Intuitionistic Euler-Venn Diagrams*

Sven Linker[0000−0003−2913−7943]

University of Liverpool, UK
s.linker@liverpool.ac.uk

Abstract. We present an intuitionistic interpretation of Euler-Venn diagrams with respect to Heyting algebras. In contrast to classical Euler-Venn diagrams, we treat shaded and missing zones differently, to have diagrammatic representations of conjunction, disjunction and intuitionistic implication. Furthermore, we need to add new syntactic elements to express these concepts. We present a cut-free sequent calculus for this language, and prove it to be sound and complete. Furthermore, we show that the rules of cut, weakening and contraction are admissible.

Keywords: intuitionistic logic · Euler-Venn diagrams · proof theory

1 Introduction

Most visualisations for logical systems, like Peirce's Existential Graphs [6] and the Venn systems of Shin [16], are dedicated to some form of classical reasoning. However, for example, within Computer Science, constructive reasoning in the form of intuitionistic logic is very important as well, due to the Curry-Howard correspondence of constructive proofs and programs, or, similarly, of formulas and types. That is, each formula corresponds to a unique type, and a proof of the formula corresponds to the execution of a function of this type. Hence, a visualisation of intutionistic logic would be beneficial not only from the perspective of formal logic, but also for visualising program types and their relations.

Typical semantics of intuitionistic logic are given in the form of Heyting algebras, a slight generalisation of Boolean algebras, and an important subclass of Heyting algebras is induced by topologies: the set of open sets of a topology forms a Heyting algebra. In particular, it is well known that intuitionistic formulas are valid, if and only if, they are valid on this subclass of Heyting algebras [15]. Hence, for a visualisation, a formalism that uses topological relations to reflect logical properties seems to be a natural choice. Due to these reasons, we will study how such a formal system of diagrams, Euler-Venn diagrams, can be used to visualise constructive reasoning based on intuitionistic logic.

Euler-Venn circles are known to be a well-suited visualisation of classical propositional logic. In previous work [9], we have presented a proof system in the style of sequent calculus [5] to reason with Euler-Venn diagrams. There, we

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speculated that, similar to sentential languages, restricting the rules and sequents in the system would allow for intuitionistic reasoning with Euler-Venn diagrams. However, further investigation showed that such a simple change is not sufficient, due to the typical use of the syntax elements of Euler-Venn diagrams.

Consider for example the diagrams in Fig. 1. In the classical interpretation, these diagrams are equivalent: the shaded zone in Fig. 1a denotes that the situation that a is true and b is false is prohibited, which is exactly what the omission of the zone included in the contour a, but not in b in Fig. 1b signifies as well. That is, shading a zone and omitting it is equivalent in classical Euler-Venn diagrams. Additionally, we can interpret these two diagrams in two ways: Fig. 1a may intuitively be read as $\neg(a \wedge \neg b)$: we do not allow for the valuations satisfying a, but not b. Fig. 1b, however, is more naturally read as $a \rightarrow b$: whenever a valuation satisfies a, it also satisfies b. While classically, these two statements are indeed equivalent, they are generally not equivalent in an intuitionistic interpretation (see the examples in Sect. 2). Hence, we want to treat missing zones and shaded zones differently. Since typically, proof systems for Euler diagrams allow to transform missing zones into shaded zones [9,7,17], this implies a stronger deviation from our sequent calculus rules than anticipated.

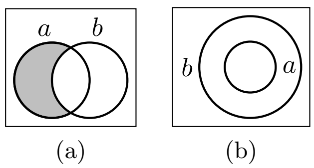


Fig. 1. Euler-Venn diagrams

We want to emphasise a constructive approach to reasoning. In particular, instead of emphasising a *negative* property by prohibiting interpretations of the diagrams, we will treat shading as a *positive* denotation. While this would not make a difference in a classical system, negation in intuitionistic systems is much weaker, and hence not suited as a basic element for the semantics of a language.

In this paper, we present an intuitionistic interpretation of Euler-Venn diagrams that takes the preceeding considerations into account. To that end, we will distinguish between *pure Venn*, *pure Euler* and *Euler-Venn* diagrams, and present semantics of these diagrams based on Heyting algebras. Pure Venn diagrams are diagrams similar to Fig. 1a, containing all possible zones of a set of contours, and shadings of some of the zones. Pure Euler diagrams only represent topological relations, for example, whether a contour is inside of another. In particular, they do not allow for any shading of zones. Hence, Fig. 1b could be seen as a pure Euler diagram. However, we will need to distinguish pure Euler diagrams from diagrams using both topological relations and (possibly) shaded zones, called general Euler-Venn diagrams. To achieve such a distinction, we draw contours with dotted lines in pure Euler diagrams, With this convention, Fig. 1b is a general Euler-Venn diagram, and not a pure Euler diagram.

Subsequently, we present a proof system in the style of sequent calculus, which we prove to be sound and complete. Furthermore, we show that the structural rules of weakening, contraction and cut are admissible. Due to space limitations, we refer for most of the proofs to the extended version of this paper [10].

Related Work. For existential graphs, there exist several visual reasoning systems for non-classical variants. For example, Bellucci et al. defined *assertive graphs* [1], including a system based on rules for iteration and deletion of graphs, among others. This logical language reflects intuitionistic logic, but the rules manipulate only single graphs, while sequent calculus systems manipulate sequents of diagrams. Ma and Pietarinen presented a graphical system for intuitionistic logic [12] and proved its equivalence with Gentzen's single succedent sequent calculus for intuitionistic logic. To that end, they translate the graphs into sentential formulas. They also extended their approach to existential graphs with quasi-Boolean algebras as their semantics [11]. Legris pointed out that structural rules of sequent calculi can be seen as special instances of rules in the proof systems for existential graphs, to analyse substructural logics [8]. de Freitas and Viana presented a calculus to reason about intuitionistic equations [4]. However, we are not aware of any intuitionistic reasoning system using Euler-Venn diagrams. Following this introduction, we briefly recall the foundations of intuitionistic logic and its semantics in terms of Heyting algebras in Sect. 2. In Sect. 3, we define the system of Euler-Venn diagrams, followed by the graphical sequent calculus system, as well as soundness and completeness proofs, in Sect. 4. Finally, we discuss our system and conclude the paper in Sect. 5.

2 Intuitionistic Logic

In this section, we give a very brief overview of the aspects of intuitionistic logic we will use. We present the underlying semantical model: Heyting algebras.

