Horses for Courses: Mean-Variance for Asset Allocation and 1/N for Stock Selection*

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Highlights:

- Large investors use a two stage process - asset allocation and then stock selection;
- Mean-variance is superior to 1/N for asset allocation;
- 1/N is superior to mean-variance for stock selection;
- Mean-variance followed by 1/N outperforms the other permutations of technique;
- This result applies to portfolios of US, Japanese, UK, and international equities.

Keywords: investment analysis; asset allocation; stock selection; mean-variance; naive diversification; portfolio theory

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Abstract

For various organizational reasons, large investors typically split their portfolio decision into two stages - asset allocation and stock selection. We hypothesise that mean-variance models are superior to equal weighting for asset allocation, while the reverse applies for stock selection, as estimation errors are less of a problem for mean-variance models when used for asset allocation than for stock selection. We confirm this hypothesis for US data using Bayes-Stein with no short sales and variance based constraints. Robustness checks with four other types of mean-variance model (Black-Litterman with three different reference portfolios, minimum variance, Bayes diffuse prior and Markowitz), and a wide range of parameter settings support our conclusions. We also replicate our core results using Japanese data, with additional replications using the Fama-French 5, 10, 12 and 17 industry portfolios and equities from seven countries. In contrast to previous results, but consistent with our empirical results, we show analytically that the superiority of mean-variance over 1/N is increased when the assets have a lower cross-sectional idiosyncratic volatility, which we also confirm in a simulation analysis calibrated to US data.

1 Introduction

Markowitz (1952) proposed the formation of portfolios as a single problem across all the available assets. However, in reality, most large investors follow a two-stage process. In the first stage (*asset allocation*) a high-level committee allocates the funds between asset classes, e.g. domestic equities, foreign equities, bonds, properties, commodities, hedge funds, private equity, etc. This often takes the form of setting a benchmark asset allocation against which performance is judged (the reference portfolio). In the second stage (*stock selection*) fund managers specialising in each asset class select individual assets from within their allocated asset class. This two-stage process has become more explicit with the shift of large investors such as pension funds and insurers away from balanced fund managers who both asset allocate and stock select, to dividing their funds between multiple specialist managers confined to stock selection within their allocated asset class.[1] Blake et al. (2013) documented this switch from balanced to multiple specialist managers, with the relative proportion using a balanced fund manager falling from 98% in 1984, to 26% in 2004.

There are a number of reasons why large investors have adopted a two-stage process. Allocating assets across a small number of asset classes is something that can be addressed by a high-level committee. Unlike fund managers, such a committee in banks, insurance companies and pension plans etc. has responsibility for the investor's asset-liability performance, and can allow for correlations between the investor's liabilities and asset classes when choosing the asset allocation and asset-liability risk exposure. Following Brinson et al. (1986) there is a widespread view that asset allocation is much more important than stock selection in determining

[1] Fiduciary management, where the investor outsources the entire investment process, has increased in popularity in recent years, and this also follows a two-stage process.

portfolio performance,[2] which also encourages investors to set the reference portfolio. However, although a high-level committee can set the reference portfolio, it is not practical for them to make a choice between thousands of stocks, bonds etc., so they delegate this to specialist managers. The high-level committee probably meets infrequently and the reference portfolio may apply for several years, while stock selection is something that changes on a much more frequent basis, again leading to a need to delegate stock selection.[3] Delegation of stock selection has the advantage that specialist fund managers have greater knowledge and expertise than a high-level committee, and splitting the money among a number of asset managers avoids any diseconomies of scale. Employing multiple specialist managers also has the advantages of diversifying across stock selection strategies and manager skills, permitting the choice of the 'best of breed' manager within each asset class, and generating competition between asset managers if two or more are allocated the same asset class.

There are a number of disadvantages to adopting a two-stage process where stock selection is delegated to multiple specialists. The agency issues involved in such delegation have been investigated by many authors (see Stracca, 2006, for a review), and involve incentivising, rewarding and monitoring the agents (fund managers) to make decisions in accordance with the principal's preferences. For example, fund managers may be rewarded for exceeding their benchmark, but not penalised for under-performance, leading them to take more risk than desired by the principal. They may also have a much shorter investment horizon than the principal due to their performance being judged over months rather than decades. Splitting portfolio selection into two-stages leads to the loss of some of the benefits of diversification (Sharpe, 1981; Van Binsbergen et al., 2008) as the correlations between individual assets in different asset classes are ignored, and effectively replaced by the average correlation between all the individual assets in each pair of asset classes. In addition, some individual assets fall within the definition of two asset classes. For example, shares in an airport may be bought by an equity manager and an infrastructure manager, and this can lead to over-weighting such assets in the investor's total portfolio. Finally the management fees and administration costs for multiple specialist managers will usually be larger than for a single balanced manager.

Given the widespread adoption of a two-stage process, investors must think the benefits outweigh the costs. Blake et al. (2013) found that, despite the diversification loss, agency problems and higher management fees; this switch to a two stage process has led to better performance due to the avoidance of diseconomies of scale, the creation of competition between fund managers, and the greater knowledge and skill of specialist managers. We argue below there is another powerful reason for this change - improved investment performance due to better portfolio weights.

While portfolio weights can be set without using any formal model, we investigate the performance of different optimization models, e.g. Markowitz, Black-Litterman (BL), Bayes-Stein (BS), Bayesian diffuse prior (BDP), minimum variance and 1/N, in making the asset allocation and stock selection decisions. This requires the investor and their fund managers to choose a portfolio optimization model, each with its own strengths and weaknesses, when used in a two-stage portfolio selection process.

We compare the out-of-sample performance of portfolios chosen using different portfolio models for each of the two-stages, and find that the appropriate portfolio model depends on the nature of the assets being analysed; with mean-variance (MV) preferable for asset allocation, and 1/N best for stock selection. We also prove analytically that when assets are permitted to have non-zero alphas the conclusion of DeMiguel et al. (2009b) is reversed; and the lower is idiosyncratic volatility, the better is the performance of MV relative to 1/N.

[2] This view has been shown to depend on the definition of the question, Ibbotson and Kaplan (2000).

[3] While the reference portfolio can be changed at any time, it is probably reviewed by pension plans on the same three year cycle as their actuarial valuation.

Section 2 develops our hypothesis. Section 3 presents the portfolio models we use in our main analysis, and Section 4 describes our data. Section 5 contains our methodology, and Section 6 has our empirical results. Section 7 presents the theoretical and simulation results, and Section 8 has robustness checks on our main results. Finally, Section 9 concludes.

2 Hypothesis Development

Solving the portfolio problem in a single stage using an optimization technique such as MV runs into data problems. When a historical estimate of the covariance matrix is used, to allow the covariance matrix to be inverted, the number of observations in the time series of asset returns must be at great as long as the number of assets under consideration, which may be over a thousand. Such a time series is unavailable for most assets; and even if a sufficiently long time series is available, it may not be stationary. Therefore only a sub-set of the individual assets can be included in such one-stage optimization models.[4] The problem of insufficient observations to estimate the covariance matrix is much less pressing for a two-stage optimization process. The number of asset classes in the first stage optimization may be in single figures, and the number of individual assets in each asset class at the second stage will be much smaller than the total number of assets across all asset classes.

Asset classes constitute grouped data, while individual assets do not. If the individual assets within a particular asset class are reasonably homogeneous, the forecast risk, return and correlations for asset classes are likely to be more accurate than those for individual assets (Elton and Gruber, 1971; Frankfurter and Phillips, 1980; Lam et al., 1994). MV and its variants is an error maximiser as it invests primarily in assets with the highest forecast returns, the lowest forecast risk and the lowest estimated correlations (Michaud and Michaud, 2008). There are far fewer asset classes than individual assets within an asset class; and so individual assets offer greater chances of risk and return forecasts that are outliers. This suggests MV is better suited to forming portfolios of assets classes (asset allocation) than portfolios of individual assets (stock selection), as estimation errors and outliers will be less of a problem for asset allocation than for stock selection.

We analyse the relative performance of a version of MV in forming portfolios in the first and second stages, relative to that of naive diversification (1/N). We use Bayes-Stein with variance based constraints (VBC) as a representative of the many variations of MV (see Section 3).We use this popular model because it is designed to address the well-known problem of estimation errors in MV portfolio model inputs.[5] The equal proportions (1/N) model is a very simple model that is unaffected by estimation errors, and recent studies have found that no other model is consistently superior, e.g., DeMiguel et al. (2009b). Therefore we use 1/N as our alternative to MV. Estimation errors are likely to be smaller at the first stage due to grouping individual assets into asset classes and, with fewer assets classes than individual assets, this leads to fewer parameters to estimate and a lower chance of outliers. Because estimation errors and outliers are expected to be less of a problem for the first than the second stage, we expect MV-based models to be better than 1/N for asset allocation. We also expect that the estimation errors and associated outliers are sufficiently large for individual assets that the performance of 1/N will be better than MV in the second stage when selecting portfolios of individual assets. Therefore our hypothesis is that using MV for asset allocation, and then 1/N for stock selection is superior to the other three

[4] If the investor is willing to impose conditions on the estimated covariance matrix, a shorter time series can be used. For example, investors can use a single-index model, various types of multi-index model, or a mean correlation model to estimate the correlation matrix for input to a single stage portfolio problem. Fan et al. (2016) find the covariance matrix to be block-diagonal, where the blocks are industry sectors.

[5] We also use the Black-Litterman, Bayesian diffuse prior, minimum variance and Markowitz models; and upper generalized constraints (UGC) in our robustness checks.

possible combinations of these two portfolio models in a two-stage procedure.

We examine this hypothesis in two different ways. First, in our main analysis we compare the out-of-sample performance of portfolios of equities (US sector indices and individual equities) formed using a two-stage process employing either MV or 1/N at each stage. We conclude that MV is superior to 1/N for the first stage (asset allocation); and 1/N is preferable to MV for the second stage (stock selection). In other words investors need to use 'different horses for different courses'.[6] Second, using a one-factor CAPM framework similar to DeMiguel et al. (2009b), we then demonstrate that in theory non-zero alphas lead to MV dominating 1/N when the average idiosyncratic volatility is small, with little difference between MV and 1/N when the average idiosyncratic volatility is large. These theoretical results shows that DeMiguel et al. (2009b)'s conclusion about the inverse relation between the idiosyncratic volatility and 1/N's outperformance over MV is restricted to scenarios where alphas are strictly zero. We also support these theoretical results with a simulation analysis calibrated using US data, a range of robustness checks using US data, and a replication of our core analysis on Japanese data. The online appendices have further robustness checks using the Fama-French 5, 10, 12 and 17 industry portfolios, international equities covering seven countries, and finally equity data for the UK.

DeMiguel et al. (2009b) state that one of the reasons why the 1/N rule performs well on their data sets is that they engage in asset allocation across asset classes, rather than forming portfolios of individual assets. They believe the loss from naive, as opposed to optimal, diversification is smaller when allocating wealth across asset classes due to the lower cross sectional idiosyncratic volatility of asset classes. This conclusion is inconsistent with our finding that, although asset classes have much lower idiosyncratic volatility than individual stocks, MV outperforms 1/N. In DeMiguel et al. (2009b)'s setting, the returns are simulated using a one-factor model with alpha being strictly zero. In our theoretical analysis, following DeMiguel et al. (2009b), we also use a one-factor model and the utility loss function defined by Kan and Zhou (2007) to quantify the out-performance of MV over 1/N. We differ from DeMiguel et al. (2009b) by allowing alpha to be a random variable with zero mean and small variance. In this more realistic setting we find that the conclusions of DeMiguel et al. (2009b) are reversed. The lower is the cross sectional idiosyncratic volatility of asset classes, the greater is the superiority of MV over 1/N for portfolios of asset classes, and vice versa.

We do not compare the performance of one versus two-stage portfolio formation procedures because the superiority of the two-stage process is affected by factors that cannot be incorporated in an analysis of past asset returns, e.g., the six factors investigated by Blake et al. (2013), and the definition of the asset classes. There is also the issue that the time series of asset returns may not be long enough to estimate the covariance matrix when MV is applied to all the individual assets.

There is a substantial literature which compares the performance of different portfolio optimization techniques. A number of studies have applied MV and 1/N to either asset allocation or stock selection problems and obtained results consistent with each of the parts of our hypothesis. For example, Allaj (2019), Durand et al. (2011), Han (2016), and Shigeta (2016) have found that MV has a bigger Sharpe ratio than 1/N for asset classes; while Barroso and Saxena (2019), Board and Sutcliffe (1994), DeMiguel et al. (2009a), Dickson (2016), Hwang et al. (2018), Jagannathan and Ma (2003), and Li (2016) have found that 1/N has a bigger Sharpe ratio than MV for individual equities.

This previous research has not drawn a clear distinction between portfolio optimization models applied to asset classes and individual stocks, nor has it argued that MV models are appropriate for asset classes, while 1/N is better suited to stock selection. Kan and Zhou (2007) demonstrate that the performance of MV models

[6] In a different context, Jarrow and Zhao (2006) show that MV is suitable for equity portfolios, while mean-lower partial moment is preferred for fixed income portfolios.

improves as $\frac{N}{2\lambda T}$ increases, where N is the number of asset classes or stocks, T is the size of the sample (the number of months for monthly data) and λ is the investor's relative risk aversion co-efficient. Usually T is much larger for stock selection than asset allocation problems as there are many more individual stocks than asset classes, and so MV performs better at asset allocation than stock selection. This could account for the previous finding that MV is superior to 1/N for asset classes, but inferior for stock selection. However, in our main empirical analysis the number of asset classes is the same as the number of individual companies in each asset class.[7] We also use the same number of observations for forming both the asset allocation and stock selection portfolios. Therefore $\frac{N}{2\lambda T}$ is identical in both cases, and comparisons of the performance of MV in selecting stocks and allocating assets are unaffected by differences in $\frac{N}{2\lambda T}$.

Frankfurter and Phillips (1980) used factor and cluster analysis to form 476 US equities into 40 homogeneous groups, or quasi-securities. Within each quasi-security the stock selection used 1/N. They then applied MV to these 40 quasi-securities to select the portfolio of quasi-securities (i.e. asset allocation). They compared the out-of-sample performance of their two-stage approach to that of MV applied directly to the 476 individual assets, i.e. a one-stage model, with the market model used to estimate the inputs, and concluded that their two-stage approach is superior to a one-stage process. However, they did not compare the performance of different portfolio models in the various stages of a two-stage process.

While Branger et al. (2019) have investigated the benefits of applying mean-variance analysis to groups of equally weighted shares, their procedures are different from the current structure and procedures of the investment management industry. Our approach is designed to be compatible with the way the industry works, and so we allow for transactions costs, exclude short sales, use data for only 24 years, exclude investment in small companies, and do not optimise the way shares are grouped. These differences from Branger et al. (2019) mean that our approach is less likely to find that a two stage approach is beneficial. Our simulation analysis looks at how alpha and idiosyncratic volatilities affect the relative performance of MV and 1/N, which is not covered by Branger et al. (2019). From looking at alpha and the idiosyncratic volatilities, our simulation results not only provide insights into the circumstances where MV is likely to beat 1/N, but also offer an explanation for why the MV-1/N two-stage procedure we propose outperforms the alternatives. The first stage deals with asset classes whose idiosyncratic volatilities are typically smaller, and alphas are more homogeneous. Therefore MV is more likely to outperform 1/N; while the second stage deals with individual stocks whose idiosyncratic volatilities are typically larger and alphas are more heterogeneous, therefore 1/N is more likely to outperform MV.

3 Portfolio Models

Mean-Variance (MV) Model The main problem with MV is estimation error. The MV portfolio framework is an optimal strategy in terms of the expected utility if asset returns follow a normal distribution and the parameter estimates are known with certainty (Hanoch and Levy, 1969). However, the future (true) parameters of the distribution of asset returns are not known with certainty, and their prediction is a difficult task. Historical estimates of returns are often used to compute the optimal asset weights, but this practice usually leads to weak out-of-sample performance with extreme asset weights. This is because marginal deviations from the future parameter estimates have a large effect on the estimated asset weights.

For this reason portfolio optimization strategies, such as Bayes-Stein, have been proposed to decrease the negative effects of estimation errors in the portfolio construction process by changing the way mean returns

[7] except for telecommunications with only three companies

and the covariance matrix of returns are estimated. Therefore, in the core part of our study we use an enhanced version of the Bayes-Stein shrinkage approach, which was initially proposed by Jorion (1985, 1986). This model is based on the shrinkage estimation of James and Stein (1961), and offers a robust way of dealing with estimation risk. Bayes-Stein has been recognized as a well-established portfolio technique that is robust to estimation risk, and has been widely used in the academic literature, e.g., Bessler et al. (2017), Board and Sutcliffe (1994), DeMiguel et al. (2009a, 2009b, 2013), Hwang et al. (2018), Kan et al. (2018), Kan and Zhou (2007), Platanakis et al. (2019), Platanakis and Sutcliffe (2017), and Tu and Zhou (2011).

