Stochastics and Dynamics

DYNAMICS OF DRAINAGE UNDER STOCHASTIC RAINFALL IN RIVER NETWORKS

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We consider a linearized dynamical system modelling the flow rate of water along the rivers and hillslopes of an arbitrary watershed. The system is perturbed by a random rainfall in the form of a compound Poisson process. The model describes the evolution, at daily time scales, of an interconnected network of linear reservoirs and takes into account the differences in flow celerity between hillslopes and streams as well as their spatial variation. The resulting stochastic process is a piece-wise deterministic Markov process of the Orstein-Uhlembeck type. We provide an explicit formula for the Laplace transform of the invariant density of streamflow in terms of the geophysical parameters of the river network and the statistical properties of the precipitation field. As an application, we include novel formulas for the invariant moments of the streamflow at the watershed's outlet, as well as the asymptotic behavior of extreme discharge events, and conditions for the statistical scaling of streamflow with respect to Horton order.

Keywords: Rainfall runoff process; Ornstein-Uhlenbeck type; Invariant distribution of discharge.

AMS Subject Classification: 60G51, 60H10, 37H10

1. Introduction

River networks are a chief example of interconnected dynamical systems operating under stochastic forcing. The emerging properties of these systems define the process by which rainfall is converted into river discharge through the accumulation of hillsolpe runoff. Therein lies the fundamental problem of hydrology. Uncertainty plays a key role too, and the pronounced temporal variability of runoff and discharge reflects the random character of key hydrologic fluxes (Botter et al., 2007a).

We propose and solve a stochastic differential model for the streamflow and subsurface runoff throughout an arbitrary watershed under a random precipitation field. The proposed conceptual model uses the linearized equations for conservation of mass and momentum on each river and hillslope proposed by Gupta and Waymire (1998); Rodriguez-Iturbe et al. (1999), and aggregates them according to the geometry of the river network. The focus of this paper is to derive the necessary mathematical details leading to the solution of the equations and some interesting initial consequences. Our results include estimates on invariant moments of discharge, asymptotics of extreme events, and statistical scaling.

The time scales of interest are of days or longer, hence precipitation events are assumed instantaneous. The spatial scale is arbitrary, with individual stream links and hillslopes considered as interconnected linear reservoirs, whereas the detailed dynamics of soil moisture, infiltration or evapotranspiration are neglected. The proposed model does take into account the kinematic delay among the contributions originating from different sub-basins. Moreover, hydraulic parameters are specified in the model at individual stream links and hillslopes, effectively incorporating variations in celerity due to location or scale.

Among the calculations presented here are explicit expressions for the Laplace transform of the densities of both, the transition probabilities and the unique invariant distribution of the process $\boldsymbol{X}$. See Proposition 4.1 below. These expressions completely characterize the distribution of the runoff and discharge within the basin for all times, as well as its behavior as $t \to \infty$. They also explicitly show how the uncertainty associated with the precipitation interacts with the geometry of the river network and is converted into the uncertainty of discharge and runoff.

As an application, we obtain formulas for the n-th moment of the streamflow at the basin outlet, and the asymptotic behavior of the probabilities of extreme discharge events. The analysis also yields a novel family of geomorphological coefficients that completely characterizes the invariant distribution of $\boldsymbol{X}$. Finally, we give conditions for the statistical scaling of the river discharge under its invariant distribution.

2. Description of the model

Natural river networks can be modeled as finite directed rooted binary trees Kovchegov and Zaliapin (2018). Let Γ denote such a tree modeling a river network as in Figure 1. The edges of Γ are called 'links', there are n_Γ of them, and each is denoted in general by the letter e. The most downstream edge, or stem, is always denoted by r. Each link is surrounded by an area a_e of hillslope that drains directly through it. We denote the vector of areas by $\boldsymbol{a}$ and the total watershed area by a:

$$\boldsymbol{a} := [a_e : e \in \Gamma]^\intercal, \quad a := \sum_{e\in\Gamma} a_e. \tag{2.1}$$

For any time $t \geqslant 0$, the quantities of interest are: the total subsurface runoff $R_e(t)$ from the hillslopes into the link e, and the streamflow $Q_e(t)$ at its downstream end, both in units of volume per unit time. Both R_e and Q_e evolve dynamically in response to a random field P_e of rainfall intensities (in units of height of water) distributed among the hillslopes of the watershed.

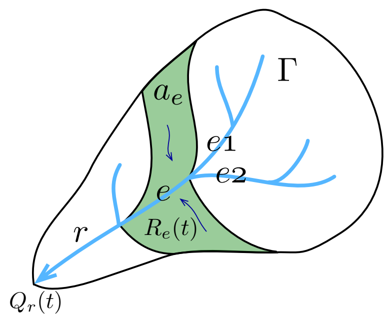

Fig. 1. Schematic representation of a river network Γ and its components with $n_\Gamma = 9$ streams.

Our main dynamical assumption rests on the linearized stream-based conservation equations proposed in Gupta and Waymire (1998) and the linearized subsurface water storage equation of Rodriguez-Iturbe et al. (1999). Namely, if the link e has tributaries $e1$ and $e2$, then

$$\frac{\mathrm{d}Q_e}{\mathrm{d}t} = K_e(-Q_e + Q_{e1} + Q_{e2} + R_e), \tag{2.2}$$

$$\frac{\mathrm{d}R_e}{\mathrm{d}t} = -H_e\left(R_e + a_e\frac{\mathrm{d}P_e}{\mathrm{d}t}\right) \tag{2.3}$$

The *inverse residence times* K_e and H_e play a pivotal role in what follows, and are obtained by supposing that the total storage in stream e and its hillslope are $\frac{1}{K_e}Q_e(t)$ and $\frac{1}{H_e}R_e(t)$ respectively. Typical values of H_e and K_e may be deduced from related parameterizations in the literature. Rodriguez-Iturbe et al. (1999) for instance, used $H_e = \frac{k_s}{nz}$ where k_s is the hydraulic conductivity of the saturated soil, n is the porosity and z the soil depth. In a more recent work, Rupp and Selker (2006) propose a power-law model for R which, in the limit of homogeneous hydraulic conductivity, can be linearized to equation (2.3) with $H_e \approx \frac{k_s L_e \sin(i)}{n a_e}$. Here L_e is the length of the stream and $\tan(i)$ is the slope of the hillslope. A common approximation to K_e is given for the classical Mukingum method of flood routing as $K_e \approx 1.5v/L_e$, where v is the average velocity in the channel Dooge (1973). See also Mantilla et al. (2011). Using typical values, one arrives at the following ranges for the non-dimensional quantities of interest (see also Botter et al., 2007b, Figure 4):

$$\frac{H_e}{K_e} \sim 10^{-3} - 10^{0}, \quad \frac{\lambda}{H_e} \sim 10^{-3} - 10^{0}. \tag{2.4}$$

The stochastic process of interest will be denoted by $\boldsymbol{X}$ and is obtained by concatenating two n_Γ-dimensional time dependent column vectors: $\boldsymbol{Q}(t) = [Q_e(t) : e \in \Gamma]^\intercal$ containing the total streamflow at the most downstream point of each link, and $\boldsymbol{R}(t) = [R_e(t) : e \in \Gamma]^\intercal$ with the total subsurface runoff from the hillslopes into each corresponding stream link. The state space of $\boldsymbol{X}$ is therefore $\mathbb{R}_+^{2n_\Gamma} := (0,\infty)^{2n_\Gamma}$. We thus write

$$\boldsymbol{X}(t) := \begin{bmatrix} \boldsymbol{Q}(t) \\ \boldsymbol{R}(t) \end{bmatrix} \in \mathbb{R}_+^{2n_\Gamma}, \quad t \geqslant 0, \tag{2.5}$$

which exemplifies our notation for vectors in $\mathbb{R}_+^{2n_\Gamma}$ in terms of their n_Γ-dimensional sub-vectors,

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} \in \mathbb{R}_+^{2n_\Gamma}, \quad \boldsymbol{x}_i = [x_{i,e} : e \in \Gamma]^\intercal \in \mathbb{R}_+^{n_\Gamma}, \quad i = 1, 2. \tag{2.6}$$

In addition, we follow the convention of having the first entries of both $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ correspond to the root edge r. In particular, $X_1(t) = Q_r(t)$ and $X_{n_\Gamma+1}(t) = R_r(t)$ in (2.5).

