Between clique-width and linear clique-width of bipartite graphs*

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Abstract

We consider hereditary classes of bipartite graphs where clique-width is bounded, but linear clique-width is not. Our goal is identifying classes that are *critical* with respect to linear clique-width. We discover four such classes and conjecture that this list is complete, i.e. a hereditary class of bipartite graphs of bounded clique-width that excludes a graph from each of the four critical classes has bounded linear clique-width.

1 Introduction

Clique-width is a graph parameter which is of primary importance in algorithmic graph theory, because many problems in this area that are generally NP-hard can be solved efficiently when restricted to graphs of bounded clique-width. This parameter generalizes tree-width in the sense that bounded tree-width implies bounded clique-width but not necessarily vice versa.

Recently, many classes of graphs have been shown to be of bounded clique-width, and for many others, the clique-width was shown to be unbounded, see e.g. [7, 8, 10, 23, 24]. Most of these studies concern hereditary classes, i.e. classes closed under taking induced subgraphs. This restriction is justified by the fact that the clique-width of a graph G can never be smaller than the clique-width of an induced subgraph of G. An important feature of hereditary classes is that they admit a description in terms of minimal forbidden induced subgraphs, i.e. minimal graphs that do not belong to the class.

In a similar way, in the study of clique-width of particular importance are *minimal* classes of graphs of unbounded clique-width. The first two hereditary classes of this type have been identified in [21] and only recently it was shown in [10] that the number of such classes is infinite. What is interesting is that all the classes found in [10] and in [21] are also minimal hereditary classes of unbounded *linear* clique-width.

Linear clique-width is a restricted version of clique-width and the relationship between these two parameters is similar to the relationship between tree-width and path-width. The notion of linear clique-width became an important ingredient in the proof of hardness of computing clique-width [13] and received considerable attention in recent years in the literature [2, 9, 17, 18, 19]. Nevertheless, our knowledge of this parameter is still restricted. In particular, we know very little about the behaviour of this parameter on graphs of

*Some results presented in this paper appeared in the extended abstract [3] published in the proceedings of the 29th International Workshop on Combinatorial Algorithms, IWOCAcoloured bi-quasi-threshold graph 2018.

bounded clique-width. In this respect, the recent paper [9] is of particular interest. It deals with the class of complement reducible graphs, also known as cographs, where the clique-width is known to be bounded, while the linear clique-width is not [17]. The authors of [9] show that there exist precisely two minimal hereditary subclasses of cographs of unbounded linear clique-width. These two subclasses are complement to each other and are known in the literature under various names, such as trivially perfect [15] or quasi-threshold graphs [26].

In Section 3, we study a bipartite analogue of cographs, known as *bi-complement reducible graphs* [14] (*bi-cographs* for short), where the clique-width is also known to be bounded. We prove that the linear clique-width is unbounded in the class of bi-cographs and, similarly to [9], show that there exist precisely two minimal hereditary subclasses of bi-cographs of unbounded linear clique-width. However, our solution differs from that in [9] in two important aspects.

Firstly, the two classes we discover in this paper had never been studied before and are of independent interest. We characterize them in terms of minimal forbidden induced subgraphs.

Secondly, and most importantly, we develop an entirely new approach to prove our results. In particular, to prove unboundedness of clique-width we introduce an auxiliary graph parameter which bounds the linear clique-width from below and provides a more flexible tool to prove results of this type. To show minimality of our classes we develop a straightforward approach, which avoids the notion of well-quasi-ordering used by the authors of [9]. Their approach is applicable only to classes well-quasi-ordered by induced subgraphs, which is not the case, for instance, for forests, where clique-width is bounded and linear clique-width is not, similarly to cographs and bi-cographs.

In the presence of infinite antichains, minimal classes may not exist, and this is precisely what happens in the class of forests, as we show in Section 4. To overcome this difficulty, we apply the notion of boundary classes, which is a relaxation of the notion of minimal classes. We identify a subclass $\mathcal{S}$ of forests, which is boundary for linear clique-width in the sense that a subclass X of forests defined by finitely many forbidden induced subgraphs has unbounded linear clique-width if and only if $\mathcal{S}$ is a subclass of X.

In Section 5 we review various other classes of bipartite graphs of bounded clique-width with the aim of identifying other obstacles for bounding linear clique-width. Our analysis suggests a conjecture that in the world of bipartite graphs of bounded clique-width the two subclasses of bi-cographs, the class $\mathcal{S}$ and the class of bipartite complements of graphs in $\mathcal{S}$ are the only critical classes for bounding linear clique-width.

2 Preliminaries

This section introduces basic terminology and notation used in the paper, as well as a number of preliminary results.

2.1 Graphs

Throughout the paper, we will be working with undirected graphs, with no loops or multiple edges. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For a vertex $x \in V(G)$ we denote by $N(x)$ the neighbourhood of x, i.e. the set of vertices of G adjacent to x. A subgraph of G induced by a subset of vertices $U \subseteq V(G)$ is denoted by $G[U]$. We use the following notation for specific graphs:

- P_n is the chordless path on n vertices.
- C_n is the chordless cycle on n vertices.
- $K_{n,m}$ is the complete bipartite graph with parts of size n and m. The graph $K_{1,n}$ is known as a star.
- The domino is the graph obtained from a C_6 by adding an edge between one pair of antipodal vertices.
- Sun_n is the graph obtained from a C_n by adding a pendant vertex to each vertex of the cycle. Sun_4 is shown in Figure 1.
- $S_{i,j,k}$ is the graph represented in Figure 1.

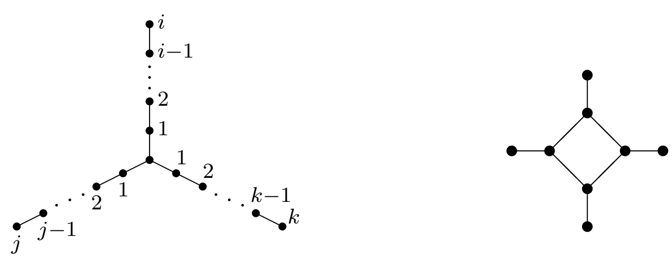

Figure 1: The graphs $S_{i,j,k}$ (left) and Sun_4 (right)

All classes in this paper are hereditary, i.e. closed under taking induced subgraphs. Every hereditary class X can be characterized by a set M of minimal forbidden induced subgraphs, in which case we say that graphs in X are M-free and write $X = Free(M)$.

2.2 Linear clique-width

The *linear clique-width* of a graph G, denoted by $\mathsf{lcw}(G)$, is the smallest number of labels needed to construct G by means of the following three operations:

- add a new vertex with label i (we denote this operation simply by i),
- add all edges between vertices labelled i and vertices labelled k, for $i \neq k$ (denoted by $i \times k$),
- relabel vertices labelled i to k (denoted by $i \to k$).

A *linear clique-width expression* A for a graph G is an ordered sequence of these three operations that constructs G. For instance, the following sequence constructs a path P_k with three different labels, showing that $\mathsf{lcw}(P_k) \leq 3$ for any value of k:

$$1, 2, 1 \times 2, (3, 2 \times 3, 2 \to 1, 3 \to 2)^{k-2}. \tag{1}$$

This example can be extended to $S_{2,2,2}$-free trees, also known as caterpillars, without increasing the number of labels. We record this fact together with one more useful observation in the following claim, the proof of which is an easy calculation.

Claim 1. *The linear clique-width of $S_{2,2,2}$-free trees is at most 3 and the linear clique-width of the graph $S_{i,j,k}$ is at most 5 for any values of i, j, k.*

For a class of graphs Y, we denote by $[Y]_k$ the class of graphs G such that $G - U$ belongs to Y for a subset $U \subseteq V(G)$ of cardinality at most k. The following lemma relates the linear clique-width of a graph class Y with the linear clique-width of $[Y]_k$.

