Explicit Kähler packings of projective complex manifolds

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Abstract

In this thesis we prove that for a projective complex manifold X equipped with an ample line bundle L the multi-point Seshadri constant is equal to the square of the multi-ball Kähler packing constant. Furthermore we suggest a general strategy to construct moment maps on open subsets of X such that the images of embedded balls tile the Newton-Okounkov body $\Delta_{Y_\bullet}(X, L)$ of X, L with respect to a general flag $Y_\bullet$ as described by the iterative dissection of $\Delta_{Y_\bullet}(X, L)$ associated to the points on X. Finally we show that this strategy works for 1 or 2 points on $\mathbb{P}^2_{\mathbb{C}}$ and discuss the problems occurring for 3 points.

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Contents

Chapter 1

Introduction.

The field of mathematics contains many long standing problems that despite the great effort of many mathematicians over the years still remain unsolved. Often these problems can be stated in such a way that at first glance they appear simple and therefore should have an obvious solution. Unfortunately this is often not the case. The motivation for many of the ideas contained in this thesis in fact arises from such a problem:

In his celebrated solution of Hilbert's Fourteenth Problem in 1959, Nagata [Nag59] stated in passing a conjecture on plane algebraic curves that in its original form predicts that the sum of the squares of the multiplicities in enough points on such a plane algebraic curve is bounded by the square of the degree.

Conjecture 1.1. *Let $P_1, \ldots, P_k$ be points of $\mathbb{P}^2_{\mathbb{C}}$ in general position and $m_1, \ldots, m_k$ be positive integers. Then for $k \geq 9$ any curve $C \subset \mathbb{P}^2_{\mathbb{C}}$ of degree d passing through each point P_i with multiplicity m_i must satisfy*

$$d \geq \frac{1}{\sqrt{k}} \sum_{i=1}^{k} m_i.$$

In the early 90s algebraic invariants call Seshadri constants were introduced by Demailly in [Dem92] as a way of studying local positivity of ample line bundles at a given point of a variety. Let X be a smooth projective variety of dimension n. Fix an ample divisor L on X and a collection of points $P_1, \ldots, P_k \in X$. If $\pi\colon Bl_{P_1,\ldots,P_k}(X) \to X$ denotes the blow up of X at $P_1 \ldots, P_k$ the *k-point Seshadri constant* of X and L at $P_1, \ldots, P_k$ is defined as

$$\epsilon(X, L; P_1, \ldots, P_k) = \sup\{\epsilon \in \mathbb{Q}_{>0} \colon \pi^* L - \epsilon \sum_{i=1}^{k} E_i \text{ is } \mathbb{Q}\text{-ample}\}.$$

Since being introduced Seshadri constants have attracted substantial attention in the field of algebraic geometry and can be used to reformulate many classical ideas. For example Nagata's conjecture in terms of Seshadri constants can be stated as follows.

Conjecture 1.2. *Let $P_1, \ldots, P_k$ be points of $\mathbb{P}^2$ in general position. Then for $k \geq 9$ the multipoint Seshadri constant*

$$\epsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), P_1, \ldots, P_k) = \frac{1}{\sqrt{k}}.$$

In general Seshadri constants are difficult to compute and relatively few examples are known. Lazarsfeld proves the existence of lower bounds for single point Seshadri constants (see [Laz04, Theorem 2.2.5]). The Ein-Lazarsfeld conjecture [Laz04, Conjecture 2.2.4] predicts that the value of the single point Seshadri constant is always greater or equal to 1 and is still an open problem. Some partial results and lower bounds are known and Küronya and Lozovanu in [KL17] and [KL18] relate single point Seshadri constants to convex bodies called Newton-Okounkov bodies (see Definition 2.62) and describe a method to determine the value of single point Seshadri constants from the geometric properties of these convex bodies. In the case of surfaces they prove a lower bound for single point Seshadri constants [KL18, Proposition 4.7].

In [DKMS16] the authors discuss the rationality of single point Seshadri constants of blow ups of the projective plane and relate this problem to a more general conjecture related to linear systems of plane curves called the SHGH conjecture (see Conjecture 2.15). The authors prove that if we assume the SHGH conjecture holds then there exists a Seshadri constant that is irrational and discuss a method to determine the rationality of Seshadri constants using functions on Newton-Okounkov bodies. For a detailed survey on single point Seshadri constants we refer the reader to [BDRH$^+$09].

Multipoint Seshadri constants have received less attention than the single point case and are in general more difficult to compute due to the more complicated geometric structure of the corresponding varieties. Some partial results are known and when X is a projective variety of dimension n equipped with a nef line bundle the multipoint Seshadri constant at points $P_1, \dots, P_k$ satisfies

$$\epsilon(X, L; P_1, \dots, P_k) \leq \sqrt[n]{\frac{L^n}{k}}.$$

For a proof of this bound see [BDRH$^+$09, Proposition 2.1]. Roé in [Roé03] gives a lower bound for $(n-1)$-dimensional Seshadri constants (see Definition 2.13) of k very general points in a variety (see [Roé03, Theorem 3]). Since for surfaces the $(n-1)$-dimensional Seshadri constant at k general points is exactly the k-point Seshadri constant in the definition given above, Roé's lower bound also provides a lower bound for Nagata's conjecture. These ideas are further discussed in [HR03] and [HR08] where the authors describe an approach for computing accurate estimates for multipoint Seshadri constants on $\mathbb{P}^2$ for points in general position. There are some results regarding Seshadri constants on abelian surfaces and it is known that single point Seshadri constants of ample line bundles on abelian surfaces are rational [BDRH$^+$09, Theorem 6.4.6]. Also due to results of Fuentes Garcìa [FG07] when we consider abelian surfaces with Picard number 1 we have some partial results for values of multipoint Seshadri constants (see [BDRH$^+$09, Theorem 6.4.8]) and moreover these results imply that the multipoint Seshadri constant of ample line bundles at the points of a finite subgroup of an abelian surface are rational.

In [MP94] McDuff and Ploterovitch connected Nagata's conjecture to a symplectic packing problem which was solved by Biran in 1997 [Bir01]. Unfortunately Biran's proof only showed the existence of such packings, and also seemed not to give any hints how to attack the original Nagata Conjecture. The work of Biran did however motivate a more restrictive packing problem.

Let (X, ω) be an n dimensional Kähler manifold with Kähler form ω and fix points $P_1, \dots, P_k \in X$. A holomorphic embedding

$$\phi = \coprod_{q=1}^{k} \phi_q \colon \coprod_{q=1}^{k} B_0(r_q) \hookrightarrow X$$

is called a *Kähler embedding of k disjoint complex flat balls in $\mathbb{C}^n$ centred in* 0*, of radius* r_q, if there exists a Kähler form ω' such that $[\omega'] = [\omega] \in H^{1,1}(X, \mathbb{R})$ and $\phi_q^*(\omega') = \omega_{std}$ is the standard Kähler form on $\mathbb{C}^n$ restricted to $B_0(r_q)$. Let

$$\gamma_k(X, \omega; P_1, \dots, P_k) = \sup\{r > 0 \colon \exists \text{ a Kähler packing as above}\}.$$

We call γ_k the *$k-$ball Kähler packing constant.*

In [Eck17] Eckl showed that Nagata's Conjecture is equivalent to a Kähler packing problem and proved that the Kähler packing constant of a projective complex surface equipped with an ample line bundle and a Kähler form ω with $[\omega] \in c_1(L)$ is equal the multipoint Seshadri constant. The main result of this thesis extends Eckl's result and proves the equality between the k-ball Kähler packing constant and the k-point Seshadri constant on arbitrary projective complex manifolds. In more details:

Theorem A. *Let X be a projective complex manifold of dimension n and L an ample line bundle on X. Fix a collection of points $P_1, \ldots, P_k$ and a Kähler form ω on X such that $[\omega] \in c_1(L)$, then the square of the k-ball packing constant is equal to the k-point Seshadri constant i.e.*

$$\gamma_k(X, \omega; P_1, \ldots, P_k) = \sqrt{\epsilon(X, L; P_1, \ldots, P_k)}.$$

Eckl already treated the surface case in [Eck17], whereas Witt-Nyström considered the one point and one ball case in [WN15]. Trussiani also proved this equality in the general case [Tru18] but proceeding by a different method. The result obtained in Theorem A offers a new approach for computing multipoint Seshadri constants on arbitrary complex projective manifolds of dimension n equipped with an ample line bundle via Káhler packings. The hope is that this may be useful to improve known bounds or determine previous unknown values of multipoint Seshadri constants.

In his paper [Oko96] as a passing remark Andrei Okounkov described a way to associate a convex body contained in $\mathbb{R}^n$ to a projective manifold of dimension n equipped with an ample line bundle. These ideas were made more precise by works of Kaveh and Khovanskii [KK12], and Lazarsfeld and Mustață [LM09]. In particular their constructions extended Okounkov's to the case when X is a projective manifold and L is a big line bundle. More precisely: For a projective manifold X of dimension n equipped with an ample line bundle L and an *admissible flag* $Y_\bullet \coloneqq X = Y_0 \supsetneq, \ldots, \supsetneq Y_n = \{pt\}$ there exists a convex body $\Delta_{Y_\bullet} \subset \mathbb{R}^n$ associated to X and L (see Definition 2.58 and 2.62 for precise definitions). Such bodies are called Newton-Okounkov bodies and one of their desirable features is they encode information about X and L. In [KLM12] the authors gave a complete classification of Newton-Okounkov bodies of surfaces building on earlier work by [LM09] and proving that the Newton-Okounkov bodies of surfaces are polygonal.

In [Eck14] Eckl used the properties of Newton-Okounkov bodies on surfaces and the classification given in [KLM12] to prove that Nagata's conjecture is equivalent to certain

Newton-Okounkov bodies having a special form. Furthermore on toric varieties these Newton-Okounkov bodies can be interpreted as the images of a moment map on the toric variety, and Haradah and Kaveh have extended this interpretation to more general varieties in [HK15] and [Kav19]. The hope is that the preimages of the Newton-Okounkov bodies yield Kähler packings, thus providing examples of explicit Kähler packings of the complex projective plane and shedding new light on Nagata's Conjecture. This research thesis provides some evidence for this hope.

First we develop an iterative construction of blow ups which we use to construct Kähler forms and prove Theorem A. We then use this iterative construction to devise a strategy for producing the desired moment maps.

In more details: Let X be a complex projective manifold of dimension n, with $P_1, \ldots, P_k$ distinct points of X and L an ample divisor on X. Denote $\pi_k \colon Bl_{P_1,\ldots,P_k}(X) \to X$ the blow-up of X at the points $P_1, \ldots, P_k$, with $\pi_i^{-1}(P_i) = E_i$ the exceptional divisor corresponding to the point P_i for all $i = 1, \ldots, k$ and set

$$\tilde{L}^{(i)} \coloneqq \pi_i^* dL - \sum_{j=1}^{i} \epsilon_j E_j.$$

For an admissible flag $Y_\bullet$ the chain of inclusions

$$\Delta_{Y_\bullet}(\tilde{L}^{(k)}) \subset \Delta_{Y_\bullet}(\tilde{L}^{(k-1)}) \subset \ldots \subset \Delta_{Y_\bullet}(L) \subset \mathbb{R}^n,$$

is called the *iterative Newton-Okounkov body dissection* associated to π_i and $\tilde{\mathcal{L}}^{(i)}$ for all $i = 1, \ldots, k$ (and the flag $Y_\bullet$). We define

$$\Delta_i \coloneqq \Delta_{Y_\bullet^{(i-1)}}(\tilde{X}_{i-1}, \tilde{L}^{(i-1)}) - \Delta_{Y_\bullet^{(i)}}(\tilde{X}_i, \tilde{L}^{(i)})$$

the *i-th piece of the iterative dissected form* of the Newton-Okounkov body. In this setting we propose the following conjecture:

Conjecture B. *Let X be a projective complex manifold of dimension n with ample line bundle L. Fix an admissible flag $Y_\bullet \coloneqq X = Y_0 \supsetneq, \ldots, \supsetneq Y_n = \{pt\}$ of X and let $\Delta_{Y_\bullet}$ be the associated Newton-Okounkov body. Fix an ordered set of points $P_1, \ldots, P_k$ that are smooth points of the flag and not contained in Y_1 and let Δ_i be the i-th piece of the iterated dissected form of the Newton-Okounkov body. Assume each Δ_i is convex, then for any $\delta > 0$ there exists a Kähler packing of X by balls $B_1, \ldots, B_k$ centred respectively at points*

$P_1, \ldots, P_k$ and moment maps $\mu_i \colon B_i \to \mathbb{R}^n$ such that $\mu_i(B_i)$ is a convex subset of Δ_i with $vol(\Delta_i - \mu_i(B_i)) < \delta$ for each $i = 1, \ldots, k$.

One of the implications of Conjecture B is that we recover the iterative dissection form of the Newton-Okounkov body directly from Kähler packings. As in general this form is difficult to compute our method may help determine multipoint Seshadri constants directly from the form of Newton-Okounkov bodies. At present we are not aware of any analogue of Küronya and Lozovanu's method for computing single point Seshadri constants in the multipoint setting. This will not be investigated further in the thesis but will be an area of future study.
Finally we explicitly construct Kähler packings and moment maps on blow ups of the complex projective plane in 1 and 2 points as well as explaining some difficulties that arise when blowing up at 3 or more points.

1.1 Organisation

The thesis is structured into 3 main chapters:

Chapter 2: This chapter contains a review of the background material needed in the construction and proofs of the main results. First we introduced multipoint Seshadri constants, state and prove some elementary results and discus the connection between multipoint Seshadri constants and Nagata's conjecture. In the following sections we define symplectic and Kähler packings and provide an example of a toric Kähler packing [Eck17] where the sections generating the Kähler form induce a moment map such that the image of the embedded balls corresponds to the cut off triangles in the moment polytope. This example motivates the results in Chapters 3 and 4. In Section 2.7 and 2.8 we introduce some machinery of symplectic geometry, namely symplectic moment maps and the construction of the symplectic blow up. In Section 2.10 we consider a construction of Kaveh [Kav19] which uses toric degenerations to construct an algebraic family whose central fiber is isomorphic to $(\mathbb{C}^*)^n$. This along with the gradient-Hamiltonian flow which is discussed in Section 2.11 allows us to construct Kähler packings and moment maps whose image under the embedding is the shadow of the embedded balls. Finally in Section 2.12 we recall facts regarding quasi-valuations. These are a weaker notation of a valuation where one of the key properties is relaxed and in this section we prove that under suitable conditions a quasi-valuation induces a filtration of vector spaces and vice versa. Later we will construct multi-ball Kähler packings and use this correspondence to construct Newton-Okounkov bodies and moment maps such that the image of the embedded balls under the moment map correspond to subdivisions of the Newton-Okounkov bodies.

Chapter 3: In this chapter we prove Theorem A. First we construct a degeneration family of iterative blow-ups and use this to construct Kähler forms on the irreducible components of the central fiber which extend to the whole of the family (see Theorems 3.5 and 3.6).

Chapter 4: Here we extend the toric example of Eckl and develop a general strategy for constructing Kähler packings equipped with moment maps whose images under the Kähler embedding corresponds to the cut off pieces in the iterative dissection form of the Newton-Okounkov body. We call these explicit Kähler packings and we prove that under suitable conditions we can construct these explicit Kähler packings. To this purpose we provide examples of $\mathbb{P}^2_{\mathbb{C}}$ blown up in 1 and 2-points and construct Kähler packings and moment maps that have the desired properties. We also consider $\mathbb{P}^2_{\mathbb{C}}$ blown up in 3-points however we find that our procedure fails and in order to be able apply our general strategy we first need to deform the central fiber further.

Chapter 5: In this section we give a very brief outlook of the project and how we would like to proceed.

Chapter 2

Preliminaries

2.1 Multipoint Seshadri constants.

Definition 2.1. Let X be an irreducible and reduced complex variety with structure sheaf $\mathcal{O}_X$. A *Cartier divisor* on X is a set $\{(U_i, f_i)\}$ where $\{U_i\}$ is a open cover of X and f_i is a non-zero rational function such that for an open U_j on X there exists some f_j such that $\frac{f_i}{f_j}$ is a nowhere vanishing regular function. The group $\text{Div}(X)$ denotes the group of all Cartier divisors on X.

Definition 2.2. A Cartier divisor $D = \sum_{i=1}^{n} a_i D_i$ on X where each D_i is an irreducible codimension 1 subvariety of X is called a $\mathbb{Q}$-*Cartier divisor* if all the coefficients a_i are elements of $\mathbb{Q}$ (a $\mathbb{R}$-*divisor* if coefficients in $\mathbb{R}$).

If X is projective then it is possible to associate every Cartier divisor with a line bundle (or invertible sheaf). To be precise: Following [Laz04] let X be a complex projective manifold with structure sheaf $\mathcal{O}_X$ and let $\text{Pic}(X)$ denote the group of isomorphism classes of line bundles on X. Each divisor D determines a line bundle $\mathcal{O}_X(D)$ and there is a canonical homomorphism $\text{Div}(X) \to \text{Pic}(X)$ that sends $D \mapsto \mathcal{O}_X(D)$. An important property of divisors (and invertible sheaves) is that of ampleness. For the classic definition of an ample invertible sheaf see [Har77, II.7]. We will give the following definition from [Laz04] as it is more conveniently formulated.

Definition 2.3 ([Laz04, Def 1.2.1])**.** Let X be a complete scheme and L a line bundle on X.

1. L is *very ample* if there exists a closed embedding $X \subseteq \mathbb{P}$ of X for some projective space $\mathbb{P} = \mathbb{P}^N$ with
$$L = \mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^N}(1)|_X.$$

2. L is *ample* if $L^{\otimes m}$ is very ample for some $m > 0$.

A Cartier divisor D is *ample* if the corresponding line bundle $\mathcal{O}_X(D)$ is ample. We call a Cartier divisor $\mathbb{Q}$-*ample* if some multiple of it is ample.

Definition 2.4. A Cartier divisor D is *nef (numerically effective)* if for all irreducible curves $C \in X$
$$D \cdot C \geq 0$$
where $\cdot$ denotes the intersection product.

Definition 2.5 ([Deb01, 1.29])**.** A nef Cartier divisor D on a proper scheme of dimension n is *big* if D^n is positive.

Definition 2.6 ([Sak83])**.** Let X be a smooth projective surface. A $\mathbb{Q}$-divisor D on X is *pseudo-effective* if $D \cdot H \geq 0$ for all ample divisors H on X.

The above definition can be generalised to an projective variety of dimension n by requiring that $D \cdot C \geq 0$ for all moving curves C on X.

Theorem 2.7 (Seshadri's criterion [Laz04, Thm 1.4.13])**.** *Let X be a projective variety and D a divisor on X. Then D is ample if and only if there exists a positive number ϵ such that for any point $p \in X$ and every irreducible curve $C \subseteq X$ we have:*
$$D \cdot C \geq \epsilon \operatorname{mult}_p C.$$

The maximal such ϵ, if it exists, is called the global Seshadri constant of D.

Definition 2.8 ([Dem92])**.** Let X be a smooth projective variety and L a nef line bundle on X. For a fixed point $p \in X$ the real number
$$\epsilon(X, L; p) = \inf \frac{L \cdot C}{\operatorname{mult}_p C}$$
is the *Seshadri constant* of L at the point p.

This definition can be generalised to work for multiple points and a nef line bundle. As this is the setting we will be most interested in we discuss these ideas in detail.

Let X be a smooth variety of dimension n, with $P_1, \dots, P_k$ distinct points of X. Let L be an ample divisor and $\tilde{X} \coloneqq Bl_{P_1,\dots,P_k}(X) \xrightarrow{\pi} X$ be the blow-up of X at the points $P_1, \dots, P_k$. Let $\pi^{-1}(P_i) = E_i$ denote the exceptional divisor corresponding to the point P_i for all $i = 1, \dots, k$ and for $\epsilon_i \in \mathbb{Q}_{>0}$ set

$$\tilde{L} \coloneqq \pi^* L - \sum_{i=1}^{k} \epsilon_i E_i \text{ for } \epsilon_i \in \mathbb{Q}_{>0}.$$

If L is a nef $\mathbb{Q}$-Cartier divisor then [Laz04] defines the multipoint Seshadri constant

$$\epsilon(X, L; P_1, \dots, P_k) = \max\{\epsilon \geq 0 \mid \pi^* L - \epsilon \sum_{i=1}^{k} E_i \text{ is nef }\}.$$

Since we require L ample to achieve the desired Kähler packing we show that there is an equivalent definition where we replace nef with ample, max with sup and ≥ 0 with > 0. The following lemmas prove that the two definitions are equivalent.

Lemma 2.9. *Assume L is ample on X. Then $\pi^* L - \epsilon' E$ is ample on $\tilde{X} = Bl_P(X) \xrightarrow{\pi} X$ if $0 < \epsilon' < \epsilon(X, L; P) = \epsilon_P$.*

Proof. By Seshadri's Criterion 2.7 if L is ample on X then there exists $\epsilon_L > 0$ such that for all points $Q \in X$ and all irreducible curves C containing Q we have that $\frac{L \cdot C}{\operatorname{mult}_Q C} > \epsilon_L$. Let $\tilde{Q} \in \tilde{X}$ such that $\pi(\tilde{Q}) = Q$ and let $\tilde{C} \subset \tilde{X}$ be an irreducible curve with $\tilde{Q} \in \tilde{C}$. Then there are the following cases. If $Q \notin E$ then

$$\frac{(\pi^* L - \epsilon' E) \cdot \tilde{C}}{\operatorname{mult}_{\tilde{Q}} \tilde{C}} = \frac{L \cdot C}{\operatorname{mult}_Q C} - \epsilon' \frac{\operatorname{mult}_P C}{\operatorname{mult}_Q C} \geq \frac{\epsilon_L}{2} \text{ if } \frac{\operatorname{mult}_P C}{\operatorname{mult}_Q C} < \frac{\epsilon_L}{2\epsilon'}.$$

Here $C = \pi \tilde{C}$ is an irreducible curve on X On the other hand if $\frac{\operatorname{mult}_P C}{\operatorname{mult}_Q C} \geq \frac{\epsilon_L}{2\epsilon'}$ then

$$\frac{(\pi^* L - \epsilon' E) \cdot \tilde{C}}{\operatorname{mult}_{\tilde{Q}} \tilde{C}} = \frac{(\pi^* L - \epsilon_P E) \cdot \tilde{C}}{\operatorname{mult}_{\tilde{Q}} \tilde{C}} + (\epsilon_P - \epsilon') \frac{E \cdot \tilde{C}}{\operatorname{mult}_{\tilde{Q}} \tilde{C}} \geq (\epsilon_P - \epsilon') \frac{\operatorname{mult}_P C}{\operatorname{mult}_Q C} \geq (\epsilon_P - \epsilon') \frac{\epsilon_L}{2\epsilon'}.$$

If $\tilde{Q} \in E$ with $\tilde{C} \not\subset E$ then $\operatorname{mult}_{\tilde{Q}} \tilde{C} \leq \operatorname{mult}_P \pi(\tilde{C})$ and

$$\frac{(\pi^*L - \epsilon' E)\cdot \tilde{C}}{\operatorname{mult}_{\tilde{Q}} \tilde{C}} \geq \frac{(\pi^*L - \epsilon' E)\cdot \tilde{C}}{\operatorname{mult}_P \pi(\tilde{C})} \geq \epsilon_P - \epsilon' > 0.$$

Finally if $\tilde{Q} \in E$ and $\tilde{C} \subset E$ then

$$\frac{(\pi^*L - \epsilon' E)\cdot \tilde{C}}{\operatorname{mult}_{\tilde{Q}} \tilde{C}} \geq -\frac{\epsilon' E \cdot \tilde{C}}{\operatorname{mult}_{\tilde{Q}} \tilde{C}} \geq \epsilon'.$$

The final inequality appears since $E \cdot \tilde{C} = \deg \tilde{C}$ on $E \cong \mathbb{P}^{n-1}$ and $\deg \tilde{C} \geq \operatorname{mult}_{\tilde{Q}} \tilde{C}$. The minus sign vanishes since $E \cdot \tilde{C} = -1$. Hence we have shown that if $0 < \epsilon' < \epsilon_P$ then $\pi^*L - \epsilon' E$ is $\mathbb{Q}$-ample by Seshadri's criterion.

□

Corollary 2.10. $\max\{\epsilon \geq 0 : \pi^*L - \epsilon E$ *is nef* $\} = \sup\{\epsilon > 0 : \pi^*L - \epsilon E$ *is* $\mathbb{Q}$*-ample*$\}$. □

Lemma 2.11. *There exists* $\epsilon_1, \ldots, \epsilon_k > 0$ *such that* $\tilde{L}$ *is* $\mathbb{Q}$*-ample for all* $i = 1, \ldots, k$.

Proof. By Seshadri's criterion and Lemma 2.9 we know there exist some $\epsilon_1 > 0$ such that $\pi_1^*L - \epsilon_1 E$ is $\mathbb{Q}$-ample on $\tilde{X}_1 = Bl_{P_1}(X)$. Choosing a second point $P_2 \in X$ Seshadri's criterion again ensures the existence of some $\epsilon_2 > 0$ such that $\pi_2^*L - \epsilon_1 E_1 - \epsilon_2 E_2$ is $\mathbb{Q}$-ample on $\tilde{X}_2 = Bl_{(P_1,P_2)}(X)$. Arguing iteratively for k points of X we obtain the claim of the lemma. □

From the statement of the above lemma it is not obvious that we can chose all the ϵ_i to be equal. This follows from the next lemma.

Lemma 2.12. *If* $\pi_k^*L - \sum_{i=1}^k \epsilon_i E_i$ *is* $\mathbb{Q}$*-ample then* $\pi^*L - \sum_{i=1}^k \epsilon_i' E_i$ *is ample if* $0 < \epsilon_i' \leq \epsilon_i$, *for all* $i = 1, \ldots, k$.

Proof. Lemma 2.9 proves the case when $k = 1$. If we show that $\pi^*L - \sum_{i=1}^k \epsilon_i E_i$ ample implies $\pi_{k-1}^*L - \sum_{i=1}^{k-1} \epsilon_i E_i$ is ample, where $\pi_{k-1} \colon \tilde{X}_{k-1} \to X$ denotes the blow up of X at the first $k-1$ points, then we can simply apply Lemma 2.9 on the last blow up. Let $C \subset \tilde{X}_{k-1}$ be an irreducible curve on $\tilde{X}_{k-1}$ and $\bar{C} \subset \tilde{X}_k$ be the strict transform of C on

$\tilde{X}_k$. Let Q be a point on C. Then

$$\begin{aligned}\frac{(\pi_{k-1}^*L - \sum_{i=1}^{k-1}\epsilon_i E_i)\cdot C}{\operatorname{mult}_Q C} &= \frac{(\pi_k^*L - \sum_{i=1}^{k}\epsilon_i E_i)\cdot \tilde{C}}{\operatorname{mult}_Q C} + \frac{\epsilon_k E_k \cdot \bar{C}}{\operatorname{mult}_Q C}\\ &= \frac{(\pi_k^*L - \sum_{i=1}^{k}\epsilon_i E_i)\cdot \tilde{C}}{\operatorname{mult}_Q C} + \epsilon_k \cdot \frac{\operatorname{mult}_{P_k} C}{\operatorname{mult}_Q C}.\end{aligned}$$

If $Q \neq P_k$ we have $\operatorname{mult}_Q C = \operatorname{mult}_Q \tilde{C}$, so the sum above is $\geq \epsilon_{\pi^*L - \sum_{i=1}^{k}\epsilon_i E_i}$, the global Seshadri constant of $\pi_k^*L - \sum_{i=1}^{k}\epsilon_i E_i$. If $Q = P_k$, the sum above is $\geq \epsilon_k$.
Thus, Seshadri's Criterion implies the ampleness of $\pi_{k-1}^*L - \sum_{i=1}^{k-1}\epsilon_i E_i$.

□

Definition 2.13. For X and L defined as above the *multipoint Seshadri constant* associated to points $P_1, \ldots P_k$ of X is

$$\epsilon(X, L; P_1, \ldots, P_k) = \sup\{\epsilon \in \mathbb{Q}_{>0} \colon \pi^*L - \epsilon \sum_{i=1}^{k} E_i \text{ is } \mathbb{Q}\text{-ample}\}.$$

There is a generalisation of the above definition called the *d-dimensional k point Seshadri constant*

$$\epsilon_d = (X, L; P_1, \ldots, P_k) = \sqrt[d]{\inf_Z \frac{L^d \cdot Z}{\sum \operatorname{mult}_{P_i} Z}}$$

where Z runs over all positive d-dimensional cycles. In this thesis we will only consider the case when $d = 1$ and will write $\epsilon_1 = (X, L; P_1, \ldots, P_k) = \epsilon(X, L; P_1, \ldots, P_k)$. For more information on d-dimensional Seshadri constants see [Dem92].
The multipoint Seshadri constant associated to an ample divisor is always > 0 by Lemma 2.12.