The uncertainty in the model comes solely from the precipitation field P_e in (2.3) which, at the daily or larger time scales of interest here, can be approximated by a n_Γ-dimensional compound Poisson process P_e of increments $\{P_{n,e} : n \geqslant 1\}$ (Rodriguez-Iturbe et al., 1999). The term $\mathrm{d}P_e/\mathrm{d}t$ in (2.3) is thus to be understood as a generalized derivative. The basin-wide precipitation process may be written as the following point process

$$\boldsymbol{P}(t) := \sum_{n=1}^{\infty} \boldsymbol{P}_n \, \delta_{T_n}(t) \tag{2.7}$$

where the storm times $\{T_n : n \geqslant 1\}$ define a Poisson process N with fixed intensity $\lambda > 0$,

$$N(t) := \sup\{n \geqslant 1 : T_n \leqslant t\}. \tag{2.8}$$

In (2.7), the symbol δ_{T_n} denotes the Dirac-delta function concentrated at instant T_n, and $\{\boldsymbol{P}_n : n \geqslant 1\}$ is a sequence of random i.i.d vectors

$$\boldsymbol{P}_n := [P_{n,e} : e \in \Gamma]^\intercal \in \mathbb{R}_+^{n_\Gamma}, \tag{2.9}$$

each with joint probability density function

$$f_{\boldsymbol{P}}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \mathbb{P}(P_{n,e} \in \mathrm{d}y_e : e \in \Gamma), \ \boldsymbol{y} \in \mathbb{R}_+^{n_\Gamma}. \tag{2.10}$$

The above formulation concerns the case in which a watershed is subject to a random, yet *statistically stationary precipitation regime*. The constant $\lambda > 0$ gives the average number of precipitation events per unit time. At time T_n, and independently of everything else, the n-th precipitation event occurs instantaneously dropping a random column of water $P_{n,e}$ onto the hillslopes surrounding link e. For fixed n, the joint distribution of the vector $[P_{n,e}, e \in \Gamma]^\intercal$ is given by $f_{\boldsymbol{P}}$. Within the specification of $f_{\boldsymbol{P}}$ one can, therefore, include any kind of statistical dependence between the precipitation intensities at different locations throughout the watershed. In particular, one may take the uniform case $P_{n,e} = P_n$ for all $e \in \Gamma$ as detailed in Remark 4.1.

We continue the mathematical formulation by introducing the following matrices

$$\mathbf{K} := \operatorname{diag}(K_e : e \in \Gamma), \quad \mathbf{H} := \operatorname{diag}(H_e : e \in \Gamma), \tag{2.11}$$

and the *incidence matrix* $\mathbf{\Lambda}_\Gamma$ of the network Γ, which is constructed as follows: $(\mathbf{\Lambda}_\Gamma)_{e,e} = 1$ for all links $e \in \Gamma$ and, if link e has tributaries e_1, e_2, then

$$(\mathbf{\Lambda}_\Gamma)_{e,e_1} = (\mathbf{\Lambda}_\Gamma)_{e,e_2} = -1, \tag{2.12}$$

with all other entries equal to zero. The matrix $\mathbf{\Lambda}_\Gamma$ encodes all the topological information of the river network.

The system of equations (2.2)-(2.3) can be written as the following linear stochastic differential equation for $\boldsymbol{X}$:

$$\mathrm{d}\boldsymbol{X}(t) = \mathbf{M}\boldsymbol{X}(t)\,\mathrm{d}t + \mathrm{d}\boldsymbol{Y}(t), \tag{2.13}$$

where $\mathbf{M}$ is the block matrix

$$\mathbf{M} := \begin{bmatrix} -\mathbf{K}\mathbf{\Lambda}_\Gamma & \mathbf{K} \\ \mathbf{O} & -\mathbf{H} \end{bmatrix} \in \mathbb{R}^{n_\Gamma \times n_\Gamma}, \tag{2.14}$$

and the driving process $\boldsymbol{Y}$ is the $2n_\Gamma$-dimensional compound Poisson process given by

$$\boldsymbol{Y}(t) := \sum_{n=1}^{N(t)} \begin{bmatrix} \mathbf{0} \\ \mathbf{H}(\boldsymbol{a} \circ \boldsymbol{P}_n) \end{bmatrix}. \tag{2.15}$$

In (2.14) and (2.15), $\mathbf{O}$ denotes the $n_\Gamma \times n_\Gamma$ matrix whose entries are all zero, $\mathbf{0}$ denotes the n_Γ-dimensional zero vector, and for vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_+^{n_\Gamma}$,

$$\boldsymbol{x} \circ \boldsymbol{y} := [x_e y_e : \ e \in \Gamma]^\intercal \tag{2.16}$$

denotes the component-wise or Hadamard product operator.

The defintion of $\boldsymbol{Y}$ in (2.15) makes explicit the assumption that rainfall falls exclusively over the hillslopes and affects $\boldsymbol{Q}$ only through $\boldsymbol{R}$. The probability density of each vector in the summation (2.15) is denoted by:

$$f_{\boldsymbol{Y}}(\boldsymbol{y}) = f_{\boldsymbol{Y}}(\boldsymbol{y}_1, \boldsymbol{y}_2) := \begin{cases} f_{\boldsymbol{Y}_2}(\boldsymbol{y}_2), & \boldsymbol{y}_1 = \mathbf{0} \\ 0, & \text{otherwise.} \end{cases}, \ \boldsymbol{y} \in \mathbb{R}_+^{2n_\Gamma}, \text{ with} \tag{2.17}$$

$$f_{\boldsymbol{Y}_2}(\boldsymbol{x}) := \frac{1}{\prod_{e \in \Gamma} H_e a_e} f_{\boldsymbol{P}}\left(\frac{\boldsymbol{x}}{\mathbf{H}\boldsymbol{a}}\right), \ \boldsymbol{x} \in \mathbb{R}_+^{n_\Gamma}, \tag{2.18}$$

with $f_{\boldsymbol{P}}$ as in (2.10). In (2.18) and below, division of a vector by $\mathbf{H}\boldsymbol{a}$ simply denotes component-wise multiplication by $[1/(H_e a_e) : e \in \Gamma]^\intercal$.

3. Analysis of $\boldsymbol{X}$

An explicit solution to (2.13) is

$$\boldsymbol{X}(t) = e^{\mathbf{M}t}\boldsymbol{X}(0) + \sum_{n=1}^{N(t)} e^{\mathbf{M}(t-T_n)} \begin{bmatrix} \mathbf{0} \\ \mathbf{H}(\boldsymbol{a} \circ \boldsymbol{P}_n) \end{bmatrix}. \tag{3.1}$$

Namely, $\boldsymbol{X}$ is a *Piecewise Deterministic Markov Process* (PDMP) (Davis, 1984). The sample paths of $\boldsymbol{X}$ evolve between storm events according to the deterministic map

$$\boldsymbol{\varphi}(t) := e^{\mathbf{M}t}, \quad t \geqslant 0, \tag{3.2}$$

and at times $\{T_n : n \geqslant 1\}$ each component of $\boldsymbol{R}(t)$ jumps by a random amount at each T_n, while those of $\boldsymbol{Q}(t)$ suffer a discontinuity in the derivative. See Figures 2a and 3b. Equivalently, the sample paths of $\boldsymbol{X}$ can be written as the following stochastic convolution integral:

$$\boldsymbol{X}(t) = \boldsymbol{X}(0) + \int_0^t \boldsymbol{\varphi}(t-s)\,\mathrm{d}\boldsymbol{Y}(s). \tag{3.3}$$

where we have conveniently defined $T_0 := 0$.

