

# Near-conformal dynamics in a chirally broken system

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Composite Higgs models must exhibit very different dynamics from quantum chromodynamics (QCD) regardless whether they describe the Higgs boson as a dilaton-like state or a pseudo-Nambu-Goldstone boson. Large separation of scales and large anomalous dimensions are frequently desired by phenomenological models. Mass-split systems are well-suited for composite Higgs models because they are governed by a conformal fixed point in the ultraviolet but are chirally broken in the infrared. In this work we use lattice field theory calculations with domain wall fermions to investigate a system with four light and six heavy flavors. We demonstrate how a nearby conformal fixed point affects the properties of the four light flavors that exhibit chiral symmetry breaking in the infrared. Specifically we describe hyperscaling of dimensionful physical quantities and determine the corresponding anomalous mass dimension. We obtain  $y_m = 1 + \gamma^* = 1.47(5)$  suggesting that  $N_f = 10$  lies inside the conformal window. Comparing the low energy spectrum to predictions of dilaton chiral perturbation theory, we observe excellent agreement which supports the expectation that the 4+6 mass-split system exhibits near-conformal dynamics with a relatively light  $0^{++}$  isosinglet scalar.

## INTRODUCTION

Experiments have discovered a 125 GeV Higgs boson [1–3] but so far, up to the range of a few TeV, no direct signs of physics beyond the standard model (BSM). The standard model (SM), however, is an effective theory and new interactions are necessary e.g. to UV complete the Higgs sector, explain dark matter or the matter-antimatter asymmetry of the universe. For BSM scenarios aiming to describe the Higgs sector, the experimental observations imply that a large separation of scales between the infrared (IR) and ultraviolet (UV) physics [4–11] is required. Systems with a large separation of scales exhibit a “walking” gauge coupling [12, 13] and can moreover provide a dynamical mechanism for electroweak (EW) symmetry breaking. They can satisfy stringent constraints from EW precision measurements but avoid unnaturally large tuning of the Higgs mass.

Interested in exploring the dynamics of systems with large scale separation and infrared dynamics different from QCD, we explicitly create such a scenario using the setup of a mass-split model [8, 14–16] where the action has two dimensionless mass parameters,  $\hat{m}_\ell$  and  $\hat{m}_h$ . The idea of mass-split models is to start with sufficiently many

fermion flavors to guarantee that the system with degenerate, massless fermions is conformal. Thus the gauge coupling is irrelevant and runs to the conformal infrared fixed point (IRFP). By giving mass to some of the flavors, we create a system with  $N_\ell$  light (or massless) flavors with mass  $\hat{m}_\ell$  and  $N_h$  heavy flavors with mass  $\hat{m}_h$ . The number of light flavors  $N_\ell$  is chosen such that the light sector on its own is chirally broken. The resulting mass-split system is governed by the conformal IRFP above the chiral symmetry breaking scale where the spectrum exhibits conformal hyperscaling. There the mass of the lightest isosinglet scalar  $0^{++}$  is expected to be comparable to the corresponding pseudoscalar mass [17, 18].

In the infrared the heavy flavors decouple, chiral symmetry for the light flavors breaks spontaneously, and the gauge coupling starts running again. The mass of the heavy flavors controls the separation of scales between the UV and IR and also takes over the role of the bare gauge coupling to set the scale [16]. Even though the low energy system is chirally broken, its properties are significantly different from a QCD-like system with  $N_\ell$  flavors. In particular a light  $0^{++}$  state may need to enter the effective chiral Lagrangian, requiring the extension to dilaton chiral perturbation theory (dChPT) [19–24].

It is favorable to keep the total number of flavors  $N_f = N_\ell + N_h$  near the onset of the conformal window to achieve a large anomalous dimension. Specifi-

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cally we study an SU(3) gauge system with four light and six heavy flavors in the fundamental representation. We explore this new, strongly coupled sector in isolation by performing large scale numerical lattice field theory simulations.

Although no consensus has been reached on the onset of the conformal window for SU(3) gauge-fermion systems with fundamental fermions, there are indications that  $N_f = 10$  is conformal [25–32]. By choosing a system with four flavors in the light, chirally broken sector, our simulations can also directly be related to different models extending the SM with a new strongly interacting sector, see e.g. [33–35]. In these models the Higgs boson is considered to be a pseudo-Nambu–Goldstone boson (pNGB) of the new strong sector [33, 34, 36–38] and acquires its mass solely from the interactions with the SM particles.

