Classical and Bayesian Estimation of Stress-strength Reliability of a Component having Multiple States

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Abstract

Purpose - This article presents the multi-state stress-strength reliability computation of a component having three states namely, working, deteriorating and failed state.
Design/Methodology/Approach - The probabilistic approach is used to obtain the reliability expression by considering the difference between the values of stress and strength of a component, say, for example, the stress (load) and strength of a power generating unit is in terms of Mega Watt. The range of values taken by the difference variable determines the various states of the component. The method of maximum likelihood and Bayesian estimation is used to obtain the estimators of the parameters and system reliability.
Findings - The maximum likelihood and Bayesian estimates of the reliability approach the actual reliability for increasing sample size.
Originality/Value - Obtained a new expression for the multi-state stress-strength

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reliability of a component and the findings are positively supported by presenting the general trend of estimated values of reliability approaching the actual value of reliability.

Keywords: Multi-state; Stress-strength reliability; Maximum likelihood estimation; Bayesian estimation; Gibbs sampling.

1 Introduction

Modern engineering systems are composed of components having a wide range of performance levels that vary from perfect functioning to complete failure. Multi-state system reliability evaluation helps in handling such situations. For example, a car engine consists of several components and the strength is defined in terms of miles to failure distribution of the components and the stress is defined in terms of usage mileage distribution. Similarly, power systems, CPU of a computer, communications network are all composed of multi-state components. The generating capacity, data processing speed and communication respectively measure the performance of these systems. These are all examples of systems having different performance levels as seen in Jana, Kumar, & Chatterjee (2016). A systematic review on multi-state system reliability can be seen in the work of Yingkui & Jing (2012).

The basic concepts in multi-state system reliability has been well addressed by Ross (1979), Lisnianski & Levitin (2003) and Natvig (2014). Multi-state reliability models are effectively studied using random processes like Markov processes, Markov reward process and semi-Markov process when failure and repair time of these components are exponentially and non-exponentially distributed as seen in Lisnianski, Frenkel, & Ding (2010), Trivedi (2008). Lisnianski (2007) presented an extended block diagram method to study a repairable multi-state system. The author combined the random process and universal generating function method to reduce the dimension of states in this multi-state model. Natvig & Mørch (2003) studied the multi-state reliability theory applied to Norwegian offshore gas pipeline network in the north sea transporting gas to Emden in Germany in 2003. More contributions to multi-state system reliability analysis can be seen in the works of Levitin, Lisnianski, Ben-Haim, & Elmakis (1998), Eryılmaz (2011), Eryılmaz (2008), Wang, Li, Huang, & Chang (2012) and Eryılmaz & İşçioğlu (2011). Recently Calik (2017) performed a study on stress-strength reliability evaluation on a multi-state component subject to two stresses.

The main objective of this study is to evaluate the multi-state stress-strength reliability of a component (system) by considering the difference between the stress and strength values of the component. This can be applied to systems like power generating systems where the stress and strength are defined in terms of the same units. The component having three states namely, working, deteriorating and failed state is considered in obtaining the reliability expression and the corresponding parameters are estimated using the method of maximum-likelihood and Bayesian estimation.

The rest of the paper is organized as follows. Section 2 describes the problem considered and presents the procedure of reliability evaluation. Sections 3 and 4 presents the method of maximum likelihood and Bayesian estimation. The illustrations using assumed data sets are given in Section 5 and Section 6 presents the conclusions.

2 Problem description and reliability evaluation

Let X denote the strength and Y denote the stress of the component. The components having multi-state characteristic can be defined by considering the difference between the stress and strength values. Consider the power generating unit whose strength is the power generated in Mega Watts. The stress is the load acting on the generating unit also in terms of Mega Watts. Let the difference $X-Y=K$. The range of values taken by K determines the states of the component, namely working, deteriorating and failed state respectively.

$$X - Y = K = \begin{cases} 0 \leq K < s_1, & \text{working state,} \\ s_1 \leq K < s_2, & \text{deteriorating state,} \\ s_2 \leq K < s_3, & \text{failed state} \end{cases} \tag{1}$$

Assume the strength and stress of the component to be random variables. Let them be statistically independent. Let $f(x)$ and $g(y)$ denote the density functions of the strength and stress random variables. Let $F(x)$ and $G(y)$ denote the distribution functions of these random variables. The reliability (R) of a multi-state component is the probability of working and deteriorating state.