Definition 1 (Heyting Algebra). *A* Heyting algebra $\mathcal{H} = (H, \sqcup, \sqcap, \mapsto, 0, 1)$ *is a bounded, distributive lattice, where* $\sqcup$ *is the join,* $\sqcap$ *the meet,* 0 *the bottom and* 1 *the top element of the lattice. Observe that such a bounded lattice possesses a natural partial order* $\leq$ *on its elements. The binary operation* $\mapsto$*,* the implication*, is defined by* $u \sqcap s \leq t$ *if, and only if,* $u \leq s \mapsto t$*. That is,* $s \mapsto t$ *is the join of all elements* u *such that* $u \sqcap s \leq t$*. We will use the abbreviation* $-s$ *for* $s \mapsto 0$*. Furthermore, we set* $\bigsqcap_{i \in \emptyset} s_i = 1$ *and* $\bigsqcup_{i \in \emptyset} s_i = 0$ *for any* s_i*.*

We collect a few basic properties of Heyting algebras that we need in the following. Proofs can be found, e.g., in the work of Rasiowa and Sikorski [15].

Lemma 1 (Properties of Heyting Algebras). *Let* $\mathcal{H}$ *be a Heyting algebra. Then for all elements* s*,* t *and* u*, we have*

$$s \sqcap (s \mapsto t) \leq t \ (1) \quad (s \mapsto t) \sqcap t = t \ (2) \quad s \mapsto (t \mapsto u) = (s \sqcap t) \mapsto u \ (3)$$

As an example, consider the set $H = \{0, a, b, 1\}$, totally ordered by $0 < b < a < 1$, and where $s \sqcap t = \min\{s, t\}$ and $s \sqcup t = \max\{s, t\}$ for $s, t \in H$. Then, we have $a \mapsto b = b$, since b is the maximal element x such that $x \sqcap a \leq b$. However, we also have $-b = b \mapsto 0 = 0$, and hence $-(a \sqcap -b) = -(a \sqcap 0) = -0 = 1$. So in this Heyting algebra $a \mapsto b$ is not the same as $-(a \sqcap -b)$.

As a different, more topological example, consider the Heyting algebra whose elements are the open subsets of the reals, as defined by the standard topology, and where the meet and join are given by the set-theoretic union and intersection operators. The bottom element is the empty set $0 = \emptyset$, the top element is $1 = \mathbb{R}$, and the implication operation is defined as $a \mapsto b = \mathsf{Int}(\bar{a} \cup b)$, where $\bar{a}$ denotes the complement of a and Int the interior operator. This implies that the negation operation corresponds to $-a = \mathsf{Int}(\bar{a})$. Now consider $a = (-1, 1)$, the open interval between -1 and 1. Then $a \sqcup -a = (-1, 1) \cup \mathsf{Int}(\bar{a}) = (-1, 1) \cup \mathsf{Int}((-\infty, -1] \cup [1, \infty)) = (-1, 1) \cup (-\infty, -1) \cup (1, \infty) = \mathbb{R} \setminus \{-1, 1\} \neq \mathbb{R}$. So, this is an example where $a \sqcup -a \neq 1$.

The syntax of propositional intuitionistic logic is similar to classical Boolean logic, with the difference that the operators are not interdefinable. Hence, the signs for conjunction, disjunction, and implication are all necessary as distinct symbols, and cannot be treated as abbreviations. We will assume a fixed, countable set of propositional variables Vars.

Definition 2 (Syntax). *Intuitionistic formulas are given by the following EBNF*

$$\varphi :\, = \bot \mid a \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \ \text{, where } a \in \mathsf{Vars}\ .$$

We let $\top \equiv \bot \rightarrow \bot$. The semantics of a formula is based on valuations, associating each variable with an element of a given Heyting algebra.

Definition 3 (Semantics). *Let $\mathcal{H}$ be a Heyting algebra and $\nu \colon \mathsf{Vars} \rightarrow H$ a* valuation*, mapping variables to elements of $\mathcal{H}$. We lift valuations to formulas.*

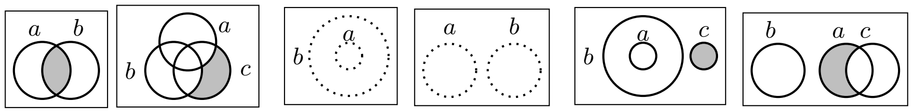
$$\nu(\bot) = 0 \qquad \nu(\varphi \wedge \psi) = \nu(\varphi) \sqcap \nu(\psi)$$
$$\nu(\varphi \vee \psi) = \nu(\varphi) \sqcup \nu(\psi) \qquad \nu(\varphi \rightarrow \psi) = \nu(\varphi) \mapsto \nu(\psi)$$

A formula φ holds *in $\mathcal{H}$, if $\nu(\varphi) = 1$. If φ holds for every valuation of $\mathcal{H}$, we write $\mathcal{H} \models \varphi$. If $\mathcal{H} \models \varphi$ for every Heyting algebra $\mathcal{H}$, we say that φ is* valid*.*

3 Euler-Venn Diagrams

In this section, we present the syntax and semantics of Euler-Venn diagrams with an intuitionistic interpretation. Generally, a diagram can be *unitary* or *compound*. A unitary diagram consists of a set of *contours* dividing the space enclosed by a bounding rectangle into different *zones*. Zones may also be shaded. Depending on how the contours may be arranged, and whether zones may be shaded, we distinguish between *Venn* diagrams, *Euler* diagrams, and *Euler-Venn* diagrams. Compound diagrams are constructed recursively. Since the structure of compound diagrams is the same, regardless of the type of unitary diagrams, we present their syntax first.

Definition 4 (Compound Diagrams). *A* compound diagram *is created according to the following syntax, $D ::= d \mid D \wedge D \mid D \vee D \mid D \rightarrow D$, where d is a unitary diagram.*



(a) Venn diagrams (b) Pure Euler diagrams (c) Euler-Venn diagrams

Fig. 2. Examples of Euler-Venn Diagrams

Definition 5 (Compound Diagram Semantics). *The semantics of compound diagrams for a Heyting algebra $\mathcal{H}$ and a valuation ν is given as follows.*

$$\nu(D_1 \wedge D_2) = \nu(D_1) \sqcap \nu(D_2)$$
$$\nu(D_1 \to D_2) = \nu(D_1) \mapsto \nu(D_2)$$
$$\nu(D_1 \vee D_2) = \nu(D_1) \sqcup \nu(D_2)$$

where D_1, D_2 are compound diagrams. If $\nu(D) = 1$, for all intuitionistic models $\mathcal{H}$ and valuations ν then we call D valid.

Observe that we did not give the semantics for unitary diagrams in the previous definition. First we present notations that are used for all types of diagrams alike. Formally, a *zone* for a finite set of contours $L \subset \mathsf{Vars}$ is a tuple $(\mathsf{in}, \mathsf{out})$, where in and out are disjoint subsets of L such that $\mathsf{in} \cup \mathsf{out} = L$. We will also write $\mathsf{in}(z)$ and $\mathsf{out}(z)$ to refer to the corresponding sets of contours in z. The set of all possible zones for a given set of contours is denoted by $\mathsf{Venn}(L)$.