Bayes-Stein is based on the idea that estimated returns closer to the norm (which is the global minimum variance portfolio when short-selling is permitted) suffer less from estimation risk, while estimated returns a long way from the norm are more likely to be due to estimation errors. Hence, Bayes-Stein computes the input parameters as the weighted sum of the sample (historical) returns for each asset in the portfolio and the norm estimate of returns. More specifically, Bayes-Stein computes the column vector of mean returns ($\boldsymbol{\mu}_{\text{BS}}$) as follows:

$$\boldsymbol{\mu}_{\text{BS}} = (1-g)\,\boldsymbol{\mu} + g\mu_G\mathbb{1} \tag{1}$$

where $\boldsymbol{\mu}$ is sample mean return vector, $\mathbb{1}$ is the column vector of ones and the shrinkage factor g, $0 \leq g \leq 1$, is computed by the following equation:

$$g = \frac{N+2}{(N+2) + T\,(\boldsymbol{\mu} - \mu_G\mathbb{1})^{\intercal}\,\boldsymbol{\Sigma}^{-1}\,(\boldsymbol{\mu} - \mu_G\mathbb{1})}, \tag{2}$$

where $\boldsymbol{\Sigma}$ is the sample variance-covariance matrix. The parameter μ_G denotes the expected return of the minimum variance portfolio when short-selling is allowed, and T represents the length of the corresponding estimation period. The covariance matrix of asset returns ($\boldsymbol{\Sigma}_{\text{BS}}$) is estimated as follows:

$$\boldsymbol{\Sigma}_{\text{BS}} = \left(\frac{T+\varphi+1}{T+\varphi}\right)\boldsymbol{\Sigma} + \frac{\varphi}{T\,(T+\varphi+1)}\frac{\mathbb{1}\mathbb{1}^{\intercal}}{\mathbb{1}^{\intercal}\boldsymbol{\Sigma}^{-1}\mathbb{1}}, \tag{3}$$

where

$$\varphi = \frac{N+2}{(\boldsymbol{\mu} - \mu_G\mathbb{1})^{\intercal}\,\boldsymbol{\Sigma}^{-1}\,(\boldsymbol{\mu} - \mu_G\mathbb{1})}. \tag{4}$$

We use the Bayes-Stein estimates ($\boldsymbol{\mu}_{\text{BS}}$ and $\boldsymbol{\Sigma}_{\text{BS}}$), and compute the vector of asset weights (decision variables), denoted by $\boldsymbol{x}$, by maximizing the following quadratic utility function:

$$\boldsymbol{x}^{\intercal}\boldsymbol{\mu}_{\text{BS}} - \frac{\lambda}{2}\boldsymbol{x}^{\intercal}\boldsymbol{\Sigma}_{\text{BS}}\boldsymbol{x} \tag{5}$$

We impose constraints to rule out short-selling ($x_i \geq 0$, $\forall i$) as well as a constraint for the normalization of portfolio weights $\left(\sum_{i=1}^{N} x_i = 1\right)$.

Several academic studies have attempted to decrease the effects of estimation risk by imposing additional constraints on the asset weights to rule out the extreme solutions produced by estimation errors in the input parameters. For example, Board and Sutcliffe (1994) report that out-of-sample portfolio performance is improved by imposing short-selling constraints, and Jagannathan and Ma (2003) enhance the performance of portfolios by imposing homogenous constraints on the asset weights. DeMiguel et al. (2009b) advocate using portfolio norm constraints, and DeMiguel et al. (2009a) impose upper generalized norm constraints in some of their models; while Li (2015) shows that performance is improved by imposing norm constraints that force small asset weights to zero. Measured by the Sharpe ratio, Levy and Levy (2014) show that their method of VBCs typically yields a better performance than ten other portfolio optimization techniques.

Performance Measure	1/N	MV
CER	0.0613	0.0712
Sharpe Ratio	0.6190	0.6939
Dowd Ratio	0.4019	0.4779
Omega Ratio	1.7999	1.9285

Table 1: Performance of 1/N and MV in Forming Portfolios of the Ten US Industrial Sectors, i.e. Stage One Asset Allocation - 12-month expanding estimation window, January 1994 to August 2017, λ = 5, η = 0.15, Transaction costs = 50 bps.

For our core analysis we impose the VBCs of Levy and Levy (2014) in an attempt to decrease further the negative effects of estimation errors on the MV input parameters. Since VBCs are anti-proportional to an asset's standard deviation, they constrain more strongly the weights on assets with larger estimation errors as measured by their standard deviation. VBCs are defined as follows

$$\left| x_i - \frac{1}{N} \right| \frac{\sigma_i}{\overline{\sigma}} \leq \eta, \forall i \tag{6}$$

where $\overline{\sigma}$ is the average standard deviation across all the assets used in the portfolio construction process. The upper bound, η, is chosen by the investor, and $\eta = 0$ is equivalent to the naïve diversification strategy (1/N). The effect of the VBCs becomes weaker as the parameter η increases, and disappears when $\eta \to \infty$. As a result, our model for optimal diversification in the core analysis is as follows:

$$\begin{aligned} &\max_{\boldsymbol{x}} \left\{ \boldsymbol{x}^{\intercal} \boldsymbol{\mu}_{\text{BS}} - \frac{\lambda}{2} \boldsymbol{x}^{\intercal} \boldsymbol{\Sigma}_{\text{BS}} \boldsymbol{x} \right\} \\ &s.t. \left| x_i - \frac{1}{N} \right| \frac{\sigma_i}{\overline{\sigma}} \leq \eta, \forall i \\ &x_i \geq 0, \forall i \\ &\sum_{i=1}^{N} x_i = 1. \end{aligned} \tag{7}$$

Naïve Diversification (1/N) A portfolio weight of 1/N is assigned to each asset, e. g., $x_i = 1/N$ for all i. We use 1/N with re-balancing, as in DeMiguel et al. (2009b), rather than 1/N (buy and hold). This method of naïve diversification is very simple, does not require any optimization, is unaffected by estimation errors, and does not depend on the value of the investor's risk aversion.

4 Data

Our hypothesis involves allocating money between asset classes, and then selecting assets within each asset class, and the same logic applies to allocating money between industries or countries within a given asset class. In our empirical analysis we consider US stocks, as the data is readily available for investing in companies in different industrial sectors. We treat value-weighted indices of ten US industrial sectors as our asset classes, with asset allocation across the ten industries; and then perform stock selection within each industry. We analyse monthly total returns from January 1994 to August 2017 on ten S&P500 sector indices - information technology, energy, utilities, discretionary consumption, materials, industrials, consumer staples, health care, telecommunications and financials from Datastream. See online appendix 1 for details of the companies. For

Industrial Sectors	CER		Sharpe Ratio		Dowd Ratio		Omega Ratio	
	1/N	MV	1/N	MV	1/N	MV	1/N	MV
Consumer Discretionary	0.0287	0.0451	0.5505	0.5604	0.3300	0.3427	1.6863	1.7099
Consumer Staples	0.0802	0.0727	0.7680	0.7050	0.5591	0.4890	2.0650	1.9403
Energy	-0.0063	0.0274	0.4614	0.4815	0.2608	0.2797	1.5321	1.5786
Financials	-0.0290	-0.0062	0.4652	0.4129	0.2619	0.2281	1.5969	1.5417
Health Care	0.0908	0.0627	0.8223	0.6382	0.6062	0.4152	2.0631	1.8131
Industrials	0.0561	0.0388	0.6395	0.5236	0.4089	0.3131	1.7792	1.6463
Information Technology	0.0057	0.0172	0.5303	0.4894	0.3115	0.2832	1.6249	1.5834
Materials	0.0347	0.0265	0.5479	0.4589	0.3295	0.2639	1.6819	1.5751
Telecommunications	-0.0034	-0.0083	0.3636	0.3493	0.1966	0.1873	1.4594	1.4427
Utilities	0.0639	0.0544	0.6461	0.5789	0.4224	0.3635	1.8064	1.7285

Table 2: Performance of 1/N and MV in Forming Portfolios of the Shares Within Each US Industrial Sector, i.e. Stage Two Stock Selection - 12-month expanding estimation window, January 1994 to August 2017, λ = 5, η = 0.15, Transaction costs = 50 bps.

the second stage we use total returns on the ten companies in each industrial sector index with the largest initial value, except that the telecommunications sector index only comprises three companies.[8] So in total we have 93 companies. For risk free returns we use 1-month T-bill returns from Ken French's web site.

5 Methodology

We use an out-of-sample procedure, which is more realistic than in-sample tests that assume investors have full knowledge of the distributional parameters of returns. We employ a 12-month expanding estimation window for both the mean return and covariance matrix estimates.[9] Since the length of the estimation window expands over time, an expanding window approach is expected to produce more stable input estimates than rolling windows. We estimate the optimal asset allocation using data available up to and including time t ($t \geq 12$), and these optimal asset weights are then used to compute the out-of-sample portfolio returns for the next month $t + 1$. This process is repeated by expanding our sample by one-month, re-estimating the portfolio inputs and re-balancing the portfolio until we reach the end of our sample period. Since our data covers the period from January 1994 to August 2017 (284 months), and we use a 12-month expanding estimation window, we start investing in January 1995 and our out-of-sample period consists of 272 months (January 1995 to August 2017).

We maximize a quadratic utility function using Bayes-Stein estimates for the means and the covariance matrix, in conjunction with no short selling constraints, normalization of portfolio weights and VBCs, as described in the portfolio optimization problem, see equation (7). For our core analysis we set the value of λ to 5, which is the value for moderately risk averse investors (e.g. Diris et al., 2015). We set the value of η involved in the VBCs to 0.15, which is the middle of the range of values used in Levy and Levy (2014). In Section 8 we conduct robustness checks for different levels of λ and η.

When assessing portfolio performance we allow for transaction costs. Each period we compute the new optimal portfolio of 93 shares, and compare this with the optimal portfolio for the previous period. The total transaction costs are computed using Equation (8), and subtracted from the out-of-sample gross return on the

[8] We chose the largest ten companies because institutional investors invest most of their money in large companies, and we need a reasonably small number of assets when estimating the portfolio input parameters.

[9] Expanding windows have been widely used, including by Branger et al. (2019).

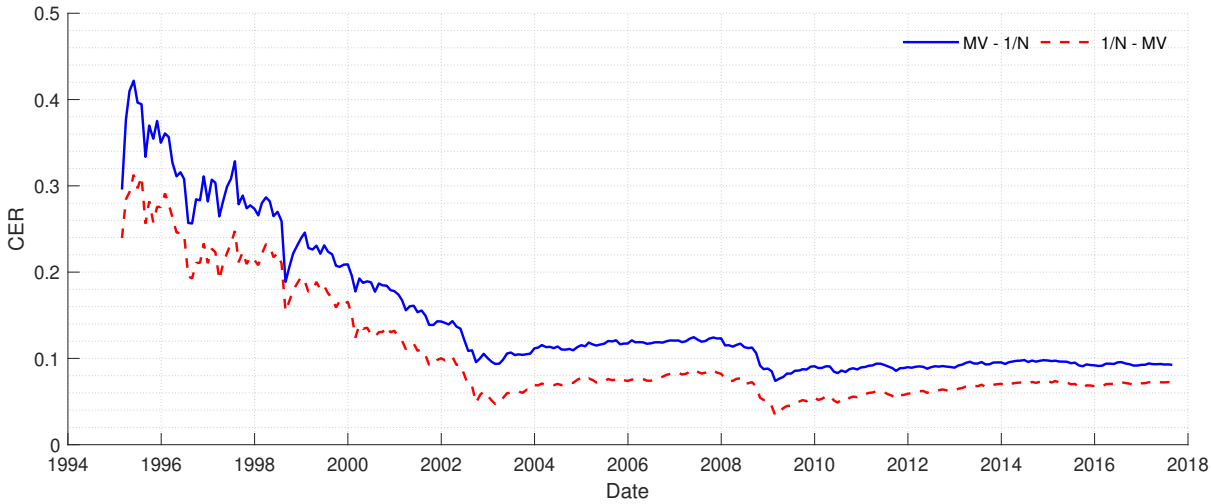

Figure 1: Time Series Plots of the CERs Comparing MV-1/N with 1/N-MV. US Data - 12-month Expanding Estimation Window, January 1995 to August 2017. λ = 5, η = 0.15, Transaction costs = 50 bps.

new portfolio to give the net return.

$$TC_t = \sum_{i=1}^{N} \pi_i \left(\left| x_{i,t} - x_{i,t-1}^{+} \right| \right) \tag{8}$$

where $x_{i,t-1}^{+}$ is the weight (proportion) of the ith asset at the end of the previous period $(t-1)$. Following DeMiguel et al. (2009b) and Kirby and Ostdiek (2012), the proportionate transaction cost (π_i) of trading US individual equities and equity indices is set at 50 bps. We assume that transaction costs are a linear function of trade value, the same for buys and sells; and we only take account of the variable costs of trading assets.

We compare the performance of MV and 1/N applied to each of the stages of a two-stage process in four different ways:

1. Apply MV to the ten asset classes, and then 1/N to the individual assets in each asset class;
2. Apply 1/N to the ten asset classes, and then 1/N to the individual assets in each asset class;
3. Apply MV to the ten asset classes, and then MV to the individual assets in each asset class;
4. Apply 1/N to the ten asset classes, and then MV to the individual assets in each asset class.

For the reasons given above, we expect that using MV for asset allocation, and then 1/N for stock selection is superior to the other three two-stage procedures, i.e., procedure 1 is dominant. For similar reasons, we also expect that the opposite strategy of using 1/N for asset allocation, and then MV for stock selection is inferior to the other three procedures, i.e., procedure 4 is dominated. Having computed the portfolio asset weights for the first and second stages using MV or 1/N, we use these weights to calculate the resulting weights of the 93 companies in the ten industrial sectors for each of the four two-stage procedures. We then compare the out-of-sample performance of the four portfolios using four different performance measures.

6 Empirical Results

In stage one we use both 1/N and MV to form a portfolio of the ten industrial sectors, i.e. asset allocation. In Table 1 we compare the out-of-sample performance using the Sharpe, Dowd, and Omega ratios, and certain

Performance Measure	MV-1/N	1/N-1/N	MV-MV	1/N-MV
CER	0.0925	0.0818	0.0745	0.0719
Sharpe Ratio	0.8614	0.7673	0.7248	0.7002
Dowd Ratio	0.6706	0.5498	0.5132	0.4854
Omega Ratio	2.2252	2.0213	2.0206	1.9331

Table 3: Performance Measures for the Four Two-stage Procedures for the US Data - 12-month Expanding Estimation Window, January 1994 to August 2017, $\lambda = 5$, $\eta = 0.15$, Transaction costs = 50 bps.

equivalent returns (CER), see online appendix 2 for details of these performance measures. This shows that, as expected, MV is superior to 1/N on every performance measure for asset allocation.

In stage two we use both 1/N and MV to form ten portfolios of individual companies for each of the industrial sectors. The out-of-sample performance of these 20 portfolios appears in Table 2 measured in four different ways. For six of the ten industrial sectors 1/N is superior to MV on all four performance measures, and for another two sectors 1/N is superior to MV on three of the measures. These results indicate that, across the ten sectors, 1/N is generally superior to MV at stock selection.

We now use the stage one results, which allocate money between sectors using either 1/N or MV, and the stage two results, which select individual companies within each of these sectors using either 1/N or MV, to compute the overall portfolio of 93 companies. The out-of-sample performance of the four ways of forming the overall portfolio is presented in Table 3. On each of our four performance measures MV-1/N is the best, and the reverse procedure of 1/N-MV is the worst. These results support our hypothesis that the two-stage procedure of MV, followed by 1/N is superior to the other three procedures. The pure strategies rank in the middle, with 1/N-1/N superior to MV-MV on the four measures.

Figure 1 plots the CERs of the best (MV-1/N) and worst (1/N-MV) two-stage procedures as we add additional out-of-sample periods.[10] These show that, in accordance with our hypothesis, MV-1/N is consistently superior to 1/N-MV.

We test for significant differences between the Sharpe ratios of these four two-stage techniques. Following DeMiguel et al. (2009b), we use the approach suggested by Jobson and Korkie (1980), as corrected by Memmel (2003). We also use the same statistic as DeMiguel et al. (2009b) to test the significance of differences between the CERs. See online appendix 3 for details of these tests.

The probability values of significance tests of differences between the four two-stage procedures appear in Table 4, with the implied quasi-ordering in Figure 2 using the three significant relationships. These show that MV-1/N is significantly superior to both MV-MV and 1/N-MV, and that 1/N-1/N is also significantly superior to 1/N-MV using both Sharpe ratios and CERs. While the benefits of using MV in the first stage and 1/N in the second stage are modest, over the two stages the benefits cumulate. This is consistent with the common finding that tests of 1/N versus MV in a one stage process are often not as clear cut as our two stage results. Kazak and Pohlmeier (2019) report that statistical tests of portfolio models for out-performance have low statistical power because such tests are heavily influenced by the length of the out-of-sample period and estimation noise. This lack of significance is more pronounced when out-of-sample performance is tested using actual data with a short investment horizon; and especially when 1/N is used as the benchmark, as it has no estimation risk.

[10] The corresponding plot for the Sharpe ratios is similar.

Superior	Inferior	Sharpe Ratio	CER‡
MV-1/N	1/N-MV	0.029**	0.026**
MV-1/N	MV-MV	0.007***	0.005***
1/N-1/N	1/N-MV	0.087*	0.085*
MV-1/N	1/N-1/N	0.155	0.198
1/N-1/N	MV-MV	0.361	0.323
MV-MV	1/N-MV	0.388	0.401

Table 4: Probability Values for the Significance Tests of the Differences Between Sharpe Ratios and CERs for the Four Two-Stage Procedures for US Data. 12-month Expanding Estimation Window, January 1994 to August 2017, λ = 5, η = 0.15, Transaction costs = 50 bps. Significance at the 10%, 5% and 1% levels is denoted by *, ** and *** respectively.

