Effects of R&D Subsidies in a Hybrid Model of Endogenous Growth and Semi-Endogenous Growth

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January 15, 2020

Abstract

We explore R&D subsidies in a hybrid growth model which may exhibit semi-endogenous growth or fully endogenous growth. We consider two types of subsidies on variety-expanding innovation and quality-improving innovation. R&D subsidies on quality-improving innovation only have effects in the fully-endogenous-growth regime, in which more subsidies cause an earlier activation of quality-improving innovation and increase the transitional/steady-state growth rate. R&D subsidies on variety-expanding innovation have contrasting effects in the two regimes. In the semi-endogenous-growth regime, more subsidies on variety-expanding innovation increase transitional growth but have no effect on steady-state growth. In the fully-endogenous-growth regime, more subsidies on variety-expanding innovation continue to increase short-run growth but delay the activation of quality-improving innovation and reduce long-run growth. Increasing subsidies on variety-expanding (quality-improving) innovation makes the semi-endogenous-growth (fully-endogenous-growth) regime more likely to emerge. Finally, we calibrate the model and find that under reasonable parameter values, the fully-endogenous-growth regime is more likely to emerge.

JEL classification: O31, O34
Keywords: R&D subsidies, innovation, endogenous growth regimes

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1 Introduction

In this study, we provide a growth-theoretic analysis on the effects of R&D subsidies. The novelty of our analysis is that we consider a hybrid growth model in which the economy may exhibit semi-endogenous growth or fully endogenous growth in the long run. The model is based on Peretto (2015), who develops a Schumpeterian growth model of endogenous takeoff. In this model, the economy initially experiences stagnation with zero growth in output per capita. As the market size of the economy becomes sufficiently large due to population growth, the economy starts to experience innovation and growth. Although the economy eventually experiences the development of new products (i.e., variety-expanding innovation), it may or may not experience the quality improvement of products (i.e., quality-improving innovation). If the economy only features variety-expanding innovation in the long run, then the balanced growth path exhibits semi-endogenous growth. If the economy features both variety-expanding innovation and quality-improving innovation, then the balanced growth path exhibits fully endogenous growth. In other words, the model in Peretto (2015) nests the semi-endogenous growth model, in which the long-run growth rate is independent of policies, and the second-generation Schumpeterian growth model, in which the long-run growth rate is fully endogenous, as special cases.

Within the above growth-theoretic framework, we consider two types of R&D subsidies on variety-expanding innovation and quality-improving innovation and obtain the following results. R&D subsidies on quality-improving innovation only have effects in the fully-endogenous-growth regime, in which a higher subsidy rate leads to an earlier activation of quality-improving innovation and increases the transitional and steady-state growth rate of output per capita. Interestingly, R&D subsidies on variety-expanding innovation have contrasting effects in the semi-endogenous-growth regime and the fully-endogenous-growth regime. Specifically, if the economy is in the semi-endogenous-growth regime, then a higher subsidy rate on variety-expanding innovation leads to a higher transitional growth rate of output per capita but has no effect on its steady-state growth rate. If the economy is in the fully-endogenous-growth regime, then a higher subsidy rate on variety-expanding innovation continues to have a positive effect on the growth rate in the short run but leads to a later activation of quality-improving innovation and a lower growth rate in the long run. Increasing R&D subsidies on variety-expanding innovation makes the semi-endogenous-growth regime more likely to emerge in equilibrium, whereas increasing R&D subsidies on quality-improving innovation makes the fully-endogenous-growth regime more likely to emerge. Finally, we calibrate the model and find that under reasonable parameter values, the fully-endogenous-growth regime is more likely to emerge in equilibrium.

This study relates to the literature on innovation and economic growth. Romer (1990) develops the variety-expanding R&D-based growth model in which innovation is driven by the creation of new products. Aghion and Howitt (1992) develop the Schumpeterian quality-ladder growth model in which innovation is driven by the quality improvement of existing products.[1] Jones (1995) argues that these seminal studies feature a counterfactual scale effect of the population size on economic growth and develops the semi-endogenous growth model,

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[1] See also Segerstrom *et al.* (1990) and Grossman and Helpman (1991a).

in which the steady-state growth rate is scale-invariant.[2] Smulders and van de Klundert (1995), Peretto (1998, 1999) and Howitt (1999) combine the two dimensions of innovation and develop a second-generation Schumpeterian model with endogenous market structure that also removes the scale effect.[3] This study explores the effects of R&D subsidies in this vintage of the Schumpeterian growth model and considers their different implications under semi-endogenous growth versus fully endogenous growth.[4]

In the literature, other studies also explore the effects of R&D subsidies in the R&D-based growth model; see for example, Segerstrom (1998), Lin (2002), Zeng and Zhang (2007), Impullitti (2010), Chu and Cozzi (2018), Yang (2018) and Hu, Yang and Zheng (2019). These studies mostly focus on either variety expansion or quality improvement. Only a few studies, such as Segerstrom (2000) and Chu, Furukawa and Ji (2016), explore the effects of R&D subsidies in the Schumpeterian growth model with both dimensions of innovation. However, none of these studies consider how R&D subsides affect the endogenous activation of the two types of innovation.

This study also relates to the literature on endogenous takeoff and economic growth. In this literature, seminal studies include Galor and Weil (2000) and Galor and Moav (2002), who develop unified growth theory.[5] Unified growth theory shows that the quality-quantity tradeoff in childrearing and human capital accumulation allow an economy to escape from the Malthusian trap and experience economic takeoff.[6] While human capital is certainly a crucial engine of economic growth, innovation is another important engine of growth. Therefore, we consider the Schumpeterian growth model in Peretto (2015) in which endogenous takeoff is driven by innovation. This model features both variety-expanding innovation and quality-improving innovation. A novel contribution of our study is to incorporate R&D subsidies into the Peretto model to explore their effects on the endogenous activation of the two types of innovation and the endogenous determination between the semi-endogenous-growth regime and the fully-endogenous-growth regime.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 explores the effects of R&D subsides. Section 4 concludes.

2 The model

We consider the Schumpeterian growth model with both variety-expanding innovation and quality-improving innovation in Peretto (2015), in which endogenous growth in the number of products gives rise to a dilution effect that removes the scale effect. In the model, labor is used as a factor input for the production of final good. Final good is consumed by households or used as a factor input for entry, in-house R&D and the production/operation of intermediate

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[2] See also Grossman and Helpman (1991b, p. 75-76) who anticipated the semi-endogenous growth model.

