Intuitionistic Euler-Venn Diagrams (extended)*

Sven Linker

University of Liverpool, UK
s.linker@liverpool.ac.uk

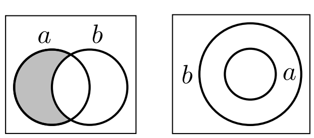
Abstract. We present an intuitionistic interpretation of Euler-Venn diagrams with respect to Heyting algebras. In contrast to classical Euler-Venn diagrams, we treat shaded and missing zones differently, to have diagrammatic representations of conjunction, disjunction and intuitionistic implication. We present a cut-free sequent calculus for this language, and prove it to be sound and complete. Furthermore, we show that the rules of cut, weakening and contraction are admissible.

Keywords: intuitionistic logic · Euler-Venn diagrams · proof theory

1 Introduction

Among diagrammatic systems to reason about logic, Euler-Venn circles have a long tradition. They are known to be a well-suited visualisation of classical propositional logic. In previous work [11], we have presented a proof system in the style of sequent calculus [5] to reason with Euler-Venn diagrams. There, we speculated that, similar to sentential languages, restricting the rules and sequents in the system would allow for intuitionistic reasoning with Euler-Venn diagrams. However, further investigation showed that such a simple change is not sufficient, due to the typical use of the syntax elements of Euler-Venn diagrams.

Consider for example the diagrams in Fig. 1. In the classical interpretation, these diagrams are equivalent: the shaded zone in Fig. 1a denotes that the situation that a is true and b is false is prohibited, which is exactly what the omission of the zone included in the contour a, but not in b in Fig. 1b signifies as well.



(a) (b)

Fig. 1. Euler-Venn diagrams

That is, shading a zone and omitting it is equivalent in classical Euler-Venn diagrams. Additionally, we can interpret these two diagrams in two ways: Fig. 1a may intuitively be read as $\neg(a \wedge \neg b)$: we do not allow for the valuations satisfying a, but not b. Fig. 1b, however, is more naturally read as $a \rightarrow b$: whenever a valuation satisfies a, it also satisfies b. While in a classical interpretation, these two statements are indeed equivalent, they are generally not equivalent in an intuitionistic interpretation. Hence, we want to treat missing zones and shaded

* This work was supported by EPSRC Research Programme EP/N007565/1 *Science of Sensor Systems Software*.

zones differently. Since typically, proof systems for Euler diagrams allow to change missing zones into shaded zones [11,8,19], this implies a stronger deviation from our sequent calculus rules than anticipated.

Furthermore, we want to emphasise a constructive approach to reasoning. In particular, instead of emphasising a *negative* property by prohibiting interpretations of the diagrams, we will treat shading as a *positive* denotation. While this would not make much of a difference in a classical system, negation in intuitionistic systems is much weaker, and hence not suited as a basic element for the semantics of a language.

In this paper, we present an intuitionistic interpretation of Euler-Venn diagrams that takes the preceeding considerations into account. To that end, we will distinguish between *pure Venn*, *pure Euler* and *Euler-Venn* diagrams, and present intuitionistic interpretations of these types of diagrams based on Heyting algebras. Subsequently, we present a proof system in the style of sequent calculus, which we prove to be sound and complete. Furthermore, we show that the structural rules of weakening, contraction and cut are admissible.

Related Work. Many reasoning systems for visualisations of classical logic have been defined over time, for example the initial work of Venn [21] and Peirce [7] and subsequently the work of Shin [18] and Hammer [6], as well Spider diagrams by Howse et al. [8]. Most of these systems are not directly comparable to sentential reasoning systems, due to very different structure of the rules, with the notable exception of the work by Mineshima et al. [14] and Takemura [20].

However, the situation is different for non-classical logics. There are several visual reasoning systems for non-classical variants of Existential Graphs. For example, Bellucci et al. defined *assertive graphs* [1], including a system based on rules for iteration and deletion of graphs, among others. This logical language reflects intuitionistic logic, but the rules manipulate only single graphs, while sequent calculus systems manipulate sequents of diagrams. Ma and Pietarinen presented a graphical system for intuitionistic logic [13] and proved its equivalence with Gentzen's single succedent sequent calculus for propositional intuitionistic logic. To that end, they translate the graphs into sentential formulas. They also extended their approach to existential graphs with quasi-Boolean algebras as their semantics [12]. Legris pointed out that structural rules of sequent calculi can be seen as special instances of rules in the proof systems for existential graphs, to analyse substructural logics [10]. de Freitas and Viana presented a calculus to reason about intuitionistic equations [4]. However, we are not aware of any intuitionistic reasoning systems using Euler-Venn-like visualisations.

Structure of the paper. Following this introduction, we briefly recall the foundations of intuitionistic logic and its semantics in terms of Heyting algebras in Sect. 2. In Sect. 3, we define the system of Euler-Venn diagrams, followed by the graphical sequent calculus system, as well as soundness and completeness proofs, in Sect. 4. Section 5 contains proofs for the admissibility of the structural rules. Finally, we discuss our system and conclude the paper in Sect. 6.

2 Intuitionistic Logic

In this section, we give a very brief overview of the aspects of propositional intuitionistic logic we will use. We start by presenting the underlying semantical model we use, Heyting algebras.

Definition 1 (Heyting Algebra). *A* Heyting algebra $\mathcal{H} = (H, \sqcup, \sqcap, \mapsto, 0, 1)$ *is a bounded, distributive lattice, where* $\sqcup$ *is the join,* $\sqcap$ *the meet,* 0 *the bottom and* 1 *the top element of the lattice. Observe that such a bounded lattice possesses a natural partial order* $\leq$ *on its elements. The binary operation* $\mapsto$*,* the implication*, is defined by* $c \sqcap a \leq b$ *if, and only if,* $c \leq a \to b$*. That is,* $a \to b$ *is the join of all elements* c *such that* $c \sqcap a \leq b$*. We will use the abbreviation* $-a$ *for* $a \mapsto 0$*. Furthermore, we set* $\bigsqcap_{i \in \emptyset} a_i = 1$ *and* $\bigsqcup_{i \in \emptyset} a_i = 0$ *for any* a_i*.*

We collect a few basic properties of Heyting algebras that we need in the following. Proofs can be found, e.g., in the work of Rasiowa and Sikorski [17].

Lemma 1 (Properties of Heyting Algebras). *Let* $\mathcal{H}$ *be a Heyting algebra. Then for all elements* a*,* b *and* c*, we have*

$$a \sqcap (a \mapsto b) \leq b \ (1) \quad (a \mapsto b) \sqcap b = b \ (2) \quad a \mapsto (b \mapsto c) = (a \sqcap b) \mapsto c \ (3)$$

The syntax of propositional intuitionistic logic is similar to classical Boolean logic, with the difference that the operators are not interdefinable. Hence, the signs for conjunction, disjunction, and implication are all necessary as distinct symbols, and cannot be treated as abbreviations. We will assume a fixed, countable set of propositional variables Vars.

Definition 2 (Syntax). *An intuitionistic formula is given by the following EBNF*

$$\varphi :\ = \bot \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \ \text{, where } p \in \mathsf{Vars} \ .$$

We will treat negation as the abbreviation $\neg\varphi \equiv \varphi \to \bot$. Furthermore, we let $\top \equiv \bot \to \bot$. The semantics of a formula is based on valuations, associating each variable with an element of a given Heyting algebra.

Definition 3 (Semantics). *Let* $\mathcal{H}$ *be a Heyting algebra and* $\nu \colon \mathsf{Vars} \to H$ *a* valuation*, mapping variables to elements of* $\mathcal{H}$*. We lift valuations to formulas.*

$$\begin{aligned} \nu(\bot) &= 0 \\ \nu(\varphi \wedge \psi) &= \nu(\varphi) \sqcap \nu(\psi) \\ \nu(\varphi \vee \psi) &= \nu(\varphi) \sqcup \nu(\psi) \\ \nu(\varphi \to \psi) &= \nu(\varphi) \mapsto \nu(\psi) \end{aligned}$$

A formula φ holds *in* $\mathcal{H}$*, if* $\nu(\varphi) = 1$*. If* φ *holds for every valuation of* $\mathcal{H}$*, we write* $\mathcal{H} \models \varphi$*. If* $\mathcal{H} \models \varphi$ *for every Heyting algebra* $\mathcal{H}$*, we say that* φ *is* valid*.*

3 Euler-Venn Diagrams

In this section, we present the syntax and semantics of Euler-Venn diagrams with an intuitionistic interpretation. Generally, a diagram can be *unitary* or *compound*. A unitary diagram consists of a set of *contours* dividing the space enclosed by a bounding rectangle into different *zones*. Zones may also be shaded. Depending on how the contours may be arranged, and whether zones may be shaded, we distinguish between *Venn* diagrams, *Euler* diagrams, and *Euler-Venn* diagrams. Compound diagrams are constructed recursively. Since the structure of compound diagrams is the same, regardless of the type of unitary diagrams, we present their syntax first.

Definition 4 (Compound Diagrams). *A* compound diagram *is created according to the following syntax,*

$$D ::= d \mid D \wedge D \mid D \vee D \mid D \rightarrow D \,,$$

where d is a unitary diagram.