Venn Diagrams A *Venn diagram* is a diagram where all possible zones for a set of contours are visible. For example, Fig. 2a shows two unitary Venn diagrams, one with the contours a and b, and the other with contours a, b, and c. Formally, a Venn diagram is of the shape $d = (L, \mathsf{Venn}(L), Z^*)$, where Z^* is the set of shaded zones and $Z^* \subseteq \mathsf{Venn}(L)$. Hence the only diagrammatic elements that may carry meaning are the presence of contours, and whether a zone is shaded. For a given diagram d, we denote the set of shaded zones also by $Z^*(d)$. We allow for the diagrams $\bot = (\emptyset, \{(\emptyset, \emptyset)\}, \emptyset)$ and $\top = (\emptyset, \{(\emptyset, \emptyset)\}, \{(\emptyset, \emptyset)\})$. A *literal* is a Venn diagram for a single contour, with exactly one shaded zone. If the zone $(\emptyset, \{c\})$ is shaded in a literal, then we call it *the negative literal for c*, otherwise it is *the positive literal for c* (see Fig. 3). Furthermore, if d is the positive literal for c, then we call the negative literal for c the *dual of d* (and vice versa). Observe that our notion of literals deviates from the original definition of Stapleton and Masthoff [17] and from our previous work [9]. The main difference between our presentation and classical Venn diagrams is the interpretation of shaded zones.

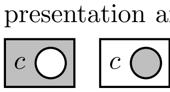


Fig. 3. Literals

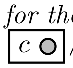
While in the traditional approach, shading denotes the *emptiness* of sets, we use shading as a *marker* of elements. That is, the semantics of a diagram consists of the join of the elements denoted by the shaded zones. This is more in line with a constructivist approach: instead of relying on a negative aspect (emptiness), we construct the semantics out of their building blocks (the shaded zones).

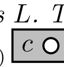
Definition 6 (Zone Semantics). *Let $\mathcal{H}$ be a Heyting algebra, ν a valuation, and z a zone. The* semantics *of z is given by $\nu(z) = \bigsqcap_{c \in \mathsf{in}(z)} \nu(c) \sqcap \bigsqcap_{c \in \mathsf{out}(z)} -\nu(c)$.*

We can now define the semantics of a Venn diagram in general.

Definition 7 (Venn Diagram Semantics). *For a Venn diagram d, a Heyting algebra $\mathcal{H}$ and a valuation ν, the semantics of d are given by $\nu(d) = \bigsqcup_{z \in Z^*(d)} \nu(z)$.*

Note that we have $\nu(\top) = 1$ and $\nu(\bot) = 0$, for any valuation ν. Furthermore, for a unitary diagram without shaded zones, i.e. $d = (L, \mathsf{Venn}(L), \emptyset)$, we have $\nu(d) = 0$. However, the semantics already diverge from the classical case for a fully shaded diagram with one contour: if $d = (\{a\}, \mathsf{Venn}(\{a\}), \mathsf{Venn}(\{a\}))$, then $\nu(d) = \nu(a) \sqcup -\nu(a)$, which in general is not equal to 1. This semantics has one consequence in particular: a zone can be decomposed into an equivalent compound diagram, and any Venn diagram into a disjunctive normal form.

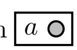
Lemma 2. *Let z be a zone for the contours L. Then the semantics of the compound diagram $d_z = \bigwedge_{c \in \mathsf{in}(z)}$* 

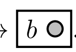
$\wedge \bigwedge_{c \in \mathsf{out}(z)}$ 

equals the semantics of z, i.e. $\nu(d_z) = \nu(z)$. For a Venn diagram d, we have $\nu(d) = \nu(\bigvee_{z \in Z^*(d)} d_z)$.

In particular, this implies that we cannot draw a unitary Venn diagram that expresses intuitionistic implication.

Lemma 3. *Let a and b be propositional variables. Then there is no unitary Venn diagram d such that $\nu(d) = \nu(a \to b)$ for all models and valuations.*

Observe however that we can trivially define a *compound* diagram 

$\to$ 

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Pure Euler Diagrams We need additional syntax if we want to express intuitionistic implication diagrammatically. This new syntax needs to be directed (since $a \to b$ is different to $b \to a$). Observe that our notion of zones already contains an asymmetry that we can understand as a direction: we distinguish between contours the zone is inside of, and contours it is outside of. So, we may treat a zone as *directed from the "in"-contours to the "out"-contours.* Furthermore, a *missing zone* expresses topological information. Following these considerations, we allow for missing zones in the diagrams. Consequently, we will now discuss pure Euler diagrams. In contrast to Venn diagrams, the semantics of a pure Euler diagram is the meet of the semantics of its missing zones.

Definition 8 (Pure Euler Diagrams). *A* pure Euler diagram *is a structure $d = (L, Z)$, where L is the set of contours and $Z \subseteq \mathsf{Venn}(L)$ the set of visible zones of d. Furthermore, the set $\mathsf{MZ}(d) = \mathsf{Venn}(L) \setminus Z$ is the set of* missing zones *of d. The* missing zone semantics *of a zone z is given by $\nu_{\mathsf{m}}(z) = \left(\bigsqcap_{c \in \mathsf{in}(z)} \nu(c)\right) \mapsto \left(\bigsqcup_{c \in \mathsf{out}(z)} \nu(c)\right)$. Then, for a pure Euler diagram d, we have $\nu(d) = \bigsqcap_{z \in \mathsf{MZ}(d)} \nu_{\mathsf{m}}(z)$.*

In contrast to Venn diagrams, pure Euler diagrams do not allow for *any* shading. To distinguish pure Euler diagrams from Venn diagrams (and Euler-Venn diagrams, see below), we draw them with dotted contours. Even with this additional syntax, we are not able to express *every* implication. A simple example would be $a \to a$, since we cannot have a zone $(\{a\}, \{a\})$. However, for this particular example, we do not lose expressivity, since $a \to a \equiv \top$ for all a. But we have a diagram equivalent to $a \to b$, as shown in the left diagram of Fig. 2b. The right diagram in Fig. 2b denotes $(a \sqcap b) \mapsto 0$, which is $-(a \sqcap b)$. Observe that in contrast to Venn diagrams without shaded zones, a pure Euler diagram without missing zones denotes 1, i.e., for $d = (L, \mathsf{Venn}(L))$, we have $\nu(d) = \nu(\top) = 1$. Furthermore, the diagram without any contours and zones denotes 0, since $\nu((\emptyset, \emptyset)) = \nu_{\mathsf{m}}((\emptyset, \emptyset)) = \bigsqcap_{c \in \emptyset} \nu(c) \mapsto \bigsqcup_{c \in \emptyset} \nu(c) = 1 \mapsto 0 = 0$. In the following, we will need to identify zones that are divided by a contour c abstractly.

Definition 9 (Adjacent Zone). *Let $z = (\mathsf{in}, \mathsf{out})$ be a zone for the contours in L and $c \in L$. The* zone adjacent to z at c*, denoted by $\overline{z}^c$ is $(\mathsf{in} \cup \{c\}, \mathsf{out} \setminus \{c\})$, if $c \in \mathsf{out}$ and $(\mathsf{in} \setminus \{c\}, \mathsf{out} \cup \{c\})$ if $c \in \mathsf{in}$.*

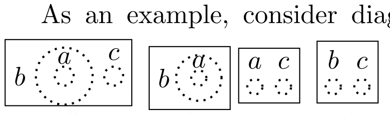
Now we can define a way to remove contours from a pure Euler diagram d. This contrasts to our previous work, where we allowed that the diagram to be reduced contains shading [9].