Lemma 1. *If the linear clique-width of graphs in a class Y is bounded by p, then the linear clique-width of graphs in $[Y]_k$ is at most $p + k + 1$.*

Proof. Let $G \in [Y]_k$ and let U be a set of at most k vertices of G such that $G - U \in Y$. Let F be a linear clique-width expression for $G[U]$ that uses labels from the set $\{1, \ldots, k\}$ such that all vertices are assigned distinct labels that never change. Let F' be a linear clique-width expression of $G - U$ that uses labels from the set $\{k + 1, \ldots, k + p\}$. Now a desired expression for G consists of two parts, where the first part is F and the second part is obtained from F' as follows. Whenever a new vertex v with label l is created in F' we instead create vertex v with a special label $k + p + 1$, then connect v to all its neighbours in U, and relabel v to l. Clearly, this is a linear clique-width expression of G and it uses at most $p + k + 1$ labels. □

One more lemma deals with the notion of modular decomposition. To define this notion, consider a graph G, a subset $U \subset V(G)$ and a vertex $v \in V(G)$ not in U. We say that v *distinguishes* U if it has both a neighbour and a non-neighbour in U. A *module* in G is a subset of vertices indistinguishable by the vertices outside of the subset. A module is *trivial* if it consists of a single vertex or includes all the vertices of the graph. A graph G is *prime* if every module of G is trivial.

It is known that the clique-width of graphs in a hereditary class X is bounded if and only if it is bounded for prime graphs in X. Unfortunately, this is not true for linear clique-width, as the example of cographs shows, because this class contains only two prime graphs: K_2 and its complement. However, if we restrict ourselves to bipartite graphs, the reduction to prime graphs is valid as well.

Lemma 2. *Let X be a hereditary class of bipartite graphs. The linear clique-width is bounded for graphs in X if and only if it is bounded for prime graphs in X.*

Proof. One first checks that if U is a non-trivial module in a connected bipartite graph G with bipartition (A, B), then either $U \subseteq A$ or $U \subseteq B$, otherwise as $V(G) \setminus U$ is non-empty and G is connected, there is $x \in V(G) \setminus U$ that is adjacent to all vertices in U, which yields a contradiction because either $N(x) \cap A = \emptyset$ or $N(x) \cap B = \emptyset$. It is then not difficult to see that a bipartite graph G with at least three vertices is prime if and only if it is connected and has no twin vertices, i.e. vertices with the same neighbourhood. Clearly, adding twins to some vertices of G does not increase its linear clique-width. Finally, it is not difficult to see that the linear clique-width of a disconnected graph does not exceed the maximum of the linear clique-widths of its connected components plus 1. This proves the lemma. □

2.3 Classes of bipartite graphs

We will be studying bipartite graphs in particular. We will distinguish between *coloured* and *uncoloured* classes of bipartite graphs. In a coloured class, all bipartite graphs come with a bipartition of their vertex set into two independent sets B and W that we will refer to as *black* and *white* vertices, respectively[1].

For a coloured bipartite graph $G = (B, W, E)$, we define the *bipartite complement* of G to be the coloured bipartite graph $\widetilde{G} = (B, W, E')$, where for any two vertices $x \in B$ and $y \in W$ we have $xy \in E$ if and only if $xy \notin E'$. Also, given two coloured bipartite graphs $G_1 = (B_1, W_1, E_1)$ and $G_2 = (B_2, W_2, E_2)$, we denote by

- $G_1 \cup G_2$ the disjoint union of G_1 and G_2, i.e. $G_1 \cup G_2 = (B_1 \cup B_2, W_1 \cup W_2, E_1 \cup E_2)$.
- $G_1 \times G_2$ the bipartite join of G_1 and G_2, i.e. the bipartite complement of $\widetilde{G}_1 \cup \widetilde{G}_2$. With a slight abuse of notation, when G_1 consists of only one black vertex v, we will write $v \times G_2$ instead of $G_1 \times G_2$.

To each coloured class of bipartite graphs corresponds an uncoloured class of bipartite graphs that we obtain by simply forgetting the colouring of all the graphs.

One of the main objects in this paper is the class of bi-complement reducible graphs that have been introduced in [14] and can be defined as follows.

Definition 1. A *coloured bi-complement reducible graph* (or *coloured bi-cograph* for short) is a coloured bipartite graph defined recursively as follows:

(i) A graph on a single black or white vertex is a coloured bi-cograph.

(ii) If G_1, G_2 are coloured bi-cographs, then so is their disjoint union $G_1 \cup G_2$.

(iii) If G is a coloured bi-cograph, then so is its bipartite complement $\widetilde{G}$.

A *bi-cograph* is a bipartite graph obtained from a coloured bi-cograph by forgetting the colouring.

It is not difficult to see that (iii) in the above definition could be replaced by:

(iii') If G_1, G_2 are coloured bi-cographs, then so is their bipartite join $G_1 \times G_2$.

In [14], an induced subgraph characterisation for bi-cographs is also shown:

Proposition 1. *A bipartite graph is a bi-cograph if and only if it is (P_7, $S_{1,2,3}$, Sun_4)-free.*

One of the goals of this paper is to characterise minimal classes of bi-cographs of unbounded linear clique-width. To prove our characterisation, we will use a modification of linear clique-width suited for coloured bipartite graphs that we define as follows.

Definition 2. The *bipartite linear clique-width* (or bi-linear clique-width, for short) of a coloured bipartite graph G, denoted by $\mathsf{blcw}(G)$, is the minimum number of labels necessary to construct G (as uncoloured) via a linear clique-width expression, but only allowing any given label to be used for either black or white vertices (we will call those labels black or white respectively).

[1] We will always explicitly say it whenever we deal with coloured bipartite graphs.

The following is clear from the definition because it suffices to copy each label, and to reserve one copy for black, and the other for white vertices.

Observation 1. *Let G' be a colouring of a bipartite graph G. Then,* $\mathsf{lcw}(G) \leq \mathsf{blcw}(G') \leq 2 \cdot \mathsf{lcw}(G)$.

We can therefore use the bi-linear clique-width of coloured bipartite graphs in order to study the linear clique-width of bipartite graphs. We start with simple lemmas describing how bi-linear clique-width behaves when taking bipartite complements, unions or joins.

Lemma 3. *If G is a coloured bipartite graph, then* $\mathsf{blcw}(\widetilde{G}) \leq \mathsf{blcw}(G) + 1$.

Proof. Let A' be an expression using $\mathsf{blcw}(G)$ labels such that each label is either black or white. We claim that we can modify A' to find a linear clique-width expression that uses $\mathsf{blcw}(G) + 1$ labels, in which vertices are connected to all of their already constructed neighbours immediately as they are inserted. Indeed, say that a new vertex v is inserted in A' with label l. Whether an already constructed vertex w is a neighbour of v only depends on its label, so we can say that the set of already constructed neighbours of v is a union $\bigcup_{k\in\Lambda}\{w : w \text{ has label } k\}$, where Λ is a set of labels. In A', the label l might already be in use, so if we tried to connect v to all its already constructed neighbours right away, we might inadvertently add some extra edges (that do not appear in G) to the already constructed graph, between vertices labelled l and some other vertices. However, using a new, reserved label to insert v allows us to go around this. We can immediately connect it to all of its neighbours without changing the already constructed graph, and afterwards change the reserved label to the original label used for inserting v in A'. Proceeding inductively allows us to modify A' to an expression giving G with the desired properties.

A linear clique-width expression for $\widetilde{G}$ can be obtained from this modified expression by instead connecting newly inserted vertices to their non-neighbours in G of opposite colour that have already been inserted. □

Lemma 4. *If $G_1, \dots, G_r$ are coloured bipartite graphs with bi-linear clique-widths at most $k_1, \dots, k_r$ respectively, then their union $\bigcup_{i=1}^{r} G_i$ and their join $\prod_{i=1}^{r} G_i$ have bi-linear clique-width at most* $\max\{k_1, k_2 + 2, k_3 + 2, \dots, k_{r-1} + 2, k_r + 2\}$.