‡ The significance tests for the CERs use the value of λ used in forming the portfolios.

7 Analytical Results

Based on parameters calibrated to individual US stocks, DeMiguel et al. (2009b) argue that, given the limited history of stock markets, the length of the estimation window needed for MV and its extensions to outperform 1/N is too large to obtain empirical results. They do not consider forming portfolios using asset classes because they believe the lower idiosyncratic volatility of asset classes leads to a higher utility loss for MV than when 1/N is used. However, we argue that this is not the case, and the lower idiosyncratic volatility of asset classes improves the performance of MV, increasing the likelihood of MV outperforming 1/N.

7.1 DeMiguel et al. (2009b)'s Relative Performance Measure

In this section we explain DeMiguel et al. (2009b)'s performance measure which we use to evaluate the relative performance of portfolio selection techniques. Following DeMiguel et al. (2009b), we assume a one factor structure for excess returns:

$$\boldsymbol{r}_{a,t} = \boldsymbol{\alpha} + \boldsymbol{\beta} r_{b,t} + \boldsymbol{\varepsilon}_t \tag{9}$$

where $\boldsymbol{r}_{a,t}$ is the excess asset returns vector, $\boldsymbol{\alpha}$ is the vector of mispricing coefficients, $\boldsymbol{\beta}$ is the factor loadings vector, $r_{b,t}$ is the excess return on the factor portfolio where $r_{b,t} \sim \mathcal{N}(\mu_b, \sigma_b^2)$, and $\boldsymbol{\varepsilon}_t$ is the noise vector assumed to be IID and $\boldsymbol{\varepsilon}_t \sim \mathcal{N}(\mathbb{0}, \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}})$, which is independent from the factor portfolio ($\mathbb{0}$ is the column vector of zeros). Given this structure, the expected excess returns vector is $\boldsymbol{\mu}_a = \boldsymbol{\alpha} + \boldsymbol{\beta}\mu_b$, and its variance-covariance matrix is $\boldsymbol{\Sigma}_a = \sigma_b^2 \boldsymbol{\beta}\boldsymbol{\beta}^\intercal + \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}$.

The expected loss function defined in Kan and Zhou (2007) and DeMiguel et al. (2009b) is given by:

$$L\left(\boldsymbol{x}^*, \boldsymbol{x}\right) = U\left(\boldsymbol{x}^*\right) - \mathbb{E}\left[U\left(\boldsymbol{x}\right)\right] \tag{10}$$

where $\boldsymbol{x}^*$ is a vector of the optimal portfolio weights, i.e., $\boldsymbol{x}^* = \boldsymbol{\Sigma}_a^{-1}\boldsymbol{\mu}_a/\lambda$, and $U(\boldsymbol{x})$ is the utility of a portfolio with weights given by $\boldsymbol{x}$, and

$$U\left(\boldsymbol{x}^*\right) = \frac{\boldsymbol{\mu}_a^\intercal \boldsymbol{\Sigma}_a^{-1}\boldsymbol{\mu}_a}{2\lambda} = \frac{\left(\boldsymbol{\alpha} + \boldsymbol{\beta}\mu_b\right)^\intercal \left(\sigma_b^2 \boldsymbol{\beta}\boldsymbol{\beta}^\intercal + \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}\right)^{-1}\left(\boldsymbol{\alpha} + \boldsymbol{\beta}\mu_b\right)}{2\lambda}$$

$$U\left(\boldsymbol{x}\right) = \boldsymbol{x}^\intercal \boldsymbol{\mu}_a - \frac{\lambda}{2}\boldsymbol{x}^\intercal \boldsymbol{\Sigma}_a \boldsymbol{x} = \boldsymbol{x}^\intercal\left(\boldsymbol{\alpha} + \boldsymbol{\beta}\mu_b\right) - \frac{\lambda}{2}\boldsymbol{x}^\intercal\left(\sigma_b^2 \boldsymbol{\beta}\boldsymbol{\beta}^\intercal + \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}\right)\boldsymbol{x}.$$

Since our goal is to compare the performance of sample-based MV and its extensions with that of 1/N, we

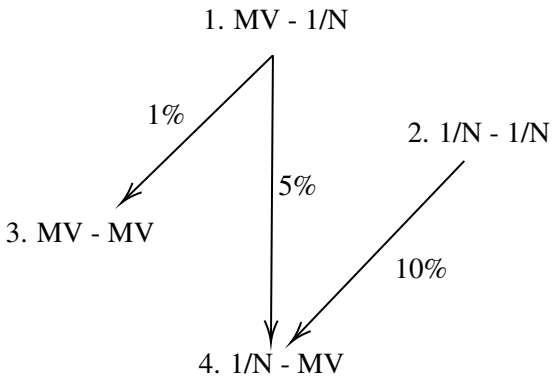

Figure 2: Quasi-Ordering of the Four Two-stage Procedures Based on the Statistically Significant Differences in Sharpe Ratios and CERs in Table 4, 12-month Expanding Estimation Window, January 1994 to August 2017, λ = 5, η = 0.15, Transaction costs = 50 bps.

focus on the loss difference between $L\left(\boldsymbol{x}^{*}, \boldsymbol{x}^{\text{ew}}\right)$ and $L\left(\boldsymbol{x}^{*}, \hat{\boldsymbol{x}}\right)$, which is defined as:

$$\Gamma\left(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}}\right) = 2\lambda\left\{\mathbb{E}\left[U\left(\hat{\boldsymbol{x}}\right)\right] - U\left(\boldsymbol{x}^{\text{ew}}\right)\right\} \tag{11}$$

where $\hat{\boldsymbol{x}}$ is the vector of portfolio weights given by an optimisation model, $\boldsymbol{x}^{\text{ew}}$ is the equally weighted policy (1/N), and[11]

$$\begin{aligned} U\left(\boldsymbol{x}^{\text{ew}}\right) &= \frac{\left(\mathbb{1}^{\intercal}\boldsymbol{\mu}_a\right)^2}{2\lambda\mathbb{1}^{\intercal}\boldsymbol{\Sigma}_a\mathbb{1}} = \frac{\Phi\left(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Sigma}_{\varepsilon}; \mu_b, \sigma_b^2\right)}{2\lambda} \\ \Phi\left(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Sigma}_{\varepsilon}; \mu_b, \sigma_b^2\right) &= \frac{\mathbb{1}^{\intercal}\left(\boldsymbol{\alpha} + \boldsymbol{\beta}\mu_b\right)\left(\boldsymbol{\alpha} + \boldsymbol{\beta}\mu_b\right)^{\intercal}\mathbb{1}}{\mathbb{1}^{\intercal}\left(\sigma_b^2\boldsymbol{\beta}\boldsymbol{\beta}^{\intercal} + \boldsymbol{\Sigma}_{\varepsilon}\right)\mathbb{1}}. \end{aligned} \tag{12}$$

We focus on $\hat{\boldsymbol{x}}$ computed using MV with sample means $\hat{\boldsymbol{\mu}}_a$ and covariances $\hat{\boldsymbol{\Sigma}}_a$, i.e., $\hat{\boldsymbol{x}} = \frac{\hat{\boldsymbol{\Sigma}}_a^{-1}\hat{\boldsymbol{\mu}}_a}{\lambda}$. Assuming the distribution of returns is jointly normal, the expected utility is given by

$$\mathbb{E}\left[U\left(\hat{\boldsymbol{x}}\right)\right] = \frac{k}{2\lambda}\Theta\left(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Sigma}_{\varepsilon}; \mu_b, \sigma_b^2\right) - \frac{h}{2\lambda}, \tag{13}$$

$$\begin{aligned} k &= \frac{T}{T-N-2}\left[2 - \frac{T^2 - 2T}{\left(T-N-1\right)\left(T-N-4\right)}\right], \\ h &= \frac{TN\left(T-2\right)}{\left(T-N-1\right)\left(T-N-2\right)\left(T-N-4\right)}, \\ \Theta\left(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Sigma}_{\varepsilon}; \mu_b, \sigma_b^2\right) &= \left(\boldsymbol{\alpha} + \boldsymbol{\beta}\mu_b\right)^{\intercal}\left(\sigma_b^2\boldsymbol{\beta}\boldsymbol{\beta}^{\intercal} + \boldsymbol{\Sigma}_{\varepsilon}\right)^{-1}\left(\boldsymbol{\alpha} + \boldsymbol{\beta}\mu_b\right), \end{aligned} \tag{14}$$

Therefore

$$\Gamma\left(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}}\right) = k\Theta\left(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Sigma}_{\varepsilon}; \mu_b, \sigma_b^2\right) - \Phi\left(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Sigma}_{\varepsilon}; \mu_b, \sigma_b^2\right) - h.$$

If $\Gamma\left(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}}\right) > 0$, the MV strategy outperforms 1/N; otherwise 1/N outperforms MV. In the next sub-sections we use this performance measure for simulation and analytical analyses of MV and 1/N.

7.2 Simulation with Non-zero Alphas

Since we have monthly data, the variables and parameters in our simulations are set to a monthly frequency. To be comparable with the simulations of DeMiguel et al. (2009b), all our settings are consistent with theirs, except

[11] See the derivation in Appendix B of DeMiguel et al. (2009b).

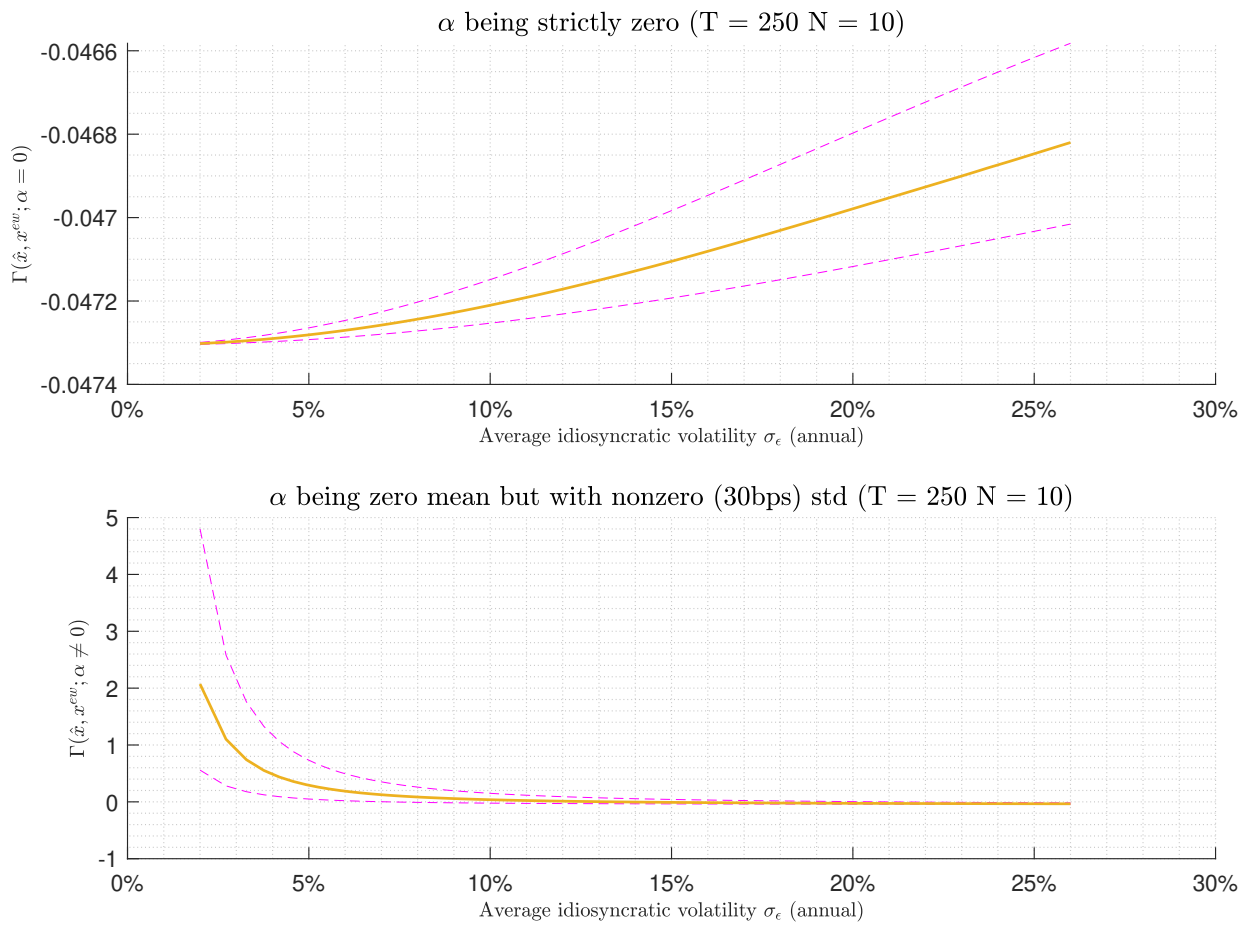

Figure 3: $\Gamma\left(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}}\right)$ versus Idiosyncratic Volatility (σ_ε). The simulation is based on $T = 240$ and $N = 10$. The dashed lines are 95% confidence intervals

for the alphas. Our betas are drawn uniformly from the 0.5 to 1.5 range using a market factor with monthly excess returns with a mean of $\mu_b = 0.67\%$, and a variance of $\sigma_b = 0.21\%$, as assumed by DeMiguel et al. (2009b). We also follow DeMiguel et al. (2009b) in assuming the covariance matrix is diagonal. DeMiguel et al. (2009b) assume the alphas are zero, but this assumption is too strong, as zero alphas only exist in perfect equilibrium. Although the estimated alphas are often statistically insignificant from zero, they are never exactly zero. As pointed out by Jarrow (2010), the conditions for a strictly zero alpha are so strong that "in the complexity of actual markets, their violation, although anathema to economists, is not logically unreasonable." Therefore we follow a more realistic approach, and draw the alphas from a normal distribution with zero mean and a standard deviation of 30 bps, which matches the cross-sectional standard deviation of the alphas of the ten stocks in the financial sector.[12] The annualised idiosyncratic volatilities vary from 2% to 26%. To compare our results with those of DeMiguel et al. (2009b) we conduct two simulations. First, the alphas are strictly zero, as assumed by DeMiguel et al. (2009b); and second, the alphas have zero mean but a non-zero standard deviation. We compute the 95% confidence intervals based on the simulation results.

The plots of the simulated values of $\Gamma\left(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}}\right)$ against average idiosyncratic volatility (σ_ε) are shown in Figure 3. These results are based on 1,000 simulations in which T = 360 and N = 10. The upper panel of Figure 3 shows the scenario depicted in DeMiguel et al. (2009b) with alpha being zero. They write that "all

[12] In a robustness check we used the health care sector, the industrial sector and the consumer discretionary sector to calibrate the alpha distribution. The results are available in online appendix 5, and show that the simulation results in Figure 3 are robust to changing the sector used to calibrate the alpha distribution.

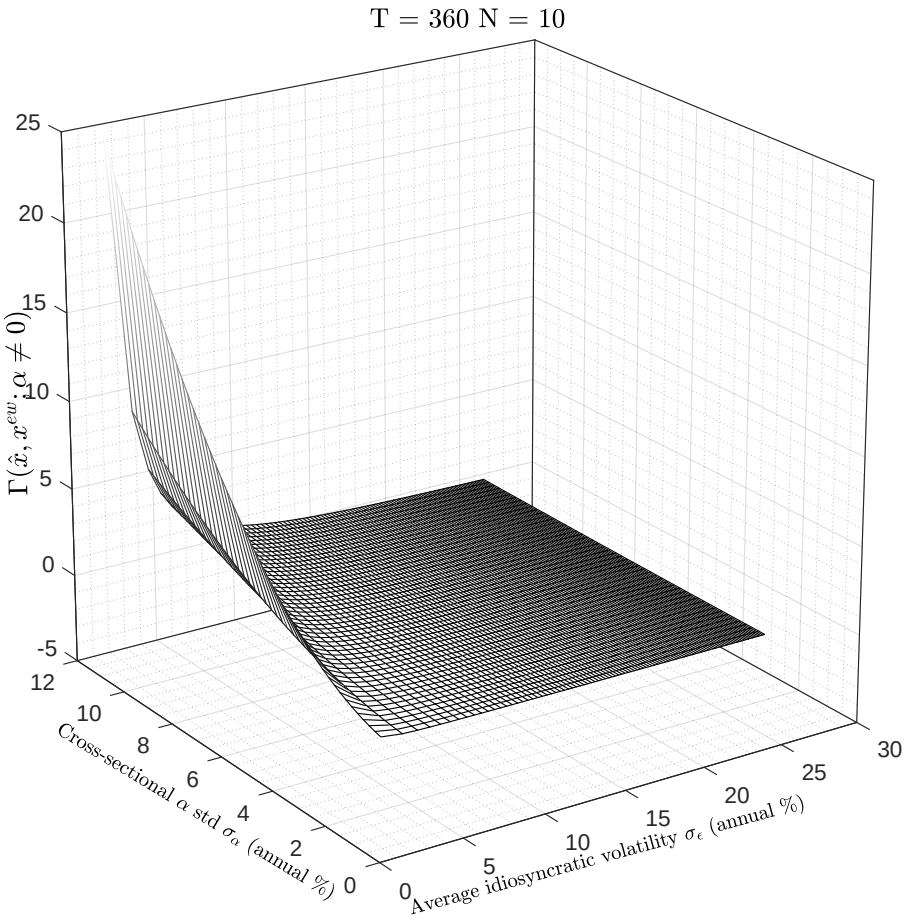

Figure 4: $\Gamma\left(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}}\right)$ versus Idiosyncratic Volatility (σ_ε) and the Cross-Sectional Standard Deviation of the Alphas (σ_α). The simulation is based on $T = 360$ and $N = 10$.

else being equal, the performance of the sample-based mean-variance (and that of the optimizing policies in general) would improve relative to that of the 1/N policy if the idiosyncratic asset volatility was much higher than 20%." However, the lower panel of Figure 3 shows that DeMiguel et al. (2009b)'s conclusion about idiosyncratic volatility is not robust to allowing the alphas to be stochastic. When the alphas are allowed to vary across assets, even a small cross-sectional variation in the alphas (30 bps standard deviation centred around zero), reverses the results of DeMiguel et al. (2009b). The smaller is idiosyncratic asset volatility (σ_ε), the better MV performs relative to 1/N. All the values of $\Gamma\left(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}}\right)$ in the upper panel of Figure 3 are negative, and so 1/N is preferred to MV for all levels of idiosyncratic volatility. In the lower panel of Figure 3 the values of $\Gamma\left(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}}\right)$ are similar at approximately zero, but for volatility below about 10%, MV becomes progressively superior to 1/N. Thus given our simulation setting, as long as the average annualised idiosyncratic volatility (σ_ε) is lower than about 10%, even with just 240 observations, MV outperforms 1/N. Of course, this also depends on other factors.