Definition 5 (Compound Diagram Semantics). *The semantics of compound diagrams for a Heyting algebra $\mathcal{H}$ and a valuation ν is given as follows.*

$$\nu(D_1 \wedge D_2) = \nu(D_1) \sqcap \nu(D_2)$$
$$\nu(D_1 \rightarrow D_2) = \nu(D_1) \mapsto \nu(D_2)$$
$$\nu(D_1 \vee D_2) = \nu(D_1) \sqcup \nu(D_2)$$

where D_1, D_2 are compound diagrams. If $\nu(D) = 1$, for all intuitionistic models $\mathcal{H}$ and valuations ν then we call D valid*.*

Observe that we did not give the semantics for unitary diagrams in the previous definition. While we will fill this gap in the next sections, we first present notations that are used for all types of diagrams alike. Formally, a *zone* for a finite set of contours $L \subset \mathsf{Vars}$ is a tuple $(\mathsf{in}, \mathsf{out})$, where in and out are disjoint subsets of L such that $\mathsf{in} \cup \mathsf{out} = L$. We will also write $\mathsf{in}(z)$ and $\mathsf{out}(z)$ to refer to the corresponding sets of contours in z. The set of all possible zones for a given set of contours is denoted by $\mathsf{Venn}(L)$.

Venn Diagrams A *Venn diagram* is a diagram where all possible zones for a set of contours are visible. Formally, a Venn diagram is of the shape $d = (L, \mathsf{Venn}(L), Z^*)$. Hence the only diagrammatic elements that may carry meaning are the presence of contours, and whether a zone is shaded. For a given diagram d, we denote the set of shaded zones also by $Z^*(d)$. We allow for the diagrams $\bot = (\emptyset, \{(\emptyset, \emptyset)\}, \emptyset)$ and $\top = (\emptyset, \{(\emptyset, \emptyset)\}, \{(\emptyset, \emptyset)\})$. A *literal* is a Venn diagram for a single contour, with exactly one shaded zone. If the zone $(\emptyset, \{c\})$ is shaded in a literal, then we call it *the negative literal for c*, otherwise it is *the positive literal for c* (see Fig. 2). Furthermore, if d is the positive literal for c, then we call the negative literal for c the *dual of d* (and vice versa). Observe that our notion of literals deviates from the original definition of Stapleton and Masthoff [19] and from our previous work [11]. The main difference between our presentation and classical Venn diagrams is the interpretation of shaded zones.

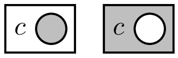


Fig. 2. Literals

While in the traditional approach, shading denotes the *emptiness* of sets, we use shading as a *marker* of elements. That is, the semantics of a diagram consists of the join of the elements denoted by the shaded zones. This is more in line with the constructivist approach we want to emphasise: instead of relying on a negative aspect (emptiness), we construct the semantics out of their building blocks (the shaded zones).

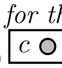
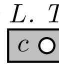
Definition 6 (Zone Semantics). *Let $\mathcal{H}$ be a Heyting algebra, ν a valuation, and z a zone. The* semantics *of z is given by $\nu(z) = \prod_{c \in \mathsf{in}(z)} \nu(c) \sqcap \prod_{c \in \mathsf{out}(z)} -\nu(c)$.*

With the semantics of single zones defined, we can now define the semantics of a Venn diagram in general.

Definition 7 (Venn Diagram Semantics). *For a Venn diagram d, a Heyting algebra $\mathcal{H}$ and a valuation ν, the semantics of d are given by $\nu(d) = \bigsqcup_{z \in Z^*(d)} \nu(z)$.*

Note that we have $\nu(\top) = 1$ and $\nu(\bot) = 0$, for any Heyting algebra $\mathcal{H}$ and valuation ν. Furthermore, for a unitary diagram with a single contour and no shaded zones, i.e. $d = (\{a\}, \mathsf{Venn}(\{a\}), \emptyset)$, we have $\nu(d) = 0$. However, the semantics already diverge from the classical case for a fully shaded diagram with one contour: if $d = (\{a\}, \mathsf{Venn}(\{a\}), \mathsf{Venn}(\{a\}))$, then $\nu(d) = \nu(a) \sqcup -\nu(a)$, which in general is not equal to 1.

Note that this semantics has one consequence in particular: we can decompose a zone into an equivalent compound diagram, and we can furthermore decompose any unitary Venn diagram into a disjunctive normal form.

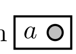
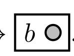
Lemma 2. *Let z be a zone for the contours L. Then the semantics of the compound diagram $d_z = \bigwedge_{c \in \mathsf{in}(z)}$*  *$\wedge \bigwedge_{c \in \mathsf{out}(z)}$*  *equals the semantics of z, i.e. $\nu(d_z) = \nu(z)$. Furthermore, for a Venn diagram $d = (L, \mathsf{Venn}(L), Z^*)$, we have $\nu(d) = \nu(\bigvee_{z \in Z^*} d_z)$.*

Proof. Immediate by the semantics in Def. 6 and Def. 7. □

In particular, this implies that we cannot draw a unitary diagram that expresses intuitionistic implication.

Lemma 3. *Let a and b be propositional variables. Then there is no unitary Venn diagram d such that $\nu(d) = \nu(a \to b)$ for all models and valuations.*

Proof. By Lemma 2, every unitary diagram d can be expressed by using $\vee$ and $\wedge$ only. However, $\to$ is not definable by any combination of $\vee$ and $\wedge$ [17]. □

Observe however that we can trivially define a *compound* diagram  $\to$ .

Pure Euler Diagrams We need additional syntax if we want to express intuitionistic implication diagrammatically. This new syntax needs to be directed (since $a \to b$ is semantically different to $b \to a$). Observe that our notion of zones is *already* directed, and expresses topological information. So, a natural consideration is to allow for *missing* zones in the diagrams. Hence, instead of using Venn diagrams we will now discuss pure Euler diagrams. In contrast to shaded zones, we will treat the missing zones as "restrictions on the construction" of the semantics. First, we give the semantics of a missing zone.

Definition 8 (Missing Zone Semantics). *For a Heyting algebra $\mathcal{H}$, a valuation ν and a zone z, the* missing zone semantics *of z is given by* $\nu(z)_{\mathsf{M}} = \left(\bigsqcap_{c \in \mathsf{in}(z)} \nu(c)\right) \mapsto \left(\bigsqcup_{c \in \mathsf{out}(z)} \nu(c)\right)$.

Definition 9 (Pure Euler Diagrams). *A* pure Euler diagram *is a structure $d = (L, Z)$, where L is the set of contours and Z the set of visible zones of d. Furthermore, the set $\mathsf{MZ}(d) = \mathsf{Venn}(L) \setminus Z$ is the set of* missing zones *of d. The semantics of pure Euler diagrams is that they require the constraints defined by their missing zones to be true. That is, for a pure Euler diagram d, we have $\nu(d) = \bigsqcap_{z \in \mathsf{MZ}(d)} \nu(z)_{\mathsf{M}}$.*

In contrast to Venn diagrams, pure Euler diagrams do not allow for *any* shading. To distinguish pure Euler diagrams from Venn diagrams (and Euler-Venn diagrams, see below), we draw them with dotted contours.

Even with this additional syntax, we are not able to express *every* implication. A simple example would be $a \to a$, since we cannot have a zone $(\{a\}, \{a\})$. However, for this particular example, we do not lose expressivity, since $a \to a \equiv \top$ for all a. But we have a diagram equivalent to $a \to b$, as shown in the left diagram of Fig. 3. The right diagram in Fig. 3 denotes $(a \sqcap b) \mapsto 0$, which is $-(a \sqcap b)$. Observe that in contrast to Venn diagrams without shaded zones, a pure Euler diagram without missing zones denotes 1, i.e., for $d = (L, \mathsf{Venn}(L))$, we have $\nu(d) = \nu(\top) = 1$. Furthermore, the diagram without any contours and zones denotes 0, since $\nu((\emptyset, \emptyset)) = \nu((\emptyset, \emptyset))_{\mathsf{M}} = \bigsqcap_{c \in \emptyset} \nu(c) \mapsto \bigsqcup_{c \in \emptyset} \nu(c) = 1 \mapsto 0 = 0$. In the following, we will need to identify zones that are divided by a contour c abstractly. Intuitively such a zone is split into two zones z and z' that only differ insofar, as c is in $\mathsf{in}(z)$ and in $\mathsf{out}(z')$.

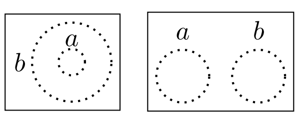


Fig. 3. Pure Euler Diagrams

Definition 10 (Adjacent Zone). *Let $z = (\mathsf{in}, \mathsf{out})$ be a zone for the contours in L and $c \in L$. The* zone adjacent to z at c, *denoted by $\overline{z}^c$ is $(\mathsf{in} \cup \{c\}, \mathsf{out} \setminus \{c\})$, if $c \in \mathsf{out}$ and $(\mathsf{in} \setminus \{c\}, \mathsf{out} \cup \{c\})$ if $c \in \mathsf{in}$.*

Now we can define a way to remove contours from a pure Euler diagram d. This contrasts to our previous work, where we allowed that the diagram to be reduced contains shading [11].