Proof. We will prove the statement for the join $\prod_{i=1}^{r} G_i$. The case of the union is similar and we omit the details.

First, we construct G_1 using labels $1, 2, \dots, k_1$ in such a way that no vertices of different colour ever receive the same label, then we relabel all black vertices to 1 and all white vertices to 2. Next, we construct G_2 using labels $3, 4, \dots, k_2+2$ (which are now unused). To construct the bipartite join, we then connect vertices labelled by 1 to all vertices labelled by white labels, except 2, and we connect vertices labelled 2 to all vertices labelled by black labels, except 1. Finally, we relabel all black vertices to 1 and all white vertices to 2. In this way we construct $G_1 \times G_2$ using at most $\max\{k_1, k_2 + 2\}$ labels where in the end all black vertices are labelled 1 and all white vertices are labelled 2. Proceeding in the same way with $G_3, G_4, \dots, G_r$ we will construct the join $\prod_{i=1}^{r} G_i$ with at most $\max\{k_1, k_2 + 2, k_3 + 2, \dots, k_{r-1} + 2, k_r + 2\}$ labels. □

3 Linear clique-width of bi-cographs

3.1 Bi-quasi-threshold graphs

The class of bi-quasi-threshold graphs is a subclass of bi-cographs.

Definition 3. A *coloured bi-quasi-threshold* graph is a coloured bipartite graph defined inductively as follows:

(i.) A graph on a single black or white vertex is a coloured bi-quasi-threshold graph.

(ii.) If G_1, G_2 are coloured bi-quasi-threshold, then so is their disjoint union $G_1 \cup G_2$.

(iii.) If G is a coloured bi-quasi-threshold graph, then the bipartite join of G with a single black vertex is a coloured bi-quasi-threshold graph.

Remark. Note the asymmetry in this definition: we do not allow white dominating vertices while constructing a coloured bi-quasi-threshold graph. However, once we have finished constructing it, we can forget the colouring, thus getting the uncoloured class of bi-quasi-threshold graphs.

Throughout the remainder of the paper, we denote by L the class of all (uncoloured) bi-quasi-threshold graphs, and by $\widetilde{L}$ the class of uncoloured bipartite graphs obtained from bipartite complements of coloured bi-quasi-threshold graphs. It is worth noting that any graph in $\widetilde{L}$ is obtained from a graph in L.

The following provides a characterisation of coloured bi-quasi-threshold graphs, that we use to provide a characterisation of bi-quasi-threshold graphs.

Lemma 5. *The following are equivalent for a coloured bipartite graph G:*

(a) G is a coloured bi-quasi-threshold graph;

(b) G contains no induced P_5 with white centre;

(c) any two black vertices of G have either comparable or disjoint neighbourhoods.

Proof.

(a)⇒(b): A P_5 with white centre is not in L, since it is not a disjoint union, and it does not have a black dominating vertex. Moreover, from the definition, L is hereditary, hence no graph in L contains a P_5 with white centre.

(b)⇒(c): If two black vertices x and y have incomparable and non-disjoint neighbourhoods, then x, y together with a private neighbour of each and with a common neighbour induce a P_5 with white centre.

(c)⇒(a): We want to show that, assuming (c), either G is disconnected, or it has a black dominating vertex (then use induction, and the fact that the condition (c) is hereditary). Suppose G is connected, and let b be a black vertex with a maximal (under set inclusion) neighbourhood. Let w be a white vertex non-adjacent to b (which exists because we assume that b is not a black dominating vertex), and consider a shortest path P from b to w (which exists, since G is connected). Write its vertices as $b = b_0, w_1, b_1, \ldots, b_{k-1}, w_k = w$ (where the vertices w_i are white, and the vertices b_i are black). If $k > 1$, then w_1 is a common neighbour to b and b_1, hence by (c) and maximality of the neighbourhood of b,

$N(b_1) \subseteq N(b)$. In particular, w_2 and b are adjacent, and we have a shorter path between b and w, contradicting the choice of P. This shows $k = 1$, i.e. b and w are in fact adjacent, so b must be a dominating vertex. $\square$

We can now give a forbidden induced subgraph characterisation of the class L.

Theorem 1. *A bipartite graph G is bi-quasi-threshold if and only if G is (P_6, C_6, domino, Sun$_4$)-free.*

Proof. The "only if" direction comes from the fact that any colouring of one of the four graphs in black and white contains a P_5 with white centre, hence by the previous lemma, none of the four graphs is bi-quasi-threshold.

Conversely, suppose G is (P_6, C_6, domino, Sun$_4$)-free. We show that there is a colouring of G in black and white such that there is no P_5 with white centre. This is clear if G is P_5-free, so assume it is not. Without loss of generality, we can assume, in addition, that G is connected. Now find a P_5 induced by a, b, c, d and e such that the neighbourhood of its middle vertex is maximal among all P_5's. Denote by S the part of G containing a, c and e, and by T the other part.

Let $x \in T$ be a neighbour of a. Then x is not a neighbour of e, otherwise a, b, c, d, e, x induce either a C_6 or a domino (depending on whether c and x are adjacent). Additionally, x must be a neighbour of c, otherwise the six vertices induce a P_6. With this in mind, let B be the set of neighbours of a and c, let D be the set of neighbours of e and c (in particular, $b \in B$ and $d \in D$), and let N_c be the set of neighbours of c that are not neighbours of a or e.

Suppose now that a vertex y in T is a non-neighbour of c (i.e. $y \notin B \cup N_c \cup D$, and then y is also a non-neighbour of a and e). Find a path from y to c. Such a path must pass through $B \cup N_c \cup D = N(c)$. Let c' be the vertex of the path just before $N(c)$, and assume without loss of generality that y is adjacent to c'. Let $z' \in N(c)$ be a neighbour of c'.

If c' has a non-neighbour b' in B, then a, b, c, z', c', y is a P_6, contradicting that G is P_6-free. Symmetrically, if c' has a non-neighbour d' in D, then y, c', z', c, d', e is a P_6, contradicting again that G is P_6-free. Therefore, $B \cup D \subseteq N(c')$. We also have that $N_c \subseteq N(c')$, otherwise if $z \in N_c \setminus N(c')$, then $G[\{a, b, c, d, e, z, c', y\}]$ induces a Sun$_4$. We can therefore conclude that $N(c) \subset N(c')$, contradicting the choice of the P_5 a, b, c, d, e because a, b, c', d, e is also a P_5.

We can thus conclude that S has a vertex c dominating T. If there was another P_5 with its centre in T, then that other P_5 cannot contain c and would induce with c a domino. Hence if we colour S black and T white, we obtain a colouring of G with no P_5's with white centre, and by Lemma 5, we can conclude that G is bi-quasi-threshold. $\square$

3.2 Unboundedness of linear clique-width

The main result of this section is the unboundedness of the linear clique-width of L and $\widetilde{L}$. To prove the result we will use an auxiliary graph parameter which provides a lower bound for linear clique-width.

Let G be a graph and A a linear clique-width expression for G. Clearly, A defines a linear order of the vertex set of G, i.e. a permutation π in the symmetric group $\mathcal{S}(V(G))$.