To have a clearer idea about how changes in the alphas can affect $\Gamma\left(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}}\right)$, in Figure 4 we plot $\Gamma\left(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}}\right)$ against the cross-sectional standard deviation of alphas (σ_α), as well as σ_ε. The larger the sample size, the more likely is MV to outperform 1/N; and so we use $T = 360$ in this simulation to produce insights useful for benchmarking the simulation results below in Section 7.4. We can see that when the alphas are more dispersed and σ_ε is smaller, MV and its extensions are more likely to outperform 1/N.

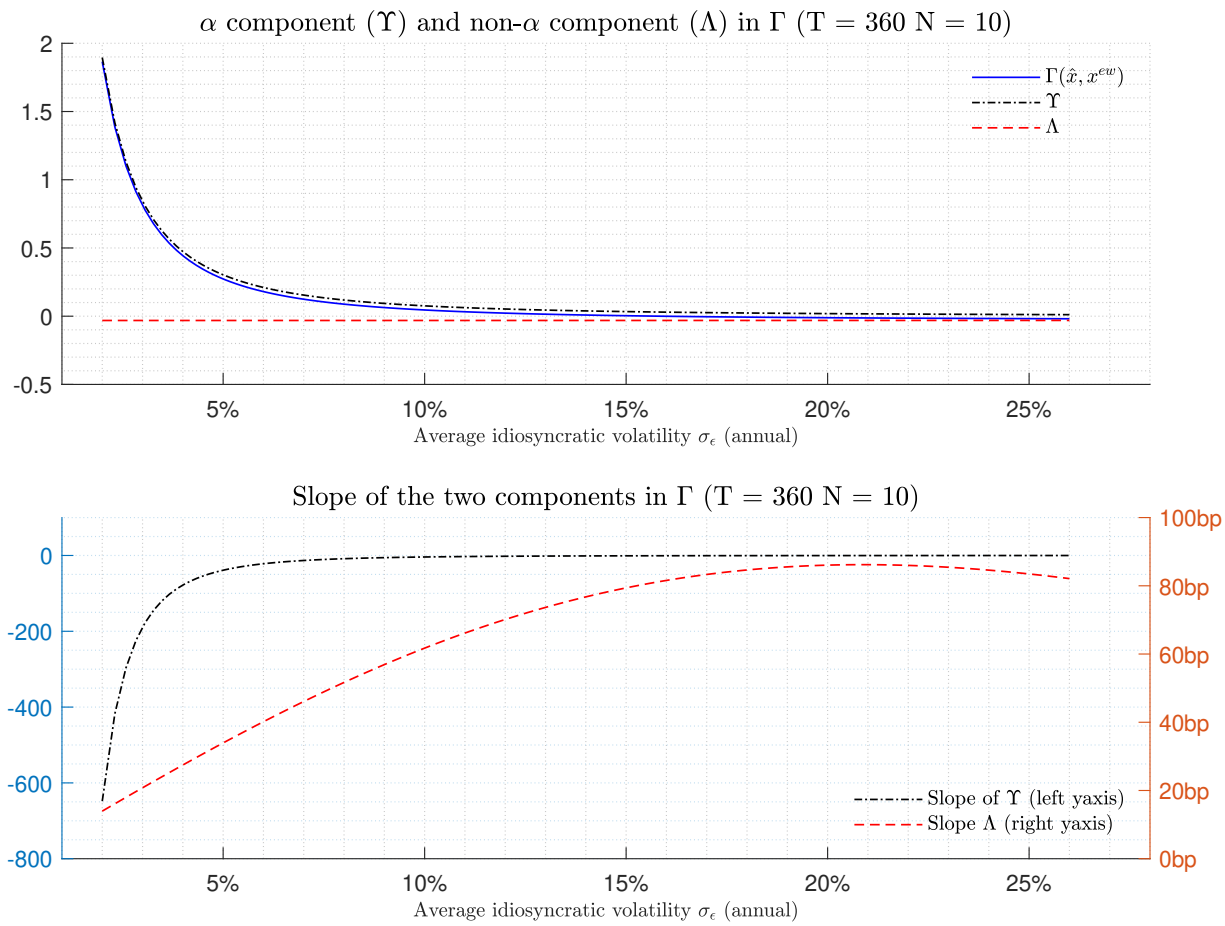

Figure 5: Magnitudes (the upper panel) and Slopes (the lower panel) of the Alpha Component (Υ) and the Non-alpha Component (Λ). The numerical calculations are based on $T = 360$ and $N = 10$.

7.3 Theoretical Results on $\Gamma\left(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}}\right)$, Alphas, and σ_ε

To further understand the effect of non-zero alpha on $\Gamma\left(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}}\right)$, we decompose $\Gamma\left(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}}\right)$ into the following two components:

$$\begin{aligned}\Upsilon &= k\Theta\left(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}; 0, \sigma_b^2\right) - \Phi\left(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}; 0, \sigma_b^2\right) \\ &= \boldsymbol{\alpha}^\intercal \left[k\left(\sigma_b^2 \boldsymbol{\beta}\boldsymbol{\beta}^\intercal + \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}\right)^{-1} - \frac{\mathbb{1}\mathbb{1}^\intercal}{\mathbb{1}^\intercal\left(\sigma_b^2 \boldsymbol{\beta}\boldsymbol{\beta}^\intercal + \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}\right)\mathbb{1}}\right]\boldsymbol{\alpha}, \qquad (15)\\ \Lambda &= k\Theta\left(0, \boldsymbol{\beta}, \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}; \mu_b, \sigma_b^2\right) - \Phi\left(0, \boldsymbol{\beta}, \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}; \mu_b, \sigma_b^2\right) - h \\ &= \mu_b^2 \boldsymbol{\beta}^\intercal \left[k\left(\sigma_b^2 \boldsymbol{\beta}\boldsymbol{\beta}^\intercal + \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}\right)^{-1} - \frac{\mathbb{1}\mathbb{1}^\intercal}{\mathbb{1}^\intercal\left(\sigma_b^2 \boldsymbol{\beta}\boldsymbol{\beta}^\intercal + \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}\right)\mathbb{1}}\right]\boldsymbol{\beta} - h, \qquad (16)\end{aligned}$$

where Υ captures the component in $\Gamma\left(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}}\right)$ that is due to alpha being non-zero, while Λ is basically $\Gamma\left(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}}\right)$ with $\boldsymbol{\alpha} = \mathbb{0}$. We have $\Gamma\left(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}}\right) = \Upsilon + \Lambda$. To indentify the role of the average idiosyncratic volatility, we assume $\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}} = \sigma_\varepsilon^2 \boldsymbol{I}$ where $\boldsymbol{I}$ is an indentity matrix. Therefore, Λ and Υ can be rewritten in explicit terms of σ_ε^2, the results are summarised in the follow proposition. Please see proof in online appendix 4.

Proposition 1 *Assuming $\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}} = \sigma_\varepsilon^2 \boldsymbol{I}$, then we have:*

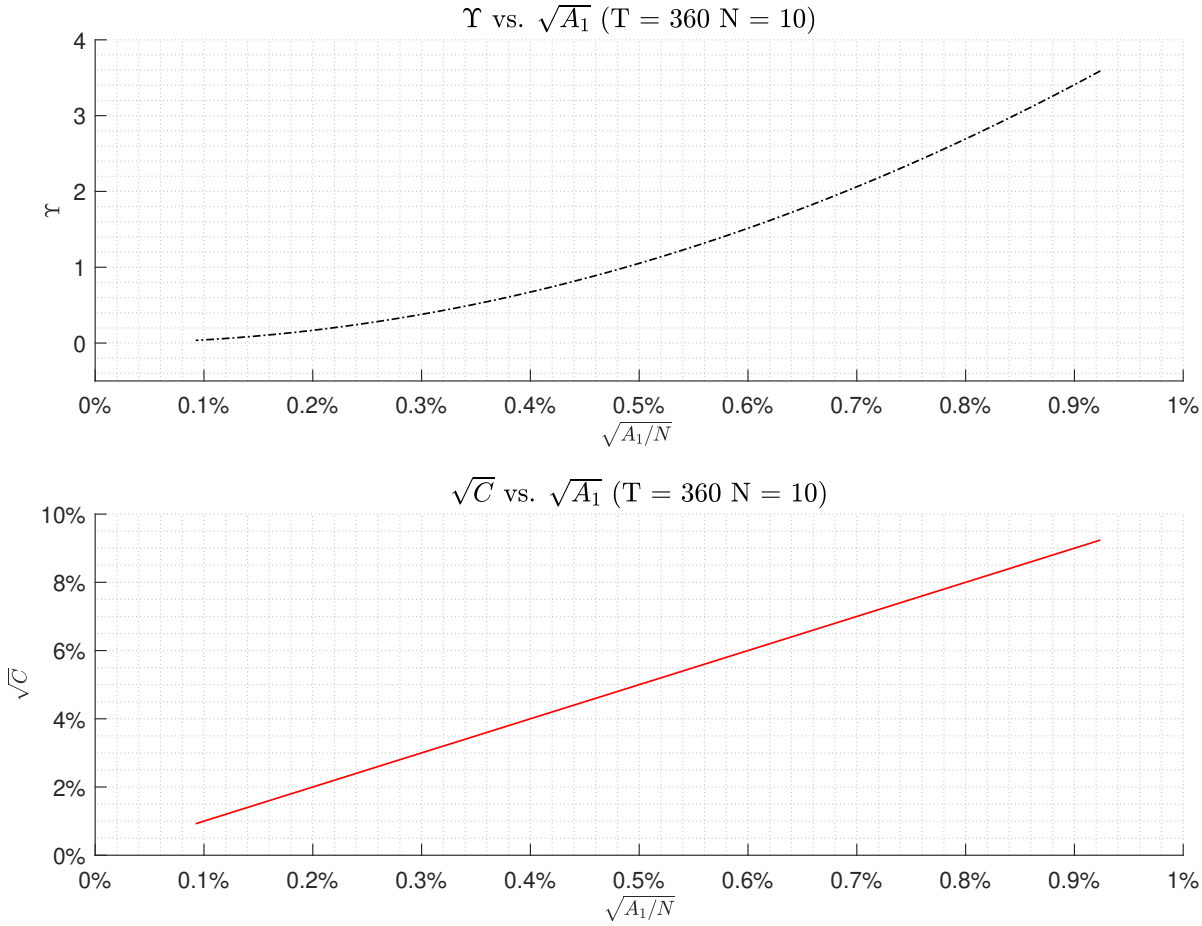

Figure 6: How Υ (the upper panel) and C (the lower panel) Change with Varying A_1. The numerical calculations are based on $T = 360$ and $N = 10$

1. *Λ and Υ can be rewritten as*

$$\Lambda = \mu_b^2 \left(\frac{kB_1}{B_1\sigma_b^2 + \sigma_\varepsilon^2} - \frac{B_2}{B_2\sigma_b^2 + N\sigma_\varepsilon^2} \right) - h, \tag{17}$$

$$\Upsilon = \frac{k\sigma_b^2 C}{(B_1\sigma_b^2 + \sigma_\varepsilon^2)\sigma_\varepsilon^2} + \frac{kA_1}{B_1\sigma_b^2 + \sigma_\varepsilon^2} - \frac{A_2}{B_2\sigma_b^2 + N\sigma_\varepsilon^2}, \tag{18}$$

$$B_1 = \boldsymbol{\beta}^\intercal\boldsymbol{\beta}, B_2 = \boldsymbol{\beta}^\intercal \left(\mathbb{1}\mathbb{1}^\intercal\right)\boldsymbol{\beta},$$
$$A_1 = \boldsymbol{\alpha}^\intercal\boldsymbol{\alpha}, A_2 = \boldsymbol{\alpha}^\intercal \left(\mathbb{1}\mathbb{1}^\intercal\right)\boldsymbol{\alpha},$$
$$C = \boldsymbol{\alpha}^\intercal \left[(\boldsymbol{\beta}^\intercal\boldsymbol{\beta} + \boldsymbol{I})^\dagger - \boldsymbol{I} \right] \boldsymbol{\alpha},$$

$(\boldsymbol{\beta}^\intercal\boldsymbol{\beta} + \boldsymbol{I})^\dagger$ is the adjugate matrix of $\boldsymbol{\beta}^\intercal\boldsymbol{\beta} + \boldsymbol{I}$;

2. *Λ is an increasing function of σ_ε^2, i.e., $\frac{\partial\Lambda}{\partial(\sigma_\varepsilon^2)} > 0$, if $B_1 > B_2\frac{k}{N} > \frac{\sigma_\varepsilon^2}{\sigma_b^2}$;*

3. *Υ is a decreasing function of σ_ε^2, i.e., $\frac{\partial\Upsilon}{\partial(\sigma_\varepsilon^2)} < 0$, if $A_1k > A_1k\left(\frac{B_2}{B_1N}\right)^2 > \frac{A_2}{N}$.*

The conditions in the second and third results in Proposition 1 are general and satisfied in all normal scenarios. Since Λ is essentially DeMiguel et al. (2009b)'s relative performance measure as they assume alpha is strictly zero. The second result in Proposition 1 explains why DeMiguel et al. conclude that the higher

Portfolio	N	Markowitz			Bayes-Stein			Idiosyncratic Volatility			Alpha			Beta		
		120	240	360	120	240	360	min	max	mean	min	max	mean	min	max	mean
SP5EFIN	10	0.0%	0.0%	0.0%	0.0%	0.0%	3.8%	20.3%	64.4%	28.8%	-6.1%	5.8%	1.1%	0.95	1.92	1.39
SP5ETEL	3	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	20.5%	24.3%	21.7%	0.7%	1.9%	1.4%	0.66	0.84	0.73
SP5EHCR	10	0.0%	0.0%	0.0%	0.0%	2.3%	23.3%	16.2%	29.4%	22.8%	4.5%	11.9%	7.6%	0.51	0.91	0.68
SP5ECST	10	0.0%	0.0%	2.5%	0.0%	9.7%	45.3%	16.2%	23.7%	18.9%	2.8%	13.6%	5.7%	0.23	0.65	0.47
SP5EIND	10	0.0%	0.0%	0.0%	0.0%	0.7%	15.8%	16.3%	24.9%	21.4%	-1.8%	6.2%	3.8%	0.62	1.30	0.98
SP5EMAT	10	0.0%	0.0%	0.0%	0.0%	0.7%	10.6%	17.4%	40.9%	24.4%	-3.0%	8.4%	3.0%	0.38	1.45	1.01
SP5ECOD	10	0.0%	0.2%	6.6%	0.0%	14.3%	49.2%	16.8%	39.0%	26.4%	-6.8%	6.7%	2.4%	0.63	1.79	1.14
SP5EUTL	10	0.0%	0.0%	0.0%	0.0%	0.0%	2.0%	16.6%	24.8%	20.0%	4.9%	9.3%	6.7%	0.08	0.39	0.29
SP5EENE	10	0.0%	0.0%	0.4%	0.0%	2.8%	23.0%	14.6%	33.5%	25.9%	0.8%	10.0%	3.5%	0.55	1.32	0.93
SP5EINT	10	0.0%	0.8%	22.5%	0.0%	33.9%	76.1%	16.9%	39.4%	28.9%	-5.6%	10.9%	3.3%	0.81	1.58	1.33
Indices	10	0.0%	11.6%	58.8%	2.9%	62.8%	89.2%	7.4%	15.2%	11.9%	-1.7%	5.2%	1.2%	0.42	1.43	0.91

Table 5: Percent Positive Value of $\Gamma(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}})$ for Markowitz and Bayes-Stein, and Data Characteristics used in the Simulations. Three Ts are considerd: $T = 120, 240$, and 360; min, max, and mean of alpha and beta are presented. SP5EFIN is Financials, SP5ETEL is Telecommunications, SP5EHCR is Health Care, SP5ECST is Consumer Staples, SP5EIND is Industrials, SP5EMAT is Materials, SP5ECOD is Consumer Discretionary, SP5EUTL is Utilities, SP5EENE is Energy, SP5EINT is Information Technology. The Idiosyncratic Volatility and alpha are in annual percentages.

the idiosyncratic asset volatilities, the better MV performs. The third result in Proposition 1 shows that under normal circumstances Υ decreases with σ_ε. To compare the magnitudes and slopes (w.r.t σ_ε) of Λ and Υ, we plot them against σ_ε using the same numerical values as those in Section 7.2. Figure 5 shows the comparison. It is clear that Υ dominates Λ in both magnitude and slope when σ_ε is small. This explains why DeMiguel et al.'s conclusion does not hold when alpha equals zero is relaxed.