2.2 Nagata's conjecture and multipoint Seshadri constants

Since being introduced, Seshadri constants have attracted substantial attention in the field of algebraic geometry and can be used to reformulate many classical ideas. A nice example of this is the famous conjecture of Nagata on plane algebraic curves, which can be stated as follows

Conjecture 2.14. *Let $P_1, \ldots, P_k$ be points of $\mathbb{P}^2$ in general position. Then for $k \geq 9$ the multipoint Seshadri constant*

$$\epsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1); P_1, \ldots, P_k) = \frac{1}{\sqrt{k}}.$$

Nagata's conjecture has received a great deal of interest, however except for a few partial results it is still an open problem. It is known to be false if $k < 9$ or if the points are in special position, however Nagata showed that if $k = n^2$ and the points are chosen in general position the above conjecture is true. Harbourne in [Har01] and Roé in [Roé03] compute lower bounds for Nagata's conjecture and for some values of m and k there are results that verify the conjecture.

There is a generalisation of Nagata's conjecture to linear systems of plane curves that arose from conjectures of Segre in 1961, Harbourne in 1986, Gimiglino in 1987 and Hirchowitz in 1988. There are many different formulations of the conjecture but we will follow the formulation given in [FHH$^+$20]. Fix integers $d \geq 1$ and $m_1, \ldots, m_k \geq 0$ and consider a linear system $\mathcal{L} = |dH - m_1E_1 - \ldots - m_kE_k|$ on a general blow up $X = Bl_{p_1,\ldots,p_k}(\mathbb{P}^2)$. The *expected dimension* of $\mathcal{L}$ is defined as

$$\mathrm{edim}\mathcal{L} = \max\left\{\binom{d+2}{2} - \sum_i \binom{m_i+1}{2} - 1, -1\right\}.$$

We call $\mathcal{L}$ *special* if $\dim\mathcal{L} > \mathrm{edim}\mathcal{L}$ and $\mathcal{L}$ *non-special* if $\dim\mathcal{L} = \mathrm{edim}\mathcal{L}$.

Conjecture 2.15 (SHGH)**.** *If $\mathcal{L}$ is special then every divisor in $\mathcal{L}$ is non-reduced. Conversely if there is a reduced curve in $\mathcal{L}$ then $\mathcal{L}$ is non-special.*

The SHGH conjecture implies Nagata's and in [CHMR13] the authors discuss the relationship between the two cases as well as other conjectures that prove to be equivalent. Unfortunately non of the equivalent formulations prove easier to solve that the original statement. One interesting results in [CHMR13] is that to prove Nagata's conjecture for $n \geq 10$ it is enough to prove for $n \leq 90$. The SGHG conjecture is known to hold when $k = 4n$ for some integer n [Eva99] and for m less than $\frac{\sqrt{k}}{2}$ [Eva98]. Also in [CM97] and [CM00] the authors determine exact values for the degree d for $m \leq 12$ and $k \geq 10$ by considering linear systems of plane curves on $\mathbb{P}^2$.

One of the implications of the Nagata's conjecture is that the value of the Seshadri constant achieved is really the maximal possible value in the sense of the Nakai-Moishezon criterion (see [Har77, Thm5.1]).

Example 2.16. Let $X = \mathbb{P}^2$, $L = \mathcal{O}_{\mathbb{P}^2}(1)$ and fix a point $P \in X$. If $\pi\colon \tilde{X} = Bl_P(X) \to X$ denotes the blow up of X at P and $E = \pi^{-1}(P)$ the exceptional divisor corresponding to P, then by definition to compute $\epsilon(X, L; P)$ we need to find the supremum of all $\epsilon > 0$ such that $\pi^*L - \epsilon E$ is ample. To do this we use the Nakai-Moishezon criterion for surfaces [Har77, Theorem 1.10] which says that $\pi^*L - \epsilon E$ is ample if and only if the following properties are satisfied.

1. $(\pi^*L - \epsilon E)^2 > 0$.

2. $(\pi^*L - \epsilon E) \cdot C > 0$ for all curves $C \subset X$.

Checking 1) we have $(\pi^*L - \epsilon E)^2 > 0 \iff \epsilon < 1$.
To check property 2) we have two cases. Let $\tilde{C} \subset \tilde{X}$ be an irreducible curve then

- If $\pi(\tilde{C}) = pt$ we have that $\tilde{C} = E$ and $(\pi^*L - \epsilon E) \cdot E = \epsilon > 0 \implies \epsilon > 0$.

- If $\pi(\tilde{C}) = C$ an irreducible curve in X, then $(\pi^*L - \epsilon E) \cdot C = \deg C - \epsilon \operatorname{mult}_P C$. Furthermore since on $\mathbb{P}^2$ $\deg C \geq \operatorname{mult}_P C$ we have that $(\pi^*L - \epsilon E) \cdot C > 0 \implies \epsilon < 1$.

We conclude that any $0 < \epsilon < 1$ will suffice, hence

$$\epsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1); P) = 1.$$

The result obtained in the above example agrees with Nagata's conjecture when extended to blow ups of just one point. Similarly we can calculate $\epsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1); P_1, \ldots, P_4) = \frac{1}{2}$ and $\epsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1); P_1, \ldots, P_9) = \frac{1}{3}$. The remaining values of $\epsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1); P_1, \ldots, P_k)$ for $k < 9$ can also be computed because the cone of curves for $\mathbb{P}^2$ blown up in at most 9 points is well understood. On $\mathbb{P}^2$ for $1 \leq k \leq 9$ values of $\epsilon_k := \epsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1); P_1, \ldots, P_k)$ are:

k	1	2	3	4	5	6	7	8	9
ϵ_k	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{3}{8}$	$\frac{6}{17}$	$\frac{1}{3}$

See [Eck14, Cor.2.8] for details of the calculations. We see from the table that only ϵ_1, ϵ_4 and ϵ_9 obtain the maximum value $\frac{1}{\sqrt{k}}$.

2.3 Seshadri constants and symplectic packing constructions

Definition 2.17. Let V be a finite dimensional real vector space and let $\omega\colon V \times V \to \mathbb{R}$ be a bilinear form. Then:

1. ω is *skew symmetric* if for all $u, v \in V$ we have $\omega(u,v) = -\omega(v,u)$.
2. ω is *non-degenerate* if for every $v \in V$, $\omega(v,w) = 0$ for all $w \in V \implies v = 0$.

A *symplectic vector space* is a pair (V, ω) where V is a finite even dimensional real vector space and ω is a non-degenerate skew symmetric bilinear form.

Remark 2.18. The condition for a bilinear form to be symplectic forces the vector space in the above definition to be of an even real dimension which we denote by $2n$. Often it is more convenient to view this as a complex vector space of dimension n so for the remainder of the report we will simply say a symplectic vector space of dimension n, referring to its complex dimension.

Definition 2.19. Let V be a finite dimensional vector space, a *complex structure* on V is an automorphism $J\colon V \to V$ such that $J^2 = -\mathbb{1}$.

When such a vector space has a complex structure J it becomes a complex vector space and J corresponds to multiplication by i.

Example 2.20. Let $M = \mathbb{R}^{2n}$, $\omega_{std} = \Sigma_{j=1}^{n} dx_j \wedge dy_j$ and fix a basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of $\mathbb{R}^{2n}$ such that $f_1 = e_{n+1}, \ldots, f_n = e_{2n}$. There is a natural identification with the basis $dx_1, \ldots, dx_n, dy_1, \ldots, dy_n$ (which is in fact the basis of the dual space to the tangent space to $\mathbb{R}^{2n}$). Identifying dx_1 with e_1 and dy_1 with f_1, we get

$$\omega_{std}(dx_1, dy_1) = dx_1 \wedge dy_1 = -dy_1 \wedge dx_1.$$

Hence ω_{std} is a skew symmetric bilinear form such that $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$ and $\omega(e_i, f_j) = \delta_{i,j}$. Therefore $\mathbb{R}^{2n}$ with Euclidean form ω_{std} is a symplectic vector space. Moreover, the automorphism

$$J_0 = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$$

gives a natural complex structure on $\mathbb{R}^{2n}$.

Definition 2.21. Let M be a smooth $\mathbb{C}^\infty$ manifold and $\omega \in \Omega^2(M)$ a closed $2-$form. We call ω *non-degenerate* if for each point $p \in M$ the bilinear form ω_p on the tangent space $T_p(M)$ is non-degenerate. Then $(T_p(M), \omega_P)$ is a symplectic vector space. A symplectic structure on M is a non-degenerate closed 2-form ω, and we call the pair (M, ω) a *symplectic manifold*.

Definition 2.22. If (M, ω_M) and (N, ω_N) are symplectic manifolds then a diffeomorphism $\phi\colon (M, \omega_M) \to (N, \omega_N)$ satisfying $\phi^*\omega_N = \omega_M$ is called a *symplectomorphism*.

We also have the following important argument known as Moser's argument (see [MS98, 3.2]). Let M be a symplectic manifold, ω_t a family of symplectic forms on M, and σ_t a smooth family of 1-forms on M satisfying the property

$$\frac{d}{dt}\omega_t = d\sigma_t \text{ for all } t \in [0,1].$$

Moser's argument constructs a family of diffeomorphisms $\phi_t \in \mathrm{Diff}(M)$ such that for all $t \in [0,1]$

$$\phi_t^*\omega_t = \omega_0.$$

Thus, (M, ω_0) is symplectomorphic to (M, ω_1).

Definition 2.23 (Symplectic packing)**.** Let (M, ω) be a n dimensional symplectic manifold and $B_0(\lambda) \subset \mathbb{C}^n$ denote a ball of radius λ centred at the origin. A symplectic embedding

$$\phi = \coprod_{q=1}^{k} \phi_q \colon \coprod_{q=1}^{k} (B_0(\lambda), \omega_{std}) \hookrightarrow (M, \omega)$$

such that $\phi_q^*\omega = \omega_{std}$ and $\phi_q(0) = P_q$ is called a *symplectic packing* of k balls of radius λ.

The following well known theorem of Darboux tells us that if the radius of the balls in the above definition is chosen sufficiently small then such a symplectic packing is guaranteed to exist. Darboux's theorem is a direct consequence of Moser's argument.

Theorem 2.24 (Darboux's Theorem [MS98, Theorem.3.15])**.** *If M is a symplectic manifold then any symplectic form ω on M is locally diffeomorphic to ω_{std} on $\mathbb{R}^{2n}$.*

The Darboux theorem tells us that in the symplectic category, locally any symplectic manifold of dimension n is symplectomorphic to $\mathbb{R}^{2n}$ equipped with the standard Euclidean form.

Another natural question to ask is since a symplectic packing always exists if a small enough radius is chosen, how much can we increase the radius of the embedded ball before we obtain an obstruction to the packing? On, $\mathbb{P}^2_{\mathbb{C}}$, this question was considered by Biran in [Bir01] where it was shown that the maximum size of the radius is determined by the Seshadri constant as conjectured in Nagata's conjecture. In more details:

Definition 2.25. Let X be a projective complex manifold and ω a symplectic form on X.

$$\gamma_s = \sup\{r \in \mathbb{R}_{>0} \colon \exists \text{ a symplectic packing of balls of radius } r\}$$

is called the *symplectic packing constant*.

Theorem 2.26 ([Bir01])**.** *Set $X = \mathbb{CP}^2$ with homogeneous coordinates $[X : Y : Z]$ and $\omega_{FS} = \frac{i}{2\pi}\partial\bar{\partial}\log(X\bar{X} + Y\bar{Y} + Z\bar{Z})$ the standard Fubini-Study form on $\mathbb{CP}^2$. Then for $P_1, \ldots, P_k$ distinct points of X and $k \geq 10$*

$$\gamma_s(\mathbb{CP}^2, \omega_{FS}; P_1, \ldots, P_k) = \frac{1}{\sqrt{k}}.$$

The proof of this theorem showed that there exists a connection between a symplectic packing of radius r and the existence of a symplectic form on the blow up in the first Chern class of $\pi^*\mathcal{O}_{\mathbb{P}^2} - r\sum_{i=1}^k E_i$. Moreover we will see later that in the Kähler setting, this is equivalent to the existence of a Kähler form in first Chern class of $\pi^*\mathcal{O}_{\mathbb{P}^2} - r\sum_{i=1}^k E_i$, which holds if and only if the line bundle is ample on the blow up.
Biran's proof of Theorem 2.26 used purely symplectic methods such as the existence of pseudo-holomorphic curves that do not translate into the language of algebraic geometry and for this reason Theorem 2.26 is often referred to as the symplectic analogue of Nagata's conjecture. However, this result of Biran's motivated the study of more restrictive packing problems, particularly Kähler packing problems to calculate Seshadri constants.

2.4 Kähler packings

Let M be a $2n$ dimensional real manifold and $\omega \in \Omega^2(M)$ a non-degenerate 2-form on M. An *almost complex structure* on M is a complex structure J on the tangent bundle TM and J is *integrable* if there exists an atlas $\{x_i, U_{x_i}\}$ such that J can be represented by the matrix J_0 (see Example 2.20) in local coordinates.

J is *compatible with* ω if

$$\langle u, v\rangle = \omega(u, Jv)$$

is a Kähler metric on M.

Definition 2.27. A *Kähler manifold* is a symplectic manifold (M, ω) equipped with an integrable almost complex structure J which is compatible with the symplectic form ω.

The standard example of a Kähler manifold is $(\mathbb{R}^{2n}, J_0, \omega_{std})$ where J_0 is the standard complex structure and $\omega_{std} = \Sigma_{j=1}^{n} dx_j \wedge dy_j$ is the standard Euclidean form on $\mathbb{R}^{2n}$.

Example 2.28 ($\mathbb{C}^n$ as a Kähler manifold)**.** Let $z_1, \ldots, z_n$ be complex coordinates of $\mathbb{C}^n$ and $\bar{z}_1, \ldots, \bar{z}_n$ the complex conjugates. Let $dz_j = dx_j + idy_j$ and $d\bar{z}_j = dx_j - dy_j$ denote complex valued 1-forms on $\mathbb{C}^n$. The standard Euclidean form on $\mathbb{C}^n$ can be written as

$$\omega_{std} = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j.$$

Example 2.29 ($\mathbb{CP}^n$ as a Kähler manifold [MS98, Example 4.21])**.** $\mathbb{CP}^n$ can be considered as the space of complex lines in $\mathbb{C}^n$ such that points in $\mathbb{C}^n$ are given by the equivalence class on a non-zero $(n+1)$ dimensional complex vector $[z_0, \ldots, z_n]$ with $[z_0, \ldots, z_n] = [\lambda z_0, \ldots, \lambda z_n]$ for $\lambda \neq 0$. In the usual way $\mathbb{CP}^n$ has charts U_i where $z_i \neq 0$ and parametrisations

$$\phi_i \colon U_i \to \mathbb{C}^n \text{ given by } [z_0, \ldots, z_n] \mapsto \left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i} \ldots \frac{z_n}{z_i}\right).$$

The transition maps $\phi_j \cdot \phi_i^{-1}$ are holomorphic. Denote the point $[z_0, \ldots, z_n] = [z] \in \mathbb{CP}^n$ then the tangent space of $\mathbb{CP}^n$ is

$$T_{[z]}\mathbb{CP}^n = \mathbb{C}^{n+1}/\mathbb{C}_z.$$

There is a natural complex structure on this given by $u \mapsto iu$ for $u \in T_{[x]}\mathbb{CP}^n$ and $i = \sqrt{-1}$ along with a natural atlas given by the charts U_i and transition maps. Hence J is an integrable complex structure and we have a 2- form

$$\omega_{FS} = \frac{i}{2}\partial\bar{\partial}\log\sum_{j=0}^{n} z_j\bar{z}_j.$$

This is the Fubini-Study form on $\mathbb{CP}^n$.

Definition 2.30. Let (X, ω) be an n dimensional Kähler manifold with Kähler form ω and fix points $P_1, \dots, P_k \in X$. A holomorphic embedding

$$\phi = \coprod_{q=1}^{k} \phi_q \colon \coprod_{q=1}^{k} B_0(r_q) \hookrightarrow X$$

is called a *Kähler embedding of k disjoint complex flat balls in $\mathbb{C}^n$ centered in* 0*, of radius* r_q, if there exists a Kähler form ω' such that $[\omega'] = [\omega] \in H^{1,1}(X, \mathbb{R})$ and $\phi_q^*(\omega') = \omega_{std}$ is the standard Kähler form on $\mathbb{C}^n$ restricted to $B_0(r_q)$. Let

$$\gamma_k(X, \omega; P_1, \dots, P_k) = \sup\{r > 0 \colon \exists \text{ a Kähler packing as above}\}.$$

We call γ_k the *$k-$ball Kähler packing constant.*

The main difference between symplectic packings and Kähler packings is that there is no notion of curvature for a symplectic packing so the pulled back symplectic form is the same as the symplectic form that we started with. Since in the Kähler setting curvature is an invariant under open holomorphic embeddings we cannot define the packing condition $\phi_q^*(\omega) = \omega_{std}$ using the original Kähler form ω on X as long as that form is not flat enough around q. However we show later that under suitable conditions it is possible to find a Kähler form flat enough around $\phi_q(0)$ in the cohomology class of ω as requested in the definition.
The following proposition shows that there is a direct correspondence between the symplectic packing constant and the Kähler packing constant.

Proposition 2.31. *Let X be a projective complex manifold, L an ample line bundle on X and ω a Kähler form such that $[\omega] = c_1(L)$. Then for points $P_1, \dots . P_k \in X$ the k-ball Kähler packing constant is always less than or equal to the k-ball symplectic packing constant i.e.*

$$\gamma_k(X, [\omega]; P_1, \dots, P_k) \leq \gamma_s(X, \omega; P_1, \dots, P_k).$$

Proof. The proof of this theorem is a direct consequence of Theorem 2.34 and will be proved later in Section 2.5. □

Remark 2.32. If the points $P_1, \dots, P_k$ are chosen to be in special position or we take fewer than 9 points then $\gamma_k < \gamma_s$. If however we take more than 9 points in general position then the truth of Nagata's conjecture and Biran's theorem would tell us that $\gamma_k = \gamma_s$.

Remark 2.33. There exists a symplectomorphism between flat Kähler balls and Fubini-Study balls given by

$$\phi\colon (B_0^{2n}(1), \omega_{std}) \hookrightarrow (\mathbb{C}^n, \omega_{FS}), \text{ such that } (z_1, \dots, z_n) \mapsto \frac{1}{(1 - \sum_{i=1}^n |z_i|^2)^{\frac{1}{2}}} \cdot (z_1, \dots, z_n).$$

This symplectomorphism along with its inverse allows us to glue Fubini-Study balls into flat Kähler balls and vice versa. The above remark is a generalisation of the 2-dimensional case (see [MS98, Ex 7.14]).

2.5 Connections between Kähler and symplectic packings

As we have seen by now there is a distinct difference between Kähler and symplectic packings. Namely, that Kähler packings only depend on the cohomology class of a Kähler form whereas symplectic packings depend on an actual symplectic form. However there is a direct connection between these two types of packing problem which is shown by the following Theorem.

Theorem 2.34. *On a complex projective manifold X a Kähler packing with respect to a Kähler form in the first Chern class of a (very) ample line bundle is a symplectic packing with respect to a Kähler form induced by a basis of sections of an ample line bundle.*

Proof. To prove the claim we show that a Kähler form ω constructed as in the proof of Theorem 3.6 can be deformed into a Kähler form allowing a packing such that Moser's argument can be used to construct a family of symplectic forms satisfying the required properties. Let $\{\tau_i\}$ and $\{\sigma_j\}$ denote bases of $H^0(X, L)$. Set

$$\omega_0 = \frac{i}{2}\pi\partial\bar{\partial}\log(\sum |\tau_i|^2) \text{ and } \omega_1 = \frac{i}{2}\pi\partial\bar{\partial}\log(\sum |\sigma_j|^2).$$

Then

$$\omega_1 - \omega_0 = \frac{i}{2}\pi\partial\bar{\partial}\log\left(\frac{\sum |\sigma_j|^2}{\sum |\tau_i|^2}\right).$$

Recall that $d = \partial + \bar{\partial}$ and $\partial^2 = \bar{\partial}^2 = 0$, hence

$$\frac{i}{2}\pi\partial\bar{\partial}\log\left(\frac{\sum |\sigma_j|^2}{\sum |\tau_i|^2}\right) = \frac{i}{2}\pi d\left(\bar{\partial}\log\left(\frac{\sum |\sigma_j|^2}{\sum |\tau_i|^2}\right)\right) = \frac{i}{2}\pi d\sigma$$

for the 1-form $\sigma = \bar{\partial} \log \left(\frac{\sum |\sigma_j|^2}{\sum |\tau_i|^2} \right)$.
Hence we can apply Moser's argument on $\omega_0 + t(\omega_1 - \omega_0)$, so (X, ω_0) and (X, ω_1) are symplectomorphic as symplectic manifolds. The Kähler form $\tilde{\omega}$ constructed in Theorem 3.6 is now obtained from a Kähler form induced by sections by (iteratively) gluing in Fubini-Study balls using a partition of unity. That allows us again to apply Moser's argument because the difference in the original and the glued in form is of the same form as the difference $\omega_0 - \omega_1$ above.
Consequently, (X, ω_0) and $(X, \tilde{\omega})$ are symplectomorphic as symplectic manifolds and the embedding of the Kähler balls into $(X, \tilde{\omega})$ can be interpreted as the embedding of symplectic balls in (X, ω_0).

□

2.6 Kähler packings for toric varieties

Toric varieties are a well studied area of mathematics and due to their rigid structure and combinatorial nature they provide a rich testing ground for many theories in algebraic geometry. In this section we first recall the basics of toric geometry and introduce notation. We then provide an example of a Kähler packing on a toric variety which motivated the result achieved in Chapter 3. The main reference for this section is [Ful93].
Let $T = (\mathbb{C}^*)^n \cong \mathbb{C}^* \times \ldots \times \mathbb{C}^*$ (where $\mathbb{C}^*$ is the multiplicative subset of $\mathbb{C}$ without the origin) denote the algebraic torus of dimension n.

Definition 2.35. A *toric variety of dimension n* is a complex normal variety X of dimension n that contains the torus $T \cong (\mathbb{C}^*)^n$ as a dense orbit under a group action $T \times X \to X$.

Let N be a lattice isomorphic to $\mathbb{Z}^n$ for some value of n and consider the real vector space $N_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} N$.

Definition 2.36. A *strongly convex rational polyhedral cone* σ is a cone with an apex at the origin which is generated by a finite number of vectors in the lattice N and contains no line passing through the origin. A *fan* Δ is a collection of convex polyhedral rational cones σ.
The *dual lattice* $N^\vee = M = \mathrm{Hom}(N, \mathbb{Z})$, where N is the lattice defined above and *dual cones* $(\sigma_i)^\vee$ are the set of all vectors which lie in the vector space $M_{\mathbb{R}}$ which are non negative on σ.

The dual cones $\sigma^\vee$ determine a commutative semigroup

$$S_\sigma = \sigma^\vee \cap M = \{u \in M_\mathbb{R} :< u, v >\geq 0 \text{ for all } v \in \sigma\}.$$

This semigroup is finitely generated and as such its corresponding group algebra $\mathbb{C}[S_\sigma]$ is a finitely generated $\mathbb{C}$-algebra. Every toric variety $X(\Delta)$ is covered by *affine toric varieties*

$$U_\sigma := \mathbb{C}[\sigma^\vee \cap M].$$

Following [Ful93, §1] let Δ be a fan constructed by glueing rational strongly convex polyhedral cones, then the toric variety $X(\Delta)$ associated to the fan Δ is constructed by gluing the collection of affine toric varieties U_σ for each σ in Δ. Two cones σ and $\sigma' \in \Delta$ are glued such that the intersection $\sigma \cap \sigma'$ is a face of both σ and σ' and $U_{\sigma\cap\sigma'}$ is a principle open subvariety of both U_σ and $U_{\sigma'}$. We construct $X(\Delta)$ by gluing affine toric varieties U_σ under this identification. The cones of maximal dimension in Δ correspond directly to the T-stable fixed points under the action of the torus. The cones of dimension $n-1$ correspond to T-stable divisors on the variety $X(\Delta)$.

Let $X(\Delta)$ be a toric variety associated to the fan Δ constructed as a collection of strongly convex rational polyhedral cones σ_i. The edges (rays) of the cones correspond to the irreducible divisors which are T-stable. Denote these rays τ_i, and for each τ_i fix a point v_i to be the the first point of the lattice we meet when moving along a given ray. The *T-stable divisors* D_i are the orbit closures given by

$$D_i = V(v_i)$$

where $V(v_i)$ denotes the vanishing locus of the point v_i. A T-stable Cartier divisor D on $X(\Delta)$ is defined by elements $u_D(\sigma) \in M$ for each $\sigma \in \Delta$ of maximal dimension such that $u_D(\sigma) - u_{D'}(\sigma') \in (\sigma \cap \sigma')^\perp$.

Definition 2.37. Let $X(\Delta)$ be a toric variety associated to a fan Δ and let $D = \sum a_i D_i$ be an associated T-stable Cartier divisor. Then the *polytope P_D associated to D* is given by

$$P_D = \{u \in M : \langle u, v_i \rangle \geq -a_i\}.$$

A T-stable Cartier divisor D defined as above is *ample* if and only if the elements $u_D(\sigma) \in M$ describing D are exactly vertices of the polytope P_D. Using these T-stable divisors we can construct a moment polytope associated to the toric variety $X(\Delta)$.

Proposition 2.38 ([Eck17]). *Let $X(\Delta)$ be a nonsingular toric variety and $\sigma \in \Delta$ a cone of maximal dimension corresponding to a T-stable point x_σ. Then the blow up of $X(\Delta)$ in x_σ is given by the morphism $X'(\Delta) \to X(\Delta)$ where Δ' is a fan constructed from Δ by subdividing σ into n cones σ_i generated by*

$$v_1, \ldots, v_{i-1}, v_1 + \ldots + v_n, v_{i+1}, \ldots, v_n$$

where $v_1, \ldots, v_n \in N$ are spanning σ and are also a $\mathbb{Z}$ basis of the lattice N. The exceptional divisor on $X(\Delta')$ is T-stable and corresponds to the ray generated by $v_1 + \ldots + v_n$.

We also have the following proposition taken from [Eck17] but based on material contained in [BDRH$^+$09, §4].

Proposition 2.39 ([Eck17, Prop 2.2]). *Let $X(\Delta)$ be an n-dimensional non-singular toric variety, σ a cone of maximal dimension n with corresponding fixed point x_σ and $\pi\colon X(\Delta') \to X(\Delta)$ the blow up of $X(\Delta)$ with exceptional divisor E_σ as defined in Prop 2.38. Let D be an ample T-stable Cartier divisor on $X(\Delta)$ with associated polyhedron P_D.*

1. *Let $v_1, \ldots, v_n \in N$ be the generators of the edges of σ and $w_1, \ldots, w_n \in M$ be generators of the edges of $\sigma^\vee$. If σ is a cone of maximal dimension n intersecting the facet spanned by $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$ then the vertices $u_D(\sigma)$ and $u_D(\sigma')$ of P_D differ by a multiple of $\epsilon_i w_i$ of $w_i, \epsilon_i > 0$.*

2. *$D_\epsilon := \pi^* D - \epsilon E_\sigma$ is an ample divisor if and only if $\epsilon < \min_{i=1,\ldots,n} \epsilon_i$ and is associated polyhedral P_{D_ϵ} is obtained from P_D by taking away the simplex with vertex $u_D(\sigma)$ and and edges ϵw_i starting in $u_D(\sigma)$.*

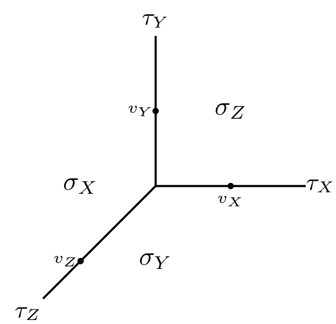
Remark 2.40. The above propositions show that blow up of a toric variety at a T-stable point is again a toric variety and the moment polytope of the blown up variety is contained within the moment polytope of the original variety.

We will not prove the above propositions (for details see [Eck17]) but we will give a simple example satisfying the claim.