[3] See Laincz and Peretto (2006) and Ha and Howitt (2007) for empirical evidence that supports the second-generation Schumpeterian model.

[4] See also Cozzi (2017a,b) who develops a general innovation specification that may yield semi-endogenous growth or fully endogenous growth in the long run.

[5] See also Jones (2001) and Hansen and Prescott (2002) for other early studies on endogenous takeoff.

[6] See Galor and Mountford (2008) and Ashraf and Galor (2011) for recent studies and empirical evidence for unified growth theory. Galor (2011) provides a comprehensive overview of unified growth theory.

goods. We extend Peretto (2015) by incorporating two types of R&D subsidies into the model and analyzing their effects on the takeoff, transitional dynamics and the balanced growth path of the economy.

2.1 Household

The representative household has the following utility function:

$$U = \int_0^\infty e^{-(\rho-\lambda)t} \ln c_t dt, \tag{1}$$

where $c_t \equiv C_t/L_t$ is (per capita) consumption of final good (numeraire) at time t, and $\rho > 0$ is the subjective discount rate. Population grows at an exogenous rate $\lambda \in (0, \rho)$. We normalize the initial population to unity (i.e., $L_t = e^{\lambda t}$). The household maximizes (1) subject to the following asset-accumulation equation:

$$\dot{a}_t = (r_t - \lambda) a_t + (1-\tau) w_t - c_t, \tag{2}$$

where $a_t \equiv A_t/L_t$ is the real value of assets owned by each member of the household, and r_t is the real interest rate. Each member supplies one unit of labor to earn w_t, and $\tau \in (0, 1)$ is an exogenous tax rate on labor income. Standard dynamic optimization yields the familiar Euler equation given by

$$\frac{\dot{c}_t}{c_t} = r_t - \rho. \tag{3}$$

2.2 Final good

Final output Y_t is produced by competitive firms. The production function is given by

$$Y_t = \int_0^{N_t} X_t^\theta (i) \left[Z_t^\alpha (i) Z_t^{1-\alpha} L_t / N_t^{1-\sigma} \right]^{1-\theta} di, \tag{4}$$

where $\{\theta, \alpha, \sigma\} \in (0, 1)$. $X_t(i)$ is the quantity of non-durable intermediate goods $i \in [0, N_t]$. The productivity of $X_t(i)$ is determined by its own quality $Z_t(i)$ and also by the average quality of all intermediate goods $Z_t \equiv \int_0^{N_t} Z_t(j) dj / N_t$, which captures technology spillovers. The parameter α determines the private return to quality, and $1-\alpha$ determines the degree of technology spillovers. The parameter $1-\sigma$ captures a congestion effect of variety, and σ determines the social return to variety as we will show.

Profit maximization yields the following conditional demand functions for L_t and $X_t(i)$:

$$L_t = (1-\theta) Y_t / w_t, \tag{5}$$

$$X_t(i) = \left(\frac{\theta}{p_t(i)} \right)^{1/(1-\theta)} Z_t^\alpha (i) Z_t^{1-\alpha} L_t / N_t^{1-\sigma}, \tag{6}$$

where $p_t(i)$ is the price of $X_t(i)$. Perfect competition implies that firms in this sector pay $\theta Y_t = \int_0^{N_t} p_t(i) X_t(i) di$ for intermediate goods.

2.3 Intermediate goods and in-house R&D

Monopolistic firms produce differentiated intermediate goods. The production process is based on a linear technology that requires $X_t\left(i\right)$ units of final good to produce $X_t\left(i\right)$ units of intermediate good $i \in [0, N_t]$. The firm in industry i also incurs $\phi Z_t^{\alpha}\left(i\right) Z_t^{1-\alpha}$ units of final good as a fixed operating cost, which is increasing in the level of technology. Furthermore, the firm devotes $I_t\left(i\right)$ units of final good to in-house R&D in order to improve the quality of its products. The innovation process is specified as

$$\dot{Z}_t\left(i\right) = I_t\left(i\right), \tag{7}$$

and the firm's (before-R&D) profit flow at time t is

$$\Pi_t\left(i\right) = \left[p_t\left(i\right) - 1\right] X_t\left(i\right) - \phi Z_t^{\alpha}\left(i\right) Z_t^{1-\alpha}. \tag{8}$$

The value of the monopolistic firm in industry i is

$$V_t\left(i\right) = \int_t^{\infty} \exp\left(-\int_t^{v} r_u du\right)\left[\Pi_v\left(i\right) - (1-s_Z) I_v\left(i\right)\right] dv, \tag{9}$$

where $s_Z \in (0,1)$ is the subsidy rate on quality-improving innovation. The monopolistic firm maximizes (9) subject to (6), (7) and (8). We solve this dynamic optimization problem in the proof of Lemma 1 and find that the profit-maximizing markup ratio is $1/\theta$. Hence, the equilibrium price is[7]

$$p_t\left(i\right) = 1/\theta. \tag{10}$$

We follow previous studies to consider a symmetric equilibrium in which $Z_t\left(i\right) = Z_t$ for $i \in [0, N_t]$ and the size of each intermediate-good firm is identical across all industries $X_t\left(i\right) = X_t$.[8] We define the following variable for the quality-adjusted firm size:

$$x_t \equiv \frac{X_t}{Z_t}. \tag{11}$$

Substituting (10) into (6) and imposing symmetry yield

$$x_t = \theta^{2/(1-\theta)} \frac{L_t}{N_t^{1-\sigma}}, \tag{12}$$

which is a state variable that determines the dynamics of the economy. In Lemma 1, we derive the rate of return on quality-improving R&D, which is increasing in x_t and s_Z.

Lemma 1 *The rate of return on quality-improving in-house R&D is*

$$r_t^q = \frac{\alpha}{1-s_Z}\frac{\Pi_t}{Z_t} = \frac{\alpha}{1-s_Z}\left(\frac{1-\theta}{\theta}x_t - \phi\right). \tag{13}$$

Proof. See the Appendix. ■

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[7] Alternatively, one can introduce a patent policy parameter to impose an upper bound on the equilibrium price. See for example Chu *et al.* (2020) for an analysis of patent policy in the Peretto model, but they focus on the fully-endogenous-growth regime.