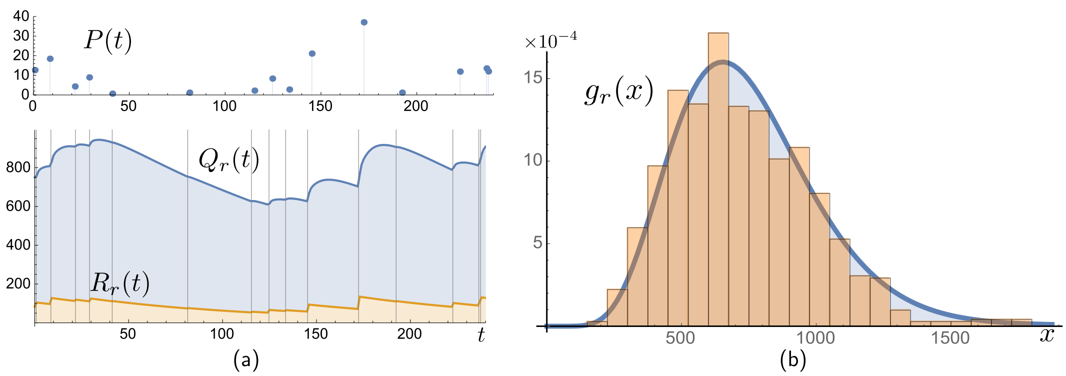

Fig. 2. (a) Simulated $P(t)$ (mm), $Q_r(t)$ and $R_r(t)$ ($\mathrm{L\,s^{-1}}$) for the river network in Figure 1, $t \in [0, 240\,\mathrm{h}]$. All hillslope areas were assumed equal $a_e = 0.6\,\mathrm{km}^2$. Values of K_e/K_r and H_e/K_r were randomly generated within the intervals $(0,1)$ and $(0, 10^{-3})$ respectively, with $K_r = 2\,\mathrm{h}^{-1}$. Storms were assumed uniform in space with $\lambda = 1/24\mathrm{h}^{-1}$, and $P_n = P_{n,e} \sim \exp(\sigma)$ with mean $\frac{1}{\sigma} = 5\,\mathrm{mm}$. The initial condition was taken to be the invariant mean $\boldsymbol{X}(0) = \mathbb{E}_g(\boldsymbol{X})$ of equation (4.2), $\mathbb{E}_g Q_r = 750\,\mathrm{L\,s^{-1}}$. (a) Using the same data: invariant probability density function of Q_r obtained by numerically inverting the Laplace transform $\tilde{g}_r$ in (4.8); and histogram obtained from samples of the simulation of $Q_r(t)$ in (a) extended to $t \in [0, 200 \times 24\mathrm{h}]$

Remark 3.1 (Connections to the unit and geomorphic hydrographs). The function $\boldsymbol{\varphi}(t) = e^{\mathbf{M}t}$ in (3.2) can be appropriately called a 'global hydrograph' both in the context of the classical Unit Hydrograph Theory of Dooge (1959), as well as within the more recent Geomorphologic Instantaneous Unit Hydrograph (GIUH) theory introduced by Gupta et al. (1980). We have added the adjective 'global' to emphasize that $\boldsymbol{\varphi}$ is a matrix function that simultaneously contains the hydrographs from all streams and hillslopes in the catchment. Let $\boldsymbol{\Theta}(t) \in \mathbb{R}_+^{2n_\Gamma}$ be the vector containing unit hydrographs at the downstream end of each link and hillslope corresponding to a unitary homogeneous precipitation event over all of Γ. Then from (3.1),

$$\boldsymbol{\Theta}(t) = e^{\mathbf{M}t} \begin{bmatrix} \mathbf{0} \\ \frac{1}{a}\mathbf{H}\boldsymbol{a} \end{bmatrix}. \tag{3.4}$$

Now, consider a GIUH model where the travel times over the hillslopes and streams corresponding to link e are taken as independent exponential random variables with respective densities

$$h_e(t) := H_e e^{-H_e t}, \quad k_e(t) := K_e e^{-K_e t}. \tag{3.5}$$

Following Gupta et al. (1980), one obtains the representation

$$\boldsymbol{\Theta}(t) = \begin{bmatrix} \left[\sum_{e' \in \Gamma_e} \frac{a_{e'}}{a}(h_{e'} * k_{e'} * \cdots * k_e)(t) : \ e \in \Gamma\right]^\intercal \\ \left[\frac{a_e}{a} h_e(t) : \ e \in \Gamma\right]^\intercal \end{bmatrix} \tag{3.6}$$

where Γ_e denotes the subnetwork with e as its outlet ($\Gamma_r := \Gamma$), and each summand contains convolutions following the flow path that starts at the hillslope surrounding e' and ends in stream e. The fact that (3.4) and (3.6) coincide can be shown by differentiating the right hand side of (3.6) with respect to t.

Remark 3.2 (Boundary behavior). Much of the classical treatment of PDMPs, e.g. Davis (1984); Rolski et al. (1999), deals with the behavior of the process at, and out of the boundary of the state space. In our case, $\partial\mathbb{R}_+^{2n_\Gamma} = \{\boldsymbol{x} \in \mathbb{R}_+^{2n_\Gamma} : x_{i,e} = 0 \text{ for some } i = 1, 2, \ e \in \Gamma\}$, which is inaccessible from $\mathbb{R}_+^{2n_\Gamma}$. Since we also refrain from considering the evolution of the process for initial conditions $\boldsymbol{X}(0) \in \partial\mathbb{R}_+^{2n_\Gamma}$, the boundary behavior of $\boldsymbol{X}$ needs not to be specified.

Denote the family of Markov transition probabilities of $\boldsymbol{X}$ by

$$p(t, \boldsymbol{x}, A) := \mathbb{P}_{\boldsymbol{x}}(\boldsymbol{X}(t) \in A), \ A \subseteq \mathbb{R}_+^{2n_\Gamma}, \tag{3.7}$$

where the subscript $\boldsymbol{x}$ on $\mathbb{P}_{\boldsymbol{x}}$ or $\mathbb{E}_{\boldsymbol{x}}$ denotes probability or expectation conditioned on $\boldsymbol{X}(0) = \boldsymbol{x}$. Let $T_t[h](\boldsymbol{x}) := \mathbb{E}_{\boldsymbol{x}}\left[h(\boldsymbol{X}(t))\right]$ denote the Markov semigroup associated to $\boldsymbol{X}$. Then T_t has extended infinitesimal generator given by the non-local operator

$$\mathcal{A}[h](\boldsymbol{x}) := \nabla h(\boldsymbol{x}) \cdot \mathbf{M}\boldsymbol{x} - \lambda h(\boldsymbol{x}) + \lambda \int_{\mathbb{R}_+^{n_\Gamma}} h(\boldsymbol{x}_1, \boldsymbol{x}_2 + \mathbf{H}(\boldsymbol{a} \circ \boldsymbol{y})) f_{\boldsymbol{P}}(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y}, \tag{3.8}$$

for functions h in the domain $\mathrm{Dom}(\mathcal{A})$ of $\mathcal{A}$, which includes all continuously differentiable and bounded functions from $\mathbb{R}_+^{2n_\Gamma}$ to $\mathbb{R}$ (Davis, 1984, page 366). The form of the infinitesimal generator $\mathcal{A}$ in (3.8) yields a second important characterization for our process: $\boldsymbol{X}$ is a conservative Feller process of the *Ornstein-Uhlenbeck Type* (OUT) as described in Sato and Yamazato (1984). In this case, the associated Lévy process of $\boldsymbol{X}$ has no Brownian component, its jump measure is $\lambda e^{-\lambda t} f_{\boldsymbol{Y}}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}t$, and the driving matrix is $-\mathbf{M}$ which has strictly positive eigenvalues.