Let us denote by $S_{\pi,i}$ the set consisting of the first i elements of the permutation, and by A_i the maximal prefix of A containing only the vertices of this set. If two vertices in $S_{\pi,i}$ have different neighbourhoods outside of the set, then they must have different labels in A_i, since otherwise in the rest of the expression we would not be able to add a neighbour to one of them without adding it to the other. Therefore, denoting by $\mu_{\pi,i}(G)$ the size of the set $\{N(x) \cap (V(G) \setminus S_{\pi,i}) \mid x \in S_{\pi,i}\}$, we conclude that A uses at least

$$\mu_\pi(G) := \max_i \mu_{\pi,i}(G)$$

different labels to construct G. As a result, the linear clique-width is bounded from below by[2]

$$\mu(G) := \min_{\pi \in \mathcal{S}(V(G))} \mu_\pi(G).$$

Therefore, to prove the main result of the section, it suffices to show that $\mu(G)$ is unbounded in the classes under consideration. In order to do that, we will need a technical lemma describing the behaviour of $\mu(G)$ in some situations.

We introduce some notation for the coming part. Given a graph G and a linear order π of its vertices, we will write $v < w$ if v appears before w in the order, and $v < S$ if v appears before every vertex of a set S. Notice that the order on a graph induces an order on all of its subgraphs in the obvious way.

Every $i \in \{1, \ldots, n\}$ corresponds to a *cut in G with respect to π*, which separates the first i vertices in π from the rest of $V(G)$. It will be useful to mark cuts for which $\mu_{\pi,i}(G)$ is large. We will insert symbols $\alpha, \beta, \ldots$ into our ordered list of vertices to mark such cuts. If α marks a cut with $\mu_{\pi,i}(G) \geq t$, then a set of t vertices in $S_{\pi,i}$ with pairwise different neighbourhoods outside of $S_{\pi,i}$ will be called a *diversity witness of size t* for α. The largest t such that there exists a diversity witness of size t for α will be called the *diversity* of (the cut at) α.

For a coloured bipartite graph G, we define $\mu(G)$ as $\mu(G')$ with G' obtained from G by forgetting the colouring.

Let H be a connected coloured bi-quasi-threshold graph with $\mu(H) = t \geq 2$. Since H is connected and has at least two vertices, it contains both white and black vertices. Let $G = v \times (H \cup H \cup H)$ for a black vertex v, and label the vertices of the three copies of H by $A = \{a_i : 1 \leq i \leq n\}$, $B = \{b_i : 1 \leq i \leq n\}$, and $C = \{c_i : 1 \leq i \leq n\}$, respectively.

Lemma 6. $\mu(G) \geq t + 1$.

Proof. To prove the lemma, we fix an arbitrary permutation π of $V(G)$ and show that $\mu_\pi(G) \geq t + 1$. Let α, β, and γ be the three cuts of diversity of at least t in the three copies of H with respect to the restrictions of π into A, B, and C, respectively. Without loss of generality we assume that $\alpha \leq \beta \leq \gamma$ in π. Let $B' \subset B$ be a diversity witness of size t for β in B, i.e. $B' < \beta$, $|B'| = t$, and the vertices of B' have pairwise different neighbourhoods in the subset of B to the right of β.

Assume first that a vertex a of A appears after β. Since $\mu(H) \geq 2$, there exist vertices of A before α (and in particular before β). Therefore, since H is connected, there must be an edge $a_i a_j$ such that $a_i < \beta < a_j$. Since, by the definition of G, none of the vertices in B' is adjacent to a_j, the set $B' \cup \{a_i\}$ is a diversity witness of size $t + 1$ for β, i.e. $\mu_\pi(G) \geq t + 1$. This conclusion allows us to assume, from now on, that

[2]the parameter $\mu(G)$ is called *group-number* in [19].

- $A < \beta$ and, by a similar argument, $\beta < C$ (we need $t \geq 2$ to make sure we do indeed have vertices of C after γ, and hence after β).

Suppose $v < \beta$. Since C has at least one white vertex, v has a neighbour in C and hence $B' \cup \{v\}$ is a diversity witness of size $t+1$ for β, i.e. $\mu_\pi(G) \geq t+1$. Therefore, in the rest of the proof we assume that

- $v > \beta$.

Assume B' contains a vertex b_i with no neighbour $b_j > \beta$ in B (observe that if such a vertex exists, then it is unique in B', since otherwise B' is not a diversity witness). If b_i is white, then for any black vertex $a_k \in A$, the set $B' \cup \{a_k\}$ is a diversity witness of size $t+1$ for β, because b_i is adjacent to v, while a_k is not (by the definition of G), and every vertex of B' different from b_i has a neighbour to the right of β, while a_k does not. Similarly, if b_i is black, then for any white vertex $a_k \in A$, the set $B' \cup \{a_k\}$ is a diversity witness of size $t+1$ for β in G, because b_i is not adjacent to v, while a_k is, and every vertex of B' different from b_i has a neighbour in B to the right of β, while a_k does not. In both cases, we have $\mu_\pi(G) \geq t+1$.

The above discussion allows us to assume that every vertex of B' has a neighbour in the subset of B to the right of β. Then for any vertex $a_k \in A$, the set $B' \cup \{a_k\}$ is a diversity witness of size $t+1$ for β in G, since a_k has no neighbours in B, i.e. $\mu_\pi(G) \geq t+1$. □

Theorem 2. *Linear clique-width is unbounded in the classes L and $\widetilde{L}$.*

Proof. Let $G_2 \simeq P_4$ given with any colouring. It is easy to see that G_2 is a connected coloured bi-quasi-threshold graph with $\mu(G_2) \geq 2$. Defining $G_k = v \times (G_{k-1} \cup G_{k-1} \cup G_{k-1})$ for $k > 2$, we conclude by Lemma 6 that G_k is a connected coloured bi-quasi-threshold graph with $\mu(G_k) \geq k$. Therefore, for each k, the class L contains a graph of linear clique-width at least k. For class $\widetilde{L}$, a similar conclusion follows from Lemma 3 and Observation 1. □

3.3 Minimality and uniqueness

The goal of this section is to show that the two classes of unbounded linear clique-width identified in the previous section are *minimal* hereditary classes where this parameter is unbounded. Moreover, we prove a more general result showing that the classes L and $\widetilde{L}$ are the *only two minimal hereditary classes* of bi-cographs where the linear clique-width is unbounded.

For a coloured bipartite graph $G = (B, W, E)$, it will sometimes be useful to work with the coloured graph we obtain by swapping the colours. To this end, we define the *reflection* G^R of G as the coloured bipartite graph (W, B, E).

In order to prove the main result of the section, we will use the notion of bi-cotrees, the bipartite analogues of cotrees also defined in [14] (our definition differs slightly in that we label the leaves as well, in order to record their colour):

Definition 4. Let G be a coloured bi-cograph. The *bi-cotree* T_G of G is the rooted labelled tree constructed as follows:

- Start with the root, which corresponds to G.

- For any internal node, label it by 0 if the corresponding subgraph is disconnected, and by 1 if its bipartite complement is disconnected. The children of the node then correspond to connected components, respectively bi-co-components.
- For any leaf, label it by 0 if the corresponding vertex in the graph is white, and by 1 if the corresponding vertex in the graph is black.

Note that the bi-cotree T_{G^R} of G^R is obtained from the bicotree T_G of G by changing the labels of the leaves from 0 to 1 and from 1 to 0. Furthermore, by definition, for any colouring H of a graph in L, its bi-cotree T_H has the property that every 1–node has at most one non-leaf child. Similarly, for any graph colouring K of a graph in $\widetilde{L}$, its bi-cotree T_K has the property that every 0–node has at most one non-leaf child.

Over the next few lemmas, we will be talking about the presence of certain trees in the bi-cotrees of coloured bi-cographs. We will use the following definition of tree containment:

Definition 5. Let S and T be two rooted trees. We say S is *contained* or *appears* in a tree T, if there is an embedding $\phi : S \hookrightarrow T$ with the following properties:

- If $x, y \in V(S)$ and x is an ancestor of y, then $\phi(x)$ is an ancestor of $\phi(y)$.
- If $x, y, z \in V(S)$ and x is the lowest common ancestor of y and z, then $\phi(x)$ is the lowest common ancestor of $\phi(y)$ and $\phi(z)$.
- If S and T are labelled, then ϕ preserves labels.