$$\begin{aligned} R &= P(working\ state) + P(deteriorating\ state) \\ &= P(0 \leq X - Y < s_1) + P(s_1 \leq X - Y < s_2) \\ &= \int_{-\infty}^{\infty}\int_{y}^{y+s_1} f(x)g(y)dxdy + \int_{-\infty}^{\infty}\int_{y+s_1}^{y+s_2} f(x)g(y)dxdy \end{aligned} \quad (2)$$

Assume the stress and strength random variables to follow exponential distribution with parameters λ and θ respectively.

$$f(x) = \lambda e^{-\lambda x}, \lambda > 0, x > 0 \quad (3)$$

$$g(y) = \theta e^{-\theta y}, \theta > 0, y > 0. \quad (4)$$

The multi-state stress strength reliability of a component having three states is obtained to be

$$R = \frac{\theta}{\lambda + \theta}(1 - e^{-\lambda s_2}) \quad (5)$$

Since the reliability expression in Eqn. (5) involves parameters of stress and strength variables, the classical and Bayesian estimation methods are used to obtain the estimators of these parameters and thereby the estimates of reliability.

3 Maximum Likelihood Estimation - Classical Estimation

Consider n copies of the multi-state component. Let $X_i,\ i = 1, 2, ...n$ be the strength of the i^{th} copy of the component and $Y_i,\ i = 1, 2, ...n$ be the stress of the component. The sample vector of random variables are $\mathbf{X} = (X_1, X_2, ...X_n)$ and $\mathbf{Y} = (Y_1, Y_2, ...Y_n)$. These random variables are independent and identical following exponential density functions with parameters λ and θ respectively.

The likelihood function is given by

$$L(\lambda, \theta; \mathbf{X}, \mathbf{Y}) = \lambda^n e^{-\lambda(\sum_{i=1}^{n} x_i)} \theta^n e^{-\theta(\sum_{i=1}^{n} y_i)},$$

$$\log L = nlog\lambda - \lambda \sum_{i=1}^{n} x_i + nlog\theta - \theta \sum_{i=1}^{n} y_i. \quad (6)$$

Taking the partial derivative with respect to λ and θ

$$(\partial/\partial\lambda)\log L \;=\; 0 \implies n/\lambda - \sum_{i=1}^{n} x_i = 0,$$

$$(\partial/\partial\theta)\log L = 0 \implies n/\theta - \sum_{i=1}^{n} y_i = 0, \text{ for } i = 1, 2, ...n \tag{7}$$

The maximum likelihood estimator of the respective parameters are obtained to be

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i} \text{ for } i = 1, 2, ..., n \text{ and} \tag{8}$$

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} y_i}, \text{for } i = 1, 2, ..., n \tag{9}$$

Since the maximum likelihood estimator is invariant the maximum likelihood estimator of system reliability is

$$\hat{R} = \frac{\hat{\theta}}{\hat{\lambda} + \hat{\theta}}(1 - e^{-\hat{\lambda}s_2}) \tag{10}$$

4 Bayesian estimation

The Bayesian estimation method considers the distribution of observations of stresses and strengths through density functions $f(x/\lambda)$ and $f(y/\theta)$, where the the parameters λ and θ are unknown. These unknown parameters are retrieved from the observations. The likelihood function represents the retrieving aspect and this function is represented by the sample density function which is rewritten as the function of the parameters λ and θ. The information apart from the current data is known as the prior information which can be obtained from past experiments. This prior information can be either informative or non-informative. The posterior information is obtained by updating this prior information observed from the current data. Considering the Gamma informative priors, $Gamma(a, b)$ and $Gamma(c, d)$ for the stress and strength parameters λ and θ as $\Pi(\lambda)$ and $\Pi(\theta)$ respectively,

$$\Pi(\lambda) = \frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda}, a > 0, b > 0, \lambda > 0 \tag{11}$$

$$\Pi(\theta) = \frac{d^c}{\Gamma(c)}\theta^{c-1}e^{-d\theta}, c > 0, d > 0, \theta > 0, \tag{12}$$

the joint posterior density is derived in the following theorem. The standard method given in Bansal (2007) is adopted in proving the following theorem.