Definition 11 (Reduction). *Let $d = (L, Z)$ be a pure Euler diagram and $c \in L$. The* reduction *of a zone $z = (\mathsf{in}, \mathsf{out})$ is $z \setminus c = (\mathsf{in} \setminus \{c\}, \mathsf{out} \setminus \{c\})$. The* reduction *of d by c is defined as $d \setminus c = (L \setminus \{c\}, Z \setminus c)$, where $Z \setminus c = \{z \setminus c \mid z \in Z\}$.*

Lemma 4 (Properties of Reduction). *We have $z \setminus c = \overline{z}^c \setminus c$. Furthermore, for each $z' \in \mathsf{MZ}(d \setminus c)$ and z with $z \setminus c = z'$, we have $z \in \mathsf{MZ}(d)$. In particular, both $z \in \mathsf{MZ}(d)$ and $\overline{z}^c \in \mathsf{MZ}(d)$.*

Proof. Immediately from the definition of reduction. □

If each missing zone in a pure Euler diagram d has a missing adjacent zone, then the reduction of d by any contour is contained in the semantics of d. In particular, the meet of all reductions equals the semantics of d. This will allow us to show soundness of some rules of the sequent calculus in Sect. 4.

Lemma 5. *Let $d = (L, Z)$ be a pure Euler diagram, where for each $z \in \mathsf{MZ}(d)$, there is a contour $\ell \in L$ such that $\overline{z}^\ell \in \mathsf{MZ}(d)$. Furthermore, let $L' = \{c \mid \mathsf{MZ}(d \setminus c) \neq \emptyset\}$. Then $\bigsqcap_{c \in L'} \nu(d \setminus c) = \nu(d)$*

Proof. Let $c \in L'$ and $z' = (\mathsf{in}, \mathsf{out}) \in \mathsf{MZ}(d \setminus c)$. Then, let $z = (\mathsf{in} \cup \{c\}, \mathsf{out})$. That is $z \setminus c = z'$ and $z, \overline{z}^c \in \mathsf{MZ}(d)$ by Lemma 4 (if $z = (\mathsf{in}, \mathsf{out} \cup \{c\})$, we can reverse the roles of z and $\overline{z}^c$ in the following). Assume $x \leq \nu(z \setminus c)_\mathsf{M}$ Then we have $x \leq \bigsqcap_{a \in \mathsf{in}} \nu(a) \mapsto \bigsqcup_{a \in \mathsf{out}} \nu(a)$, if, and only if, $x \sqcap \bigsqcap_{a \in \mathsf{in}} \nu(a) \leq \bigsqcup_{a \in \mathsf{out}} \nu(a)$. This implies $x \sqcap \bigsqcap_{a \in \mathsf{in}} \nu(a) \leq \bigsqcup_{a \in \mathsf{out}} \nu(a) \sqcup \nu(c)$, which is equivalent to $x \leq \bigsqcap_{a \in \mathsf{in}} \nu(a) \mapsto \bigsqcup_{a \in \mathsf{out}} \nu(a) \sqcup \nu(c) = \nu(\overline{z}^c)_\mathsf{M}$. Also, from $x \leq \nu(z \setminus c)_\mathsf{M}$, we have $x \sqcap \bigsqcap_{a \in \mathsf{in}} \nu(a) \sqcap \nu(c) \leq \bigsqcap_{a \in \mathsf{in}} \nu(a) \leq \bigsqcup_{a \in \mathsf{out}} \nu(a)$, which gives us $x \leq \bigsqcap_{a \in \mathsf{in}} \nu(a) \sqcap \nu(c) \mapsto \bigsqcup_{a \in \mathsf{out}} \nu(a) = \nu(z)_\mathsf{M}$. Hence, we have $x \leq \nu(z)_\mathsf{M} \sqcap \nu(\overline{z}^c)_\mathsf{M}$. That is, for each $z' \in \mathsf{MZ}(d \setminus c)$, we have a $z \in \mathsf{MZ}(d)$ such that $\nu(z')_\mathsf{M} = \nu(z \setminus c)_\mathsf{M} \leq \nu(z)_\mathsf{M} \sqcap \nu(\overline{z}^c)_\mathsf{M}$. Thus, we have $\nu(d \setminus c) \leq \nu(d)$ for each $c \in L'$, that is $\bigsqcap_{c \in L'} \nu(d \setminus c) \leq \nu(d)$.

Conversely, let $x \leq \nu(d)$, i.e. $x \leq \bigsqcap_{z \in \mathsf{MZ}(d)} \left(\bigsqcap_{a \in \mathsf{in}(z)} \nu(a) \mapsto \bigsqcup_{a \in \mathsf{out}(z)} \nu(a)\right)$. For an arbitrary $z \in \mathsf{MZ}(d)$, choose $c \in \mathsf{in}(z)$ and $c \in L'$, i.e., the zone $z \setminus c$ is missing in at least one diagram (namely $d \setminus c$). Of course, we have $x \leq \nu(z)_\mathsf{M}$, from which we get by Lemma 1 (3) $x \leq \bigsqcap_{a \in \mathsf{in}(z) \setminus \{c\}} \nu(a) \mapsto \left(\nu(c) \mapsto \bigsqcup_{a \in \mathsf{out}(z)} \nu(a)\right)$, which is equivalent to $x \sqcap \bigsqcap_{a \in \mathsf{in}(z) \setminus \{c\}} \nu(a) \leq \nu(c) \mapsto \bigsqcup_{a \in \mathsf{out}(z)} \nu(a)$. Furthermore, from $x \leq \nu(\overline{z}^c)_\mathsf{M}$, we also have $x \sqcap \bigsqcap_{a \in \mathsf{in}(z) \setminus \{c\}} \nu(a) \leq \nu(c) \sqcup \bigsqcup_{a \in \mathsf{out}(z)} \nu(a)$. By the properties of a distributive lattice, and Lemma 1 (1) and (2), we then get $x \sqcap \bigsqcap_{a \in \mathsf{in}(z) \setminus \{c\}} \nu(a) \leq \left(\nu(c) \sqcup \bigsqcup_{a \in \mathsf{out}(z)} \nu(a)\right) \sqcap \left(\nu(c) \mapsto \bigsqcup_{a \in \mathsf{out}(z)} \nu(a)\right) \leq \left(\nu(c) \sqcap \left(\nu(c) \mapsto \bigsqcup_{a \in \mathsf{out}(z)} \nu(a)\right)\right) \sqcup \left(\bigsqcup_{a \in \mathsf{out}(z)} \nu(a) \sqcap \left(\nu(c) \mapsto \bigsqcup_{a \in \mathsf{out}(z)} \nu(a)\right)\right) \leq \bigsqcup_{a \in \mathsf{out}(z)} \nu(a)$, which is equivalent to $x \leq \bigsqcap_{a \in \mathsf{in}(z) \setminus \{c\}} \nu(a) \mapsto \bigsqcup_{a \in \mathsf{out}(z)} \nu(a) = \nu(z \setminus c)_\mathsf{M}$. Now, since z was arbitrary, this reasoning holds for all $z \in \mathsf{MZ}(d)$ (possibly with the roles of z and $\overline{z}^c$ reversed), and thus $x \leq \bigsqcap_{c \in L'} \nu(d \setminus c)$, and hence $\nu(d) \leq \bigsqcap_{c \in L'} \nu(d \setminus c)$. □

For an example, consider the derivation in Sect. 5. The diagram d_C^* as shown in Table 1 can be reduced to the three diagrams shown in the application of rule $L\mathsf{r}$ in derivation Π_1 presented in Fig. 10.

Euler-Venn Diagrams In this section, we combine pure Euler diagrams with the central syntactic aspect of Venn diagrams: shading. Our main idea can be summarised as follows: We treat the information given by a pure Euler diagram as a condition for the construction of the combinations of atomic propositions denoted by the shading. That is, whenever we have constructions as indicated by the spatial relations of contours in a diagram d, we also have a construction of the elements denoted by the shaded zones of the diagram. Since we use the syntactic elements of pure Euler diagrams and Venn diagrams, we will subsequently call such diagrams *Euler-Venn diagrams*.

The abstract syntax of Euler-Venn diagrams is similar to Venn diagrams. A diagram is a tuple $d = (L, Z, Z^*)$ consisting of a set of contours L, a set of visible zones Z over L, and a set of shaded zones $Z^* \subseteq Z$. We will often need to refer to the pure Euler or Venn aspects of an Euler-Venn diagram separately. Hence, we introduce some additional notation. For an Euler-Venn diagram $d = (L, Z, Z^*)$ we will write $\mathsf{Venn}(d) = (L, \mathsf{Venn}(L), Z^*)$ for the Venn diagram with the same set of shaded zones as d, and $\mathsf{Euler}(d) = (L, Z)$ for the pure Euler diagram with the same set of visible zones as d. Similarly to pure Venn and Euler diagrams, we will refer to the missing zones of d by $\mathsf{MZ}(d)$ and to its shaded zones by $Z^*(d)$.

Definition 12 (Euler-Venn Diagram Semantics). *The semantics of a unitary* Euler-Venn diagram *for a Heyting algebra $\mathcal{H}$ and a valuation ν is $\nu(d) = \nu(\mathsf{Euler}(d)) \mapsto \nu(\mathsf{Venn}(d))$.*

Observe that with this definition, the semantics for the case $\mathsf{MZ}(d) = \emptyset$ and $Z^*(d) \neq \emptyset$ yields $\nu(d) = 1 \mapsto \bigsqcup_{z \in Z^*(d)} \nu(z) = \bigsqcup_{z \in Z^*(d)} \nu(z)$. Furthermore, we get $\nu(\bot) = 1 \mapsto 0 = 0$ and $\nu(\top) = 1 \mapsto 1 = 1$.

