

The combinatorial geometry of stresses in frameworks

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Abstract Consider a realization of a graph in the space with straight segments representing edges. Let us assign a stress for every its edge. In case if at every vertex of the graph the stresses sum up to zero, we say that the realization is a tensegrity. Some realizations possess non-zero tensegrities while the others do not. In this paper we study necessary and sufficient existence conditions for tensegrities in the plane. For an arbitrary graph we write down these conditions in terms of projective “meet-join” operations.

Keywords Tensegrity · self-tensional equilibrium frameworks

Introduction

Background. The study of tensegrities goes back to a classic paper [22] by J.C. Maxwell, where he has developed the basic elements of the theory of stresses on graph realizations. Numerous tensegrities were implemented in real life by artists and engineers several decades later. A famous *Needle Tower* by K. Snelson was one of the first huge tensegrity constructions. It was erected in 1968 in Washington, D.C. (see [28]). For a historical overview on tensegrity constructions we refer to the book [23] by R. Motro.

The word “tensegrity” (which is “tension” + “integrity”) is due to R. Buckminster Fuller who was inspired by the beauty of tensegrity constructions. Nowadays tensegrities can be spotted in architecture (when a light structural support is needed) and in arts. A good example here is a tensegrity sculpture

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TensegrityTree. It was erected in 2015 was erected in Kent (UK) to celebrate the 50th Anniversary of the University of Kent (see [11]). Tensegrities are now widely used in different areas of research, for instance they appear in the study of cells [13, 14], viruses [1, 26], deployable mechanisms [27, 31], etc.

The mathematical interest to tensegrities was revived in early 80s of the last century when B. Roth and W. Whiteley in [24], R. Connelly and W. Whiteley in [4], and W. Whiteley [38] studied and summarized various rigidity and flexibility properties of tensegrities. For a very recent development in the field are, we refer to papers [32] by M. Wang and M. Sitharam (on minimal rigidity) and [16] by B. Jackson, T. Jordán, B. Servatius and H. Servatius (on mechanical properties of tensegrities). See also a very nice short introductory paper on tensegrities in mathematics by R. Connelly [6].

Later tensegrities were studied in various different settings. For instance, F.V. Saliola and W. Whiteley in [25] considered tensegrities spherical and projective geometries; D. Kitson and S.C. Power in [19] and later D. Kitson and B. Schulze in [20] studied tensegrities in various normed spaces; B. Jackson and A. Nixon in [17] introduced tensegrities to surfaces in \mathbb{R}^3 , etc.

What precisely is a tensegrity? In the framework of this paper we mostly work with the following projective extension of tensegrities.

Definition 1 Let G be an arbitrary graph without loops and multiple edges. Let n be the number of vertices of G .

- Let $V(G) = \{v_1, \dots, v_n\}$ and $E(G)$ denote the sets of vertices and edges of G respectively. Denote by $(v_i; v_j)$ the edge joining v_i and v_j .
- A *framework* $G(P)$ in $\mathbb{R}P^2$ is a map of the set of vertices $\{v_1, \dots, v_n\}$ of G onto a finite point configuration $P = (p_1, \dots, p_n)$ in \mathbb{R}^2 (or respectively $\mathbb{R}P^2$), such that $G(P)(v_i) = p_i$ for $i = 1, \dots, n$. We say that there is an *edge* between p_i and p_j if $(v_i; v_j)$ is an edge of G and denote it by $(p_i; p_j)$.
- A *stress* on an edge $(p_i; p_j)$ is an assignment of two forces $F_{i,j}$ and $F_{j,i}$ whose line of forces coincide with the line $p_i p_j$ such that $F_{i,j} + F_{j,i} = 0$. For the case of $p_i = p_j$ we set the line of force for $F_{i,j} = -F_{j,i}$ to be any line passing through $p_i = p_j$.

Remark 1 The notion of force in projective geometry is introduced in terms of exterior decomposable forms. We recall basics of projective statics later in Subsection 2.1.1.

Now everything is ready for the definition of tensegrity.

Definition 2 Let G be an arbitrary graph with no loops and multiple edges. Let n be the number of vertices of G .

- A *force-load* F on a framework is an assignment of stresses for all edges. Additionally we set $F_{i,j} = 0$ if $(v_i; v_j)$ is not an edge of G .

- A force-load F is called an *equilibrium* force-load if, in addition, at every vertex p_i the following equilibrium condition is fulfilled:

$$\sum_{\{j|j \neq i\}} F_{i,j} = 0.$$

- A pair $(G(P), F)$ is called a *tensegrity* if F is an equilibrium force-load for the framework $G(P)$.
- A force-load F is said to be *nonzero* if there exist integers $1 \leq i, j \leq n$ such that $F_{i,j} \neq 0$.

We will return to the above definitions several times in the text in various contexts.

Description of frameworks admitting nonzero tensegrities. In papers [33,34] N.L. White and W. Whiteley introduced algebraic conditions for the existence of nonzero tensegrities. They have expressed these conditions in terms of bracket rings using the determinants of extended rigidity matrices (see also [36]). Further they have examined around 20 different examples of graphs, writing the corresponding conditions both in terms of bracket rings and in terms of Cayley algebra. In papers [9,10] M. de Guzmán and D. Orden introduced atom decomposition techniques and described various conditions for particular graphs.

Currently there are two main approaches to the study of existence conditions for nonzero tensegrities. The first is coordinate-based ([33,34,36], etc.); dealing with bracket ring expressions. The second approach is geometric ([7, 9], etc.); its main tools are the “meet-join” operations of projective geometry.

The coordinate-base approach is based on a simple algorithm to write down bracket ring expressions describing existence of nontrivial tensegrities. However these conditions do not give a geometric insight of configurations. For instance, it is not known how many connected components a certain bracket ring expression has. Here we should mention that the problem on bracket ring expression factorization is open even in the planar case (see [33]).

The geometric approach via Cayley algebra is more intuitive but at the same time it is more complicated technically (for further details see [33,34]). Cayley algebra tensegrity existence conditions have been found for numerous particular examples of graphs (see, e.g., in [7,33]). It was believed that there exists a simple geometric description of such conditions for arbitrary graphs. This description should be written in terms of the “meet-join” operations of projective geometry of Cayley algebra.

Our aim is to provide the geometric description for the planar case. In current paper we construct systems of geometric conditions for tensegrities in the plane for an arbitrary graph G (see Theorem 1 and Subsection 6.3). The conditions in this system are written in terms of slightly extended “meet-join” operations of Cayley algebra. These operations are applied to the vertices of frameworks and to the lines containing the edges of frameworks. Each condition corresponds to a generator of the first homology group $H_1(G)$.

Some preliminary work has been done in papers [7] and [18] where we have studied numerous examples and have introduced surgeries on graphs that are used in current paper.

Organization of the paper. We start in Section 1 with main definitions and preliminary discussions. We discuss projective tensegrities in Section 2. First we study basics of projective statics and give necessary definitions regarding projective tensegrities. Then in Subsection 2.5 we formulate the main result of this paper (see Theorem 1). Further in Section 3 we consider framed cycles and define monodromy operators for them. We prove that the monodromy operators for a framed cycle in general position are trivial if and only if there exists a nonzero equilibrium force load on it (see Theorem 2). In Section 4 we describe the techniques of resolution diagrams in order to study local properties of equilibrium force-loads at vertices. Further in Section 5 we introduce the notion of quantizations and prove a necessary and sufficient condition of existence of non-parallelizable tensegrities in terms of quantizations (see Theorem 3). In Section 6 we prove the main result of the current paper on the necessary and sufficient geometric conditions of existence of non-parallelizable tensegrities (i.e., Theorem 1). In this section we also introduce a technique to write these conditions explicitly. We conclude this paper in Section 7 with a conjecture on strong geometric conditions for nonzero tensegrities and some related discussions.

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1 Preliminary definitions and discussions

In this section we very briefly and rather informally explain major difficulties in the study of geometric conditions of tensegrities in \mathbb{R}^2 . As we will see later these difficulties can be partially resolved by extending the notion of tensegrities to the projective settings.

In what follows we use the following general notation. Consider two distinct points p, q of the Euclidean (or projective) plane and denote the line passing through p and q by pq .

Let us now work through a particular example.

Example 1 Consider the graph G shown on Figure 1 on the left. Let us describe a geometric condition on 6-tuples of points $P = (p_1, \dots, p_6)$ in the plane for the framework $G(P)$ to admit nonzero self-stresses. The complete answer is given by the following 6 cases.

Generic cases.

- 1) the lines p_1p_2, p_3p_4, p_5p_6 are concurrent (see Figure 1 in the middle);
- 2) the lines p_1p_2, p_3p_4, p_5p_6 are parallel (see Figure 1 on the right);

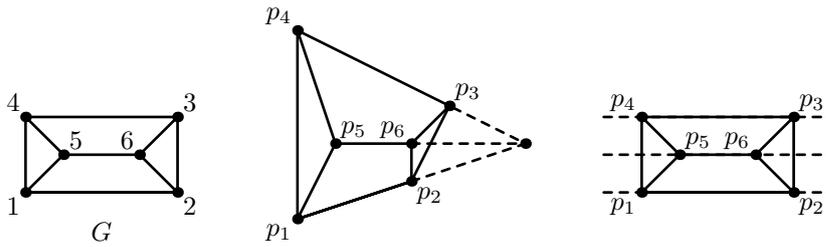


Fig. 1 The graph G (on the left) and two generic configurations of G admitting nonzero self-stresses (in the middle and on the right).

Non-generic cases.

- 3) the points p_1, p_4, p_5 are in a line;
- 4) the points p_2, p_3, p_6 are in a line;
- 5) the points p_1, p_2, p_3, p_4 are in a line;
- 6) the points p_1, p_2, p_5, p_6 are in a line;
- 7) the points p_3, p_4, p_5, p_6 are in a line.

Note that in generic cases any nonzero tensegrities is nonzero at every edge, while in non-generic ones this is not the case. We discuss genericity more detailed in Section 2.4. For further information on theory of stratifications of the configuration spaces of tensegrities we refer to [7].

Remark 2 The geometric conditions for Case 1) defines a Desargues configuration $(p_1, \dots, p_6, q, r_1, r_2, r_3)$, where

- q is the intersection point of p_1p_2, p_3p_4, p_5p_6 (*center of perspective*);
- r_1 is the intersection point of the lines p_1p_4 and p_2p_3 ;
- r_2 is the intersection point of the lines p_4p_5 and p_2p_6 ;
- r_3 is the intersection point of the lines p_1p_5 and p_3p_6 .

(The points q, r_1, r_2, r_3 are well defined in the complement to Cases 2–7.)

Recall that by Desargues' theorem the points r_1, r_2 , and r_3 are in a line, this line is called the *axis of perspectivity*.

The above example illustrates the following fact: even for graphs with a relatively small number of edges one should study numerous different geometrical cases. So it would be desirable to remove some non-important cases and to merge important ones. It is worthy to notice that all non-generic cases are generic cases for some subgraphs of a given graphs, and therefore can be studied in simpler settings for that subgraph.

Remark 3 While constructing actual tensegrities in real life it is usually expected that the stresses are of the same level of magnitude. In this case one should not consider tensegrities within some neighbourhood of a non-generic case.

Suppose now we are in a generic situation. We still have the following feature: *the number of generic cases has at least a quadratic growth in terms of the number of edges in the graphs.*

So we are faced with the following two difficulties:

I. *How to remove non-generic cases?*

II. *How to merge generic cases?*

We answer the first question providing a description of generic cases in Subsection 2.4. The second question is resolved by considering projectivization of tensegrities. Here we use a projective version of classical statics, where forces in the projective plane are represented by 2-forms in \mathbb{R}^3 (see Subsection 2.1 for more details).

Remark 4 Alternatively, one can consider spherical tensegrities for merging generic cases. The interplay of spherical, hyperbolic, and Euclidean tensegrities is discussed in [25].

Remark 5 Currently there is not much known about geometric descriptions of nonzero tensegrities in n -dimensional space for $n \geq 3$. Algebraic conditions in terms of bracket rings are provided in [38] by N.L. White and W. Whiteley, but their factorization is known to be a hard problem. In this context we would like to mention the paper [3] by J. Cheng, M. Sitharam, I. Streinu where the authors study nonzero stresses on arbitrary frameworks.

In this paper we introduce the techniques of graph quantization and $H\Phi$ -surgeries. The first of these techniques has a straightforward generalization to n -dimensional tensegrities, while the second is essentially planar. One would need to construct n -dimensional analogs of $H\Phi$ -surgeries in order to approach the combinatorial geometry of tensegrities in higher dimensions.

2 Projective geometry and tensegrities

In this section we collect main notions and definitions, and formulate the main result of this paper. We provide basic definitions of projective statics in Subsection 2.1. Further in Subsection 2.2 we describe elementary geometric operations and geometric equations on configurations of points and lines. We give the definition of a non-parallelizable tensegrity in Subsection 2.3. Next we define frameworks in general position in Subsection 2.4. Finally, in Subsection 2.5 we formulate the main result of this paper. We conclude Section 2 with geometric surgeries on frameworks that do not alter the property of admitting a nonzero tensegrity.

2.1 Basics of projective statics

Let us recall basics of projective statics and its relation to classical Euclidean statics (see also [2], [37], and [15]).

2.1.1 Forces in projective spaces and their lines of forces

Denote by $\Lambda^2(\mathbb{R}^3)$ the space of exterior 2-forms on \mathbb{R}^3 .

Definition 3 We say that a decomposable 2-form in $\Lambda^2(\mathbb{R}^3)$ is a *force* in $\mathbb{R}P^2$.