We say that S is contained in T *internally* if no vertex of S is mapped to a leaf of T.

Our proof strategy for minimality and uniqueness is as follows: we first show that the bi-cotrees of coloured bi-cographs of large bi-linear clique-width must contain large perfect binary trees. We then show that, in particular, certain labelled perfect binary trees must appear in those bi-cotrees in a very specific way. Finally, we show that the latter implies that a family of coloured bi-cographs of unbounded bi-linear clique-width contains either colourings of graphs in L or in $\widetilde{L}$.

For $h \in \mathbb{N}$, let B_h denote the unlabelled perfect binary tree of height h (i.e. the binary tree where every internal node has two children, and all leaves have the same depth h). Let $B_{h,0}$ and $B_{h,1}$ further denote perfect binary trees of height h, with all vertices labelled 0 or 1 respectively.

Lemma 7. *Let G be a coloured bi-cograph and h be a natural number. If T_G does not contain B_h, then the bi-linear clique-width of G is at most $2h$.*

Proof. We prove the lemma by induction on h. The result holds for $h = 1$, since forbidding B_1 means no node has two children, and G is trivial. Suppose the statement holds for some $h \geq 1$. We will prove that the bi-linear clique-width of any graph G whose bi-cotree T_G does not contain B_{h+1} is at most $2h+2$. We proceed by induction on the height of the bi-cotree. Clearly, the statement holds for any graph with the bi-cotree of height at most $h-1$, as in this case the bi-cotree does not contain B_h, and the bi-linear clique-width of the graph is at most $2h \leq 2h+2$ by the induction hypothesis for h. Assume now that the statement holds of any graph with bi-cotree of height at most $r \geq 0$ and suppose that the height of T_G is $r+1$. Let x be the root of T_G. Then at most one of the subtrees rooted at the children of x contains B_h, otherwise T_G would contain a B_{h+1}. If none of those subtrees do, we are done, since by the inductive hypothesis for h, the subgraphs corresponding to each child of x have bi-linear clique-width at most $2h$, and their join or disjoint union can be constructed using two additional labels. Otherwise, let b be the bi-linear clique-width of G, let x_1 be the child whose induced subtree contains a B_h, and b_1 the bi-linear clique-width of the graph corresponding to x_1. Then, by Lemma 4, we have $b \leq \max\{b_1, 2h+2\}$. Since the height of the tree rooted at x_1 is at most $r-1$, by the inductive hypothesis for r we have $b_1 \leq 2h+2$, which implies $b \leq 2h+2$, and hence the lemma. □

We next consider $B_{h,0}$ and $B_{h,1}$. We first show that if one of those trees appears in a certain way in the bi-cotree of a coloured bi-cograph, then that coloured bi-cograph contains either all coloured bi-quasi-threshold graphs up to a certain size, or their bipartite complements.

Definition 6. Let G be a coloured bi-cograph, h be a natural number, and $i \in \{0,1\}$. We say $B_{h,i}$ is *meaningfully embedded* in T_G, if the following hold:

- $B_{h,i}$ is internally contained in T_G with embedding ϕ.
- Let x be a node in $B_{h,i}$ and let y be a child of x. Let P be the path in T_G between $\phi(x)$ and $\phi(y)$. Then there exists a vertex z on P labelled by $1-i$ such that the subtree rooted at z excluding the branch containing $\phi(y)$ has a leaf in T_G corresponding to a black vertex (i.e. a leaf labelled 1).

Lemma 8. *Let G be a coloured bi-cograph, let h be a natural number, and $i \in \{0,1\}$. Furthermore, suppose that $B_{h,i}$ is meaningfully embedded in T_G. Then if $i=0$, G contains all coloured bi-quasi-threshold graphs on at most h vertices as an induced subgraph, and if $i=1$, G contains bipartite complements of coloured bi-quasi-threshold graphs on at most h vertices as an induced subgraph.*

Proof. We assume that $i=0$. The case of $i=1$ is analogous and we omit the details.

We will prove by induction on h that G contains every coloured bi-quasi-threshold graph on at most h vertices as a coloured induced subgraph. If $B_{1,0}$ is meaningfully embedded in T_G, write x for the embedding of the root and y_1, y_2 for the embeddings of its two children. By definition, there is a vertex labelled 1 on the path between x and y_1, and that vertex is not a leaf. It follows that G has at least one edge, and hence it contains both a black and a white vertex, i.e. it contains every coloured bi-quasi-threshold graph on 1 vertex as a coloured induced subgraph.

Assuming the statement holds for some $h \geq 1$, suppose that $B_{h+1,0}$ is meaningfully embedded in T_G. Like before, write x for the embedding of the root, and write y_1, y_2 for the embeddings of its two children. Each of the subtrees of T_G rooted at y_1 and y_2 have a meaningfully embedded $B_{h,0}$, so the corresponding induced subgraphs of G contain all coloured bi-quasi-threshold graphs on h vertices. Since x is labelled 0, G contains the disjoint union of any two such subgraphs, and the second condition in the definition of meaningful embeddings implies that G contains the join of any such subgraph with a single black vertex. The recursive construction of bi-quasi-threshold graphs then implies, as required, that G contains every coloured bi-quasi-threshold graph on $h+1$ vertices. □

The next two lemmas give a Ramsey type result on the presence of large meaningfully embedded $B_{h,i}$.

Lemma 9. *Let $r \geq 1$. There exists $n = n(r) \in \mathbb{N}$ such that any red-blue colouring of B_n contains internally a monochromatic B_r.*

Proof. We will show by induction on r that the recursion $n(r+1) = n(r)+r+2$, $n(1) = 3$, defines a desired function.

To prove the base case $r = 1$, let x be the root of a coloured B_3, y_1, y_2 its children, and z_j $(1 \leq j \leq 4)$ its grandchildren (z_1 and z_2 are the children of y_1, and z_3 and z_4 are the children of y_2). All of those nodes are internal, since they all have descendants (the nodes on the last level of the B_3). Without loss of generality, we may assume that x is red. If a vertex in $\{y_1, z_1, z_2\}$ and a vertex in $\{y_2, z_3, z_4\}$ are also red, we are done, so assume not. Then in one of the two triples, all vertices are blue, and we are also done.

For the induction step, assume that for some $r \geq 1$ any red-blue colouring of $B_{n(r)}$ contains internally a monochromatic B_r, and consider $B_{n(r+1)} = B_{n(r)+r+2}$. By the induction hypothesis, the top $n(r)$ levels of the $B_{n(r+1)}$ contain without loss of generality a red internal B_r. The leaves of the red B_r are embedded at level at most $n(r) - 1$, so their children are at level at most $n(r)$, and the subtrees rooted at those children have height at least $r + 2$. Either those subtrees each contain a red internal node, in which case we have a red internal B_{r+1}, or there is one such subtree with all internal nodes being blue, in which case we have a blue internal B_{r+1}. □

Lemma 10. *Let $r \geq 1$, and let G be a coloured bi-cograph. There exists $m = m(r) \in \mathbb{N}$ such that if T_G contains B_m, then either T_G or T_{G^R} contains a meaningfully embedded $B_{r,0}$ or $B_{r,1}$.*

Proof. The proof consists of two applications of Lemma 9. First, we colour red the nodes of T_G labelled 0 and blue the nodes labelled 1. Since containment is transitive, this guarantees that if T_G contains $B_{n(r)}$, then it contains an internal monochromatic B_r, i.e. a $B_{r,i}$ for some $i \in \{0, 1\}$.