Definition 10 (Reduction). *Let $d = (L, Z)$ be a pure Euler diagram and $c \in L$. The* reduction *of a zone $z = (\mathsf{in}, \mathsf{out})$ is $z \setminus c = (\mathsf{in} \setminus \{c\}, \mathsf{out} \setminus \{c\})$. The* reduction *of d by c is defined as $d \setminus c = (L \setminus \{c\}, Z \setminus c)$, where $Z \setminus c = \{z \setminus c \mid z \in Z\}$.*

Lemma 4 (Properties of Reduction). *We have $z \setminus c = \overline{z}^c \setminus c$. Furthermore, for each $z' \in \mathsf{MZ}(d \setminus c)$ and z with $z \setminus c = z'$, we have $z \in \mathsf{MZ}(d)$. In particular, both $z \in \mathsf{MZ}(d)$ and $\overline{z}^c \in \mathsf{MZ}(d)$.*

If each missing zone in a pure Euler diagram d has a missing adjacent zone, then the reduction of d by any contour is contained in the semantics of d. In particular, the meet of all reductions equals the semantics of d. This will allow us to show soundness of some rules of the sequent calculus in Sect. 4.

Lemma 5. *Let $d = (L, Z)$ be a pure Euler diagram, where for each $z \in \mathsf{MZ}(d)$, there is a contour $\ell \in L$ such that $\overline{z}^\ell \in \mathsf{MZ}(d)$. Furthermore, let $L' = \{c \mid \mathsf{MZ}(d \setminus c) \neq \emptyset\}$. Then $\bigsqcap_{c \in L'} \nu(d \setminus c) = \nu(d)$.*



d_C^* d_c d_b d_a

Fig. 4. Example of a reduction.

As an example, consider diagram d_C^* of Fig. 4. Intuitively, this diagram contains the information that contour c is disjoint from both a and b, and that a is contained in b. Now, if the diagram satisfies the precondition of the previous lemma, then we can reduce d_C^* to diagrams reflecting exactly these properties. The set of missing zones of d_C^* is $\mathsf{MZ}(d_C^*) = \{(\{a\}, \{b, c\}), (\{a, c\}, \{b\}), (\{b, c\}, \{a\}), (\{a, b, c\}, \emptyset)\}$, and indeed, each of these missing zones has at least one adjacent missing zone. For example, if $z = (\{a\}, \{b, c\})$, then $\overline{z}^c = (\{a, c\}, \{b\})$. So, d_C^* can be reduced according to the lemma. The set of visible zones is $Z(d_C^*) =$

$\{(\emptyset, \{a, b, c\}), (\{c\}, \{a, b\}), (\{b\}, \{a, c\}), (\{a, b\}, \{c\})\}$. Reducing this diagram by the contour c yields the set $Z(d_c) = \{(\emptyset, \{a, b\}), (\{b\}, \{a\}), (\{a, b\}, \emptyset)\}$, which is visualised in Fig. 4 as the pure Euler diagram d_c. Similarly, it can be checked that reducing d^*_C by b indeed yields d_b, and respectively for d_a. By Lemma 5, the conjunction of these three diagrams is equivalent to the original diagram d^*_C.

Euler-Venn Diagrams In this section, we combine pure Euler diagrams with shading. Our main idea is as follows: we treat the information given by a pure Euler diagram as a condition for the construction of the combinations of atomic propositions denoted by the shading. That is, whenever we have constructions as indicated by the spatial relations of contours in a diagram d, we also have a construction of the elements denoted by the shaded zones of the diagram. Since we use the syntactic elements of pure Euler diagrams and Venn diagrams, we will subsequently call such diagrams *Euler-Venn diagrams*. Figure 2c shows two Euler-Venn diagrams that omit some of the possible zones and contain shading.

The abstract syntax of Euler-Venn diagrams is similar to Venn diagrams. A diagram is a tuple $d = (L, Z, Z^*)$ consisting of a set of contours L, a set of visible zones Z over L, and a set of shaded zones $Z^* \subseteq Z$. We will often need to refer to the pure Euler or Venn aspects of an Euler-Venn diagram separately. Hence, we introduce some additional notation. For an Euler-Venn diagram $d = (L, Z, Z^*)$ we will write $\mathsf{Venn}(d) = (L, \mathsf{Venn}(L), Z^*)$ for the Venn diagram with the same set of shaded zones as d, and $\mathsf{Euler}(d) = (L, Z)$ for the pure Euler diagram with the same set of visible zones as d. Similarly to pure Venn and Euler diagrams, we will refer to the missing zones of d by $\mathsf{MZ}(d)$ and to its shaded zones by $Z^*(d)$.

Definition 11 (Euler-Venn Diagram Semantics). *The semantics of a unitary* Euler-Venn diagram *for a valuation ν is $\nu(d) = \nu(\mathsf{Euler}(d)) \mapsto \nu(\mathsf{Venn}(d))$.*

Observe that with this definition, the semantics for the case $\mathsf{MZ}(d) = \emptyset$ and $Z^*(d) \neq \emptyset$ yields $\nu(d) = 1 \mapsto \bigsqcup_{z \in Z^*(d)} \nu(z) = \bigsqcup_{z \in Z^*(d)} \nu(z)$. Furthermore, we get $\nu(\bot) = 1 \mapsto 0 = 0$ and $\nu(\top) = 1 \mapsto 1 = 1$. The language of compound Euler-Venn diagrams can be seen as a subset of intuitionistic logic. In particular, we can translate every diagram into a formula, which we call its *canonical formula*. This translation is very similar to the translation of spider diagrams into monadic first-order logic with equality [18].

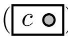
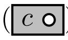
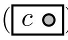
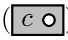
Definition 12 (Canonical Formula). *The* canonical formula *of any diagram is given by the following recursive definition. We start with the definition of the canonical formula of shaded and missing zones.*

$$\chi^z(z) = \bigwedge_{c \in \mathsf{in}(z)} c \wedge \bigwedge_{c \in \mathsf{out}(z)} -c \qquad \chi^m(z) = \bigwedge_{c \in \mathsf{in}(z)} c \to \bigvee_{c \in \mathsf{out}(z)} c$$

For a pure Euler diagram d_e, a Venn diagram d_v, an Euler-Venn diagram d and compound diagrams D and E, the canonical formula is given as

$$\chi(d_e) = \bigwedge_{z \in \mathsf{MZ}(d_e)} \chi^m(z) \qquad \chi(d_v) = \bigvee_{z \in Z^*(d_v)} \chi^z(z)$$

$$\chi(d) = \chi(\mathsf{Euler}(d)) \to \chi(\mathsf{Venn}(d)) \quad \chi(D \otimes E) = \chi(D) \otimes \chi(E) \ \ , \otimes \in \{\wedge, \vee, \to\}$$

Remark 1. Observe that according to Def. 12, we get $\chi($$) = c \wedge \top$ and $\chi($$) = \top \wedge -c$. However, for simplicity, we will assume that the canonical formula construction omits superfluous occurences of $\top$ and $\bot$. Hence, $\chi($$) = c$ and $\chi($$) = -c$. Similarly, e.g., $\chi^m((\emptyset, L)) = \bigvee_{c \in L} c$.