Observe that the language of compound Euler-Venn diagrams can be seen as a subset of intuitionistic logic. In particular, we can translate every diagram into a formula, which we call its *canonical formula*.

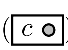
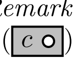
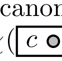
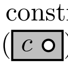
Definition 13 (Canonical Formula). *The* canonical formula *of an Euler-Venn diagram is given by the following recursive definition. We start with the definition of the canonical formula of shaded and missing zones.*

$$\chi^z(z) = \bigwedge_{c \in \mathsf{in}(z)} c \wedge \bigwedge_{c \in \mathsf{out}(z)} -c \qquad \chi^m(z) = \bigwedge_{c \in \mathsf{in}(z)} c \rightarrow \bigvee_{c \in \mathsf{out}(z)} c$$

For a pure Euler diagram d_e, a Venn diagram d_v, an Euler-Venn diagram d and compound diagrams D and E, the canonical formula is given as

$$\chi(d_e) = \bigwedge_{z \in \mathsf{MZ}(d_e)} \chi^m(z) \qquad \chi(d_v) = \bigvee_{z \in Z^*(d_v)} \chi^z(z)$$

$$\chi(d) = \chi(\mathsf{Euler}(d)) \rightarrow \chi(\mathsf{Venn}(d)) \quad \chi(D \otimes E) = \chi(D) \otimes \chi(E) \ \ , \otimes \in \{\wedge, \vee, \rightarrow\}$$

Remark 1. Observe that according to Def. 13, we get $\chi($$) = c \wedge \top$ and $\chi($$) = \top \wedge -c$. However, for simplicity, we will assume that the canonical formula construction omits superfluous occurences of $\top$ and $\bot$. Hence, $\chi($$) = c$ and $\chi($$) = -c$. Similarly, e.g., $\chi^m((\emptyset, L)) = \bigvee_{c \in L} c$.

4 Sequent Calculus

Sequent calculus, as defined by Gentzen [5] is closely related to natural deduction. It is based on *sequents*, which are decomposed by rule applications. In the following, we will define a multi-succedent version of sequent calculus for Euler-Venn diagrams called EDim. This version is inspired by the work of Dragalin [3], while following the more modern presentation of Negri et al. [15].

Definition 14 (Sequent). *A* sequent $\Gamma \Rightarrow \Delta$ *consists of multisets* Γ *and* Δ *of Euler diagrams. The multiset* Γ *is called the* antecedent *and* Δ *the* succedent*.*

If Γ *(*Δ*) is the empty multiset, we write* $\Rightarrow \Delta$ *(*$\Gamma \Rightarrow$*, respectively). If a sequent is of the form* $p, \Gamma \Rightarrow \Delta, p$ *where* p *is a positive literal, then it is called an* axiom*. A sequent* $D_1, \ldots, D_k \Rightarrow E_1, \ldots, E_l$ *is valid, if, and only if,* $\nu(D_1) \sqcap \ldots \sqcap \nu(D_k) \leq \nu(E_1) \sqcup \ldots \sqcup \nu(E_l)$ *for all valuations* ν *in all Heyting algebras. We will often abbreviate* $\nu(D_1) \sqcap \ldots \sqcap \nu(D_k)$ *by* $\nu(\Gamma)$ *and* $\nu(E_1) \sqcup \ldots \sqcup \nu(E_l)$ *by* $\nu(\Delta)$*. That is, for the multiset* Γ *we always mean the meet, while for* Δ *we always refer to the join of the diagrams it consists of.*

A *deduction* for a sequent $\Gamma \Rightarrow \Delta$ is a tree, where the root is labelled by $\Gamma \Rightarrow \Delta$, and the children of each node are labelled according to the rules defined below. If the validity of the premisses of a rule imply the validity of its conclusion, we call the rule *sound*. A deduction where the leaves are labelled with axioms, or instances of $L\bot$ and $R\top$, is called a *proof* for $\Gamma \Rightarrow \Delta$. We will write $\vdash \Gamma \Rightarrow \Delta$ to denote the existence of a proof for $\Gamma \Rightarrow \Delta$. In all rules, we call the diagram in the conclusion that is being decomposed the *principal diagram* of the rule. For example, in $L\wedge$, the principal diagram is $D \wedge E$, and in the rule Ls it is d. For a given proof of $\Gamma \Rightarrow \Delta$, its *height* is the highest number of successive proof rule applications [15]. We will write $\vdash_n \Gamma \Rightarrow \Delta$ if $\Gamma \Rightarrow \Delta$ is provable with a proof of height at most n.

We now turn to define and explain the rules of EDim. The rules to treat compound diagrams, as shown in Fig. 4, are directly taken from sequent calculus for intuitionistic propositional logic and are sound.

Lemma 6 (Soundness). *The rules for sentential operators are sound.*

Proof. A straightforward adaptation of the proofs shown by Ono [16]. □

Remark 2. If we take the placeholders D, E and F as formulas according to Def. 2 and both Γ and Δ as multisets of such formulas, then the rules of Fig. 4 together with axioms $p, \Gamma \Rightarrow \Delta, p$ form the sentential sequent calculus G3im [15]. Provability in G3im is equivalent to provability in Gentzen's system LJ. The system LJ is sound and complete [16]. Hence, G3im is sound and complete as well. Furthermore, the structural rules of weakening, contraction and cut are admissible [15]. Observe that we treat $L\bot$ as a *rule*, and not as an axiom.

$$\frac{D, E, \Gamma \Rightarrow \Delta}{D \wedge E, \Gamma \Rightarrow \Delta}\, L\wedge \qquad \frac{D, \Gamma \Rightarrow \Delta \quad E, \Gamma \Rightarrow \Delta}{D \vee E, \Gamma \Rightarrow \Delta}\, L\vee \quad \frac{\Gamma, D \to E \Rightarrow D \quad E, \Gamma \Rightarrow \Delta}{D \to E, \Gamma \Rightarrow \Delta}\, L{\to}$$

$$\frac{\Gamma \Rightarrow \Delta, D \quad \Gamma \Rightarrow \Delta, E}{\Gamma \Rightarrow \Delta, D \wedge E}\, R\wedge \qquad \frac{\Gamma \Rightarrow \Delta, D, E}{\Gamma \Rightarrow \Delta, D \vee E}\, R\vee \qquad \frac{D, \Gamma \Rightarrow E}{\Gamma \Rightarrow \Delta, D \to E}\, R{\to}$$

$$\frac{}{\Gamma, \bot \Rightarrow \Delta}\, L\bot$$

Fig. 4. Proof Rules for Sentential Operators

Rules for Venn Diagrams. The rules in 5a let us reduce negative to positive literals. Observe that we may introduce arbitrary sets of formulas into the succedent. This ensures admissability of the structural rules (cf. Lemma 13 and 14). Furthermore, the rule $R\top$ lets us finish a proof similarly to $L\bot$. Let $d = (L, \mathsf{Venn}(L), Z^*)$ be a Venn diagram with $|Z^*| > 1$, and let $d_i = (L, \mathsf{Venn}(L), Z_i^*)$, for $i \in \{1, 2\}$, such that $Z^* = Z_1^* \cup Z_2^*$. Then the rules $L\mathsf{s}$ and $R\mathsf{s}$ in Fig. 5b *separate* d into d_1 and d_2. These rules are closely related to the *Combine* equivalence rule for Spider diagrams [8]. For a Venn diagram d with $Z^*(d) = \{z\}$, where $z = (\{n_1, \ldots, n_k\}, \{o_1, \ldots, o_l\})$, the rules $L\mathsf{dec}$ and $R\mathsf{dec}$ of Fig. 5c *decompose* the single zone z into literals.

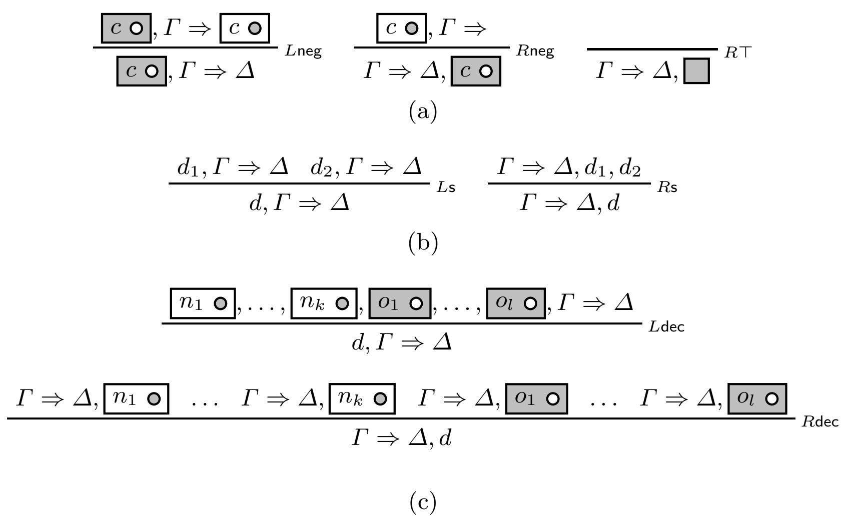


Fig. 5. Rules for Unitary Venn Diagrams

Lemma 7. *The rules shown in Fig. 5 are sound.*

Proof. In all of the following cases, let ν be an arbitrary valuation. The rule $R\top$ is clearly sound, since $\nu(\top) = 1$ for any valuation. For $L\mathsf{neg}$, assume $\nu(\boxed{c\ \circ}) \sqcap$