From (18) in the first result of Proposition 1, we can also shed light on the how the cross-sectional alpha standard deviation affects $\Gamma(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}})$. Since $B_2\sigma_b^2 + N\sigma_\varepsilon^2$ is much larger than $B_1\sigma_b^2 + \sigma_\varepsilon^2$, and A_1 and A_2 are of similar magnitude,[13] Υ is primarily controlled by the first two terms in the right hand side of (18):

$$\frac{k\sigma_b^2 C}{(B_1\sigma_b^2 + \sigma_\varepsilon^2)\sigma_\varepsilon^2} + \frac{kA_1}{B_1\sigma_b^2 + \sigma_\varepsilon^2}.$$

By the definitions of C and A_1, the cross-sectional alpha standard deviation is $\sqrt{\frac{A_1}{N}}$ and is positively related to C. Therefore, we can see from (18) that when the cross-sectional alpha standard deviation is larger, $\Gamma(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}})$ is more positive, or less negative. This is consistent with the simulation results in Section 7.2. Using the same parameters, we also plot Υ and $\sqrt{C}$ against $\sqrt{\frac{A_1}{N}}$ in Figure 6. The upper panel of Figure 6 essentially confirms the results of $\Gamma(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}})$ vs cross-sectional alpha standard deviation in Figure 4. The lower panel shows the positive relation between $\sqrt{C}$ and $\sqrt{\frac{A_1}{N}}$ is nearly linear.

7.4 Simulation Based on Real World Parameters

To gain more insight into when MV and its extensions outperform 1/N, we calibrate the simulation to match our monthly return data on ten sector indices and 93 companies, and continue to use $\Gamma(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}})$ as a proxy for the relative performance of MV versus 1/N. A positive value of $\Gamma(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}})$ can be regarded as the likelihood of MV outperforming 1/N, and the higher this value, the better is the relative performance of MV. If the value of $\Gamma(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}})$ is higher than 50%, we say MV outperforms 1/N.

To gain a broader sense of the relative performance of extensions to MV, we also consider the Bayes-Stein model, in addition to the Markowitz sample-based MV model. For Bayes-Stein the vector of portfolio weights

[13] When alpha has both positive and negative values, A_1 is typically slightly larger than A_2 as the latter could include negative cross terms dominating the positive cross terms.

MV Model	λ	η	Initial Window	Constraints	Performance Measure	1	2	3	4
						MV-1/N	1/N-1/N	MV-MV	1/N-MV
Bayes-Stein	2	0.15	12	VBCs	CER	0.1124	0.1113	0.0913	0.0932
					Sharpe Ratio	0.7974	0.7673	0.6599	0.6815
Bayes-Stein	10	0.15	12	VBCs	CER	0.0513	0.0328	0.0436	0.0342
					Sharpe Ratio	0.8375	0.7673	0.7492	0.7108
Bayes-Stein	5	0.1	12	VBCs	CER	0.0897	0.0818	0.0787	0.0745
					Sharpe Ratio	0.8413	0.7673	0.7610	0.7198
Bayes-Stein	5	0.2	12	VBCs	CER	0.0912	0.0818	0.0710	0.0710
					Sharpe Ratio	0.8487	0.7673	0.6942	0.6936
Bayes-Stein	5	0.15	36	VBCs	CER	0.0745	0.0666	0.0579	0.0583
					Sharpe Ratio	0.7410	0.6793	0.6111	0.6151
Bayes-Stein	5	0.15	60	VBCs	CER	0.0620	0.0541	0.0462	0.0458
					Sharpe Ratio	0.6733	0.6118	0.5358	0.5372
Bayes-Stein	5	0.15	120‡	VBCs	CER	0.0752	0.0666	0.0657	0.0689
					Sharpe Ratio	0.8520	0.7460	0.7854	0.7935
Bayes-Stein	5	—	12	UGCs	CER	0.0874	0.0818	0.0647	0.0702
					Sharpe Ratio	0.8150	0.7673	0.6425	0.6877
Bayes-Stein	5	—	12	LGC	CER	0.0870	0.0818	0.0776	0.0774
					Sharpe Ratio	0.8156	0.7673	0.7473	0.7413
Bayes-Stein	5	0.15	12	VaR	CER	0.0923	0.0818	0.0738	0.0719
					Sharpe Ratio	0.8593	0.7673	0.7183	0.7003
Black-Litterman (Min Variance)	5	0.15	12	VBCs	CER	0.0873	0.0818	0.0804	0.0743
					Sharpe Ratio	0.8350	0.7673	0.7909	0.7198
Black-Litterman (1/N)	5	0.15	12	VBCs	CER	0.0839	0.0818	0.0787	0.0779
					Sharpe Ratio	0.7869	0.7673	0.7523	0.7421
Black-Litterman (Risk Parity)	5	0.15	12	VBCs	CER	0.0868	0.0818	0.0788	0.0762
					Sharpe Ratio	0.8154	0.7673	0.7614	0.7319
Markowitz	5	0.15	12	VBCs	CER	0.0838	0.0818	0.0630	0.0659
					Sharpe Ratio	0.7862	0.7673	0.6291	0.6523
Bayesian Diffuse Prior	5	0.15	12	VBCs	CER	0.0840	0.0818	0.0634	0.0661
					Sharpe Ratio	0.7878	0.7673	0.6322	0.6537
Min Variance	5	0.15	12	VBCs	CER	0.0876	0.0818	0.0823	0.0768
					Sharpe Ratio	0.8379	0.7673	0.8061	0.7383

Table 6: CERs and Sharpe Ratios for the Robustness Checks. All the models allow for transactions costs and rule out short sales. US Data, January 1994 to August 2017. BL = Black-Litterman. VBCs = Variance based constraints, VaR = Value at risk (99%) constraint, λ = Risk aversion parameter, η = VBC upper bound. ‡ = Rolling window of 5 years.

is $\hat{\boldsymbol{x}} = \frac{\boldsymbol{\Sigma}_{\text{BS}}^{-1}\boldsymbol{\mu}_{\text{BS}}}{\lambda}$. When Bayes-Stein is used $\mathbb{E}[U(\hat{\boldsymbol{x}})]$ no longer has a closed form expression. Therefore, we compute the simulated values of $U(\hat{\boldsymbol{x}})$ (250 values are simulated based on equations (1) to (4)) and use the resulting average to approximate $\mathbb{E}[U(\hat{\boldsymbol{x}})]$.

In this simulation we use the S&P500 index as our single factor whose sample monthly excess return (μ_b) has a mean of 0.64%, and a variance (σ_b^2) of 0.18%. We consider 11 portfolios. The first ten consist of the constituent stocks in the ten industrial sectors, i.e. the second stage stock selection problem. The last portfolio includes the ten sector indices, i.e. the first stage asset allocation problem. The values of alpha and beta are drawn from a uniform distribution matching the first two moments of the sample cross-sectional distribution of alphas and betas in the corresponding portfolio.[14] Consistent with DeMiguel et al. (2009b), $\boldsymbol{\Sigma}_\varepsilon$ is a diagonal matrix, with its diagonal elements drawn from a uniform distribution matching the minimum and maximum of the cross-sectional σ_ε of the corresponding portfolio. Using the full sample, we estimate the idiosyncratic asset volatilities of the shares available for inclusion in the ten sector portfolios and the ten sector indices. Table 5 summarises the values of $\Gamma\left(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}}\right)$ for both the Markowitz and Bayes-Stein models computed from 1,000

[14] We also assumed normal distributions for the values of alpha and beta, and the results are similar to those for uniform distributions.

Models	Superior	MV-1/N	MV-1/N	1/N-1/N	MV-1/N	1/N-1/N	MV-MV
	Inferior	1/N-MV	MV-MV	1/N-MV	1/N-1/N	MV-MV	1/N-MV
Bayes-Stein, 2, 0.15, 12, VBCs	CER	0.023**	0.003***	0.008***	0.431	0.099*	0.438
	SR	0.055*	0.009***	0.049**	0.330	0.181	0.414
Bayes-Stein, 10, 0.15, 12, VBCs	CER	0.056*	0.124	0.430	0.079*	0.250	0.200
	SR	0.067*	0.050**	0.117	0.217	0.443	0.329
Bayes-Stein, 5, 0.10, 12, VBCs	CER	0.034**	0.021**	0.111	0.202	0.424	0.301
	SR	0.034**	0.033**	0.115	0.145	0.493	0.264
Bayes-Stein, 5, 0.20, 12, VBCs	CER	0.039**	0.006***	0.090*	0.234	0.276	0.485
	SR	0.044**	0.008***	0.091*	0.198	0.297	0.482
Bayes-Stein, 5, 0.15, 36, VBCs	CER	0.089*	0.011**	0.140	0.311	0.268	0.435
	SR	0.090*	0.010***	0.093*	0.299	0.240	0.427
Bayes-Stein, 5, 0.15, 120‡, VBCs	CER	0.339	0.092*	0.402	0.331	0.473	0.392
	SR	0.337	0.154	0.216	0.239	0.415	0.459
Bayes-Stein, 5, 0.15, 12, VaR	CER	0.026**	0.005***	0.090*	0.191	0.322	0.413
	SR	0.029**	0.007***	0.091*	0.152	0.354	0.402
BL (Reward to VaR), 5, 0.15, 12, VBCs	CER	0.051*	0.017**	0.049**	0.359	0.235	0.439
	SR	0.055*	0.021**	0.045**	0.314	0.261	0.417
BL (Reward to Risk Parity), 5, 0.15, 12, VBCs	CER	0.046**	0.013**	0.058*	0.331	0.238	0.454
	SR	0.047**	0.018**	0.054*	0.269	0.272	0.419
BL (Risk Parity), 5, 0.15, 12, VBCs	CER	0.046**	0.042**	0.132	0.230	0.401	0.330
	SR	0.045**	0.063*	0.139	0.160	0.487	0.267
Markowitz, 5, 0.15, 12, VBCs	CER	0.038**	0.005***	0.020**	0.403	0.123	0.412
	SR	0.041**	0.006***	0.017**	0.384	0.125	0.412
Bayes Diffuse Prior, 5, 0.15, 12, VBCs	CER	0.037**	0.005***	0.021**	0.397	0.128	0.420
	SR	0.041**	0.006***	0.018**	0.377	0.130	0.420
Minimum Variance, 5, 0.15, 12, VBCs	CER	0.088*	0.137	0.198	0.309	0.494	0.273
	SR	0.060*	0.218	0.223	0.191	0.359	0.175

Table 7: Probability Values for the Significance Tests of the Differences Between Sharpe Ratios and CERs for the Four Two-stage Procedure Robustness Checks. The order of the numbers after the name of each model is λ, η, initial window, and constraint. All the models allow for transactions costs and rule out short sales. US Data, January 1994 to August 2017. BL = Black-Litterman. SR = Sharpe Ratio, VBCs = Variance based constraints, VaR = Value at risk (99%) constraint, λ = Risk aversion parameter, η = VBC upper bound. ‡ = Rolling window of 5 years. Significance at the 10%, 5% and 1% levels is denoted by *, ** and *** respectively. The significance tests for CERs use the value of λ used in forming the portfolio.

simulations.

From Table 5 we see that in these 11 portfolios, all the average annualised σ_ε of portfolios of individual equities, except for consumer staples (SP5ECST) at 18.9%, are higher than or equal to 20%. Only the portfolio of sector indices has a much lower average annualised σ_ε of 11.9%. The low values of $\Gamma(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}})$ show that Markowitz is clearly inferior to 1/N for stage two stock selection as most $\Gamma(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}})$ values are zero, and the highest is only 22.5%. However for first stage asset allocation, when $T = 360$, $\Gamma(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}})$ is 58.8%, and Markowitz is superior to 1/N. These results are consistent with Figure 5. The 11.9% average annual σ_ε of the indices is much smaller than that of the stock portfolios, but still higher than the 10% critical level. When $T = 240$, the positive rate for indices using Markowitz is only 11.6%; and only when $T = 360$ do we observe the much higher positive rate of 58.8%. This is because a larger sample leads to better estimates of the MV input parameters.

The results for Bayes-Stein in Table 5 reinforce the view that shrinkage estimators are superior to sample-based MV estimates in the portfolio optimisation context (Jobson and Korkie, 1980), and provide stronger support for our hypothesis that MV is superior for stage one, and 1/N is superior for stage two. Even with only 240 observations, Bayes-Stein has a $\Gamma(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}})$ value of 62.8% for the index portfolio; and with 360 observations $\Gamma(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}})$ is close to 90%, indicating that MV is clearly superior to 1/N for first stage asset allocation. For the second stage of stock selection, the only sector for which Bayes-Stein is superior to 1/N is information technology (SP5INT) with a $\Gamma(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}})$ value of 76.1%. Although σ_ε in this sector is high (around

Performance Measure	1/N	MV
CER	-0.0384	-0.0072
Sharpe Ratio	0.1269	0.1955
Dowd Ratio	0.0609	0.0982
Omega Ratio	1.2020	1.2889

Table 8: Performance of 1/N and MV in Forming Portfolios of the 13 Japan Industrial Sectors, i.e. Stage One Asset Allocation - 12-month expanding estimation window, January 1994 to August 2017, λ = 5, η = 0.15, Transaction costs = 50 bps.

29%), the alphas in this sector are distributed across a relatively wide range (16.5%). This observation is consistent with the results in Figure 4, in that the cross-sectional standard deviation of the alphas has a positive influence on Bayes-Stein's performance, relative to 1/N.

These simulation results, calibrated to the real world, are consistent with our empirical results in Section 6 and theoretical results in Section 7.3, and support our hypothesis that MV is superior for asset allocation, and 1/N is superior for stock selection. They also show that relaxing DeMiguel et al.'s assumption of zero alphas reverses their conclusions concerning the effect of idiosyncratic volatility on the relative performance of MV and 1/N.

8 Robustness Checks

We now run a set of robustness checks on our core results presented in Section 6 and vary the choice of the optimal diversification model and the various parameters involved. We use the Black and Litterman (1992) with the risk-parity, reward to risk timing and reward to VaR timing portfolios as the reference portfolio, Markowitz, minimum variance, and Bayes diffuse prior (BDP) models as alternatives to Bayes-Stein.

Black-Litterman This is a popular portfolio optimization method for dealing with estimation risk, and has recently attracted considerable interest in both academia and the financial services industry, (see, e.g., Bessler and Wolff, 2015; Harris et al., 2017; Kolm et al., 2014; Oikonomou et al., 2018; Platanakis et al., 2019; Platanakis and Sutcliffe, 2017; Platanakis and Urquhart, 2019; Silva et al., 2017). This model combines two sources of information: the investors' 'views' on asset returns (subjective return estimates), and the reference (or benchmark) portfolio used for estimating the 'neutral' (or 'implied') return estimates. The original Black-Litterman model computes the column vector of implied excess-returns, denoted by $\boldsymbol{z}$, as follows,

$$\boldsymbol{z} = \lambda \boldsymbol{\Sigma} \boldsymbol{x}^{\text{reference}} \tag{19}$$

where $\boldsymbol{x}^{\text{reference}}$ is a column vector with the weights of the benchmark (reference) portfolio. In turn we set $\boldsymbol{x}^{\text{reference}}$ to the risk parity, reward to risk parity and reward to VaR portfolios.