Example 2.41 ([Eck17, Exa.2.4]). Let $X = \mathbb{P}^2_{\mathbb{C}}$ and consider the action of the torus $T \cong (\mathbb{C}^*)^2$ given by

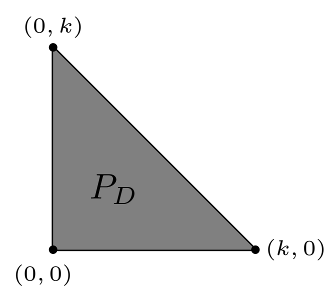
$$(t_1, t_2) \cdot [X : Y : Z] = [t_1 X : t_2 Y : Z].$$

The three cones of maximal dimension σ_Z, σ_Y and σ_X in the fan Δ describing $\mathbb{P}^2_{\mathbb{C}} = X(\Delta)$ are separated by the rays τ_X, τ_Z and τ_Y and are spanned by the points $v_X = (1,0)$, $v_Z = (-1,-1)$ and $v_Y = (0,1)$ in $N \cong \mathbb{Z}^2$, respectively. The cones σ_Z, σ_Y and σ_X correspond to the three T-stable points $x_Z = [0:0:1]$, $x_Y = [0:1:0]$ and $x_X = [1:0:0]$ of $\mathbb{P}^2_{\mathbb{C}}$, respectively.

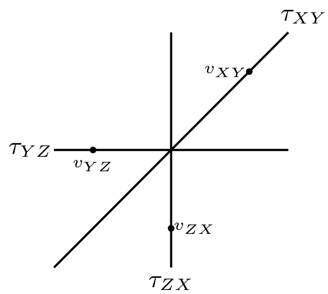


The rays τ_X, τ_Z and τ_Y correspond to the T_N-stable divisors $D_X = \{X=0\}$, $D_Z = \{Z = 0\}$ and $D_Y = \{Y=0\}$ respectively and are lines in $\mathbb{P}^2_{\mathbb{C}}$, hence linearly equivalent divisors. For $k \in \mathbb{Z}_{>0}$ the moment polytope P_D of the divisor $D := kD_Z$ is

$$P_D = \{(u_1,u_2) \in \mathbb{R}^2 : u_1, u_2 \geq 0, u_1 + u_2 \leq k\}.$$

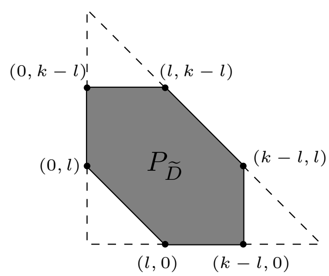


Blowing up the T-stable points x_Z, x_Y and x_X yields a toric variety $\widetilde{X} = X(\widetilde{\Delta})$ where the fan $\widetilde{\Delta}$ is obtained from the fan Δ by splitting up the cones of maximal dimension σ_Z, σ_Y and σ_X with rays spanned by $v_{XY} = (1,1)$, $v_{ZX} = (0,-1)$ and $v_{YZ} = (-1,0)$, respectively.



The rays spanned by v_{XY}, v_{ZX} and v_{YZ} correspond to the exceptional divisors E_Z, E_Y and E_X, of the blow up map $\pi : \widetilde{X} \to \mathbb{P}^2_{\mathbb{C}}$, and the rays τ_X, τ_Z and τ_Y correspond to the strict transforms of D_X, D_Z and D_Y, respectively.
Hence, for $\widetilde{D} = k\pi^* D_Z - lE_X - lE_Z - lE_Y$ the moment polytope $P_{\widetilde{D}}$ is

$$P_{\widetilde{D}} = \{(u_1, u_2) \in \mathbb{R}^2 : 0 \leq u_1 \leq k - l, 0 \leq u_2 \leq k - l, l \leq u_1 + u_2 \leq k\}.$$



The divisor $\tilde{D}$ can be associated with the line bundle $L = \mathcal{O}_{\tilde{D}}$ and by considering the T-stable global sections of high enough multiples of L it is possible to construct Kähler packings on $X(\tilde{\Delta})$.

We now present an important example which is discussed in [Eck17] and illustrates this idea further.

Example 2.42 ([Eck17, Thm.2.5])**.** Let $X = \mathbb{P}^2$ be provided with the standard $(\mathbb{C}^*)^2$-action as Example 2.41. Recall there exists a fan of X denoted Δ which is generated by three cones of maximal dimension σ_x, σ_y and σ_z. Each of these cones corresponds to an affine toric variety U_σ having defined affine coordinates. These coordinates correspond directly to the generators of the edges of $\sigma^\vee \cap M$ for each cone of maximal dimension.

On $\mathbb{P}^2$ there exists a T-invariant basis of global sections of $\mathcal{O}_{\mathbb{P}^2}(k)$ which can be described by monomials of the form $\{C_{a,b}x^a y^b \colon |a+b| \leq k\}$ in T-invariant coordinates around the T-stable point x_σ. These global sections induce a Kähler form on $\mathbb{P}^2$ which by a careful choice of the coefficients $C_{a,b}$ restrict to the Kähler form

$$\omega_\sigma|_{U_\sigma} = \frac{1}{k}\cdot\frac{i}{2\pi}\partial\bar{\partial}\log\Big((\delta^2+|x|^2+|y|^2)^l + \text{ terms of higher order } > l\Big)$$

on U_σ: If $a+b \leq l$ we choose the coefficients of $x^a y^b$ to be the square root of the coefficient of $|x|^{2a}|y|^{2b}$ in the expansion of $(\delta^2+x^2+y^2)^l$. For the remaining T-stable sections that do not correspond to the cones of maximum dimension we choose the coefficients to be 1.

Consider the holomorphic embedding

$$\phi_R : B_0(R) \hookrightarrow U_{\sigma_z} \subset X(\Delta) \text{ such that } 0 \mapsto (0,0) \text{ and } z \mapsto \delta \cdot z.$$

Restricting to U_σ and pulling back the induced Kähler form ω_σ along this embedding gives

$$\phi_R^*\omega_\sigma = \frac{1}{k}\cdot\frac{i}{2\pi}\partial\bar{\partial}\log\Big(\delta^{2l}(1+|x|^2+|y|^2)^l + \text{ terms of order } > l \text{ in } \delta^2\Big)$$

which expanded term wise equals

$$\frac{1}{k}\cdot\frac{i}{2\pi}\partial\bar{\partial}[\log\delta^{2l} + \log\left(1+|x|^2+|y|^2\right)^l + \ldots\ldots].$$

Finally taking the limit as delta tends to zero (and using the fact that $\partial\bar{\partial}\log\delta^{2l} = 0$) we obtain

$$\frac{1}{k}.l\frac{i}{\pi}\partial\bar{\partial}\log\left((1+|x|^2+|y|^2)\right) = \frac{l}{k}\cdot\omega_{FS}$$

where ω_{FS} is the Fubini-Study form on $\mathbb{P}^2_{\mathbb{C}}$. A gluing constructing as described in Chapter 3 shows that there exists a holomorphic embedding of Fubini-Study balls on $X(\Delta)$. Using the existence of the symplectomorphism between flat Kähler balls and Fubini-Study balls

$$\phi\colon (B_0(1), \omega_{std}) \to (\mathbb{C}^2, \omega_{FS})$$

we can rescale and we obtain an embedding of flat Kähler balls on to the toric surface $X(\Delta)$.

Furthermore the T-invariant sections constructed above along with coefficients induce a moment map which when restricted to the affine subset U_σ is

$$\mu_{|U_\sigma} = \frac{1}{k}\frac{1}{\Sigma_u |c_u|^2 |z|^{2u}} \cdot \Sigma_u |c_u|^2 |z|^{2u} \cdot u.$$

Let $\phi_R : B_0(R) \to U_\sigma \subset \tilde{X}(\Delta)$ such that $0 \mapsto (0,0)$ and $z \mapsto \delta \cdot z$. Consequently, as δ tends to zero the composition $\mu \circ \phi_R$ tends to

$$z \mapsto \frac{1}{k} \cdot \frac{1}{(1+|x|^2+|y|^2)^l} \cdot \Sigma_{|u|\leq l} \frac{|c_u|^2}{\delta^2} |z|^{2u} \cdot u.$$

This is a toric moment map generated by the global sections $\binom{l}{u-1+u_2}\binom{u_1+u_2}{u_1} z_1^{u_1} z_2^{u_2}$ with $0 \leq u_1 + u_2 \leq l$ and is hence the symplectic moment map with respect to $\frac{l}{k} \cdot \omega_{FS}$.

At the same time, if we blow up $\mathbb{P}^2$ in the T-stable point $x_{\sigma_z} = [0:0:1]$, the blown up variety is also a toric variety, with T-stable divisor $L_{k,l} = \pi^* \mathcal{O}_{\mathbb{P}^2}(k) - lE$, where $\pi \colon \tilde{X} = Bl_{x_{\sigma_z}}(\mathbb{P}^2) \to \mathbb{P}^2$ is the blow up map and E the exceptional divisor. The image of the toric moment map associated to $L_{k,l}$ is obtained from the image of the toric moment associated to $\mathcal{O}_{\mathbb{P}^2}(k)$ on $\mathbb{P}^2$ by cutting away the image of the moment map on the Kähler ball above.

Eckl in [Eck17] was able to generalise Example 2.41 somewhat and prove that for any projective complex surface equipped with an ample line bundle the multipoint Seshadri constant is equal the Kähler packing constant. Together with Example 2.42 this motivates the following question:

Question 2.43. *For a projective complex manifold of dimension n is the multipoint Seshadri constant equal to the multi-ball Kähler packing constant?*

The main result of this thesis answers Question 2.43 positively.

2.7 Symplectic moment maps

In this section we recall some definitions and facts from [MS98] regarding symplectic group actions and symplectic moment maps. The material in this section has been collected from [MS98] and [Sil08] and we refer the reader there for further details.

Let (M, ω) be a symplectic manifold and G a compact Lie group. A symplectic action of G on M is a group homomorphism $G \to \mathrm{Symp}(M, \omega)$ such that $g \mapsto \psi_g$, and $\psi_g \colon (M, \omega) \to$

(M, ω) is a symplectomorphism (i.e. $g^*\omega = g$) for all $g \in G$, satisfying:

$$\psi_{g \circ h} = \psi_g \circ \psi_h, \text{ and } \psi_1 = id.$$

Denote by $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{g}^*$ its dual, then for each $\xi \in \mathfrak{g}$ there exists an induced vector field X_ξ on M such that

$$X_\xi = \frac{d}{dt}|_{t=0} \exp(t\xi).$$

Here $\exp\colon \mathfrak{g} \to G$ denotes the usual exponential map.
We call the action of G weakly Hamiltonian if for each $\xi \in \mathfrak{g}$ there exists a corresponding Hamiltonian function $H_\xi\colon M \to \mathbb{R}$ such that

$$dH_\xi = \omega(X_\xi, \ . \ ).$$

Definition 2.44. We call the induced group action on (M, ω) *Hamiltonian* if the map

$$\mathfrak{g} \to C^\infty(M, \mathbb{R}) \ , \ \xi \mapsto H_\xi$$

is a Lie algebra homomorphism between $\mathfrak{g}$ and the Poisson Lie algebra on $C^\infty(M, \mathbb{R})$, given by $\{F, G\} = \omega(X_F, X_G)$.

Let$< -, - >\colon \mathfrak{g} \times \mathfrak{g}^* \to \mathbb{R}$ denote pairing of $\mathfrak{g}$ and $\mathfrak{g}^*$. Then a moment map is defined as follows:

Definition 2.45. Assume that the action of G on (M, ω) is Hamiltonian. A map

$$\mu\colon M \to \mathfrak{g}^*$$

such that

$$H_\xi(p) = \langle \mu(p), \xi \rangle$$

is the Lie algebra homomorphism $\xi \to H_\xi$ is called a *moment map* of the action.

Example 2.46 ([MS98, Ex.5.20])**.** Let $G = U(n)$ and consider the action on $(\mathbb{C}^n, \omega_{std})$. There exists a Lie algebra homomorphism

$$\mathfrak{u}(n) \to C^\infty(\mathbb{R}^n) \text{ such that } \eta \mapsto H_\eta$$

given by the moment map

$$\mu\colon \mathbb{C}^n \to \mathfrak{u}(n)\ ,\ z \mapsto \frac{i}{2}zz^*.$$

Here z^* denotes the conjugate transpose of z.

Example 2.47. The natural $U(n+1)$-action on $\mathbb{CP}^n$ is Hamiltonian, with corresponding moment map

$$\mu\colon \mathbb{CP}^n \to \mathfrak{u}(n+1) \text{ given by } \mu(z) = \frac{i}{2}\frac{zz^*}{|z|^2}.$$

From the subset $(\mathbb{C}^*)^n \subset U(n+1)$ we obtain a moment map

$$\mu\colon \mathbb{CP}^n \to \mathbb{R}^n$$

by composing the moment map for the $U(n+1)$-action with the Lie algebra homomorphism $\mathfrak{u}(n+1)^* \to \mathbb{R}^n$, whose image is the convex closed hull of the characters.

Example 2.48. Consider the action of $(\mathbb{C}^*)^n$ on $\mathbb{P}^N_{\mathbb{C}}$ given by

$$(t_1, \ldots, t_n) \cdot [Z_0 : \ldots : Z_N] = [Z_0 : t^{\beta_1} Z_1 : \ldots : t^{\beta_n} Z_n : \ldots : t^{\beta_N} Z_N].$$

Identify the Lie algebra $\mathfrak{g} = \mathbb{R}^n$ of $(\mathbb{C}^*)^n$ with its dual $\mathfrak{g}^* = \mathbb{R}^n$ by the standard inner product. Then there is a moment map associated to this group action,

$$\mu\colon \mathbb{P}^N_{\mathbb{C}} \to \mathbb{R}^n$$

$$[z_0 : \ldots : z_N] \mapsto \frac{\pi}{2}\frac{\sum_{i=1}^N |z_1|^2\beta_i}{\sum_{i=0}^N |z_i|^2}.$$

The image of μ is the convex hull of the β_i for $i = 0, \ldots, N$ and the $(n+1)$-fixed points under the action of $(\mathbb{C}^*)^n$ on $\mathbb{P}^N_{\mathbb{C}}$ are mapped to the vertices of Δ.

Example 2.49. Consider the m-th Segre embedding

$$\sigma_m\colon \mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^N_{\mathbb{C}}$$

$$[X_0 : \ldots : X_N] \mapsto [\ldots : X^\alpha : \ldots]$$

where $\alpha \in \mathbb{N}^n$ denotes an exponent tuple with $|\alpha| = \alpha_1 + \ldots + \alpha_n = m$. Fix a weight vector $\gamma \in \mathbb{N}^n$ and a shift vector $\beta \in \mathbb{N}^n$ and let Z_α denote the coordinate corresponding to X^α. Take a $(\mathbb{C}^*)^n$-action on $\mathbb{P}^N_{\mathbb{C}}$

$$(t_1, \ldots, t_n) \cdot [\ldots : Z_\alpha : \ldots] = [\ldots : t^{\gamma\cdot\alpha+\beta} Z_\alpha : \ldots]$$

It is easy to check that this group action consists of isomorphisms of $\mathbb{P}^N_{\mathbb{C}}$ mapping the image $\sigma_m(\mathbb{P}^n_{\mathbb{C}})$ of the Segre embedding onto itself. The $(\mathbb{C}^*)^n$-action descends onto an action on $\mathbb{P}^n_{\mathbb{C}}$ and the pulled back moment map of the previous example along σ_m is a moment map on $\mathbb{P}^n_{\mathbb{C}}$.

2.8 Symplectic blow ups

The constructions of symplectic blow ups (and blow downs) allow us to relate Kähler packings to the ampleness of line bundles on the blown-up manifolds. The following constructions are taken from [MP94] and [MS98] and we refer the reader to those expositions for further details.

Consider the triples $(\mathbb{C}^n, J_0, \omega_{std})$ and $(\mathbb{CP}^{n-1}, J_0, \omega_{FS})$, where ω_{std} is the standard Euclidean form on $\mathbb{C}^n$, J_0 is the standard complex structure on both $\mathbb{C}^n$ and $\mathbb{CP}^{n-1}$, and ω_{FS} is the Fubini-Study form on $\mathbb{CP}^{n-1}$. Let $l \in \mathbb{CP}^{n-1}$ be a line and fix a point $z \in \mathbb{C}^n$. Then the complex manifold $\mathcal{M}$

$$\begin{array}{ccc} \mathcal{M} = \{(z,l) \mid z \in l\} \subset \mathbb{C}^n \times \mathbb{CP}^{n-1} & \xrightarrow{q} & \mathbb{CP}^{n-1} \\ \Big\downarrow{\scriptstyle\pi} & & \\ \mathbb{C}^n & & \end{array}$$

is the blow up of $\mathbb{C}^n$ at the origin. To help the reader visualise this one can imagine replacing the origin of $\mathbb{C}^n$ by the set of all complex lines passing through the origin.

Let $B^{2n}_\lambda(0) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \colon |z_1|^2 + \ldots |z_n|^2 \leq \lambda^2\}$ denote a n-dimensional complex ball of radius λ and centred at the origin. For $\lambda > 0$ the set $\mathcal{M}(\lambda) := \pi^{-1}(B^{2n}_\lambda(0))$ is the sub-manifold obtained by taking the inverse image of $B^{2n}_\lambda(0)$ under the projection π. We call the fiber $\pi^{-1}(0)$ over 0 the exceptional divisor and the projection π the blow-up map. It is worth noting that away from the exceptional divisor π is biholomorphic onto $\mathbb{C}^n$.

Proposition 2.50. *On $\mathcal{M}$ there exists a Kähler form*

$$\rho(1,\lambda) := \pi^*\omega_{std} + \lambda^2 q^*\omega_{FS} \text{ for } \lambda > 0.$$

Proof. To prove that the bilinear form $\rho(1,\lambda)$ is a Kähler form we must show that it is closed, skew symmetric, that it is compatible with the complex structure on $\mathcal{M}$ and the associated Riemannian metric is positive definite. First we note $\rho(1,\lambda)$ is closed and skew symmetric, as pullbacks of closed and skew symmetric forms are closed and skew symmetric. Since ω_{std} and ω_{FS} are compatible with the complex structure on $\mathbb{C}^n$ and $\mathbb{CP}^{n-1}$ and the projections π and q are holomorphic, we can deduce that the associated bilinear form is compatible with the complex structure on $\mathcal{M}$. Finally the associated Riemannian metric is positive definite as the pullbacks are positive-semi definite and furthermore $\pi^*\omega_{std}$ is positive definite away from the exceptional divisor, and positive definite on the tangent vectors mapped to 0 by the differential of q. Moreover on the other tangent vectors of $\mathcal{M}$, $q^*\omega_{FS}$ is positive definite. $\square$

Corollary 2.51. *On the exceptional divisor $\pi^{-1}(0)$ the (1,1)-form $\rho(1,\lambda)|_{\pi^{-1}(0)} = \lambda^2 q^*\omega_{FS}$.*

Definition 2.52. An embedding $F\colon \mathbb{C}^n - \{0\} \to \mathbb{C}^n$ is called *monotone* if in spherical coordinates $(u,r) \in S^{2n-1} \times (0,\infty) \cong \mathbb{C}^n - \{0\}$ it can be written as $(u,r) \mapsto (u, f(r))$, where f is a strictly increasing function.

Lemma 2.53. *There exists a smooth family of monotone embeddings*

$$h_\lambda\colon \mathbb{C}^n - \{0\} \to \mathbb{C}^n - B(\lambda)$$

such that $\pi^* h_\lambda^* \omega_{std} = \rho(1,\lambda)$ on $\mathcal{M} - \pi^{-1}(0))$.

Proof. Using spherical coordinates we can choose $h_\lambda\colon (u,v) \mapsto (u, (r^2+\lambda^2)^{\frac{1}{2}})$. Fix a point $P = ((c,0,\dots,0)\ ,\ [1:0:\dots:0]) \in \mathcal{M}$ for $c \in \mathbb{R}_{>0}$. Recall that around the point $(c,0,\dots,0)$ if we take local coordinates $z_j = x_j + iy_j$ of $\mathbb{C}^n$ then $\omega_{std} = \sum_{j=1}^n dx_j \wedge dy_j$. Furthermore if $t_j = u_j + iv_j$ are local coordinates of $\mathbb{CP}^{n-1}$ around $[1:0:\dots:0]$ then the Fubini-Study form in P is given as $\omega_{FS}(P) = \sum_{j=2}^n du_j \wedge dv_j$. Since $t_j = \frac{z_j}{z_1}$ on $\mathcal{M}$ around P, the z-coordinates provide a chart of $\mathcal{M}$ around P. Calculations show that in these coordinates

$$\begin{aligned}\pi^*\omega_{std}+\lambda^2 q^*\omega_{FS}(P) &= \sum_{j=1}^n dx_j\wedge dy_j+\lambda^2\sum_{j=2}^n du_j\wedge dv_j\\ &= \sum_{j=1}^n dx_j\wedge dy_j+\frac{\lambda^2}{c^2}\left(\sum_{j=2}^n dx_j\wedge dy_j\right)\\ &= dx_1\wedge dy_1+\sum_{j=1}^n d\frac{(r^2+\lambda^2)^{\frac{1}{2}}}{r}x_j\wedge d\frac{(r^2+\lambda^2)^{\frac{1}{2}}}{r}y_j(P)=(\pi^*h_\lambda^*\omega_{std})(P).\end{aligned}$$

Then in P we have that $\rho(1,\lambda)(P)=\pi^*h_\lambda^*\omega_{std}(P)$. Now since h_λ and π are compatible under the natural $U(n)$-action and ω_{std} and $\rho(1,\lambda)$ are invariant under the same action we can deduce that $\pi^*h_\lambda^*\omega_{std}=\rho(1,\lambda)$ everywhere on $\mathcal{M}$ away from $\pi^{-1}(0)$.

□

Lemma 2.54. *For every monotone embedding $F\colon\mathbb{C}^n-\{0\}\to\mathbb{C}^n$ the form $F^*\omega_{std}$ is Kähler.*

Proof. Since F is monotone it is once again compatible with the natural $U(n)$- actions on $\mathbb{C}^n$ and $\mathbb{C}^n-\{0\}$. Furthermore, since

$$(dF)|_{(F(1),0,\ldots,0)}\left(\frac{\partial}{\partial x_1}\right)=\frac{dF}{dr}(F(1),0,\ldots,0)\cdot\left(\frac{\partial}{\partial x_1}\right),$$

and $\frac{dF}{dr}(F(1),0,\ldots,0)>0$ as F is strictly increasing and $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial y_k}$ are left invariant by F at $(F(1),0,\ldots,0)$, for $j=2,\ldots,n$ and $k=1,\ldots,n$, we can deduce that $F^*\omega_{std}(F(1),0,\ldots,0)$ is compatible with the complex structure. □

Proposition 2.55 ([MP94, Prop.5.1A])**.** *For every $\eta>0,\lambda>0$ there exists a Kähler form $\bar\tau=\bar\tau(\eta,\lambda)$ on $\mathcal{M}$ such that:*

1. $\bar\tau|_{\mathcal{M}-\mathcal{M}(1+\eta)}=\pi^*\omega_{std}$.

2. $\bar\tau|_{\mathcal{M}(\delta)}=\rho(1,\lambda)$*, for some* $\delta>0$.

Proof. Take $\delta<1+\eta$ such that $h_\lambda(B_\delta^{2n}(0))\subset B^{2n}_{\lambda(1+\frac{\eta}{2})}(0)$. Then we can use a smoothing procedure to find a monotone embedding F, such that:

1. $F(z) = \lambda z$ for $|z| > 1 + \eta$.

2. $F(z) = h_\lambda(z)$ for $|z| < \delta$.

The form $\bar{\tau} = \pi^*(F^*(\omega_{std}))$ extended to $\pi^{-1}(0)$ has the required properties. $\square$

Construction 2.56 (Symplectic blow up of a Kähler manifold)**.** Let (M, ω) be an n dimensional Kähler manifold with a Kähler form ω and fix a point $P \in M$. Assume that

$$\phi\colon (B^{2n}_{(1+2\eta)}(0), \omega_{std}) \hookrightarrow (M, \omega)$$

is a holomorphic embedding with $\phi(0) = P$ and $\phi^*\omega = \lambda^2\omega_{std}$ for $\lambda \in \mathbb{R}_{>0}$.
Since $\mathcal{M}(1+2\eta)$ and $B^n_{1+2\eta}(0)$ are holomorphic away from 0 respectively $\pi^{-1}(0)$ we can glue in $\mathcal{M}(1+2\eta)$ instead of $B^n_{1+2\eta}(0)$ and obtain the blown up complex variety $\tilde{M}$, with blow up map $\Pi\colon \tilde{M} \to M$ and exceptional divisor $E = \pi^{-1}(0)$:

$$\begin{array}{ccccc}
E = \pi^{-1}(0) & \subset & \mathcal{M}(1+2\eta) & \xrightarrow{\tilde{\phi}} & \tilde{M} \\
\big\downarrow & & \pi\big\downarrow & & \big\downarrow\Pi \\
0 & \in & B^{2n}_{1+2\eta}(0) & \xrightarrow{\phi} & M
\end{array}$$

We can construct a Kähler form $\tilde{\omega}$ on $\tilde{M}$ with $\phi^*\tilde{\omega} = \omega_{std}$ as follows:

$$\tilde{\omega} = \begin{cases} \Pi^*\omega \text{ on } \tilde{M} - \tilde{\phi}(\mathcal{M}(1+2\eta) \\ (\tilde{\phi}^{-1})^*\bar{\tau}(\eta, \lambda) \text{ on } \tilde{\phi}(\mathcal{M}(1+2\eta)) \end{cases}$$

where $\bar{\tau}(\eta, \lambda)$ is defined as above. This is possible since $\tilde{\phi}^*\tilde{\omega}$ coincides with $\phi^*\omega = \omega_{std}$ on $\mathcal{M}(1+2\eta) - \mathcal{M}(1)$ by construction of $\tilde{\tau}(\eta, \lambda)$. Moreover $\tilde{\omega}$ is a Kähler form as the two glued components are Kähler forms.

In the setting of Construction 2.56 if M is a projective complex manifold and $[\tilde{\omega}]$ the ample class $[\pi^*\omega] - \lambda[E]$ on M, where $[E]$ is the cohomology class of (the Poincaré dual of) the exceptional divisor E then $\tilde{\omega}$ represents a ample class on $\tilde{M}$. By construction $[\tilde{\omega}]$ is in the interior of a cone generated by integral Kähler classes. Kodaira's embedding Theorem says that Kähler forms representing such integral Kähler classes are curvature forms of Kähler metrics on ample line bundles.

There is also a symplectic blow down of Kähler manifolds relying on the following construction:

Proposition 2.57 ([MP94, Prop.5.1B]). *For every $\eta > 0, \lambda > 0, \delta > 0$ there exists a Kähler form $\tau = \tau(\eta, \delta, \lambda)$ on $\mathbb{C}^n$ such that the following hold:*

1. $\pi^*\tau = \rho(\delta, \lambda)$ *on* $\mathcal{M} - \mathcal{M}(1 + \eta)$.
2. $\tau = \lambda^2\omega$ *on the unit ball* $B_1^n(0)$.

Proof. Note that $\rho(\delta, \lambda) = \delta^2\rho(1, \gamma)$ for $\gamma = \frac{\lambda}{\delta}$. Using a smoothing procedure it is possible to construct a monotone embedding G such that

$$G(z) = \gamma \cdot z \text{ for } |z| \leq 1 \text{ and } G(z) = h_v(z) \text{ for } |z| \geq 1 + \eta.$$

The form $\tau = \delta^2 G^*\omega$ has the required properties. □

Since we will not use the symplectic blow down later on we will not state the analogue to Construction 2.56
From Construction 2.56 it is clear that to define a symplectic form on the blow up of a symplectic manifold M at points $P_1, \ldots, P_k$ of X is equivalent to specifying a holomorphic embedding

$$\phi\colon\ \amalg_{i=1}^k \left(B_i(\lambda_i), \omega_{std}\right) \hookrightarrow (M, \omega)$$

which sends the center of the i-th ball to P_i for every $i = 1, \ldots, k$. Then one can cut out images of the embedded balls and collapse their boundaries to copies of $\mathbb{CP}^{n-1}$ called exceptional divisors. Conversely given a symplectic form on the blow up by blowing down we obtain a symplectic form ω on M and a packing of (M, ω) by balls. Both constructions preserve Kähler forms. In terms of the blow down this amounts to determining if the cohomology class on the blow up of M can be represented by symplectic forms.