We naturally set the sum of forces as the sum of 2-forms in the linear space $\Lambda^2(\mathbb{R}^3)$. Notice that the sum of any number of forces is a force, since all 2-forms in $\Lambda^2(\mathbb{R}^3)$ are decomposable.

For a point $p = (a_1, a_2, a_3) \in \mathbb{R}^3$ set

$$dp = a_1 dx_1 + a_2 dx_2 + a_3 dx_3.$$

Set also $\tilde{p} = (a_1 : a_2 : a_3)$.

Definition 4 Consider a non-zero force $Fd = p_1 \wedge dp_2$. Let \tilde{p}_1, \tilde{p}_2 be two distinct projective points corresponding to p_1 and p_2 respectively. Then the projective line $\tilde{p}_1\tilde{p}_2$ is the *line of force*.

Note that by Definition 3 any nonzero force F is a decomposable 2-form. So there exists a pair of nonproportional vectors $p_1, p_2 \in \mathbb{R}^3$ such that

$$F = dp_1 \wedge dp_2.$$

2.1.2 Summation of the lines of force

Let us discuss a simple way to construct the line of the sum of forces. Consider two nonzero forces F_1 and F_2 . Let the lines of forces for F_1 and F_2 intersect in a point p . Hence they can be written as follows

$$F_1 = \alpha_1 dp \wedge dq_1 \quad \text{and} \quad F_2 = \alpha_2 dp \wedge dq_2.$$

Hence

$$F_1 + F_2 = dp \wedge d(\alpha_1 q_1 + \alpha_2 q_2),$$

and therefore the line of force for $F_1 + F_2$ is passes through points p and $\alpha_1 q_1 + \alpha_2 q_2$.

2.1.3 Links to classical Euclidean statics

Let us embed classical statics into projective statics. (For an elementary introduction to Euclidean statics see [30].)

First of all we consider the affine chart $x_3 = 0$. All points in this chart are in one-to-one correspondence with points of type $(u_1, u_2, 1) \in \mathbb{R}^3$.

Any force F defined by two points of this affine chart can be written as

$$F = \lambda d((a_1, a_2, 1)) \wedge d((b_1, b_2, 1)).$$

Then the *vector of force* for F should be set as follows

$${}^{\iota}F = \lambda \left(d((b_1, b_2, 1)) - d((a_1, a_2, 1)) \right) = \lambda d((b_1 - a_1, b_2 - a_2, 0)).$$

In classical statics one considers the first two coordinates of this form as the corresponding vector of force:

$$\lambda(b_1 - a_1, b_2 - a_2).$$

In the case if $\lambda \neq 0$, the *line of force* in classical statics is the line passing through (a_1, a_2) and (b_1, b_2) .

2.2 Geometry on lines and points

In this subsection we introduce elementary geometric operations and geometric equations on points and lines in $\mathbb{R}P^2$. Geometric equations here have much in common with algebraic equations. Later systems of geometric equations will be used in the formulation of main results.

2.2.1 Elementary geometric operations on lines and points

Let us start with four basic operations.

Operation I (2-line operation).

- Denote by $\ell_1 \cap \ell_2$ the intersection point of two lines ℓ_1, ℓ_2 , where $\ell_1 \neq \ell_2$.
- In the case if $\ell_1 = \ell_2$ we write $\ell_1 \cap \ell_2 = \text{true}$.
- Finally we formally set

$$\ell \cap \text{true} = \text{true} \cap \ell = \text{true} \cap \text{true} = \text{true}.$$

Operation II (2-point operation).

- Denote by (p_1, p_2) the line passing through p_1 and p_2 , where $p_1 \neq p_2$.
- In case if the points coincide we write $(p_1, p_2) = \text{true}$.
- Finally we formally set

$$(p, \text{true}) = (\text{true}, p) = (\text{true}, \text{true}) = \text{true}.$$

If there is no ambiguity we write simply $p_1 p_2$ instead of (p_1, p_2) .

Operation III (Generic point operation).

Given a line ℓ and a finite $S \subset \ell$, we choose a point of $\ell \setminus S$.

In fact, for this paper it is sufficient to consider two-point sets S .

Operation IV (Generic line operation).

Given a point p and a line ℓ , we choose a line passing through p and distinct to ℓ .

Remark 6 Let us say a few words about the notion of Cayley algebra in symbolic computation (not to be confused with octonions). Cayley algebra is an algebra whose elements are subspaces of a given space. It has two natural operations: the intersection and the sum. Traditionally these two operations are denoted by standard logic operators OR and AND, respectively:

$$\begin{aligned} L_1 \wedge L_2 &= L_1 \cap L_2; \\ L_1 \vee L_2 &= L_1 + L_2. \end{aligned}$$

In some texts they are referred as *meet* and *join* operations, respectively.

Notice that Operations I and II are particular cases of meet and join operations respectively. In order to avoid confusion with exterior products “ \wedge ” for exterior forms we prefer to follow the set theoretic notation in this paper. For more information on Cayley algebras we refer to [8], [35], and [21].

2.2.2 Elementary geometric equations

Now we introduce the following three elementary geometric equations.

Equation I (3-line equation).

- We write $\ell_1 \cap \ell_2 \cap \ell_3 = true$ if the lines ℓ_1 , ℓ_2 , and ℓ_3 are concurrent.
- We formally set

$$\ell_1 \cap \ell_2 \cap true = \ell_1 \cap true \cap \ell_3 = true \cap \ell_2 \cap \ell_3 = true,$$

(here ℓ_1 , ℓ_2 , ℓ_3 are allowed to be “*true*” as well).

Equation II (3-point equation).

- We write $(p_1, p_2, p_3) = true$ if three points p_1 , p_2 , and p_3 are in a line.
- We formally set

$$(p_1, p_2, true) = (p_1, true, p_3) = (true, p_2, p_3) = true,$$

(where p_1 , p_2 , p_3 can be “*true*”).

Equation III (Point-line equation).

- We write $(p \in \ell) = true$ if p is a point of ℓ .
- Here we formally set again

$$(p \in true) = (true \in \ell) = true,$$

(where ℓ and p can be “*true*”).

2.2.3 Geometric equations on configuration spaces of points and lines

In what follows we work with configuration spaces of lines and points of special types. All points of such configuration spaces are fixed and each line of the configuration space is not fixed but passes through a prescribed fixed point.

Definition 5 Consider the following data.

- Let P be an n -tuple of points (multiple points are allowed).
- Let L be an m -tuple of lines (multiple lines are allowed as well).
- Let S_P be a finite set of indices for P .
- Let S_L be a finite set of indices for L .
- Set finally $S \subset S_P \times S_L$.

Denote by R_S a collection of formal inclusions $p_i \in \ell_j$ for $(i, j) \in S$.

- We say that the pair (P, L) is a *realization* of R_S if all the inclusions in R_S are fulfilled for the corresponding points of P and lines of L .
- Denote by $(P, L|R_S)$ the *configuration space of all realizations* of R_S .

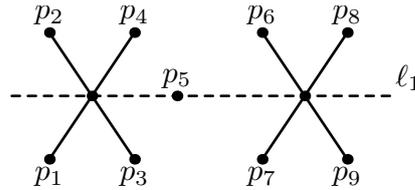
Note that if L is empty then R_S is empty as well. Hence the configuration space $(P, L|R_S)$ coincides with the configuration space of all n -tuples of points in the plane, i.e. with $(\mathbb{R}^2)^n$.

Definition 6 We say that a system of geometric equations on $(P, L|R_S)$ is *fulfilled* at a collection of points P' if there exists a choice of non-fixed lines in L' such that every geometric equation in the system is “*true*” for (P', L') . (Alternatively, we say that P' *satisfies* the system of geometric equations on $(P, L|R_S)$.)

We say that two systems of geometric equations on $(P, L|R_S)$ are *equivalent* if for every point configuration either both these systems are fulfilled or neither of them are fulfilled.

Remark 7 Note that if at some iteration of a geometric equation calculation we arrive at “*true*”, then this geometric equation is fulfilled as well.

Example 2 Consider $((p_1, \dots, p_9), (\ell_1)|p_5 \in \ell_1)$. Then the following configuration



fulfills the system of geometric equations

$$\begin{cases} p_1p_4 \cap p_2p_3 \cap \ell_1 = \text{true} \\ p_6p_9 \cap p_7p_8 \cap \ell_1 = \text{true} \end{cases}.$$

See also Examples 9, 10, and 11 below.

2.3 Non-parallelizable tensegrity

Our main result (Theorem 1) has a natural genericity condition that is described in the following definition.

Definition 7 An equilibrium force-load F on $G(P)$ is said to be *non-parallelizable* at vertex p , if the following two conditions hold.

- Suppose that the forces of F at all edges adjacent to p are F_1, \dots, F_s .
- Let $a_i \in \{0, 1\}$ for $i = 1, \dots, s$. Then

$$\sum_{i=1}^s a_i F_i = 0 \quad \text{implies} \quad a_1 = \dots = a_s.$$

- All the lines of forces defined by

$$F_1 + \sum_{i=2}^s a_i F_i, \quad \text{where} \quad (a_2, \dots, a_s) \in \{0, 1\}^s \setminus \{(1, \dots, 1)\}$$

are distinct (recall that we have $2^{s-1} - 1$ of such lines).

We say that a tensegrity $(G(P), F)$ is *non-parallelizable* if it is non-parallelizable at every vertex.

Remark 8 Let us assume that the first condition of non-parallelizability for $(G(P), F)$ is not fulfilled at p . Then either some edge will have a zero force, and therefore it can be removed; or p can be split into two vertices (each edge is adjacent either to one copy of p or to another assuming that each copy is adjacent to at least one edge). In both cases the new framework admits a tensegrity whose stresses coincide with the stresses of the original tensegrity at the corresponding edges.

Assume that we split vertex p (i.e., we consider the second case), then we have the following two situations. If p is a cut vertex for a graph, then the resulting graph might have several connected components. If p is not a cut vertex, then the resulting graph has a smaller first Betti number. So in all cases the tensegrity realizability question is reduced to a simpler graph.

2.4 Frameworks in general position

General position of frameworks is the last ingredient for the formulation of the main result of this paper.

First of all we give the definition of a collection of lines in general position.

Definition 8 An n -tuple of lines in the projective plane is said to be *in general position* if the union of their pairwise intersections is discrete and contains precisely $\frac{n(n-1)}{2}$ distinct points.

We continue with the following general definition.

Definition 9 A *simple cycle* in a graph G is a subgraph of G homeomorphic to the circle.

Here we use the word “simple” to avoid confusion with closed walks also known as cycles in graphs.

We have the following genericity condition for cycles.

Definition 10 Let $C(P)$ be a realization of a cycle C in the projective plane, where $P = (p_1, \dots, p_n)$. The realization $C(P)$ is said to be in *general position* if the lines passing through the edges of $C(P)$ are in general position.

Remark 9 One might consider the following immediate graph simplifications.

- A degree 1 vertex can be removed.
- A degree 2 vertex can be removed if the adjacent vertices are already joined by an edge. If the adjacent vertices are not joined by an edge then the degree 2 vertex together with two edges emanating from it can be replaced by an edge joining the adjacent vertices.
- In case if the graph is not connected, one can consider its connected components separately.

Keeping in mind the above remark, we now entirely restrict ourself to graphs whose all edges are of degree greater or equal to 3. Let us now define a framework in general position for such graphs.

Definition 11 Let G be a connected graph on n vertices whose vertices are all of degree at least 3. We say that $G(P)$ is a framework *in general position* if every simple cycle with at most $n - 1$ vertices is in general position.

To get a flavor of non-parallelizable tensegrities for frameworks in general position let us observe the following simple property.

Proposition 1 *Let a framework $G(P)$ in general position admit a non-parallelizable tensegrity. Then it does not have zero edges (i.e., for every edge $(p_i; p_j)$ we have $p_i \neq p_j$).* \square

2.5 Formulation of main result

Consider a graph G on n vertices and let $G(P)$ be its framework on an n -tuple of points $P = (p_1, \dots, p_n)$. Consider the following data:

- enumerated *fixed points* P : all points of $P = (p_1, \dots, p_n)$.
- enumerated *non-fixed lines* L : for each point $p_i \in P$ we consider $\deg p_i - 3$ lines. We denote them by $\ell_{i,1}, \dots, \ell_{i,\deg p_i - 3}$.
- *inclusion conditions* R : $p_i \in \ell_{i,j}$ for all admissible pairs (i, j) .

For P , L , and R as above we set

$$\Xi_G = (P, L|R).$$

Finally, for a fixed n -tuple P_0 denote by $\Xi_G(P_0)$ all the elements of Ξ_G whose point set is P_0 .

Notice that for a fixed n -tuple P_0 the configuration space $\Xi_G(P_0)$ is homeomorphic to the product of k circles S^1 where

$$k = \sum_{i=1}^n (\deg(p_i) - 3),$$

since each line of L is parameterized by $\mathbb{R}P^1 = S^1$.

Example 3 In the case if all vertices are of degree 3, the list of non-fixed lines L is empty. Here $\Xi_G(P_0)$ consists of one element $(P_0, ())$.

The central results of this paper is as follows.

Theorem 1 *A framework $G(P)$ in general position admits a non-parallelizable tensegrity if and only if P satisfies an associated system of geometric equations on Ξ_G derived from the collection of simple cycles in $G(P)$ which do not contain all vertices of P .*

Remark 10 Associated systems of geometric equations on Ξ_G will be defined much later (in Definition 43) after the notion of quantizations is introduced. For the moment the theorem should be treated as an existence statement. One should keep in mind that the proof of this theorem is constructive, one can write the system of geometric equations explicitly (see Subsection 6.1).