The choice of  $N_f = 10$  improves over a pilot study using four light and eight heavy flavors [8, 14, 15, 39–43] by moving the system closer to the onset of the conformal window. In addition we perform the numerical simulations using chiral domain wall fermions (DWF) [44–47] which preserve the flavor structure. While numerically more costly, DWF provide a theoretically clean environment to perform investigations of strongly coupled systems near a conformal IR fixed point.

We briefly introduce the details of the numerical simulations before we demonstrate hyperscaling and determine the mass anomalous dimension. This allows us to explore implications for a possible effective description at low energies. Finally we give an outlook on our future calculations of phenomenologically important quantities. Preliminary results have been reported in [48, 49].

## NUMERICAL SIMULATIONS

Simulations are performed on hypercubic lattices using  $(L/a)^3 \times T/a$  volumes with  $L/a = 24$  or  $32$  and  $T/a = 64$  where  $a$  indicates the lattice spacing. We simulate the SU(3) gauge system with four light and six heavy flavors using the Symanzik gauge action [50, 51] with 3-times stout-smearing ( $\rho = 0.1$ ) [52] Möbius domain wall fermions [47] ( $b_5 = 1.5$ ,  $c_5 = 0.5$ ). DWF are simulated by adding a fifth dimension of extent  $L_s$  which separates the physical modes of four dimensional space-time. For practical reasons  $L_s$  needs to be finite i.e. DWF exhibit a small, residual chiral symmetry breaking, conventionally parameterized as an additive mass term  $am_{\text{res}}$ . In our simulations we choose  $L_s = 16$  and set the domain wall height  $M_5 = 1$ . We determine the residual chiral symmetry breaking numerically and find small values of  $O(10^{-3})$ . To correctly refer to the dimensionless lattice masses, we introduce the notation

$$\hat{m}_x \equiv a\tilde{m}_x = a(m_x + m_{\text{res}}) \quad \text{with } x = \ell, h. \quad (1)$$

Based on insight from our accompanying step-scaling investigation [31, 32], we set the bare gauge coupling

$\beta = 6/g_0^2 = 4.03$ , close to the IRFP of the underlying conformal theory with ten degenerate flavors. The hybrid Monte Carlo (HMC) update algorithm [53] with a trajectory length of  $\tau = 2$  MDTU (molecular dynamics time units) is used to generate ensembles of dynamical gauge field configurations with 1 – 3k (0.3 – 0.5k) thermalized trajectories for  $am_\ell \leq 0.04$  ( $am_\ell > 0.04$ ). Using input heavy flavor mass  $am_h = 0.200, 0.175$ , and  $0.150$ , we explore the 4+6 system choosing five or seven values for the input light flavor mass in the range  $0.015 \leq am_\ell \leq 0.100$ . Measurements are performed every 20 (10) MDTU for  $am_\ell < 0.04$  ( $am_\ell \geq 0.04$ ).

## HYPERSCALING

To understand the properties of mass-split systems, we refer to Wilsonian renormalization group (RG). In the UV both mass parameters are much lighter than the cutoff  $\Lambda_{\text{cut}} = 1/a$ :  $\hat{m}_\ell \ll 1$ ,  $\hat{m}_h \ll 1$ . As the energy scale  $\mu$  is lowered from the cutoff, the RG flowed lattice action moves in the infinite parameter action space as dictated by the fixed point structure of the  $N_f$  flavor conformal theory. The masses are increasing according to their scaling dimension  $y_m$ ,  $\hat{m}_{\ell,h} \rightarrow \hat{m}_{\ell,h}(a\mu)^{-y_m}$ , but we assume that they are still small so the system remains close to the conformal critical surface. The gauge couplings run toward the IRFP and stay there.