$\nu(\Gamma) \leq \nu(\boxed{c\,\bullet})$. Then, we have $\nu(\colorbox{lightgray}{$c\,\circ$}) \sqcap \nu(\Gamma) = \nu(\colorbox{lightgray}{$c\,\circ$}) \sqcap \nu(\Gamma) \sqcap \nu(\colorbox{lightgray}{$c\,\circ$}) \sqcap \nu(\Gamma) \leq \nu(\colorbox{lightgray}{$c\,\circ$}) \sqcap \nu(\Gamma) \sqcap \nu(\boxed{c\,\bullet}) \leq \nu(\Gamma) \sqcap 0 = 0 \leq \nu(\Delta)$, where the first inequality is an application of the assumption, and the second is due to Lemma 1 (1). For $R\mathsf{neg}$, assume $\nu(\boxed{c\,\bullet}) \sqcap \nu(\Gamma) \leq 0$. Then we get, by the definition of the implication, the lattice properties, and the semantics of literals, $\nu(\Gamma) \leq \nu(\boxed{c\,\bullet}) \mapsto 0 = \nu(\colorbox{lightgray}{$c\,\circ$}) \leq \nu(\Delta) \sqcup \nu(\colorbox{lightgray}{$c\,\circ$})$.

Consider $R\mathsf{s}$. Assume $\nu(\Gamma) \leq \nu(\Delta) \sqcup \nu(d_1) \sqcup \nu(d_2)$, we have in particular $\nu(\Gamma) \leq \nu(\Delta) \sqcup \bigsqcup_{z \in Z_1^*} \nu(z) \sqcup \bigsqcup_{z \in Z_2^*} \nu(z)$. Since $Z_1^* \cup Z_2^* = Z^*$, and since we can ignore duplicate contour semantics by the lattice properties of Heyting algebras, $\nu(\Gamma) \leq \nu(\Delta) \sqcup \bigsqcup_{z \in Z^*} \nu(z)$, i.e., $\nu(\Gamma) \leq \nu(\Delta) \sqcup \nu(d)$. Now consider $L\mathsf{s}$. We have both $\nu(d_1) \sqcap \nu(\Gamma) \leq \nu(\Delta)$ and $\nu(d_2) \sqcap \nu(\Gamma) \leq \nu(\Delta)$, i.e.,

$$
\begin{aligned}
&\Big(\bigsqcup_{z \in Z_1^*} \nu(z) \sqcap \nu(\Gamma)\Big) \sqcup \Big(\bigsqcup_{z \in Z_2^*} \nu(z) \sqcap \nu(\Gamma)\Big) && \leq \nu(\Delta) \sqcup \nu(\Delta) \\
\Longleftrightarrow\ &\Big(\bigsqcup_{z \in Z_1^*} \nu(z) \sqcup \bigsqcup_{z \in Z_2^*} \nu(z)\Big) \sqcap \nu(\Gamma) && \leq \nu(\Delta) \\
\Longleftrightarrow\ &\Big(\bigsqcup_{z \in Z^*} \nu(z)\Big) \sqcap \nu(\Gamma) && \leq \nu(\Delta)
\end{aligned}
$$

which is exactly $\nu(d) \sqcap \nu(\Gamma) \leq \nu(\Delta)$.

Now consider $L\mathsf{dec}$. By Def. 14, the premiss denotes $\nu(n_1) \sqcap \ldots \sqcap \nu(n_k) \sqcap -\nu(o_1) \sqcap \ldots \sqcap -\nu(o_l) \sqcap \nu(\Gamma) \leq \nu(\Delta)$. But since z is the only shaded zone of d, this is exactly the semantics of $d, \Gamma \Rightarrow \Delta$, by Def. 6 and Def. 14. Finally, consider $R\mathsf{dec}$. Then, we have $\nu(\Gamma) \leq \nu(\Delta) \sqcup \nu(n_i)$ and $\nu(\Gamma) \leq \nu(\Delta) \sqcup -\nu(o_j)$ for all $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, l\}$. By the lattice properties, we get $\nu(\Gamma) \leq (\nu(\Delta) \sqcup \nu(n_1)) \sqcap \ldots \sqcap (\nu(\Delta) \sqcup \nu(n_k)) \sqcap (\nu(\Delta) \sqcup -\nu(o_1)) \sqcap \ldots \sqcap (\nu(\Delta) \sqcup -\nu(o_l))$, which is, by distributivity and since z is the only shaded zone in d, the same as $\nu(\Gamma) \leq \nu(\Delta) \sqcup \nu(d)$. □

Rules for pure Euler Diagrams. Now let $d = (L, Z)$ be a pure Euler diagram, where for each $z \in \mathsf{MZ}(d)$ there is a contour $\ell \in L$, such that $\overline{z}^{\ell} \in \mathsf{MZ}(d)$. Furthermore, let $\{c_1, \ldots, c_k\} \subseteq L$ be the maximal set of contours such that $\mathsf{MZ}(d \setminus c_i) \neq \emptyset$ for every $i \leq k$. Then we can *reduce* d according to the rules $L\mathsf{r}$ and $R\mathsf{r}$ shown in Fig. 6a. Let $d = (L, Z)$ be a pure Euler diagram with more than one missing zone, i.e., $|\mathsf{MZ}(d)| > 1$, and let $d_1 = (L, Z_1)$ and $d_2 = (L, Z_2)$ be two pure Euler diagrams such that $Z_1 \cap Z_2 = Z$. Then the rules $L\mathsf{MZ}$ and $R\mathsf{MZ}$ of Fig. 6b *separate* the diagram z at its missing zones. If d is a pure Euler diagram with a single missing zone, i.e. $\mathsf{MZ}(d) = \{z\}$ and $z = (\{n_1, \ldots, n_k\}, \{o_1, \ldots, o_\ell\})$, then the rules of Fig. 6c decompose z into literals.

Lemma 8. *The rules shown in Fig. 6 are sound.*

Proof. The soundness of the rules $L\mathsf{r}$ and $R\mathsf{r}$ is immediate by Lemma 5. For rules $L\mathsf{MZ}$ and $R\mathsf{MZ}$ observe that by the condition on d_1 and d_2, we have $\mathsf{MZ}(d_1) \cup \mathsf{MZ}(d_2) = \mathsf{MZ}(d)$. That is, $\nu(d_1) \sqcap \nu(d_2) = \nu(d)$ for all valuations and

$$\frac{d \setminus c_1, \ldots, d \setminus c_k, \Gamma \Rightarrow \Delta}{d, \Gamma \Rightarrow \Delta}\, L\mathsf{r} \qquad \frac{\Gamma \Rightarrow \Delta, d \setminus c_1 \quad \ldots \quad \Gamma \Rightarrow \Delta, d \setminus c_k}{\Gamma \Rightarrow \Delta, d}\, R\mathsf{r}$$

(a)

$$\frac{d_1, d_2, \Gamma \Rightarrow \Delta}{d, \Gamma \Rightarrow \Delta}\, L\mathsf{MZ} \qquad \frac{\Gamma \Rightarrow \Delta, d_1 \quad \Gamma \Rightarrow \Delta, d_2}{\Gamma \Rightarrow \Delta, d}\, R\mathsf{MZ}$$

(b)

$$\frac{d, \Gamma \Rightarrow \boxed{n_1 \bullet} \quad \ldots \quad d, \Gamma \Rightarrow \boxed{n_k \bullet} \quad \boxed{o_1 \bullet}, \Gamma \Rightarrow \Delta \quad \ldots \quad \boxed{o_l \bullet}, \Gamma \Rightarrow \Delta}{d, \Gamma \Rightarrow \Delta}\, L\mathsf{Idec}$$

$$\frac{\Gamma, \boxed{n_1 \bullet}, \ldots, \boxed{n_k \bullet} \Rightarrow \boxed{o_1 \bullet}, \ldots, \boxed{o_l \bullet}}{\Gamma \Rightarrow \Delta, d}\, R\mathsf{Idec}$$

(c)

Fig. 6. Proof Rules for pure Euler Diagrams

Heyting algebras. The soundness of both $L\mathsf{MZ}$ and $R\mathsf{MZ}$ follows by straightforward computations. For the rule $R\mathsf{Idec}$, the proof is straightforward by the definition of $\mapsto$ and the lattice properties. The rule $L\mathsf{Idec}$ can be proven sound similarly to $L\mathsf{s}$. $\square$

Rules for Euler-Venn Diagrams. Let d be an Euler-Venn diagram. Then the rules $L\mathsf{det}$ and $R\mathsf{det}$ of Fig. 7 *detach* the spatial relations from the shading.

$$\frac{d, \Gamma \Rightarrow \mathsf{Euler}(d) \quad \mathsf{Venn}(d), \Gamma \Rightarrow \Delta}{d, \Gamma \Rightarrow \Delta}\, L\mathsf{det} \qquad \frac{\mathsf{Euler}(d), \Gamma \Rightarrow \mathsf{Venn}(d)}{\Gamma \Rightarrow \Delta, d}\, R\mathsf{det}$$

Fig. 7. Proof Rules For Euler-Venn Diagrams

Lemma 9. *The rules shown in Fig. 7 are sound.*

Proof. Consider $R\mathsf{det}$, and assume $\nu(\mathsf{Euler}(d)) \sqcap \nu(\Gamma) \leq \nu(\mathsf{Venn}(d))$. Then, by Def. 1, this is equivalent to $\nu(\Gamma) \leq \nu(\mathsf{Euler}(d)) \mapsto \nu(\mathsf{Venn}(d))$, which by Def. 12 and the lattice properties implies $\nu(\Gamma) \leq \nu(\Delta) \sqcup \nu(d)$. So consider $L\mathsf{det}$, and assume both $\nu(d) \sqcap \nu(\Gamma) \leq \nu(\mathsf{Euler}(d))$ and $\nu(\mathsf{Venn}(d)) \sqcap \nu(\Gamma) \leq \nu(\Delta)$. We then have $\nu(d) \sqcap \nu(\Gamma) = \nu(d) \sqcap \nu(\Gamma) \sqcap \nu(d) \sqcap \nu(\Gamma) \leq \nu(d) \sqcap \nu(\Gamma) \sqcap \nu(\mathsf{Euler}(d)) \leq \nu(\mathsf{Venn}(d)) \sqcap \nu(\Gamma) \leq \nu(\Delta)$. The inequalities are correct due to the first premiss, Lemma 1 (1) and the second premiss, respectively. $\square$

By an induction on the height of proofs, we get the soundness theorem for EDim, using Lemma 6, 7, 8, and 9.