Example 4 Let G be the 6-vertex graph on the left of Figure 2. If $G(P)$ is a framework in general position then the associated geometric equation on the configuration space Ξ_G is as follows

$$\left((p_1, p_2) \cap (p_4, p_5), (p_2, p_3) \cap (p_5, p_6), (p_3, p_4) \cap (p_1, p_6) \right) = \text{true}.$$

This geometric equation is satisfied by the framework indicated in Figure 2 (center) and so, by Theorem 1, $G(P)$ admits a non-parallelizable tensegrity. Due to Pascal's theorem this geometric equation is equivalent to the following condition: *the points p_1, \dots, p_6 are on a conic* (see Figure 2 on the right).

Remark 11 In the proof of Theorem 1 we use a technique of quantizations and resolution diagrams. We give necessary notions, definitions and formulate several related propositions in Sections 3–5. We return to the proof of Theorem 1 in Subsection 6.2.

2.6 $H\Phi$ -surgeries on frameworks

$H\Phi$ -surgeries on frameworks do not alter the property of admitting a nonzero tensegrity. They provide matching between geometric conditions for the corresponding graphs, and therefore they are of great importance for the study of geometric existence and uniqueness conditions of nonzero tensegrities.

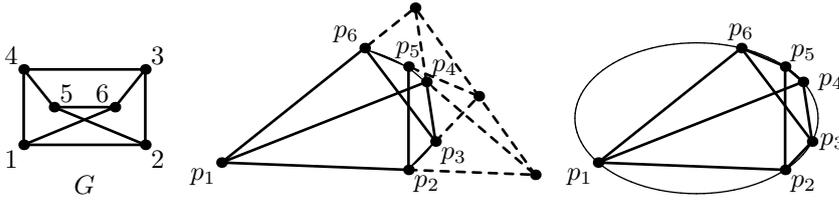


Fig. 2 The graph G (on the left), corresponding geometric condition G (in the middle) and equivalent algebraic condition (on the right).

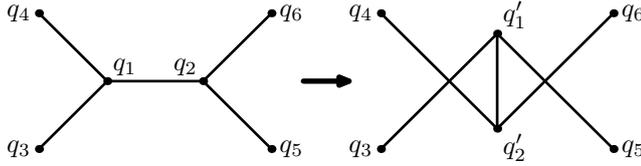


Fig. 3 $H\Phi$ -surgery.

Consider a graph G_H on 6 vertices v_1, \dots, v_6 with edges

$$(v_1; v_2), (v_1; v_3), (v_1; v_4), (v_2; v_5), \text{ and } (v_2; v_6).$$

Denote by G_Φ the graph on vertices $v'_1, v'_2, v_3, v_4, v_5, v_6$ with edges

$$(v'_1; v'_2), (v'_1; v_3), (v'_1; v_5), (v'_2; v_4), \text{ and } (v'_2; v_6).$$

Definition 12 Let G be an arbitrary graph and let G_H be a subgraph in G . Let $G(P)$ be a framework on G and let $G_H(Q) \subset G(P)$ have vertices q_1, \dots, q_6 . Suppose that

- the vertices q_1 and q_2 have degree 3 in G ;
- $q_1 \neq q_2$;
- the triples of points (q_1, q_3, q_4) and (q_2, q_5, q_6) are not in a line.
- $q_1q_3 \neq q_2q_5$ and $q_1q_4 \neq q_2q_6$;

Consider $G_\Phi(Q')$ on points $q'_1, q'_2, q_3, q_4, q_5, q_6$ where

$$q'_1 = q_1q_3 \cap q_2q_5 \quad \text{and} \quad q'_2 = q_1q_4 \cap q_2q_6.$$

Finally we assume that

- $q_3 \neq q'_1$, $q_5 \neq q'_1$, $q_4 \neq q'_2$, and $q_6 \neq q'_2$.

We say that the operation of replacing the subframework $G_H(Q)$ with $G_\Phi(Q')$ on the framework $G(P)$ is an $H\Phi$ -surgery on $G(P)$ at the edge q_1q_2 . (See Figure 3.)

The main property of $H\Phi$ -surgeries is as follows.

Proposition 2 *An $H\Phi$ -surgery on a framework does not change the dimension of the space of equilibrium force-loads.* \square

Note that $H\Phi$ -surgeries are projective analogs of surgeries of type II from [7], so we skip the proof of this proposition here.

3 Monodromies of framed cycles and equilibrium force-loads

In this section we study static properties of framed cycles. We start in Subsection 3.1 with basic definitions of shift operators and monodromy operators. Further in Subsection 3.2 we define equilibrium force-loads for framed cycles. In Subsection 3.3 we study the projection operation on framed cycles (then we use them in the proof of the next subsection). Finally, in Subsection 3.4 we prove that the existence of nonzero equilibrium force-loads on a framed cycle in general position is equivalent to triviality of monodromy operators for this cycle.

3.1 Framed cycles in general position, shift operators, monodromies

We start with the notion of framed cycles in general position. Further we introduce the notions of shift operators and monodromy operators.

3.1.1 Framed cycles in general position

Let us now introduce framed cycles. Framed cycles extracted from frameworks will be of use in construction of geometric equations for non-zero tensegrities. Namely (as we show later) every framed cycle in a graph G , contributes with one geometric equation to the geometric system on Ξ_G .

In this subsection we consider all indices mod k .

Definition 13 Consider a realization of a cycle $C(P)$ with $P = (p_1, \dots, p_k)$ in the projective plane. We say that $C(P)$ has a *framing* if every vertex p_i is equipped with a line ℓ_i passing through p_i . The framework $C(P)$ together with its framing is called the *framed cycle*. We denote

$$C(P, L) = ((p_1, \dots, p_k), (\ell_1, \dots, \ell_k)).$$

Definition 14 A framed cycle $C(P, L)$ is in *general position* if the following conditions are fulfilled.

- The cycle $C(P)$ is in general position.
- For every admissible i the line ℓ_i does not contain p_{i-1} and p_{i+1} .

3.1.2 Shift operators

Let us introduce shift maps for the lines of a framed cycle $C(P, L)$ along simple paths. In fact each such shift operator is a linear mapping between the lines. Later we use the compositions of such operators to define monodromy operators. Monodromy operators detect existence of nonzero tensegrities at framed cycles.

First of all we define a shift operator for consecutive framed lines. We assume that $C(P, L)$ has k vertices (here the summation of indices is mod k).

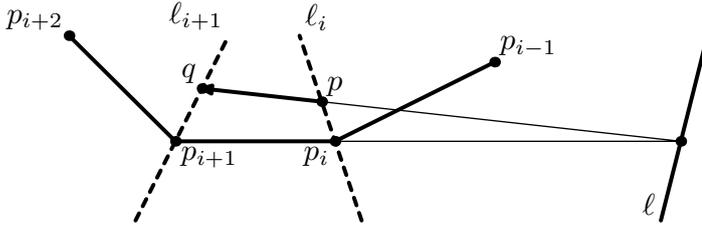


Fig. 4 A shift operation $\xi_\ell[p_i p_{i+1}, \ell_i, \ell_{i+1}]$. Here $q = \xi_\ell[p_i p_{i+1}, \ell_i, \ell_{i+1]}(p) = \ell_{i+1} \cap ((p_i p_{i+1} \cap \ell), p)$.

Definition 15 Let p_i and p_{i+1} be two consecutive points of the cycle, and let ℓ be any line that does not contain p_i and p_{i+1} . The mapping

$$\xi_\ell[p_i p_{i+1}, \ell_i, \ell_{i+1}] : \ell_i \rightarrow \ell_{i+1}$$

where for every $p \in \ell_i$ we set

$$\xi_\ell[p_i p_{i+1}, \ell_i, \ell_{i+1]}(p) = \ell_{i+1} \cap ((p_i p_{i+1} \cap \ell), p)$$

is called the *shift operator* from ℓ_i to ℓ_{i+1} along $C(P, L)$.

On Figure 4 we show an example of a shift operator.

Remark 12 One might consider the line ℓ as the line at infinity for some affine chart. In this affine chart the line $((p_i p_{i+1} \cap \ell), p)$ is simply the line through p parallel to $p_i p_{i+1}$.

Now we define the shift operator for a path (summation of indices is mod k as above).

Definition 16 Let $p_i \dots p_{i+s}$ be a path in a framed cycle $C(P, L)$ in general position. Consider a line ℓ that does not contain vertices of the path. We define

$$\begin{aligned} \xi_\ell[p_i \dots p_{i+s}; \ell_i, \dots, \ell_{i+s}] = \\ \xi_\ell[p_{i+s-1} p_{i+s}; \ell_{i+s-1}, \ell_{i+s}] \circ \dots \circ \xi_\ell[p_i p_{i+1}; \ell_i, \ell_{i+1}] : \ell_i \rightarrow \ell_{i+s}. \end{aligned}$$

3.1.3 Monodromy conditions for nonzero tensegrities on framed cycles

Using combinations of shift operators we can construct the following monodromy operators for framed cycles in general position. These operators will be used to describe existence of nonzero tensegrities for these framed cycles.

Definition 17 Consider a framed cycle $C(P, L)$ in general position, where $P = (p_1, \dots, p_k)$ and $L = (\ell_1, \dots, \ell_k)$. Let ℓ be a line in $\mathbb{R}P^2$ which does not pass through vertices of the framed cycle $C(P, L)$.

- The *monodromy* of $C(P, L)$ at $\ell_i \in L$ is the operator

$$\xi_\ell[p_i p_{i+1} p_{i+2} \cdots, p_{i-1} p_i; \ell_i, \ell_{i+1}, \ell_{i+2}, \dots, \ell_{i-1}, \ell_i] : \ell_i \rightarrow \ell_i.$$

We denote it by $M_\ell(\ell_i, C(P, L))$.

- A monodromy $M_\ell(\ell_i, C(P, L))$ is said to be *trivial* if it is an identity map on ℓ_i .
- We say that a framed cycle in general position satisfies the *monodromy cycle condition* if it has a trivial monodromy.

Let us collect some basic properties of monodromy operators.

Proposition 3 *The following four statements hold.*

- *The monodromy operator acts as a linear operator on ℓ_i with the origin at p_i .*
- *The property of a monodromy to be trivial is a projective invariant.*
- *Let $C(P, L)$ be a framed cycle in general position and let ℓ' and ℓ'' be a pair of lines neither of which contain the intersection point of any pair of distinct edges in the cycle. Then $M_{\ell'}(\ell_i, C(P, L))$ is trivial if and only if $M_{\ell''}(\ell_i, C(P, L))$ is trivial.*
- *Suppose that there exists i such that the monodromy $M_\ell(\ell_i, C(P, L))$ is trivial. Then $M_\ell(\ell_j, C(P, L))$ is trivial for all $j \in \{1, \dots, k\}$. \square*

3.2 Equilibrium force-loads on framed cycles

Let us now discuss equilibrium force-loads for framed cycles.

Definition 18 Let $C(P, L)$ be a framed cycle in general position. A *force-load* F on a framed cycle $C(P, L)$ is an assignment of

- stresses $F_{i, i+1} = -F_{i+1, i}$ for every edge $(p_i; p_{i+1})$ where $1 \leq i \leq k$;
- framing forces F_i (for $1 \leq i \leq k$) whose lines of forces are ℓ_i respectively.

Remark 13 The notion of an equilibrium force-load is as in Definition 2.

Definition 19 A force-load F is called an *almost equilibrium* force-load on $C(P, L)$ if the equilibrium condition is fulfilled at every vertex of $C(P, L)$ except one, say p_j , and at p_j we have either $F_{j, j-1} + F_j = 0$ or the line of force for $F_{j, j-1} + F_j$ is $p_j p_{j+1}$.

The following proposition justifies the usage of term “almost equilibrium” for force-loads.

Proposition 4 *Consider a framed cycle $C(P, L)$ in general position. Assume that $C(P, L)$ admits a nonzero equilibrium force load. Then every almost equilibrium force load on this cycle is an equilibrium force-load. \square*

The proof of this proposition is straightforward, so we omit it.

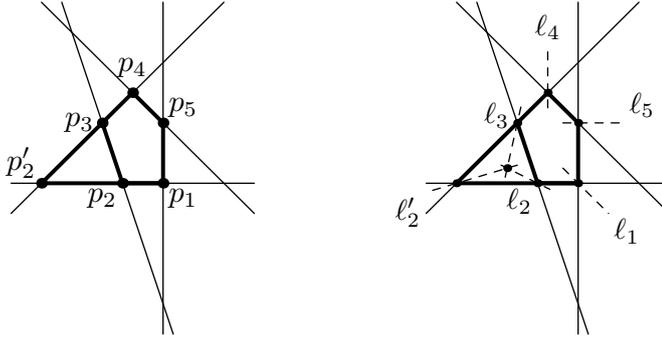


Fig. 5 A projection operation ω_2 . (The point $p'_2 = p_1p_2 \cap p_3p_4$ is on the left, and the line $\ell'_2 = (p'_2, \ell_2 \cap \ell_3)$ is on the right.)

3.3 Projection operations on framed cycles and their properties

In this subsection we introduce the notion of a projection operation on a framed cycle, which we use in the proof of Proposition 5 below. Further we study basic properties of projection operations.

3.3.1 Projection operations on framed cycles

Let us give the definition projection operations on framed cycles.

Definition 20 Consider a framed cycle $C(P, L)$ with $k \geq 4$ vertices. Let also $i \in \{1, 2, \dots, k\}$. Denote

$$\begin{aligned} p'_i &= p_{i-1}p_i \cap p_{i+1}p_{i+2}; \\ \ell'_i &= p'_i(\ell_i \cap \ell_{i+1}). \end{aligned}$$

(Here as before, we set $p_0 = p_k$, $p_{k+1} = p_1$, and $p_{k+2} = p_2$.)