The transition probabilities of OUT processes have been explicitly characterized, and those of $\boldsymbol{X}$ exhibit one important caveat: they are not absolutely continuous with respect to Lebesgue measure. In fact, this can be directly seen from the strong Markov property, as

$$p(t,\boldsymbol{x},\,\mathrm{d}\boldsymbol{y}) = \boldsymbol{\delta}_{\boldsymbol{\varphi}(t)\boldsymbol{x}}(\,\mathrm{d}\boldsymbol{y})e^{-\lambda t} + \int_0^t \int_{\mathbb{R}_+^{2n_\Gamma}} \lambda e^{-\lambda\tau} f_{\boldsymbol{Y}_2}(\boldsymbol{z}_2 - (\boldsymbol{\varphi}(\tau)\boldsymbol{x})_2)\, p(\tau,\boldsymbol{z},\,\mathrm{d}\boldsymbol{y})\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}\tau$$

where $\boldsymbol{\delta}$ denotes the Dirac delta measure, and $\boldsymbol{\varphi}$ is as in (3.2). Namely, starting at $\boldsymbol{x}$, the atomic event $[\boldsymbol{X}(t) = \boldsymbol{\varphi}(t)\boldsymbol{x}]$ has positive probability.

Sato and Yamazato (1984, Theorem 3.1) provide an explicit formula for the characteristic function of $\boldsymbol{X}(t)$, which we include for completeness. Their formula and most of the subsequent analysis is given here in terms of the multidimensional spatial Laplace transforms of $p(t,\boldsymbol{x},\cdot)$ and $f_{\boldsymbol{Y}_2}$:

$$\tilde{p}(t,\boldsymbol{x},\boldsymbol{s}) := \int_{\mathbb{R}_+^{2n_\Gamma}} e^{-\boldsymbol{y}\cdot\boldsymbol{s}} p(t,\boldsymbol{x},\,\mathrm{d}\boldsymbol{y}),\ \ \boldsymbol{s} \in \mathbb{R}_+^{2n_\Gamma}, \tag{3.9}$$

$$\tilde{f}_{\boldsymbol{Y}_2}(\boldsymbol{s}) := \int_{\mathbb{R}_+^{n_\Gamma}} e^{-\boldsymbol{y}\cdot\boldsymbol{s}} f_{\boldsymbol{Y}_2}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \tilde{f}_{\boldsymbol{P}}(\mathbf{H}\boldsymbol{a}\circ\boldsymbol{s}),\ \ \boldsymbol{s} \in \mathbb{R}_+^{n_\Gamma}. \tag{3.10}$$

The Laplace transform of the transition probability density is:

$$\tilde{p}(t,\boldsymbol{x},\boldsymbol{s}) = \exp\left\{ -\boldsymbol{\varphi}(t)\boldsymbol{x}\cdot\boldsymbol{s} - \frac{\lambda}{H_r}\int_{e^{-H_r t}}^1 \frac{1-\tilde{f}_{\boldsymbol{Y}_2}((u^{-\mathbf{M}^\intercal/H_r}\boldsymbol{s})_2)}{u}\,\mathrm{d}u \right\}, \tag{3.11}$$

where, for $\boldsymbol{s} \in \mathbb{R}_+^{2n_\Gamma}$ and $u > 1$, the vector $(u^{-\mathbf{M}^\intercal/H_r}\boldsymbol{s})_2$ is obtained by computing the matrix exponential $\exp\left(\frac{\log u}{H_r}\mathbf{M}^\intercal\right)$, right-multiplying by $\boldsymbol{s}$, and extracting the second half of the resulting vector. A more convenient formula is given in Corollary 4.1 below.

4. Invariant density of $\boldsymbol{X}$

A probability density function $g : \mathbb{R}_+^{2n_\Gamma} \to \mathbb{R}_+$ is 'invariant' or 'stationary' for the process $\boldsymbol{X}$ if

$$\mathbb{P}_g(\boldsymbol{X}(t) \in A) := \int_{\mathbb{R}_+^{2n_\Gamma}} g(\boldsymbol{x})p(t,\boldsymbol{x},A)\,\mathrm{d}\boldsymbol{x} = \int_A g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \mathbb{P}_g(\boldsymbol{X}(0) \in A) \tag{4.1}$$

for all measurable $A \subseteq \mathbb{R}_+^{2n_\Gamma}$ and $t > 0$. Note that for notational convenience and without risk of confusion, in (4.1) and below we are using the symbol g to denote a probability density and the measure determined by it. Also, the subscript g on $\mathbb{P}_g$ or $\mathbb{E}_g$ denotes probabilities or expectations with respect to the measure

g, namely conditioned on $\boldsymbol{X}(0)$ distributed as g. For example, taking expectations throughout the stochastic differential equation (2.13) and using invariance in the form $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}_g\boldsymbol{X}(t) = 0$, we get the invariant mean streamflow and runoff

$$\mathbb{E}_g\boldsymbol{X} = -\lambda\,\mathbf{M}^{-1}\begin{bmatrix}\mathbf{0}\\ \mathbf{H}(\boldsymbol{a}\circ\mathbb{E}\boldsymbol{P}_1)\end{bmatrix} = \lambda\begin{bmatrix}\boldsymbol{\Lambda}_\Gamma^{-1}(\boldsymbol{a}\circ\mathbb{E}\boldsymbol{P}_1)\\ \boldsymbol{a}\circ\mathbb{E}\boldsymbol{P}_1\end{bmatrix}. \tag{4.2}$$

Conditions for the existence, and the characterization of invariant distributions for OUT processes are given in (Sato and Yamazato, 1984, Theorems 4.1-4.2). We now apply their result to $\boldsymbol{X}$.

Proposition 4.1 (Sato and Yamazato (1984)). *A necessary and sufficient condition for the existence of a unique invariant measure for $\boldsymbol{X}$ is*

$$\int_{\boldsymbol{y}\in\mathbb{R}_+^{n_\Gamma}:|\boldsymbol{y}|\geqslant 1}\log(|\boldsymbol{y}|)f_{\boldsymbol{Y}_2}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} < \infty. \tag{4.3}$$

If (4.3) *holds, then the invariant distribution has Laplace transform given by*

$$\tilde{g}(\boldsymbol{s}) = \exp\left\{-\frac{\lambda}{H_r}\int_0^1\frac{1-\tilde{f}_{\boldsymbol{Y}_2}((u^{-\mathbf{M}^\intercal/H_r}\boldsymbol{s})_2)}{u}\,\mathrm{d}u\right\}. \tag{4.4}$$

Moreover, $p(t,\boldsymbol{x},\cdot)$ converges to g as $t\to\infty$ for all $\boldsymbol{x}\in\mathbb{R}_+^{2n_\Gamma}$.

The random variable of most interest in $\boldsymbol{X}(t)$ is its first entry $Q_r(t)$: the discharge at the watershed's outlet. Let g_r denote its invariant density

$$g_r(x) := \mathbb{P}_g(Q_r(t)\in\,\mathrm{d}x),\quad x>0. \tag{4.5}$$

Crucially, the Laplace transform of g_r is easily obtained from that of g as

$$\tilde{g}_r(s) = \tilde{g}([s,0,\ldots,0]^\intercal). \tag{4.6}$$

We then have the following Corollary.