[8] Symmetry also implies $\Pi_t\left(i\right) = \Pi_t$, $I_t\left(i\right) = I_t$ and $V_t\left(i\right) = V_t$.

2.4 Entrants

We follow the standard treatment in the literature to assume that entrants have access to aggregate technology Z_t, which in turn ensures symmetric equilibrium at any time t. A new firm pays βX_t units of final good to develop a new variety of intermediate goods and set up its operation. $\beta > 0$ is a cost parameter, and X_t captures the scale of the initial operation. The asset-pricing equation implies that the rate of return on assets is

$$r_t = \frac{\Pi_t - (1 - s_Z) I_t}{V_t} + \frac{\dot{V}_t}{V_t}. \tag{14}$$

When entry is positive (i.e., $\dot{N}_t > 0$), the no-arbitrage condition is given by

$$V_t = (1 - s_N)\beta X_t, \tag{15}$$

where $s_N \in (0, 1)$ is the subsidy rate on variety-expanding innovation. Substituting (7), (8), (10), (12) and (15) into (14) yields the rate of return on entry as

$$r_t^e = \frac{1}{(1 - s_N)\beta}\left[\frac{1-\theta}{\theta} - \frac{\phi + (1 - s_Z) z_t}{x_t}\right] + \frac{\dot{x}_t}{x_t} + z_t, \tag{16}$$

where $z_t \equiv \dot{Z}_t / Z_t$ is the growth rate of aggregate quality.

2.5 Government

The government collects income tax T_t from the representative household. The amount of tax revenue is

$$T_t = \tau w_t L_t = \tau(1 - \theta) Y_t. \tag{17}$$

The balanced-budget condition is given by

$$T_t = G_t + s_Z \int_0^{N_t} I_t(i) di + s_N \dot{N}_t \beta X_t, \tag{18}$$

where G_t is unproductive government spending. We follow Peretto (2007) to assume that G_t changes endogenously to balance the fiscal budget.

2.6 Equilibrium

The equilibrium is a time path of allocations $\{A_t, Y_t, C_t, X_t, I_t, G_t\}$ and prices $\{r_t, w_t, p_t, V_t\}$ such that

- the household maximizes utility taking $\{r_t, w_t\}$ and the tax rate τ as given;
- competitive firms produce Y_t and maximize profits taking $\{w_t, p_t\}$ as given;
- incumbents for intermediate goods produce X_t and choose $\{p_t, I_t\}$ to maximize V_t taking r_t and the subsidy rate s_Z as given;

- entrants make entry decisions taking V_t and the subsidy rate s_N as given;
- the government balances the fiscal budget in (18);
- the value of all existing monopolistic firms adds up to the value of the household's assets such that $A_t = N_t V_t$; and
- the following market-clearing condition of final good holds:

$$Y_t = G_t + C_t + N_t \left(X_t + \phi Z_t + I_t \right) + \dot{N}_t \beta X_t. \tag{19}$$

2.7 Aggregation

Substituting (6) and (10) into (4) and imposing symmetry yield aggregate output as

$$Y_t = \theta^{2\theta/(1-\theta)} N_t^{\sigma} Z_t L_t. \tag{20}$$

The growth rate of per capita output $y_t \equiv Y_t / L_t$ is

$$g_t \equiv \dot{y}_t / y_t = \sigma n_t + z_t, \tag{21}$$

which is determined by the variety growth rate $n_t \equiv \dot{N}_t / N_t$ and the quality growth rate z_t.

2.8 Dynamics of the economy

The dynamics of the economy is determined by the firm size $x_t = \theta^{2/(1-\theta)} L_t / N_t^{1-\sigma}$. Its initial value is $x_0 = \theta^{2/(1-\theta)} / N_0^{1-\sigma}$. In the first stage of the economy, there is neither variety expansion nor quality improvement. At this stage, x_t increases solely due to population growth. When x_t becomes sufficiently large, innovation occurs. The following inequality ensures the case in which the creation of products (i.e., variety-expanding innovation) occurs prior to the improvement of products (i.e., quality-improving innovation):[9]

$$\alpha < \frac{\left(1 - s_Z\right) \left[\left(1-\theta\right)/\theta - \left(1 - s_N\right) \left(\rho - \lambda\right) \beta \right]}{\left(1 - s_N\right) \left(\rho - \lambda\right) \beta \phi} \left\{ \rho + \frac{\theta^2 \left[\left(1-\theta\right)/\theta - \left(1 - s_N\right) \left(\rho - \lambda\right) \beta \right] \lambda}{1 - \theta^2 \left[1/\theta - \left(1 - s_N\right) \left(\rho - \lambda\right) \beta \right] - \tau \left(1 - \theta\right)} \right\}. \tag{22}$$

Variety-expanding innovation happens (i.e., $n_t > 0$) when x_t reaches the first threshold x_N:

$$x_N \equiv \frac{\phi}{(1-\theta)/\theta - (\rho - \lambda)(1 - s_N)\beta}, \tag{23}$$

which is decreasing in s_N. Then, quality-improving innovation also happens (i.e., $z_t > 0$) if the firm size x_t reaches the second threshold x_Z defined as

$$x_Z = \underset{x}{\arg} \operatorname{solve} \left\{ \left(\frac{1-\theta}{\theta} x - \phi \right) \left[\frac{\alpha}{1 - s_Z} - \frac{\sigma}{\left(1 - s_N\right) \beta x} \right] = \left(1 - \sigma\right) \left(\rho - \lambda\right) + \lambda \right\}, \tag{24}$$

which is decreasing in s_Z and increasing in s_N. The inequality in (22) implies $x_N < x_Z$.

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[9] Peretto (2015) shows that if quality-improving innovation occurs before variety-expanding innovation, then the model features both types of innovation in the long run and never exhibits semi-endogenous growth.

The firm size x_t must eventually reach x_N, at which point variety-expanding innovation occurs. However, x_t may or may not reach x_Z. If x_t never reaches x_Z, then the economy features only variety-expanding innovation and exhibits semi-endogenous growth in the long run as we will show in the next section. If x_t reaches x_Z, then the economy features quality-improving innovation in addition to variety-expanding innovation and exhibits fully endogenous growth in the long run. The following proposition adapted from Peretto (2015) summarizes the dynamics of x_t.