For the second application of the lemma, we start with an internal copy of $B_{n(r),i}$ in T_G. It follows from the definition of the bi-cotree, that for any internal node labelled by i, any of its internal children is labelled by $1-i$. This implies that for a node $x \in B_{n(r),i}$ and a child y of x, the path P between $\phi(x)$ and $\phi(y)$ contains at least one node labelled $1-i$, and for every such node, the subtree rooted at it has leaves outside the branch containing $\phi(y)$. Now, for each non-root node $y \in B_{n(r),i}$ with parent x, pick a vertex z labelled $1-i$ on the path between $\phi(x)$ and $\phi(y)$; if at least one of the leaves in the tree rooted at z excluding the branch containing $\phi(y)$ is black, colour y red. Otherwise colour it blue. Colour the root arbitrarily. It can be checked that a red internal B_r corresponds to a $B_{r,i}$ meaningfully embedded in T_G, while a blue internal B_r corresponds to a $B_{r,i}$ meaningfully embedded in T_{G^R}.

Thus putting $m(r) = n(n(r))$ completes the proof. □

We are ready to prove the main result of the section.

Theorem 3. *Let $H \in L$ and $K \in \widetilde{L}$. The class of (H, K)–free bi-cographs has bounded linear clique-width.*

Proof. Suppose that G is an (H, K)–free bi-cograph, and let $r = \max\{|V(H)|, |V(K)|\}$. Let G' be a colouring of G and let $T_{G'}$ be the bi-cotree of G'. By Lemma 8, there is no meaningfully embedded $B_{r,i}$ in $T_{G'}$ or in $T_{G'^R}$. By Lemma 10, $T_{G'}$ contains no $B_{m(r)}$. Finally, by Lemma 7, $\mathsf{blcw}(G') \leq 2m(r)$, and by Observation 1 we can conclude that $\mathsf{lcw}(G) \leq 2m(r)$. □

4 Linear clique-width of forests

Forests constitute one more class of bipartite graphs, where clique-width is bounded, but linear clique-width is not. Unboundedness of linear clique-width can be shown similarly to Lemma 6, using the parameter $\mu(G)$ introduced in Section 3.2. Alternatively, the reader can refer to [27] where it is proved that the tree B_h has path-width $\lceil h/2 \rceil$, and this combined with the following allows us to state Corollary 1. We do not define *path-width* because we do not manipulate path-decompositions, but refer to [2, 12] for the definition.

Theorem 4. *Let T be a tree. Then:*

- *T has path-width at most k if and only if for every node x of T, at most two of the subtrees of $T \setminus \{x\}$ have path-width at most k, and all the others have path-width at most $k - 1$ [12].*
- *its linear clique-width equals its path-width + 2, unless the maximum vertex degree of T is at most 1, in which case its linear clique-width equals its path-width + 1 [2].*

Corollary 1. *Linear clique-width is unbounded in the set of binary trees.*

We now prove the following result, where by H_i we denote the tree represented in Figure 2. Notice that we allow index i to be equal to 0, in which case H_i coincides with the star $K_{1,4}$. We call any graph of the form H_i an H-graph.

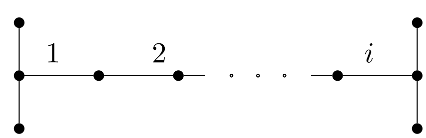

Figure 2: Graph H_i

Lemma 11. *For each positive integer k, the class of $(H_0, H_1, \ldots, H_k)$-free forests has unbounded linear clique-width.*

Proof. For each h and ℓ, let B_h^ℓ be obtained from B_h by subdividing each edge ℓ times. We denote by $\mathcal{T}^k$ the set $\{B_h^k \ : \ h \geq 2\}$. It is easy to check that any binary tree on at least 3 nodes is a minor of a tree in $\mathcal{T}^k$, and then the path-width of the set $\mathcal{T}^k$ is unbounded. From Corollary 1 and Theorem 4, we can conclude that linear clique-width are unbounded in the set $\mathcal{T}^k$. We claim now that graphs in $\mathcal{T}^k$ are $(H_0, H_1, \ldots, H_k)$-free. Indeed, if for some $\ell \leq k$, H_ℓ is an induced subgraph of some B_h^k, then B_h^k contains two nodes of degree 3 and an induced path of length ℓ between both. However, any path between two degree-3 nodes of B_h^k has length at least $k+1$. $\square$

For k tending to infinity, the sequence of classes of $(H_0, H_1, \ldots, H_k)$-free forests converges to the class of forests containing no H-graphs, i.e. to the class of $(H_0, H_1, \ldots)$-free forests. We denote this class by $\mathcal{S}$.

Clearly, all graphs in $\mathcal{S}$ have vertex degree at most 3, since otherwise a forbidden H_0 arises. Also, every connected component of a graph from $\mathcal{S}$ contains at most one vertex of degree 3, since otherwise a forbidden H_i with $i > 0$ arises. Therefore, $\mathcal{S}$ contains graphs every connected component of which is an induced subgraph of some $S_{i,j,k}$ represented in Figure 1. We call graphs in $\mathcal{S}$ *tripods*.

By Claim 1 the linear clique-width of connected tripods is bounded. Therefore, by Lemma 2, the linear clique-width of $\mathcal{S}$ is bounded. As a result, we obtain an infinite sequence of hereditary subclasses of forests and in the limit, linear clique-width jumps from infinity to a finite value without meeting any minimal class of unbounded linear clique-width.

To overcome this difficulty, we will employ the notion of boundary classes. This notion is a relaxation of the notion of minimal classes and it plays a similar role in the universe of classes defined by finitely many forbidden induced subgraphs (finitely defined classes, for short). In recent years, the notion of boundary classes was applied to various problems of both algorithmic [1, 22] and combinatorial [20, 25] nature. Now we introduce it in the context of linear clique-width.

4.1 Boundary classes of graphs

To simplify our discussion, we will call any hereditary class of unbounded linear clique-width *bad* and any hereditary class of bounded linear clique-width *good*.

Definition 7. Given a sequence $X_1 \supseteq X_2 \supseteq X_3 \supseteq \ldots$ of graph classes, we will say that the sequence *converges* to a class X if X is the intersection of classes of the sequence. A class X of graphs is a *limit class* if there is a sequence of bad classes converging to X.

Observe that we do not require the class in the sequence $X_1 \supseteq X_2 \supseteq X_3 \supseteq \ldots$ to be distinct. This means that any bad class is a limit class. The converse statement is false, as the example of classes converging to tripods shows.

The above definition provides a relaxation for the notion of bad classes, replacing it with limit classes. Now we relax the notion of minimal bad classes, replacing it with minimal limit classes, which we call boundary.

Definition 8. A minimal limit class, i.e. a limit class that does not properly contain any other limit class, is called a *boundary* class.

The power of this notion comes from the fact that every limit class contains a minimal limit class. This can be shown through a series of short lemmas that are standard in the theory of boundary classes. To make the paper self-contained, we prove them below. A class of graphs is *finitely defined* if it is equal to $Free(M)$ for a finite set of graphs M.