If the gauge couplings take their IRFP value, only the two masses change under RG flow. We can use standard hyperscaling arguments [54–56] to show that any physical quantity  $aM_H$  of mass dimension one follows, at leading order, the scaling form [15]

$$aM_H = \hat{m}_h^{1/y_m} \Phi_H(\hat{m}_\ell/\hat{m}_h), \quad (2)$$

where  $y_m = 1 + \gamma_m^*$  is the universal scaling dimension of the mass at the IRFP and  $\Phi_H$  some function of  $\hat{m}_\ell/\hat{m}_h$ .  $\Phi_H$  depends on the observable  $H$  and could be qualitatively different for different  $H$ .<sup>1</sup> The scaling relation Eq. (2) is valid as long as the gauge couplings remain at the IRFP and lattice masses are small, i.e. even in the  $\hat{m}_\ell = 0$  chiral limit. As a consequence, ratios of masses

$$\frac{M_{H1}}{M_{H2}} = \frac{\Phi_{H1}(\hat{m}_\ell/\hat{m}_h)}{\Phi_{H2}(\hat{m}_\ell/\hat{m}_h)} \quad (3)$$

depend only on  $\hat{m}_\ell/\hat{m}_h$ . The heavy flavors decouple when  $\hat{m}_h(a\mu)^{-y_m} \approx 1$ . At that point the light flavors condense and spontaneously break chiral symmetry. This allows us to define the hadronic or chiral symmetry breaking scale

$$\Lambda_H = \hat{m}_h^{1/y_m} a^{-1}. \quad (4)$$

<sup>1</sup> Equivalent to Eq. (2) is the hyperscaling relation,  $aM_H = \hat{m}_\ell^{1/y_m} \Phi_H(\hat{m}_\ell/\hat{m}_h)$ , given in Ref. [15]. Depending on the observable and scaling test, one or the other form might be preferable.

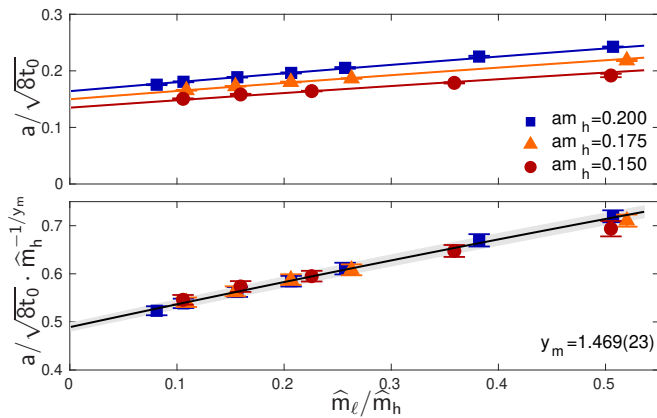


FIG. 1. The inverse Wilson flow scale  $a/\sqrt{8t_0}$  and interpolating fit according to Eq. (2) as function of  $\hat{m}_\ell/\hat{m}_h$ . The bottom panel shows “curve collapse” for  $\Phi_{\sqrt{8t_0}}(\hat{m}_\ell/\hat{m}_h) = a/\sqrt{8t_0} \cdot \hat{m}_h^{-1/y_m}$ .

As the energy scale  $\mu$  is lowered below  $\Lambda_H$ , the gauge coupling starts running again. However, properties of the IRFP are already encoded in hadronic observables. We have established hyperscaling of ratios in the 4+8 flavor system [8, 15] and preliminary results for the 4+6 system are reported in [48, 49].

In Fig. 1 we illustrate hyperscaling and the determination of  $y_m$  by considering the inverse Wilson flow scale  $a/\sqrt{8t_0}$  as the quantity  $aM_H$  in Eq. (2). The dimensionful quantity  $1/\sqrt{8t_0}$  is proportional to the energy scale where the renormalized running coupling in the gradient flow scheme equals a reference value ( $g_{GF}^2 \approx 16$ ) [57]. The top panel shows  $a/\sqrt{8t_0}$  as the function of  $\hat{m}_\ell/\hat{m}_h$ . While the data corresponding to our three different  $am_h$  values are different, each set on its own follows a smooth, almost linear curve. This suggests to parametrize the unknown function  $\Phi_{\sqrt{8t_0}}(\hat{m}_\ell/\hat{m}_h)$  using a low-order polynomial and perform a combined fit to all 17 data points in Fig. 1 using the Ansatz given in Eq. (2). A fit with a quadratic polynomial describes our data well. Small deviations of very precise  $a/\sqrt{8t_0}$  values lead to  $\chi^2/\text{d.o.f.} \approx 3$  and  $y_m = 1.469(23)$  with likely underestimated statistical uncertainties.