Risk-parity (RP) is a heuristic technique that does not require optimization and achieves an equal contribution by each asset to total portfolio risk. We follow Oikonomou et al. (2018) and Platanakis et al. (2019) and use a simplified version of the risk-parity method, where the portfolio weights are computed as follows:

$$\boldsymbol{x}^{\text{RP}} = \frac{1/\sigma_i^2}{\sum_{i=1}^{N} \left(1/\sigma_i^2\right)}, \forall i \tag{20}$$

where σ_i^2 represents the sample variance of asset i. The reward-to-risk timing (RRT) heuristic approach assigns greater weights to assets with a higher reward-to-risk ratio, and is often a more stable method (lower turnover)

Industries	CER		Sharpe Ratio		Dowd Ratio		Omega Ratio	
	1/N	MV	1/N	MV	1/N	MV	1/N	MV
Basic materials	-0.04	-0.02	0.29	0.31	0.15	0.16	1.33	1.35
Basic resources	-0.09	-0.09	0.20	0.09	0.10	0.04	1.23	1.14
Consumer discretionary	0.00	0.00	0.32	0.30	0.17	0.16	1.38	1.38
Consumer products	0.00	0.00	0.37	0.33	0.20	0.17	1.43	1.39
Consumer staples	0.00	-0.01	0.30	0.24	0.16	0.12	1.40	1.34
Financials	-0.11	-0.08	0.03	-0.05	0.02	-0.02	1.10	1.05
Financial services	-0.16	-0.12	0.24	0.23	0.12	0.12	1.27	1.27
Foods, beverages, tobacco	0.00	-0.01	0.30	0.21	0.16	0.11	1.39	1.30
Health care	0.01	0.01	0.41	0.39	0.23	0.21	1.51	1.47
Industrial goods	-0.05	-0.05	0.12	-0.05	0.06	-0.02	1.19	1.07
Technology	-0.08	-0.03	0.19	0.23	0.09	0.12	1.23	1.30
Telecommunications	-0.07	-0.06	0.23	0.22	0.12	0.11	1.28	1.27
Utilities	-0.07	-0.07	0.03	-0.04	0.01	-0.02	1.11	1.06

Table 9: Performance of 1/N and MV in Forming Portfolios of the Shares Within Each Japanese Industry, i.e. Stage Two Stock Selection - 24-month expanding estimation window, January 1994 to August 2017, λ = 5, η = 0.15, Transaction costs = 50 bps.

than other portfolio construction approaches. The RRT computes the asset weights as follows:

$$\boldsymbol{x}^{\text{RRT}} = \frac{\mu_i^+/\sigma_i^2}{\sum_{i=1}^{N}\left(\mu_i^+/\sigma_i^2\right)}, \forall i \tag{21}$$

where $\mu_i^+ = \max(0, \mu_i)$ to prohibit short-selling. We also propose and use a reward-to-VaR timing (RVT) heuristic technique, which is an extension of the RRT heuristic, where we use the VaR (99%) as a risk measure instead of the sample variance. Hence, the asset weights for RVT are computed as follows:

$$\boldsymbol{x}^{\text{RVT}} = \frac{\mu_i^+/VaR_{99\%,i}}{\sum_{i=1}^{N}\left(\mu_i^+/VaR_{99\%,i}\right)}.\forall i \tag{22}$$

The Black-Litterman model computes the column vector of mean returns as follows

$$\boldsymbol{\mu}_{\text{BL}} = \left[(c\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}^{\intercal}\boldsymbol{\Omega}^{-1}\boldsymbol{P}\right]^{-1}\left[(c\boldsymbol{\Sigma})^{-1}\boldsymbol{z} + \boldsymbol{P}^{\intercal}\boldsymbol{\Omega}^{-1}\boldsymbol{q}\right], \tag{23}$$

where $\boldsymbol{P}$ is a binary diagonal matrix with a value of unity if a subjective return estimate exists for the asset. The column vector $\boldsymbol{q}$ contains the investor's views (subjective return estimates), and the parameter c is a measure of the reliability of the implied excess-returns in $\boldsymbol{z}$, which we set to 0.1625 as in Bessler and Wolff (2015), Platanakis et al. (2019), and Platanakis and Sutcliffe (2017), amongst others. We follow Meucci (2010) and many others, and compute the diagonal matrix $\boldsymbol{\Omega}$ as follows

$$\boldsymbol{\Omega} = \frac{\boldsymbol{P}\boldsymbol{\Sigma}\boldsymbol{P}^{\intercal}}{\delta}. \tag{24}$$

Following Meucci (2010) and Platanakis and Sutcliffe (2017), amongst others, we set $1/\delta$ to unity, and use mean asset returns as the subjective return estimates in the column vector $\boldsymbol{q}$. We follow Satchell and Scowcroft (2000) and many other studies, and compute the posterior covariance matrix ($\boldsymbol{\Sigma}_{\text{BL}}$) as follows

$$\boldsymbol{\Sigma}_{\text{BL}} = \boldsymbol{\Sigma} + \left[(c\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}^{\intercal}\boldsymbol{\Omega}^{-1}\boldsymbol{P}\right]^{-1}.$$

Markowitz The Markowitz (1952) model with VBCs appears in equation (7), where the inputs are sample mean and covariance.

Bayesian Diffuse Prior Barry (1974) and Klein and Bawa (1976), among many others, have shown that the predictive distribution of asset returns has the following estimates for the mean returns and covariance matrix (student's t-distribution)

$$\left[\boldsymbol{\mu}, \left(1+\frac{1}{T}\right)\boldsymbol{\Sigma}\right]$$

subject to a diffuse prior $\left[p\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \propto |\boldsymbol{\Sigma}|^{-\frac{N+1}{2}}\right]$ and a normal conditional likelihood. As a result, this portfolio construction technique inflates the sample covariance matrix by a factor of $\left(1+\frac{1}{T}\right)$.

Minimum Variance We used equation (7) with a historical covariance matrix,[15] and the expected means set to zero.

We varied six aspects of the analysis - the MV portfolio model (Bayes-Stein, Black-Litterman, Markowitz, minimum variance or Bayesian diffuse prior), the risk aversion parameter, (λ = 2, 5 or 10), the VBC parameter (η = 0.10, 0.15 and 0.20), the initial length of the expanding window (12, 36 and 60 months), a rolling window of 120 months, and the type of constraint (VBC, UGC,[16] LGC or VaR[17]). Table 6 shows the CERs and Sharpe ratios for the four robustness checks. In every case the MV-1/N model is preferable to the other three procedures, which is consistent with our hypothesis.

Table 7 presents significance tests on the differences in performance between the four pairs of procedures. These 13 robustness checks generally confirm our hypothesis, with MV-1/N significantly superior to 1/N-MV and MV-MV; and 1/N-1/N superior to 1/N-MV. The significance results for the CERs and Sharpe ratios in Table 7 are almost always the same.

As a further robustness check, we repeated our analysis of US data using the corresponding monthly value weighted total returns for Japan for January 1994 to August 2017 expressed in $US. We analysed data from DataStream on 13 industry indices, and the ten firms in each industry with the largest initial market capitalization.[18] The risk free rate of return is one-month T-bill returns from Ken French's web site. The stage one out-of-sample results for asset allocation across the 13 industries using Bayes-Stein with VBC constraints appear in Table 8, and show that MV is superior on all four performance measures.

Table 9 has the out-of-sample stage two results, where Bayes-Stein with VBC constraints and 1/N are used to form 26 portfolios of the ten largest firms in each industry. On three of the performance measures 1/N is superior to MV for all the industries, except basic materials and technology. For CERs the results are more mixed, but with the majority of industry portfolios favouring 1/N. These results support the superiority of 1/N for stock selection.

We now use the stage one and two results to compute the overall portfolios of 120 firms, and these appear in Table 10. MV-1/N is the best on all four performance measures, and 1/N-MV is the worst on three measures, which supports our hypothesis. Table 11 presents tests of the significance of the differences between the CERs and Sharpe ratios in Table 10. These significance tests imply a strong ordering of the four ways of performing a two stage process: MV-1/N is significantly better than 1/N-1/N, which is significantly better than MV-MV, which is significantly better than 1/N-MV, which accords with our hypothesis.

In a Monte Carlo simulation similar to that in Section 7.4, Table 12 shows the proportion of times that MV

[15] Carroll et al. (2017) find that time-varying estimates of the correlation matrix improve performance.
[16] UGC is an upper generalised constraint, and LGC is a lower generalized constraint, see Platanakis et al. (2019).
[17] VaR is the value at risk at the 99% level.
[18] Basic material has nine firms, financial services has eight, and telecommunications has three. So in total we have 120 firms.

Performance Measure	1	2	3	4
	MV-1/N	1/N-1/N	MV-MV	1/N-MV
CER	0.02	0.00	0.02	0.00
Sharpe Ratio	0.40	0.28	0.37	0.24
Dowd Ratio	0.22	0.15	0.20	0.13
Omega Ratio	1.51	1.36	1.48	1.34

Table 10: Performance Measures for the Four Two-stage Procedures for the Japanese data - 24 month Expanding Estimation Window, January 1994 to August 2017, λ = 5, η = 0.15, Transaction costs = 50 bps.

Superior	Inferior	Sharpe Ratio	CER‡
MV-1/N	1/N-MV	0.0334**	0.0796*
MV-1/N	MV-MV	0.3371	0.4966**
1/N-1/N	1/N-MV	0.2759	0.2611
MV-1/N	1/N-1/N	0.0877*	0.0454**
1/N-1/N	MV-MV	0.2166	0.0983*
MV-MV	1/N-MV	0.0702*	0.0729*

Table 11: Probability Values for the Significance Tests of the Differences Between Sharpe Ratios and CERs for the Four Two-Stage Procedures - Japan. 24-month Expanding Estimation Window, January 1994 to August 2017, λ = 5, η = 0.15, Transaction costs = 50 bps. Significance at the 10%, 5% and 1% levels is denoted by *, ** and *** respectively.

‡ The significance tests for the CERs use the value of λ used in forming the portfolios.

(Bayes-Stein with VBC) was superior to 1/N for the ten stage two stock selections, and for the stage one asset allocation between industrial sectors. When $T = 360$ MV is clearly superior to 1/N at the first stage, with scores of 79% and 86%; and 1/N is highly superior to MV in stock selection in the second stage, with many scores for individual industries close to zero, particularly when T is small. These results are consistent with our hypothesis and theoretical results.

We conduct further robustness checks on six more data sets. We applied our core analysis to the four Fama-French 5, 10, 12 and 17 industry portfolios, and found that MV-1/N is superior to the other strategies for each of these industry portfolios using our four performance measures. We also applied our core analysis to the market indices for the UK, USA, Germany, Switzerland, France, Canada and Brazil, and the ten largest firms in each index. These results confirm our hypothesis and show Finally we repeated our core analysis with ten UK industrial sectors, and the largest ten largest firms in each sector. Again MV-1/N is the best two stage strategy. The full results are available in online appendices 6, 7 and 8.

9 Conclusions

For a range of organizational reasons, large investors typically split their portfolio decision into two stages - asset allocation and stock selection. We find that mean-variance models are superior to 1/N for asset allocation, while the reverse applies for stock selection. This is primarily because estimation errors are lower for asset classes than for individual assets, and so are less of a problem for mean-variance models when used for asset allocation than stock selection. We identify three reasons for the superiority of mean-variance over 1/N for asset allocation. First, due to the small number of asset classes, a shorter time series is required to estimate the covariance matrix for asset classes, which offers more degrees of freedom when estimating the inputs for asset allocation. Second, asset classes are grouped data, resulting in smaller estimation errors for asset classes than

Industrial Sector	N	Normal Distribution			Uniform Distribution		
		120	240	360	120	240	360
Basic Materials	9	0.0%	10.5%	36.3%	0.0%	7.1%	44.2%
Basic Resources	10	0.0%	0.1%	1.3%	0.0%	0.0%	0.1%
Consumer Discretionary	10	0.1%	12.4%	43.5%	0.0%	8.9%	50.4%
Consumer Products	10	0.0%	2.7%	19.1%	0.0%	1.0%	18.5%
Consumer Staples	10	0.0%	0.8%	12.1%	0.0%	0.1%	9.6%
Financials	10	0.0%	0.8%	9.8%	0.0%	0.0%	4.3%
Financial Services	8	0.0%	0.7%	7.8%	0.0%	0.4%	4.7%
Foods, beverages, tobacco	10	0.0%	2.0%	15.8%	0.0%	0.8%	12.3%
Health Care	10	0.0%	2.9%	19.9%	0.0%	1.2%	20.8%
Industrial Goods	10	0.0%	1.9%	18.0%	0.0%	0.1%	13.7%
Technology	10	0.0%	5.5%	22.8%	0.0%	1.2%	20.2%
Telecommunications	3	0.0%	3.1%	8.1%	0.0%	1.8%	8.1%
Utilities	10	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Japanese industry indices	13	1.2%	42.4%	78.8%	0.3%	42.9%	85.5%

Table 12: Percent Positive Value of $\Gamma\left(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}}\right)$ for Bayes-Stein. The simulations are based on Japanese Data (January 1994 to August 2017). The results are reported for the parameters simulated from both the normal and uniform distributions.

individual assets. Finally, as there are fewer asset classes than individual assets, this leads to a smaller chance of outliers and the error maximization problem for asset allocation. 1/N is unaffected by estimation errors, and so its performance, relative to mean-variance, is greater for stock selection than for asset allocation.

Using a sample of US equities for ten asset classes and the 93 underlying individual assets, we confirm this hypothesis using four different types of mean-variance model and a range of parameter settings; suggesting that our hypothesis applies across a wide range of mean-variance models. Following DeMiguel et al. (2009b)'s framework and parameter setting, but relaxing their assumption that equities have alphas that are strictly zero, our theoretical and simulation results find the conclusions of DeMiguel et al. (2009b) are reversed - the smaller is idiosyncratic asset volatility, the better mean-variance performs relative to 1/N. This finding is consistent with our empirical results that mean-variance is superior to 1/N for asset classes. Finally we calibrate the simulation analysis to our sample of US equities, and show that the Markowitz and Bayes-Stein models are superior to 1/N for asset classes, while 1/N is superior for stock selection. We support our conclusions with 13 robustness checks on our S&P500 industry portfolios, and the replication of our core analysis on data for Japan, with further replication for four US data sets, UK data and international data.

Future research could investigate the use of other portfolio techniques in both the asset allocation and stock selection stages. It could also examine the asset allocation decision as between the main types of asset, e.g. domestic equities, foreign equities, government debt, corporate debt, property, infrastructure, hedge funds, etc.

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A1 S&P Sectors and the Largest Constituents

The details of the S&P sectors and stocks are summarised in Table A1.1.

Sector	Constituents
SP4EFIN	JPM, BAC, WFC, C (Citi Group), MS, AXP(American Express), PNC, AIG, MMC(Marsh & McLennan Cos.), ALL(Allstate Corp.)
SP5ETEL	T(AT&T Inc.), VZ(Verizon Communications Inc.), CTL(CenturyLink Inc.)
SP5EHCR	JNJ(Johnson & Johnson), PFE(Pfizer Inc.), UNH(UnitedHealth Group Inc.), MRK(Merck & Co. Inc.), AMGN(Amgen Inc.), BMY(Bristol-Myers Squibb Co.), ABT(Abbott Laboratories), LLY(Eli Lilly & Co.), AET(Aetna Inc.), BAX(Baxter International Inc.)
SP5ECST	WMT, PG(Procter & Gamble Co.), KO(Coca-Cola Co.), PEP(PepsiCo Inc.), MO(Altria Group Inc.), CL(Colgate-Palmolive Co.), KMB(Kimberly-Clark Corp.), GIS(General Mills Inc.), K(Kellogg Co.), CPB(Campbell Soup Co.)
SP5EIND	GE, BA(Boeing Co.), MMM(3M), HON(Honeywell International Inc.), UNP(Union Pacific Corp.), CAT(Caterpillar Inc.), RTN(Raytheon Co.), CSX(CSX Group), EMR(Emerson Electric Co.), NSC(Norfolk Southern Corp.)
SP5EMAT	PX(Praxair Inc.), SHW(Sherwin-Williams Co.), APD(Air Products & Chemicals Inc.), PPG(PPG Industries Inc.), IP(International Paper Co.), NEM(Newmont Mining Corp.), NUE(Nucor Corp.), FMC(FMC Corp.), EMN(Eastman Chemical Co.), IFF(International Flavors & Fragrances Inc.)
SP5ECOD	HD(Home Depot Inc.), DIS(Walt Disney Co.), MCD(McDonald's Corp.), F(Ford Motor Co.), CCL(Carnival Corp.), TGT(Target Corp.), GPC(Genuine Parts Co.), LB(L Brands Inc.), GPS(Gap Inc.), GT(Goodyear Tire & Rubber Co.)
SP5EUTL	NEE(NextEra Energy Inc.), DUK(Duke Energy Corp.), D(Dominion Energy Inc.), SO(Southern Co.), AEP(American Electric Power Co. Inc.), EXC(Exelon Corp.), PCG(PG&E Corp.), ED(Consolidated Edison Inc.), EIX(Edison International), PEG(Public Service Enterprise Group Inc.)
SP5EENE	XOM(Exxon Mobil Corp.), CVX(Chevron Corp.), SLB(Schlumberger Ltd.), COP(ConocoPhillips), EOG(EOG Resources Inc.), OXY(Occidental Petroleum Corp.), HAL(Halliburton Co.), BHGE(Baker Hughes, a GE co.), HES(Hess Corp.), MRO(Marathon Oil Corp.)
SP5EINT	MSFT, ORCL(Oracle Corp.), INTC(Intel Corp.), CSCO(Cisco Systems Inc.), IBM, ADP(Automatic Data Processing Inc.), HPQ(HP Inc.), CA(CA Inc.), MSI(Motorola Solutions Inc.), XRX(Xerox Corp.)

Table A1.1: S&P Sectors and the Largest Constituents

A2 Performance Metrics

The Sharpe ratio (Sharpe, 1966) is one of the most popular metrics for measuring risk-adjusted returns, and is defined as:

$$SR = \frac{\overline{\mu}_P - \overline{r}_f}{\sigma_P},$$

where $\overline{\mu}_P - \overline{r}_f$ represents the out-of-sample mean excess portfolio return, and σ_P is the portfolio standard deviation over the entire out-of-sample (investment) period. The Sharpe ratio has its limitations, and for this reason we also employ several other metrics.