Proposition 1 *Suppose the initial condition of the economy satisfies*[10]

$$\phi\theta/\left(1-\theta\right) < x_0 < x_N$$

and the following inequality holds:[11]

$$\min\left\{\frac{1-\theta}{\left(1-s_N\right)\beta\theta}, \frac{\phi}{1-s_Z}\right\} > \frac{1}{1-\alpha}\left(\rho+\frac{\sigma\lambda}{1-\sigma}\right). \tag{25}$$

If $x_Z \geq \bar{x}^*$, then the dynamics of x_t is given by

$$\dot{x}_t = \begin{cases} \lambda x_t > 0 & x_0 \leq x_t \leq x_N \\ \bar{v}\left(\bar{x}^* - x_t\right) \geq 0 & x_N < x_t \leq \bar{x}^* \end{cases}, \tag{26}$$

where

$$\bar{v} = \frac{1-\sigma}{\left(1-s_N\right)\beta}\left[\frac{1-\theta}{\theta} - \left(1-s_N\right)\beta\left(\rho+\frac{\sigma\lambda}{1-\sigma}\right)\right],$$

$$\bar{x}^* \equiv \frac{\phi}{\left(1-\theta\right)/\theta - \left(1-s_N\right)\beta\left[\rho+\sigma\lambda/\left(1-\sigma\right)\right]}.$$

If $x_Z < \bar{x}^*$,[12] *then the dynamics of x_t is given by*

$$\dot{x}_t = \begin{cases} \lambda x_t > 0 & x_0 \leq x_t \leq x_N \\ \bar{v}\left(\bar{x}^* - x_t\right) > 0 & x_N < x_t \leq x_Z \\ v\left(x^* - x_t\right) \geq 0 & x_Z < x_t \leq x^* \end{cases}, \tag{27}$$

where

$$v = \frac{1-\sigma}{\left(1-s_N\right)\beta}\left[\left(1-\alpha\right)\frac{1-\theta}{\theta} - \left(1-s_N\right)\beta\left(\rho+\frac{\sigma\lambda}{1-\sigma}\right)\right],$$

$$x^* = \frac{\left(1-\alpha\right)\phi - \left(1-s_Z\right)\left(\rho+\sigma\lambda/\left(1-\sigma\right)\right)}{\left(1-\alpha\right)\left(1-\theta\right)/\theta - \left(1-s_N\right)\beta\left(\rho+\sigma\lambda/\left(1-\sigma\right)\right)}.$$

Proof. See the Appendix. ■

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[10] The inequality $x_0 > \phi\theta/\left(1-\theta\right)$ implies that $\Pi_0 > 0$.

[11] Together with the initial condition, the inequality in (25) ensures that $x_N \in (0, \bar{x}^*)$, where $\bar{x}^*$ is the steady-state value of x_t under the semi-endogenous-growth regime.

[12] Together with (22), (25) and the initial condition, this inequality implies that $0 < x_N < x_Z < \bar{x}^* < x^*$.

3 Effects of R&D subsidies

In this section, we explore the different effects of R&D subsidies under the two growth regimes. First, we consider the semi-endogenous-growth regime (i.e., $x_Z \geq \bar{x}^*$). Then, we consider the fully-endogenous-growth regime (i.e., $x_Z < \bar{x}^*$). Finally, we consider how R&D subsidies determine which growth regime emerges in equilibrium.

3.1 Semi-endogenous growth

When the market size of the economy is not large enough (i.e., $x_t \leq x_N$), there are insufficient incentives for firms to develop new products. In this case, output per capita is

$$y_t = \theta^{2\theta/(1-\theta)} N_0^\sigma Z_0, \tag{28}$$

and the growth rate of y_t is $g_t = 0$. At this stage, an increase in the subsidy rate s_N on variety-expanding innovation affects neither the level of output per capita nor its growth rate. However, it leads to an earlier takeoff by decreasing x_N, so that x_t crosses this threshold at an earlier time. Intuitively, a higher subsidy rate s_N increases the return r_t^e to entry in (16), and hence, a smaller firm size x_t is required for variety-expanding innovation to occur.

When the market size becomes sufficiently large (i.e., $x_t > x_N$), the economy experiences variety-expanding innovation. In this case output per capita is

$$y_t = \theta^{2\theta/(1-\theta)} N_t^\sigma Z_0, \tag{29}$$

and the growth rate of y_t is $g_t = \sigma n_t$. In the Appendix, we show that whenever $n_t > 0$, the consumption-output ratio c_t/y_t always jumps to a steady state. Therefore, we can substitute r_t^e in (16) into the Euler equation $r_t = \rho + g_t = \rho + \sigma n_t$ in (3) and also use (12) to derive the variety growth rate as

$$n_t = \frac{1}{(1-s_N)\beta}\left[\frac{1-\theta}{\theta} - \frac{\phi}{x_t}\right] + \lambda - \rho, \tag{30}$$

which is increasing in the subsidy rate s_N for a given level of x_t. Intuitively, a higher subsidy rate s_N increases the return r_t^e to entry and increases the variety growth rate.

In the semi-endogenous-growth regime (i.e., $x_Z \geq \bar{x}^*$), the economy never experiences quality-improving innovation because the firm size x_t reaches its steady state at $\bar{x}^*$ and stops growing. In this case, the economy only experiences variety-expanding innovation even in the long run. Substituting (12) into (30) yields

$$\frac{\dot{N}_t}{N_t} = \frac{1}{(1-s_N)\beta}\left[\frac{1-\theta}{\theta} - \frac{\phi N_t^{1-\sigma}}{\theta^{2/(1-\theta)} L_t}\right] + \lambda - \rho, \tag{31}$$

which shows that the growth rate of N_t is decreasing in the level of N_t as in the semi-endogenous growth model in Jones (1995). When the economy reaches the balanced growth path, the ratio $N_t^{1-\sigma}/L_t$ becomes stationary. In this case, the steady-state variety growth rate is $n^* = \lambda/(1-\sigma)$, which in turn determines the steady-state growth rate $g^* = \sigma n^*$

that is independent of s_N. A higher subsidy rate s_N on variety-expanding innovation instead increases the balanced growth path of N_t given by

$$N_t^* = \left[\theta^{2/(1-\theta)} \frac{L_t}{\bar{x}^*}\right]^{1/(1-\sigma)}, \tag{32}$$

where

$$\bar{x}^* = \frac{\phi}{(1-\theta)/\theta - (1-s_N)\beta\left[\rho + \sigma\lambda/(1-\sigma)\right]}, \tag{33}$$

which is obtained by setting n_t in (30) to $n^* = \lambda/(1-\sigma)$. Equation (33) shows that $\bar{x}^*$ is decreasing in s_N. These effects of R&D subsidies are quite common in the semi-endogenous growth model; see for example Segerstrom (1998).