Lemma 12. *A finitely defined class is a limit class if and only if it is bad.*

Proof. Every bad class is a limit class by definition. Now let $X = Free(G_1, \dots, G_k)$ be a limit class and let $X_1 \supseteq X_2 \supseteq X_3 \supseteq \dots$ be a sequence of bad classes converging to X. There must exist a number n such that X_n is $(G_1, \dots, G_k)$-free. But then for each $i \geq n$, we have $X_i = X$ and therefore X is bad. □

Lemma 13. *If a class Y contains a limit class X, then Y is also a limit class.*

Proof. Let $X_1 \supseteq X_2 \supseteq X_3 \dots$ be a sequence of bad classes converging to X. Then the sequence $(X_1 \cup Y) \supseteq (X_2 \cup Y) \supseteq (X_3 \cup Y) \dots$ consists of bad classes and it converges to Y. □

Lemma 14. *If a sequence $X_1 \supseteq X_2 \supseteq X_3 \dots$ of limit classes converges to a class X, then X also is a limit class.*

Proof. Let $\{G_1, G_2, \dots\}$ be the set of minimal forbidden induced subgraphs of X. For each natural k, define $X^{(k)}$ to be the class $Free(G_1, \dots, G_k)$. Clearly, for every k, there must exist an n such that X_n does not contain $G_1, \dots, G_k$, implying that $X_n \subseteq X^{(k)}$. Therefore, by Lemma 13, $X^{(k)}$ is a limit class, and by Lemma 12, $X^{(k)}$ is bad. This is true for all natural k, and therefore, $X^{(1)} \supseteq X^{(2)} \supseteq X^{(3)} \dots$ is a sequence of bad classes converging to X, i.e. X is a limit class. □

Lemma 15. *Every limit class X contains a minimal limit class.*

Proof. Let X be a limit class. To reveal a minimal limit class contained in X, let us fix an arbitrary linear order $\mathcal{L}$ of the set of all graphs and let us define a sequence $X_1 \supseteq X_2 \supseteq \dots$ of graph classes as follows. We define X_1 to be equal to X. For $i > 1$, let G be the first graph in the order $\mathcal{L}$ such that G belongs to X_{i-1} and $X_{i-1} \cap Free(G)$ is a limit class. If there is no such graph G, we define $X_i := X_{i-1}$. Otherwise, $X_i := X_{i-1} \cap Free(G)$.

Denote by Y the intersection of classes $X_1 \supseteq X_2 \supseteq X_3 \dots$. Clearly, $Y \subseteq X$. By Lemma 14, Y is a limit class. Let us show that Y is a minimal limit class contained in X. For contradiction, assume there exists a limit class Z which is properly contained in Y. Let H be a graph in Y which does not belong to Z. Then $Z \subseteq Y \cap Free(H) \subseteq X_k \cap Free(H)$ for each k. Therefore, by Lemma 13, $X_k \cap Free(H)$ is a limit class for each k. For some k, the graph H becomes the first graph in the order $\mathcal{L}$ such that $X_k \cap Free(H)$ is a limit class. But then $X_{k+1} := X_k \cap Free(H)$, and H belongs to no class X_i with $i > k$, which contradicts the fact that H belongs to Y. □

The importance of the notion of boundary classes is due to the following theorem.

Theorem 5. *A finitely defined class is bad if and only if it contains a boundary class.*

Proof. From Lemma 15, we know that every bad class contains a boundary class. To prove the converse, consider a finitely defined class X containing a boundary class. Then, by Lemma 13, X is a limit class, and therefore, by Lemma 12, X is bad. □

From Lemma 11 we conclude that the class $\mathcal{S}$ of tripods is a limit class. In the next section, we show that $\mathcal{S}$ is a minimal limit, i.e. a boundary class. Moreover, we show that $\mathcal{S}$ is the *only* boundary subclass of forests.

4.2 Minimality and uniqueness

To prove the minimality and the uniqueness of the class $\mathcal{S}$ of tripods, we will show that any hereditary subclass of forests excluding at least one tripod has bounded linear clique-width. We consider forbidden tripods of the form $pS_{i,i,i}$ only, because any tripod is an induced subgraph of $pS_{i,i,i}$ for some values of p and i. We start with the base case, $p = 1$, in Lemma 16 and then complete the proof in Theorem 6. Before, let us state the following easy fact that can be proved from Theorem 4.

Observation 2. *The path-width of any rooted tree T of height h is at most h.*

Lemma 16. *For each positive integer i, the class of $S_{i,i,i}$-free forests has bounded linear clique-width.*

Proof. As in Lemma 2, we need to prove boundedness only for connected $S_{i,i,i}$-free forests. Let T be an $S_{i,i,i}$-free tree and let $P = (1, 2, \ldots, k)$ be a longest path in T. If $k \leq 2i$, then it is not hard to check that any pending subtree has height at most $i - 1$, and then path-width at most $i-1$ by Observation 2. By Theorem 4 we can conclude that the path-width of T is at most i and then its linear clique-width is bounded by $i + 2$. So assume $k > 2i$ and denote by T_j the subtree of $T - (V(P) \setminus \{j\})$ rooted at vertex j of P. Then each T_j has height at most $i-1$. For the first i and the last i vertices of P, this is because P is a longest path, and for the remaining vertices of P this is because T is $S_{i,i,i}$-free. Therefore, each T_j has path-width bounded by i by Observation 2, and by Theorem 4 T has path-width at most i, hence has linear clique-width bounded by $i + 2$. □

Theorem 6. *For positive integers i and p, the class of $pS_{i,i,i}$-free forests has bounded linear clique-width.*

Proof. As in Lemma 2 we need only to prove it for trees. We prove it by induction on p and claim that the linear clique-width of any $pS_{i,i,i}$-free tree T is bounded by $f_i + 4(p-1)$, where f_i is a constant bounding the linear clique-width of $S_{i,i,i}$-free forests. The base case $p = 1$ follows from Lemma 16. For larger values of p, we prove separately the case $p = 2$ and $p > 2$.

Let $p = 2$ and let T be a $2S_{i,i,i}$-free tree. If T is $S_{i,i,i}$-free, we are done by Lemma 16. Otherwise, take any vertex r as a root in T and let v be a farthest vertex from r such that the subtree T_v rooted at v contains a copy of $S_{i,i,i}$. Then, by assumption, for any child u of v, the tree T_u rooted at u is $S_{i,i,i}$-free. By Lemma 16, T_u has linear clique-width at most f_i for each child u of v. Also, if w is a parent of v, then in the graph $T-w$ every connected component not containing v is $S_{i,i,i}$-free, since otherwise an induced $2S_{i,i,i}$ arises. Then, each connected component of $T \setminus \{v, w\}$ has linear clique-width at most f_i. Applying Lemmas 1 and 2, we conclude that T has linear clique-width bounded by $f_i + 4$.

Now let $p > 2$ and let T be a $pS_{i,i,i}$-free tree. If T is $2S_{i,i,i}$-free, we are done by the previous paragraph. If T contains an induced $2S_{i,i,i}$, then let v be any internal vertex of the path connecting the two copies of $S_{i,i,i}$. Then every connected component of $T - v$ is $(p-1)S_{i,i,i}$-free, since otherwise an induced $pS_{i,i,i}$ arises. Therefore, by induction, each of these subtrees has linear clique-width bounded by $f_i + 4(p-2)$. Applying Lemmas 1 and 2 once again, we conclude that T has linear clique-width bounded by $f_i + 4p - 5 \leq f_i + 4(p-1)$. □

5 Towards other classes with a disparity between clique-width and linear clique-width

In the previous sections, we analyzed two classes of bipartite graphs where clique-width is bounded but linear clique-width is not, and identified three critical classes, i.e. three minimal obstacles for bounding linear clique-width. Two of them are minimal classes and the third is a boundary class. According to Lemma 3, the class of bipartite complements of forests (co-forests, for short) is one more class with a disparity between clique-width and linear clique-width, and the bipartite complements of graphs in $\mathcal{S}$ constitute one more obstacle for bounding linear clique-width. The class of bi-cographs is self-complementary in the bipartite sense, and hence the complementary arguments do not lead to new critical classes.

Clearly, any extension of forests, co-forests or bi-cographs of bounded clique-width possesses the same sort of disparity, in which case it is natural to ask whether these extensions contain new minimal or boundary classes for linear clique-width. Another line of research is identifying *new* classes with the disparity between clique-width and linear clique-width, i.e. classes containing neither bi-cographs, nor forests or co-forests, and looking for new critical classes. In this section, we review some classes of bipartite graphs of bounded clique-width with the aim of identifying new candidates exhibiting the disparity and new candidates for critical classes. As a result of our review, we form a conjecture that in case of bipartite graphs there are no more obstacles for bounding linear clique-width.