2.9 Newton-Okounkov bodies

First introduced by Andrei Okounkov in [Oko96] a Newton-Okounkov body is a convex body associated to a projective variety equipped with an ample line bundle. Küronya, Lozovanu and Maclean provide a complete classification of Newton-Okounkov bodies of surfaces in [KLM12] however no such classification currently exists for higher dimensions.

Newton-Okounkov bodies have started to generate lots of interest from algebraic geometers in particular in relation to determining ampleness of given line bundles. Progress in this direction has shown that Newton-Okounkov bodies encode intrinsic information about ample line bundles such as the multipoint Seshadri constant. Küronya and Lozovanu in [KL17] and [KL18] show that is is possible to determine single point Seshadri constants directly from Newton-Okounkov bodies and Eckl in [Eck14] shows that this is also possible for multipoint Seshadri constants when we consider blow ups of the projective complex plane in up to 9 points. Kaveh in [Kav19] constructed symplectic packings on a projective variety using Newton-Okounkov bodies and connected these to single point Seshadri constants. Trusiani in [Tru18] gave a new description of Newton-Okounkov bodies (called multipoint Okounkov bodies), showing they are directly related to Kähler packing constructions and multipoint Seshadri constants.

Let X be a projective, complex manifold and L an ample line bundle on X. Fix points $P_1, \ldots, P_k \in X$.

Definition 2.58. A chain of inclusions $Y_\bullet := X = Y_0 \supsetneq, \ldots, \supsetneq Y_n = \{pt\}$ such that each Y_i is an irreducible subvariety of co-dimension i, is smooth at Y_n and $P_i \notin Y_1$, is called an *admissible flag* on X.

Definition 2.59. Let R be a $\mathbb{C}$-algebra. Choose a total order on $\mathbb{Z}^n$, then a function $v\colon R \to \mathbb{Z}^n$ is called a *valuation* on R if, for all $a, b \in R \setminus \{0\}$:

1. $v(\lambda \cdot b) = v(b)$, for all $\lambda \in \mathbb{C}$.
2. $v(a \cdot b) = v(a) + v(b)$.
3. $v(a + b) \geq \min(v(a), v(b))$.

In the setting above the valuation $v_{Y_\bullet}\colon H^0(X, L) \to \mathbb{Z}^n$ associated to the flag $Y_\bullet$ is defined by the following recipe:

Construction 2.60. Let $s \in H^0(X, L) \setminus \{0\}$ be a global section of L on X.

1. Start with $D_0 = \text{zero}(s)$ i.e. D_0 is the divisor in X which is the zero locus of s.
2. For $i \geq 1$ set $v_i(s) = ord_{Y_i}(D_{i-1})$.
3. Set $D_i = (D_{i-1} - v_i(s)Y_i|_{Y_i})$.

We denote $v_{Y\bullet}(s) = (v_1(s), v_2(s), \ldots, v_n(s)) \in \mathbb{Z}^n$ the valuation vector of s associated to the admissible flag $Y_\bullet$.

Remark 2.61. On the $\mathbb{C}$-algebra $\bigoplus_{N=0}^{\infty} H^0(X, NL)$ the map $v_{Y\bullet}$ defines a valuation in the sense of Definition 2.60. This can be extended to the extended valuation $\tilde{v}_{Y\bullet}$ with values in $\mathbb{Z} \times \mathbb{Z}^n$, by setting

$$v_{Y_\bullet}(s) = (N, v_{Y_\bullet}(s)) \text{ for } s \in H^0(X, NL).$$

Definition 2.62. Let X be a projective manifold and L an ample line bundle on X. Then if $Y_\bullet \coloneqq X = Y_0 \supsetneq, \ldots, \supsetneq Y_n = \{pt\}$ is an admissible flag for X we define the *Newton-Okounkov body associated to* L w.r.t $Y_\bullet$ to be:

$$\Delta_{Y_\bullet}(L) \coloneqq conv\Big\{\frac{v_{Y_\bullet}(s)}{N} : s \in H^0(X, NL) \setminus \{0\}, N \geq 1\Big\}$$

where *conv* stands for the closed convex hull and $v_{Y_\bullet}(s)$ is the valuation of the section s with respect to the flag $Y_\bullet$.

In general Newton-Okounkov bodies are difficult to compute but when X is a projective surface Lazarsfeld and Mustaţă gave a description of the Newton-Okounkov body associated to a big $\mathbb{R}$-divisor (see [LM09, Theorem 6.4]) that relies on a variation of the Zariski decomposition.

Definition 2.63 (Zariski Decomposition)**.** Let D be a pseudo-effective $\mathbb{R}$-divisor on a smooth projective algebraic surface X. A decomposition $D = P + N$ into a nef $\mathbb{R}$-divisor P (the positive part) and an effective $\mathbb{R}$-divisor N (the negative part) is called a *Zariski decomposition of* D if $P \cdot C_i = 0$ for all $i = 1, \ldots, q$ and the intersection matrix $(C_i \cdot C_j)_{1 \leq i,j \leq q}$ is negative-definite, where $C_1, \ldots, C_q$ are the reduced and irreducible components of the support of N.

Let D be a big $\mathbb{R}$-divisor on a smooth projective algebraic surface X, and let $Y_\bullet : X \supset C \supset \{x\}$ be an admissible flag on X, with C an irreducible and reduced curve on X and $x \in C$ a nonsingular point on C. Set

$$\mu = \mu(D; C) := \sup\{s > 0 \mid D - sC \text{ is big}\}$$

and for $t \in [\nu, \mu]$ we set $D_t = D - tC$ and write $D_t = P_t + N_t$ for its Zariski decomposition.

There then exist two continuous functions $\alpha, \beta : [\nu, \mu] \to \mathbb{R}_+$ defined as follows

$$\alpha(t) = \mathrm{ord}_x(N_t|_C), \quad \beta(t) = \mathrm{ord}_x(N_t|_C) + P_t \cdot C.$$

In this setting Lazarsfeld and Mustaţă prove the following result (see [LM09, Theorem 6.4]).

Theorem 2.64. *With the notation as above there exist continuous functions $\alpha, \beta : [a, \mu] \to \mathbb{R}_+$ for some $0 \leq a \leq \mu$ with α convex and increasing, β concave, $\alpha \leq \beta$, and both α and β piecewise linear with rational slopes and only finitely many breakpoints such that the Newton-Okounkov body $\Delta_{Y_\bullet}(D) \subset \mathbb{R}^2_+$ is the region bounded by the graphs of α and β,*

$$\Delta_{Y_\bullet}(D) = \{(t, y) \in \mathbb{R}^2_+ | a \leq t \leq \mu, \alpha(t) \leq y \leq \beta(t)\}.$$

In [KLM12] the authors were able to strengthen this result to provide a complete classification of Newton-Okounkov bodies of surfaces and prove that they are all polygonal.

Theorem 2.65 ([KLM12, Theorem B])**.** *The Newton-Okounkov body of an $\mathbb{R}$-divisor on a smooth projective surface with respect to some flag is a finite polygon. Up to translation, a real polygon $\Delta \subseteq \mathbb{R}^2_+$ is the Newton-Okounkov body of an $\mathbb{R}$-divisor D on a smooth projective surface S with respect to a complete flag $Y_\bullet : X \supset C \supset \{x\}$ if and only if*

$$\Delta = \{(t, y) \in \mathbb{R}^2 \mid \nu \leq t \leq \mu, \alpha(t) \leq y \leq \beta(t)\}$$

for certain real numbers $0 \leq \nu \leq \mu$ and certain continuous piecewise linear functions $\alpha, \beta : [\nu, \mu] \to \mathbb{R}_+$ with rational slopes such that β is concave and α is increasing and convex.

It is known that no such classification can exist in higher dimensions and in [LM09] the authors give an example of a Newton-Okounkov body in higher dimensions which is not polyhedral. This is also discussed in [KLM12] where the authors give 2 examples of Mori dream spaces whose associated Newton-Okounkov bodies are not polyhedral in most cases. When X is $\mathbb{P}^2_{\mathbb{C}}$ Eckl in [Eck14] gave a description of Newton-Okounkov bodies of blow ups of X in up to 9 points with respect to an admissible flag in terms of the multipoint Seshadri constant.

Theorem 2.66 ([Eck14, Theorem 3.4])**.** *Let $X = \mathbb{P}^2$ with ample line bundle L and fix points $x_1, \ldots, x_n \in X$. Let $\pi_n \colon \tilde{X} = Bl_{\mathbb{P}^2}(x_1, \ldots, x_n)$ be the blow up of X at the points*

$x_1, \dots, x_n$ and denote $E_i = \pi_n^{-1}(x_i)$ the exceptional divisor corresponding to the point x_i. Let $\epsilon_n = \sup\{t > 0 \mid \pi^* nL - t\sum_{i=1}^n E_i$ is ample $\}$ denote the multipoint Seshadri constant then the Newton-Okounkov body of $D_n = \pi_n^* L - \epsilon_n \sum_{i=1}^n E_i$

$$\Delta_{Y_\bullet}(D) = \{t_1 \cdot (D_n^2, 0) + t_2 \cdot (0,1) \mid 0 \le t_1, t_2, t_1 + t_2 \le 1\},$$

the convex hull of the points $(0,0), (0, D_n^2)$ and $(0,1) \in \mathbb{R}^2$.

Eckl's proof relies on the characterisation of Newton-Okounkov bodies of surfaces in [KLM12] via the Zariski decomposition. Since no such characterisation exists in higher dimensions it is not clear if it is possible to determine multipoint Seshadri constants directly from Newton-Okounkov bodies. One problem is that for multiple points it will not just be simplexes contained in the interior of the Newton-Okounkov body and other shapes such as quadrilaterals may exist.

Recall the setting in Section 2.1: Let X be a complex projective manifold of dimension n, with $P_1, \dots, P_k$ distinct points of X and L an ample divisor on X. Denote $\pi_k \colon \tilde{X}_k = Bl_{P_1,\dots,P_k}(X) \xrightarrow{\pi} X$ the blow-up of X at the points $P_1, \dots, P_k$, with $\pi^{-1}(P_i) = E_i$ the exceptional divisor corresponding to the point P_i for all $i = 1, \dots, k$ and set

$$\tilde{L}^{(i)} := \pi_i^* dL - \sum_{j=1}^{i} \epsilon_i E_i.$$

Proposition 2.67. *Let X, L and $\tilde{L}^{(i)}$ be defined as above and denote $\tilde{X}_i$ the blow up of X at the first i points and fix a point $P \in X$ and sub varieties Y_i that for each $i = 1, \dots, k$ are of co-dimension i and are smooth at P. Take an admissible flag*

$$Y_\bullet := X = Y_0 \supsetneq, \dots, \supsetneq Y_k = \{pt\}$$

on X that can also be taken as an admissible flag on each $\widetilde{X}_i$. Then for any $k \in \mathbb{N}$:

$$\Delta_{Y_\bullet}(\tilde{L}^{(k)}) \subset \Delta_{Y_\bullet}(\tilde{L}^{(k-1)}) \subset \dots \subset \Delta_{Y_\bullet}(L) \subset \mathbb{R}^n.$$

Proof. We follow the same method as Eckl in in [Eck14, Prop.3.1] where the 2-dimensional case of the above proposition is proved. Consider the inclusions

$$H^0(\tilde{X}_1, \mathcal{O}_{\tilde{X}_1}(\tilde{L}^{(1)})) \hookrightarrow H^0(X, \mathcal{O}_X(dL)),$$

where sections of $\mathcal{O}_{\tilde{X}_1}(\tilde{L}^{(1)})$ are identified with sections of $\mathcal{O}_X(dL)$ having multiplicity $\geq m$ in P_1. Identified sections are equal when identifying the line bundles $\mathcal{O}_{\tilde{X}_1}(\tilde{L}^{(1)})$ and $\mathcal{O}_X(dL)$ on $X - \{P_1\} \cong \tilde{X}_1 - E_1$. These equal sections have the same valuations hence the Newton-Okounkov body associated to $\tilde{L}^{(1)}$ is contained in the Newton-Okounkov body associated to dL with respect to the flag $Y_\bullet$. Arguing iteratively for each $\tilde{L}^{(i)}$ with $i = 1, \dots, k$ we obtain the chain of inclusions as claimed. □

Definition 2.68 ([Eck14])**.** With notation as in the above proposition the chain of inclusions

$$\Delta_{Y_\bullet}(\tilde{L}^{(k)}) \subset \Delta_{Y_\bullet}(\tilde{L}^{(k-1)}) \subset \ldots \subset \Delta_{Y_\bullet}(L) \subset \mathbb{R}^n,$$

is called the *iterative Newton-Okounkov body dissection* associated to π_i and $\tilde{\mathcal{L}}^{(i)}$ for all $i = 1, \dots, k$ (and the flag $Y_\bullet$).

One of the nice features of Newton-Okounkov bodies is that they encode invariants and Küronya and Lozovanu show in [KL18] that when X is a smooth projective surface that Newton-Okounkov bodies of an ample divisor D are bound to contain a standard simplex

$$\Delta_\lambda \coloneqq \{(s,t) \in \mathbb{R}^2_+ \mid \lambda s + \lambda t \leq \lambda^2\}$$

of some length $\lambda > 0$ (see [KL18, Theorem A] for details). The authors define the *largest simplex constant*

$$\lambda(D,x) \coloneqq \sup_{(C,x)} \sup\{\lambda > 0 \mid \Delta_\lambda \subseteq \Delta_{(C,x)}(D)\},$$

where the first supremum runs through all admissible flags $(C,x) = X \supset C \supset \{x\}$ centred at the point $x \in X$ and $\Delta_{(C,x)}(D)$ is the Newton-Okounkov body associated to the ample divisor D on X with respect to the flag (C,x). Using this they observe that $\epsilon(D;x) \geq \lambda(D,x)$ where the right hand side is the single point Seshadri constant (see [KL18, Proposition 4.7]].) The authors were able to use similar ideas to give a characterisation of (moving) Seshadri constants for any smooth projective variety of any dimension. The moving point Seshadri constant is a generalisation of the Seshadri constant for big line bundles. We will not define it here but the interested reader can find a definition in [BDRH$^+$09, Definition 1.16].

Remark 2.69. When D is nef but not big the moving Seshadri constant agrees with the classical Seshadri constant.

Let X be a smooth projective surface, D a big divisor on X and denote $\pi\colon \tilde{X} \to X$ the blow up of X at a point $x \in X$, with $\pi^{-1}(x) = E$ the exceptional divisor. Define

$$\Delta_{\xi}^{-1}\{(s,t)) \in \mathbb{R}_{+}^{2} \mid 0 \leq s \leq \xi, 0 \leq t \leq s\}$$

the inverted simplex of length ξ. Küronya and Lozovanu define a geometric invariant

$$\xi(\pi^*(D); y) \coloneqq \sup\{\xi \geq 0 \mid \Delta_{\xi}^{-1}\Delta_{(C,x)}(D)\}$$

called the *largest inverted simplex constant.* Furthermore they prove the following:

Theorem 2.70 ([KL18, Theorem D]). *Let X be a smooth projective surface and D be a big $\mathbb{R}$-divisor on X. If $x \notin Neg(D)$ then*

$$\epsilon(||D||; x) = \xi(\pi^*(D); y),$$

Where $\epsilon(||D||; x)$ denotes the moving Seshadri constant.

In [KL17] the authors generalise these ideas and produce the following result for a smooth projective variety of dimension n equipped with a big $\mathbb{R}$-divisor.

Theorem 2.71 ([KL17, Corollary 3.2]). *Let X be a smooth projective variety and D a big $\mathbb{R}$-divisor on X. Then the following are equivalent:*

1. *D is ample.*

2. *For every point $x \in X$ there exists an admissible flag $Y_\bullet$ centered at x with Y_1 ample such that $\Delta_{\epsilon_0} = \{(x_0, \ldots, x_n) \in \mathbb{R}_{+}^{n} \mid x_1 + \ldots x_n \leq \epsilon_0\} \subseteq \Delta_{Y_\bullet}(D)$ for some $\epsilon_0 > 0$.*

3. *For every admissible flag $Y_\bullet$ there exists some $\epsilon > 0$ (possibly depending on $Y_\bullet$) such that $\Delta_{\epsilon} = \{(x_0, \ldots, x_n) \in \mathbb{R}_{+}^{n} \mid x_1 + \ldots x_n \leq \epsilon\} \subseteq \Delta_{Y_\bullet}(D)$.*

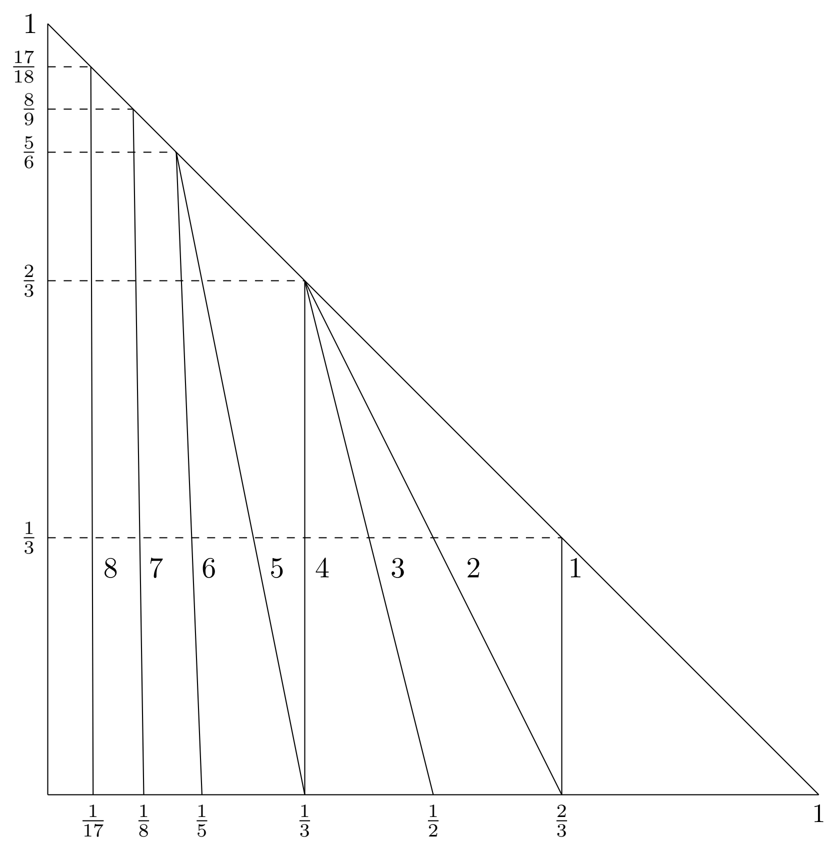
As remarked in [KL17] the above theorem can be seen as a variant of Seshadri's criterion for ampleness in the language of convex geometry and in dimension 2 the requirement for Y_1 to be ample can be dropped.

We will now present an example of Eckl which illustrates the iterative dissection form of the Newton-Okounkov body of $\mathbb{P}^2$ blown up in up to 8 points of general position. In Chapter 4 we will use the form given in the following example to construct moment maps and Kähler packings such that the images of the embedded balls with respect to these maps correspond to the cut off triangles in the iterative dissection form.

Example 2.72. Let $X = \mathbb{P}^2_{\mathbb{C}}$, L an ample line on $\mathbb{P}^2$ and fix points $P_1, \ldots, P_k \in X$. Denote $\pi_k \colon \tilde{X}_k = Bl_{P_1,\ldots,P_k}(X) \to X$ the blow-up of X at the points $P_1, \ldots, P_k$, with $\pi^{-1}(P_i) = E_i$ the exceptional divisor corresponding to the point P_i and fix integers $d, m_i > 0$ such that

$$\tilde{L}^{(i)} \coloneqq \pi_i^* dL - \sum_{j=1}^{i} \epsilon_i E_i$$

is ample for all $i = 1, \ldots, k$. The iterative dissection form of the Newton-Okounkov body associated to $\tilde{X}_k$ and $\tilde{L}^{(k)}$ for $1 \leq k \leq 8$ is given by the following convex body (see [Eck14, Thm.3.6] for details).



Iterative dissection form of the Newton-Okounkov body associated to blow ups of $\mathbb{P}^2_{\mathbb{C}}$ in up to 8-points. The whole triangle corresponds to $\Delta_{Y_\bullet}(L)$ and the areas to the left of the i-th lines correspond to $\Delta_{Y_\bullet}(\tilde{L}^{(i)})$ for all $i = 1, \ldots, 8$.

2.10 Toric degenerations of projective complex manifolds

Anderson first introduced toric degenerations in connection with Newton-Okounkov bodies in [And13]. To be more precise let X be an irreducible projective variety, then a toric degeneration of X is a family of varieties $\pi\colon \mathcal{X} \to \mathbb{C}$ where $\mathcal{X}$ is a variety and π is a morphism satisfying the following properties:

1. The family is trivial over $(\mathbb{C})^*$ and each fiber $X_t = \pi^{-1}(t), t \neq 0$ is isomorphic to X.
2. The central fiber $X_0 = \pi^{-1}(0)$ is a projective toric variety (possibly non-normal).
3. All the fibers are irreducible and reduced as schemes.
4. The family $\mathcal{X}$ is flat over $\mathbb{C}$.

Anderson's construction is dependent on a choice of a $\mathbb{Z}^n$ valued valuation v on the field of rational functions of X and as such the construction requires that the value semigrouop of v is finitely generated. These ideas were extended by Harada and Kaveh in [HK15] where the authors use a given degeneration $\mathcal{X}$ for a projective variety X to construct an integrable system on X such that the image of the integrable system is the Newton-Okounkov body associated to X and the valuation v used to build the toric degeneration. Finally Kaveh in [Kav19] provides an alternative construction where the requirement for the value semigroup to be finitely generated is dropped. Kaveh uses this to construct moment maps on non-toric projective complex manifolds mapping to the Newton-Okounkov body. This is achieved by introducing an algebraic family which degenerates to a central fiber isomorphic to $(\mathbb{C}^*)^n$ together with a family of Kähler forms which restricts to a toric-Kähler form on the central fiber, with an associated moment map to the Newton-Okounkov body. Finally, he uses the gradient-Hamiltonian flow to deform this moment map to non-central fibers. The method of Kaveh works in a more general setting than the previous constructions and this is the method that we will follow.

In more details: Let X be a projective complex variety of dimension n equipped with an ample line bundle L and fix a point $P \in X$. Choose an open subset $U \subset X$ along with local coordinates $u_1, \ldots, u_n$ of X around P such that $u_1(P) = u_2(P) = \ldots = u_n(P) = 0$ and fix a vector $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n_{>0}$.

Proposition 2.73 ([Kav19, Prop 3.1])**.** *The map $\phi\colon U \times \mathbb{C}^* \to \mathbb{C}^{n+1}$ such that*

$$\phi(x,t) = (t^{-\gamma_1} u_1(x), \ldots, t^{-\gamma_n} u_n(x), t)$$

satisfies the following properties:

1. *ϕ is a biholomorphism between $U \times \mathbb{C}^*$ and its image.*

2. *The image $\phi(U \times \mathbb{C}^*)$ is open in $\mathbb{C}^{n+1}$.*

Let $f = \sum c_\alpha u^\alpha \in \mathcal{O}_{X,P}$ be function regular at P, expressed as a power series in $u = (u_1, \dots, u_n)$. Note that f is meromorphic on U and holomorphic at P. Fix a total order on $\mathbb{Z}^n$ (we take the lexicographical ordering) and let

$$v(f) = \{\alpha \mid c_\alpha \neq 0\} = \beta$$

be the valuation of f.

Remark 2.74. These valuations are exactly those corresponding to an admissible flag $Y_\bullet \coloneqq X \supset Y_1 \supset \dots \supset Y_n = \{P\}$ on X where the co-dimension i sub-varieties Y_i are:

$$Y_i = \{u_1 = 0\},\ Y_2 = \{u_1 = u_2 = 0\},\ \dots, Y_n = \{u_1 = \dots = u_n = 0\} = P.$$

In this way the leading term exponents that generate the valuation $v_{Y_\bullet}$ with respect to the flag $Y_\bullet$ are equal $v(f)$ as defined above. The flag is only locally defined but intersects transversally with local coordinates $u_1 \dots, u_n$ in the above construction. When we use a variation of the method described in this section in Chapter 4 we will define a flag of the above form and consider valuations with respect to this flag rather than considering valuations of the form $v(f)$ as defined above.

Consider the subset

$$\tilde{U}_\gamma \coloneqq \phi(U \times \mathbb{C}^*) \cup ((\mathbb{C}^*)^n \times \{0\}) \subset \mathbb{C}^{n+1}.$$

Kaveh proves that for a suitable choice of γ the subset $\tilde{U}_\gamma$ is open (in the analytic topology) in $\mathbb{C}^{n+1}$ and this along with the biholomorphism ϕ allows one to construct an algebraic family $\mathcal{X}_\gamma$ constructed as the union of $X \times \mathbb{C}^*$ and $\tilde{U}_\gamma$ glued via ϕ along the open subsets $U \times \mathbb{C}^*$ and $\phi(U \times \mathbb{C}^*)$. To be more precise, let $\sim$ be the equivalence relation that identifies the point (x, t) with $(y_1, \dots, y_n, t)$ whenever $\tilde{\phi}(x, t) = (y_1, \dots, y_n, t)$. Then set

$$\mathcal{X}_\gamma \coloneqq ((X \times \mathbb{C}^*) \ \amalg \ \tilde{U}_\gamma) \ / \ \sim .$$

There exists a well defined map $\pi\colon \mathcal{X}_\gamma \to \mathbb{C}$ such that $\pi(x,t) = t$ for $(x,t) \in X \times \mathbb{C}^*$, and $\pi(\tilde{u}, t) = t$ for all $(\tilde{u}, t) \in \tilde{U}_\gamma$.

$$\begin{tikzcd} X \times \mathbb{C}^* \arrow[r, hook] \arrow[dr] & \mathcal{X}_\gamma \arrow[d, "\pi"] \\ & \mathbb{C} \end{tikzcd}$$

Proposition 2.75 ([Kav19, Prop 3.3])**.** *The family $\pi\colon \mathcal{X}_\gamma \to \mathbb{C}$ satisfies the following properties:*

1. *$\mathcal{X}_\gamma$ has the structure of a complex manifold.*
2. *The family $\mathcal{X}_\gamma$ is trivial over $(\mathbb{C})^*$ hence $\pi^{-1}((\mathbb{C})^*) \cong X \times (\mathbb{C})^*$ and the fibers $\mathcal{X}_{\gamma,t} = \pi^{-1}(t)$ are isomorphic to X for all $t \neq 0$.*
3. *The special fiber $\mathcal{X}_{\gamma,0} = \pi^{-1}(0) \cong (\mathbb{C}^*)^n$.*
4. *The map $\pi\colon \mathcal{X}_\gamma \to \mathbb{C}$ is holomorphic and has no critical points.*

The aim now is to embed the family $\mathcal{X}_\gamma$ into $\mathbb{P}^N$ and use this embedding to construct a Kähler form on $\mathcal{X}_\gamma$.

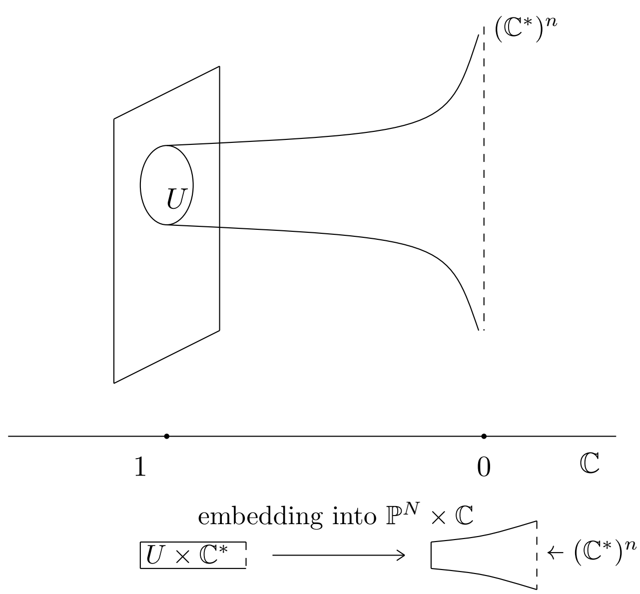


Figure 2.1

We construct the embedding as follows: Let $V := H^0(X, dL)$ denote the $(N+1)$-dimensional complex vector space (for $d \in \mathbb{Z}_+$) and consider the embedding of X into the projective space $\mathbb{P}(V)$. Fix a linear system $E \subset V$ of global sections of dL and a section $\tau \in E$ such that $\tau(P) \neq 0$ and consider the map

$$E \setminus \{0\} \to \mathbb{Z}^n, \ \sigma \mapsto v\Big(\frac{\sigma}{\tau}\Big).$$

Let $\mathcal{A} = \{v(\frac{\sigma}{\tau}) | \sigma \in E \setminus \{0\}\}$ and choose an orthonormal basis $\eta_0, \ldots, \eta_N$ of E such that for $i = 1, \ldots, N$ we can write $f_i = \frac{\eta_i}{\tau} = \sum_\alpha c_{\alpha,i} u^\alpha$, and furthermore

$$\Big\{(v\Big(\frac{\eta_i}{\tau}\Big), \ldots . v\Big(\frac{\eta_N}{\tau}\Big))\Big\} = \{(v(f_1), \ldots, v(f_N))\} = \mathcal{A}.$$

The existence of such a basis is detailed in [Kav19, Section 5]. Also assume that the pairwise distinct valuations $\beta_j = v(f_j)$ have differences $\beta_0 - \beta_1, \ldots, \beta_{N-1} - \beta_N$ generating $\mathbb{Z}^n$.