A *projection operation* ω_i for a fixed index i is a mapping that sends $C(P, L)$ to the cycle

$$\omega_i(C(P, L)) = C(P', L')$$

where

$$\begin{aligned} P' &= (p_1, p_2, \dots, p_{i-1}, p'_i, p_{i+2}, \dots, p_k), \quad \text{and} \\ L' &= (\ell_1, \ell_2, \dots, \ell_{i-1}, \ell'_i, \ell_{i+2}, \dots, \ell_k). \end{aligned} \tag{1}$$

In Figure 5 we consider an example of a projection operation ω_2 applied to the framed cycle $C((p_1, \dots, p_5), (\ell_1, \dots, \ell_5))$. We show the points p_1, \dots, p_5, p'_2 on the left, and the lines $\ell_1, \dots, \ell_5, \ell'_2$ on the right.

3.3.2 Basic properties of projection operators

Let us bring together several basic properties of projective operations. Further we use them in the proof of Theorem 2 on the existence of a nonzero equilibrium force-load for a given framed cycle.

Proposition 5 *Let $C(P, L)$ be a framed cycle in general position on $k \geq 4$ vertices and let ω_i be one of its projections. Then the following three statements hold.*

(i) *The cycle $\omega_i(C(P, L))$ is in general position.*

(ii) *Let $j \notin \{i, i+1\}$ and let ℓ be a line that contains neither vertices of $C(P, L)$ nor p'_i . Then we have*

$$M_\ell(\ell_j, C(P, L)) = M_\ell(\ell_j, \omega_i(C(P, L))).$$

(iii) *The existence of a nonzero equilibrium force-load for $C(P, L)$ is equivalent to the existence of a nonzero equilibrium force-load for $\omega_i(C(P, L))$.*

Proof Let $\omega_i(C(P, L)) = C(P', L')$, here we follow the notation of (1) above.

Item (i). Since $C(P, L)$ is in general position, the cycle $C(P)$ is in general position. The set of lines through all edges in $C(P')$ coincide with the set of lines through edges in $C(P)$ minus the line $p_i p_{i+1}$. Hence the number of intersection points of lines through all edges in $C(P')$ is

$$\frac{k(k-1)}{2} - (k-1) = \frac{(k-1)(k-2)}{2},$$

So these lines are in general position. Therefore, $C(P')$ is in general position.

Recall that the only new line in the framing L' is the line

$$\ell'_i = (p'_i, \ell_i \cap \ell_{i+1})$$

through the point $p'_i = p_{i-1} p_i \cap p_{i+1} p_{i+2}$. Denote $B = \ell_i \cap \ell_{i+1}$.

First, let us show that the line ℓ'_i does not contain p_{i-1} . Since the line ℓ_{i+1} does not contain p_i , we have $B \neq p_i$. Further since the line ℓ_i does not contain p_{i-1} , the point B is not in the line $p_{i-1} p_i$. Therefore ℓ'_i does not contain the edge (p_{i-1}, p_i) . Finally, since $C(P)$ is in general position, we have $p'_i = \ell'_i \cap p_{i-1} p_i \neq p_{i-1}$ and therefore the point p_{i-1} is not in ℓ'_i .

Secondly, by the same reasons ℓ'_i does not contain p_{i+2} . Finally, all the other genericity conditions for the other lines of $C(P', L')$ are as for the lines of the framed cycle $C(P, L)$. Hence all genericity conditions are fulfilled. Therefore, the cycle $C(P', L')$ is in general position.

Item (ii). Consider two operators sending ℓ_{i-1} to ℓ_{i+2} :

$$\xi_\ell[p_{i-1} p_i p_{i+1} p_{i+2}; \ell_{i-1}, \ell_i, \ell_{i+1}, \ell_{i+2}], \quad \text{and} \quad \xi_\ell[p_{i-1} p'_i p_{i+2}; \ell_{i-1}, \ell'_i, \ell_{i+2}].$$

Let us prove that they coincide.

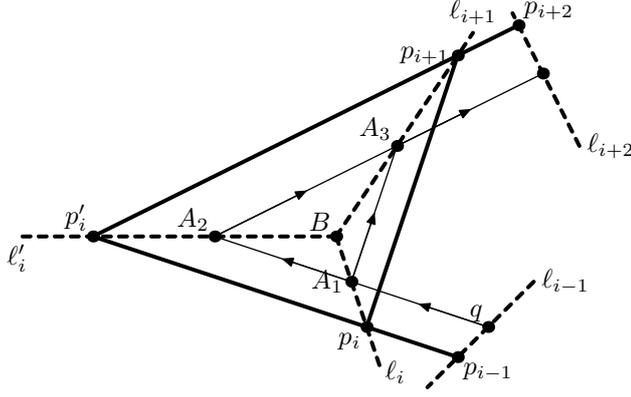


Fig. 6 Projection operation does not change monodromy operators, since $\xi_\ell[p_{i-1}p'_i p_{i+2}; \ell_{i-1}, \ell'_i, \ell_{i+2}](q) = \xi_\ell[p_{i-1}p_i p_{i+1} p_{i+2}; \ell_{i-1}, \ell_i, \ell_{i+1}, \ell_{i+2}](q)$ for every $q \in \ell_{i-1}$.

Denote

$$q_1 = \ell \cap (p_{i-1}, p_i), \quad q_2 = \ell \cap (p_i, p_{i+1}), \quad q_3 = \ell \cap (p_{i+1}, p_{i+2}).$$

Let q be a point of ℓ_{i-1} . Set

$$A_1 = \ell_i \cap qq_1, \quad A_2 = \ell'_i \cap qq_1, \quad A_3 = \ell_{i+1} \cap A_2 q_3.$$

Recall also that $B = \ell'_i \cap \ell_i = \ell'_i \cap \ell_{i+1}$.

Without loss of generality we consider the affine chart with the line ℓ at infinity. Now $A_1 A_2$ is parallel to $p_{i-1} p_i$ and $A_2 A_3$ is parallel to $p_{i+1} p_{i+2}$. Let us prove that $A_1 A_3$ is parallel to $p_i p_{i+1}$. (All the points and lines of the affine chart are shown on Figure 6. Note that the points q_1 , q_2 , and q_3 are at the line ℓ at infinity, hence they are not in the affine chart.)

The triangle $p'_i p_i B$ is homothetic to the triangle $A_2 A_1 B$, and the coefficient of homothety is $|p'_i B|/|A_2 B|$. The triangle $p'_i p_{i+1} B$ is homothetic to the triangle $A_2 A_3 B$, and the coefficient of homothety is $|p'_i B|/|A_2 B|$. Hence the quadrangle $p_i p'_i p_{i+1} B$ is homothetic to the quadrangle $A_1 A_2 A_3 B$, and the coefficient of homothety is $|p'_i B|/|A_2 B|$. Therefore, $p_i p_{i+1}$ is parallel to $A_1 A_3$. Hence we have

$$\begin{aligned} \xi_\ell[p_{i-1} p'_i p_{i+2}; \ell_{i-1}, \ell'_i, \ell_{i+2}](q) &= \xi_\ell[p'_i p_{i+2}; \ell'_i \ell_{i+2}] \circ \xi_\ell[p_{i-1} p'_i; \ell_{i-1} \ell'_i](q) \\ &= \xi_\ell[p_{i+1} p_{i+2}; \ell_{i+1}, \ell_{i+2}] \circ \xi_\ell[p_i p_{i+1}; \ell_i, \ell_{i+1}] \\ &\quad \circ \xi_\ell[p_{i-1} p_i; \ell_{i-1}, \ell_i](q) \\ &= \xi_\ell[p_{i-1} p_i p_{i+1} p_{i+2}; \ell_{i-1}, \ell_i, \ell_{i+1}, \ell_{i+2}](q). \end{aligned}$$

Therefore,

$$\xi_\ell[p_{i-1} p'_i p_{i+2}; \ell_{i-1}, \ell'_i, \ell_{i+2}] = \xi_\ell[p_{i-1} p_i p_{i+1} p_{i+2}; \ell_{i-1}, \ell_i, \ell_{i+1}, \ell_{i+2}].$$

Since the remaining shift operators defining the monodromy $M_\ell(\ell_j, C(P, L))$ are invariant under the projection ω_i , we have

$$M_\ell(\ell_j, C(P, L)) = M_\ell(\ell_j, \omega_i(C(P, L))).$$

Item (iii). Here the cycle $C(P', L')$ is obtained from $C(P, L)$ via an $H\Phi$ -surgery sending a graph $G_H(p_i, p_{i+1}, p_{i-1}, A_1, p_{i+2}, A_3)$ to the corresponding graph $G_\Phi(p'_i, B, p_{i-1}, A_1, p_{i+2}, A_3)$. (Here A_1 and A_3 are some points on ℓ_i and ℓ_{i+1} , and $B = \ell_i \cap \ell_{i+1}$, see also Figure 6). Now the statement follows directly from Proposition 2. \square

3.4 Monodromy condition for a nonzero equilibrium force-load

Let us formulate a necessary and sufficient condition of the existence of a nonzero equilibrium force-load for a given framed cycle. This theorem contributes to the essential stage in the proof of the main result (of Theorem 3).

Theorem 2 *Let $C(P, L)$ be a framed cycle in general position; let ℓ be a line that does not contain intersection points of any pair of edges for $C(P, L)$; and let $1 \leq i \leq k$. Then the monodromy operator $M_\ell(\ell_i, C(P, L))$ is trivial if and only if there exists a nonzero equilibrium force-load for $C(P, L)$.*

We start the proof of Theorem 2 with the following lemma.

Lemma 1 *Consider a triangular cycle $C(P, L)$ in general position, and let ℓ be a line that does not contain the vertices of $C(P, L)$. Then the following three statements are equivalent:*

- (a) *the lines ℓ_1, ℓ_2, ℓ_3 meet in a point;*
- (b) *there exists a nonzero equilibrium force-load for $C(P, L)$;*
- (c) *the monodromy operator $M_\ell(\ell_i, C(P, L))$ is trivial for every $i \in \{1, 2, 3\}$.*

Proof (a) \iff (b). The equivalence of the first and second statements is a classical statement. Suppose that there exists a nonzero equilibrium force-load F on $C(P, L)$. Hence

$$F_1 + F_2 + F_3 = 0, \quad \text{or, equivalently,} \quad F_1 = -F_2 - F_3.$$

As we know the intersection point of force lines F_2 and F_3 belongs to the force line $F_2 + F_3$ and hence to F_1 . Therefore, ℓ_1, ℓ_2 and ℓ_3 intersect in a common point.

Conversely, let ℓ_1, ℓ_2 and ℓ_3 meet in a point B . Consider

$$F_i = a_i dp_i \wedge dB \quad \text{for } i = 1, 2, 3$$

with nonzero real numbers a_1, a_2, a_3 such that $F_1 + F_2 + F_3 = 0$ which is equivalent to

$$(a_1 dp_1 + a_2 dp_2 + a_3 dp_3) \wedge dB = 0. \quad (2)$$

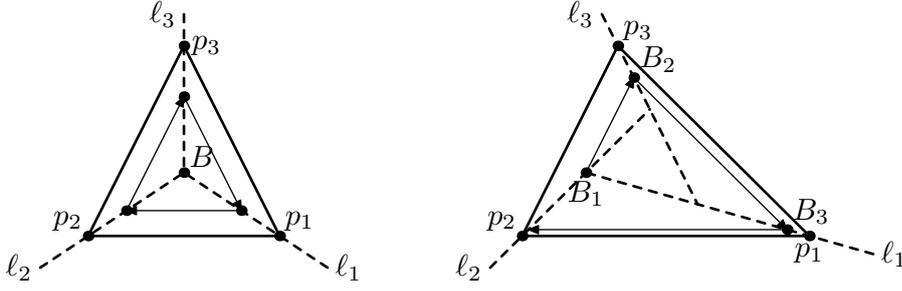


Fig. 7 Framed triangular cycles with trivial monodromy operators (on the left) and with non-trivial monodromy operators (on the right).

Since p_1 , p_2 , and p_3 are not in a line, we have $a_1 dp_1 + a_2 dp_2 + a_3 dp_3 \neq 0$. Therefore, Equation (2) implies

$$dB = \alpha(a_1 dp_1 + a_2 dp_2 + a_3 dp_3)$$

for some nonzero α . Set

$$F_{i,j} = -\alpha a_i a_j dp_i \wedge dp_j.$$

Then at every edge we have

$$F_{i,i+1} + F_{i+1,i} = -\alpha a_i a_{i+1} dp_i \wedge dp_{i+1} - \alpha a_{i+1} a_i dp_{i+1} \wedge dp_i = 0,$$

and at every vertex we have

$$\begin{aligned} F_i + F_{i,i-1} + F_{i,i+1} &= a_i dp_i \wedge (-\alpha a_{i-1} dp_{i-1} - \alpha a_{i+1} dp_{i+1} + dB) \\ &= \alpha a_i dp_i \wedge (-a_{i-1} dp_{i-1} - a_{i+1} dp_{i+1} \\ &\quad + (a_{i-1} dp_{i-1} + a_i dp_i + a_{i+1} dp_{i+1})) \\ &= \alpha a_i dp_i \wedge a_i dp_i \\ &= 0. \end{aligned}$$

Hence F is a nonzero equilibrium force-load on $C(P, L)$.

(a) \iff (c). Denote $B = \ell_1 \cap \ell_2$. Suppose $B \in \ell_3$, then B is a fixed point for every monodromy (see Figure 7, on the left). Therefore, all monodromies are trivial.

