}_n(X_v)) = F_1(u, v) + F_2(u, v),$$

where we have written, see Theorem 2.1,

$$\begin{aligned}
F_1(u, v) &= \rho_\nu(\mathcal{P}_m(X_u), \mathcal{V}_n(X_v))1_{\{u \geq v\}} = \kappa_\nu^{-1}(m)\delta_{mn}e^{-\lambda_m(u-v)}1_{\{u \geq v\}}, \\
F_2(u, v) &= \rho_\nu(\mathcal{P}_m(X_u), \mathcal{V}_n(X_v))1_{\{u < v\}}.
\end{aligned}$$

Then, for any $t \geq s > 0$ and $m, n \in \mathbb{N}$, we have

$$(4.21) \quad \rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{V}_n(X_{L_s})) = \int_0^\infty \int_0^\infty F_1(u, v)H_{t,s}(du, dv) + \int_0^\infty \int_0^\infty F_2(u, v)H_{t,s}(du, dv).$$

Now, recalling the bivariate integration by parts formula (4.13), one has

$$\begin{aligned}
\int_0^\infty \int_0^\infty F_1(u, v)H_{t,s}(du, dv) &= \int_0^\infty \int_0^\infty \bar{H}_{t,s}(u, v)F_1(du, dv) + \int_0^\infty \bar{H}_t(u)F_1(du, 0) \\
&\quad + \kappa_\nu^{-1}(m)\delta_{mn},
\end{aligned}$$

where we used that $\int_0^\infty \bar{H}_s(v)F_1(0, dv) = 0$ and $F_1(0, 0) = \kappa_\nu^{-1}(m)\delta_{mn}$. Now, following the same pattern as in the proof of the first part of Theorem 2.1 above, and since on $\{u < v\}$, $F_1(du, dv) = 0$, one gets

$$\begin{aligned}
\int_0^\infty \int_0^\infty F_1(u, v)H_{t,s}(du, dv) &= \int_0^\infty \int_{u=v}^\infty \bar{H}_{t,s}(u, v)F_1(du, dv) + \int_0^\infty \int_v^\infty \bar{H}_{t,s}(u, v)F_1(du, dv) \\
&\quad + \int_0^\infty \bar{H}_t(u)F_1(du, 0) + \kappa_\nu^{-1}(m)\delta_{mn} \\
&= \kappa_\nu^{-1}(m)\delta_{mn}\lambda_m \mathbb{E}[L_s] \\
&\quad + \kappa_\nu^{-1}(m)\delta_{mn} \left(\lambda_m \int_0^s \eta_{t-r}(\lambda_m)U(dr) - \lambda_m \mathbb{E}[L_s] \right) \\
&\quad + \kappa_\nu^{-1}(m)\delta_{mn} (\eta_t(\lambda_m) - 1) \\
&= \kappa_\nu^{-1}(m)\delta_{mn} \left(\lambda_m \int_0^s \eta_{t-r}(\lambda_m)U(dr) + \eta_t(\lambda_m) \right).
\end{aligned}$$

Next, we turn to the computation of the second integral on the right-hand side of (4.21). As the functions $(u, v) \mapsto F_2(u, v)$, $u \mapsto F_2(u, v)$ and $v \mapsto F_2(u, v)$ are of bounded variation since by (4.20), $F_2(u, v)$ is a difference of two functions of bounded variation, then, by means of the bivariate integration by parts formula (4.13), we get

$$(4.22) \quad \int_0^\infty \int_0^\infty F_2(u, v) H_{t,s}(du, dv) = \int_0^\infty \int_0^\infty \bar{H}_{t,s}(u, v) F_2(du, dv) + \int_0^\infty \bar{H}_s(v) F_2(0, dv),$$

where we used that $\int_0^\infty \bar{H}_t(u) F_2(du, 0) = 0$ and $F_2(0, 0) = 0$. Now, since on $\{u > v\}$, $F_2(du, dv) = 0$, then, for $t \geq s$, (4.22) reduces to

$$\begin{aligned} \int_0^\infty \int_0^\infty F_2(u, v) H_{t,s}(du, dv) &= \int_0^\infty \int_0^v \bar{H}_{t,s}(u, v) F_2(du, dv) \\ &+ \int_0^\infty \int_{u=v}^\infty \bar{H}_{t,s}(u, v) F_2(du, dv) + \int_0^\infty \bar{H}_s(v) F_2(0, dv) \\ &= \int_0^\infty \int_0^v \bar{H}_s(v) F_2(du, dv) + \int_0^\infty \int_{u=v}^\infty \bar{H}_s(v) F_2(du, dv) \\ &+ \int_0^\infty \bar{H}_s(v) F_2(0, dv). \end{aligned}$$

Thus, $\int_0^\infty \int_0^\infty F_2(u, v) H_{t,s}(du, dv)$ does not depend on t . On the other hand, taking $t = s$ in (4.21), we get, for any $s \geq 0$,

$$\kappa_\nu^{-1}(m) \delta_{mn} = \kappa_\nu^{-1}(m) \delta_{mn} + \int_0^\infty \int_0^\infty F_2(u, v) H_{s,s}(du, dv),$$

where we used that by taking $t = s$ in (4.19), Lemma 2.1 yields that for any $t \geq 0$,

$$\lambda_m \int_0^t \eta_{t-r}(\lambda_m) U(dr) + \eta_t(\lambda_m) = 1.$$

This can also be independently proven as in Remark 4.1. Hence, for any $s \geq 0$,

$$\int_0^\infty \int_0^\infty F_2(u, v) H_{s,s}(du, dv) = 0,$$

and we deduce that for any $t \geq s > 0$,

$$\int_0^\infty \int_0^\infty F_2(u, v) H_{t,s}(du, dv) = \int_0^\infty \int_0^\infty F_2(u, v) H_{s,s}(du, dv) = 0.$$