The Certainty Equivalent Return (CERs) for mean-variance investors can be approximated and computed as follows:

$$CER = \overline{\mu}_P - \frac{\lambda \sigma_P^2}{2},$$

where λ is the relative risk aversion parameter, and $\overline{\mu}_P$ and σ_P have been defined above.

The Omega ratio (Keating and Shadwick, 2002) with a target return of zero, also known as the average gain to the average loss ratio, is computed as follows:

$$Omega = \frac{\sum_{t=1}^{\tau} \max\left(0, R_{p,t}\right)}{\sum_{t=1}^{T} \max\left(0, -R_{p,t}\right)}$$

where τ is the sample size of the out of sample observations, $R_{p,t}$ denotes the out-of-sample portfolio return at time t. The main advantage of the Omega ratio is that it does not require any assumption about the distribution of portfolio returns.

The Dowd ratio is the out-of-sample mean excess portfolio return, divided by the portfolio value-at-risk (Prigent, 2007), and is computed as follows:

$$Dowd = \frac{\overline{\mu}_P - \overline{r}_f}{VaR_{95\%}}.$$

The VaR in the Dowd ratio has been computed at the 95% confidence level over the entire investment (out-of-sample) period.

A3 Significance Tests for Sharpe Ratio and CER

A3.1 For Comparing Sharpe Ratios

Given two portfolios k and n, with $\hat{\mu}_k$, $\hat{\mu}_n$, $\hat{\sigma}_k^2$, $\hat{\sigma}_n^2$, and $\hat{\sigma}_{k,n}$ as their estimated means, variances, and covariance over a sample of size τ, the test of the hypothesis H_0:

$$\frac{\hat{\mu}_k}{\hat{\sigma}_k} = \frac{\hat{\mu}_n}{\hat{\sigma}_n}$$

is obtained by the test statistic Z_{JK}, which is asymptotically distributed as a standard normal:

$$Z_{\text{JK}} = \frac{\hat{\sigma}_n \hat{\mu}_k - \hat{\sigma}_k \hat{\mu}_n}{\sqrt{\vartheta}}$$

where

$$\vartheta = \frac{2\hat{\sigma}_k^2 \hat{\sigma}_n^2 - 2\hat{\sigma}_k \hat{\sigma}_n \hat{\sigma}_{k,n} + \frac{\hat{\sigma}_n^2 \hat{\mu}_k^2}{2} - \frac{\hat{\mu}_n \hat{\mu}_k \hat{\sigma}_{k,n}^2}{\hat{\sigma}_k \hat{\sigma}_n}}{\tau}.$$

A3.2 For Comparing CERs

If ν denotes the vector of moments, $\nu = (\mu_k, \mu_n, \sigma_k^2, \sigma_n^2)$, then $\hat{\nu}$, its empirical counterpart, is obtained from a sample of size τ. The difference between the CERs of the two strategies k and n is,

$$f(\nu) = \left(\mu_k - \frac{\lambda \sigma_k^2}{2}\right) - \left(\mu_n - \frac{\lambda \sigma_n^2}{2}\right)$$

and the asymptotic distribution of $f(\nu)$ is

$$\sqrt{T}\left[f(\hat{\nu}) - f(\nu)\right] \to N\left(0, \frac{\partial f^{\intercal}}{\partial \nu} \Xi \frac{\partial f}{\partial \nu}\right)$$

where

$$\Xi = \begin{bmatrix} \sigma_k^2 & \sigma_{k,n} & 0 & 0 \\ \sigma_{k,n} & \sigma_n^2 & 0 & 0 \\ 0 & 0 & 2\sigma_k^4 & 2\sigma_{k,n}^2 \\ 0 & 0 & 2\sigma_{k,n}^2 & 2\sigma_n^4 \end{bmatrix}.$$

A4 Proof of Proposition 1

For Λ, we derive $\beta^{\intercal}\left(\sigma_b^2\beta\beta^{\intercal}+\sigma_\varepsilon^2 I\right)^{-1}\beta$ and $\beta^{\intercal}\frac{\mathbb{1}\mathbb{1}^{\intercal}}{\mathbb{1}^{\intercal}\left(\sigma_b^2\beta\beta^{\intercal}+\sigma_\varepsilon^2 I\right)\mathbb{1}}\beta$ separately:

$$\beta^{\intercal}\left(\sigma_b^2\beta\beta^{\intercal}+\sigma_\varepsilon^2 I\right)^{-1}\beta=\frac{\beta^{\intercal}\left(\sigma_b^2\beta\beta^{\intercal}+\sigma_\varepsilon^2 I\right)^{\dagger}\beta}{\det\left(\sigma_b^2\beta\beta^{\intercal}+\sigma_\varepsilon^2 I\right)}=\frac{B_1}{B_1\sigma_b^2+\sigma_\varepsilon^2};$$
$$\beta^{\intercal}\frac{\mathbb{1}\mathbb{1}^{\intercal}}{\mathbb{1}^{\intercal}\left(\sigma_b^2\beta\beta^{\intercal}+\sigma_\varepsilon^2 I\right)\mathbb{1}}\beta=\frac{\mathbb{1}^{\intercal}\beta\beta^{\intercal}\mathbb{1}}{\sigma_b^2\mathbb{1}^{\intercal}\beta\beta^{\intercal}\mathbb{1}+N\sigma_\varepsilon^2}=\frac{B_2}{B_2\sigma_b^2+N\sigma_\varepsilon^2}.$$

For Υ, we derive $\alpha^{\intercal}\left(\sigma_b^2\beta\beta^{\intercal}+\sigma_\varepsilon^2 I\right)^{-1}\alpha$ and $\alpha^{\intercal}\frac{\mathbb{1}\mathbb{1}^{\intercal}}{\mathbb{1}^{\intercal}\left(\sigma_b^2\beta\beta^{\intercal}+\sigma_\varepsilon^2 I\right)\mathbb{1}}\alpha$ separately:

$$\begin{aligned}\alpha^{\intercal}\left(\sigma_b^2\beta\beta^{\intercal}+\sigma_\varepsilon^2 I\right)^{-1}\alpha&=\frac{\alpha^{\intercal}\left(\sigma_b^2\beta\beta^{\intercal}+\sigma_\varepsilon^2 I\right)^{\dagger}\alpha}{\det\left(\sigma_b^2\beta\beta^{\intercal}+\sigma_\varepsilon^2 I\right)}\\&=\frac{\sigma_\varepsilon^2\alpha^{\intercal}\left(\sigma_b^2\left[\left(\beta\beta^{\intercal}+I\right)^{\dagger}-I\right]+\sigma_\varepsilon^2 I\right)\alpha}{\det\left(\sigma_b^2\beta\beta^{\intercal}+\sigma_\varepsilon^2 I\right)}\\&=\frac{k\sigma_b^2 C}{(B_1\sigma_b^2+\sigma_\varepsilon^2)\sigma_\varepsilon^2}+\frac{kA_1}{B_1\sigma_b^2+\sigma_\varepsilon^2};\\\alpha^{\intercal}\frac{\mathbb{1}\mathbb{1}^{\intercal}}{\mathbb{1}^{\intercal}\left(\sigma_b^2\beta\beta^{\intercal}+\sigma_\varepsilon^2 I\right)\mathbb{1}}\alpha&=\frac{\mathbb{1}^{\intercal}\alpha\alpha^{\intercal}\mathbb{1}}{\sigma_b^2\mathbb{1}^{\intercal}\beta\beta^{\intercal}\mathbb{1}+N\sigma_\varepsilon^2}=\frac{A_2}{B_2\sigma_b^2+N\sigma_\varepsilon^2}.\end{aligned}$$

To prove $\frac{\partial\Lambda}{\partial(\sigma_\varepsilon^2)}>0$, we can easily compute

$$\begin{aligned}\frac{\partial\Lambda}{\partial\left(\sigma_\varepsilon^2\right)}&=\mu_b^2\left[N\frac{B_2}{\left(B_2\sigma_b^2+N\sigma_\varepsilon^2\right)^2}-k\frac{B_1}{\left(B_1\sigma_b^2+\sigma_\varepsilon^2\right)^2}\right]\\&>\mu_b^2\left[\frac{\frac{B_2}{N}}{\left(\frac{B_2}{N}\sigma_b^2+\frac{\sigma_\varepsilon^2}{k}\right)^2}-\frac{\frac{B_1}{k}}{\left(\frac{B_1}{k}\sigma_b^2+\frac{\sigma_\varepsilon^2}{k}\right)^2}\right].\end{aligned}$$

The inequation above is by the fact $k<1$ which is straightforward to verify (see (21) in Kan and Zhou, 2007). Define $f(x)$ as

$$f(x)\triangleq\frac{x}{\left(x\sigma_b^2+\frac{\sigma_\varepsilon^2}{k}\right)^2},$$

then we have

$$\frac{\partial\Lambda}{\partial\left(\sigma_\varepsilon^2\right)}>\mu_b^2\left[f\left(\frac{B_2}{N}\right)-f\left(\frac{B_1}{k}\right)\right].$$

Since $\frac{\partial f(x)}{\partial x}=\frac{k^2}{\left(xk\sigma_b^2+\sigma_\varepsilon^2\right)^2}\left(\sigma_\varepsilon^2-xk\sigma_b^2\right)$ and $\frac{B_1}{k}>\frac{B_2}{N}$ (from the condition $B_1>B_2\frac{k}{N}>\frac{\sigma_\varepsilon^2}{\sigma_b^2}$), $f(x)$ is decreasing in $\frac{B_2}{N}\leq x\leq\frac{B_1}{k}$ (when $x\geq\frac{B_2}{N}$, $\sigma_\varepsilon^2-xk\sigma_b^2<0$ by the condition $B_1>B_2\frac{k}{N}>\frac{\sigma_\varepsilon^2}{\sigma_b^2}$),

i.e., $f\left(\frac{B_2}{N}\right) > f\left(\frac{B_1}{k}\right) \implies \frac{\partial \Lambda}{\partial(\sigma_\varepsilon^2)} > 0$. This completes the proof of $\frac{\partial \Lambda}{\partial(\sigma_\varepsilon^2)} > 0$.

To prove $\frac{\partial \Upsilon}{\partial(\sigma_\varepsilon^2)} < 0$, we focus on $\frac{kA_1}{B_1\sigma_b^2+\sigma_\varepsilon^2} - \frac{A_2}{B_2\sigma_b^2+N\sigma_\varepsilon^2}$ first:

$$\begin{aligned}\frac{\partial\left(\frac{kA_1}{B_1\sigma_b^2+\sigma_\varepsilon^2} - \frac{A_2}{B_2\sigma_b^2+N\sigma_\varepsilon^2}\right)}{\partial\left(\sigma_\varepsilon^2\right)} &= \frac{NA_2}{\left(B_2\sigma_b^2+N\sigma_\varepsilon^2\right)^2} - \frac{kA_1}{\left(B_1\sigma_b^2+\sigma_\varepsilon^2\right)^2} \\ &= \frac{1}{\left(\frac{B_2}{N}\sqrt{\frac{N}{A_2}}\sigma_b^2 + \sqrt{\frac{N}{A_2}}\sigma_\varepsilon^2\right)^2} - \frac{1}{\left(\frac{B_1}{\sqrt{kA_1}}\sigma_b^2 + \frac{1}{\sqrt{kA_1}}\sigma_\varepsilon^2\right)^2}.\end{aligned}$$

Given the condition $A_1k > A_1k\left(\frac{B_2}{B_1N}\right)^2 > \frac{A_2}{N}$, we can readily verify that

$$\frac{B_2}{N}\sqrt{\frac{N}{A_2}} > \frac{B_1}{\sqrt{kA_1}} \text{ and } \frac{N}{A_2} > \frac{1}{kA_1},$$

meaning

$$\frac{\partial\left(\frac{kA_1}{B_1\sigma_b^2+\sigma_\varepsilon^2} - \frac{A_2}{B_2\sigma_b^2+N\sigma_\varepsilon^2}\right)}{\partial\left(\sigma_\varepsilon^2\right)} < 0.$$

Next, with C being positive due to its quadratic form, it is clear that

$$\frac{\partial\left(\frac{k\sigma_b^2C}{(B_1\sigma_b^2+\sigma_\varepsilon^2)\sigma_\varepsilon^2}\right)}{\partial\left(\sigma_\varepsilon^2\right)} < 0,$$

therefore $\frac{\partial \Upsilon}{\partial(\sigma_\varepsilon^2)} < 0$ is proved.

A5 Robustness Check on Using the Financial Sector to Calibrate the US S&P Simulations

For the simulations in section 7 we calibrated the distribution of alphas using the US finance sector. To test the robustness of using the financial sector, we also ran these simulations using the health care sector, the industrial sector and the consumer discretional sector to calibrate the alpha distribution. The results appear in Figures A5.1 to A5.3; and show that the simulation results in the paper are robust to changing the sector used to calibrate the alpha distribution.

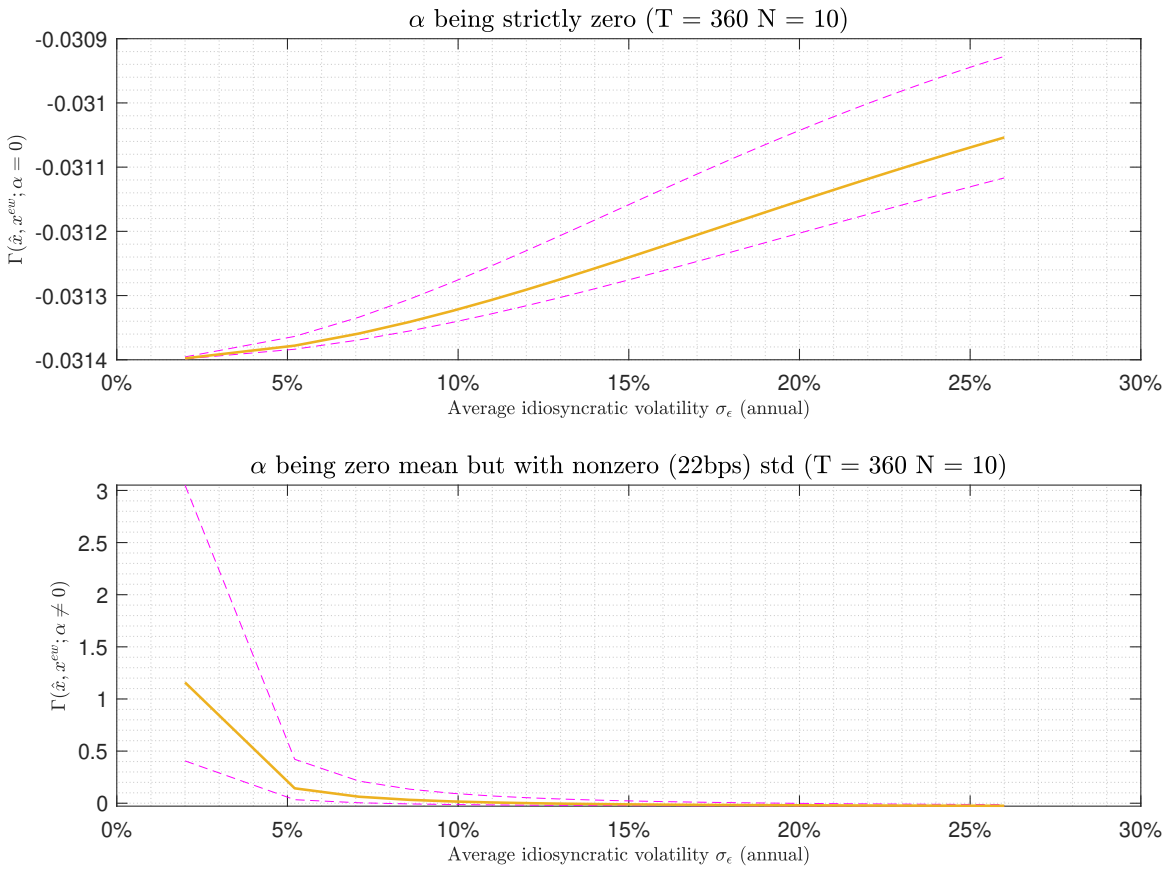

Figure A5.1: $\Gamma\left(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}}\right)$ versus Idiosyncratic Volatility (σ_ε). The simulation is based on $T = 240$, $N = 10$ and alpha distribution from the Health Care sector. The dashed lines are 95% confidence intervals