Theorem 1 (Soundness). *If $\Gamma \Rightarrow \Delta$ is provable in* EDim*, then $\Gamma \Rightarrow \Delta$ is valid.*

To prove completeness of the system, we first show that certain rules are invertible. Even stronger, a rule is *height-preserving invertible*, if whenever we have a proof of height n for its conclusion, its premisses are provable with a proof of at most height n.

Lemma 10 (Inversions).

1. *All of the rules $L\wedge$, $R\wedge$, $L\vee$ and $R\vee$ are height-preserving invertible.*
2. *All of the rules Ldec, Rdec, Ls, Rs, Lr, Rr, LMZ, and RMZ are height-preserving invertible.*
3. *If $\vdash_n d, \Gamma \Rightarrow \Delta$ for an Euler-Venn diagram d, then also $\vdash_n \mathsf{Venn}(d), \Gamma \Rightarrow \Delta$.*
4. *If $\vdash_n d, \Gamma \Rightarrow \Delta$ for a pure Euler diagram with one missing zone $z = (\{n_1, \dots, n_k\}, \{o_1, \dots, o_l\})$, then also $\vdash_n \boxed{o_i \;\bullet}, \Gamma \Rightarrow \Delta$ for all $1 \leq i \leq l$.*

Proof. The propositional operator rules are height-preserving invertible as shown by Negri et al. [15] (Chap. 5, Lemma 5.3.4). For the rules Ldec, Rdec, Ls, Rs, Lr, Rr, LMZ and RMZ, similar arguments during an induction on the height of the proof yield the result. Case 3 and 4 can be shown by an induction similar to the case of $R \to$. ☐

That these rules can be used in an inverse manner is used in the following lemma, where we connect provability of a sequent $\Gamma \Rightarrow \Delta$ within EDim with the provability of the corresponding sequent $\chi(\Gamma) \Rightarrow \chi(\Delta)$ consisting of the canonical formulas of the antecedent and the succedent.

Lemma 11. *Let $\Gamma \Rightarrow \Delta$ be a sequent of compound diagrams. Then $\Gamma \Rightarrow \Delta$ is provable in* EDim *if, and only if, $\chi(\Gamma) \Rightarrow \chi(\Delta)$ is provable in* G3im.

Proof. Let $\Gamma \Rightarrow \Delta$ be provable in EDim. By Theorem 1, the sequent is valid, and hence the sequent $\chi(\Gamma) \Rightarrow \chi(\Delta)$ is valid as well. Since G3im is complete (cf. Remark 2), the sequent is provable in G3im.

For the other direction, we proceed by induction on the height n of the proof of $\chi(\Gamma) \Rightarrow \chi(\Delta)$. If $n = 0$, then $\chi(\Gamma) \Rightarrow \chi(\Delta)$ is an axiom $p, \Gamma' \Rightarrow \Delta', p$ or an instance of $L\bot$. In the first case, since the only diagram D with $\chi(D) = p$ is a positive literal, $\Gamma \Rightarrow \Delta$ is an axiom as well. Similarly, in the second case, it is an instance of $L\bot$ of EDim. Now assume that the statement is true for all sequents with proofs of height less than n. We proceed by a case distinction on the last rule applied in the proof of $\chi(\Gamma) \Rightarrow \chi(\Delta)$.

If the last rule is $R \to$, then the sequent is of the form $\chi(\Gamma) \Rightarrow \chi(\Delta'), \chi(D)$, where D is either a compound diagram $D = E \to F$, a pure Euler diagram $D = d_e$ with a single missing zone, an Euler-Venn diagram with missing zones and shaded zones $D = d$, a single negative literal for a contour c, or $D = \top$. In the first case, the premiss is then $\chi(E), \chi(\Gamma) \Rightarrow \chi(F)$, which by the induction hypothesis implies that $E, \Gamma \Rightarrow F$ is provable in EDim. An application of $R \to$ then proves $\Gamma \Rightarrow \Delta$. Since all cases, where the principal diagram is compound are treated exactly like this, we will ignore these possibilities in the following. For the case where d is an Euler-Venn diagram, we have $\chi(d) = \mathsf{Euler}(d) \to \mathsf{Venn}(d)$. and hence the premiss of the last step is $\chi(\mathsf{Euler}(d)), \chi(\Gamma) \Rightarrow \chi(\mathsf{Venn}(d))$. By the induction hypothesis, we get that $\mathsf{Euler}(d), \Gamma \Rightarrow \mathsf{Venn}(d)$ is provable, and by applying $R\mathsf{det}$, $\Gamma \Rightarrow \Delta, d$ as well. Now assume that the principal diagram is a pure Euler diagram d_e with a single missing zone $z = (\{n_1, \dots, n_k\}, \{o_1, \dots, o_l\})$. Hence, the premiss of the last step in $\mathsf{G3im}$ is $\bigwedge_{1 \leq i \leq k} n_i, \chi(\Gamma) \Rightarrow \bigvee_{1 \leq i \leq l} o_i$. Since both $L\wedge$ and $R\vee$ are height-preserving invertible, the provability of this sequent is equivalent to the provability of $n_1, \dots, n_k, \chi(\Gamma) \Rightarrow o_1, \dots, o_l$, with height less than n. Since the canonical formula is only atomic for diagram literals, we have that $\boxed{n_1 \circ}, \dots, \boxed{n_k \circ}, \Gamma \Rightarrow \boxed{o_1 \circ}, \dots, \boxed{o_l \circ}$ is provable by the induction hypothesis, and hence by applying $R\mathsf{ldec}$ also $\Gamma \Rightarrow \Delta, d_e$. If the principal formula was a negative literal for c, then the proven sequent is of the form $\chi(\Gamma) \Rightarrow \chi(\Delta), \chi(\fcolorbox{black}{lightgray}{$c \circ$})$. Since $\chi(\fcolorbox{black}{lightgray}{$c \circ$}) = -c = c \to \bot$, the premiss is $c, \chi(\Gamma) \Rightarrow \bot$, which is exactly $\chi(\boxed{c \circ}), \chi(\Gamma) \Rightarrow \bot$. By induction hypothesis, we get a proof for $\boxed{c \circ}, \Gamma \Rightarrow$ in EDim. Thus an application of $R\mathsf{neg}$ yields a proof for $\Gamma \Rightarrow \Delta, \fcolorbox{black}{lightgray}{$c \circ$}$. Finally, if the principal formula was $\top$, then $\chi(D) = \top$, and an application of $R\top$ yields a proof for $\Gamma \Rightarrow \Delta', D$. Observe that $\chi(\Gamma) \Rightarrow \chi(\Delta'), \chi(\top)$ is also provable since the premiss of applying $R \to$ is an instance of $L\bot$.

If the last application in the proof of $\chi(\Gamma) \Rightarrow \chi(\Delta)$ was $L \to$, the arguments are similar, with appropriate applications of $L\mathsf{det}$, $L\mathsf{ldec}$, $L\mathsf{neg}$, and the invertibility of $R\wedge$ and $L\vee$.