Under these conditions the choice of functions f_i and β guarantees the existence of $\gamma \neq 0 \in \mathbb{Z}^n$ such that $\langle \gamma, \beta_i \rangle < \langle \gamma, \alpha \rangle$ for every monomial u^α with $\beta_i < \alpha$ appearing in the Taylor series of $f_i = \frac{\eta_i}{\tau}$. For each $f_i = \sum_\alpha c_{\alpha,i} u^\alpha$ define a meromorphic function $\tilde{f}_i = t^{-\gamma \cdot \beta_i + a_i} f_i = t^{-\gamma \cdot \beta_i + a_i} \sum_\alpha c_\alpha u^\alpha \in X \times \mathbb{C}^*$ for some $a_i < 0, \ i = 0, \ldots, N$. Moreover let

$$\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_n) \text{ where each } \tilde{u}_j = t^{-\gamma_j} u_j.$$

The $\tilde{u}_i$ form a local coordinate system on an open neighbourhood of the central fiber $\mathcal{X}_{\gamma,0} \cong (\mathbb{C}^*)^n \times \{0\}$, such that for $a_i \geq 0$ we can write $\tilde{f}_i$ in this neighbourhood as

$$\tilde{f}_i = t^{\langle \gamma, \alpha \rangle \ - \ \langle \gamma, \beta_i \rangle + a_i} \sum_\alpha c_{\alpha,i} \tilde{u}^\alpha$$

which is regular on the central fiber $\mathcal{X}_{\gamma,0}$, thus extending $\tilde{f}_i$ to all of $\mathcal{X}_\gamma$.

Hence the map

$$F \colon \mathcal{X}_\gamma \to \mathbb{P}^N_{\mathbb{C}} \times \mathbb{C}$$

that sends (x, t) to $([\tilde{f}_0(x) : \ldots : \tilde{f}_N(x)], t)$ is a holomorphic map of $\mathcal{X}_\gamma$ into $\mathbb{P}^N_{\mathbb{C}} \times \mathbb{C}$.

Proposition 2.76. *Assume that $\beta_1 - \beta_0, \ldots, \beta_n - \beta_0$ generate $\mathbb{Z}^n$, and that $a_i = 0$ for $i = 1, \ldots, n$. Then the map F is an immersion, i.e. its derivative at every point has maximum rank.*

Proof. As above define $\tilde{f}_i = t^{-\gamma\cdot\beta+a_i} f = t^{-\gamma\cdot\beta+a_i} \sum_\alpha c_\alpha u^\alpha$. The case when $a_i = 0$ for all $i = 1, \ldots, N$ is proved in [Kav19, Thm. 6.2] so all we need to do is show that for $a_i > 0$, for $i > n$ Kaveh's proof runs through the same. This is the case because the maximality of the rank of the derivative can be derived from the $\beta \ldots, \beta_n$ where a_i is assumed to be 0. Note that by construction of the local coordinate system $\tilde{u}$ on a subset of $\tilde{U}_\gamma$ containing $\mathcal{X}_{\gamma,0}$, if $a_i > 0$ then a function $\tilde{f}_i|_{\mathcal{X}_{\gamma,0}} = 0$.

□

The result of this is that if we consider the Kähler form ω on $\mathbb{P}^N \times \mathbb{C}$ which is the product of the Fubini-Study form on $\mathbb{P}^N$ and ω_{std} on $\mathbb{C}$ then there exists a Kähler form on $\mathcal{X}$ given by

$$\tilde{\omega} = F^*\omega.$$

On the central fiber $\mathcal{X}_{\gamma,0}$ the Kähler form $\tilde{\omega}_0 = \tilde{\omega}|_{\mathcal{X}_{\gamma,0}}$ is a toric Kähler form i.e. a Kähler form generated by sections that are all stable under the action of $(\mathbb{C}^*)^n$, namely the leading monomials of the f_i.

The toric Kähler form $\tilde{\omega}_0$ has an associated moment map to the convex hull of the set of differences of $\beta_j - \beta_0$ and we can extend $\tilde{\omega}_0$ and its associated moment map to a Kähler form and moment map on the whole of $\mathcal{X}_\gamma$ using the gradient-Hamiltonian flow.

2.11 The gradient-Hamiltonian flow

Let $\pi \colon \mathcal{X} = X \times \mathbb{C} \to (\mathbb{C})^*$ denote an algebraic family and consider a Kähler form $\tilde{\omega}$ on $\mathcal{X}$ as defined in the previous section. Let g be the Riemann metric associated to $\tilde{\omega}$ given by $g(X, Y) = \tilde{\omega}(X, JY)$, for X and Y real vector fields. There exists a gradient-vector field denoted by $\nabla_\mathcal{X}\mathrm{Re}(\pi)$ satisfying

$$g(\nabla_\mathcal{X}\mathrm{Re}(\pi), X) = d(\mathrm{Re}(\pi))(X).$$

We can consider this gradient vector field $\nabla_\mathcal{X}\mathrm{Re}(\pi)$ as a Hamiltonian vector field $-\xi_{\mathrm{Im}(\pi)}$ associated to the function $\mathrm{Im}(\pi) \colon \mathcal{X} \to \mathbb{R}$ i.e.

$$\nabla_\mathcal{X}\mathrm{Re}(\pi) = -\xi_{\mathrm{Im}(\pi)}.$$

This equality is due to the Cauchy-Riemann equations $\bar{\partial}\pi = 0$, since

$$\tilde{\omega}(\xi_{\mathrm{Im}(\pi)}, X) = d(\mathrm{Im}(\pi))(X).$$

Let Z be the set where $\nabla_{\mathcal{X}}(\mathrm{Re}(\pi)) = 0$. Then we define the *gradient-Hamiltonian vector field* V_π *on* $\mathcal{X} \setminus Z$ by

$$V_\pi := -\frac{\nabla_{\mathcal{X}}(\mathrm{Re}(\pi))}{||\nabla_{\mathcal{X}}(\mathrm{Re}(\pi))||^2}.$$

It is easy to see that

$$V_\pi(\nabla_{\mathcal{X}}(\mathrm{Re}(\pi))) = -\frac{1}{||\nabla_{\mathcal{X}}(\mathrm{Re}(\pi))||^2}\langle \nabla_{\mathcal{X}}(\mathrm{Re}(\pi)), \nabla_{\mathcal{X}}(\mathrm{Re}(\pi))\rangle = -1$$

and furthermore since V_π is a gradient-Hamiltonian vector field (where defined) we can consider the time flow associated to V_π which we denote ϕ_t for $t \in \mathbb{R}_{>0}$. We call ϕ_t the gradient-Hamiltonian flow associated to the Kähler form $\tilde{\omega}$.

Proposition 2.77 ([Kav19, Prop 7.1])**.** *With notation as in Section 2.10:*

1. *Suppose $s, t \in \mathbb{R}$ with $s \geq t > 0$. Where defined the flow ϕ_t takes $X_s \cap (\mathcal{X}_\gamma \setminus Z)$ to X_{s-t}.*

2. *Where defined the flow ϕ_t preserves the symplectic structure, i.e. for a point $x \in X_s \cap (\mathcal{X} \setminus Z)$ where $\phi_t(x)$, then $\phi_t^*(\omega_{s-t})_{\phi_t(x)} = (\omega_s)_x$.*

Lemma 2.78 ([Kav19, Lem.8.2])**.** *On $\mathcal{X}_\gamma$ the gradient-Hamiltonian flow ϕ_t is defined for all $(\tilde{u}, 0) \in (\mathbb{C}^*)^n \times \{0\}$ and all $t \geq 0$.*

Proposition 2.79. *The gradient-Hamiltonian vector field on the product family $X \times \mathbb{C}$ for any Kähler manifold (X, ω) provided with the Kähler form $\tilde{\omega} = \omega \otimes \omega_{std}$ is given by $\frac{\partial}{\partial t}$.*

Proof. This is a consequence of the definition of the gradient-Hamiltonian flow using that the Riemannian metric associated to $\tilde{\omega}$ makes $\frac{d}{dt}$ at tangent vectors to the fibers X orthogonal (note that $d(\mathrm{Re}(\pi)) = dt$). $\square$

Proposition 2.80. *Let $\mathcal{Y} \subset \mathcal{X} \xrightarrow{\pi} \mathbb{A}^1_{\mathbb{C}}$ be a complex submanifold of the complex manifold $\mathcal{X}$ equipped with a Kähler form $\tilde{\omega}$. Where defined the gradient-Hamiltonian vector field $\nabla_{\mathcal{Y}}(Re(\pi))$ on $\mathcal{Y}$ associated to the restriction of $\tilde{\omega}_{\mathcal{Y}}$ is the symplectic projection of the gradient-Hamiltonian vector field $\nabla_{\mathcal{X}}(Re(\pi))$ of $(\mathcal{X}, \tilde{\omega})$ to the tangent bundle of $\mathcal{Y}$.*

Proof. The symplectic projection throws away the part of the vector field $\nabla_{\mathcal{X}}(\mathrm{Re}(\pi))$ perpendicular to the tangent space of the subvariety with respect to the associated Riemannian metric. The claim follows since for vector fields X tangent to $\mathcal{Y}$, we have $g(\nabla_{\mathcal{Y}}(\mathrm{Re}(\pi), X) = g(\nabla_{\mathcal{X}}(\mathrm{Re}(\pi), X) = d(\mathrm{Re}(\pi))(X)$, as desired. □

Theorem 2.81 ([Kav19, Thm.8.1])**.** *With notation as above, there exists an open (in the analytic topology) subset $U \subset X$ such that (U, ω) is symplectomorphic to $((\mathbb{C}^*)^n, \tilde{\omega}_0)$.*

Applying these results on the toric degeneration described in Section 2.10 allows to us deform a toric moment map in the central fiber with moment polytope the Newton-Okounkov body, to a moment map on an open subset of X, again with image equal to the Newton-Okounkov body.

2.12 Quasi-valuations

Later on we will deform valuations into weaker quasi-valuations. In this section we collect the basic facts on such quasi-valuations.

Definition 2.82. Let R be a $\mathbb{C}-$algebra and $(G, \geq)$ be a totally ordered abelian group which is well ordered for $\geq$. Then a map $v \colon R \setminus \{0\} \to G$ is a *quasi-valuation* if it satisfies:

1. $v(\lambda f) = v(f)$ for all $\lambda \in \mathbb{C}$ and $f \in R \setminus \{0\}$.
2. $v(f + g) \geq \min(v(f), v(g))$ for all $f, g \in R \setminus \{0\}$ with $f + g \in R \setminus \{0\}$.
3. $v(f \cdot g) \geq v(f) + v(g)$.

Definition 2.83. Let R be a $\mathbb{C}-$algebra and $(G, \geq)$ be a totally ordered abelian group which is well ordered for $\geq$. A *filtration* $(R_g)_{g \in G}$ of R is a chain of inclusions of $\mathbb{C}-$vector subspaces R_g such that for all $g, g', h \in G$:

1. $R_g \subset R$ is a $\mathbb{C}$-vector subspace of R.
2. $R_g \subset R_{g'}$ if $g' \leq g$.
3. $R_g \cdot R_h \subset R_{g+h}$.
4. $\bigcup_{g \in G} R_g = R$.

Proposition 2.84.

1. *Given a quasi-valuation* $v\colon R\setminus\{0\}\to G$ *the collection:*

$$\{R_g^v\}_{g\in G} \quad \textit{such that} \quad R_g^v=\{r\in R\colon v(r)\geq g\}$$

is a G-filtration of R.

2. *Given a* $G-$*filtration* $(R_g)_{g\in G}$ *of R the map*

$$v_{R_g}\colon R\setminus\{0\}\to G \textit{ such that } r\mapsto \ \textit{max}\{g\in G\colon r\in R_g\}$$

is a quasi-valuation.

3. $v_{R_g^v}=v$ *and* $(R_g^{v_{R_g}})_{g\in G}=R_g$*, for all quasi-G-valuations v and all* $G-$*filtration* $(R_g)_g\in G$*.*

Proof. First assume that there exists a quasi-valuation $v\colon R\setminus\{0\}\to G$ and define the $\mathbb{C}-$ vector space

$$R_g\coloneqq\{f\in R\setminus\{0\}\colon v(f)\geq g\}\cup\{0\}.$$

It is obvious that each R_g is a vector subspace of R and that $\bigcup_{g\in G}R_g=R$. Furthermore for $f,h\in R\setminus\{0\}$ with $f\neq f'$ by property 3) of Definition 2.82 we have that $v(f\cdot f')\geq v(f)+v(f')$ hence $R_g\cdot R_h\subset R_{g+h}$. Finally if $g\geq g'$ then

$$R_g=\{f\in R\setminus\{0\}\colon v(f)\geq g \ \text{ and } R_{g'}=\{f\in R\setminus\{0\}\colon v(f)\geq g'\}$$

hence we conclude that $R_g\subset R_g'$.
To show the converse now assume that there exists a filtration $(R_g)_{g\in G}$ of R and define a map

$$v\colon R\setminus\{0\}\to G\ ,\ f\mapsto v(f)=\max\{g\in G\colon f\in R_g\}.$$

For $\lambda\in\mathbb{C}$ we have $\lambda\cdot v(f)=\max\{g\in G\colon f\in R_g\}=v(f)$. If $f\in R_g$ and $f'\in R_{g'}$ with $R_g\subset R_{g'}$, then $v(f)\geq g'$ and

$$v(f+f')=\max\{g'\in G\colon f+f'\in R_{g'}\}\geq\min\{v(f),v(f')).$$

Finally since for $g,h\in G$ we have $R_g\cdot R_h\subset R_{g+h}$ it is easy to see that for $f\in R_g,f'\in R_h$ we have $v(f\cdot f')\geq v(f)+v(f')$. Hence v is a quasi-valuation in the sense of Definition 2.82.

All that remains is to show that the composites

- quasi-valuation → filtration → quasi-valuation
- filtration → quasi-valuation → filtration

are identities. To prove the first claim we see that

$$v_{R_g^v}(f) = \max\{g \in G \colon f \in R_g^v\} = \max\{g \in G \colon v(f) \geq g\} = v(f).$$

For the second claim we have

$$\begin{aligned} f \in R^{v_{R_g}} &\iff v_{R_g}(f) \geq g \\ &\iff \max\{h \colon f \in R_h^v\} \geq g \\ &\iff \max\{h \colon v(f) \geq h\} \geq g \\ &\iff v(f) \geq g \\ &\iff f \in R_g. \end{aligned}$$

□

Definition 2.85. Let $\tilde{v}_g \colon R_g \setminus \{0\} \to \mathbb{Z} \times \mathbb{Z}^n$ define a quasi-valuation and fix a total order on $\mathbb{Z} \times \mathbb{Z}^n$. There are vectors $u_1, u_2, \ldots, u_n$ such that v_g induces a quasi-filtration

$$0 \subset F_{u_0}^{R_g} \subset F_{u_1}^{R_g} \subset \ldots \subset F_{u_n}^{R_g} \subset F_{u_{n+1}}^{R_g} \subset R_g.$$

We say that the quasi valuation has *one dimensional leaves* if the associated quasi filtration satisfies

$$\dim(F_{u_{i+1}}^{R_g}/F_{u_i}^{R_g}) = 1.$$

Chapter 3

Kähler packings of projective complex manifolds

Motivated by work of Biran [Bir01], who proved a correspondence between a symplectic analogue of Nagata's conjecture and symplectic packing problems, Eckl proved in [Eck17] that Nagata's conjecture is in fact equivalent to a more restricted packing problem, namely a Kähler packing problem. More generally in dimension 2 there is a direct correspondence between sizes of Kähler packings and multipoint Seshadri constants. A similar result was obtained by David Witt-Nyström in [WN15] but this time for a variety of any dimension blown up at a single point. The aim of this section is to generalise these results to projective complex manifolds of arbitrary dimension blown up at any number of points. During the writing of this thesis a similar result was achieved by Trusiani [Tru18, Cor.5.2] however there are some differences in the formulation of the statement and its proof.

For completeness we briefly dicuss Trusiani's result, however for full details refer to [Tru18]. Trusiani extends an earlier definition of Witt-Nyström in [WN15] and defines torus invariant domains of $\mathbb{C}^n$ called multipoint Okounkov domains (see [Tru18, Definition 4.1]). When X is a projective manifold and L an ample line bundle Trusiani proves that the collection of all these Okounkov domains each equipped with the standard Euclidean form on $\mathbb{C}^n$ packs into (X, L) (see [Tru18, Theorem C] for precise statements). A similar result for big line bundles is also proved in the same paper. Using these ideas Trusiani proves the following:

Theorem 3.1 ([Tru18, Corollary 5.17])**.** *Let X be a complex manifold of dimension n and*

L a big line bundle on X. Then

$$\epsilon(X, ||L||; P_1, \ldots, P_k) = \max\{0, \sup\{r > 0 \colon B_r(0) \subset D_j(L) \text{ for any } j = 1, \ldots, N\}\}$$

where $\epsilon(X, ||L||; P_1, \ldots, P_k)$ denotes the multipoint moving Seshadri constant and $D_j(L)$ denotes the Okounkov domain associated to the point P_j.

The main difference in our construction is that we use an iterative method to obtain our multiball Kähler packing. Making use of iterative blow ups of a projective complex manifold we are able to obtain a k-ball packing where for each ball we only have to ensure that the Kähler form is smooth enough around the point that we are blowing up. Trusiani describes a method that constructs Kähler metrics using sections of dL (for some suitable integer $d > 0$) that, locally around the points corresponding to the centres of the embedded balls approximates the Fubuni-Study metric after some appropriate scaling. In Trusiani's method it is important to ensure the resulting Kähler form is smooth around all the points simultaneously. The advantages of our iterative approach will become clearer in Chapter 4 as it will be possible to discuss the embedding of each ball separately, for example constructing a moment map on them.

In this chapter we prove Theorem A which for the readers convenience we restate here.

Theorem A. *Let X be a projective complex manifold of dimension n and L an ample line bundle on X Fix a collection of points $P_1, \ldots, P_k$ and a Kähler form ω on X such that $[\omega] \in c_1(L)$, then the square of the k-ball packing constant is equal to the k-point Seshadri constant i.e.*

$$\gamma_k(X, \omega; P_1, \ldots, P_k) = \sqrt{\epsilon(X, L; P_1, \ldots, P_k)}.$$

3.1 Degeneration of projective complex manifolds to multi-point blow up.

Let X be a projective complex manifold of dimension n equipped with an ample line bundle L. Fix points $P_1, \ldots, P_k$ on X and let $\pi \colon Bl_{P_1,\ldots,P_k}(X) \to X$ be the blow up of X, where $E_i = \pi^{-1}(P_i)$ denotes the exceptional divisor corresponding to P_i. Consider the product manifold

$$\begin{tikzcd} \mathcal{X} := X \times \mathbb{A}^k_{\mathbb{C}} \arrow[r, "p"] \arrow[d, "q"] & X \\ \mathbb{A}^k_{\mathbb{C}} & \end{tikzcd}$$

If $t_1, \ldots, t_k$ represent coordinates of $\mathbb{A}^k_{\mathbb{C}}$ we can consider the centres $Z_i := \{P_i\} \times \{t_i = 0\}$ as coordinate hyperplanes cut out by $t_i = 0$ over the points P_i.

Remark 3.2. Since the points P_i are distinct we have that $Z_i \cap Z_j = \emptyset$ for all $1 \leq i < j \leq k$.

By blowing up the family $\mathcal{X}$ over the union of all the centres defined above we obtain a new algebraic family

$$\begin{tikzcd} \tilde{\mathcal{X}} := Bl_{\cup_{i=1}^k Z_i}(\mathcal{X}) \arrow[r, "\Pi"] \arrow[dr, "(q \circ \Pi)"'] & \mathcal{X} \arrow[r, "p"] \arrow[d, "q"] & X \\ & \mathbb{A}^k_{\mathbb{C}} & \end{tikzcd}$$

such that the exceptional divisor corresponding to the centre Z_i is $\mathcal{E}_i := \Pi^{-1}(Z_i)$.

Remark 3.3. 1. $\tilde{\mathcal{X}}$ is a well defined projective, complex manifold over $\mathbb{A}^k_{\mathbb{C}}$.

2. $\mathcal{E}_i \cong Z_i \times \mathbb{P}^n_{\mathbb{C}}$.

3. $(q \circ \Pi)^{-1}\big((t_1, \ldots, t_k)\big)$ is the blowup of X in the points P_i where $t_i = 0$.

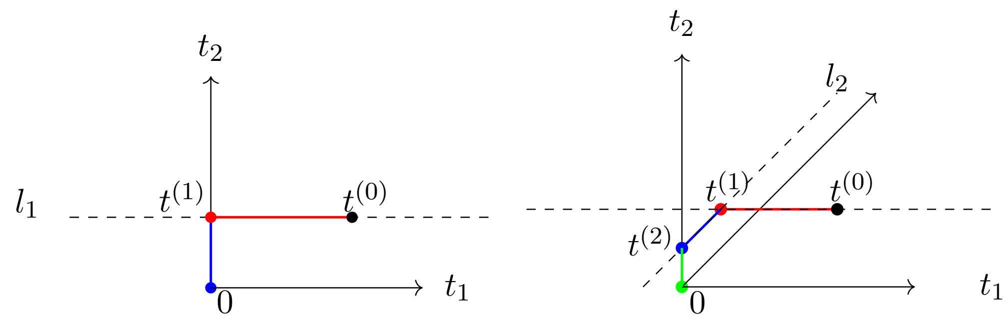
To describe the iterative blow ups of one point after the other, we choose a path through the parameter space $\mathbb{A}^k_{\mathbb{C}}$. Let $\delta_1, \ldots, \delta_k$ be positive real numbers and define a line in $\mathbb{A}^k_{\mathbb{C}}$ by

$$l_i := \{(0, \ldots, 0, t \cdot \delta_i, \delta_{i+1}, \ldots, \delta_k) \colon t \in \mathbb{R}\}$$

such that the first $(i-1)$ coordinates are zero, the i-th coordinate is equal $t \cdot \delta_i$ and the remaining $(i+1)$ coordinates are unchanged. See Figure 3.1 for a visualisation of the path. Let $t^{(i)} = (0, \ldots, 0, \delta_i, \delta_{i+1}, \ldots, \delta_k)$ be a point on l_i, then the preimage of $t^{(i)}$, for $i = 1, \ldots, k$ is

$$(q \circ \Pi)^{-1}(t^{(i)}) = Bl_{P_1, \ldots, P_{i-1}}(X) \cup \left(\bigcup_{j=1}^{i-1} \mathcal{E}_j \cap (q \circ \Pi)^{-1}(t^{(i)}) \right)$$

where $\mathcal{E}_j = \Pi^{-1}(Z_j)$. In this way we trace a path through the parameter space and we see that the preimage of $t^{(0)}$ is simply X, the preimage of $t^{(1)}$ is simply the blow up of X at P_1 with some contributions from the exceptional divisor and so on until we get that the preimage of $t^{(k)} = (0, \ldots, 0)$ is the blow up of X at $P_1, \ldots, P_k$.



(a) Parameter space for k=2

(b) Parameter space for k=3

Figure 3.1

Similarly for $l_i \in \mathbb{A}^n_{\mathbb{C}}$ we have

$$(q \circ \Pi)^{-1}(l_i) = \Big(\bigcup_{j=1}^{i-1} \mathcal{E}_j \cap (q \circ \Pi)^{-1}(l_i) \Big) \cup \tilde{\mathcal{X}}_i = \bigcup_{j=1}^{i-1} \mathcal{E}_j^{(j)} \cup \tilde{\mathcal{X}}_i,$$

where $\mathcal{E}_j^{(j)} = \mathbb{P}^n_{\mathbb{C}}$ and $\tilde{\mathcal{X}}_i$ is the blow up of $\tilde{\mathcal{X}}_{i-1}$ in the point $(P_i, 0)$:

$$\begin{tikzcd}
\tilde{\mathcal{X}}_i \arrow[r, "\Pi_i"] \arrow[rr, bend left, "\Pi^{(i)}"] & \mathcal{X}_i = \tilde{X}_{i-1} \times l_i \arrow[r, "q_{i-1}"] \arrow[d, "p_{i-1}"] & l_i \cong \mathbb{A}^1_{\mathbb{C}} \\
& \tilde{X}_{i-1} &
\end{tikzcd}$$

On $\tilde{\mathcal{X}}_i$ there exists a family of divisors

$$\tilde{\mathcal{L}}^{(i)}_{d,m_1,\ldots,m_i} := \Pi_i^* p_{i-1}^* \tilde{L}^{i-1}_{d;m_1,\ldots,m_i} - m_i \mathcal{E}_i^{(i)}$$

where $\mathcal{E}_i^{(i)} \cong \mathbb{P}^n_{\mathbb{C}}$ is the exceptional divisor of the blow up Π_i.

If we want to consider the specific divisor associated to a particular fiber we use the notation $\mathcal{L}^{(i)}_{d,m,t}$, where t denotes the parameter over $\mathbb{A}^1_{\mathbb{C}}$ and $d, \underline{m} = (m_1, \ldots, m_i)$ record the degree and multiplicity (we will often just write $\tilde{\mathcal{L}}^{(i)}_t$ for short if the multiplicity and degree are fixed).

Remark 3.4. $\tilde{\mathcal{X}}_{i,0} = \tilde{X}_i \cup \mathcal{E}^{(i)}_i$, where $\mathcal{E}^{(i)}_i \cong \mathbb{P}^n_{\mathbb{C}}$ and $\tilde{\mathcal{X}}_{i,t} = \Pi^{(i)\ -1}(t)$ is the fiber over $l_i \cong \mathbb{A}^1_{\mathbb{C}}$.

Now that we have defined a algebraic family $\tilde{\mathcal{X}}_i$ and a family of divisors $\tilde{\mathcal{L}}^{(i)} = \tilde{\mathcal{L}}^{(i)}_{d;\underline{m}}$ we would like to construct sections of $\tilde{\mathcal{L}}^{(i)}$ that restrict to a basis of the global sections of $\tilde{\mathcal{L}}^{(i)}$ restricted to each fiber of $\Pi^{(i)}$ and to particular nice sections on the exceptional divisor $\mathcal{E}^{(i)}_i$. These sections can be used to produce a nice family of Kähler forms on the family $\tilde{\mathcal{X}}_i$.

Theorem 3.5.
For all $i = 1, \ldots, n$ there exists global sections $\sigma^{(i)}_0, \ldots, \sigma^{(i)}_{N_i}$ of $\Pi^{(i)}_* \tilde{\mathcal{L}}^{(i)}$ such that:

1. *$\sigma^{(i)}_0, \ldots, \sigma^{(i)}_{N_i}$ trivialise $\Pi^{(i)}_* \tilde{\mathcal{L}}^{(i)}$, in particular, $\Pi^{(i)}_* \mathcal{L}^{(i)}$ is a vector bundle over $\mathbb{A}^1_{\mathbb{C}}$.*

2. *If 0_i denotes the zero of the line l_i then the sections*

$$\sigma^{(i)}_{0,0_i}|_{\mathcal{E}^{(i)}_i}, \ldots, \sigma^{(i)}_{N_i,0_i}|_{\mathcal{E}^{(i)}_i}$$

   generate the Kähler form $m_i \cdot \omega_{FS}$ on $\mathcal{E}^{(i)}_i \cong \mathbb{P}^n_{\mathbb{C}}$.