Proposition 2 summarizes the effects of R&D subsidies on variety-expanding innovation in the semi-endogenous-growth regime. Figure 1 shows that an increase in the subsidy rate s_N on variety-expanding innovation leads to an earlier takeoff of the economy and a higher transitional growth rate before converging to the steady-state growth rate $g^* = \sigma\lambda/(1-\sigma)$, which is independent of s_N.[13]

Proposition 2 *In the semi-endogenous-growth regime, an increase in the subsidy rate* s_N *on variety-expanding innovation has the following effects. When* $x_t \leq x_N$*, it has no effect on the level of output per capita and its growth rate; however, it leads to an earlier activation of variety-expanding innovation. When* $x_t \in (x_N, \bar{x}^*)$*, it leads to a higher growth rate* $g_t = \sigma n_t$ *for a given* x_t*. When* $x_t = \bar{x}^*$*, it has no effect on the steady-state growth rate* $g^* = \sigma\lambda/(1-\sigma)$ *but increases the balanced growth path of* N_t*.*

Proof. Use (28) to show that y_t and g_t are independent of s_N when $x_t \leq x_N$. Use (23) to show that x_N is decreasing in s_N. Use (30) to show that $g_t = \sigma n_t$ is increasing in s_N for a given x_t when $x_t \in (x_N, \bar{x}^*)$. Use (32) and (33) to show that N_t^* is increasing in s_N when $x_t = \bar{x}^*$. ■

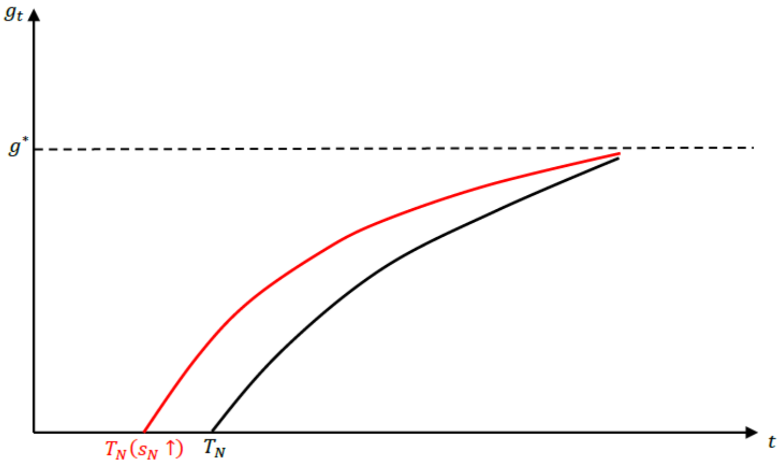

Figure 1: Effects of s_N under semi-endogenous growth

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[13] T_N is the time when variety-expanding innovation is activated.

As for R&D subsidies on quality-improving innovation, they have no effect on the economy. The reason is that quality-improving innovation is never activated in the semi-endogenous-growth regime. However, they may make the semi-endogenous-growth regime less likely and the fully-endogenous-growth regime more likely to emerge in equilibrium as we will show in Section 3.3.

3.2 Fully endogenous growth

In the fully-endogenous-growth regime (i.e., $x_Z < \bar{x}^*$), the economy eventually experiences quality-improving innovation. At this stage, output per capita is

$$y_t = \theta^{2\theta/(1-\theta)} N_t^{\sigma} Z_t, \tag{34}$$

and the growth rate of y_t is $g_t = \sigma n_t + z_t$. An increase in the subsidy rate s_Z on quality-improving innovation leads to an earlier activation of quality-improving innovation by decreasing x_Z, so that x_t crosses this threshold at an earlier time. Intuitively, a higher subsidy rate s_Z increases the return r_t^q to quality improvement in (13), in which case a smaller firm size x_t is required for quality-improving innovation to occur.

When quality-improving innovation is activated in the economy, we can substitute r_t^q in (13) into the Euler equation $r_t = \rho + g_t = \rho + \sigma n_t + z_t$ in (3) to derive the quality-growth rate as

$$z_t = \frac{\alpha}{1-s_Z}\left(\frac{1-\theta}{\theta}x_t - \phi\right) - \rho - \sigma n_t. \tag{35}$$

Equation (35) shows that for a given level of x_t, the equilibrium growth rate $g_t = \sigma n_t + z_t = r_t^q - \rho$ is independent of the variety growth rate n_t and the variety subsidy rate s_N but increasing in the quality subsidy rate s_Z. Intuitively, a higher subsidy rate s_Z increases the return r_t^q to quality improvement and leads to a higher rate of quality-improving innovation.[14]

In the long run, x_t converges to x^*. Then, the steady-state quality growth rate is

$$z^* = \frac{\alpha}{1-s_Z}\left(\frac{1-\theta}{\theta}x^* - \phi\right) - \rho - \sigma n^*, \tag{36}$$

where $n^* = \lambda/(1-\sigma)$ and

$$x^* = \frac{(1-\alpha)\,\phi - (1-s_Z)\left[\rho + \sigma\lambda/\left(1-\sigma\right)\right]}{(1-\alpha)\left(1-\theta\right)/\theta - (1-s_N)\,\beta\left[\rho + \sigma\lambda/\left(1-\sigma\right)\right]}, \tag{37}$$

which is decreasing in the subsidy rate s_N on variety-improving innovation. Intuitively, raising R&D subsidies on variety-expanding innovation increases the number of products, which in turn reduces the market size of each product. This smaller firm size x^* decreases the incentives for quality-improving innovation and the steady-state equilibrium growth rate $g^* = \sigma n^* + z^*$.[15] This result generalizes the result in Chu *et al.* (2016), who assume zero social

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[14] The equilibrium growth rate is also given by $g_t = r_t^e - \rho$, but r_t^e depends on z_t as (16) shows.

[15] See Peretto and Connolly (2007) for a discussion on why quality improvement must be the main engine of innovation in the long run.

return to variety (i.e., $\sigma = 0$). In contrast, R&D subsidies on quality-improving innovation continue to have a positive effect on quality-improving innovation z^* and the steady-state equilibrium growth rate g^*.