4 Sequent Calculus

Sequent calculus, as defined by Gentzen [5] is based on *sequents*, which are composed by rule applications. In the following, we will define a multi-succedent version of sequent calculus for Euler-Venn diagrams called EDim, inspired by the work of Dragalin [3], but following the modern presentation of Negri and von Plato [13].

Definition 13 (Sequent). *A* sequent $\Gamma \Rightarrow \Delta$ *consists of multisets* Γ *and* Δ *of Euler-Venn diagrams, where* Γ *(*Δ*) is the* antecedent *(*succedent*, respectively).*

If Γ *(*Δ*) is the empty multiset, we write* $\Rightarrow \Delta$ *(*$\Gamma \Rightarrow$*, respectively).* Axioms *are sequents of the form* $p, \Gamma \Rightarrow \Delta, p$ *where* p *is a positive literal. A sequent* $D_1, \ldots, D_k \Rightarrow E_1, \ldots, E_l$ *is valid, if, and only if,* $\nu(D_1) \sqcap \ldots \sqcap \nu(D_k) \leq \nu(E_1) \sqcup \ldots \sqcup \nu(E_l)$ *for all valuations* ν *in all Heyting algebras. We will often abbreviate* $\nu(D_1) \sqcap \ldots \sqcap \nu(D_k)$ *by* $\nu(\Gamma)$ *and* $\nu(E_1) \sqcup \ldots \sqcup \nu(E_l)$ *by* $\nu(\Delta)$*.*

A *deduction* for a sequent $\Gamma \Rightarrow \Delta$ is a tree, where the root is labelled by $\Gamma \Rightarrow \Delta$, and the children of each node are labelled according to the rules defined below. If the validity of the premisses of a rule imply the validity of its conclusion, we call the rule *sound*. A deduction where the leaves are labelled with axioms, or instances of $L\bot$ and $R\top$, is called a *proof* for $\Gamma \Rightarrow \Delta$. We will write $\vdash \Gamma \Rightarrow \Delta$ to denote the existence of a proof for $\Gamma \Rightarrow \Delta$. In all rules, we call the diagram in the conclusion that is being composed the *principal diagram*. For example, in $L\wedge$, the principal diagram is $D \wedge E$, and in the rule Ls it is d. For a given proof of $\Gamma \Rightarrow \Delta$, its *height* is the highest number of successive proof rule applications [13]. We will write $\vdash_n \Gamma \Rightarrow \Delta$ if $\Gamma \Rightarrow \Delta$ is provable with a proof of height at most n.

We now turn to define and explain the rules of EDim. The rules to treat compound diagrams, shown in Fig. 5, are directly taken from sequent calculus for intuitionistic logic and can be proven sound by adapting the proofs by Ono [14].

Lemma 6 (Soundness). *The rules for sentential operators are sound.*

Remark 2. If we take the placeholders D, E and F as formulas according to Def. 2 and both Γ and Δ as multisets of such formulas, then the rules of Fig. 5 together with axioms $p, \Gamma \Rightarrow \Delta, p$ form the sentential sequent calculus G3im [13]. Provability in G3im is equivalent to provability in Gentzen's system LJ. The system LJ is sound and complete [14]. Hence, G3im is sound and complete as well. Furthermore, the structural rules of weakening, contraction and cut are admissible [13]. Observe that we treat $L\bot$ as a *rule*, and not as an axiom.

$$\frac{D, E, \Gamma \Rightarrow \Delta}{D \wedge E, \Gamma \Rightarrow \Delta}\,L\wedge \qquad \frac{D, \Gamma \Rightarrow \Delta \quad E, \Gamma \Rightarrow \Delta}{D \vee E, \Gamma \Rightarrow \Delta}\,L\vee \qquad \frac{\Gamma, D \to E \Rightarrow D \quad E, \Gamma \Rightarrow \Delta}{D \to E, \Gamma \Rightarrow \Delta}\,L{\to}$$

$$\frac{\Gamma \Rightarrow \Delta, D \quad \Gamma \Rightarrow \Delta, E}{\Gamma \Rightarrow \Delta, D \wedge E}\,R\wedge \qquad \frac{\Gamma \Rightarrow \Delta, D, E}{\Gamma \Rightarrow \Delta, D \vee E}\,R\vee \qquad \frac{D, \Gamma \Rightarrow E}{\Gamma \Rightarrow \Delta, D \to E}\,R{\to}$$

$$\frac{}{\Gamma, \bot \Rightarrow \Delta}\,L\bot$$

Fig. 5. Proof Rules for Sentential Operators

Rules for Venn Diagrams. The rules in Fig. 6a let us reduce negative to positive literals. Observe that we may introduce arbitrary sets of formulas into the succedent. Rule $R\top$ lets us finish a proof similarly to $L\bot$. Let d, d_1 and d_2 be Venn diagrams with the same contours such that $|Z^*(d)| > 1$, and $Z^*(d) = Z^*(d_1) \cup Z^*(d_2)$. Then the rules $L\mathsf{s}$ and $R\mathsf{s}$ in Fig. 6b *separate* d into d_1 and d_2. These rules are closely related to the *Combine* equivalence rule for Spider diagrams [7]. For a Venn diagram d with $Z^*(d) = \{z\}$, where $z = (\{n_1, \ldots, n_k\}, \{o_1, \ldots, o_l\})$, the rules $L\mathsf{dec}$ and $R\mathsf{dec}$ of Fig. 6c *decompose* the single zone z. .

$$\frac{\colorbox{lightgray}{$c\ \circ$}, \Gamma \Rightarrow \boxed{c\ \circ}}{\colorbox{lightgray}{$c\ \circ$}, \Gamma \Rightarrow \Delta}\,L\mathsf{neg} \qquad \frac{\boxed{c\ \circ}, \Gamma \Rightarrow}{\Gamma \Rightarrow \Delta, \colorbox{lightgray}{$c\ \circ$}}\,R\mathsf{neg} \qquad \frac{}{\Gamma \Rightarrow \Delta, \colorbox{lightgray}{}}\,R\top$$

(a)

$$\frac{d_1, \Gamma \Rightarrow \Delta \quad d_2, \Gamma \Rightarrow \Delta}{d, \Gamma \Rightarrow \Delta}\,L\mathsf{s} \qquad \frac{\Gamma \Rightarrow \Delta, d_1, d_2}{\Gamma \Rightarrow \Delta, d}\,R\mathsf{s}$$