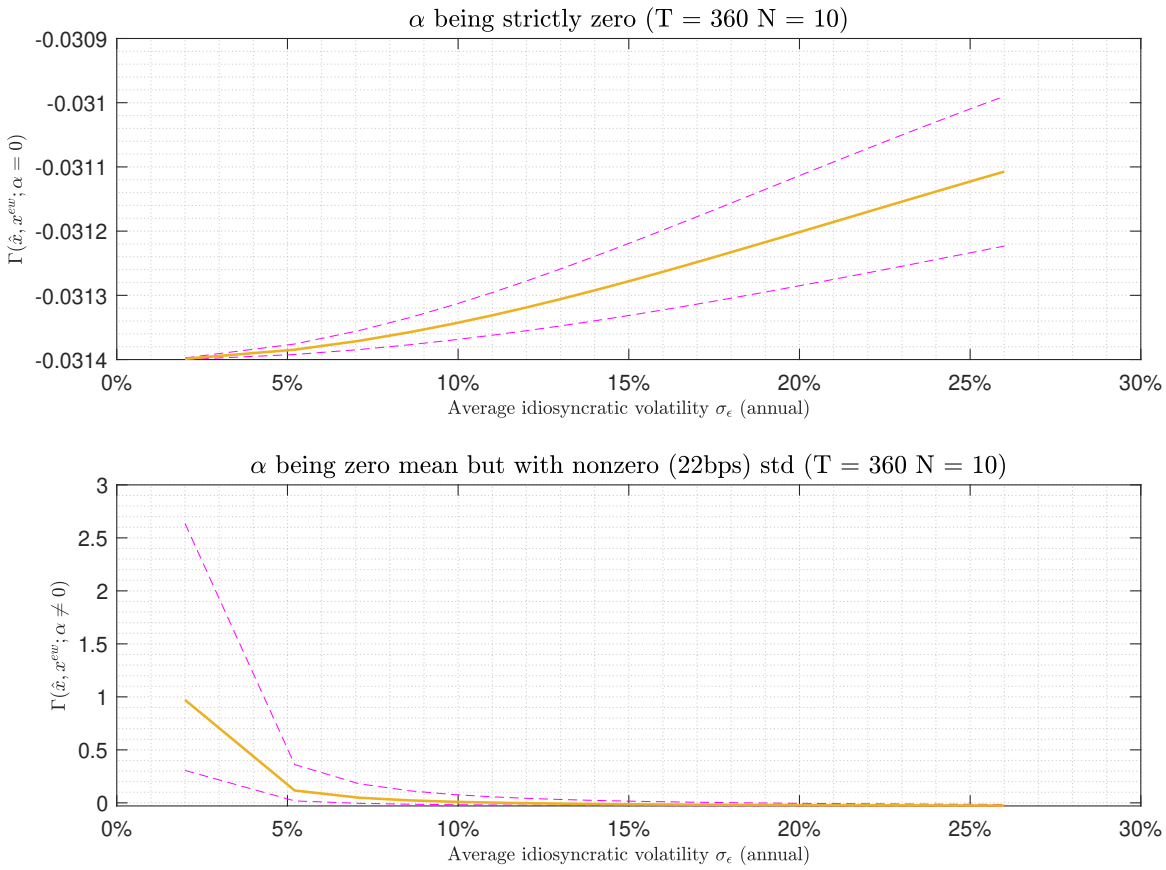

Figure A5.2: $\Gamma\left(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}}\right)$ versus Idiosyncratic Volatility (σ_ε). The simulation is based on $T = 240$, $N = 10$ and alpha distribution from the Industrial sector. The dashed lines are 95% confidence intervals

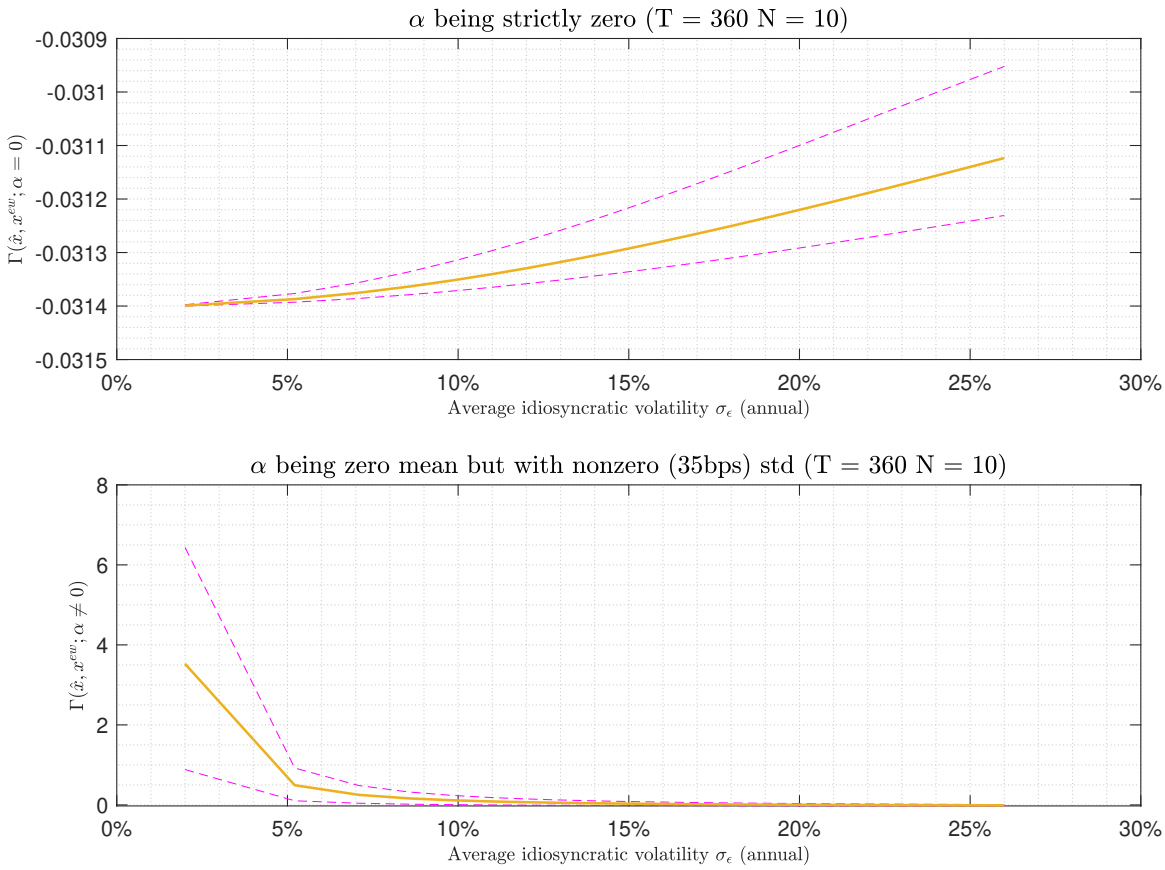

Figure A5.3: $\Gamma\left(\hat{\boldsymbol{x}}, \boldsymbol{x}^{\text{ew}}\right)$ versus Idiosyncratic Volatility (σ_ε). The simulation is based on $T = 240$, $N = 10$ and alpha distribution from the Consumer Discretionary sector. The dashed lines are 95% confidence intervals

A6 Fama-French 5, 10, 12 and 17 Industries

In a robustness check we applied our core analysis to the Fama-French 5, 10, 12 and 17 industry portfolios using monthly data for these companies from January 1994 to August 2017. In each case we analysed the ten largest companies in January 1994 and, allowing for less than ten companies in some industries, the total number of companies we analysed is 50, 80, 115 and 131, respectively. The risk free rate of return is 1-month T-bill returns from French's web site. In the asset allocation stage, Table A6.1 shows that MV is superior for all four industry portfolios and every performance measure. Overall the performance measures in Table A6.2 indicate that, as hypothesised, MV-1/N is superior for all four Fama-French portfolios. 1/N-MV and MV-MV are the worst performers.

Performance Measure		FF = 5	FF = 10	FF = 12	FF = 17
CER	MV	0.0718	0.0741	0.0673	0.0561
	1/N	0.0640	0.0618	0.0604	0.0411
Sharpe Ratio	MV	0.6960	0.7178	0.6631	0.5871
	1/N	0.6404	0.6247	0.6147	0.5158
Dowd Ratio	MV	0.4759	0.5037	0.4464	0.3736
	1/N	0.4202	0.4059	0.3968	0.3080
Omega Ratio	MV	1.8853	1.9506	1.8588	1.7531
	1/N	1.8052	1.8014	1.7903	1.6422

Table A6.1: Performance of 1/N and MV in Forming Portfolios of Four Fama and French Industry Portfolios, i.e. Stage One Asset Allocation - 12-month expanding estimation window (24 months for FF = 17), January 1994 to August 2017, λ = 5, Transaction costs = 50 bps.

	MV Model	λ	η	Initial Window	Constraints	Performance Measure	1	2	3	4
							MV-1/N	1/N-1/N	MV-MV	1/N-MV
1	Bayes-Stein FF=5	5	0.15	12	VBCs	CER	0.0851	0.0833	0.0639	0.0665
						Sharpe Ratio	0.7919	0.7768	0.6363	0.6565
2	Bayes-Stein FF=10	5	0.15	12	VBCs	CER	0.0924	0.0889	0.0841	0.0826
						Sharpe Ratio	0.8696	0.8189	0.8067	0.7830
3	Bayes-Stein FF=12	5	0.15	12	VBCs	CER	0.0886	0.0841	0.0749	0.0775
						Sharpe Ratio	0.8355	0.7815	0.7313	0.7410
4	Bayes-Stein FF=17	5	0.15	24	VBCs	CER	0.0756	0.0722	0.0645	0.0686
						Sharpe Ratio	0.7449	0.7124	0.6566	0.6834

Table A6.2: Performance Measures for the Four Two-stage Procedures for the Four Fama French (FF) Industry Portfolios - 12-month Expanding Estimation Window (24 months for FF = 17), January 1994 to August 2017, λ = 5, Transaction costs = 50 bps.

A7 International Data

As a further robustness check, we repeated the two stage methodology of our core analysis on international data for the UK, USA, Germany, Switzerland, France, Canada and Brazil. This consists of value-weighted total return equity market indices for seven countries - UK (FTSE 100), US (S&P500), Germany (DAX 30), Switzerland (SMI), France (CAC 40), Canada (S&P/TSX Composite), and Brazil (Bovespa); with monthly data from December 1994 to August 2017 expressed in $US. We also analysed the ten companies with the largest market capitalization in each index in December 1994; so in total we have 70 companies. We used 1-month T-bill returns from Ken French?s web site as the riskless rate.

Table A7.1 shows that for asset allocation across these countries MV remains dominant, while Table A7.2 indicates that 1/N is generally preferred for stock selection, although not for Switzerland or Brazil. Table A7.3 confirms our main hypothesis, as MV-1/N is the best strategy for selecting international portfolios. We also find that, on balance, its reverse, 1/N-MV, is the worst strategy.

Performance Measure	1/N	MV
CER	0.0126	0.0272
Sharpe Ratio	0.4178	0.4233
Dowd Ratio	0.2328	0.2398
Omega Ratio	1.4994	1.5305

Table A7.1: Performance of 1/N and MV in Forming Portfolios of the Seven Countries, i.e. Stage One Asset Allocation - 12-month expanding estimation window, December 1994 to August 2017, $\lambda = 5$, $\eta = 0.15$, Transaction costs = 50 bps.

Countries	CER		Sharpe Ratio		Dowd Ratio		Omega Ratio	
	1/N	MV	1/N	MV	1/N	MV	1/N	MV
UK	0.0446	0.0345	0.5480	0.4632	0.3326	0.2690	1.6889	1.5755
US	0.0731	0.0615	0.7182	0.6275	0.4989	0.4117	1.9767	1.8392
Germany	-0.0444	-0.0495	0.3660	0.2818	0.1951	0.1442	1.4163	1.3261
France	0.0062	0.0144	0.4599	0.4021	0.2603	0.2229	1.5338	1.4755
Canada	0.0619	0.0535	0.6736	0.6181	0.4388	0.3897	1.8320	1.7582
Switzerland	0.0113	0.0432	0.4609	0.5274	0.2617	0.3173	1.5487	1.6300
Brazil	-0.2499	-0.1657	0.5368	0.5759	0.3077	0.3386	1.5987	1.6363

Table A7.2: Performance of 1/N and MV in Forming Portfolios of the Shares Within Each County, i.e. Stage Two Stock Selection - 12-month expanding estimation window, December 1994 to August 2017, λ = 5, η = 0.15, Transaction costs = 50 bps.

Performance Measure	1	2	3	4
	MV-1/N	1/N-1/N	MV-MV	1/N-MV
CER	0.0632	0.0510	0.0590	0.0552
Sharpe Ratio	0.6512	0.6232	0.6110	0.6117
Dowd Ratio	0.4244	0.3922	0.3928	0.3863
Omega Ratio	1.8318	1.7642	1.7783	1.7554

Table A7.3: Performance Measures for the Four Two-stage Procedures for the International Data - 12-month Expanding Estimation Window, December 1994 to August 2017, λ = 5, η = 0.15, Transaction costs = 50 bps.

Country	Constituents
UK - FTSE100	Barclays Bank, British Petroleum, Unilever, GlaxoSmithKlein, British American Tobacco, Diageo, Rio Tinto, Royal Dutch Shell, British Telecommunications, Marks & Spencer
USA - S&P500	Walmart, Exxon-Mobil, Coca-Cola, IBM, General Electric, Proctor & Gamble, Merck, Pepsico, Altria, Bristol Myers Squibb.
Germany - DAX 30	Deutsche Bank, BMW, Allianz, Siemens, BASF, Bayer, RWE, Munich Re, E.ON, ThyssenKrupp.
France - CAC 40	BNP Paribas, L'Oreal, Total, Societe Generale, AXA, Danone, LVMH, Air Liquide, Carrefour, Vivendi.
Canada - S&P/TSX Composite	Royal Bank of Canada, Bank of Montreal, Toronto Dominion Bank, Bank of Nova Scotia, Canadian Imperial Bank of Commerce, Bell Canada Enterprises, Imperial Oil, Barrick Gold, Encana, Thomson-Reuters.
Switzerland - SMI	Nestle, UBS, Roche, Credit Suisse, Novartis, ABB, Zurich Insurance, Richemont, Swiss Re, Swatch.
Brazil - Bovespa	Vale, Petrobras, Companhia Siderúrgica Nacional, Usiminas, Eletrobras, CEMIG, ITAU Unibanco, Banco do Brazil, Bradesco, Lojas Americanas

Table A7.4: Largest Ten Companies in the Seven Country Indices

A8 UK Data

We analysed monthly value-weighted total returns from DataStream on ten UK industry indices, and the ten largest firms in each sector in January 1994. Some sectors had less than ten firms, and so the total number of firms analysed is 56. Our data is from January 1994 to August 2017, and the risk free asset is the Thomson Reuters UK Government Benchmark Yield 1 Month. Table A8.1 shows that in the first stage MV produces better out-of-sample asset allocation results on all four performance measures, as expected.

Performance Measure	1/N	MV
CER	0.0692	0.0783
Sharpe Ratio	0.6017	0.6658
Dowd Ratio	0.4050	0.4725
Omega Ratio	1.8550	1.9519

Table A8.1: Performance of 1/N and MV in Forming Portfolios of the 10 UK Industrial Sectors, i.e. Stage One Asset Allocation - 24-month expanding estimation window, January 1994 to August 2017, $\lambda = 5$, $\eta = 0.15$, Transaction costs = 50 bps

In Table A8.2 1/N produces better out-of-sample results on all four performance measures for the second stage (stock selection) for seven industries, and superior performance by 1/N on three measures for two or three industries. This supports the hypothesis that 1/N is preferable for stock selection.

Table A8.3 compares the overall performance of the four strategies. MV-1/N is the best on all four performance measures and 1/N-MV is the worst, as expected.

Industries	N	CER		Sharpe Ratio		Dowd Ratio		Omega Ratio	
		1/N	MV	1/N	MV	1/N	MV	1/N	MV
Basic resources	6	0.0525	0.0525	0.6789	0.6722	0.4497	0.4438	1.8983	1.8994
Consumer discretionary	10	0.0510	0.0426	0.6259	0.5226	0.4040	0.3207	1.8495	1.6960
Consumer products	8	0.0271	0.0267	0.6222	0.4990	0.3946	0.2986	1.8018	1.6777
Energy	6	-0.0154	0.0072	0.4326	0.4200	0.2455	0.2396	1.5395	1.5416
Financial services	10	0.0572	0.0402	0.5318	0.4106	0.3379	0.2432	1.7478	1.6036
Industrial goods	10	0.0177	0.0012	0.4236	0.2911	0.2438	0.1568	1.5796	1.4381
Real estate	10	0.0248	0.0262	0.4083	0.3824	0.2351	0.2187	1.5661	1.5466
Technology	4	-0.0118	-0.0152	0.5304	0.5140	0.3163	0.3036	1.6772	1.6578
Telecommunications	3	-0.0241	-0.0137	0.3767	0.3471	0.2076	0.1896	1.4910	1.4658
Utilities	5	0.0665	0.0550	0.5977	0.5219	0.3948	0.3288	1.8473	1.7381

Table A8.2: Performance of 1/N and MV in Forming Portfolios of the Shares Within Each UK Industry, i.e. Stage Two Stock Selection - 24-month expanding estimation window, January 1994 to August 2017, $\lambda = 5$, $\eta = 0.15$, Transaction costs = 50 bps.

Performance Measure	1	2	3	4
	MV-1/N	1/N-1/N	MV-MV	1/N-MV
CER	0.0921	0.0853	0.0867	0.0784
Sharpe Ratio	0.7620	0.7185	0.7259	0.6664
Dowd Ratio	0.5652	0.5155	0.5330	0.4705
Omega Ratio	2.1119	2.0455	2.0842	1.9837

Table A8.3: Performance Measures for the Four Two-stage Procedures for the UK Data - 24 month Expanding Estimation Window, January 1994 to August 2017, $\lambda = 5$, $\eta = 0.15$, Transaction costs = 50 bps.

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