Corollary 4.1. *The discharge Q_e has limiting invariant density g_e given by*

$$\tilde{g}_e(s) = \exp\left\{-\frac{\lambda}{H_r}\int_0^1\frac{1-\tilde{f}_{\boldsymbol{P}}(\mathbf{H}\boldsymbol{a}\circ\boldsymbol{m}_e(u)\,s)}{u}\,\mathrm{d}u\right\},\quad e\in\Gamma,\ s>0, \tag{4.7}$$

where $\boldsymbol{m}_e(u)$ is the second half of the column of $u^{-\mathbf{M}^\intercal/H_r}$ corresponding to Q_e.

Once $\tilde{g}$ has been computed, the Laplace transform of the invariant distribution of any of the components of $\boldsymbol{X}$ can be numerically inverted to get the corresponding approximate density function. See Figures 2b and 3b for example. For all computations reported here, we use the classical algorithm by Zakian (1969).

Remark 4.1 (Spatially uniform rainfall). Note that the precipitation intensity vector $\boldsymbol{P}_n$ in (2.9) contains the rainfall amounts that fall during the n-th event over each of the hillslopes, and explicitly allows for a spatially heterogeneous precipitation field. In fact, one could construct the Poisson process $\boldsymbol{P}$ in (2.7) by

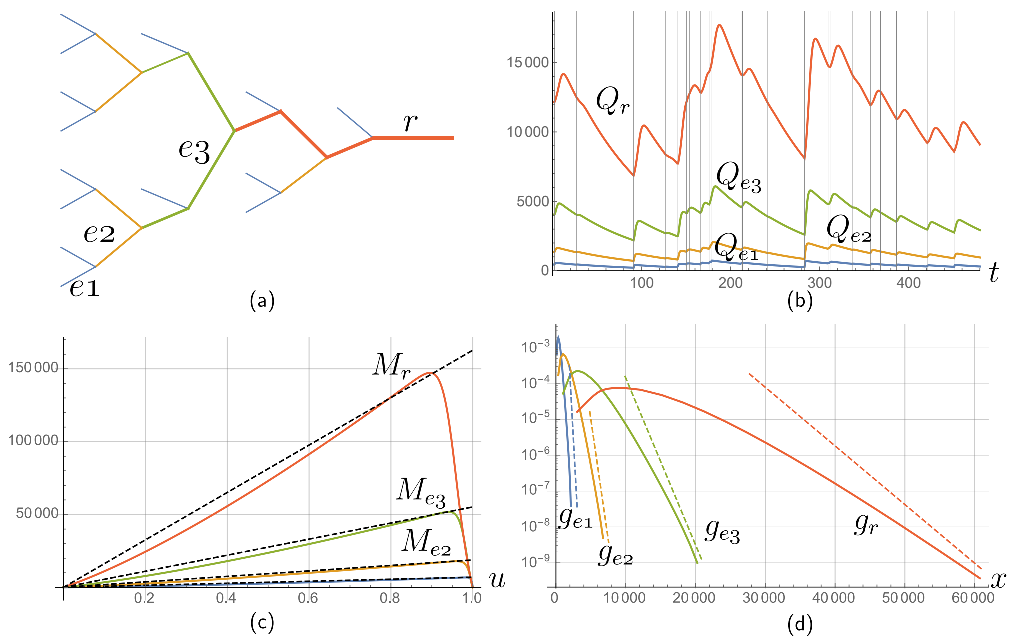

Fig. 3. Simulation and invariant densities for a watershed of fourth order. The precipitation field satisfies $\lambda = \frac{1}{24}\mathrm{h}^{-1}$, $P_n = P_{n,e} \sim \exp(\sigma)$ with $\frac{1}{\sigma} = 5\,\mathrm{mm}$. All hillslope areas were assumed equal to $a_e = 0.6\,\mathrm{km}^2$. Values of K_e, H_e were randomly generated as $K_e = \epsilon_K K$, $H_e = \epsilon_H H$ with $H = 0.017$, $H/K = 0.008$, $\epsilon_K, \epsilon_H \sim \mathrm{Unif}[0.5, 1.5]$ (a) Structure of the river network with four selected stream links of increasing Horton order. (b) Simulated discharge $Q_e(t)$ in $\mathrm{L\,s^{-1}}$ at selected streams for $t \in [0, 20 \times 24\mathrm{h}]$. Vertical lines mark storm events. (c) Plots of $M_e(u) := \mathbf{H}\boldsymbol{a} \cdot \boldsymbol{m}_e(u)$ compared with the linear approximation $\mathbf{H}\boldsymbol{a} \cdot \boldsymbol{m}_e^0(u)$ in (4.11). (d) Logplots of the invariant densities $g_e(x)$ for selected streams and $x \in [0.25, 6] \times \mathbb{E}_g Q_e\ \mathrm{L\,s^{-1}}$. The straight dashed lines have the slope predicted in Proposition 5.3.

merging several independent Poisson processes with different statistical properties, and modeling rainfall over different parts of the watershed. On the other hand, the case where $P_{n,e} = P_n$ for all $n \geqslant 1$ and $e \in \Gamma$, is important for its simplicity, as it represents a scenario where on every storm, all hillslopes receive the same random amount P_n of rainfall. In this case it suffices to consider an i.i.d. sequence $\{P_n : n \geqslant 1\}$ of rainfall depths with common probability density f_P instead of the joint density $f_{\boldsymbol{P}}$. The description of the process is obtained by replacing $\mathbf{H}(\boldsymbol{a} \circ \boldsymbol{P}_n)$ by $\mathbf{H}\boldsymbol{a}P_n$ in (2.15). Similarly, the integrands in (3.11), (4.4) and (4.7) must be modified by replacing Hadamard products with dot products: using expression (3.10) for $\tilde{f}_{\boldsymbol{Y}_2}$, gives

$$\tilde{g}_e(s) = \exp\left\{-\frac{\lambda}{H_r}\int_0^1 \frac{1 - \tilde{f}_P(\mathbf{H}\boldsymbol{a} \cdot \boldsymbol{m}_e(u)\, s)}{u}\,\mathrm{d}u\right\}. \tag{4.8}$$

Remark 4.2 (The functions $\boldsymbol{m}_e$: calculation and approximation). The functions $\boldsymbol{m}_e$, $e \in \Gamma$ in (4.7) and (4.8) encapsulate the role played by the network geomorphology on the asymptotic distribution of discharge. Some remarks are in order. Note first that the value of H_r in (4.4) and (4.7) can be changed to any H_e (or any other positive rate) by making the change of variables $u \mapsto u^{H_e/H_r}$ in the integral. Secondly, if we denote the lower left block matrix of $u^{-\mathbf{M}^\intercal/H_r}$ by $\boldsymbol{m}(u) \in \mathbb{R}^{n_\Gamma \times n_\Gamma}$, then $\boldsymbol{m}_e(u)$ is the column of $\boldsymbol{m}(u)$ corresponding to link e. By (2.14) the matrix $\boldsymbol{m}(u)$ is given by

$$\boldsymbol{m}(u) = \sum_{n=0}^{\infty} -\frac{\log(u)^n}{H_r^n n!} \left\{ \mathbf{K} \left[\mathbf{I} - \boldsymbol{\Lambda}_\Gamma \mathbf{K} \mathbf{H} \right]^{-1} \left[\mathbf{I} - (\boldsymbol{\Lambda}_\Gamma \mathbf{K} \mathbf{H}^{-1})^n \right] \mathbf{H}^{n-1} \right\}^\intercal . \tag{4.9}$$