Suppose now, that ℓ_3 does not contain B (see Figure 7, on the right). Denote

$$B_1 = B, \quad B_2 = \xi_\ell[p_2 p_3; \ell_2, \ell_3](B_1), \quad \text{and} \quad B_3 = \xi_\ell[p_3 p_1; \ell_3, \ell_1](B_2).$$

Since $\ell_3 \neq \ell_1$, the point B_2 is not in ℓ_1 . Further, since $p_1 \notin \ell$ we have

$$p_3 p_1 \cap \ell \neq p_3 p_2 \cap \ell.$$

So we have:

$$\begin{aligned}
 M_\ell(\ell_1, C(P, L))(B_1) &= \xi_\ell[p_1 p_2 p_3 p_1; \ell_1, \ell_2, \ell_3, \ell_1](B_1) \\
 &= \xi_\ell[p_2 p_3 p_1; \ell_2, \ell_3, \ell_1](B_1) \\
 &= \xi_\ell[p_3 p_1; \ell_3, \ell_1](B_2) \\
 &= B_3 \neq B_1.
 \end{aligned}$$

(The second equation holds since $B_1 = \ell_1 \cap \ell_2$.) Therefore, $M_\ell(\ell_1, C(P, L))$ is not trivial. Then by Remark 3 all monodromies are not trivial. \square

Proof of Theorem 2. Let us prove the statement of the theorem by induction in the number of vertices in the cycle.

Base of induction. If the cycle $C(P, L)$ is triangular then the statement of Theorem 2 follows from Lemma 1.

Induction step. Suppose the statement holds for every framed cycle in general position on n vertices. Let us prove the statement for an arbitrary framed cycle in general position on $n+1$ vertices.

Let $C(P, L)$ be a framed cycle in general position on $n+1$ vertices, and let ω_i be one of its projection operations. Then on the one hand, by Proposition 5(ii) the monodromy at every edge of $C(P, L)$ is trivial if and only if the monodromy at the corresponding edge of the projection $\omega_i(C(P, L))$ is trivial. On the other hand by Proposition 5(iii) the existence of a nonzero equilibrium force-load for $C(P, L)$ is equivalent to the existence of a nonzero equilibrium force-load for $\omega_i(C(P, L))$.

Now the statement of the theorem follows directly from the induction assumption, since the framed cycle $\omega_i(C(P, L))$ is a framed cycle in general position on n vertices (see Proposition 5(i)). \square

4 Resolution diagrams at vertices

In this section we describe a technique of resolution diagrams at vertices. It will be used to generate framed cycles of quantizations in the next section. In Subsection 4.1 we introduce resolution diagrams and define force-loads at them. Further in Subsection 4.2 we formulate the notions of weakly and strongly generic resolution diagrams. We define $H\Phi$ -surgeries for resolution diagrams and a corresponding equivalence relation in Subsection 4.3. Finally, in Subsection 4.4 we prove finiteness of equivalent strongly generic resolution diagrams (and also provide their actual number).

4.1 Resolution diagrams and force-loads on them

We say that an edge of a tree is a *leaf* if one of its vertices is of degree 1. All other edges are said to be *interior*.

Denote the set of all lines in the projective plane by $\text{Gr}(1, \mathbb{R}P^2)$. Note that the set $\text{Gr}(1, \mathbb{R}P^2)$ is naturally isomorphic to the Grassmannian of 2-dimensional planes in \mathbb{R}^3 , i.e., to $\text{Gr}(2, 3)$.

Definition 21 We say that a tree T is an unrooted full binary tree if the degree of every its vertex is either 1 or 3.

Definition 22 Consider an unrooted full binary tree T . Let

$$\mathcal{L} : E(T) \rightarrow \text{Gr}(1, \mathbb{R}P^2).$$

We say that a pair (T, \mathcal{L}) is a *resolution diagram* at point $p \in \mathbb{R}P^2$ if for every edge $e \in T$ we have $p \in \mathcal{L}(e)$. Denote it by $(T, \mathcal{L})_p$.

Definition 23 Let $(T, \mathcal{L})_p$ be a resolution diagram. Assume that T has vertices ν_1, \dots, ν_k . Consider an edge $(\nu_i; \nu_j)$ of T . A *stress* on $(\nu_i; \nu_j)$ of the resolution diagram $(T, \mathcal{L})_p$ is a pair of forces $(F_{i,j}, F_{j,i})$ satisfying:

- the line of force $F_{i,j}$ coincides with the line $\mathcal{L}(\nu_i; \nu_j)$;
- $F_{i,j} + F_{j,i} = 0$.

Remark 14 For the stresses of Definition 23 we set force-loads, equilibrium force-loads, and nonzero force loads as in Definition 2.

Remark 15 Let us justify the usage of the term “resolution” in our settings. In some sense an equilibrium force-load on a resolution diagram can be seen as a part of an *infinitesimally perturbed* tensegrity at point p . Here we add extra *infinitesimal* edges along certain directions. Note that it is not always possible to add finite edges, due to metric reasons (some directions will have to change in this case). The situation is analogous to resolutions of singularities in singularity theory, where one adds exceptional lines in order to simplify a singularity.

4.2 Resolution diagrams in general position

In this subsection we study two types of genericity for resolution diagrams: weakly and strongly generic resolution diagrams. We will use both of them for the further studies. The notion of weak genericity is a very natural general notion to work with, while the notion of strong genericity is dictated by the natural restrictions for the main results of this paper.

4.2.1 Weakly generic resolution diagrams

Let us start with weakly generic resolution diagrams.

Definition 24 We say that a resolution diagram $(T, \mathcal{L})_p$ is *weakly generic* if for every pair of adjacent edges $(\nu_i; \nu_j)$ and $(\nu_j; \nu_s)$ of T we have

$$\mathcal{L}(\nu_i; \nu_j) \neq \mathcal{L}(\nu_j; \nu_s).$$

Basic properties of weakly generic resolution diagram are as follows.

Proposition 6 *Let $(T, \mathcal{L})_p$ be a weakly generic resolution diagram, then the following three items hold.*

- (i) *The diagram $(T, \mathcal{L})_p$ has a nonzero equilibrium force-load.*
- (ii) *If an equilibrium force-load is nonzero at some edge, then it is nonzero at every edge.*
- (iii) *All equilibrium force-loads on $(T, \mathcal{L})_p$ are proportional.*

Proof Let us construct an equilibrium force-load on $(T, \mathcal{L})_p$ starting from an arbitrary edge e . Fix an arbitrary nonzero stress at the edge e . Let us consider all edges adjacent to e . The equilibrium conditions at vertices of degree 3 uniquely define the stresses at all adjacent edges. Inductively attaching adjacent vectors further we uniquely extend the collection of constructed stresses at edges to the equilibrium force-load on the entire tree T . This can be done for the entire tree T , since it has only vertices of degree 1 and of degree 3. This concludes the proof of Item (i).

At each step of the induction discussed above we obtain nonzero stresses at edges of T , hence the statement of Item (ii) holds.

Finally, the linear combination of two equilibrium force-loads is an equilibrium force-load. Therefore, Items (ii) and (iii) are equivalent. \square

4.2.2 Strongly generic resolution diagrams

Let us give a more restrictive definition of genericity for resolution diagrams.

Definition 25 Let $(T, \mathcal{L})_p$ be a weakly generic resolution diagram and let F be a nonzero equilibrium force-load at it. Suppose that the forces of F at all the leaves of T are F_1, \dots, F_s . We say that $(T, \mathcal{L})_p$ is *strongly generic* if the following two conditions hold.

- Let $a_i \in \{0, 1\}$ for $i = 1, \dots, s$. Then

$$\sum_{i=1}^s a_i F_i = 0 \quad \text{implies} \quad a_1 = \dots = a_s.$$

- All $2^{s-1} - 1$ lines of forces defined by

$$F_1 + \sum_{i=2}^s a_i F_i, \quad \text{where} \quad (a_2, \dots, a_s) \in \{0, 1\}^{s-1} \setminus \{(1, \dots, 1)\}$$

are distinct.

Remark 16 It is clear that if $(T, \mathcal{L})_p$ is strongly generic then the forces

$$F_1 + \sum_{i=2}^s a_i F_i \quad \text{and} \quad \sum_{i=2}^s (1 - a_i) F_i \quad \text{where} \quad (a_2, \dots, a_s) \in \{0, 1\}^{s-1} \setminus \{(1, \dots, 1)\}$$

are always negative to each other and hence their lines of forces coincide. In general the definition of “strongly generic” is independent of the choice of equilibrium force-load since, by Proposition 6, they are all proportional.

Example 5 Suppose that the lines for the leaves of a resolution diagram with s leaves are distinct. If $s = 3$ then a resolution diagram is strongly generic. Further if $s = 4, 5$ then a resolution diagram is strongly generic if and only if \mathcal{L} is injective. For $s > 5$ we have more complicated conditions on strongly generic resolution diagrams.

4.2.3 On nonzero equilibrium force-loads at weakly and strongly generic resolution schemes

Let us observe the following static property of lines at interior edges of weakly generic resolution diagrams.

Proposition 7 *Let $(T, \mathcal{L})_p$ be a weakly generic resolution diagram and let e be an interior edge. Let $\{e_1, \dots, e_r\}$ be the subset of all leaves of T that are in one of the connected components for $T \setminus \{e\}$. Consider a nonzero equilibrium force-load F on $(T, \mathcal{L})_p$, and let F_1, \dots, F_r be the forces acting along the edges e_1, \dots, e_r . Set*

$$\bar{F} = F_1 + \dots + F_r.$$

If $\bar{F} \neq 0$ then the line of force for \bar{F} coincides with $\mathcal{L}(e)$.

Proof Denote by F_e the force along the edge e . Since F is an equilibrium force-load on $(T, \mathcal{L})_p$, it is an equilibrium force-load on each of the connected components for $T \setminus \{e\}$ considered together with e . Hence

$$\bar{F} + F_e = F_1 + \dots + F_r + F_e = 0.$$

Since $\bar{F} \neq 0$, it is proportional to F_e . Hence the line of forces for \bar{F} is $\mathcal{L}(e)$. \square

In particular, Proposition 7 implies the following result for strongly generic resolution diagrams, for which \bar{F} never vanish.

Corollary 1 *A strongly generic resolution diagram $(T, \mathcal{L})_p$ is uniquely determined by the binary tree T and the forces F_1, \dots, F_s acting on its leaves given by any non-zero equilibrium force-load F on $(T, \mathcal{L})_p$. \square*

4.3 Equivalent resolution diagrams

Our next goal is to introduce an equivalence relation on strongly generic resolution diagrams.

First of all we define $H\Phi$ -surgeries for strongly generic resolution diagrams. In the next definition we consider graphs G_H with vertices ν_1, \dots, ν_6 , and G_Φ with vertices $\nu'_1, \nu'_2, \nu_3, \nu_4, \nu_5, \nu_6$ as in Definition 12 (see also Figure 8).

Definition 26 Let $(T, \mathcal{L})_p$ be a strongly generic resolution diagram. Consider $G_H \subset T$ and G_Φ such that ν'_1 and ν'_2 are not vertices of T . Consider a nonzero force-load F on $(T, \mathcal{L})_p$. Set ℓ as a line of force for $F_{3,1} + F_{2,5}$.

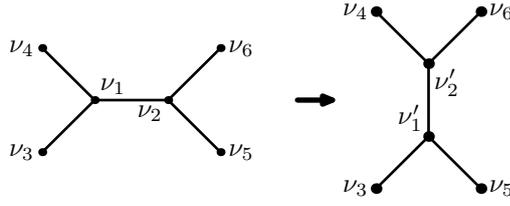


Fig. 8 $H\Phi$ -surgery at an interior edge $\nu_1\nu_2$ of a resolution diagram.

We say that the operation of replacing the subgraph G_H with G_Φ and changing \mathcal{L} to \mathcal{L}' defined as

$$\begin{aligned} \mathcal{L}'(\nu'_1; \nu'_2) &= \ell; & \mathcal{L}'(\nu'_1; \nu_3) &= \mathcal{L}(\nu_1; \nu_3); \\ \mathcal{L}'(\nu'_1; \nu_5) &= \mathcal{L}(\nu_2; \nu_5); & \mathcal{L}'(\nu'_2; \nu_4) &= \mathcal{L}(\nu_1; \nu_4); \\ \mathcal{L}'(\nu'_2; \nu_6) &= \mathcal{L}(\nu_2; \nu_6); & \mathcal{L}'(e) &= \mathcal{L}(e) \text{ for any other edge } e \end{aligned}$$

is an $H\Phi$ -surgery on $(T, \mathcal{L})_p$ at the interior edge $(\nu_1; \nu_2)$. (See Figure 8.)

Later in Subsection 6.1.2 we show how to define the surgery using Operations I–IV.

Remark 17 Notice that $H\Phi$ -surgery is not defined if $F_{3,1} + F_{2,5} = 0$.

Let us collect basic properties of strongly generic resolution diagrams in the following proposition.

Proposition 8 *Let $(T, \mathcal{L})_p$ be a strongly generic resolution diagram. Then for every $G_H \subset T$ the following three statements hold.*

- (i) *The corresponding $H\Phi$ -surgery is well-defined.*
- (ii) *The $H\Phi$ -surgery does not change equilibrium force-loads for all the edges of $(T, \mathcal{L})_p$ except for the edge $(\nu_1; \nu_2)$ where the surgery takes place. In particular all equilibrium force-loads at leaves are preserved by every $H\Phi$ -surgery.*
- (iii) *The resulting resolution diagram is strongly generic.*

Proof Item (i). By Proposition 7, since $(T, \mathcal{L})_p$ is strongly generic, the lines corresponding to the edges of $G_H \subset T$ are defined by forces with distinct lines of forces. This implies that $F_{3,1} + F_{2,5} \neq 0$ and thus ℓ is uniquely defined (where ℓ is the line of force for $F_{3,1} + F_{2,5}$). Therefore, $H\Phi$ -surgery is well-defined.