5.1 Subclasses of chordal bipartite graphs

A graph G is *chordal bipartite* if it is $(C_3, C_5, C_6, C_7, \ldots)$-free. Clique-width is known to be unbounded in the class of chordal bipartite graphs, because it is unbounded even for bipartite permutation graphs [8], which form a proper subclass of chordal bipartite graphs. On the other hand, clique-width is bounded for

- chordal bipartite graphs excluding a bi-clique $K_{p,p}$ [11],
- *domino*-free chordal bipartite graphs, since this is a subclass of distance-hereditary graphs [16],
- F_k-free chordal bipartite graphs [23], where an F_k is a k-fork, i.e. a graph obtained from a star $K_{1,k+1}$ by subdividing one of its edges exactly once.

The class of $K_{p,p}$-free chordal bipartite graphs contains the class of forests for each $p \geq 2$ and hence linear clique-width is unbounded in this class. We conjecture that if additionally we exclude a tripod, then we obtain a class of graphs of bounded linear clique-width, i.e. $K_{p,p}$-free chordal bipartite graphs do not contain any obstacles for bounding linear clique-width different from the class $\mathcal{S}$ of tripods.

The class of *domino*-free chordal bipartite graphs contains all forests and all bi-quasi-threshold graphs, and hence it contains two critical classes. But, again, we conjecture that no new obstacles for bounding linear clique-width can be found in this class.

The class of F_k-free chordal bipartite graphs generalizes $K_{1,k+1}$-free chordal bipartite graphs, i.e. chordal bipartite graphs of vertex degree at most k. For $k = 2$, this class consists of graphs every connected component of which is either a path or an almost complete bipartite graph, i.e. a graph in which every vertex has at most one non-neighbour in the opposite part. Therefore, for $k = 2$ the linear clique-width of F_k-free chordal bipartite graphs is bounded. For larger values of k, the linear clique-width is unbounded, since in this case the class contains all binary trees. Once again, we conjecture that by excluding a tripod (in addition to F_k) we obtain a class of graphs of bounded linear clique-width.

5.2 Monogenic classes of bipartite graphs

A class of bipartite graphs is called *monogenic* if it is defined by a single forbidden induced *bipartite* subgraph. In [24], it was shown that if H is a bipartite graph with both parts being non-empty, then the clique-width of H-free bipartite graphs is bounded if and only if H is an induced subgraph of one of the following four graphs (see Figure 3): $S_{1,2,3}$, $K_{1,3} + 3K_1$, $K_{1,3} + e$, $S_{1,1,3} + v$.

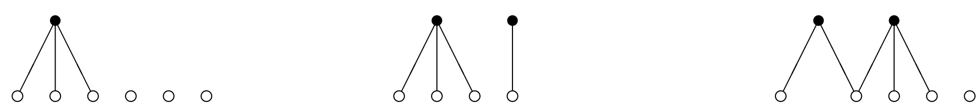

Figure 3: Graphs $K_{1,3} + 3K_1$ (left), $K_{1,3} + e$ (middle) and $S_{1,1,3} + v$ (right)

The class of $S_{1,2,3}$-free bipartite graphs extends the class of bi-cographs, but we conjecture that this extension does not introduce any new obstacle for bounding linear clique-width, i.e. a subclass of $S_{1,2,3}$-free bipartite graphs has bounded linear clique-width if and only if it excludes a bi-quasi-threshold graph and the bipartite complement of a bi-quasi-threshold graph.

In the other three cases, we conjecture that the linear clique-width is bounded. We prove it for the class of $K_{1,3} + 3K_1$-free bipartite graphs and leave the remaining two classes for future research, since the proof is more involved for these classes.

Theorem 7. *The linear clique-width of $(K_{1,3} + 3K_1)$-free bipartite graphs is bounded.*

Proof. Let $G = (W, B, E)$ be a $(K_{1,3} + 3K_1)$-free bipartite graph. Denote by W_l and B_l the sets of white and black vertices of degree at most 2 (low degree) and by W_h and B_h the remaining vertices in W and B, respectively (high degree).

Clearly, every vertex of G has at most two neighbours or at most two non-neighbours in the opposite part, since otherwise a forbidden induced subgraph arises. Therefore, the number of edges of G between W_l and B_h is at most $2|W_l|$ and at least $|B_h|(|W_l|-2)$. Thus $|B_h|(|W_l| - 2) \leq 2|W_l|$, which implies that either $|W_l| \leq 4$ or $|B_h| \leq 4$. Similarly, either $|B_l| \leq 4$ or $|W_h| \leq 4$. Therefore, by deleting from G at most 8 vertices, one can obtain a graph G' such that either G' or its bipartite complement is of vertex degree at most 2. In either case, the linear clique-width of G' is bounded, and so is the linear clique-width of G by Lemma 1. $\square$

5.3 Bigenic classes of bipartite graphs

We say that a class of bipartite graphs is *bigenic* if it is defined by two forbidden induced *bipartite* subgraphs. We are aware of only two bigenic classes of bounded clique-width, which are not subclasses of monogenic classes. In both classes, one of the two forbidden graphs is the graph obtained from Sun_4 by deleting two consecutive vertices of degree 1, i.e. two vertices of degree 1 of distance 3 from each other. We denote this graph by A.

One of bigenic classes of bounded clique-width is the class of $(S_{2,2,2}, A)$-free bipartite graphs. A structural characterization of graphs in this class was proposed in [6] as follows, where a long circular caterpillar is a graph that becomes a cycle of length more than 4 after removing the pendant vertices.

Theorem 8. *A connected prime $(S_{2,2,2}, A)$-free bipartite graph is either a caterpillar or a long circular caterpillar or the bipartite complement of a graph of vertex degree at most 1.*

Corollary 2. *The linear clique-width of $(S_{2,2,2}, A)$-free bipartite graphs is bounded.*

Proof. By Lemma 2 it suffices to prove the result for caterpillar, long circular caterpillars and bipartite complements of graphs of vertex degree at most 1.

Caterpillars are $S_{2,2,2}$-free trees and hence have bounded linear clique-width by Claim 1. A long circular caterpillar becomes a caterpillar after removing any vertex on the cycle. Together with Lemma 1 this implies that these graphs also have bounded linear clique-width. Finally, graphs of vertex degree at most 1 have linear clique-width bounded by 3 and hence the bipartite complements of these graphs have bounded linear clique-width too by Lemma 3. □

One more bigenic class of bipartite graphs of bounded clique-width is the class of (P_k, A)-free bipartite graphs, and now we prove that linear clique-width is bounded in this class too.

Claim 2. *The linear clique-width is bounded in the class of (P_k, A)-free bipartite graphs.*

Proof. It was shown in [4] that any prime A-free bipartite graph containing a C_4 is the bipartite complement of an induced matching (a 1-regular graph). Therefore, (P_k, A)-free bipartite graphs containing a C_4 have bounded linear clique-width.

Now consider a (P_k, C_4)-free bipartite graph G. It was shown in [5] that for any k, p and t, there is a constant $c = c(k, p, t)$ such that any graph containing a path of length c contains either an induced path P_k or an induced biclique $K_{p,p}$ or a clique K_t. Since G is $K_{2,2}$- and K_3-free, we conclude that G does not contain a path P_c as a (not necessarily induced) subgraph. Therefore, path-width and hence linear clique-width is bounded for (P_k, C_4)-free bipartite graphs. □

5.4 Conjecture

We summarize our review of various subclasses of bipartite graphs of bounded clique-width in the following conjecture.

Conjecture 1. A hereditary class of bipartite graphs of bounded clique-width that excludes a tripod, the bipartite complement of a tripod, a bi-quasi-threshold graph and the bipartite complement of a bi-quasi-threshold graph has bounded linear clique-width.

Acknowledgments

Viktor Zamaraev acknowledges support from EPSRC, grant EP/P020372/1. A part of this work was done while V. Lozin visited Université Clermont Auvergne, supported by the ISITE CAP 20-25 Visiting Scholar Fellowship.

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