In particular, in the spatially uniform case where $K_e = K_r$ and $H_e = H_r$ for all $e \in \Gamma$, one can write

$$\boldsymbol{m}(u) = \left\{ [\boldsymbol{\Lambda}_\Gamma - \beta \mathbf{I}]^{-1} \left[u\mathbf{I} - u^{\frac{1}{\beta}\boldsymbol{\Lambda}_\Gamma} \right] \right\}^\intercal , \quad \beta := \frac{H_r}{K_r}. \tag{4.10}$$

If $H_e = H_r \ll K_r = K_e$ for all $e \in \Gamma$, the following linear approximation holds (see Figure 3c)

$$\boldsymbol{m}(u) \approx \lim_{\beta \to 0} \boldsymbol{m}(u) = (\boldsymbol{\Lambda}_\Gamma^{-1})^\intercal u =: \boldsymbol{m}^0(u) \tag{4.11}$$

As illustrated in Figure 4, expression (4.10) for homogeneous self-similar networks seems to yield self-similar invariant distributions for the discharge. See also Section 6. Deviations from homogeneity in the parameters H_e, K_e show much richer behavior.

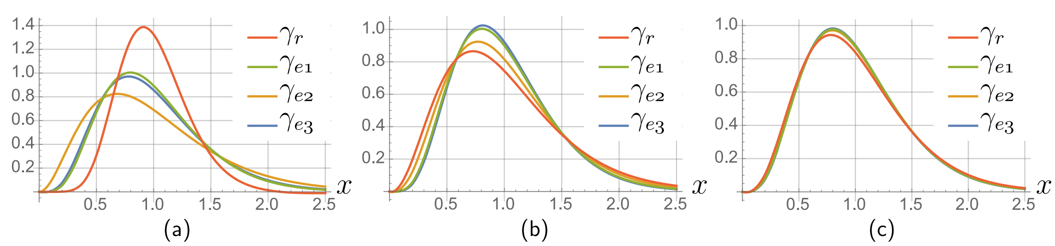

Fig. 4. (The effect of heterogeneity) Scaled densities γ_e of the normalized discharge $Q_e/\mathbb{E}_g Q_e$ for selected links of the river network and precipitation regime shown in Figure 3. The values of H_e and K_e were assigned as in Figure 3 but with random multipliers ϵ of decreasing variance: a) $\epsilon_K, \epsilon_H \sim$ Unif$[0.25, 1.75]$, b) $\epsilon_K, \epsilon_H \sim$ Unif$[0.5, 1.5]$, c) $\epsilon_K, \epsilon_H \sim$ Unif$[0.9, 1.1]$.

5. Moments and extreme events of Q_r

In this section we exploit the properties of the Laplace transform of the density g_r to derive important characteristics of the invariant distribution of $\boldsymbol{X}$ and of Q_r. For simplicity, we restrict our attention to the case of uniform rainfall as described in Remark 4.1.

5.1. *Calculation of moments*

Denote the n-th invariant moment of Q_r by

$$M_{Q_r}^{(n)} := \mathbb{E}_g Q_r^n = (-1)^n \frac{\mathrm{d}^n \tilde{g}_r}{\mathrm{d}s^n}(0), \; n = 1, 2, \dots \tag{5.1}$$

If we denote the function inside the exponential in (4.4) by

$$h(s) := -\frac{\lambda}{H_r} \int_0^1 \frac{1 - \tilde{f}_P(\mathbf{H}\boldsymbol{a} \cdot \boldsymbol{m}_r(u)s)}{u} \, \mathrm{d}u, \tag{5.2}$$

then Faa di Bruno's formula for the n-th derivative of $e^{h(s)}$ gives

$$\frac{\mathrm{d}^n \tilde{g}_r}{\mathrm{d}s^n} = \tilde{g}_r \sum_{k=1}^{n} B_{n,k} \left(\left\{ \frac{\mathrm{d}^i h}{\mathrm{d}s^i} \right\}_{i=1}^{n-k+1} \right) \tag{5.3}$$

where $B_{n,k}$ is the Bell polynomial

$$B_{n,k}(\{x_i\}_{i=1}^{n-k+1}) = n! \sum_{\boldsymbol{j} \in I(n,k)} \prod_{i=1}^{n-k+1} \frac{1}{j_i!} \left(\frac{x_i}{i} \right)^{j_i}. \tag{5.4}$$

Here, $\boldsymbol{j} \in I(n,k)$ denotes that the sum is taken over all vectors $\boldsymbol{j} = \{j_1, \dots, j_{n-k+1}\}$ of indices such that:

$$\sum_{i=1}^{n-k+1} j_i = k, \qquad \sum_{i=1}^{n-k+1} i j_i = n. \tag{5.5}$$

Letting $s \downarrow 0$ in (5.3), and writing $M_P^{(n)}$ for the n-th moment of P_1, one arrives at the following useful expression (see Figure 5 for a numerical example).

Proposition 5.1. *The n-th invariant moment of Q_r is*

$$M_{Q_r}^{(n)} = (aK_r)^n \sum_{k=1}^{n} \left(\frac{\lambda}{H_r} \right)^k B_{n,k} \left(\left\{ M_P^{(i)} c_i \right\}_{i=1}^{n-k+1} \right) \tag{5.6}$$

where the coefficients c_i are given by

$$c_\alpha := \int_0^1 \frac{[\boldsymbol{\beta} \tilde{\boldsymbol{a}} \cdot \boldsymbol{m}_r(u)]^\alpha}{u} \, \mathrm{d}u, \quad \alpha > 0, \tag{5.7}$$

with $\boldsymbol{\beta} := \left[\frac{H_e}{K_r} : e \in \Gamma \right]^\intercal$ and $\tilde{\boldsymbol{a}} := \frac{1}{a} \boldsymbol{a}$.

Remark 5.1. There are two important implications of the Proposition 5.1. First, that under the invariant distribution, the discharge will have exactly as many moments as each P_n. Secondly, the non-dimensional constants $\{c_n : n = 1, 2, \dots\}$ constitute a set of parameters, depending only on the geomorphology of the

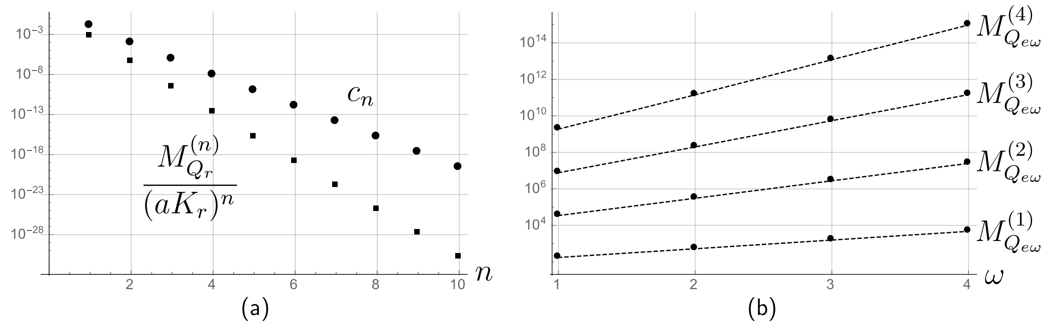

Fig. 5. For the example of figure 3: (a) logarithmic plots of the sequence of c_n (unit-less), and scaled moments $M_{Q_r}^{(n)}/(aK_r)^n$ in units of m^n, $n = 1, \ldots, 10$. (b) First four invariant moments of Q_e for the selected links in Figure 3 which have consecutive Horton order $\omega = 1, 2, 3, 4$, $e_4 := r$.

watershed, that completely determine the invariant distribution of the discharge Q_r.