(b)

$$\frac{\boxed{n_1\ \circ}, \ldots, \boxed{n_k\ \circ}, \colorbox{lightgray}{$o_1\ \circ$}, \ldots, \colorbox{lightgray}{$o_l\ \circ$}, \Gamma \Rightarrow \Delta}{d, \Gamma \Rightarrow \Delta}\,L\mathsf{dec}$$

$$\frac{\Gamma \Rightarrow \Delta, \boxed{n_1\ \circ} \quad \ldots \quad \Gamma \Rightarrow \Delta, \boxed{n_k\ \circ} \quad \Gamma \Rightarrow \Delta, \colorbox{lightgray}{$o_1\ \circ$} \quad \ldots \quad \Gamma \Rightarrow \Delta, \colorbox{lightgray}{$o_l\ \circ$}}{\Gamma \Rightarrow \Delta, d}\,R\mathsf{dec}$$

(c)

Fig. 6. Rules for Unitary Venn Diagrams

Lemma 7. *The rules shown in Fig. 6 are sound.*

Rules for pure Euler Diagrams. Now let $d = (L, Z)$ be a pure Euler diagram, where for each $z \in \mathsf{MZ}(d)$ there is a contour $\ell \in L$, such that $\overline{z}^{\ell} \in \mathsf{MZ}(d)$. Furthermore, let $\{c_1, \ldots, c_k\} \subseteq L$ be the maximal set of contours such that $\mathsf{MZ}(d \setminus c_i) \neq \emptyset$ for every $i \leq k$. Then we can *reduce* d according to the rules $L\mathsf{r}$ and $R\mathsf{r}$ shown in Fig. 7a. Let $d = (L, Z)$ be a pure Euler diagram with more than one missing zone, i.e., $|\mathsf{MZ}(d)| > 1$, and let $d_1 = (L, Z_1)$ and $d_2 = (L, Z_2)$ be two pure Euler diagrams such that $Z_1 \cap Z_2 = Z$. Then the rules $L\mathsf{MZ}$ and $R\mathsf{MZ}$ of Fig. 7b *separate* the diagram z at its missing zones. If d is a pure Euler diagram with a single missing zone, i.e. $\mathsf{MZ}(d) = \{z\}$ and $z = (\{n_1, \ldots, n_k\}, \{o_1, \ldots, o_\ell\})$, then the rules of Fig. 7c decompose z into literals.

$$\frac{d \setminus c_1, \ldots, d \setminus c_k, \Gamma \Rightarrow \Delta}{d, \Gamma \Rightarrow \Delta}\ L\mathsf{r} \qquad \frac{\Gamma \Rightarrow \Delta, d \setminus c_1 \quad \ldots \quad \Gamma \Rightarrow \Delta, d \setminus c_k}{\Gamma \Rightarrow \Delta, d}\ R\mathsf{r}$$

(a)

$$\frac{d_1, d_2, \Gamma \Rightarrow \Delta}{d, \Gamma \Rightarrow \Delta}\ L\mathsf{MZ} \qquad \frac{\Gamma \Rightarrow \Delta, d_1 \quad \Gamma \Rightarrow \Delta, d_2}{\Gamma \Rightarrow \Delta, d}\ R\mathsf{MZ}$$

(b)

$$\frac{d, \Gamma \Rightarrow \boxed{n_1 \ \circ} \quad \ldots \quad d, \Gamma \Rightarrow \boxed{n_k \ \circ} \quad \boxed{o_1 \ \circ}, \Gamma \Rightarrow \Delta \quad \ldots \quad \boxed{o_l \ \circ}, \Gamma \Rightarrow \Delta}{d, \Gamma \Rightarrow \Delta}\ L\mathsf{ldec}$$

$$\frac{\Gamma, \boxed{n_1 \ \circ}, \ldots, \boxed{n_k \ \circ} \Rightarrow \boxed{o_1 \ \circ}, \ldots, \boxed{o_l \ \circ}}{\Gamma \Rightarrow \Delta, d}\ R\mathsf{ldec}$$

(c)

Fig. 7. Proof Rules for pure Euler Diagrams

Lemma 8. *The rules shown in Fig. 7 are sound.*

Rules for Euler-Venn Diagrams. Let d be an Euler-Venn diagram. Then the rules $L\mathsf{det}$ and $R\mathsf{det}$ of Fig. 8 *detach* the spatial relations from the shading.

$$\frac{d, \Gamma \Rightarrow \mathsf{Euler}(d) \quad \mathsf{Venn}(d), \Gamma \Rightarrow \Delta}{d, \Gamma \Rightarrow \Delta}\ L\mathsf{det} \qquad \frac{\mathsf{Euler}(d), \Gamma \Rightarrow \mathsf{Venn}(d)}{\Gamma \Rightarrow \Delta, d}\ R\mathsf{det}$$

Fig. 8. Proof Rules For Euler-Venn Diagrams

Lemma 9. *The rules shown in Fig. 8 are sound.*

By an induction on the height of proofs, we get the soundness theorem for EDim, using Lemma 6, 7, 8, and 9.

Theorem 1 (Soundness). *If* $\Gamma \Rightarrow \Delta$ *is provable in* EDim*, then* $\Gamma \Rightarrow \Delta$ *is valid.*

A rule is *height-preserving invertible*, if whenever we have a proof of height n for its conclusion, its premisses are provable with a proof of at most height n.

Lemma 10 (Inversions).

1. *All of the rules* $L\wedge$, $R\wedge$, $L\vee$, $R\vee$ $L\mathsf{dec}$, $R\mathsf{dec}$, $L\mathsf{s}$, $R\mathsf{s}$, $L\mathsf{r}$, $R\mathsf{r}$, $L\mathsf{MZ}$, *and* $R\mathsf{MZ}$ *are height-preserving invertible.*
2. *If* $\vdash_n d, \Gamma \Rightarrow \Delta$ *for an Euler-Venn diagram* d*, then also* $\vdash_n \mathsf{Venn}(d), \Gamma \Rightarrow \Delta$*.*
3. *If* $\vdash_n d, \Gamma \Rightarrow \Delta$ *for a pure Euler diagram with one missing zone* $z = (\{n_1, \ldots, n_k\}, \{o_1, \ldots, o_l\})$*, then also* $\vdash_n \boxed{o_i \bullet}, \Gamma \Rightarrow \Delta$ *for all* $1 \leq i \leq l$*.*

Invertibility is used in the following lemma, where we connect provability of a sequent $\Gamma \Rightarrow \Delta$ within EDim with the provability of the sequent $\chi(\Gamma) \Rightarrow \chi(\Delta)$ consisting of the canonical formulas of the antecedent and the succedent.