The bottom panel of Fig. 1 shows the data points for  $a/\sqrt{8t_0} \cdot \hat{m}_h^{-1/y_m}$  and the quadratic fit function  $\Phi_{\sqrt{8t_0}}(\hat{m}_\ell/\hat{m}_h)$ , exhibiting the expected “curve collapse.” We find similar curve collapse for other observables and show in Fig. 2 the result for a combined, correlated fit to the light-light ( $\ell\ell$ ), heavy-light ( $h\ell$ ), and heavy-heavy ( $hh$ ) pseudoscalar decay constant  $aF_{ps}$ . Since the determination of  $aF_{ps}$  is equally precise for  $\ell\ell$ ,  $h\ell$ , or  $hh$  states, this fit provides a representative determination of  $y_m$  with a good  $p$ -value. Subsequently we use

$$y_m = 1 + \gamma_m^* = 1.470(52), \quad (5)$$

as our reference value and note it is consistent within uncertainties to determinations from other observables like

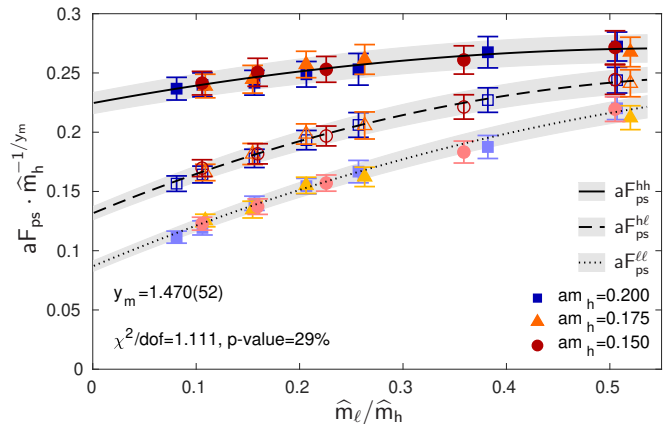


FIG. 2. Hyperscaling of the light-light (shaded symbols), heavy-light (open symbols), and heavy-heavy (filled symbols) pseudoscalar decay constant as the function of  $\hat{m}_\ell/\hat{m}_h$ . A combined fit based on Eq. (2) determines  $y_m$ .

vector or pseudoscalar masses. Further  $y_m$  is in agreement to an independent determination based on gradient flow [58]. The predicted  $\gamma_m^*$  is substantially below 1, the value expected for a system close to the sill of the conformal window [12, 59]. Since dChPT analysis of the  $N_f = 8$  data [60, 61] predicts  $\gamma_m^*$  near 1 [19–24], this indicates the sill of the conformal window lies between  $N_f = 8$  and 10, whereas the 12 flavor system ( $\gamma_m^* \approx 0.24$  [62–66]) is even deeper in the conformal regime.

The scaling of  $a/\sqrt{8t_0}$  is particularly interesting because it shows that the lattice spacing in the  $\hat{m}_\ell = 0$  chiral limit has a simple dependence on the heavy flavor mass

$$a = (\hat{m}_h)^{1/y_m} \cdot \Phi_{\sqrt{8t_0}}(0) \cdot \sqrt{8t_0}|_{m_\ell=0}, \quad (6)$$

where  $\Phi_{\sqrt{8t_0}}(0)$  is a finite number,  $\approx 0.48$ , in the 4+6 system. This confirms the expectation that the continuum  $a = 0$  limit is approached as  $\hat{m}_h$  decreases. Combined with Eq. (4) it predicts the hadronic scale

$$\Lambda_H^{-1} = \Phi_{\sqrt{8t_0}}(0) \cdot \sqrt{8t_0}|_{m_\ell=0}. \quad (7)$$

## LOW ENERGY EFFECTIVE DESCRIPTION

In the low energy infrared limit our system exhibits spontaneous chiral symmetry breaking. It should be described by a chiral effective Lagrangian which smoothly connects to the hyperscaling relation Eq. (2), valid at the hadronic scale  $\mu = \Lambda_H$ . In order to combine data sets with different  $\hat{m}_h$ , we express the lattice scale  $a$  in terms of the hadronic scale  $\Lambda_H$

$$M_H/\Lambda_H = (aM_H) \cdot \hat{m}_h^{-1/y_m} = \Phi_H(\hat{m}_\ell/\hat{m}_h). \quad (8)$$