3. *The restricted sections $\sigma^{(i)}_{0,\delta_i}, \ldots, \sigma^{(i)}_{N_i,\delta_i}$ on $\tilde{\mathcal{X}}_{i,\delta_i} \cong \tilde{X}_{i-1}$ coincide with the sections $\sigma^{(i-1)}_{0,0_{i-1}}|_{\tilde{X}_{i-1}}, \ldots, \sigma^{(i-1)}_{N_i,0_{i-1}}|_{\tilde{X}_{i-1}}$ from the previous family $\tilde{\mathcal{X}}_{i-1}$.*

Proof. We first show that the dimension of the space of global sections of $\tilde{\mathcal{L}}^{(i)}$ restricted to each fiber of $\Pi^{(i)}$ is the same. By Grauert's semi-continuity theorem [Har77, Thm.12.8] that implies immediately that $\Pi^{(i)}_* \tilde{\mathcal{L}}^{(i)}$ is locally free, hence it is free by the Quillen-Suslin Theorem [Lan02, Thm.3.7]. Then we describe the trivialising sections directly and prove the properties of the theorem.
We start by calculating a basis of sections of $H^0(X, \mathcal{O}_X(dL))$ characterised by their vanishing behaviour at the points $P_1, \ldots, P_k$ so that subsets of this basis can be interpreted

as bases for $H^0(\tilde{X}, \tilde{L}^{(i)})$. Let $\mathfrak{m}_{X,P_i}$ denote the maximal ideal of X at P_i and consider the short exact sequence

$$0 \to \bigcap_{i=1}^{k} \mathfrak{m}_{X,P_i}^{m_i+1} \otimes \mathcal{O}_X(dL) \to \mathcal{O}_X(dL) \to \bigoplus_{i=1}^{k} \mathcal{O}_X/\mathfrak{m}_{X_i,P_i}^{m_i+1} \to 0.$$

Taking the long exact sequence of cohomology with respect to the above short exact sequence gives

$$\begin{array}{l} 0 \to H^0(X, \bigcap_{i=1}^k \mathfrak{m}_{X,P_i}^{m_i+1} \otimes \mathcal{O}_X(dL)) \to H^0(X, O_X(dL)) \to H^0(X, \bigoplus_{i=1}^k \mathcal{O}_X/\mathfrak{m}_{X,P_i}^{m_i+1}) \\ \to H^1(X, \bigcap_{i=1}^k \mathfrak{m}_{X,P_i}^{m_i+1} \otimes \mathcal{O}_X(dL)) \to H^1(X, \mathcal{O}_X(dL))) \longrightarrow \ldots\ldots\ldots\ldots\ldots \end{array}$$

The claim is then that all the $H^1(X, \bigcap_{i=1}^k \mathfrak{m}_{X,P_i}^{m_i+1} \otimes \mathcal{O}_X(dL))$ vanishes: Note that the projection formula yields

$$H^1(X, \bigcap_{i=1}^{k} \mathfrak{m}_{X,P_i}^{m_i+1} \otimes \mathcal{O}_X(dL)) = H^1(\tilde{X}, \mathcal{O}_X(\tilde{L}_{d,m_i})).$$

Since we assume $\tilde{L}_{d,m_i}$ is ample and $d, m_i >> 0$ Serre vanishing applies and $\phi\colon H^0(X, \mathcal{O}_X(dL)) \to \bigoplus_{i=1}^k \mathcal{O}_X/\mathfrak{m}_{X,P_i}^{m_i+1}$ is surjective. This gives a basis of sections of $H^0(X, \mathcal{O}_X(dL))$ which is the union of several sets:

- A basis of $H^0(X, \bigcap_{i=1}^k \mathfrak{m}^{m_i+1} \otimes \mathcal{O}_X(dL))$, which are sections of $\mathcal{O}_X(dL)$ vanishing to multiplicity at least $m_i + 1$ in P_i for $i = 1, \ldots, k$.
- A set B_i of sections of $\mathcal{O}_X(dL)$ mapped to 0 in $\mathcal{O}_X/\mathfrak{m}_{X,P_j}^{m_j+1}$ for $j \neq i$ and to a basis of homogeneous polynomials of degree $\leq m_i$ in $\mathcal{O}_X/\mathfrak{m}_{X,P_i}^{m_i+1}$, in variables given by local coordinates around P_i.
- A set $\tilde{B}_i$ of sections of $\mathcal{O}_X(dL)$ as above, but with homogeneous polynomials of degree $= m_i$. Then, $\tilde{B}_i \subset B_i$.

Hence there exists a basis of $H^0(\tilde{X}_i, \mathcal{O}_{\tilde{X}_i}(\tilde{L}^{(i)}))$ given by

$$B^{(i)} := B_0 \cup \bigcup_{j=1}^{i} \tilde{B}_j \cup \bigcup_{j=i+1}^{k} B_j.$$

To change between a basis of X_{i-1} and X_i we simply skip those sections which vanish to order less than m_i at P_i.
Since $\mathcal{X}_i = \tilde{X}_{i-1} \times \mathbb{A}^1_{\mathbb{C}}$ we can use these sections to construct the sections of $H^0(\tilde{\mathcal{X}}_i, \tilde{\mathcal{L}}^{(i)})$ then analyse these sections restricted to different fibers. This is done using the following procedure:

Step 1: Pull back a section $s \in H^0(\tilde{X}_{i-1}, \mathcal{O}_{X_{i-1}}(\tilde{L}^{(i-1)}))$ along p_{i-1} to get a section of $p^*_{i-1}\tilde{L}^{(i-1)} = \mathcal{L}^{(i-1)}$ on $\mathcal{X}_i = \tilde{X}_{i-1} \times \mathbb{A}^1_{\mathbb{C}}$.

Step 2: If $\mathrm{mult}_{P_i} s < m_i$ then $t^{m_i - \text{ mult } _{P_i} s} p^*_{i-1} s$ has multiplicity m_i in $(P_i, 0)$. If $\mathrm{mult}_{P_i} s \geq m_i$ then $\mathrm{mult}_{(P_i,0)} p^*_{i-1} s \geq m_i$.

Step 3: In both cases, we can subtract m_i copies of the exceptional divisor $\mathcal{E}_i$ from the pullback of this section along Π_i, thus obtaining sections of $H^0(\tilde{\mathcal{X}}_i, \tilde{\mathcal{L}}^{(i)})$.

When starting with the basis sections in $B^{(i-1)}$, restricting to the general fibers $\tilde{\mathcal{X}}_{i,t} \cong \tilde{X}_{i-1}$ of $\Pi^{(i)}$ (that is $t \neq 0$) we get back the sections in $B^{(i-1)}$ possibly multiplied with some power of t. Thus all the sections $\sigma^{(i)}_1, \dots \sigma^{(i)}_{N_i}$ obtained from $B^{(i-1)}$ in the way described above are trivializing $\Pi^{(i)}_* \tilde{\mathcal{L}}^{(i)}$ outside the central fiber. To understand the restriction of these sections to the central fiber we need to analyse $\tilde{\mathcal{L}}^{(i)}|_{\tilde{\mathcal{X}}_{i,0}}$ and its global sections. Note, $\tilde{\mathcal{X}}_{i,0} = \tilde{X}_i \cup \mathcal{E}^{(i)}_i$ and $\tilde{X}_i \cap \mathcal{E}^{(i)}_i = E_i$. Furthermore $\tilde{\mathcal{L}}^{(i)}|_{\tilde{X}_i} = \tilde{L}^{(i)}$, and $\tilde{\mathcal{L}}^{(i)}|_{\mathcal{E}^{(i)}_i} = \mathcal{O}_{\mathcal{E}^{(i)}_i}(-m_i \mathcal{E}^{(i)}_i) \cong \mathcal{O}_{\mathbb{P}^n}(m_i)$ via the isomorphism $\mathcal{E}^{(i)}_i \cong \mathbb{P}^n_{\mathbb{C}}$. This means that global sections of $\tilde{\mathcal{L}}^{(i)}|_{\tilde{\mathcal{X}}_{i,0}}$ consist of a section of $\mathcal{L}^{(i)}|_{\tilde{X}_i}$ and a section of $\mathcal{L}^{(i)}|_{\mathcal{E}^{(i)}_i}$ coinciding when restricted to E_i.

- Global sections of $\tilde{\mathcal{L}}^{(i)}|_{\tilde{X}_i}$ and their restriction to E_i:
  These are all sections of dL on X which vanish to multiplicity greater or equal to m_j in P_j for $j = 1, \dots, i$. Equivalently these are global sections of $\tilde{L}^{(i-1)}$ on $\tilde{X}_{i-1}$ vanishing with multiplicity $\geq m_i$ in P_i. To obtain a section of $\tilde{L}^{(i)}$ from a section of $\tilde{L}^{(i-1)}$ we pull back the section along the map $\pi^{(i-1)}\colon \tilde{X}_i \to \tilde{X}_{i-1}$ then subtract m_i copies of the exceptional divisor. If a section of $\tilde{\mathcal{L}}^{(i)}|_{\tilde{X}_i} = \tilde{L}^{(i)}$ corresponds to a section of $\tilde{L}^{(i-1)}$ with multiplicity greater than m_i at P_i the restriction of this section to E_i is zero. If the multiplicity is exactly m_i at P_i then $\tilde{\mathcal{L}}^{(i)}|_{E_i} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(m_i)$ shows that the restriction of such a section of $\tilde{\mathcal{L}}^{(i)}$ to $E_i \cong \mathbb{P}^n_{\mathbb{C}}$ will be described by a non-zero homogeneous polynomials of degree m_i in homogeneous coordinates $Y_1, \dots, Y_n$ on $\mathcal{E}_i$ (see below).

- Global sections of $\tilde{\mathcal{L}}^{(i)}|_{\mathcal{E}_i^{(i)}}$ and their restriction to E_i:

  To describe these we introduce homogeneous coordinates $[T : Y_1 : \ldots : Y_n]$ on $\mathcal{E}_i^{(i)}$. The coordinates $Y_1, \ldots, Y_n$ come from local coordinates $y_1, \ldots, y_n$ around P_i, the T coordinate comes from the affine base parameter t, and $T = 0$ describes $E_i \subset \mathcal{E}_i^{(i)}$. Hence the restriction of a section of $\tilde{\mathcal{L}}^{(i)}|_{\mathcal{E}_i^{(i)}}$ to E_i is obtained by setting $T = 0$. Then sections of $\tilde{\mathcal{L}}^{(i)}|_{\mathcal{E}_i^{(i)}}$ are non-zero homogeneous polynomials of degree m_i on $\mathbb{P}^n$ in $T, Y_1, \ldots, Y_n$.

A basis of global sections of $\tilde{\mathcal{L}}^{(i)}|_{\tilde{\mathcal{X}}_{i,0}}$ can therefore be built as the union of

- pairs of basis sections on $\tilde{X}_i$ vanishing on E_i and the zero section of $\mathcal{O}_{\mathbb{P}^n}(m)$ on $\mathcal{E}_i^{(i)}$,
- pairs of the zero section on $\tilde{X}_i$ and a basis section of $\mathcal{O}_{\mathbb{P}^n}(m)$ on $\mathcal{E}_i^{(i)}$ vanishing on E_i and
- pairs of basis sections of $\tilde{X}_i$ and $\mathcal{E}_i^{(i)}$ that restrict to the same non-zero homogeneous polynomial of degree m_i on E_i.

Now we want to show that the restriction of the $\sigma_1^{(i)} \ldots \sigma_{N_i}^{(i)}$ to $\tilde{\mathcal{X}}_{i,0}$ yields a basis of $\tilde{\mathcal{L}}^{(i)}|_{\tilde{\mathcal{X}}_{i,0}}$. To this purpose, we follow steps 1-3 to construct $\sigma_j^{(i)}$ from a section of $\tilde{L}^{(i-1)}$ and then identify the pair of sections describing the restriction to $\tilde{\mathcal{X}}_{i,0}$. If a section of $\tilde{L}^{(i-1)}$ vanishes to multiplicity strictly greater than m_i at p_i then it corresponds to a pair of sections on $\tilde{\mathcal{L}}_0^{(i)}$ consisting of a section of $\tilde{\mathcal{L}}^{(i)}$ that restricts to zero on E_i, and the zero section on $\mathcal{E}_i^{(i)}$. If the section of $\tilde{L}^{(i-1)}$ on $\tilde{X}_{i-1}$ vanishes to exactly multiplicity m_i at P_i then pulling back and subtracting m_i copies of the exceptional divisor we obtain non-zero sections on $\tilde{X}_i$ and on $\mathcal{E}_i^{(i)}$. Such a section of $\tilde{L}^{(i-1)}$ corresponds to a pair of sections on $\tilde{\mathcal{X}}_{i,0}$ that restrict to the same non-zero homogeneous monomial on E_i. Finally if a section s of $\tilde{L}^{(i-1)}$ on $\tilde{X}_{i-1}$ vanishes to multiplicity strictly less than m_i at P_i then after pulling back along p_{i-1} we must multiply by $t^{m_i - \text{ mult }_{P_i} s}$ before we can subtract copies of the exceptional divisor. This type of section corresponds to a pair of sections on $\tilde{\mathcal{X}}_{i,0}$ consisting of the zero section on $\tilde{X}_i$ and a section on $\mathcal{E}_i^{(i)}$ that restricts to zero on E_i.
In terms of the sets B_0, B_j and $\tilde{B}_j$, making up the basis $B^{(i)}$ of $\tilde{L}^{(i)}$ we find that a basis of global sections of $\tilde{L}^{(i-1)}$ vanishing to multiplicity $> m_i$ at P_i can be written as $B_0 \cup \bigcup_{j=1}^{i-1} \tilde{B}_j \cup \bigcup_{i+1}^{k} B_j$. A basis of global sections vanishing to multiplicity exactly m_i at P_i is $\tilde{B}_i$ and finally a basis of global sections vanishing to multiplicity less than m_i at P_i is

given by $B_i - \tilde{B}_i$. The union of all three bases provides a basis $B^{(i-1)}$ for $\tilde{\mathcal{X}}_{i,t} \cong \tilde{X}_{i-1}$. The union of the basis corresponding to all sections of $\tilde{L}^{(i-1)}$ vanishing to multiplicity less than or equal to m_i at P_i is isomorphic to a basis of global sections of $\tilde{\mathcal{L}}^{(i)}|_{\mathcal{E}_i^{(i)}}$ where the identification is given by the correspondence between homogeneous polynomials in local coordinates around P_i and homogeneous coordinates of $\mathcal{E}_i^{(i)} \cong \mathbb{P}^n_{\mathbb{C}}$.

Thus, the sections $\sigma_0^{(i)}, \ldots, \sigma_{N_i}^{(i)}$ also restrict to a basis of $\tilde{\mathcal{L}}^{(i)}|_{\tilde{\mathcal{X}}_{i,0}}$, hence trivialize $\Pi_*^{(i)}\tilde{\mathcal{L}}^{(i)}$ over all fibers of the family.

Now let us construct a Kähler form using the trivializing sections of $\tilde{\mathcal{L}}^{(i)}$ restricted to $\mathcal{E}_i^{(i)} \cong \mathbb{P}^n_{\mathbb{C}}$. On this exceptional divisor we can choose homogeneous coordinates $T, Y_1, \ldots Y_n$ as above. Then

$$\begin{aligned} m_i \cdot \omega_{FS} &= \frac{i}{2\pi}\partial\bar{\partial} m_i \cdot log(Y_1\bar{Y}_1 + \ldots + Y_n\bar{Y}_n + T\bar{T}) \\ &= \frac{i}{2\pi}\partial\bar{\partial} log(Y_1\bar{Y}_1 + \ldots + Y_n\bar{Y}_n + T\bar{T})^{m_i} \\ &= \frac{i}{2\pi}\partial\bar{\partial} log(Y_1^{m_i}\bar{Y}_1^{\,m_i} + m_i Y_1^{m_i-1}\bar{Y}_1^{m_i-1} Y_2\bar{Y}_2 + \ldots) \\ &= \frac{i}{2\pi}\partial\bar{\partial} log(\sum_{\alpha+\beta=m_i} c_{\alpha,\beta} Y^\alpha \bar{Y}^\alpha T^\beta \bar{T}^\beta) = m_i \cdot \omega_{FS} \end{aligned}$$

where $c_{|\alpha|,\beta}$ are positive integers.

If we choose the sections $\sigma_j^{(i)}$ that do not vanish on the exceptional divisor (i.e. coming from sections of $\tilde{L}^{(i-1)}$ vanishing with multiplicity $\leq m_i$ in P_i) such that they restrict to the basis of monomials $\sqrt{c_{\alpha,\beta}} Y^\alpha T^\beta$ of the homogeneous polynomials of degree m_i, then $m_i \cdot \omega_{FS}$ is the Kähler form on $\mathcal{E}^{(i)}$ generated by the restriction of these sections $\sigma_j^{(i)}$. Note that $c_{\alpha,\beta}$ is a positive integer, so $\sqrt{c_{\alpha,\beta}}$ is just the usual real square root.

To achieve property (3) we have to construct the trivializing sections of $\tilde{\mathcal{L}}^{(i)}$ on $\tilde{\mathcal{X}}_i$ iteratively, starting with $i = k$. On $\tilde{\mathcal{X}}_k$ we construct sections $\sigma_1^{(k)}, \ldots, \sigma_{N_k}^{(k)}$ as above, which restricted to $\tilde{\mathcal{X}}_{k,\delta_k}$ provides a basis of sections of $\tilde{L}^{(k-1)}$ on $\tilde{X}_{k-2}$ vanishing with multiplicity $\geq m_{k-1}$ in P_{k-1}. We can complete these sections to a basis of all sections of $\tilde{L}^{(k-2)}$ on $\tilde{X}_{k-2}$ by adding sections which vanish with multiplicity $< m_{k-1}$ in P_{k-1}. This basis can be used to construct trivializing sections of $\tilde{\mathcal{L}}^{(i)}$ on $\tilde{\mathcal{X}}_{k-1}$ as above, because by its construction it can be split up into the subsets $B_0, B_i, \tilde{B}_i$ and $B^{(i)}$. Iterating this process for $\tilde{\mathcal{X}}_{k-2}, \ldots, \tilde{\mathcal{X}}_1$ we deduce property (3) for each $i = 1 \ldots, k$. $\square$

3.2 Main result

Theorem 3.6. *Let X be a projective complex manifold of dimension n and L an ample line bundle on X. Fix points $P_1, \dots, P_k \in X$ and denote $\epsilon_0 = \epsilon(X; L, P_1, \dots, P_k)$ the multipoint Seshadri constant of L on X in $P_1, \dots, P_k$. Then, for any radius $r < \sqrt{\epsilon_0}$ there exists a Kähler packing of k flat Kähler balls of radius r into X.*

Proof. First we construct families $\tilde{\mathcal{X}}_i$ as in the previous section. Then we embed Fubini-Study Kähler balls of large enough volume on $\mathcal{E}_i^{(i)} - E_i$ provided with the Kähler metric $\omega_i^{(i)}$ induced by the global sections of $\tilde{\mathcal{L}}^{(i)}$ constructed in Theorem 3.5 on $\mathcal{E}_i^{(i)}$, for $i = k, k-1, \dots, 1$. These balls can be deformed to Kähler balls on non-central fibers $\tilde{\mathcal{X}}_{i,\delta_i} \cong \tilde{X}_{i-1}$, and then iteratively to non-central fibers $\tilde{\mathcal{X}}_{j,\delta_j} \cong \tilde{X}_{j-1}$ for $j = i-1, \dots, 1$ if the δ_j are chosen small enough. Doing this carefully the deformed balls will not intersect on $\tilde{\mathcal{X}}_{i,\delta_i} \cong X$, so after gluing in standard Kähler balls into the Fubini-Study Kähler balls we obtain the claim. In more details:

1. Let $\Delta_\delta \subset \mathbb{A}^1_{\mathbb{C}}$ denote the open disk of radius δ, with affine parameter t. Choose local coordinates $y_1, \dots, y_n$ of X around P_i, so $t, y_1, \dots, y_n$ are local coordinates around $(0, P_i)$ in $\tilde{\mathcal{X}}_i$. Then over the open subset $\mathcal{U}_t \subset \tilde{\mathcal{X}}_i$ where these coordinates are defined there is a chart of the blow up of $\tilde{\mathcal{X}}_i$ in $(0, P_i)$ with coordinates $t, z_1, \dots, z_n$ such that the blow up map to $\mathcal{U}_t$ is described by $(t, z_1, \dots, z_n) \mapsto (t, tz_1, \dots, tz_n) = (t, y_1, \dots, y_n)$. The central fiber of the induced projection of $\mathcal{U}_t$ onto $\mathbb{A}^1_{\mathbb{C}}$ is $\mathcal{E} - E \cong \mathbb{A}^n_{\mathbb{C}}$. The non-central fibers are not quite isomorphic to $\mathbb{A}^n_{\mathbb{C}}$ but contain balls $B_R(0)$ with R arbitrarily large close to $t = 0$ (see Figure 3.2). Thus for R arbitrarily large we can find δ sufficiently small and an embedding $\iota\colon \Delta_\delta \times B_R(0) \hookrightarrow \tilde{\mathcal{X}}_i$.

2. Choosing in (1) R large enough and δ small enough implies that for $R' < R$ there exists embeddings of Fubini-Study Kähler balls $B_{R'}(0)$ of volume arbitrarily close to the volume of $(\mathcal{E}_i^{(i)}, \omega_i^{(i)})$ in all fibers of $\Delta_\delta \times B_R(0)$ over $t \in \Delta_\delta$ with respect to the same Fubini-Study Kähler form $\omega_i^{(i)}$.

3. By continuity, for t small enough these Fubini-Study forms $\omega_i^{(i)}$ differ by an arbitrarily small amount from the Kähler form $\omega_{i,t}$ on $\tilde{\mathcal{X}}_{i,t}$ obtained from the trivializing sections of $\tilde{\mathcal{L}}^{(i)}$ pulled back via the embedding ι_i. This allows us to glue in the Fubini-Study Kähler balls of step (2) into non-central fibers $\tilde{\mathcal{X}}_{i,t}$ provided with the Kähler form $\omega_{i,t}$ for $t \ll 1$ small enough. Assume that $\omega_i^{(i)} = \frac{i}{2\pi}\partial\bar{\partial} log(s_i^{(i)})$ and $\omega_{i,t} = \frac{i}{2\pi}\partial\bar{\partial} log s_{i,t}$

on $\{t\} \times B_R(0)$, where $s_i^{(i)}, s_{i,t}$ are functions constructed in the usual way from the appropriate sections. Then choose a partition of unity (ρ_1, ρ_2) such that $\rho_2|_{B_{R'}(0)} \equiv 1$ and $\rho_2|_{B_{R'}(0)} \equiv 0$. The glued 2-form $\tilde{\omega}_{i,t}$ is given by $\frac{i}{2\pi}\partial\bar{\partial}log(\rho_1 s_i^{(i)} + \rho_2 s_{i,t}) = \frac{i}{2\pi}\partial\bar{\partial}log(\rho_1(s_i^{(i)} - s_{i,t}) + s_{i,t})$. This form is obviously closed, and it is non-degenerate because $s_i^{(i)} - s_{i,t}$ gets arbitrarily small for $t \ll 0$, thus $\rho_1(s_i^{(i)} - s_{i,t}) + s_{i,t}$ is arbitrarily close to $s_{i,t}$. Consequently $\tilde{\omega}_{i,t}$ is a Kähler form.

4. The i^{th} Fubini-Study Kähler ball on $\tilde{\mathcal{X}}_{i,\delta_i}$ does not intersect the $(i+1)^{st}, \ldots, k^{th}$ Kähler ball constructed before: On the central fiber $\tilde{\mathcal{X}}_{i,0}$ these balls lie on $\mathcal{E}_i^{(i)} - E_i$ and $\tilde{X}_i - E_i \cong \tilde{X}_{i-1} - P_i$, so do not intersect. This will not change when we deform the balls to $\tilde{\mathcal{X}}_{i,\delta_i}$ if we choose δ_i small enough.

5. Let $\int \omega_{FS}^n = 1$, and $\int_{B_1(0)} \omega_{std} = 1$. Then there exists a Kähler embedding $(B_r(0), \omega_{std}) \hookrightarrow (\mathbb{CP}^n, \omega_{FS})$, for all $r < 1$ (for more details on this embedding see [Eck17]). Hence for all $r < \sqrt{m_i}$ there exists a Kähler embedding $(B_r(0), \omega_{std}) \hookrightarrow (\mathbb{CP}^n, m_i\omega_{FS})$. Rescaling by d_i we obtain a Kähler embedding $(B_r(0), \omega_{std}) \hookrightarrow (\mathbb{CP}^n, \frac{m_i}{d_i}\omega_{FS})$ for all $r < \sqrt{\frac{m_i}{d_i}}$. Since $\omega_i^{(i)} = m_i\omega_{FS}$ the embeddings constructed above imply that $\frac{m_i}{d_i} < \epsilon_0$, but since $\frac{m_i}{d_i}$ can be chosen arbitrarily close to ϵ_0 we can conclude that $\sqrt{\epsilon_0} \leq \gamma_k$, where γ_k is the $k-$ball packing constant.

□

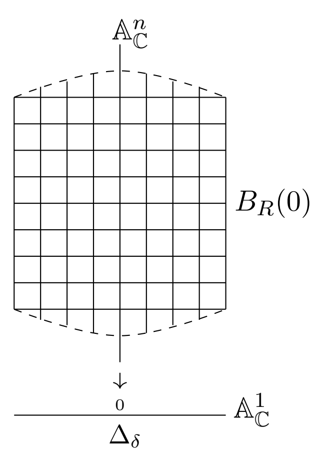


Figure 3.2

Proof of Theorem A. The claim that the k-point Seshadri constant is less or equal to the Kähler packing constant is a direct consequence of Theorem 3.6. The converse argument is a consequence of the symplectic blow up construction [MP94] detailed in Section 2.8. The embedded balls allow us to construct Kähler forms on the blow up π of the centres whose curvature lies in the first Chern class of $\pi^*L - \sum \gamma_k^2 E_i$. $\square$

Remark 3.7. The method to prove Theorem 3.6 also allows us to construct Kähler packings of balls with radius $r_1, \ldots, r_k$ arbitrarily close to $\epsilon_1, \ldots, \epsilon_k$ as long as $\pi^*L - \sum_{i=1}^k \epsilon_i E_i$ is nef.

Chapter 4

Explicit Kähler packings of the projective complex plane

When X is a toric surface and L a toric invariant divisor on X, sections generating the Kähler form also induce a moment map whose image is the well known toric polytope associated to X and L. From earlier work on symplectic cutting it is known that there is a direct correspondence between symplectic cuts of manifolds and cuts of moment polytopes (see [Ler95, Section 1.1]). Furthermore in [Eck17] Eckl constructs a Kähler packing such that the cut off triangles of the moment polytope are the images under the moment map of the embedded balls. In this chapter we develop a framework that generalises this observation, by replacing the moment polytopes with Newton-Okounkov bodies and a non-toric moment map (i.e. a moment map not directly associated to the action of the torus). To illustrate these ideas we present examples of $\mathbb{P}^2_{\mathbb{C}}$ blown up at 1 and 2 points and construct Kähler packings and moment maps with the desired properties. Finally we discuss what difficulties must be overcome on $\mathbb{P}^2_{\mathbb{C}}$ blown up in 3 or more points.

4.1 Dissection of Newton-Okounkov bodies

We first identify natural candidates for the images of the embedded balls under a moment map. Let X be a projective complex manifold of dimension n and fix the points $P_1, \dots P_k$ and an ample line bundle L on X. If $\pi_i \colon \tilde{X}_i = Bl_{P_1,\dots,P_i}(X) \to X$ denotes the blow up of X at the points $P_1, \dots, P_i$ for $i = 1, \dots, k$ and $E_j = \pi^{-1}(P_j)$ is the exceptional divisor corresponding to P_j then set $\epsilon_j = \frac{m_j}{d}$ a positive rational number such that for each

$j = 1, \dots k$ the line bundle $\tilde{L}^{(i)} = d\pi_i^* L - \sum_{j=1}^{i} m_j E_j$ is ample on $\tilde{X}_i$.
Moreover fix an admissible flag

$$Y_\bullet \colon X = Y_0 \supsetneq Y_1 \supsetneq \dots \supsetneq Y_n = \{pt\}$$

and points $P_1, \dots, P_K$ which are smooth points of the flag. Let $v_{Y_\bullet}$ be the extended valuation with respect to the reverse lexicographical ordering on $\bigoplus_{N=0}^{\infty} H^0(X, NdL)$ (see Remark 2.61). The flag $Y_\bullet$ pulls back to an admissible flag $Y_\bullet^{(i)}$ on $\tilde{X}_i$, also centred in $Y_n \subset \tilde{X}_i$ and the extended valuation $v_{Y_\bullet^{(i)}}$ given by this flag on $\bigoplus_{N=0}^{\infty} H^0(\tilde{X}_i, N\tilde{L}^{(i)})$ is the restriction of $v_{Y_\bullet}$ under the natural inclusions $H^0(\tilde{X}, N\tilde{L}^{(i)}) \hookrightarrow H^0(X, NdL)$. Let $\Delta_{Y_\bullet}(\tilde{X}_i, \tilde{L}^{(i)})$ denote the Newton-Okounkov body associated to $\tilde{X}_i$ and $\tilde{L}^{(i)}$ with respect to the flag $Y_\bullet$, for each $i = 1, \dots, k$.