Lemma 11. *Let* $\Gamma \Rightarrow \Delta$ *be a sequent of compound diagrams. Then* $\Gamma \Rightarrow \Delta$ *is provable in* EDim *if, and only if,* $\chi(\Gamma) \Rightarrow \chi(\Delta)$ *is provable in* $\mathsf{G3im}$*.*

Proof. Let $\Gamma \Rightarrow \Delta$ be provable in EDim. By Theorem 1, the sequent is valid, and hence the sequent $\chi(\Gamma) \Rightarrow \chi(\Delta)$ is valid as well. Since $\mathsf{G3im}$ is complete (cf. Remark 2), the sequent is provable in $\mathsf{G3im}$.

For the other direction, we proceed by induction on the height n of the proof of $\chi(\Gamma) \Rightarrow \chi(\Delta)$. If $n = 0$, then $\chi(\Gamma) \Rightarrow \chi(\Delta)$ is an axiom $p, \Gamma' \Rightarrow \Delta', p$ or an instance of $L\bot$. In the first case, since the only diagram D with $\chi(D) = p$ is a positive literal, $\Gamma \Rightarrow \Delta$ is an axiom as well. The second case is trivial.

The induction step is mostly straightforward. We partially present one of the cases, and refer to the extended version for the full proof [10]. If the last rule is $R \rightarrow$ then the sequent is of the form $\chi(\Gamma) \Rightarrow \chi(\Delta'), \chi(D)$, where D is either a compound diagram $D = E \rightarrow F$, a pure Euler diagram $D = d_e$ with a single missing zone, an Euler-Venn diagram with missing zones and shaded zones $D = d$, a single negative literal for a contour c, or $D = \top$. The first case is straightforward. For the case where d is an Euler-Venn diagram, we have $\chi(d) = \mathsf{Euler}(d) \rightarrow \mathsf{Venn}(d)$. and hence the premiss of the last step is $\chi(\mathsf{Euler}(d)), \chi(\Gamma) \Rightarrow \chi(\mathsf{Venn}(d))$. By the induction hypothesis, we get that $\mathsf{Euler}(d), \Gamma \Rightarrow \mathsf{Venn}(d)$ is provable, and by applying $R\mathsf{det}$, $\Gamma \Rightarrow \Delta, d$ as well. Now assume that the principal diagram is a pure Euler diagram d_e with a single missing zone $z = (\{n_1, \ldots, n_k\}, \{o_1, \ldots, o_l\})$. Hence, the premiss of the last step in $\mathsf{G3im}$ is $\bigwedge_{1 \leq i \leq k} n_i, \chi(\Gamma) \Rightarrow \bigvee_{1 \leq i \leq l} o_i$. Since both $L\wedge$ and $R\vee$ are height-preserving invertible, the sequent $n_1, \ldots, n_k, \chi(\Gamma) \Rightarrow o_1, \ldots, o_l$ is provable with height less than n. Since the canonical formula is only atomic for diagram literals, we have that $\boxed{n_1 \bullet}, \ldots, \boxed{n_k \bullet}, \Gamma \Rightarrow \boxed{o_1 \bullet}, \ldots, \boxed{o_l \bullet}$ is provable by the induction hypothesis, and hence by applying $R\mathsf{ldec}$ also $\Gamma \Rightarrow \Delta, d_e$. The other cases are proven using suited applications of $R\mathsf{neg}$ and $R\top$. □

Since every valid sequent is derivable in $\mathsf{G3im}$, we get the completeness result for EDim directly from Lemma 11.

Theorem 2 (Completeness). *If* $\Gamma \Rightarrow \Delta$ *is valid, then* $\Gamma \Rightarrow \Delta$ *is provable.*

We show that some rules are admissible. To that end, we define the *weight* of diagrams, to order them by the number of their syntactic elements.

Definition 14. *The* weight $\omega(d)$ *of a diagram is defined inductively. The base cases are given by* $\omega(\bot) = 0$, $\omega($[c ○]$) = 0$, *and* $\omega($[c ○ shaded]$) = 1$. *Otherwise we set*

$$\omega(d) = \begin{cases} |Z^*(d)| + 1 & \text{, if } d \text{ is a Venn diagram} \\ |\mathsf{MZ}(d)| + 1 & \text{, if } d \text{ is a pure Euler diagram} \\ \omega(\mathsf{Euler}(d)) + \omega(\mathsf{Venn}(d)) + 1 & \text{, if } d \text{ is an Euler-Venn diagram} \\ \omega(d_1) + \omega(d_2) + 1 & \text{, if } d = d_1 \otimes d_2 \text{ for } \otimes \in \{\wedge, \vee, \to\} \end{cases}$$

Lemma 12 (Structural Rules).

(1) For any diagram D*, the sequent* $D, \Gamma \Rightarrow \Delta, D$ *is provable in* EDim.
(2) Weakening:
  i) If $\vdash_n \Gamma \Rightarrow \Delta$*, then also* $\vdash_n D, \Gamma \Rightarrow \Delta$.
  ii) If $\vdash_n \Gamma \Rightarrow \Delta$*, then also* $\vdash_n \Gamma \Rightarrow \Delta, D$.
(3) Contraction:
  i) If $\vdash_n D, D, \Gamma \Rightarrow \Delta$*, then also* $\vdash_n D, \Gamma \Rightarrow \Delta$.
  ii) If $\vdash_n \Gamma \Rightarrow \Delta, D, D$*, then also* $\vdash_n \Gamma \Rightarrow \Delta, D$.
(4) Cut: If $\vdash \Gamma \Rightarrow D, \Delta$ *and* $\vdash D, \Gamma' \Rightarrow \Delta'$*, then* $\vdash \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$.

Proof. (1) can be proven by a straightforward induction on the weight of D. Items (2), and (3) can be proven by induction on the height of the proofs using Lemma 10 and arguments similar to Negri and von Plato [13]. For (4), we use soundness and completeness of EDim. If both sequents are provable, they are also valid, by soundness. So choose an arbitrary valuation ν. Then $\nu(\Gamma) \leq \nu(D) \sqcup \nu(\Delta)$ and $\nu(D) \sqcap \nu(\Gamma') \leq \nu(\Delta')$. Now we have $\nu(\Gamma) \sqcap \nu(\Gamma') \leq (\nu(D) \sqcup \nu(\Delta)) \sqcap \nu(\Gamma') = (\nu(D) \sqcap \nu(\Gamma')) \sqcup (\nu(\Delta) \sqcap \nu(\Gamma')) \leq \nu(\Delta') \sqcup (\nu(\Delta) \sqcap \nu(\Gamma')) \leq \nu(\Delta') \sqcup \nu(\Delta)$. This is due to the first premiss, distributivity, the second premiss and the fact $a \sqcap b \leq a$. Since ν was arbitrary, $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ is valid, and due to completeness, $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ is provable. □

Remark 3. It is also possible to prove cut admissibility with a purely syntactic argument by adapting the inductive proof for the system G3im [13].