Below the hadronic scale  $\Lambda_H$ , the 4+6 system reduces to a chirally broken  $N_f = 4$  system. The low energy effective theory (EFT) expresses the dependence of physical

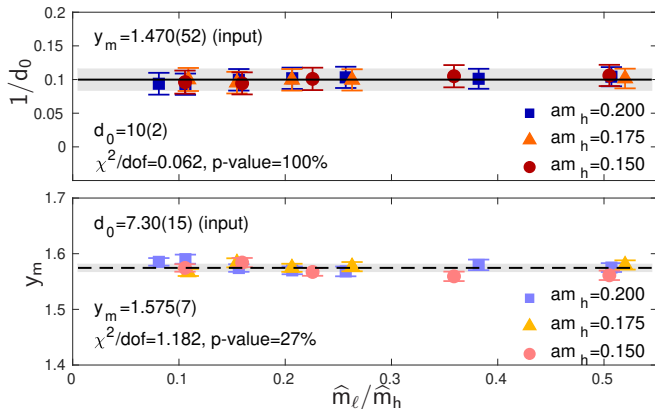


FIG. 3. Scaling test of dChPT exploiting Eq. (11). The top panel uses  $y_m$  (Eq. (5)) as input to fit  $1/d_0$ , the bottom panel shows a fit for  $y_m$  from a scan over  $d_0$  values used as input.

quantities on the running fermion mass  $m_f$  of the light flavors. At the hadronic energy scale the light flavor mass in lattice units is  $\hat{m}_\ell(a\Lambda_H)^{-y_m}$ , predicting

$$m_f \propto \hat{m}_\ell(a\Lambda_H)^{-y_m} \cdot \Lambda_H = (\hat{m}_\ell/\hat{m}_h) \cdot \Lambda_H. \quad (9)$$

The continuum limit is taken by tuning  $\hat{m}_h \rightarrow 0$  while keeping  $\hat{m}_\ell/\hat{m}_h$  fixed.

For  $\hat{m}_\ell/\hat{m}_h \lesssim 1$ , we expect the  $0^{++}$  ground state to be dominated by the light fermions. It is confined at scales of order  $\Lambda_H$  as are the other states, but its mass could well be small, comparable to the  $\ell\ell$  pseudoscalar mass. An EFT describing the small mass regime then needs to incorporate the light scalar state together with the pseudoscalars. In the  $m_f = 0$  limit, only the pseudoscalar states are massless. The  $0^{++}$  decouples at very low energies and  $N_f = 4$  ChPT should describe the data.

The dChPT Lagrangian incorporates the effect of a light dilaton state [19–24]. While derived for a chirally broken system with degenerate fermions just below the conformal window, we explore its application to our near-conformal mass-split system.

dChPT predicts the scaling relation

$$d_0 \cdot F_{ps}^{2-y_m} = M_{ps}^2/m_f, \quad (10)$$

which is a general result first discussed in Refs. [19, 21] and independent of the specific form of the dilaton effective potential. The quantity  $d_0$  is a combination of low energy constants. Using Eq. (8) we express this relation in terms of lattice quantities of the light sector (dropping the superscripts  $\ell\ell$ )

$$d_0 \cdot (aF_{ps})^{2-y_m} = (aM_{ps})^2/\hat{m}_\ell. \quad (11)$$

From Eq. (2) we can deduce that  $d_0 = (aM_{ps})^2 \cdot (aF_{ps})^{-2+y_m}/\hat{m}_\ell$  may only depend on  $\hat{m}_\ell/\hat{m}_h$ , whereas Eq. (10) states  $d_0$  is a constant.