If the last application was $R\wedge$, then the last sequent is of the form $\chi(\Gamma) \Rightarrow \chi(\Delta'), \chi(D)$, where either $D = d_e$ is an Euler diagram with more than one missing zone, or $D = d$ is a Venn diagram with exactly one shaded zone. In the first case, this means $\chi(\Gamma) \Rightarrow \chi(\Delta'), \bigwedge_{z' \in \mathsf{MZ}(d_e)} \chi^m(z')$ was proved, and the premisses are $\chi(\Gamma) \Rightarrow \chi(\Delta'), \chi^m(z)$ and $\chi(\Gamma) \Rightarrow \chi(\Delta'), \bigwedge_{z' \in \mathsf{MZ}(d_e) \setminus \{z\}} \chi^m(z')$ for some $z \in \mathsf{MZ}(d_e)$. Now consider the Euler diagrams $d_1 = (L, \mathsf{Venn}(L) \setminus \{z\})$ and $d_2 = (L, (\mathsf{Venn}(L) \setminus \mathsf{MZ}(d)) \cup \{z\})$. Then $\chi(d_1) = \chi^m(z)$ and $\chi(d_2) = \bigwedge_{z' \in \mathsf{MZ}(d_e) \setminus \{z\}} \chi^m(z')$. Hence, we get by the induction hypothesis that $\Gamma \Rightarrow \Delta', d_1$ and $\Gamma \Rightarrow \Delta', d_2$ are provable, and thus an application of $R\mathsf{MZ}$ yields a proof of $\Gamma \Rightarrow \Delta$. For the second case, assume $D = d$ is a Venn diagram with exactly one shaded zone $z = (\{n_1, \dots, n_k\}, \{o_1, \dots, o_l\})$, i.e., the sequent is in the form $\chi(\Gamma) \Rightarrow \chi(\Delta'), \bigwedge_{1 \leq i \leq k} n_i \wedge \bigwedge_{1 \leq i \leq l} -o_i$. Assume without loss of generality that n_1 is part of the outer conjunction, i.e., the conjunction in the succedent is of the form $n_1 \wedge \left(\bigwedge_{2 \leq i \leq k} n_i \wedge \bigwedge_{1 \leq i \leq l} -o_i\right)$. Hence, the premisses are of the form $\chi(\Gamma) \Rightarrow \chi(\Delta'), n_1$ and $\chi(\Gamma) \Rightarrow \chi(\Delta'), \bigwedge_{2 \leq i \leq k} n_i \wedge \bigwedge_{1 \leq i \leq l} -o_i$. Since $R\wedge$ is height-preserving invertible, all sequents of the form $\chi(\Gamma) \Rightarrow \chi(\Delta'), n_i$ and $\chi(\Gamma) \Rightarrow \chi(\Delta'), -o_i$ are provable with a proof of height less than n. From the induction hypothesis, and Remark 1, we get that all of the sequents $\Gamma \Rightarrow \Delta', \boxed{n_i \circ}$ and $\Gamma \Rightarrow \Delta', \fcolorbox{black}{lightgray}{$o_i \circ$}$ are provable, and hence $\Gamma \Rightarrow \Delta$ is provable with an application of $R\mathsf{dec}$.

If the last rule applied in the proof is $L\wedge$, the arguments are similar, with suited applications of $L\mathsf{MZ}$ and $L\mathsf{dec}$.

Now, assume that the last rule applied was $R\vee$. Then, the only possibility is that the principal diagram is a Venn diagram with more than one shaded zone, i.e., the sequent is $\chi(\Gamma) \Rightarrow \chi(\Delta'), \bigvee_{z \in Z^*(d)} \chi^z(z)$. So without loss of generality assume that the premiss is $\chi(\Gamma) \Rightarrow \chi(\Delta'), \chi^z(z_i), \bigvee_{z \in Z^*(d)\setminus\{z_i\}} \chi^z(z)$. Consider the Venn diagrams $d_1 = (L, Zd, \{z_i\})$ and $d_2 = (L, Zd, Z^*d \setminus \{z_i\})$, and observe that $\chi(d_1) = \chi^z(z_i)$ and $\chi(d_2) = \bigvee_{z \in Z^*(d)\setminus\{z_i\}} \chi^z(z)$. That is, by the induction hypothesis, we have that $\Gamma \Rightarrow \Delta', d_1, d_2$ is provable, and hence by an application of $R\mathsf{s}$, we can prove $\Gamma \Rightarrow \Delta$.

The case for $L\vee$ is similar, with an appropriate application of $L\mathsf{s}$. □

Since every valid sequent is derivable in G3im, we get the completeness result for EDim directly from Lemma 11.

Theorem 2 (Completeness). *If $\Gamma \Rightarrow \Delta$ is valid, then $\Gamma \Rightarrow \Delta$ is provable.*

Figure 8 consists of a simple proof containing only Venn diagrams with a single contour. It shows how disjunction and shaded zones interact. That is, the presence of several shaded zones can be proven from simpler diagrams. In particular, this proof shows the similarity between the separation rules ($L\mathsf{s}$ and $R\mathsf{s}$) and the rules for disjunction. Furthermore, we can see how the rules $L\mathsf{neg}$ and $R\mathsf{neg}$ can be used to reduce a sequent with negative literals to an axiom.

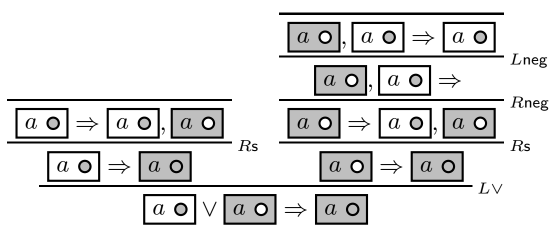


Fig. 8. Example of a Simple Proof

5 Admissible Rules

We show that some rules are admissible. To that end, we define the *weight* of diagrams, to order them by the number of their syntactic elements.

Definition 15. *The* weight $\omega(d)$ *of a diagram is defined inductively. The base cases are given by* $\omega(\bot) = 0$, $\omega($ [c ●] $) = 0$, *and* $\omega($ [c ○, shaded] $) = 1$. *Otherwise we set*

$$\omega(d) = \begin{cases} |Z^*(d)| + 1 & \text{, if } d \text{ is a Venn diagram} \\ |\mathsf{MZ}(d)| + 1 & \text{, if } d \text{ is a pure Euler diagram} \\ \omega(\mathsf{Euler}(d)) + \omega(\mathsf{Venn}(d)) + 1 & \text{, if } d \text{ is an Euler-Venn diagram} \\ \omega(d_1) + \omega(d_2) + 1 & \text{, if } d = d_1 \otimes d_2 \text{ for } \otimes \in \{\wedge, \vee, \rightarrow\} \end{cases}$$

Lemma 12. *For any diagram D, the sequent $D, \Gamma \Rightarrow \Delta, D$ is provable in* EDim.

Proof. A straightforward induction on the weight of D. □

Lemma 13 (Admissibility of Weakening). *i) If $\vdash_n \Gamma \Rightarrow \Delta$, then also $\vdash_n D, \Gamma \Rightarrow \Delta$. ii) If $\vdash_n \Gamma \Rightarrow \Delta$, then also $\vdash_n \Gamma \Rightarrow \Delta, D$.*

Proof. By induction on the height of the proof for $\Gamma \Rightarrow \Delta$. For i), we can add a new diagram into the antecedent of the sequent at the inductive step, since Γ is kept from the premisses to the conclusion. In case ii), this works for most rules as well, except, where the succecedent of the premiss is restricted (e.g. $R\mathsf{neg}$). In these cases, the weakening diagram D is simply added to the multiset Δ in the rule's conclusion. □

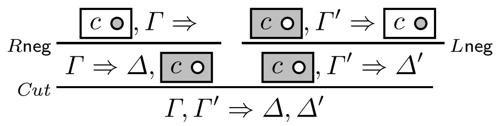
Lemma 14 (Admissibility of Contraction). *i) If $\vdash_n D, D, \Gamma \Rightarrow \Delta$, then also $\vdash_n D, \Gamma \Rightarrow \Delta$. ii) If $\vdash_n \Gamma \Rightarrow \Delta, D, D$, then also $\vdash_n \Gamma \Rightarrow \Delta, D$.*

Proof. Both cases can be proven by an induction on the height of proofs using Lemma 10 and arguments similar to Negri et al. [15]. In case ii), the only special case are rules with restricted right context in the premisses (e.g. $R\mathsf{det}$), where the contraction is done by changing the right context appropriately. □

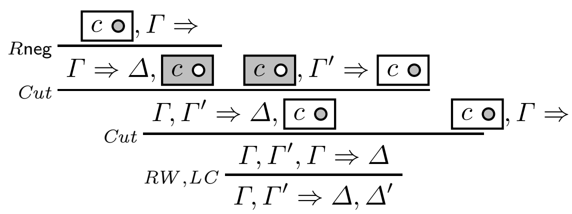
Lemma 15 (Admissibility of Cut). *If both $\Gamma \Rightarrow D, \Delta$ and $D, \Gamma' \Rightarrow \Delta'$ are provable, then also $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ is provable.*

Proof. We use a semantic proof, employing both soundness and completeness of EDim. If both sequents are provable, they are also valid, by soundness. So choose an arbitrary valuation ν. Then $\nu(\Gamma) \leq \nu(D) \sqcup \nu(\Delta)$ and $\nu(D) \sqcap \nu(\Gamma') \leq \nu(\Delta')$. Now we have $\nu(\Gamma) \sqcap \nu(\Gamma') \leq (\nu(D) \sqcup \nu(\Delta)) \sqcap \nu(\Gamma') = (\nu(D) \sqcap \nu(\Gamma')) \sqcup (\nu(\Delta) \sqcap \nu(\Gamma')) \leq \nu(\Delta') \sqcup (\nu(\Delta) \sqcap \nu(\Gamma')) \leq \nu(\Delta') \sqcup \nu(\Delta)$. These relations are due to the first premiss, distributivity, the second premiss and the fact $a \sqcap b \leq a$, respectively. Since ν was arbitrary, $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ is valid, and due to the completeness of EDim, we have that $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ is provable. □

Remark 3. It is also possible to prove cut admissibility with a purely syntactic argument by adapting the inductive proof for the system $\mathsf{G3im}$ given by Negri et al.[15]. The proof consists of a replacement of each cut application with a derivation, where each cut either posesses a lower cut-height, or the weight of the cut diagram is lower. Within that proof, most cases are straightforward, where $L\mathsf{dec}$, $R\mathsf{dec}$, $L\mathsf{r}$, $R\mathsf{r}$, $L\mathsf{MZ}$ and $R\mathsf{MZ}$ are treated similarly to the rules $L\wedge$ and $R\wedge$, while $L\mathsf{s}$ and $R\mathsf{s}$ play roles similar to $L\vee$ and $R\vee$. The rules $L\mathsf{neg}$, $R\mathsf{neg}$, $L\mathsf{det}$, $R\mathsf{det}$, $L\mathsf{Idec}$ and $R\mathsf{Idec}$ need special attention, since they restrict the succedent in the premiss. However, the proof proceeds in these cases along the lines of the the treatment of $L\rightarrow$ and $R\rightarrow$ in $\mathsf{G3im}$. While the number of cases to consider increases, the arguments and constructions are similar. As an example, we present the case where the cut formula is principal in both premisses, and is a negative literal. That is, we have a derivation of the following form:



Observe that the cut-height of this cut application is $m + n + 2$, where m is the height of the proof of the left premiss and n the height of the proof of the right premiss. Then, we can replace this derivation with the following.