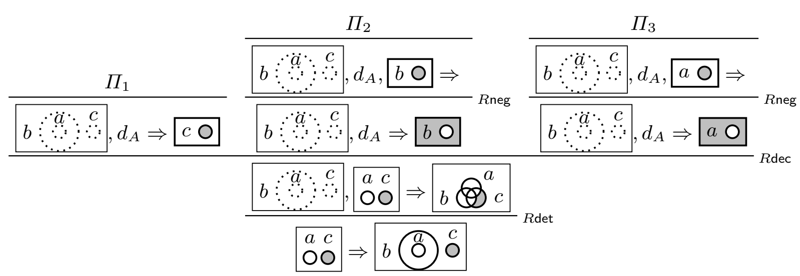


Fig. 9. Proof using Euler-Venn diagrams

A derivation that uses all three types of diagrams can be found in Fig. 9. We explain parts of the proof from bottom to top. The last applied rule detaches the pure Euler part from the Venn part of the succedent, so that we can then decompose the single shaded zone into literals. This splits the proof into three branches, which we treat in the sub-derivations Π_1, Π_2 and Π_3, respectively (see Fig. 11). For reasons of brevity, we use the abbrevations for diagrams as shown in Fig. 10.

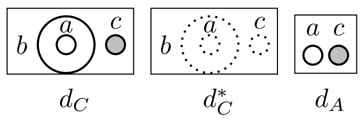


Fig. 10. Diagram Abbreviations

Now, the two right proof branches contain a negative literal in the succedent, which we move to the antecedent with an application of Rneg. Then, all three proof branches proceed similarly: we reduce the pure Euler diagram d_C^* into smaller diagrams, as explained in Sect. 3. Π_1 proceeds by detaching the Euler and Venn aspects of the diagram d_A, which immediately closes the left branch, due to Lemma 12 (1). The right branch ends in an axiom after decomposing the single shaded zone in the antecedent. Within Π_2 there is a similar structure, denoted by the derivation Π_1', where the antecedent contains slightly different diagrams, but the application of rules is similar. The other branches proceed similarly. This example shows how the reduction rules lead to smaller diagrams, and how the rules of Lemma 12 may reduce the size of the proofs, here in the form of the generalised axioms.

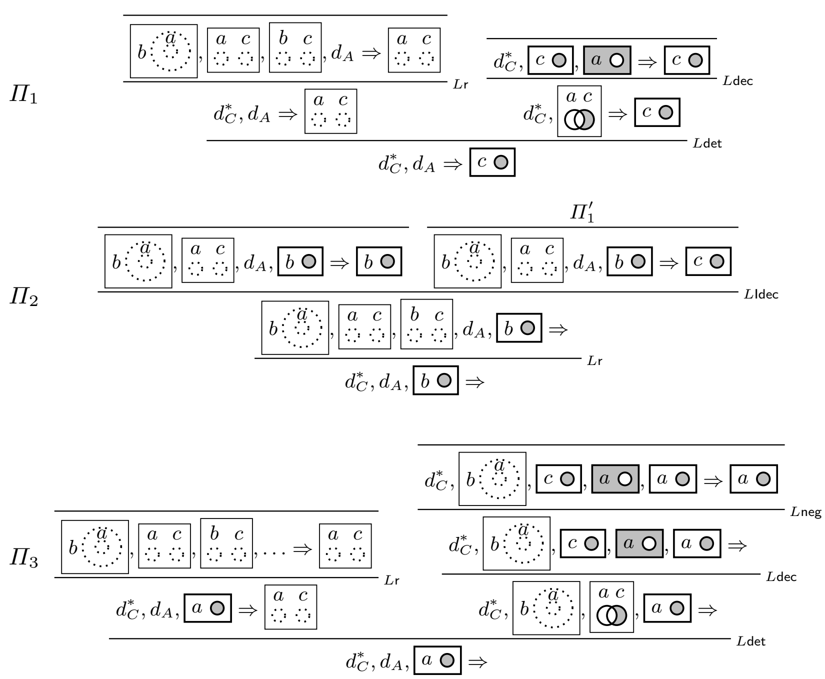


Fig. 11. Auxiliary Derivations for Fig. 9

5 Conclusion

In this paper, we presented an intuitionistic interpretation of Euler-Venn diagrams, based on a semantics of Heyting algebras. We then defined a cut-free sequent calculus EDim, which we have proven to be sound and complete with respect to this semantics. Furthermore, we have shown that the structural rules of contraction, weakening and cut are admissible. We deviated from classical Euler-Venn diagrams in two ways: we did not treat missing zones and shaded zones as equivalent, and we introduced the new syntactic element of dotted contours.

The first deviation is due to the basic restrictions of intuitionistic reasoning. More specifically, intuitionistic implication cannot be treated as an abbreviation of the other operators. To have a syntax explicitly for implications, we need to increase the number of distinct syntactic elements of Euler-Venn diagrams. Hence, distinguishing these two elements is a natural choice. Of course, it can be argued that shading should be used to reflect implications. However, we think that since the representation of missing zones (or rather their absence) introduces a direction into the diagram, in the form of inclusions, this choice is justified.

The introduction of dotted diagrams is more debatable. Arguably, the need for distinguishing pure Euler diagrams arises, since we interpret the missing zones of Euler-Venn diagrams as a precondition for the construction of the elements denoted by the shaded zones. That is, in the constructive interpretation of intuitionistic reasoning, an Euler-Venn diagram means that, given a construction as indicated by the missing zones, we have another construction for the assertions given by the shaded zones. Hence, there is an additional implication within the semantics of Euler-Venn diagrams, as can also be seen in the rules of EDim to detach the pure Euler from the Venn aspects of a diagram. These rules behave similarly to the rules for implication in sentential intuitionistic sequent calculus.

However, the introduction of new syntactic elements is necessary, due to the independence of the operators. Compare for example the intuitionistic systems based on Existential Graphs (EGs). While the operations in classical EGs are denoted by juxtaposition and cuts, reflecting conjunction and negation, respectively, the assertive graphs [1] explicitly introduce notation for disjunction, and also treat the "scroll" as a distinct element. Similarly, the intuitionistic EGs [12] include the notion of *n-scrolls* for each $n > 0$. We think that our system stretches the idea of Euler-Venn diagrams quite far. In particular, logics that need even more independent operators, for example substructural logics, may not be well-matched for such a diagrammatic system. While it may be possible to define such an interpretation, the necessary syntax is far from obvious, if we want to keep the diagrammatic structure of Euler-Venn diagrams. Of course, it is always possible to add new operators to the compound part of the system, but we think that such an addition misses the point of a diagrammatic reasoning system.

Still, there are future directions this work can be taken into. For example, our sequent calculus resembles sentential sequent calculus, while typical Euler-Venn reasoning systems work by adding syntax to single diagrams, and then removing unnecessary parts [2]. It is interesting to see, if we can define such a system for intuitionistic Euler-Venn diagrams. We assume that for the rules to introduce and remove contours, or to copy contours, the reduction of a pure Euler diagram (cf. Def 10 and Lemma 5) will play a significant role.

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