Item (ii) follows directly from the definition of $H\Phi$ -surgery.

Item (iii). First, let us show that the resulting resolution diagram is weakly generic. Assume the converse. Suppose the resulting resolution diagram is not strongly generic. Then there exist two adjacent edges in it such that we have $\mathcal{L}(\nu_i; \nu_j) = \mathcal{L}(\nu_j; \nu_s)$. Denote by Q_i and Q_s the subsets of all leaves of T in the components of $T \setminus \nu_j$ containing ν_i and ν_s respectively. It is clear that the sum of all the forces at the leaves of Q_i has the line of force $\mathcal{L}(\nu_i; \nu_j)$, and the sum of forces at the leaves of Q_s has the same line of force $\mathcal{L}(\nu_j; \nu_s) = \mathcal{L}(\nu_i; \nu_j)$. Therefore, the second condition of strong genericity for the resulting resolution

diagram at leaves of T is not fulfilled (by Definition 25 and Remark 16). It remains to say that the $H\Phi$ -surgery does not change the forces on the leaves. Hence the second condition of strong genericity for the original resolution diagram $(T, \mathcal{L})_p$ is not fulfilled. Therefore, $(T, \mathcal{L})_p$ is not strongly generic. We come to the contradiction.

Hence the resulting resolution diagram is weakly generic. Since the forces at leaves before and after the $H\Phi$ -surgery coincide, two conditions of strong genericity are fulfilled. Together with weak genericity this implies strong genericity of the resulting resolution diagram. \square

Definition 27 Consider two strongly generic resolution diagrams at a common point p . We say that these resolution diagrams are *equivalent* if there exists a sequence of $H\Phi$ -surgeries taking one of them to another.

4.4 Finiteness of equivalent resolution diagrams

Finiteness of equivalent resolution diagrams follows directly from the following statement.

Proposition 9 *Let $(T, \mathcal{L})_p$ be a strongly generic resolution diagram (where T has n leaves). Then the set of resolution diagrams equivalent to $(T, \mathcal{L})_p$ is in one-to-one correspondence with the set of all unrooted binary full trees with n marked leaves.*

Proof By Proposition 8(ii) the set of restrictions of equilibrium force-loads to the leaves are the same for equivalent resolution diagrams. By Proposition 6(i) there exists a nonzero equilibrium force-load F on $(T, \mathcal{L})_p$, hence the set of restrictions contains nonzero elements. By Corollary 1 the equilibrium force-load on leaves and the type of the tree uniquely defines the resolution diagram. Hence for every unrooted binary full tree T' with n leaves there exists at most one resolution diagram equivalent to $(T, \mathcal{L})_p$. We have injectivity.

By Proposition 8(i) the $H\Phi$ -surgery at every interior edge of a strongly generic resolution diagram is well-defined, and by Proposition 8(iii) the image is also a strongly generic resolution diagram. Hence for every unrooted binary full tree T' with n leaves there exists a resolution diagram $(T', \mathcal{L}')_p$ equivalent to $(T, \mathcal{L})_p$. This implies surjectivity. \square

Remark 18 Note that the number of rooted binary full trees with n labeled leaves is precisely $(2n - 5)!!$ (e.g., see ex. 5.2.6 in [29]). Here the root may be considered as a vertex with an additional leaf with “empty” color. Hence the number of unrooted binary full trees is $(2n - 5)!!$ as well.

5 Quantization of a graph

The main goal of this section is to prove a necessary and sufficient condition of existence of non-parallelizable tensegrities in terms of quantizations.

We start in Subsection 5.1 with the notion of graph quantizations. Further in Subsection 5.2 we introduce framed cycles of frameworks related to quantizations. We generalize the notion of tensegrity to the case of quantizations in Subsection 5.3. In Subsection 5.4 we study force-loads for framed cycles of quantizations. Finally, in Subsection 5.5 we formulate and prove a necessary and sufficient condition for existence of non-parallelizable tensegrities (Theorem 3).

5.1 Definition of a quantization

The notion of quantizations is central for this paper, we essentially use quantizations in the proof of the main theorem. Roughly speaking quantization of a graph is another graph (with infinitesimal edges) whose all vertices are trivalent, all the edges of the original graph stays, while some “infinitesimal edges” are added.

The term quantization here is inspired by Feynman diagrams in the string field theory, where the generic case contain only 3-valent vertices describing three propagators merging together.

Let us start with the following general definition.

Definition 28 Consider a graph G on vertices v_1, \dots, v_n . Let \mathcal{T} be a disjoint collection of unrooted binary full trees T_1, \dots, T_n such that the leaves of the tree T_i are in one-to-one correspondence with the edges adjacent to v_i (where $i = 1, \dots, n$).

- Denote by $G_{\mathcal{T}}$ the graph obtained from \mathcal{T} by gluing together pairs of leaves of trees corresponding to the same edges of G . We say that $G_{\mathcal{T}}$ is a *resolution graph* for G with respect to \mathcal{T} .
- The tree T_i is a *resolution tree* for $G_{\mathcal{T}}$ at vertex v_i .
- We say that an edge of $G_{\mathcal{T}}$ is a *leaf* if it is obtained by gluing leaves in \mathcal{T} . All the other edges of $G_{\mathcal{T}}$ are said to be *interior* edges.

Now we are ready to define quantizations.

Definition 29 Consider a framework $G(P)$ for a graph G , and let \mathcal{T} be a collection of trees indexed by vertices of $G(P)$. A *quantization* $G_{\mathcal{T}}(\mathcal{L}, P)$ is a pair $(G_{\mathcal{T}}, \mathcal{L})$ where $G_{\mathcal{T}}$ is a resolution graph for G with respect to \mathcal{T} and

$$\mathcal{L} : E(G_{\mathcal{T}}) \rightarrow \text{Gr}(1, \mathbb{R}P^2)$$

such that the following two conditions are fulfilled.

- **Leaf condition:** If $e \in G_{\mathcal{T}}$ is a leaf corresponding to the edge $(p_i; p_j) \in G(P)$, then

$$\mathcal{L}(e) = p_i p_j.$$

If $p_i = p_j$, we let $\mathcal{L}(e)$ be any line passing through $p_i = p_j$.

- **Interior edge condition:** If $e \in G_{\mathcal{T}}$ is an interior edge corresponding to the point $p_i \in G(P)$, then

$$p_i \in \mathcal{L}(e).$$

Remark 19 The idea of quantization is similar to the idea of atlases for manifolds, where one has charts and maps between them. The only difference here is that in each “chart” of the atlas we consider a resolution diagram for a single vertex rather than this vertex itself.

Definition 30 Let $G_{\mathcal{T}}(\mathcal{L}, P)$ be a quantization of G . Consider a point $p_i \in P$ and the corresponding tree $T_i \in \mathcal{T}$. Here we consider T_i as a tree naturally embedded in $G_{\mathcal{T}}$. Denote by $\mathcal{L}|_{T_i}$ the restriction of \mathcal{L} to T_i . We say that $(T_i, \mathcal{L}|_{T_i})_{p_i}$ is the *resolution diagram of the quantization $G_{\mathcal{T}}(\mathcal{L}, P)$ at vertex p_i* .

Let us continue with a particular example.

Example 6 In Figure 9 we consider a framework $G(P)$, a collection of resolution trees \mathcal{T} , and the corresponding quantization $G_{\mathcal{T}}(\mathcal{L}, P)$. The values of the function \mathcal{L} at edges of the resolution diagrams and the quantization are shown on the edges.

Further we will use the following notion of genericity for quantizations.

Definition 31 A quantization is called *generic* if all its resolution diagrams are strongly generic.

Remark 20 A collection of strongly generic resolution diagrams uniquely defines the corresponding quantization.

5.2 Framed cycles associated to quantizations

First, we define the framing for two adjacent edges of a frameworks. It will be used further in Definition 41.

Definition 32 Let $G(P)$ be a framework in general position and let $G_{\mathcal{T}}(\mathcal{L}, P)$ be a generic quantization. Consider two edges e_{ij} and e_{ik} in $G(P)$ with a common vertex p_i , let also $(T_i, \mathcal{L})_{p_i}$ be the resolution diagram at p_i . Consider a resolution diagram $(T'_i, \mathcal{L}')_{p_i}$ equivalent to $(T_i, \mathcal{L})_{p_i}$ such that the leaves corresponding to e_{ij} and e_{ik} are adjacent to the same vertex ν in T'_i , and let e be the third edge adjacent to ν . We say that the line $\mathcal{L}'(e)$ is the *associated framing for the pair of edges (e_{ij}, e_{ik}) at p_i* and denote it by ℓ_{jik} .

The above definition leads to the natural notion of framed cycles associated to quantizations.

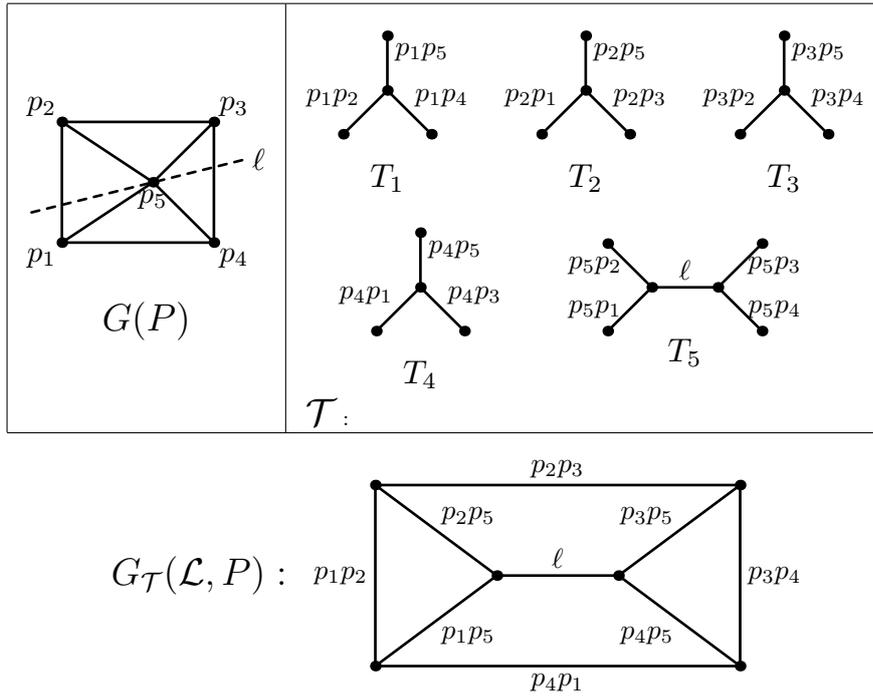


Fig. 9 A framework $G(P)$, a collection of resolution diagrams \mathcal{T} and the corresponding quantization $G_{\mathcal{T}}(\mathcal{L}, P)$.

Definition 33 Let $G(P)$ be a framework in general position and let $G_{\mathcal{T}}(\mathcal{L}, P)$ be a generic quantization. Consider a simple cycle $C = q_1 \dots q_s$ in $G(P)$ that does not contain all vertices of P . Denote by $C(G, \mathcal{T}, P, \mathcal{L})$ the framed cycle with consecutive vertices $q_1 \dots q_s$ and framing lines $\ell_{i-1, i, i+1}$ at q_i (for $i = 1, \dots, s$). We say that this cycle is a *framed cycle associated to quantizations*.

We should mention the following basic property of the framings.

Proposition 10 *The framing ℓ_{jik} as in Definition 32 does not depend on the choice of the equivalent resolution diagram $(T'_i, \mathcal{L}')_{p_i}$ with adjacent leaves $\mathcal{L}'^{-1}(e_{ij})$ and $\mathcal{L}'^{-1}(e_{ik})$.*

Proof Let $(T'_i, \mathcal{L}')_{p_i}$ and $(T''_i, \mathcal{L}'')_{p_i}$ be two equivalent resolution diagrams satisfying the condition of the proposition. Let also e' and e'' be the edges in T'_i and T''_i respectively adjacent to both leaves representing e_{ij} and e_{ik} in T'_i and T''_i . By Proposition 6 since $(T'_i, \mathcal{L}')_{p_i}$ is strongly generic (and hence weakly generic), $(T'_i, \mathcal{L}')_{p_i}$ has a nonzero equilibrium force-load, denote it by F' . Since $(T'_i, \mathcal{L}')_{p_i}$ and $(T''_i, \mathcal{L}'')_{p_i}$ are equivalent, we can connect them by a sequence of $H\Phi$ -surgeries. By Proposition 8(ii) any sequence of $H\Phi$ -surgeries does not change the equilibrium force-loads on leaves. Therefore, $(T''_i, \mathcal{L}'')_{p_i}$ has a force-load F'' whose forces at leaves coincide with the forces of F' . Therefore, the

forces of F' and F'' coincide on the edges e' and e'' as well (since both F' and F'' are equilibrium force-loads). By Proposition 6, since $(T'_i, \mathcal{L}')_{p_i}$ is strongly generic, the force on the edge e' is nonzero, and hence the line of force is the same. Therefore, the line ℓ_{jik} is uniquely defined. \square

Remark 21 In order to construct a resolution diagram $(T'_i, \mathcal{L}')_{p_i}$ starting from $(T_i, \mathcal{L})_{p_i}$ we propose the following. Let $\nu_1 \dots \nu_s$ be a simple path connecting the leaf $(\nu_1; \nu_2)$ where $\mathcal{L}(\nu_1; \nu_2) = p_i p_j$ and the leaf $(\nu_{s-1}; \nu_s)$ where $\mathcal{L}(\nu_{s-1}; \nu_s) = p_i p_k$. Then we consequently perform $s-3$ $H\Phi$ -surgeries along the edges $(\nu_2; \nu_3)$, $(\nu_3; \nu_4), \dots, (\nu_{s-2}; \nu_{s-1})$. As a result we have a resolution diagram $(T'_i, \mathcal{L}')_{p_i}$ whose leaves $\mathcal{L}'^{-1}(e_{ij})$ and $\mathcal{L}'^{-1}(e_{ik})$ have a common vertex.