Proposition 4.1. $\Delta_{Y_\bullet}(\tilde{X}_i, \tilde{L}^{(i)}) \subset \Delta_{Y_\bullet}(\tilde{X}_{i-1}, \tilde{L}^{(i-1)})$.

Proof. The proof of this claim is a direct consequence of Proposition 2.67 and Definition 2.68. $\square$

Definition 4.2. We call

$$\Delta_i \coloneqq \Delta_{Y_\bullet^{(i-1)}}(\tilde{X}_{i-1}, \tilde{L}^{(i-1)}) - \Delta_{Y_\bullet^{(i)}}(\tilde{X}_i, \tilde{L}^{(i)}).$$

the *i-th piece of the iterative dissected form* of the Newton-Okounkov body.

Definition 4.3. Let Δ_i be defined as above and let $\partial\Delta_i$ denote the boundary of Δ_i. Let

$$\mu_i \colon B_R(0) \to \mathbb{R}^n$$

denote an $\epsilon_j \cdot \omega_{FS}$-moment map such that the image $\mu_i(B_R(0))$ is convex for each $i = 1, \dots, k$, then we say that $\Delta_i - \mu_i(B_R(0))$ is *δ-close to* $\partial\Delta_i$ if every point in $\Delta_i - \mu_i(B_R(0))$ has distance $\leq \delta$ to $\partial\Delta_i$.

Definition 4.4. We say that the collection $\amalg_{i=1}^{k} \mu_i(B_R(0)$ *δ-tile* $\Delta_{Y_\bullet}$ if the $\Delta_i - \mu_i(B_R(0)$ are δ-close to $\partial\Delta_i$ for all $i = 1, \dots, k$.

4.2 Aim and general strategy

With the notation as in Section 4.1 the aim of this section is to develop a strategy to:

1. Construct a Kähler packing

   $\phi = \amalg_{i=1}^{k} \phi_i \colon \amalg_{i=1}^{k} (B_R(0), \epsilon_j \cdot \omega_{FS}) \hookrightarrow (X, \omega)$, such that $[\omega] = c_1(L)$ and $\phi_i(0) = P_i$

   and an $\epsilon_j \cdot \omega_{FS}$-moment map
   $$\mu_i \colon B_R(0) \to \mathbb{R}^n$$
   such that $\mu_i(B_R(0)) \subset \Delta_i$ and $\Delta_i - \mu_i(B_R(0))$ is arbitrarily close to $\partial\Delta_i$ (see Figure 4.1).

2. Construct a ω-moment map $\mu \colon U \to \mathbb{R}$ on an open (with respect to the analytic topology) subset $U \subset X - \bigcup_{i=1}^{k} \phi_i(B_R(0))$ such that
   $$\mu(U) \subset \Delta_{Y_\bullet}(X, L) - \bigcup_{i=1}^{k} \Delta_i = \Delta_{Y_\bullet^{(k)}}$$
   and $\Delta_{Y_\bullet^{(k)}} - \mu(U)$ is arbitrarily close to $\partial\Delta_{Y_\bullet^k}$.

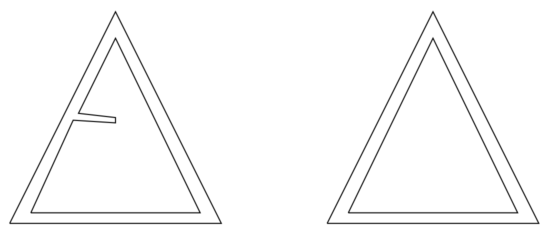
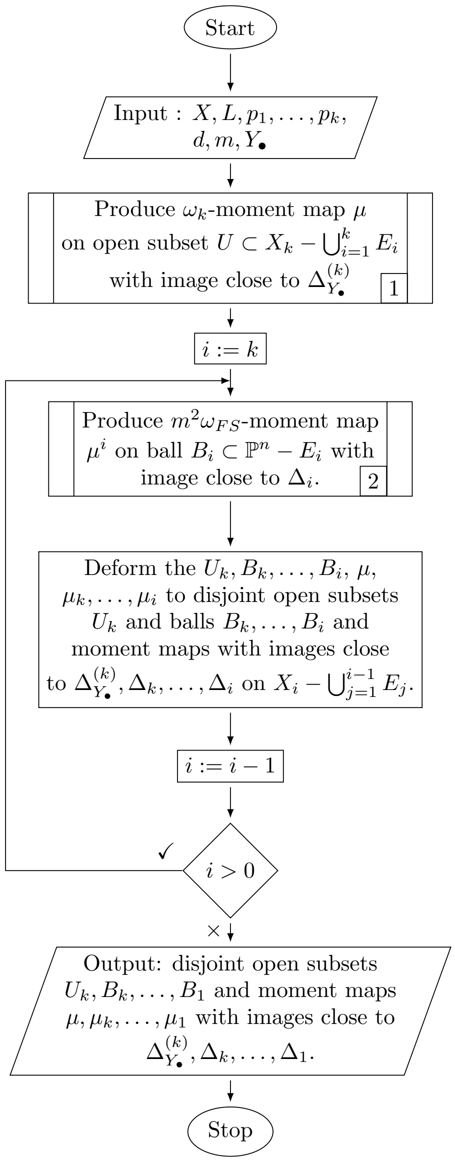
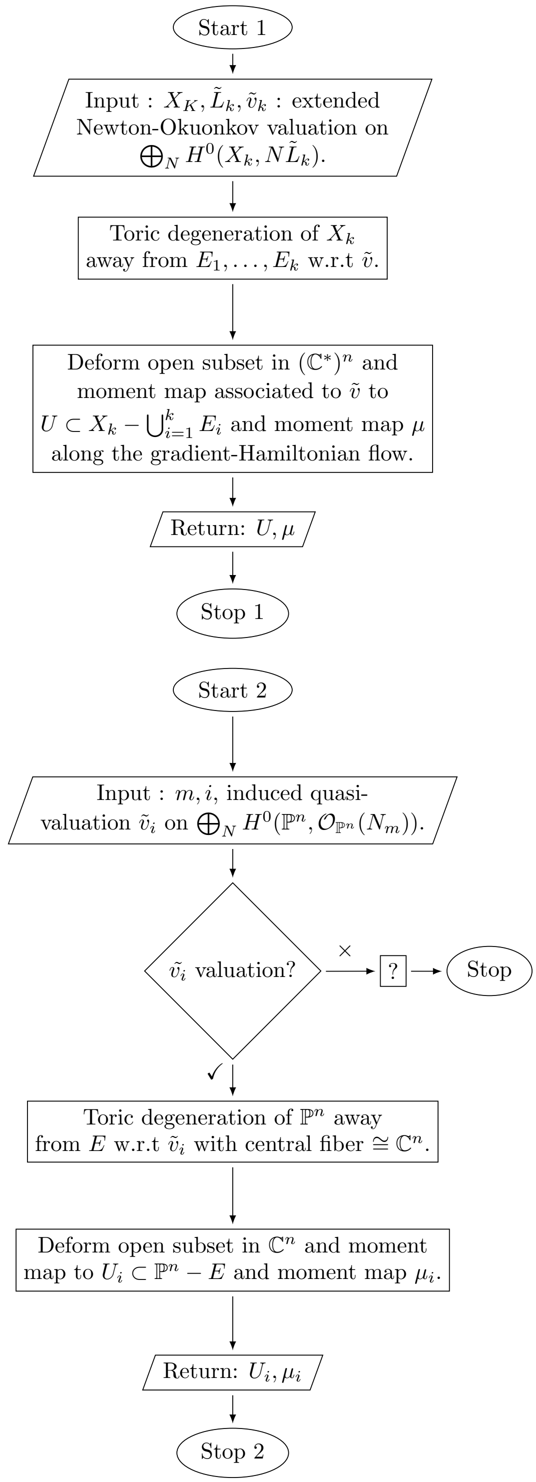


Figure 4.1: Approximation of $(\mu(U_R))$ to Δ_i: Left hand side image is not allowed due to the indent.

First we produce k-ball Kähler packings as detailed in Theorem 3.6. To construct the desired moment maps we will use the procedure detailed in the following flow chart (using the ISO norm 5807 convention for the symbols). Ellipses denote starts and stops, rectangles denote processes, parallelograms denote inputs and outputs and diamonds represent decisions. A rectangle with double-struck vertical edges describes a predefined process (sub-program) and we number these in the bottom right corner to show which process is used. Our main procedure is detailed on the left track of the flow chart and the subprograms are detailed on the right. The notation used in the flow chart is explained at the start of this chapter.


Start
Input : $X, L, p_1, \dots, p_k, d, m, Y_\bullet$
Produce ω_k-moment map μ on open subset $U \subset X_k - \bigcup_{i=1}^k E_i$ with image close to $\Delta^{(k)}_{Y_\bullet}$
1
$i := k$
Produce $m^2\omega_{FS}$-moment map μ^i on ball $B_i \subset \mathbb{P}^n - E_i$ with image close to Δ_i.
2
Deform the $U_k, B_k, \dots, B_i$, μ, $\mu_k, \dots, \mu_i$ to disjoint open subsets U_k and balls $B_k, \dots, B_i$ and moment maps with images close to $\Delta^{(k)}_{Y_\bullet}, \Delta_k, \dots, \Delta_i$ on $X_i - \bigcup_{j=1}^{i-1} E_j$.
$i := i - 1$
$i > 0$
✓
×
Output: disjoint open subsets $U_k, B_k, \dots, B_1$ and moment maps $\mu, \mu_k, \dots, \mu_1$ with images close to $\Delta^{(k)}_{Y_\bullet}, \Delta_k, \dots, \Delta_1$.
Stop


Start 1
Input : $X_K, \tilde{L}_k, \tilde{v}_k$: extended Newton-Okuonkov valuation on $\bigoplus_N H^0(X_k, N\tilde{L}_k)$.
Toric degeneration of X_k away from $E_1, \dots, E_k$ w.r.t $\tilde{v}$.
Deform open subset in $(\mathbb{C}^*)^n$ and moment map associated to $\tilde{v}$ to $U \subset X_k - \bigcup_{i=1}^k E_i$ and moment map μ along the gradient-Hamiltonian flow.
Return: U, μ
Stop 1
Start 2
Input : m, i, induced quasi-valuation $\tilde{v}_i$ on $\bigoplus_N H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(N_m))$.
$\tilde{v}_i$ valuation?
×
?
Stop
✓
Toric degeneration of $\mathbb{P}^n$ away from E w.r.t $\tilde{v}_i$ with central fiber $\cong \mathbb{C}^n$.
Deform open subset in $\mathbb{C}^n$ and moment map to $U_i \subset \mathbb{P}^n - E$ and moment map μ_i.
Return: U_i, μ_i
Stop 2

The procedure detailed in the flow chart shows that it is enough to construct Kähler packings and moment maps on the irreducible componets of the degeneration family that satisfy prescribed conditions. To make this more precise we have the following:

Theorem 4.5. *Take notation as defined above. If there exist moment maps $\mu_i\colon B_R(0)\to\mathbb{R}^n$ on each $(\mathcal{E}_i - E_i, \tilde{\omega}_0|_{\mathcal{E}_i})$ and $(\tilde{X}_i, \tilde{\omega}_0)$ whose moment polytopes δ-tile the Newton-Okounkov body of (X, L) with respect to $Y_\bullet$, then there exist moment maps associated to a non-toric action on open (with respect to the analytic topology) subsets of X whose images are the tiling of the pieces of the moment polytopes corresponding to the irreducible components of $\mathcal{E}_i$ and $\tilde{X}_i$.*

Proof. To construct the desired moment maps it is enough to construct moment maps $\mu_i\colon B_R(0)\to\mathbb{R}^n$ on relatively compact open (in the analytic topology) subsets $\tilde{U}_i\subset\mathcal{E}_i - E_i$ for $i = 1,\dots,k$ and a moment map μ on a relatively compact open (in the analytic topology) subset $\tilde{U}\subset\tilde{X}_k - \bigcup_{i=1}^k E_k$ such that:

1. $\mu_i(\tilde{U}_i)\subset\Delta_i$ and $\Delta_i - \mu_i(\tilde{U}_i)$ is arbitrarily close to $\partial\Delta_i$.

2. $\mu(\tilde{U})\subset\Delta_{Y_k^{(i)}}(\tilde{X}_k, \tilde{L}_k)$ and $\Delta_{Y_k^{(i)}}(\tilde{X}_k, \tilde{L}_k) - \tilde{\mu}(\tilde{U})$ is arbitrarily close to $\partial\Delta_{Y_k^{(i)}}(\tilde{X}_k, \tilde{L}_k)$.

We construct these maps following the same method as Kaveh in [Kav19] described in Section 2.10. Start by constructing a k-ball Kähler packing as detailed in the proof of Theorem 3.6. Recall that in the iterative construction detailed in the proof the Kähler ball embeddings on X come from Kähler ball embeddings on the $\mathcal{E}_k - E_k$, provided with multiples of Fubini-Study Kähler forms. The sections generating this Kähler form induce a moment map $\tilde{\mu}_i\colon B_R(0)\to\mathbb{R}^n$ with the desired properties. On nearby fibers of $\tilde{\mathcal{X}}_{i,0}$ the balls keep arbitrarily close to P_i (see product structure on fibers, Figure 4.2). Away from the P_i the Kähler form is not a pullback of the Kähler form on $\tilde{U}\subset\tilde{X}_k$, so we use instead the gradient-Hamiltonian flow discussed in Section 2.11 to move $\tilde{U}$ to an open (with respect to the analytic topology) subset U on a nearby fiber away from P_i and to move the moment map $\tilde{\mu}$ to a moment map on U. Here, we need a nearby fiber because otherwise we cannot be sure that $P_i\notin U$. $\square$

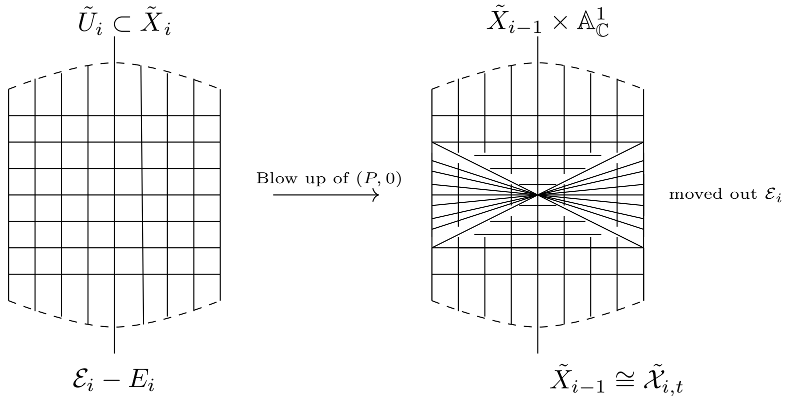


Figure 4.2

The proof of the above theorem leaves us with two tasks to achieve the aim and explicitly construct the desired moment maps.

1. First we construct moment maps on $\mathcal{E}_i \cong \mathbb{P}^n$ with the prescribed image in Δ_i. For this we use the filtration induced on the graded ring $S := \bigoplus_{N=0}^{\infty} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(Nm))$ by the degeneration constructed in Chapter 3. The filtration has 1-dimensional steps, so it corresponds to a quasi-valuation with 1-dimensional leaves (see Definition 2.85). If it is a valuation then we can use the toric degeneration and the gradient-Hamiltonian flow as described in Section 2.11 to construct the desired moment map. This is the case for $\mathbb{P}^2$ blown up in 1 and 2 points, however this fails already for $\mathbb{P}^2$ blown up in 3 points in general position, as we show in some examples below.

2. Construct moment maps on an open (with respect to the analytic topology) subset $\tilde{U} \subset \tilde{X}$ not intersecting the exceptional divisors E_i. We do this in the examples by using the variation of the toric degeneration described in Section 2.10 along with the gradient-Hamiltonian flow.

The construction of the filtration needed in (1) works as follows: Construct the algebraic families $\tilde{\mathcal{X}}_i$ along with line bundles $\tilde{\mathcal{L}}^{(i)}$ as detailed in Chapter 3.

$$\begin{tikzcd} \tilde{\mathcal{X}}_i \arrow[r, "\Pi_i"] \arrow[rr, bend left, "\Pi^{(i)}"] & \mathcal{X}_i = \tilde{X}_{i-1} \times l_i \arrow[r, "q_{i-1}"] \arrow[d, "p_{i-1}"] & l_i \cong \mathbb{A}^1_{\mathbb{C}} \\ & \tilde{X}_{i-1} & \end{tikzcd}$$

Fix $N > 0$ and consider the vector bundle $\Pi_* N\tilde{\mathcal{L}}^{(i)}$. This is a finite rank vector bundle over $\mathbb{A}^1_{\mathbb{C}}$ and as such trivial. As shown in the proof of Theorem 3.5 there exists trivialising sections σ_i that restrict to sections of $H^0(\tilde{X}_{i-1}, N\tilde{L}^{(i-1)})$ on non-central fibers and a subset of sections of $H^0(\mathcal{E}_i, \mathcal{O}_{\mathcal{E}_i}(Nm)) \oplus H^0(\tilde{X}_i, N\tilde{L}^{(i)})$ on the central fiber. We construct these sections as follows:

- Choose a basis of sections of $N\tilde{L}^{(i-1)}$ with multiplicity $\geq m_i$ in P_i. Pull back to obtain a section of $N\tilde{\mathcal{L}}^{(i)}$ by removing m_i copies of $\mathcal{E}_i$. If $\mathrm{mult}_{P_i} > m_i$ the restriction of the section to $\mathcal{E}_i$ is 0.

- Choose a basis of sections of $NL^{(i-1)}$ with multiplicity $\leq m_i$ in P_i. Multiply with $t^{m_i - \mathrm{mult}_{P_i}}$ then pull back to sections of $N\tilde{\mathcal{L}}^{(i)}$ and remove m_i copies of $\mathcal{E}_i$. If $\mathrm{mult}_{P_i} < m_i$ then the restriction to $\tilde{\mathcal{X}}_i$ is 0.

But this time, we choose the basis to be a basis of the filtration given by the Newton-Okounkov valuations with respect to the flag $Y_\bullet$. In more details, define the graded rings:

$$R := \bigoplus_{N=0} H^0(X, \mathcal{O}_X(NL)) \ni f$$

$$\tilde{R}^{(i)} := \bigoplus_{N=0} H^0(\tilde{X}_i, \mathcal{O}_{\tilde{X}_i}(N\tilde{L}^{(i)})) \ni f_{\tilde{R}}$$

$$S = \bigoplus_{N=0}^{\infty} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(Nm)) \ni f_S$$

$$T := \bigoplus_{N=0}^{\infty} H^0(E, \mathcal{O}_E(-NmE)).$$

Then as discussed in Section 3.2, there is an isomorphism

$$R_0^{(i)} \coloneqq \bigoplus_{N=0}^{\infty} H^0(\tilde{\mathcal{X}}_{i,0}, \mathcal{O}_{\tilde{\mathcal{X}}_{i,0}}(N\tilde{\mathcal{L}}^{(i)}) = \tilde{R}^{(i)} \times_T S \ni (f_{\tilde{R}}, f_S).$$

Where the fibered product is given by the restriction maps $\tilde{R}^{(i)} \to T$ and $S \to T$. Fix an admissible flag $Y_\bullet$ on X and as before let $\tilde{v}_{Y_\bullet}$ be the extended valuation on $R \setminus \{0\}$ with values in $\mathbb{Z} \times \mathbb{Z}^n$. The flag $Y_\bullet$ extends to a flag on each $\tilde{X}_{i-1}$ by pulling back along the $(i-1)^{st}$ blow up map π_{i-1}. Hence it defines a valuation

$$\tilde{v}_{i-1} \colon R^{(i-1)} \setminus \{0\} \to \mathbb{Z} \times \mathbb{Z}^n.$$

Fixing a total order on $\mathbb{Z} \times \mathbb{Z}^n$ there are vectors $u_1 > u_2 > \ldots > u_n > u_{n+1} > \ldots$ on $\mathbb{Z} \times \mathbb{Z}^n$ such that the valuation $\tilde{v}_{i-1}$ induces a filtration of vector spaces

$$0 \subset F_{\geq u_1}^{\tilde{R}^{(i-1)}} \subset F_{\geq u_2}^{\tilde{R}^{(i-1)}} \subset \ldots \subsetneq F_{\geq u_n}^{\tilde{R}^{(i-1)}} \subset F_{\geq u_{n+1}}^{\tilde{R}^{(i-1)}} \subset \ldots \subset \tilde{R}^{(i-1)}$$

such that each step in the filtration is 1-dimensional (i.e. $\dim({F_{\geq}^{\tilde{R}^{(i-1)}}}_{u_j} / {F_{\geq}^{\tilde{R}^{(i-1)}}}_{u_{j+1}}) = 1$). If we choose a basis of this filtration and use the degeneration method described above we obtain a filtration

$$0 \subset F_{\geq u_1}^{\tilde{R}_0^{(i)}} \subset F_{\geq u_2}^{\tilde{R}_0^{(i)}} \subset \ldots \subsetneq F_{\geq u_n}^{\tilde{R}_0^{(i)}} \subset F_{\geq u_{n+1}}^{R_0^{(i)}} \subset \ldots \subset \tilde{R}_0^{(i)}$$

also with 1-dimensional steps.

Consider the projection

$$\psi \colon R \times_T S \to S \text{ such that } \psi((f_{\tilde{R}}, f_S)) = f_S.$$

The surjectivity of ψ implies the existence of a filtration of $S \cong \bigoplus_{N=0}^{\infty} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(Nm))$

$$0 \subsetneq F_{\geq u_1}^S \subset F_{\geq u_2}^S \subset \ldots \subset F_{\geq u_n}^S \subset F_{\geq u_{n+1}}^S \subset \ldots \subset S$$

which also has 1-dimensional steps after omitting all $F_{\geq u_{i-1}}^s$ with $F_{\geq u_{i-1}}^s = F_{\geq u_i}^s$. As described in Section 2.12 [Proposition 2.84] such a filtration induces a quasi-valuation by setting

$$\tilde{v}_S(f_S) = \max \{u_i \colon f_S \in F_{\geq u_i}^S\}.$$

Thus, the map $\tilde{v}_S$ satisfies the following properties of a quasi-valuation:

- $\tilde{v}_S(\lambda \cdot f_S) = \tilde{v}_i(f_S)$.
- $\tilde{v}_S(f_S + g_S) \geq min(\tilde{v}_i(f_S), \tilde{v}_i(g_S))$, for all $f_S, g_S \in F^S$.
- $\tilde{v}_S(f_S \cdot g_S) \geq \tilde{v}_i(f_S) + \tilde{v}_i(g_S)$, for all $f_S, g_S \in F^S$.

Note that further unpacking the construction of $\tilde{v}_S$ shows that

$$\tilde{v}_S = \max\{\tilde{v}_i(f_{\tilde{R}}) \colon (f_{\tilde{R}}, f_S) \in R_0^{(i)}\}.$$

When the above quasi-valuation is in fact a valuation we can use the toric degeneration method (detailed in Section 2.10) directly to construct the corresponding Newton-Okounkov body and a moment map with the desired properties.
All this provides strong evidence of the following conjecture.

Conjecture 4.6. *Let X be a projective complex manifold of dimension n with ample line bundle L. Fix an admissible flag $Y_\bullet \coloneqq X = Y_0 \supsetneq, \dots, \supsetneq Y_n = \{pt\}$ of X and let $\Delta_{Y_\bullet}$ be the associated Newton-Okounkov body. Fix an ordered set of points $P_1, \dots, P_k$ that are smooth points of the flag and not contained in Y_1 and let Δ_i be the i-th piece of the iterated dissected form of the Newton-Okounkov body. Assuming each Δ_i is convex, for any $\delta > 0$ there exists a Kähler packing of X by balls $B_1, \dots, B_k$ centered respectively at points $P_1, \dots, P_k$ and moment maps $\mu_i \colon B_i \to \mathbb{R}^n$ such that $\mu_i(B_i)$ is a convex subset of Δ_i with $vol(\Delta_i - \mu_i(B_i)) < \delta$ for each $i = 1, \dots, k$.*

We do not know of any case when the Δ_i in the setting of the above conjecture are not convex but we think that they may exist.

4.3 1-point case on $\mathbb{P}^2$

Let $X = \mathbb{P}^2_{\mathbb{C}}$, with homogeneous coordinates $[X : Y : Z]$ and fix the point $P_1 = [1 : 0 : 0]$. Let $\pi \colon \tilde{X}_1 = Bl_{P_1}(\mathbb{P}^2) \to X$ be the blow up of X at P_1, with $E_1 = \pi^{-1}(P_1)$ the exceptional divisor. Choose positive integers m and d such that $\tilde{L}^{(1)} = d\pi^*L - mE$ is ample (in particular, $m < d$) and fix the admissible flag $Y_\bullet \coloneqq \{X = 0\} \supsetneq \{X = Y = 0\}$ on X. Now construct the families $\mathcal{X}$ and $\tilde{\mathcal{X}}$ as in Chapter 3 (see Figure 4.3).

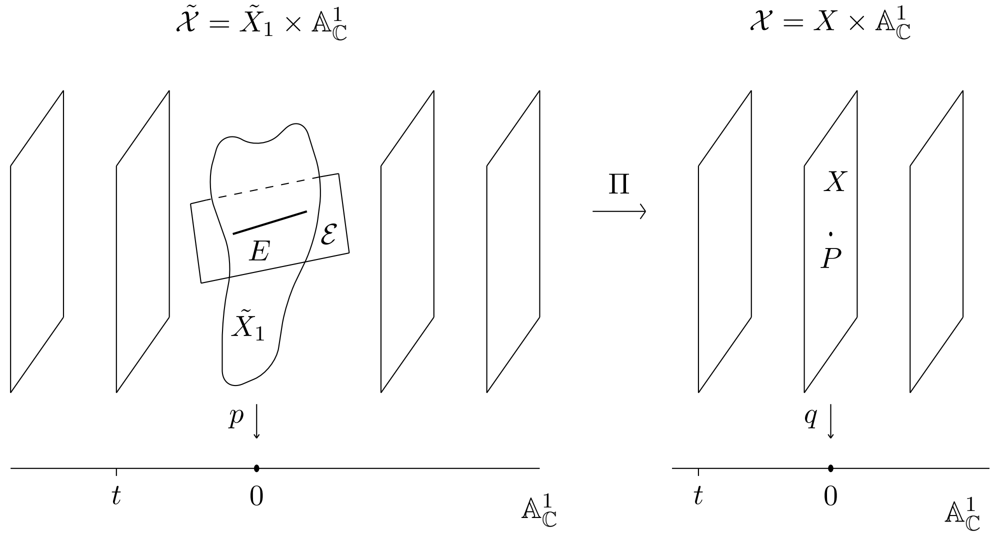


Figure 4.3

Recall that on $\tilde{\mathcal{X}}$ there exists a family of ample line bundles $\tilde{\mathcal{L}}$ and the central fiber $\tilde{\mathcal{X}}_0 \cong \tilde{X}_1 \cup \mathcal{E}_1$ where $\mathcal{E}_1 \cong \mathbb{P}^2$.

Using the method in Theorem 3.5 we determine a basis of sections of $H^0(\tilde{\mathcal{X}}_0, \tilde{\mathcal{L}}|_{\tilde{\mathcal{X}}_0})$ characterised by vanishing behaviour at the point $[1:0:0]$.