Since our main goal is to study Eq. (10), we simply fix  $y_m$  from Eq. (5) and determine  $d_0$  using Eq. (11). As

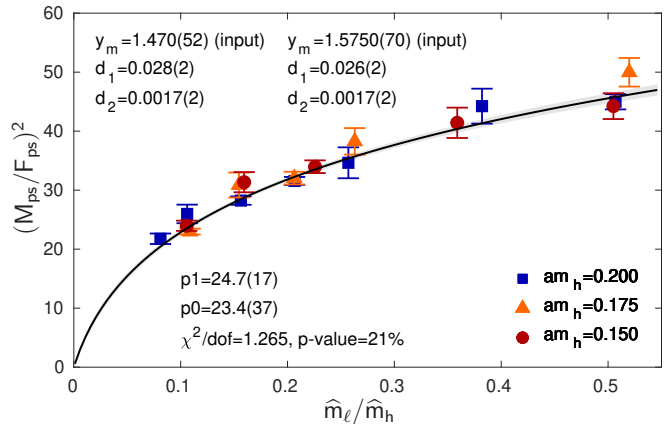


FIG. 4. Test of dChPT using Eq. (12). The black line with gray band is fitting our ratios  $(M_{ps}/F_{ps})^2$  to the function  $y = p_0 W_0(p_1 \cdot \hat{m}_\ell/\hat{m}_h)$ .

shown in the top panel of Fig. 3, our data form a flat line without dependence on  $\hat{m}_\ell/\hat{m}_h$ . A direct fit of our data to Eq. (11) to determine  $y_m$  and  $d_0$  simultaneously is troublesome because  $aF_{ps}$  and  $aM_{ps}$  have similar size uncertainties, are highly correlated, and the relation is nonlinear. Instead we perform a second test scanning a range of input values for  $d_0$  and fit for  $y_m$ . At a minimum  $\chi^2/\text{d.o.f.}$  we obtain a  $y_m = 1.575(7)$  within  $2\sigma$  of our reference value and shown in the lower panel of Fig. 3. In summary, our data are consistent with Eq. (11) and we obtain a rough estimate of  $y_m$  and  $d_0$ .

Assuming a specific form of the dilaton potential leads to another dChPT relation [24]

$$\frac{M_{ps}^2}{F_{ps}^2} = \frac{1}{y_m d_1} W_0 \left( \frac{y_m d_1}{d_2} m_f \right) \quad (12)$$

where  $W_0$  is the Lambert W-function and  $d_1, d_2$  are mass independent constants. Figure 4 shows a fit of our data to Eq. (12). The fit has an excellent  $p$ -value and allows us to determine the constants  $d_1$  and  $d_2$ . Relations of  $N_f = 4$  ChPT at leading and next-to-leading order exhibit a mass dependence different from Eqs. (10) and (12) and do not describe our data.

Finally we comment on the mass dependence of  $\sqrt{8t_0}/a$ . In ChPT this quantity has a linear mass dependence and corrections enter only at NNLO [67]. So far dChPT does not provide a useful description for  $\sqrt{8t_0}/a$  [24]. Our results in Fig. 1 show however that  $a/\sqrt{8t_0}$  obeys the usual hyperscaling relation in mass-split systems and  $a\sqrt{8t_0} \cdot \hat{m}_h^{-1/y_m}$  is well described by a linear mass dependence.

## CONCLUSION

In this work we highlight the unique features of the 4+6 mass split system built on a conformal IRFP. We show that physical masses exhibit hyperscaling and determine

the universal mass scaling dimension of the corresponding  $N_f = 10$  system  $y_m = 1 + \gamma_m^* = 1.47(5)$ . This value is smaller than expected for a theory near the edge of the conformal window suggesting that  $N_f = 9$  or 8 flavor models could be closer to the sill of the conformal window.

We compare our numerical results to predictions based on dChPT relations and find good agreement. Leading and next-to-leading order standard  $N_f = 4$  ChPT is, however, not consistent with our data. This strongly suggests that the  $0^{++}$  isosinglet scalar of the 4+6 mass-split system is a light state for the investigated parameter range.

There are many important questions to be studied in the future. Numerically determining the  $0^{++}$  scalar mass has the highest priority. Investigation of the baryonic anomalous dimension, relevant for partial compositeness, is already in progress [58]. Calculations of the  $S$  parameter and the Higgs potential are planned as well. Finite temperature studies could identify phase transitions with potentially significant implications for the early universe.

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<sup>2</sup> <https://github.com/paboyle/Grid>

<sup>3</sup> <https://usqcd.lns.mit.edu/w/index.php/QLUA>

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