5.3 Tensegrities on quantizations

Let us extend the definition of tensegrity to the case of quantizations.

Definition 34 Consider a quantization $G_{\mathcal{T}}(\mathcal{L}, P)$.

- A *stress* on an edge $(\nu_i; \nu_j) \in G_{\mathcal{T}}$ is an assignment of two forces $F_{i,j}$ and $F_{j,i}$ whose line of forces coincide with the line $\mathcal{L}(\nu_i; \nu_j)$ and such that $F_{i,j} + F_{j,i} = 0$.
- A *force-load* and an *equilibrium force-load* are set as in Definition 2.
- A force-load \hat{F} on $G(P)$ is called a force-load *induced* by F on the quantization $G_{\mathcal{T}}(\mathcal{L}, P)$ if for every leaf $(\nu_i; \nu_j)$ the stress on $(\nu_i; \nu_j)$ for F coincides with the stress on the corresponding edge of $G(P)$ for \hat{F} .

We have the following natural property of the induced force-loads.

Proposition 11 *If F is an equilibrium force-load on a quantization $G_{\mathcal{T}}(\mathcal{L}, P)$ then the induced force-load \hat{F} on the framework $G(P)$ is also an equilibrium force-load.*

Proof It is enough to check that at every vertex p_i the forces of the framework sum up to zero. This follows from the fact that the sum of all forces on the leaves of any equilibrium force-load on a tree is always zero. \square

Further we define quantizations associated to non-parallelizable equilibrium force-loads on frameworks.

Definition 35 Let F be a non-parallelizable equilibrium force-load F on a framework $G(P)$ and let $G_{\mathcal{T}}$ be a resolution graph for G . Consider

$$\mathcal{L}_F : E(G_{\mathcal{T}}) \rightarrow \text{Gr}(1, \mathbb{R}P^2)$$

defined as follows. Consider an edge $e \in G_{\mathcal{T}}$. Then there exists i such that $e \in T_i$ for $T_i \in \mathcal{T}$. Let $F_{i_1, j_1}, \dots, F_{i_s, j_s}$ be the forces at all leaves of one of the connected components of $T_i \setminus e$ (we assume that the second indices for the forces correspond to 1-valent vertices). Set $\mathcal{L}_F(e)$ as the line of force for

$$F_{i_1, j_1} + \dots + F_{i_s, j_s}.$$

The quantization $G_{\mathcal{T}}(\mathcal{L}_F, P)$ is called the *quantization associated to the pair* $(G(P), F)$.

We have the following simple statements for the associated quantizations.

Proposition 12 (i) *The quantization $G_{\mathcal{T}}(\mathcal{L}_F, P)$ is uniquely defined for every non-parallelizable equilibrium force-load F on $G(P)$.*

(ii) *If two non-parallelizable equilibrium force-loads on $G(P)$ are proportional then they define the same quantization for a given \mathcal{T} .* \square

5.4 Framed cycles of quantizations and force-loads on them

We start this subsection with several notions and definitions related to framed cycles in quantizations. Such cycles will be further used to write down geometric conditions for existence of nonzero tensegrities in quantization, basing on an important static property. We formulate and prove this statement later in this subsection.

5.4.1 Basic definition

Let us define cycle resolutions, their framings, and corresponding force-loads for the framed cycles.

Definition 36 Consider a graph G with a resolution graph $G_{\mathcal{T}}$. Let C be a simple cycle in G . Let a simple cycle (denote it by $C_{\mathcal{T}}$) in $G_{\mathcal{T}}$ satisfying the following two properties:

- $C_{\mathcal{T}}$ contains all the leaves corresponding to the edges of C ;
- $C_{\mathcal{T}}$ does not contain the leaves corresponding to the edges of $G \setminus C$.

Then the cycle $C_{\mathcal{T}}$ is called the *cycle resolution* of C .

Notice that the cycle $C_{\mathcal{T}}$ is uniquely defined by C .

Definition 37 Consider a cycle resolution $C_{\mathcal{T}}$ of C in a generic quantization $G_{\mathcal{T}}(\mathcal{L}, P)$. A *framed cycle* $C_{\mathcal{T}}(G, \mathcal{T}, P, \mathcal{L})$ is a cycle $C_{\mathcal{T}}$ with the property that each vertex ν_i in $C_{\mathcal{T}}$ is equipped with the line $\mathcal{L}(\nu_i; \nu_j)$, where $(\nu_i; \nu_j)$ is the only edge in $E(G_{\mathcal{T}}) \setminus E(C_{\mathcal{T}})$ adjacent to ν_i .

Definition 38 Let $C_{\mathcal{T}}(G, \mathcal{T}, P, \mathcal{L})$ be a framed cycle of a generic quantization $G_{\mathcal{T}}(\mathcal{L}, P)$.

- A *force-load* F on a framed cycle $C_{\mathcal{T}}(G, \mathcal{T}, P, \mathcal{L})$ is an assignment of
 - stresses $F_{i,i+1} = -F_{i+1,i}$ for each edge $(\nu_i; \nu_{i+1})$, whose lines of forces coincide with $\mathcal{L}(\nu_i; \nu_{i+1})$;
 - framing forces F_i (for $1 \leq i \leq k$), whose lines of forces coincide with the lines of the framing.

- A force-load F is said to be an *equilibrium* force-load on $C_{\mathcal{T}}(G, \mathcal{T}, P, \mathcal{L})$ if at every vertex ν_i we have:

$$F_{i,i-1} + F_{i,i+1} + F_i = 0.$$

- A force-load F is called an *almost equilibrium* force-load on $C_{\mathcal{T}}(G, \mathcal{T}, P, \mathcal{L})$ if the equilibrium condition is fulfilled at every vertex of $C_{\mathcal{T}}(G, \mathcal{T}, P, \mathcal{L})$ except one, say ν_j , and at ν_j we have either $F_{j,j-1} + F_j = 0$ or the line of force for $F_{j,j-1} + F_j$ is $\mathcal{L}(\nu_j; \nu_{j+1})$.

5.4.2 Almost equilibrium force-loads for consistent framed cycles

We continue with the definitions of consistency for framed cycles of frameworks and for frameworks themselves.

Definition 39 Let $G(P)$ be a framework in general position and let C be a simple cycle of G that does not contain all the vertices of G . A generic quantization of the framework $G(P)$ is said to be *consistent* at the cycle C if the framed cycle of $C(G, \mathcal{T}, P, \mathcal{L})$ satisfies the monodromy cycle condition (see Definition 17).

Note that the framed cycle of $C(G, \mathcal{T}, P, \mathcal{L})$ as in Definition 39 is always in general position.

Definition 40 A generic quantization of a framework $G(P)$ in general position is said to be *consistent* if it is consistent at each simple cycle that does not contain all the vertices of P .

Consistent framed cycles of a framework satisfy the following property.

Proposition 13 *Let $G(P)$ be a framework in general position and let $G_{\mathcal{T}}(\mathcal{L}, P)$ be a generic quantization of $G(P)$. Consider a framed simple cycle $C(G, \mathcal{T}, P, \mathcal{L})$ of $G(P)$ that does not contain all the vertices of $G(P)$. Assume that C is consistent. Then any almost equilibrium force-load at $C_{\mathcal{T}}(G, \mathcal{T}, P, \mathcal{L})$ is an equilibrium force-load.*

This proposition is used further in the proof of Theorem 3.

Proof Suppose that a cycle $C_{\mathcal{T}}(G, \mathcal{T}, P, \mathcal{L})$ has interior edges of resolution diagrams. Then we remove such edges by applying $H\Phi$ -surgeries at these edges. By Proposition 2 $H\Phi$ -surgeries do not change the property of a cycle to have an (almost) equilibrium nonzero force-load.

Now the question is reduced to the framed cycle of $C_{\tilde{\mathcal{T}}}(G, \tilde{\mathcal{T}}, P, \mathcal{L})$ whose edges are all leaves. In this case equilibrium (almost equilibrium) force-loads for the framed cycle $C_{\tilde{\mathcal{T}}}(G, \tilde{\mathcal{T}}, P, \mathcal{L})$ coincide with equilibrium (almost equilibrium) force-loads for the framed cycle

$$C(G, \tilde{\mathcal{T}}, P, \mathcal{L}) = C(G, \mathcal{T}, P, \mathcal{L})$$

of $G(P)$. By Theorem 2, since $C(G, \mathcal{T}, P, \mathcal{L})$ is consistent (i.e., it has a trivial monodromy), $C(G, \mathcal{T}, P, \mathcal{L})$ has a nonzero equilibrium force-load. Now the statement of this proposition follows directly from Proposition 4. \square

5.5 A necessary and sufficient condition for a tensegrity with nonzero stresses

The following statement is one of the main ingredient for the proof of Theorem 1.

Theorem 3 *A framework in a general position admits a non-parallelizable tensegrity if and only if there exists a consistent generic quantization of the framework.*

Remark 22 The choice of trees in the resolution diagrams of the quantization does not change the set of frameworks that admit nonzero tensegrities. However an appropriate choice can simplify the expressions for systems of geometric equations defining this set (for further details see in Subsection 6.1 below).

Remark 23 The monodromy operators are multiplicative with respect to cycle addition, in particular the monodromies do not depend on how one adds cycles. Hence the monodromies are well-defined on elements of the first homology group of the graph G (i.e., on $H_1(G)$).

Proof of sufficiency. Suppose that there exists a non-parallelizable tensegrity $(G(P), F)$ (note that this tensegrity has nonzero stresses at all edges). By Proposition 12(i) the tensegrity $(G(P), F)$, together with a collection \mathcal{T} of resolution trees, defines a quantization $G_{\mathcal{T}}(\mathcal{L}_F, P)$. Let us show that $G_{\mathcal{T}}(\mathcal{L}_F, P)$ is consistent.

First of all, let us show that $G_{\mathcal{T}}(\mathcal{L}_F, P)$ is generic. This is equivalent to the fact that every resolution diagram (T_i, \mathcal{L}_F) of this quantization is strongly generic. The last directly follows from non-parallelizability of the force F at every point p_i .

Secondly, we show that every simple framed cycle $C(G, \mathcal{T}, P, \mathcal{L}_F)$ associated to the quantization and not passing through all vertices of G is in general position. Since $G(P)$ is in general position, and the cycle does not pass through all edges of G the cycle $C(P)$ is in general position. Now let us check the conditions for framed lines at vertices. Let p_{i-1} , p_i , and p_{i+1} be arbitrary consecutive vertices of the cycle $C(P)$ and let $\ell_{i-1, i, i+1}$ be the corresponding framing at p_i . Then the direction of the line $\ell_{i-1, i, i+1}$ is defined by the sum of two forces representing edges $(p_i; p_{i-1})$ and $(p_i; p_{i+1})$. Since F is non-parallelizable, both forces are nonzero and the edges are non-parallel. Hence the line $\ell_{i-1, i, i+1}$ contains neither p_{i-1} nor p_{i+1} . Therefore, by Definition 14 the cycle $C(G, \mathcal{T}, P, \mathcal{L}_F)$ is in general position.

Assume that a simple framed cycle $C(G, \mathcal{T}, P, \mathcal{L}_F)$ does not pass through all vertices of G . Let us prove that it has trivial monodromies. Consider a force-load \tilde{F} on the cycle $C(G, \mathcal{T}, P, \mathcal{L}_F)$ defined as follows:

- The stresses at the edges of the cycle $C(G, \mathcal{T}, P, \mathcal{L}_F)$ coincide with the stresses at the edges of the tensegrity $(G(P), F)$;
- The force at the framed line at vertex p_i coincides with the sum of all the forces at the adjacent edges to p_i in $G \setminus C$.

By Definition 35, the line of the framing at vertex p_i coincides with the line of the sum of the forces at the adjacent edges at p_i , hence \tilde{F} is a force-load on $C(G, \mathcal{T}, P, \mathcal{L}_F)$ for all admissible i . Since F is an equilibrium force-load for $G(P)$, an equilibrium condition for \tilde{F} at each vertex of $C(G, \mathcal{T}, P, \mathcal{L}_F)$ is fulfilled. Thus \tilde{F} is a nonzero equilibrium force-load on $C(G, \mathcal{T}, P, \mathcal{L}_F)$. As we have shown above, the cycle $C(G, \mathcal{T}, P, \mathcal{L}_F)$ is in general position. Therefore, by Theorem 2 for any generic line ℓ (as in Theorem 2) the monodromies of the cycle are trivial. Hence by Definition 39 the cycle $C(G, \mathcal{T}, P, \mathcal{L}_F)$ satisfies the monodromy cycle condition.

Therefore, $G_{\mathcal{T}}(\mathcal{L}_F, P)$ is consistent.

Proof of necessity. Suppose now that $G_{\mathcal{T}}(\mathcal{L}, P)$ is a consistent generic quantization of $G(P)$. Let us construct an equilibrium force-load by induction on vertices of the resolution graph $G_{\mathcal{T}}$. We construct a sequence

$$((G_0, F), (G_1, F), \dots, (G_N, F)),$$

where G_i is a collection of vertices and edges of $G_{\mathcal{T}}$ and F is an assignment of forces to all edges of G_i . In addition this collection satisfies all the following properties:

- we have $G_0 \subset G_1 \subset \dots \subset G_N$;
- for every $i < N$ the set G_{i+1} has exactly one vertex more than G_i ;
- each vertex of G_i is adjacent to exactly three edges of G_i ;
- each edge e of G_i is equipped with a nonzero stress whose direction coincides with $\mathcal{L}(e)$;
- at every edge $e \in G_i \cap G_j$ the forces of the force-loads (G_i, F) and (G_j, F) coincide;
- the forces at every vertex of G_i sum up to zero;
- the graph G_0 contains no vertices and one edge of $G_{\mathcal{T}}$;
- $G_N = G_{\mathcal{T}}$.