Theorem 4.1. *The joint posterior density of λ and θ, given the stress and strength data follow exponential distribution with parameters following $Gamma(a, b)$ and $Gamma(c, d)$ informative prior is $Gamma(n + a, b + \sum_{i=1}^{n} x_i) \times Gamma(n + c, d + \sum_{i=1}^{n} y_i)$.*

Proof. The joint posterior distribution of λ, θ given the data is derived using

$$\begin{aligned}
\Pi(\lambda, \theta/X, Y) &= \frac{L(\lambda, \theta/X, Y)\Pi(\lambda)\Pi(\theta)}{\int_0^\infty \int_0^\infty L(\lambda, \theta/X, Y)\Pi(\lambda)\Pi(\theta)d\lambda d\theta} \\
&= \frac{\lambda^n e^{-\lambda(\sum\limits_{i=1}^{n} x_i)} \theta^n e^{-\theta(\sum\limits_{i=1}^{n} y_i)} \frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda}\frac{d^c}{\Gamma(c)}\theta^{c-1}e^{-d\theta}}{\int_0^\infty \int_0^\infty \lambda^n e^{-\lambda(\sum\limits_{i=1}^{n} x_i)} \theta^n e^{-\theta(\sum\limits_{i=1}^{n} y_i)} \frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda}\frac{d^c}{\Gamma(c)}\theta^{c-1}e^{-d\theta}d\lambda d\theta}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\lambda^{n+a-1}e^{-\lambda(b+\sum_{i=1}^{n} x_i)}\theta^{n+c-1}e^{-\theta(d+\sum_{i=1}^{n} y_i)}}{\int_0^\infty \int_0^\infty \lambda^{n+a-1}e^{-\lambda(b+\sum_{i=1}^{n} x_i)}\theta^{n+c-1}e^{-\theta(d+\sum_{i=1}^{n} y_i)}d\lambda d\theta} \\
&= \frac{(b + \sum_{i=1}^{n} x_i)^{n+a}}{\Gamma(n+a)}\lambda^{n+a-1}e^{-\lambda(b+\sum_{i=1}^{n} x_i)} \times \\
&\quad \frac{(d + \sum_{i=1}^{n} y_i)^{n+c}}{\Gamma(n+c)}\theta^{n+c-1}e^{-\theta(d+\sum_{i=1}^{n} y_i)}
\end{aligned}$$

Hence joint posterior distribution of the parameters λ and θ follows $Gamma(n + a, b + \sum_{i=1}^{n} x_i) \times Gamma(n + c, d + \sum_{i=1}^{n} y_i)$. □

The conditional distribution of the parameters λ and θ given the data are $Gamma(n + a, b + \sum_{i=1}^{n} x_i)$ and $Gamma(n + c, d + \sum_{i=1}^{n} y_i)$. Under the squared error loss function the Bayes estimator of the parameter λ is the posterior mean which implies

$$\hat{\lambda} = \frac{n + a}{b + \sum_{i=1}^{n} x_i}. \tag{13}$$

Similarly the Bayes estimator of the parameter θ is

$$\hat{\theta} = \frac{n + c}{d + \sum_{i=1}^{n} y_i}. \tag{14}$$

The Bayes estimator of multi-state stress-strength reliability of the component is the posterior expectation of R.

$$\begin{aligned}
\hat{R} &= E(R/X,Y) \\
&= \int_0^\infty \int_0^\infty R \times \Pi(\lambda,\theta/X,Y) d\lambda d\theta, \\
&= \int_0^\infty \int_0^\infty \frac{\theta}{\lambda+\theta}(1-e^{-\lambda s_2})\frac{(b+\sum_{i=1}^n x_i)^{n+a}}{\Gamma(n+a)}\lambda^{n+a-1}e^{-\lambda(b+\sum_{i=1}^n x_i)} \times \\
&\quad \frac{(d+\sum_{i=1}^n y_i)^{n+c}}{\Gamma(n+c)}\theta^{n+c-1}e^{-\theta(d+\sum_{i=1}^n y_i)} d\lambda d\theta.
\end{aligned}$$

The Gibbs sampling algorithm is used to simplify the above double integral.