Direct integration of (5.6) shows that

$$c_1 = \frac{H_r}{K_r}, \quad \mathbb{E}_g Q_r = a\lambda\mathbb{E}(P_1) \tag{5.8}$$

which coincides with the expression for $\mathbb{E}_g \boldsymbol{X}$ in (4.2), and makes explicit the fact that under the invariant distribution, the discharge is in a state of average equilibrium with the precipitation.

5.2. *Asymptotics of extreme events*

We now turn our attention to the asymptotic behavior of the probabilities of extreme events of discharge, namely $\mathbb{P}_g(Q_e > x)$ as $x \to +\infty$. We are particularly interested on how this decay relates to both, the geomorphological properties of the network, and the probablity density f_P of rainfall. First of all, as noted in Remark 5.1, if the distribution of P_1 has n finite moments, then it follows from Proposition 5.1 that Q_r will also have exactly n finite moments. In that case, general theory (see for example Chung, 2001, Excercise 3.2.5) gives

$$\lim_{x\to\infty} x^n \mathbb{P}_g(Q_r > x) = 0. \tag{5.9}$$

We now give precise results for the asymptotics of $\mathbb{P}_g(Q_e > x)$ in two contrasting types of distributions of P_1: a heavy-tailed distribution with no moments, and the exponential distribution. In both cases, we prove that the invariant distribution of Q_e preserves the general asymptotic behavior as that of P_1. See Figure 3d.

Proposition 5.2. *Suppose $P_1 \sim Pareto(\alpha, k)$ for some $k > 0$ and $0 < \alpha < 1$. Then*

$$\mathbb{P}_g(Q_r > x) \sim \frac{\lambda(kaK_r)^\alpha}{H_r} c_\alpha \, x^{-\alpha} \quad \textit{as } x \to \infty \tag{5.10}$$

where c_α *is given by* (5.7).

Proof. □

In this case $f_P(x) \sim x^{-1-\alpha}$ as $x \to \infty$. Equation (5.6) along with the expansion of $\tilde{f}_{\boldsymbol{P}}$ in Taylor series, yields

$$\tilde{g}_r \sim \exp(-Cs^\alpha) \text{ where } C = \frac{\lambda}{H_r} k^\alpha \Gamma(1-\alpha) \int_0^1 \frac{(\mathbf{H}\boldsymbol{a} \cdot \boldsymbol{m}_r(u))^\alpha}{u} \, \mathrm{d}u$$

as $s \downarrow 0$. Denote $\Psi(x) := \mathbb{P}(Q_r > x)$. Then $\tilde{\Psi}(s) = \frac{1}{s}(1 - \tilde{g}(s)) \sim Cs^{\alpha-1}$ as $s \downarrow 0$. The Karamata Tuberian theorem gives that $\int_0^x \Psi(y)\,\mathrm{d}y \sim \frac{C}{\Gamma(2-\alpha)} x^{1-\alpha}$ as $x \to \infty$. Differentiation yields the desired asymptotics for $\Psi(x)$. See Bingham et al. (1989), Theorems 1.7.1, 1.7.2.

Proposition 5.3. *Suppose* $P_1 \sim \exp(\sigma)$ *for* $\sigma > 0$. *Let* $M_r^* = \max\{\mathbf{H}\boldsymbol{a} \cdot \boldsymbol{m}_r(u) : u \in [0,1]\}$, *then*

$$\lim_{x \to \infty} \frac{1}{x} \log \mathbb{P}_g(Q_r > x) = -\frac{\sigma}{M_r^*}. \tag{5.11}$$

Proof. The tails of $f_P(y)$ decay exponentially which is manifested in $\tilde{f}_P(s) = \sigma/(s+\sigma)$ as a pole at $s = -\sigma$. We now show the same a pole also exists for $\tilde{g}_r$. Note first that for $u \in [0,1]$, the matrix $-\frac{\log u}{H_r}\mathbf{M}^\intercal$ has non-negative off-diagonal entries, and therefore every entry of $u^{-\mathbf{M}^\intercal/H_r}$ is non-negative. This implies that $M_r(u) := \mathbf{H}\boldsymbol{a} \cdot \boldsymbol{m}_r(u)$, $u \in [0,1]$ is a non-negative function with $M_r(0) = M_r(1) = 0$. See Figure 3d. Moreover by (4.9), M_r is a bounded and differentiable function of u with a positive maximum at $u = u^* \in (0,1)$ where $M_r'(u^*) = 0$. The convergence of $\int_0^1 \frac{1}{u}(1 - \tilde{f}_{\boldsymbol{P}}(M_r(u)s))\,\mathrm{d}u$ for $s > 0$ is guaranteed by Proposition 4.1. For $-\frac{\sigma}{M_r^*} < s \leqslant 0$ the following estimate holds

$$\int_0^1 \tfrac{1}{u}(1 - \tilde{f}_P(M_r(u)s))\,\mathrm{d}u \geqslant \frac{sM_r^*}{u^*} \log\left(1 + \frac{u^*}{\sigma + sM_r^*}\right)$$

and therefore $\tilde{g}_r(s) \to \infty$ as $s \downarrow -\sigma/M_r^*$. The asymptotic formula (5.11) now follows from Nakagawa (2005, Theorem 3) □

Note that the presence of the constants c_α and M_r in equations (5.10) and (5.11) show explicitly how the geometry of the river network and the spatial distribution of the geomorphic variables impact the asymptotic behavior of extreme discharge events.

6. Statistical scaling of discharge

Since its first introduction, the river network ordering scheme of Horton (1945), and its relation to river network self-similarity and the statistical scaling of geomorphic variables, have played a key role in the contemporary development of the hydrological sciences. See Kovchegov and Zaliapin (2016) and Barndorff-Nielsen (1993) for mathematically minded surveys. It is now abundantly clear that statistical scaling properties of river flows have a common, mechanistic origin related to conservation principles and the geometry of river networks (see Gupta et al., 2007, and references therein).

Within our framework we can study one of the most remarkable of Horton's laws: the statistical scaling of discharge. As formulated by Peckham and Gupta (1999), this refers to the concept of invariance of the probability distribution of stream discharge with respect to the basin's scale measured by its Horton order. To be definite, let $\omega = \omega(e)$ denote the Horton order of edge $e \in \Gamma$, and Ω the Horton order of the network's stem stream. We restrict our analysis to random variables X that exhibit 'statistical simple scaling' with respect to ω, namely

$$X_{\omega+1} \stackrel{d}{=} R_X X_\omega, \quad \omega = 1, 2, \ldots, \Omega \tag{6.1}$$

where $R_X > 0$ is the corresponding Horton ratio and $\stackrel{d}{=}$ denotes equality in distribution. For example, it is well-known that the total upstream area A_ω of streams of order ω satisfies

$$A_{\omega+1} \stackrel{d}{=} R_A A_\omega, \quad \omega = 1, 2, \ldots, \Omega \tag{6.2}$$

with $4 < R_A < 5$ (Schumm, 1956). Equation (6.2) has been shown by Veitzer and Gupta (2000) to hold for a large class of random self-similar river networks, including those generated via the Galton-Watson and the Tokunaga models. See Tokunaga (1978) and Kovchegov and Zaliapin (2016).

Our aim here is to establish a scaling relationship of the form (6.1) for the spatially averaged discharge

$$\bar{Q}_\omega(t) := \frac{1}{N_\omega} \sum_{\{e \in \Gamma:\, \omega(e) = \omega\}} Q_e(t) \tag{6.3}$$

under the invariant distribution g for $\boldsymbol{X}$. Namely, we would like to find conditions under which there exists a positive R_Q such that the invariant distribution of $\bar{Q}_{\omega+1}(t)$ is the same as that of $R_Q \bar{Q}_\omega(t)$ for all $t > 0$ and all $\omega = 1, \ldots, \Omega$.