Therefore, putting pieces together, (4.21) reduces to

$$\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{V}_n(X_{L_s})) = \kappa_\nu^{-1}(m) \delta_{mn} \left(\lambda_m \int_0^s \eta_{t-r}(\lambda_m) U(dr) + \eta_t(\lambda_m) \right).$$

Now we are ready to study the right-hand side of (2.4) and (2.5) for large t when $s > 0$ is fixed under the assumption that $\lim_{t \rightarrow \infty} \frac{\eta_{t-s}(\lambda_m)}{\eta_t(\lambda_m)} = 1$. Since $t \mapsto \eta_t(\lambda_m)$ is decreasing on \mathbb{R}^+ , we have

$$\int_0^s \eta_{t-r}(\lambda_m) U(dr) \geq \eta_t(\lambda_m) U(0, s) = \eta_t(\lambda_m) \mathbb{E}[L_s]$$

and

$$\int_0^s \eta_{t-r}(\lambda_m) U(dr) \leq \eta_{t-s}(\lambda_m) U(0, s) \leq \eta_{t-s}(\lambda_m) \mathbb{E}[L_s].$$

Consequently,

$$\begin{aligned} c_\nu(n, m) \eta_t(\lambda_m) (\lambda_m \mathbb{E}[L_s] + 1) &\leq \rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_s})) \leq c_\nu(n, m) \eta_{t-s}(\lambda_m) (\lambda_m \mathbb{E}[L_s] + 1), \\ \kappa_\nu^{-1}(m) \delta_{mn} \eta_t(\lambda_m) (\lambda_m \mathbb{E}[L_s] + 1) &\leq \rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{V}_n(X_{L_s})) \leq \kappa_\nu^{-1}(m) \delta_{mn} \eta_{t-s}(\lambda_m) (\lambda_m \mathbb{E}[L_s] + 1). \end{aligned}$$

Now, if for a fixed $s > 0$, there exists a constant $C = C(s, \lambda_m) > 0$ such that $\overline{\lim}_{t \rightarrow \infty} \frac{\eta_{t-s}(\lambda_m)}{\eta_t(\lambda_m)} = C$, then there exists $t_0 > 0$ such that for $t \geq t_0$, $\frac{\eta_{t-s}(\lambda_m)}{\eta_t(\lambda_m)} \leq C$, and thus

$$\begin{aligned}\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_s})) &\stackrel{t_0}{\gtrsim} c_\nu(n, m)\eta_t(\lambda_m)(\lambda_m\mathbb{E}[L_s] + 1), \\ \rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{V}_n(X_{L_s})) &\stackrel{t_0}{\gtrsim} \kappa_\nu^{-1}(m)\delta_{mn}\eta_t(\lambda_m)(\lambda_m\mathbb{E}[L_s] + 1).\end{aligned}$$

In particular, if for a fixed $s > 0$, $\lim_{t \rightarrow \infty} \frac{\eta_{t-s}(\lambda_m)}{\eta_t(\lambda_m)} \equiv 1$, we have

$$\begin{aligned}\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_s})) &\stackrel{t \rightarrow \infty}{\sim} c_\nu(n, m)\eta_t(\lambda_m)(\lambda_m\mathbb{E}[L_s] + 1), \\ \rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{V}_n(X_{L_s})) &\stackrel{t \rightarrow \infty}{\sim} \kappa_\nu^{-1}(m)\delta_{mn}\eta_t(\lambda_m)(\lambda_m\mathbb{E}[L_s] + 1),\end{aligned}$$

and this concludes the proof of Theorem 2.2. ■

Remark 4.1. One can easily check that for any $n, m \in \mathbb{N}$, when $t = s$ in (2.4), we have for any $t \geq 0$,

$$\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_t})) = c_\nu(n, m),$$

i.e.

$$\lambda_m \int_0^t \eta_{t-r}(\lambda_m)U(dr) + \eta_t(\lambda_m) = 1.$$

Indeed, let us plug in $t = s$ in (2.4). Then, noting that the convolution, we get that, for any $q > 0$,

$$\begin{aligned}\mathcal{L}_{\int_0^\cdot \eta_{\cdot-r}(\lambda_m)U(dr)}(q) &= \mathcal{L}_{\eta \cdot (\lambda_m)}(q)\mathcal{L}_U(q) \\ &= \frac{\varphi(q)}{q(\lambda_m + \varphi(q))} \frac{1}{\varphi(q)} \\ &= \frac{1}{q(\lambda_m + \varphi(q))}.\end{aligned}$$

Next, using (4.9), one has

$$\mathcal{L}_{\lambda_m \int_0^\cdot \eta_{\cdot-r}(\lambda_m)U(dr) + \eta \cdot (\lambda_m)}(q) = \frac{\lambda_m}{q(\lambda_m + \varphi(q))} + \frac{\varphi(q)}{q(\lambda_m + \varphi(q))} = \frac{1}{q}.$$

Thus, by the injectivity of the Laplace transform we conclude that

$$\lambda_m \int_0^t \eta_{t-r}(\lambda_m)U(dr) + \eta_t(\lambda_m) = 1.$$

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SCHOOL OF OPERATIONS RESEARCH AND INFORMATION ENGINEERING, CORNELL UNIVERSITY, ITHACA, NY 14853.

Email address: `pp396@orie.cornell.edu`

CENTER FOR APPLIED MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NY 14853.

Email address: `as3348@cornell.edu`