4.0.1 Method of simulation

This method is used to evaluate the double integral using the posterior distribution of the parameters involved in the obtained expression for reliability (Eqn. (5)). When the posterior distribution is in a known form the following Gibbs sampling procedure may be employed in computing the system reliability. The following steps are used to obtain the component reliability.

Gibbs sampling procedure

1. Assume initial value for the parameter $\lambda^{(0)}, \theta^{(0)}$, .

2. Let p=1.

3. Generate random value for the parameter ($\lambda^{(p)}$ and $\theta^{(p)}$), from the obtained distribution of the parameter.

4. Compute the system reliability (R^p), for the value of the parameter obtained in step 3.

5. Set p=p+1.

6. Repeat the steps 3 to 5 for K number of times.

7. Obtain the approximate posterior mean of system reliability as

$$\hat{R} = E(R/t) = \frac{1}{K} \sum_{p=1}^{K} R^p \tag{15}$$

This algorithm gives the Bayesian estimates of reliability.

5 Illustration

This section presents the verification of a statistically known fact in sampling theory that the estimated values of multi-state stress-strength system reliability approaches the actual values of reliability for increasing sample size. This method of illustration justifies the obtained expressions of system reliability under maximum likelihood (Eqn,(3)) and Bayesian estimation methodology (Eqn. (4)). The illustration is presented for an assumed data set which consists of three sets of values of parameters λ, θ and s_2. The values are given in Table 1 for maximum likelihood and Table 3 for Bayesian estimation methodology respectively. It may be noted that the actual values of system reliability are computed using Eqn (5) for each case in the assumed data given in Table 1. Table 2 shows the general trend of maximum likelihood estimates of system reliability approaching actual reliability for increasing sample size $n = \{20, 40, 60, 100\}$. Similarly, the illustration for Bayesian estimates of system reliability is presented in Table 4. The calculations are performed using MATLAB Programming. The standard deviation and 95% confidence interval are also presented. The graphical representations of the same are given in Figs. 1 and 2.

6 Conclusions

This article presents the reliability computation of stress-strength reliability of a component (system) having three states. The evaluations are performed by assuming the stress and strength random variables to follow an exponential distribution. The maximum likelihood estimator and Bayesian estimator of the parameters of stress, strength and thereby the estimates of system reliability are computed. The Bayesian estimation is provided by assuming informative Gamma prior for the stress and strength parameters. The posterior distribution of the strength and stress parameters are obtained to follow Gamma distribution with parameters $(n+a, b+\sum_{i=1}^{n} x_i)$ and $(n+c, d+\sum_{i=1}^{n} y_i)$. The maximum likelihood and Bayesian estimates of reliability approaching actual values of reliability for increasing sample size are presented in Tables 2, 4 and Figs. 1 and 2 respectively.

Table 1: Stress and strength parameters of exponential distribution

Cases	λ	θ	s_2
1	0.05	0.9	40
2	0.1	2.0	40
3	0.3	6.0	40

Table 2: Maximum Likelihood estimates of system reliability using Exponential distribution (using values of parameters given in Table 1)

$Cases$	n	R	$\hat{R}$	$S.D$	$95\% - C.I$
1	20	0.809618	0.819156	0.044385	0.793292 0.825945
	40	0.811562	0.819156	0.038026	0.801672 0.821452
	60	0.815633	0.819156	0.026896	0.809921 0.821345
	100	0.815640	0.819156	0.021338	0.812130 0.819150
2	20	0.924841	0.934937	0.016881	0.918631 0.931050
	40	0.930228	0.934937	0.009457	0.927768 0.932688
	60	0.931967	0.934937	0.008534	0.930154 0.933779
	100	0.932133	0.934937	0.006507	0.931063 0.933204
3	20	0.950294	0.952375	0.014615	0.944918 0.955670
	40	0.951887	0.952375	0.010503	0.949155 0.954619
	60	0.952707	0.952375	0.008077	0.950992 0.954422
	100	0.951981	0.952375	0.006324	0.950940 0.953021