We now show that at least in the spatially uniform case, the invariant distribution of discharge exhibits simple statistical scaling.

Theorem 6.1. *Suppose that $H_e = H_r \ll K_r = K_e$, and that the precipitation field is uniform, $P_{n,e} = P_n$, for all $e \in \Gamma$. Suppose further that* (6.2) *holds exactly in the sense that*

$$A_e = A_{\omega(e)} = R_A^{\omega(e)-1} A_1 \quad \textit{for all } e \in \Gamma. \tag{6.4}$$

Then, under the invariant distribution,

$$\bar{Q}_{\omega+1}(t) \stackrel{d}{=} R_A \bar{Q}_\omega(t). \tag{6.5}$$

Proof. The invariant distribution of Q_e/A_e has Laplace transform $\tilde{g}_e(s/A_e)$. Under the spatially uniform rainfall hypothesis, $\tilde{g}_e$ is given by (4.8) and since $\beta = H_e/H_e \ll 1$, the matrix $\boldsymbol{m}$ may be given approximately by (4.11). The relevant term is the argument of $\tilde{f}_P$:

$$\frac{\mathbf{H}\boldsymbol{a} \cdot \boldsymbol{m}_e(u)\, s}{A_e} = \frac{(\boldsymbol{m}(u)^{\intercal}\mathbf{H}\boldsymbol{a})_e}{A_e} s = \frac{H_r(\boldsymbol{\Lambda}_\Gamma^{-1}\boldsymbol{a})_e}{A_e} us = H_r us,$$

which is independent of e. This that under g, the Q_e/A_e, $\omega = 1, \dots, \Omega$ are identically distributed. Equation (6.5) follows from making $A_e = A_{\omega(e)}$ and taking spatial average. $\square$

Figure 6 shows numerical experiments seemingly indicating that the conditions of Theorem 6.1 are also necessary for scaling of the invariant distribution of discharge. Specializing to the case of uniform precipitation, $P_{n,e} = P_n$ for all $e \in \Gamma$, and networks that satisfy (6.4), $\mathbb{E}_g \bar{Q}_\omega = \lambda \mathbb{E}(P_1) A_\omega$. Under this conditions, the scaling relationship (6.5) is equivalent to a collapse of the γ_ω. Figure 4 shows that if $\beta_e = H_e/K_e$ is not spatially constant, scaling may not occur. Moreover, even for spatially homogeneous networks, scaling fails for large values of β_e.

7. Conclusion and outlook

In this work we have presented the detailed mathematical solution for the equations (2.2)-(2.3) of mass and momentum balance in an arbitrary watershed at the hillslope scale. The solution covers the deterministic case through the global hydrograph map φ in (3.2), as well as the case of rainfall given by a Poisson point process. In its most basic form, our main result gives an approximation for the distribution of runoff and streamflow within a watershed given the geometry of the river network and a set of hillslope-scale physical parameters.

An immediate consequence of Proposition 4.1 is that $\boldsymbol{X}$ is an 'ergodic' process (see (Kallenberg, 2002, Section 20) for Feller ergodic processes, or Costa and Dufour (2008) for the specific case of ergodic PDMPs). It follows in particular, that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(\boldsymbol{X}(s))\, \mathrm{d}s = \lim_{t\to\infty} T_t[f](\boldsymbol{x}) = \int_{\mathbb{R}_+^{2n_\Gamma}} f(\boldsymbol{x}) g(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \tag{7.1}$$

g-almost surely for all suitable $f : \mathbb{R}_+^{2n_\Gamma} \to \mathbb{R}$ and any initial condition $\boldsymbol{X}(0) = \boldsymbol{x} \in \mathbb{R}_+^{2n_\Gamma}$. If the statistical properties of the precipitation field do not change for a sufficiently long period of time, our model predicts that the watershed will attain a statistical invariant regime determined by g.

At larger time scales, our model thus allows us to consider the effect of inter-seasonal variations on the statistical properties of rainfall. Namely, if a changing climate imposes perturbations in the frequency λ of rainfall, or in the distribution f_P of storm intensities, our results yield the corresponding statistical response of discharge. This and further insights about the effect of climatic change on streamflow uncertainty will be included on a separate note.

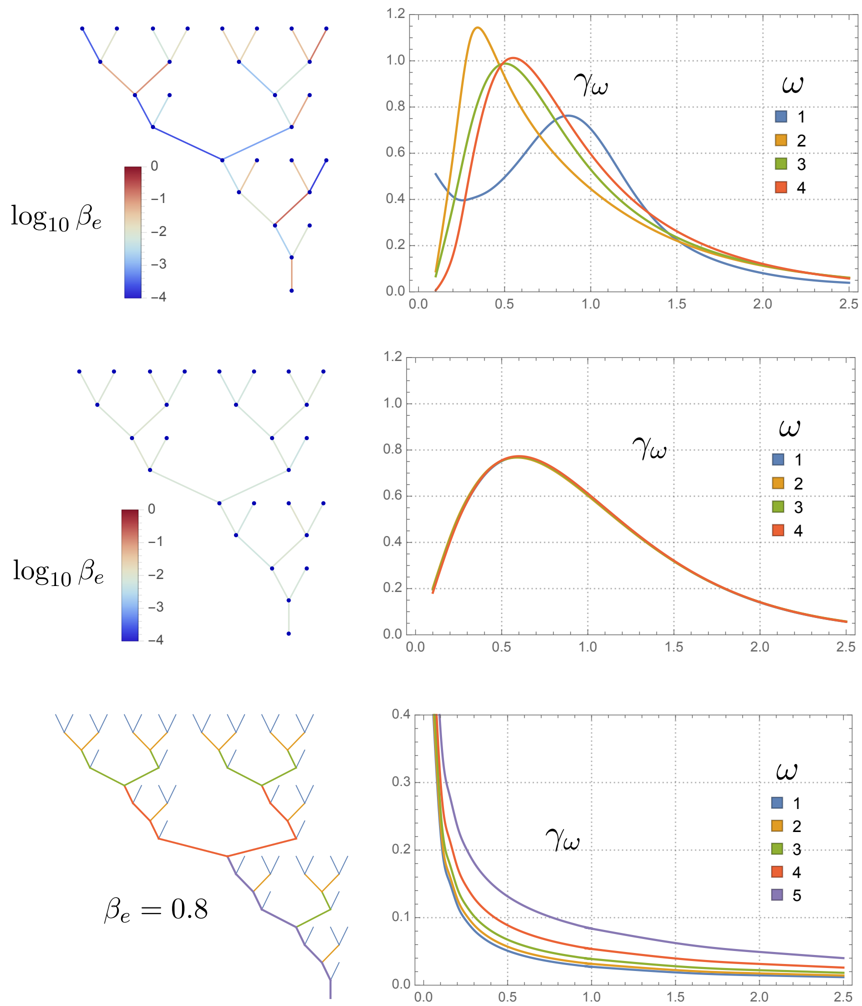

Fig. 6. Numerically reconstructed densities γ_ω of $\bar{Q}_\omega/\mathbb{E}_g\bar{Q}_\omega$ for different networks. The value of λ and the distribution of P_1 are as in Figure 2 for all cases. Top: fully heterogeneous network of order $\Omega = 4$ with $\beta_e = 0.008 \times 10^{\pm 4}$. Middle: same network as top figure but with almost homogeneous and small $\beta_e = 0.008 \times 10^{\pm 0.4}$. Bottom: network of order $\Omega = 5$ with $\beta_e = 0.8$ for all e.

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