Proposition 3 summarizes the effects of R&D subsidies on variety-expanding innovation in the fully-endogenous-growth regime. Figure 2 shows that an increase in the subsidy rate s_N on variety-expanding innovation leads to an earlier takeoff of the economy but a lower growth rate in the long run.[16]

Proposition 3 *In the fully-endogenous-growth regime, an increase in the subsidy rate* s_N *on variety-expanding innovation has the following effects. When* $x_t \leq x_N$*, it has no effect on the level of output per capita and its growth rate; however, it leads to an earlier activation of variety-expanding innovation. When* $x_t \in (x_N, x_Z]$*, it leads to a higher growth rate* $g_t = \sigma n_t$ *for a given* x_t*. When* $x_t \in (x_Z, x^*)$*, it does not affect the growth rate* $g_t = \sigma n_t + z_t$ *for a given* x_t*. When* $x_t = x^*$*, it lowers the steady-state growth rate* g^* *by reducing* x^**.*

Proof. Use (28) to show that y_t and g_t are independent of s_N when $x_t \leq x_N$. Use (23) to show that x_N is decreasing in s_N. Use (30) to show that $g_t = \sigma n_t$ is increasing in s_N for a given x_t when $x_t \in (x_N, x_Z]$. Use (35) to show that $g_t = \sigma n_t + z_t$ is independent of s_N for a given x_t when $x_t \in (x_Z, x^*)$. Use (36) and (37) to show that $g^* = \sigma n^* + z^*$ is decreasing in s_N when $x_t = x^*$. ■

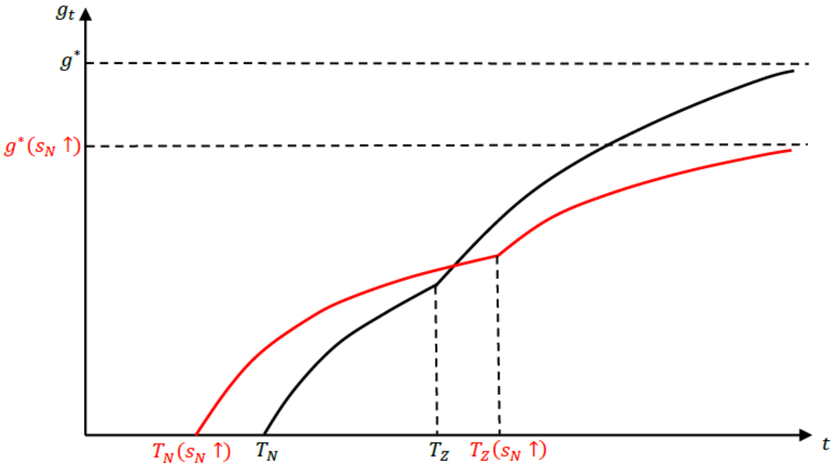

Figure 2: Effects of s_N under fully endogenous growth

Proposition 4 summarizes the effects of R&D subsidies on quality-improving innovation in the fully-endogenous-growth regime. Figure 3 shows that an increase in the subsidy rate s_Z leads to an earlier activation of quality-improving innovation in the economy and a higher growth rate in the long run.

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[16] T_Z (T_N) is the time when quality-improving (variety-expanding) innovation is activated.

Proposition 4 *In the fully-endogenous-growth regime, an increase in the subsidy rate s_Z on quality-improving innovation has the following effects. When $x_t \leq x_N$, it has no effect on the level of output per capita and its growth rate; furthermore, it does not affect the activation date of variety-expanding innovation. When $x_t \in (x_N, x_Z]$, it does not affect the growth rate $g_t = \sigma n_t$ for a given x_t; however, it leads to an earlier activation of quality-improving innovation. When $x_t \in (x_Z, x^*)$, it increases the growth rate $g_t = \sigma n_t + z_t$ for a given x_t. When $x_t = x^*$, it raises the steady-state growth rate g^* by increasing z^*.*

Proof. Use (28) to show that y_t and g_t are independent of s_Z when $x_t \leq x_N$. Use (23) to show that x_N is independent of s_Z. Use (30) to show that $g_t = \sigma n_t$ is independent of s_Z for a given x_t when $x_t \in (x_N, x_Z]$. Use (24) to show that x_Z is decreasing in s_Z. Use (35) to show that $g_t = \sigma n_t + z_t$ is increasing in s_Z for a given x_t when $x_t \in (x_Z, x^*)$. Use (36) and (37) to show that $g^* = \sigma n^* + z^*$ is increasing in s_Z when $x_t = x^*$. ∎

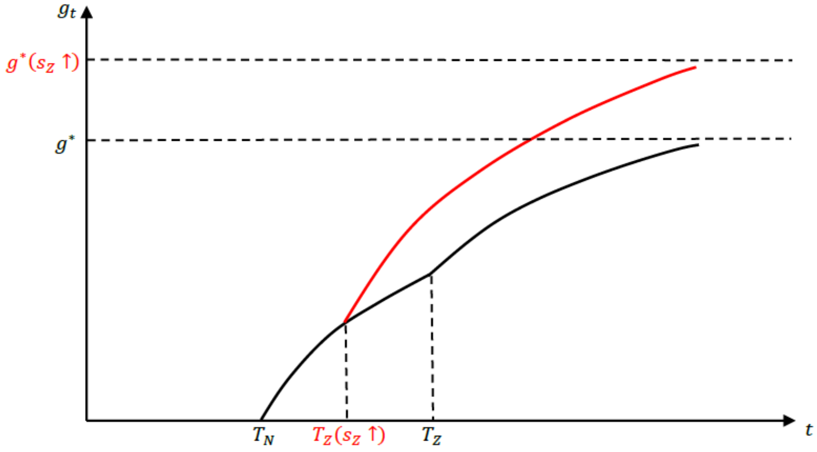

Figure 3: Effects of s_Z under fully endogenous growth

3.3 Endogenous switching between the growth regimes

Whether the semi-endogenous-growth regime or the fully-endogenous-growth regime emerges in equilibrium depends on the relative value of x_Z and $\bar{x}^*$. Specifically, if $x_Z \geq \bar{x}^*$, then the semi-endogenous-growth regime emerges in equilibrium. If $x_Z < \bar{x}^*$, then the fully-endogenous-growth regime emerges in equilibrium. Therefore, an increase in $x_Z/\bar{x}^*$ makes the semi-endogenous-growth regime more likely to emerge in equilibrium, whereas a decrease in $x_Z/\bar{x}^*$ makes the fully-endogenous-growth regime more likely to emerge in equilibrium.