First we compute sections of $H^0(\tilde{X}_1, \mathcal{O}_{\tilde{X}}(\tilde{L}^{(1)}))$, which are homogeneous monomials of degree d in coordinates $[X : Y : Z]$ vanishing to order greater or equal m in P_1, that is monomials of the form

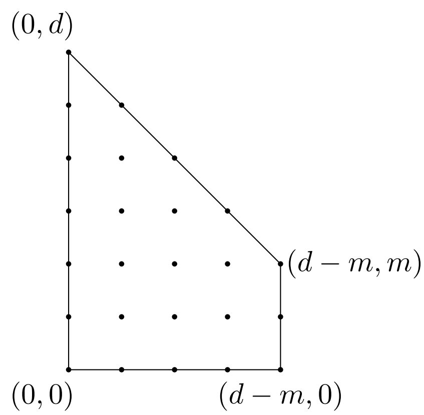
$$X^a Y^b Z^c$$

such that $a+b+c=d$ and $b+c \geq m$. Sections of $H^0(\mathcal{E}_1, \mathcal{O}_{\mathcal{E}_1}(-mE_1))$ are simply homogeneous monomials of degree m in coordinates $[U_1 : V_1 : W_1]$ on $\mathbb{P}^2$ under the isomorphism $\mathcal{E}_1 \cong \mathbb{P}^2$. The monomial $U^a V^b W^c$ restricts to $V_1^f W_1^g$ on $E_1 = \{U_1 = 0\}$ if $f+g=m$ and to 0 otherwise. Similarly the section of $\tilde{L}^{(1)}$ corresponding to $X^a Y^b Z^c$ restricts to $V_1^b W_1^c$ if $b+c=m$ and zero otherwise.

Now let $v_{Y_\bullet}(X^a Y^b Z^c) = (a,b)$ be the valuation vector associated to a section of $H^0(\tilde{X}_1, \mathcal{O}_{\tilde{X}}(\tilde{L}^{(1)}))$ with respect to the flag $Y_\bullet$ and set $\tilde{v}_{Y_\bullet} = (N, v_{Y_\bullet})$, for $N \in \mathbb{Z}$ the extended valuation with values in $\mathbb{Z} \times \mathbb{Z}$. Using $\tilde{v}_{Y_\bullet}$ we construct the Newton-Okounkov body associated directly from the sections.

The Newton-Okounkov body $\Delta_{Y_\bullet}(\tilde{X}, \tilde{L})$ associated to $\tilde{X}$ and $\tilde{L}$ contains

$$conv\{(0,0), \dots, (0,d), \dots, (d-m, m), \dots (d-m, 0), \dots, (1,0)\}.$$



By Lazarsfeld and Mustaţă's description of Newton-Okounkov bodies of surfaces (see Theorem 2.64) the volume of the convex body is equal to half of the volume $d^2 - m^2$ of $dL - mE_1$, hence the convex body is equal to the Newton-Okounkov body of $dL - mE_1$ on $\tilde{X}_0 = X$.

According to the strategy stated in Theorem 4.5 we now have to construct a moment map on $\tilde{X}_1 - E_1$, mapping into $\Delta_{Y_\bullet}(\tilde{X}_1, \tilde{L}^{(i)})$. We do this following the toric degeneration method introduced in Section 2.10 and applying the gradient-Hamiltonian flow but we need to be careful to avoid the exceptional divisor E_1. To this purpose we need a closer look at the construction of the degeneration.
The degeneration of a given section $X^a Y^b Z^c \in H^0(X, dL)$ to a section on $\tilde{\mathcal{X}}_0$ works as follows:

1. Take a section of the form $X^a Y^b Z^c$, such that $a + b + c = d$, and de-homogenise by $X \neq 0$ to obtain a section $y^b z^c$ in affine coordinates $y = \frac{Y}{X}, z = \frac{Z}{X}$.

2. Use the coordinate transformation $y = tv_1$ and $z = tw_1$ to rewrite as $t^{b+c} v_1 w_1$ and divide out m copies of the exceptional divisor (in these charts this is given by $t = 0$). Note that if $b + c > m$ then this sections degenerates to 0 on $\mathcal{E}_1$ and if $b + c < m$ we have sections on $\mathcal{E}_1$ that lie away from the exceptional divisor E_1 hence we multiply by an appropriate power of t so that when we remove m copies of the exceptional divisor we obtain a non-zero section on $\mathcal{E}_1$.

3. Re-homogenise affine monomials $v_1^b w_1^c$ to homogeneous monomials of degree m of the form $U_1^a V_1^b W_1^c$. All non-zero sections on $\mathcal{E}_1$ can be described this way.

We can choose the weight vector $\gamma = (1,1)$ because the Taylor series expansion of the sections $X^aY^bZ^c$ around P only consists of monomials x^ay^b. Then the image of $E \times \{t\}$ under the embedding F is independent of $t \neq 0$: The sections defining the embedding are given by $t^{a+b}X^aY^bZ^c$. If $b+c > m$ these sections vanish on E_1. If $b+c = m$ then $a+b = d-c$, hence $t^{a+b}X^aY^bZ^c = t^dX^aY^b(t^{-1}Z)^c$. So we can cancel t^d, and the remaining section restricts to $Y^b(t^{-1}Z)^c$ on E_1. These sections map E_1 to the image of the m-th Segre embedding of $\mathbb{P}^1 \cong E_1$. Hence by general statements on the gradient-Hamiltonian flow (see Section 2.11, [Lemma 2.78] and [Proposition 2.80]) the flow is trivial on $E_1 \times (\mathbb{A}^1_{\mathbb{C}} - \{0\})$. Finally the central fiber is embedded by F away from the image of $E_1 \times \{t\}$: Z^d is mapped to 0 on E_1 but degenerates to the constant function 1 on the central fiber of the toric degeneration. This means the gradient-Hamiltonian flow moves to an open (with respect to the analytic topology) subset away from the $E_1 \times \{t\}$ on the non-central fiber over t, as requested.

For the other part of the strategy we need to construct moment maps on the $\mathcal{E}_i$: As described in the general strategy of Section 4.2 the $U_1^aV_1^bW_1^c$ are the basis of a filtration on $H^0(\mathbb{P}^2_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(m))$ inducing a quasi-valuation given by

$$v_0(U_1^aV_1^bW_1^c) = (d-b-c, b) = (d-m+a, b).$$

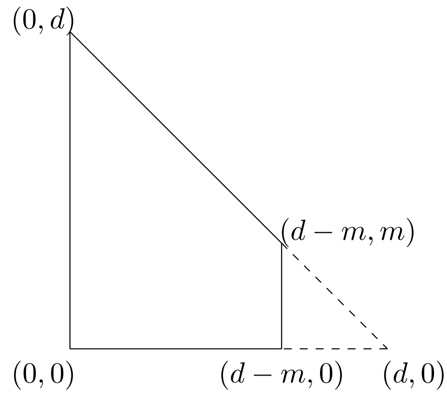
In the usual way we can define an *extended quasi-valuation*

$$\tilde{v}_0(U_1^aV_1^bW_1^c) = (N, Nd - Nm + a, b)$$

for $Nm = a+b+c$. This quasi-valuation is obviously a valuation on $S = \bigoplus H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(Nm))$, and the associated Newton-Okounkov body is the triangle with vertices at

$$(d-m, 0), (d-m, m) \text{ and } (d, 0).$$

Combining these results we obtain the iterative dissected form of the Newton-Okounkov body $\Delta_{Y_\bullet}(\tilde{X}, \tilde{L}^{(1)})$ associated to $\tilde{X}$ and $\tilde{L}^{(1)}$.



The Newton-Okounkov body constructed above is a translated version of the standard toric moment polytope associated to $\mathbb{P}^2$ blow up at a T-invariant point. More precisely the standard moment polytope is the image of the symplectic moment map associated to the $(\mathbb{C}^*)^2$-action on $\mathbb{P}^2_{\mathbb{C}}$ given by

$$(t_1, t_2) \cdot [X : Y : Z] = [t_1 X : t_2 Y : Z].$$

Where as the Newton-Okounkov body constructed above is the image of the symplectic moment map associated to the $(\mathbb{C}^*)^2$-action on $\mathbb{P}^2_{\mathbb{C}}$ given by

$$(t_1, t_2) \cdot [\ldots : U_1^a V_1^b W_1^c : \ldots] = [\ldots : t_1^{d-m+a} t_2^b U_1^a V_1^b W_1^c : \ldots]$$

on the i-th Segre embedding of $\mathbb{P}^2_{\mathbb{C}}$. The factor t_1^{d-m} distinguishes this action from the action induced by the standard action on the m-th Segre embedding of $\mathbb{P}^2$. The associated moment map therefore translates the moment polytope horizontally by $d - m$.

4.4 2-point case on $\mathbb{P}^2$

Following the general strategy presented in Section 4.2 we proceed in an iterative manner and blow up a second point, being careful to choose a point that is not invariant under the $(\mathbb{C}^*)^2$ action. This ensures that the resulting blown up variety is no longer toric and as such the moment map we obtain is an example of a non-toric moment map.

Assume the setting of the previous section and fix $P_2 = [1 : 1 : 0]$. Let $\pi_2 \colon \tilde{X}_2 = Bl_{P_1,P_2}(\mathbb{P}^2) \to \tilde{X}_1$ denote the blow up of $\tilde{X}_1$ at P_2 and $E_2 = \pi_2^{-1}(P_2)$ the exceptional divisor. Set

$$\tilde{L}^{(2)} = d\pi^* L - mE_1 - mE_2 = \pi_1^* \tilde{L}^{(1)}$$

and fix $m, d \in \mathbb{Z}_{>0}$ such that $\tilde{L}^{(2)}$ is ample on $\tilde{X}_2$.

First we construct the algebraic families $\mathcal{X}_2$ and $\tilde{\mathcal{X}}_2$ as before, such that

$$\Pi^{(2)} \colon \tilde{\mathcal{X}}_2 = \tilde{X}_2 \times \mathbb{A}^1_{\mathbb{C}} \to \mathcal{X}_2 = \tilde{X}_1 \times \mathbb{A}^1_{\mathbb{C}}.$$

There exists a family of ample line bundles $\tilde{\mathcal{L}}^{(2)}$ on $\tilde{\mathcal{X}}_2$ and the central fiber $\tilde{\mathcal{X}}_{1,0} \cong \tilde{X}_2 \cup \mathcal{E}_2$ where $\mathcal{E}_2 \cong \mathbb{P}^2_{\mathbb{C}}$.

By the general strategy we first need to construct a basis of the Newton-Okounkov filtration on the global sections of $\tilde{L}^{(2)}$. Then using the same procedure used in the 1-point case we degenerate these sections to sections on $\mathcal{E}_2 \cong \mathbb{P}^2_{\mathbb{C}}$. Next we construct (quasi)-valuations on sections of $\mathcal{E}_2$ and a basis of the induced filtration. We check that the graded ring is the same as the ring of sections, take the induced $(\mathbb{C}^*)^n$-action and the associated moment map.

To achieve the first aim we use construct a basis of sections of $H^0(\tilde{X}_2, \mathcal{O}_{\tilde{X}_2}(\tilde{L}^{(2)}))$ described by homogeneous polynomials of degree d in coordinates $[X : Y : Z]$, vanishing to order $\geq m$ in both P_1 and P_2. To construct these sections we first look for linear forms that vanish in $P_1 = [1 : 0 : 0]$ and $P_2 = [1 : 1 : 0]$. The simplest choice is

$$Z = 0 \ , \ Y = 0 \ , \ X - Y = 0.$$

Hence we construct homogeneous polynomials of degree d of the form

$$X^a Y^b Z^c (X - Y)^e$$

such that

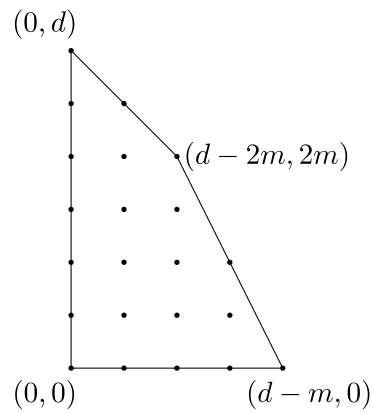
$$\begin{aligned} a + b + c + e &= d \\ b + c &\geq m \\ b + e &\geq m \end{aligned}$$

To determine a basis of these sections we need only those that are linearly independent and this means fixing some choices. We do this by fixing the order of vanishing at P_1 to always be greater or equal P_2 (in this construction this means fixing $b \geq e$ for all our choices of polynomial sections). Taking valuations of sections of this type with respect to the flag $Y_\bullet$ we have

$$v_{Y_\bullet}(X^a Y^b Z^c (X-Y)^e) = (a, b+c).$$

Hence the Newton-Okounkov body associated to $\tilde{X}_1$ and $\tilde{L}_1$ is therefore is contained in in the convex hull of

$$conv\{(0,0), \ldots, (0,d), \ldots, (d-2m, 2m), \ldots (d-m, 0), \ldots, (1,0)\}.$$



In this case the volume of the convex body is equal to half of the volume $d^2 - 2m^2$ of $dL - mE_1 - mE_2$, hence the convex body is equal to the Newton-Okounkov body of $dL - mE_1 - mE_2$ on $\tilde{X}_2$.

The aim now is to construct the desired moment map on $\tilde{X}_2 - E_2$, mapping into $\Delta_{Y_\bullet}(\tilde{X}_1, \tilde{L}^{(2)})$. This time to achieve this goal we choose the weight vector $\gamma = (2,1)$ and consider the embedding $F\colon \mathcal{X}_\gamma \to \mathbb{P}^N$ detailed in Section 2.10 [Proposition 2.76] sending $x \mapsto [\ldots : \tilde{f}_i : \ldots]$, where $\tilde{f}_i = t^{-\gamma\cdot\beta_i + a_i} \cdot f_i$ for each $i = 0, \ldots, N$. To determine the a_i we first exhibit 3 sections whose differences generate $\mathbb{Z}^2$: Z^d with valuation $(0,0)$, XZ^{d-1} with valuation $(1,0)$ and YZ^{d-1} with valuation $(0,1)$ will do. We choose $a_i = 0$ for these 3 sections. Note also that they vanish on E_1 and E_2. For those f_i not vanishing on E_1 and E_2 we are free to choose the a_i such that the powers of t cancel. Hence the image of $E_2 \times \{t\}$ under the embedding F is again independent of $t \neq 0$. The gradient-Hamiltonian flow on $F(E_i \times \{t\})$ is given by $\frac{\partial}{\partial t}$ and by properties of the gradient-Hamiltonian flow (see [Lemma 2.78] and [Proposition

2.80]) is trivial on $E_2 \times (\mathbb{A}^1_{\mathbb{C}} - \{0\})$. Since the central fiber is embedded by F away from the image of $E_2 \times \{0\}$ (i.e. $F(E_2 \times \{0\})$ does not intersect $\mathcal{X}_{\gamma,0}$) as Z^d degenerates to 1 on $\mathcal{X}_{\gamma,0}$, the gradient-Hamiltonian flow moves to an open (with respect to the analytic topology) subset away from the $E_2 \times \{t\}$ on the non-central fiber over t, as requested.

For the second aim we first construct a basis of $\mathcal{E}_2$ by degenerating sections of $H^0(X, dL)$ to sections of $H^0(\mathcal{E}_2, \mathcal{O}_{\mathcal{E}_2}(-mE_2))$ to obtain homogeneous monomials of degree m in coordinates homogeneous coordinates $[U_2 : V_2 : W_2]$ on $\mathbb{P}^2$ under the isomorphism $\mathcal{E}_2 \cong \mathbb{P}^2$. Using the procedure detailed in Theorem 3.5 the section $U_2^a V_2^b W_2^C$ is obtained by degenerating the section on $\tilde{X}_2$ which vanish in multiplicity $< m$ in P_2 and multiplicity $\geq m$ in P_1. Finally sections that lie on the intersection $\tilde{X} \cap \mathcal{E}_2 = E_2$ are homogeneous monomials on $\tilde{X}_2$ which degenerate to a non-zero section $U_2^a V_2^b W_2^c$ on $\mathcal{E}_2$

In more details, Fix d and m as in 4.3, then sections of dL can be categorised as follows:

1. Sections of dL vanishing to order $\leq m$ in P_1 and $< m$ in P_2. These are homogeneous polynomials of the form $X^a Y^b Z^c (X - Y)^e$ such that $a + b + c + e = d$, $b + c < m$ and $c + e < m$.

2. Sections of dL vanishing to order $\geq m$ in P_1 and $\leq m$ in P_2. These are homogeneous polynomials of the form $X^a Y^b Z^c (X - Y)^e$ such that $a + b + c + e = d$, $b + c \geq m$ and $c + e < m$.

3. Sections of dL vanishing to order $\geq m$ in P_1 and order $= m$ in P_2. These are homogeneous polynomials of the form $X^a Y^b Z^c (X - Y)^e$ such that $a + b + c + e = d$, $b + c \geq m$ and $c + e = m$.

Sections of (1) degenerate into homogeneous monomials on $\mathcal{E}_1$ in the same way as the 1-point case, however as $\mathcal{E}_i \cap \mathcal{E}_2 = \emptyset$ these sections are zero on $\mathcal{E}_2$. The sections of type (2) and (3) degenerate to non zero sections on $\mathcal{E}_2$ in a similar way to the 1-point case, but this time we need a different coordinate system as it is the vanishing behaviour at P_2 that we care about. We use the following algorithm to construct these sections:

- Take a section $X^a Y^b Z^c (X - Y)^e$ and de-homogenise by $X \neq 0$ to obtain an affine polynomial $y^b z^c (1 - y)^e$, where $y = \frac{Y}{X}$, $z = \frac{Z}{X}$ and $(1 - y) = \frac{(X-Y)}{X}$.

- Set $y' = (1 - y)$ and substitute in the previous expression to obtain $(1 - y')^b z^c y'^e$.

- Let $y' = tu_2$ and $z = tw_2$ and rewrite as an affine polynomial of the form
$$(1 - tu_2)^b t^c w_2^c t^e u_2^e.$$

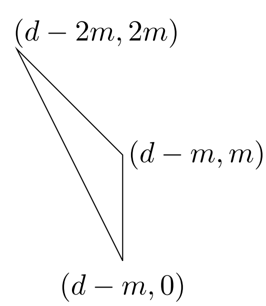
- Divide out m copies of t (the coordinates of the exceptional divisor).

- Set $t = 0$ to obtain an affine monomial $v_2^b w_2^c$ and re-homogenise using homogeneous coordinates $[U_2 : V_2 : W_2]$ on $\mathbb{P}^2_{\mathbb{C}}$ to produce homogeneous monomials $U_2^a V_2^b W_2^c$ such that $a + b + c = m$.

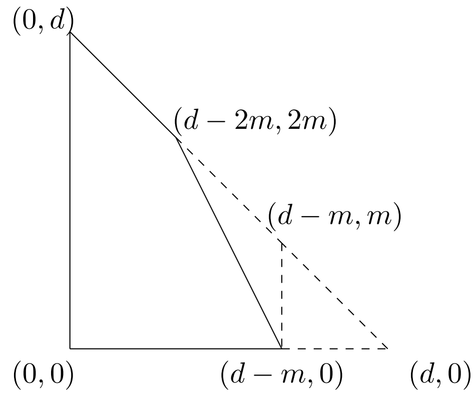
Homogeneous monomials constructed in this way form basis of a filtration on $H^0(\mathbb{P}^2_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(m))$ inducing a quasi-valuation given by
$$\tilde{v}_0(U_2^a V_2^b W_2^c) = (d - 2a - b - c, 2a + b) = (d - m - a, 2a + b).$$

As before define an extended quasi-valuation
$$\tilde{v}_0(U_2^a V_2^b W_2^c) = (N, Nd - Nm - a, 2a + b)$$

for $Nm = a + b + c$. Again the quasi valuations we obtain are obviously a valuations on $S^{(2)} = \bigoplus H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(Nm)))$, and the associated Newton-Okounkov body is the triangle



Combining this with the result obtained in Section 4.3 we obtain the iterative dissected form of Newton-Okounkov body $\Delta_{Y_\bullet}(\tilde{X}_2, \tilde{L}^{(2)})$ associated to $\tilde{X}_2$ and $\tilde{L}^{(2)}$.



Consider the $(\mathbb{C}^*)^2$-action on $\mathcal{E}_2 \cong \mathbb{P}^2_{\mathbb{C}}$ given by

$$(t_1, t_2) \cdot [\ldots : U_2^a V_2^b W_2^c : \ldots] = [\ldots : t_1^{d-m-a} t_2^{2a+b} U_2^a V_2^b W_2^c : \ldots]$$

on the i-th Segre embedding of $\mathbb{P}^2_{\mathbb{C}}$. We then take the moment map associated to this toric action on $\mathbb{P}^2_{\mathbb{C}}$ with respect to $\mathcal{E}_2$ in the same way as in Example 2.49. This satisfies the required properties.

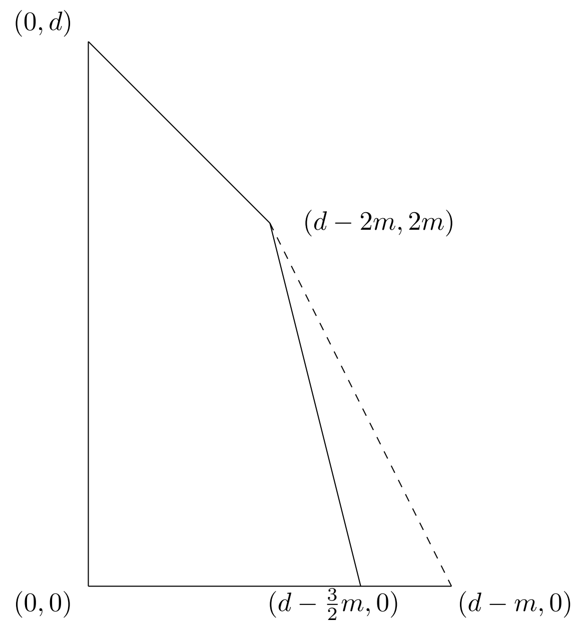
4.5 3-point case on $\mathbb{P}^2$ and problems that arise

In this section we continue with our example blowing up the projective complex plane. Unfortunately we find that blowing up at the third point introduces some difficulties which we will explain and present some intuition on how to proceed. The main problem we encounter is that the deformation family we construct degenerates to a central fiber that may contain multiple non-reduced components and any basis of global sections of the filtration only corresponds to the weaker notion of quasi-valuations. So our method to produce the moment maps does not work as described.

Let $\tilde{X}_2$ be defined as in the previous section, fix $P_3 = [2 : 1 : 1]$ and let $\pi_2 \colon \tilde{X}_3 \to \tilde{X}_2$ denote the blow up of $\tilde{X}_2$ at the point P_3.

Remark 4.7. The third point must be chosen carefully to ensure that the resulting Newton-Okounkov body is of the correct shape. To ensure this condition is met we must choose points such that no two points lie on a line passing the flag point P and that no three points are chosen co-linearly.

As we have seen in the previous examples the iterative dissected form of the Newton-Okounkov body is known for $\mathbb{P}^2_{\mathbb{C}}$ blown up in up to 9 points (see Figure 2.68 for details). Hence we know that the Newton-Okounkov body $\Delta_{Y_\bullet}(\tilde{X}_3, \tilde{L}^3)$ will have the form:



Moreover due to our previous calculations we need only calculate sections whose quasi-valuation generate the convex closed hull

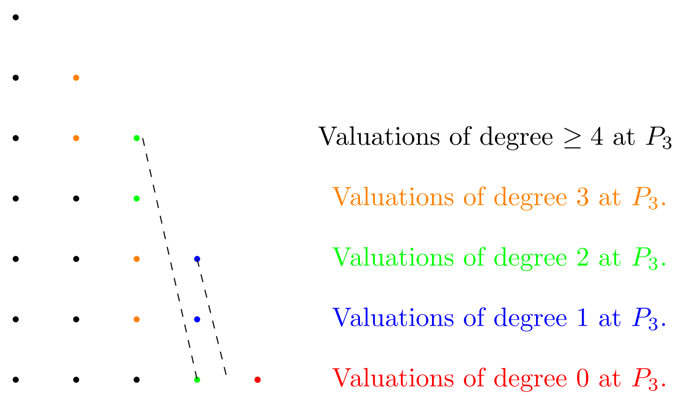
$$conv\{(0,0), \ldots, (0,d), \ldots, (d-2m, 2m), \ldots (d - \frac{3}{2}m, 0), \ldots, (1,0)\}.$$

A simple calculation shows that the above convex body can be obtained using the formulas:

$$\Delta_{Y_\bullet}(d\pi^* L - mE_1 - mE_2 - lE_3) - \Delta_{Y_\bullet}(d\pi^* L - mE_1 - mE_2 - (l+1)E_3) = \begin{cases} (d-m-l'-k, 4k) & l' = 2l \text{ and,} \\ (d-m-l'-k, 4k-1) & 0 \leq k \ \leq l' \end{cases}$$

$$\Delta_{Y_\bullet}(d\pi^* L - mE_1 - mE_2 - lE_3) - \Delta_{Y_\bullet}(d\pi^* L - mE_1 - mE_2 - (l+1)E_3) = \begin{cases} (d-m-l'-k, 4k+2) & l' = 2l-1 \text{ and,} \\ (d-m-l'-k, 4k+1) & 0 \leq k \ \leq l'-1 \end{cases}$$

To determine the correct order of vanish of the remaining sections we use the following procedure:



There are $n+1$ valuations points of degree n for all $n = 1, \ldots, m$. Each valuation point corresponds to one of the $1 + 2 + ... + (m+2)$ homogeneous polynomials of degree m in a basis of $H^0(\mathbb{P}^2, \mathcal{O}(m))$.

From the above picture we already see that the associated filtration only leads to a quasi-valuation as the square of functions corresponding to functions not on the dashed lines must equal 0 in the associated graded ring. To make this clearer we fix $d = 6$ and $m = 2$ and calculate the associated graded ring to show that its Proj has only multiple components.

For these values of d, m we know that the Newton-Okounkov body of sections of $\mathcal{E}_3$ is contained in the convex closed hull

$$conv\{(4,0), (3,0), (3,1), (3,2), (2,4), (2,3)\}.$$

Hence we take generators of the associated graded ring corresponding to the central fiber,

$$X_{0,0}, X_{1,0}, X_{1,1}, X_{1.2}, X_{2,3}, X_{2,4}$$

which correspond to the integer lattices points as follows:

$$\begin{array}{ccc}
\underset{X_{2,4}}{\bullet} & & \\
\underset{X_{2,3}}{\bullet} & & \\
\bullet & \underset{X_{1,2}}{\bullet} & \\
\bullet & \underset{X_{1,1}}{\bullet} & \\
\bullet & \underset{X_{1,0}}{\bullet} & \underset{X_{0,0}}{\bullet}
\end{array}$$

Products of generators are either 0 or equal to products of other generators with the same valuations, so in the $d = 6, m = 2$ case we compute the following relations:

$$\begin{aligned}
X_{1,1}^2 &= 0 \\
X_{2,3}^2 &= 0 \\
X_{1,1} \cdot X_{2,3} &= 0 \\
X_{1,2}^2 &= X_{2,4} \cdot X_{0,0} \\
X_{1,1} \cdot X_{2,4} &= X_{2,3} \cdot X_{12} \\
X_{1,1} \cdot X_{1,2} &= X_{2,3} \cdot X_{0,0}
\end{aligned}$$

Assume $X_{1,2} \neq 0$ and de-homogenise to obtain affine generators

$$x_{0,0},\ x_{1,0},\ x_{1,1},\ x_{2,4},\ x_{2,3}$$

and relations

$$\begin{aligned}
x_{1,1}^2 &= 0 \\
x_{2,3}^2 &= 0 \\
x_{1,1} \cdot x_{2,3} &= 0 \\
x_{2,4} \cdot x_{0,0} &= 1 \\
x_{2,3} \cdot x_{0,0} &= x_{1,1} \\
x_{1,1} \cdot x_{2,4} &= x_{2,3}.
\end{aligned}$$

Let I denote the ideal associated to the above affine generators then the coordinate ring of the affine piece of the central fiber is given by $\mathbb{C}[x_{0,0}, x_{1,0}, x_{2,3}, x_{2,4}]/I$. It is easy to see that the Proj of the associated graded ring has double structure.
To circumvent these problems one can try to deform the central fiber with multiple components further, hopefully to a toric variety, maybe not normal. This seems to be possible because if the monomial generators above are deformed to binomials we obtain a toric ideal whose variety describes (possibly non-normal) toric varieties. If this is successful we produce an explicit symplectic packing of $\mathbb{P}^2$ with 3 balls not described up to now (in contrast to the 2-point case see [Tra95, Figure 2]).

Chapter 5

Outlook

The aim now is to develop a strategy to deform the central fiber of the algebraic family further to a nicer scheme such that the associated quasi-valuation is really a valuation. Then we can apply the procedure detailed in Chapter 4 and construct the desired Kähler packing and moment maps. Thus we can produce explicit Kähler packings for $\mathbb{P}^2_{\mathbb{C}}$ blown up in up to 8 points that are new even for symplectic topologists. For 9 or more points it is expected the current method will fail but we hope that we can at least produce moment maps on balls whose image is the shape conjectured by an interpretation of Nagata's conjecture as described in [Eck14] and Example 2.72.

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