In this derivation, the uppermost cut has a lower cut-height, while the second cut uses a cut diagram of lower weight. Here, it is crucial that the negative literal has a higher weight than the positive literal. The last step in the derivation is a sequence of weakening and contraction. The treatment of the other cases is analogous.

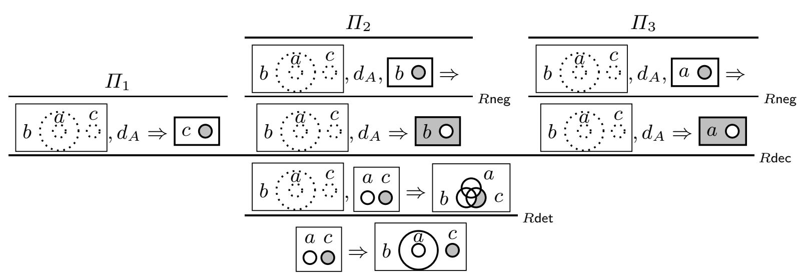
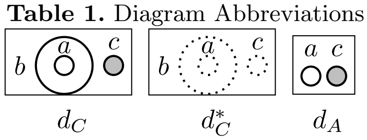


Fig. 9. Proof using Euler-Venn diagrams

A derivation that uses all three types of diagrams can be found in Fig. 9. We explain parts of the proof from bottom to top. The last applied rule detaches the pure Euler part from the Venn part of the succedent, so that we can then decompose the single shaded zone into literals. This splits the proof into three branches, which we treat in the sub-derivations Π_1, Π_2 and Π_3, respectively. For reasons of brevity, we use the abbrevations for diagrams as shown in Table 1. Now, the two right proof branches contain a negative literal in the succedent, which we move to the antecedent with an application of

Rneg. Then, all three proof branches proceed similarly: we reduce the pure Euler diagram d_C^* into smaller diagrams. The set of missing zones is $\mathsf{MZ}(d_C^*) = \{(\{a\},\{b,c\}),(\{a,c\},\{b\}),(\{b,c\},\{a\}),(\{a,b,c\},\emptyset)\}$, and each of these missing zones has at least one adjacent missing zone. For example, $\overline{(\{a\},\{b,c\})}^c = (\{a,c\},\{b\})$. In particular, the reduction of d_C^* with respect to any of the contours a, b and c still contains missing zones. It is easy to check that the three diagrams shown in the derivations are indeed these reductions. Then, Π_1 proceeds by detaching the Euler and Venn aspects of the diagram d_A, which immediately closes the left branch, due to Lemma 12. The right branch ends in an axiom after decomposing the single shaded zone in the antecedent. Within Π_2 there is a similar structure, denoted by the derivation Π_1', where the antecedent contains slightly different diagrams, but the application of rules is similar. The other branches proceed similarly. This example shows, how the reduction rules lead to smaller diagrams, and, as we claim, better readable diagrams, due to the reduced clutter [9]. Furthermore, it shows how the admissible rules may reduce the size of the proofs, here in the form of the generalised axioms proven admissible in Lemma 12.

Table 1. Diagram Abbreviations



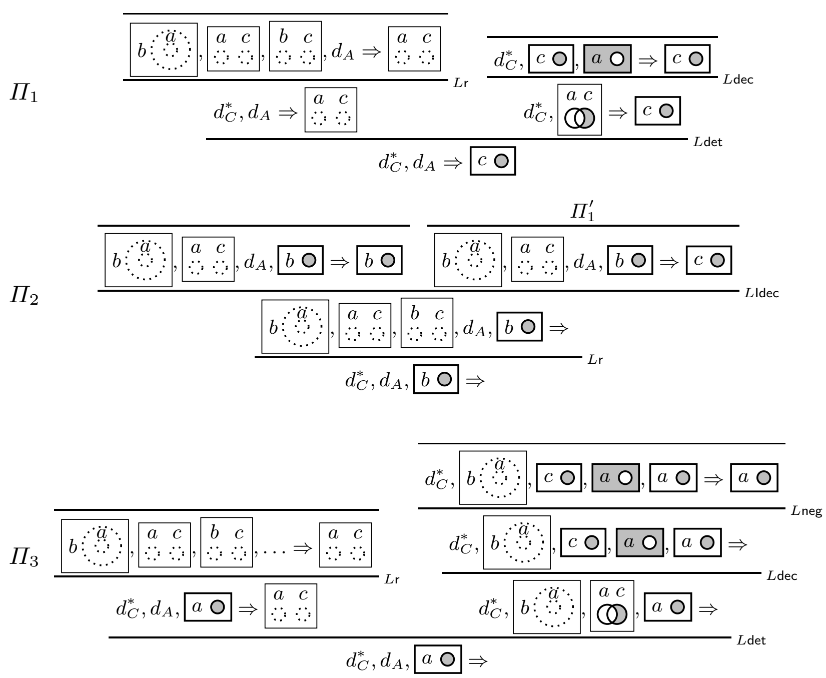


Fig. 10. Auxiliary Derivations for Fig. 9

6 Conclusion

In this paper, we presented an intuitionistic interpretation of Euler-Venn diagrams, based on a semantics of Heyting algebras. We then defined a cut-free sequent calculus EDim, which we have proven to be sound and complete with respect to this semantics. Furthermore, we have shown that the structural rules of contraction, weakening and cut are admissible.

For this visualisation, we deviated from classical Euler-Venn diagrams in two ways: we did not treat missing zones and shaded zones as equivalent, and we introduced the new syntactic element of dashed contours.

The first deviation is due to the basic restrictions of intuitionistic reasoning. More specifically, intuitionistic implication cannot be treated as an abbreviation of the other operators. To have a syntax explicitly for implications, we need to increase the number of distinct syntactic elements of Euler-Venn diagrams. Hence, distinguishing these two elements is a natural choice. Of course, it can be argued that the choice we made is not the correct one, and that shading should be used to reflect implications. However, we think that since the representation of missing zones (or rather their absence) introduces a direction into the diagram, in the form of inclusions, this choice is justified.

The introduction of dashed diagrams is more debatable. Arguably, the need for distinguishing pure Euler diagrams by dashing arises, since we interpret the missing zones of Euler-Venn diagrams as a kind of "constructive precondition" for the construction of the elements denoted by the shaded zones. That is, in the constructive interpretation of intuitionistic reasoning, an Euler-Venn diagram means that, given a construction as indicated by the missing zones, we have another construction for the assertions given by the shaded zones. Hence, there is an additional implication within the semantics of Euler-Venn diagrams, as can also be seen in the rules of EDim to detach the pure Euler aspects from the Venn aspects of a diagram. These rules behave similarly to the rules for implication in sentential intuitionistic sequent calculus.

However, the introduction of new syntactic elements is necessary, due to the independence of the operators, and the restrictive nature of Euler-Venn diagrams makes this need even more overt. Compare for example the intuitionistic systems based on Existential Graphs (EGs). While the operations in classical EGs are denoted by juxtaposition and cuts, reflecting conjunction and negation, respectively, the assertive graphs [1] explicitly introduce notation for disjunction, and also treat the "scroll" as a distinct element. Similarly, the intuitionistic EGs [13] include the notion of *n-scrolls* for each $n > 0$.

We think that our system stretches the idea of Euler-Venn diagrams quite far. In particular, logics that need even more independent operators, for example substructural logics and modal logics, may not be well-matched for such a diagrammatic system. While it may be possible to define such an interpretation, the type of new syntactic elements is far from obvious, if we want to keep the diagrammatic structure of Euler-Venn diagrams. Of course, it is always possible to add new operators to the compound part of the reasoning system, but we think that such an addition misses the point of a diagrammatic reasoning system.

Still, there are future directions this work can be taken into. For example, our sequent calculus resembles sentential sequent calculus, while typical Euler-Venn reasoning systems work by adding syntax to single diagrams, and then removing unnecessary parts [2]. It is interesting to see, if we can define such a system for intuitionistic Euler-Venn diagrams. We assume that for the rules to introduce and remove contours, or to copy contours from one diagram into another, the reduction of a pure Euler diagram (cf. Def 11 and Lemma 5) will play a significant role.

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