An increase in the subsidy rate s_Z on quality-improving innovation reduces x_Z but does not affect $\bar{x}^*$. Therefore, increasing R&D subsidies on quality-improving innovation makes the fully-endogenous-growth regime more likely to emerge in equilibrium. Intuitively, the fully-endogenous-growth regime depends on the *presence* of quality-improving innovation.

Therefore, an increase in the subsidy rate s_Z that raises the return to quality-improving innovation makes the fully-endogenous-growth regime more likely to emerge.

An increase in the subsidy rate s_N on variety-expanding innovation reduces $\bar{x}^*$ and raises x_Z. Therefore, increasing R&D subsidies on variety-expanding innovation makes the semi-endogenous-growth regime more likely to emerge. Intuitively, the semi-endogenous-growth regime depends on the *absence* of quality-improving innovation. Therefore, an increase in the subsidy rate s_N that raises the return to variety-expanding innovation ends up crowding out resources for quality-improving innovation and making the semi-endogenous-growth regime more likely to emerge. Proposition 5 summarizes these results.

Proposition 5 *An increase in the subsidy rate on quality-improving innovation makes the fully-endogenous-growth regime more likely to emerge in equilibrium. An increase in the subsidy rate on variety-expanding innovation makes the semi-endogenous-growth regime more likely to emerge in equilibrium.*

Proof. One can use (24) and (33) to show that $x_Z < \bar{x}^*$ can be expressed as

$$\frac{\alpha\phi}{1-s_Z} > \frac{1-\theta}{(1-s_N)\,\beta\theta} - \left(\rho + \frac{\sigma\lambda}{1-\sigma}\right),$$

which is equivalent to $z^* > 0$ in (36). This inequality holds if and only if s_Z is sufficiently large or s_N is sufficiently small. ■

3.4 Quantitative analysis

We calibrate the model to quantitatively examine which growth regime is more likely to emerge. The model features the following parameters: $\{\rho, \alpha, \sigma, \lambda, \tau, s_N, s_Z, \theta, \beta, \phi\}$. We set the discount rate ρ to 0.05. We follow Iacopetta *et al.* (2019) to set the degree of technology spillovers $1-\alpha$ to 0.833 and the degree of congestion $1-\sigma$ to 0.75. We consider a long-run population growth rate λ of 1%. According to the OECD, the average tax rate τ on wage income in the US is 23.8%. Given that the US has a uniform rate of R&D subsidies, we consider $s_N = s_Z$ and follow Impullitti (2010) to set the rate of subsidies to 18.8%. Then, we calibrate $\{\theta, \beta\}$ by matching the following moments: (a) the ratio of labor income to output is 60%; and (b) the ratio of consumption to output is 60%.[17] Finally, we compute a range of values for the operating cost ϕ under which the quality growth rate z^* is positive and examine whether this range of values for ϕ is empirically plausible.

Table 1: Calibration and simulation

ρ	α	σ	λ	τ	s_N	s_Z	θ	β	ϕ	z^*
0.050	0.167	0.250	0.010	0.238	0.188	0.188	0.400	27.478	0.068	0%
0.050	0.167	0.250	0.010	0.238	0.188	0.188	0.400	27.478	0.070	1%
0.050	0.167	0.250	0.010	0.238	0.188	0.188	0.400	27.478	0.072	2%

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[17] The equilibrium expressions for $\{wL/Y, C/Y\}$ are the same across the two growth regimes.

Table 1 shows that the quality growth rate z^* is increasing in the operating cost ϕ. Intuitively, a larger operating cost ϕ increases the average firm size x^* in (37), which in turn increases the incentives for quality-improving innovation. Also, when $\phi > 0.068$, the quality growth rate z^* is positive (i.e., the fully-endogenous-growth regime emerges in equilibrium). Ferraro *et al.* (2019) estimate the operating cost parameter ϕ and find that its mean estimate is 0.125 with a standard error of 0.027.[18] Therefore, we can conclude that under reasonable parameter values, the fully-endogenous-growth regime with both quality-improving innovation and variety-expanding innovation is more likely to emerge than the semi-endogenous growth regime. Although this finding depends on the specific structure of our model, it is consistent with empirical studies, such as Laincz and Peretto (2006), Ha and Howitt (2007), Madsen (2008, 2010) and Ang and Madsen (2011), which also find supportive evidence for endogenous growth in the second-generation Schumpeterian model.

4 Conclusion

This study explores the effects of R&D subsidies in a hybrid growth model that may exhibit semi-endogenous growth or fully endogenous growth in equilibrium. Whether the semi-endogenous-growth regime or the fully-endogenous-growth regime emerges in equilibrium is endogenously determined. Within this growth-theoretic framework, we obtain the following novel results. First, R&D subsidies have different effects on the endogenous activation of variety-expanding innovation and that of quality-improving innovation. Second, R&D subsidies have different effects on economic growth in the semi-endogenous-growth regime and in the fully-endogenous-growth regime. Finally, R&D subsidies determine which growth regime emerges in equilibrium. Therefore, previous studies that restrict their analysis to either growth regime may not capture the complete effects of R&D subsidies.

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[18] See also Ferraro and Peretto (2020) and Iacopetta *et al.* (2019) in which the calibrated values of ϕ are 0.262 and 0.715, respectively.

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Appendix

Proof of Lemma 1. The current-value Hamiltonian for monopolistic firm i is

$$H_t(i) = \Pi_t(i) - (1 - s_Z)I_t(i) + \eta_t(i)\dot{Z}_t(i), \tag{A1}$$

where $\eta_t(i)$ is the multiplier on $\dot{Z}_t(i) = I_t(i)$. Substituting (6)-(8) into (A1), we can derive

$$\frac{\partial H_t(i)}{\partial p_t(i)} = 0 \Rightarrow \frac{\partial \Pi_t(i)}{\partial p_t(i)} = 0, \tag{A2}$$

$$\frac{\partial H_t(i)}{\partial I_t(i)} = 0 \Rightarrow \eta_t(i) = 1 - s_Z, \tag{A3}$$

$$\frac{\partial H_t(i)}{\partial Z_t(i)} = \alpha \left\{ [p_t(i) - 1] \left[\frac{\theta}{p_t(i)}\right]^{1/(1-\theta)} \frac{L_t}{N_t^{1-\sigma}} - \phi \right\} Z_t^{\alpha-1}(i)\, Z_t^{1-\alpha} = r_t \eta_t(i) - \dot{\eta}_t(i). \tag{A4}$$

First, $\partial \Pi_t(i) / \partial p_t(i) = 0$ in (A2) yields

$$p_t(i) = 1/\theta. \tag{A5}$$

Then, substituting (A3), (A5) and (12) into (A4) and imposing symmetry yield

$$r_t^q = \frac{\alpha}{1 - s_Z} \frac{\Pi_t}{Z_t} = \frac{\alpha}{1 - s_Z} \left(\frac{1-\theta}{\theta} x_t - \phi \right), \tag{A6}$$

which is the rate of return on quality-improving in-house R&D. ■

Before we prove Proposition 1, we first derive the dynamics of the consumption-output ratio C_t/Y_t when $n_t > 0$.