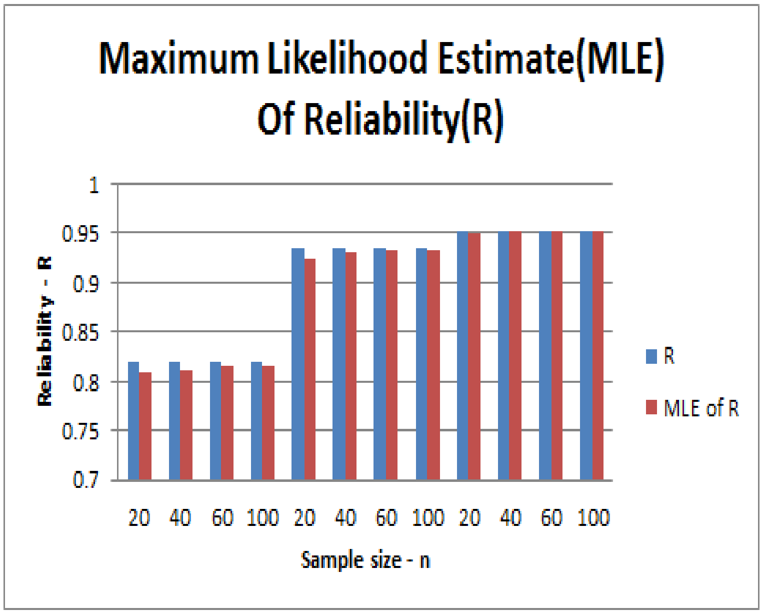

Figure 1: Exponential model - Graphical representation of maximum likelihood estimates of system reliability approaching the value of system reliability with increasing sample size.

Table 3: Values of stress and strength hyperparameters under informative prior distribution.

Cases	a, b	c, d
1	2,3	2,4
2	4,9	8,10
3	1,5	2,5

Table 4: Bayesian estimates of system reliability using Exponential distribution - Modern approach (using values of parameters given in Table 3).

Cases	n	$\hat{R}$	R	$S.D$	$95\% - C.I$	
1	20	0.821876	0.819156	0.055536	0.801448	0.842304
	40	0.815164	0.819156	0.053835	0.801162	0.829167
	60	0.816061	0.819156	0.033653	0.808914	0.823208
	100	0.820662	0.819156	0.032318	0.815346	0.825979
2	20	0.904346	0.934937	0.022801	0.895959	0.912733
	40	0.918616	0.934937	0.015858	0.914491	0.922741
	60	0.923566	0.934937	0.011364	0.921153	0.925979
	100	0.927805	0.934937	0.007396	0.926589	0.929022
3	20	0.889853	0.952375	0.040515	0.874950	0.904756
	40	0.920305	0.952375	0.023324	0.914238	0.926371
	60	0.931918	0.952375	0.015946	0.928531	0.935304
	100	0.939558	0.952375	0.011284	0.937702	0.941414

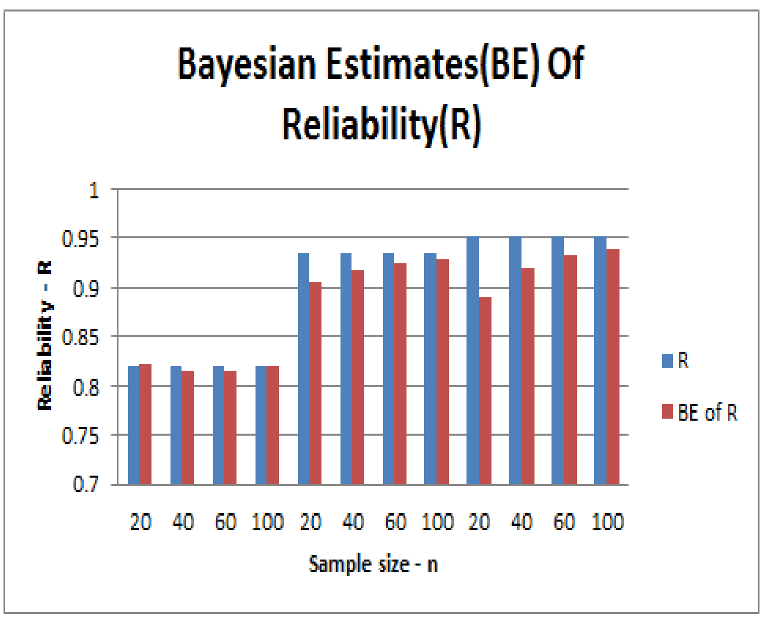

Figure 2: Exponential model - Graphical representation of Bayesian estimates of system reliability approaching the value of system reliability with increasing sample size.

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