Lemma 2 *When $n_t > 0$, the consumption-output ratio always jumps to*

$$C_t/Y_t = (1 - s_N)(\rho - \lambda)\beta\theta^2 + (1 - \tau)(1 - \theta). \tag{A7}$$

Proof. The total value of assets owned by the household is

$$A_t = N_t V_t. \tag{A8}$$

When $n_t > 0$, the no-arbitrage condition for entry in (15) holds. Then, substituting (15) and $X_t N_t = \theta^2 Y_t$ into (A8) yields

$$A_t = N_t (1 - s_N) \beta X_t = (1 - s_N) \beta \theta^2 Y_t, \tag{A9}$$

which implies that the asset-output ratio A_t/Y_t is constant. Substituting (A9), (2), (3) and (5) into $\dot{A}_t/A_t = \dot{a}_t/a_t + \lambda$ yields

$$\begin{aligned} \frac{\dot{Y}_t}{Y_t} &= \frac{\dot{A}_t}{A_t} = r_t + (1 - \tau)\frac{w_t L_t}{A_t} - \frac{C_t}{A_t} \\ &= \rho + \frac{\dot{C}_t}{C_t} - \lambda + \frac{(1-\tau)(1-\theta)}{(1-s_N)\beta\theta^2} - \frac{1}{(1-s_N)\beta\theta^2}\frac{C_t}{Y_t}, \end{aligned} \tag{A10}$$

which can be rearranged as

$$\frac{\dot{C}_t}{C_t} - \frac{\dot{Y}_t}{Y_t} = \frac{1}{(1-s_N)\,\beta\theta^2}\frac{C_t}{Y_t} - \frac{(1-\tau)\,(1-\theta)}{(1-s_N)\,\beta\theta^2} - (\rho-\lambda)\,. \tag{A11}$$

Therefore, the dynamics of C_t/Y_t is characterized by saddle-point stability, such that C_t/Y_t jumps to its steady-state value in (A7). ■

Proof of Proposition 1. Using (12), we can derive the growth rate of x_t as

$$\frac{\dot{x}_t}{x_t} = \lambda - (1-\sigma)\,n_t. \tag{A12}$$

When $x_0 \leq x_t \leq x_N$, we have $n_t = 0$ and $z_t = 0$. In this case, the dynamics of x_t is given by

$$\dot{x}_t = \lambda x_t. \tag{A13}$$

When $x_N < x_t \leq x_Z$, we have $n_t > 0$ and $z_t = 0$. In this case, Lemma 2 implies that C_t/Y_t is constant and $\dot{c}_t/c_t = \dot{y}_t/y_t$. Therefore, we can substitute r_t^e in (16) and (A12) into $r_t = \rho + \sigma n_t$ in (3) to obtain (30). Substituting (30) into (A12) yields the dynamics of x_t as

$$\dot{x}_t = \frac{1-\sigma}{(1-s_N)\,\beta}\left\{\phi - \left[\frac{1-\theta}{\theta} - (1-s_N)\,\beta\left(\rho + \frac{\sigma}{1-\sigma}\lambda\right)\right]x_t\right\}. \tag{A14}$$

Defining $\overline{v} \equiv \frac{1-\sigma}{(1-s_N)\beta}\left[\frac{1-\theta}{\theta} - (1-s_N)\,\beta\left(\rho + \frac{\sigma\lambda}{1-\sigma}\right)\right]$ and $\overline{x}^* \equiv \frac{\phi}{(1-\theta)/\theta-(1-s_N)\beta[\rho+\sigma\lambda/(1-\sigma)]}$, we can express (A14) as

$$\dot{x}_t = \overline{v}(\overline{x}^* - x_t). \tag{A15}$$

If $\overline{x}^* < x_Z$, then x_t reaches its steady state at $x_t = \overline{x}^*$.

However, it is also possible for $x_Z < \overline{x}^*$. In this case, when $x_t > x_Z$, we have $n_t > 0$ and $z_t > 0$. Given $n_t > 0$, C_t/Y_t is constant, and $\dot{c}_t/c_t = \dot{y}_t/y_t$. Then, substituting r_t^e in (16) and (A12) into $r_t = \rho + \sigma n_t + z_t$ in (3) yields

$$n_t = \frac{1}{(1-s_N)\,\beta}\left[\frac{1-\theta}{\theta} - \frac{\phi + (1-s_Z)\,z_t}{x_t}\right] - \rho + \lambda. \tag{A16}$$

We substitute (35) into (A16) to derive

$$n_t = \frac{[(1-\alpha)\,(1-\theta)\,/\theta - (1-s_N)\,(\rho-\lambda)\,\beta]\,x_t - (1-\alpha)\,\phi + (1-s_Z)\,\rho}{(1-s_N)\,\beta x_t - (1-s_Z)\,\sigma}. \tag{A17}$$

Substituting (A17) into (A12) yields the dynamics of x_t as

$$\dot{x}_t = v\,(x^* - x_t)\,, \tag{A18}$$

where

$$v \equiv \frac{1-\sigma}{(1-s_N)\,\beta - (1-s_Z)\,\sigma/x_t}\left[(1-\alpha)\,\frac{1-\theta}{\theta} - (1-s_N)\,\beta\left(\rho + \frac{\sigma}{1-\sigma}\lambda\right)\right] \tag{A19}$$

and x^* is in (37). Finally, we approximate $(1-s_Z)\,\sigma/x_t \cong 0$ for $x_t > x_Z$, so v becomes a constant. ■