If such sequence is constructed, then F is an equilibrium force-load for $G_{\mathcal{T}}(\mathcal{L}, P)$. And therefore the induced force-load \tilde{F} is an equilibrium force-load for $G(P)$.

Base of induction. Without loss of generality we assume that v_n is not a cut vertex of G (each graph has vertices that are not cut vertices). Let us start with any edge $(\nu_{i_0}; \nu_{j_0})$ of $G_{\mathcal{T}}(\mathcal{L}, P)$ where ν_{i_0} and ν_{j_0} do not represent v_n , and a nonzero stress $(F_{i_0, j_0}, F_{j_0, i_0})$ on it whose lines of force coincide with $\mathcal{L}(\nu_{i_0}; \nu_{j_0})$. We set this as (G_0, F) .

Step of induction. It is important to choose a correct order for adding new vertices. Denote by $V'(G_{\mathcal{T}})$ the set of all vertices of $G_{\mathcal{T}}$ that are not adjacent to interior edges of a resolution tree T_n for p_n corresponding to v_n where as usual $P = (p_1, \dots, p_n)$. In other words $V'(G_{\mathcal{T}})$ is the set of all the vertices that do not represent p_n in any resolution tree.

Suppose that we have already constructed the pair (G_s, F) (we assume that $s < N$). Let us construct (G_{s+1}, F) . If $V'(G_{\mathcal{T}})$ is not a subset of vertices of G_s we choose a new vertex ν_i from $V'(G_{\mathcal{T}})$ such that ν_i is adjacent to at

least one of the edges $(\nu_i; \nu_j) \in G_s$. It is always possible to do so, since v_n is not a cutting vertex. If $V'(G_{\mathcal{T}})$ is already a subset of vertices of G_s then we pick any remaining vertex v_i adjacent to at least one of the edges $(\nu_i; \nu_j) \in G_s$.

Let $(\nu_k; \nu_i)$ and $(\nu_l; \nu_i)$ be the remaining two edges adjacent to ν_i in the quantization, add them to $G_{s+1}(F)$ in case if they are not already in it. Note that ν_i is added to G_s when constructing G_{s+1} but the vertices ν_k and ν_l are not.

We keep all the stresses at all the edges of $G_s \subset G_{s+1}$. In particular the edge $(\nu_i; \nu_j)$ has a stress $(F_{i,j}, F_{j,i})$. Define the stresses $(F'_{i,k}, F'_{k,i})$ and $(F'_{i,l}, F'_{l,i})$ at $(\nu_i; \nu_k)$ and $(\nu_i; \nu_l)$ respectively using the following three conditions

- 1) $F_{i,j} + F'_{i,k} + F'_{i,l} = 0$;
- 2) $F'_{i,k} = -F'_{k,i}$ and $F'_{i,l} = -F'_{l,i}$;
- 3) the lines of forces for $F'_{i,k}$ and $F'_{i,l}$ coincide with the lines $\mathcal{L}(\nu_i; \nu_k)$ and $\mathcal{L}(\nu_i; \nu_l)$.

It is clear that $F'_{i,k}$ and $F'_{i,l}$ are uniquely defined by these three conditions. In the case if ν_k or ν_l are not in G_s we set respectively $F_{i,k} = F'_{i,k}$ or $F_{i,l} = F'_{i,l}$.

In the case if ν_k is in G_s , we have already defined another stress while adding the vertex ν_k . In this case G_{s+1} contains a simple framed cycle $C_{\mathcal{T}}$ with a consecutive triple of vertices $\nu_j \nu_i \nu_k$.

If $V'(G_{\mathcal{T}})$ is not a subset of vertices of G_s then by construction the set of vertices of G_s is a subset of $V'(G_{\mathcal{T}})$. Hence the framed cycle $C(G, \mathcal{T}, P, \mathcal{L})$ does not pass through p_n (since the vertices of $V'(G_{\mathcal{T}})$ are not represented by p_n). Hence $C(G, \mathcal{T}, P, \mathcal{L})$ is consistent. We set

$$\tilde{C}(G, \mathcal{T}, P, \mathcal{L}) = C(G, \mathcal{T}, P, \mathcal{L}).$$

Suppose now $V'(G_{\mathcal{T}})$ is already contained in G_s . Then it might happen that the cycle $C \subset G$ corresponding to $C_{\mathcal{T}} \subset G_{\mathcal{T}}$ runs through all vertices of G . Since all vertices of G are at least of degree 3, we have at least $\frac{3n}{2}$ edges. Therefore, if $n > 3$ then there is an edge e which is not passing through v_n . Using the edge e one can make another simple cycle \tilde{C} that has less than n vertices and such that $\tilde{C}_{\mathcal{T}} \cap T_n = C_{\mathcal{T}} \cap T_n$ (see an example on Figure 10). The obtained cycle \tilde{C} is consistent, since $G_{\mathcal{T}}(\mathcal{L}, P)$ is consistent.

By construction, the force-load F at $\tilde{C}_{\mathcal{T}}(G, \mathcal{T}, P, \mathcal{L})$ is almost equilibrium. Then by Proposition 13 the force-load F at $\tilde{C}_{\mathcal{T}}(G, \mathcal{T}, P, \mathcal{L})$ is an equilibrium force-load, since cycle $\tilde{C}(G, \mathcal{T}, P, \mathcal{L})$ is consistent.

Therefore, all forces of $G_{s+1}(F)$ sum up to zero at every vertex. All the other required conditions are fulfilled by construction. Hence we are done with the step of induction.

As the output of the inductive process described before we get a nonzero equilibrium force-load F at $G_N = G_{\mathcal{T}}$. Since the quantization is generic, the resulting equilibrium force-load is non-parallelizable. This concludes the proof of the necessary condition. \square

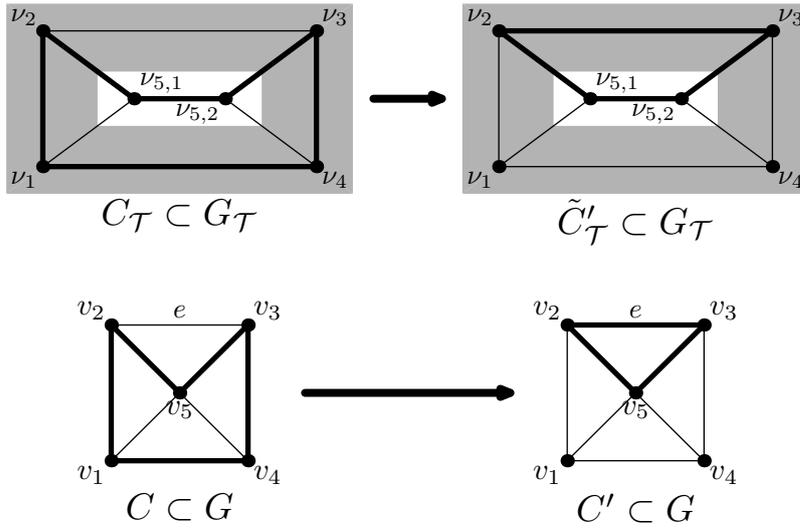


Fig. 10 Cycles $C, C' \subset G$ and the corresponding cycles $C_{\mathcal{T}}, C'_{\mathcal{T}} \subset G_{\mathcal{T}}$. Here $V'(G_{\mathcal{T}})$ is the intersection of gray regions with the vertices of $G_{\mathcal{T}}$.

6 Geometric conditions on realizability of tensegrities

In this section we study algorithmic questions related to explicit construction of the systems of geometric equations for tensegrities and prove Theorem 1.

6.1 Construction of geometric equations for systems defining tensegrities

Let us discuss here the main building blocks for geometric equations. We start with a natural correspondence between the elements of $\Xi_G(P)$ and quantizations of $G(P)$. Further we show how to construct all the lines in simple framed cycles starting from the lines of the quantizations. After that we give the algorithm to rewrite $H\Phi$ -surgeries on resolution diagrams in terms of Operations I–IV. Then we discuss how to express monodromy cycle conditions for framed cycles in general position in terms of Operations I–IV. Finally, we define geometric equations for the framed cycles associated to quantizations. The results of this subsection are used in the next two subsections.

6.1.1 A natural correspondence between the elements of $\Xi_G(P)$ and the set of all quantizations for $G(P)$

Consider a framework $G(P)$ and one of its quantization graphs $G_{\mathcal{T}}$. For every point $p_i \in P$ we enumerate the interior edges of $G_{\mathcal{T}}$ corresponding to the resolution diagram T_i as follows: $e_{i,1}, \dots, e_{i, \deg p_i - 3}$.

Once enumeration of edges of $G_{\mathcal{T}}$ is fixed, the choice of \mathcal{L} in a quantization uniquely defines an element of $\Xi_G(P)$. Namely we pick the non-fixed lines

according to the rule

$$\ell_{i,j} = \mathcal{L}(e_{i,j}),$$

where $1 \leq i \leq n$ and $1 \leq j \leq \deg p_i - 3$. It is clear that this correspondence is bijective.

6.1.2 Construction of lines in the framed cycles associated to quantizations in terms of elementary geometric operations

Consider an $\mathbb{H}\Phi$ -surgery on a resolution diagram, here we follow the notation of Definition 26. Let us show how to construct the new line $\ell = \mathcal{L}'(\nu'_1; \nu'_2)$ using Operations I–IV.

Step 1. First of all, let us choose an affine chart. Namely we pick a point $p_\infty \neq p$ on a line (say at $\mathcal{L}(\nu_1; \nu_2)$) and a line $\ell_\infty \neq \mathcal{L}(\nu_1; \nu_2)$ through p_∞ (which is a combination of Operations III and IV).

Step 2. Take an arbitrary point $p' \notin \{p, p_\infty\}$ in the affine chart on the line $\mathcal{L}(\nu_1; \nu_3)$ (Operation III).

Step 3. Consider the line $\hat{\ell}$ passing through p' and parallel to $\mathcal{L}(\nu_1; \nu_2)$. Namely,

$$\hat{\ell} = p'(\mathcal{L}(\nu_1; \nu_2) \cap \ell_\infty).$$

This is a combination of Operations I and II.

Step 4. Set

$$p'' = \hat{\ell} \cap \mathcal{L}(\nu_2; \nu_5).$$

Notice that $p'' \neq p$. Here we use Operation I.

Step 5. Draw lines ℓ' and ℓ'' through the points p' and p'' parallel to $\mathcal{L}(\nu_1; \nu_4)$ and $\mathcal{L}(\nu_2; \nu_6)$ respectively:

$$\begin{aligned} \ell' &= p'(\mathcal{L}(\nu_1; \nu_4) \cap \ell_\infty); \\ \ell'' &= p''(\mathcal{L}(\nu_2; \nu_6) \cap \ell_\infty); \end{aligned}$$

Here we have used two Operations I and two Operations II.

Step 6. Consider $p''' = \ell' \cap \ell''$ (Operation I), note that by construction we have $p''' \neq p$.

Step 7. Finally, we get $\ell = \mathcal{L}'(\nu'_1; \nu'_2) = pp'''$ (Operation II).

Remark 24 Any line ℓ_{jik} in the framing associated to a quantization $G_{\mathcal{T}}(\mathcal{L}, P)$ is defined by a certain composition of $\mathbb{H}\Phi$ -surgeries (see Definitions 32 and 27), and hence by the above it is defined by a composition of Operations I–IV on the lines of $\mathcal{L}(E(G_{\mathcal{T}}))$.

Definition 41 For simplicity we fix one of the choices of sequences of Operations I–IV defining the framing ℓ_{jik} and call it the *sequence of elementary geometric operations* defining the line ℓ_{jik} .

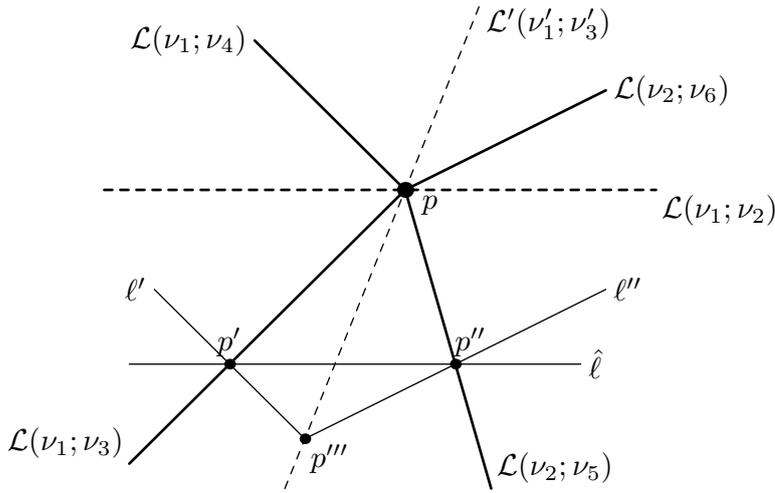
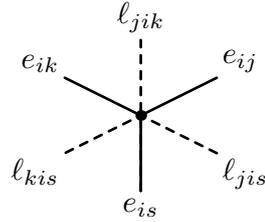
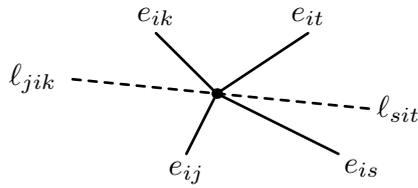


Fig. 11 Geometric construction of $H\Phi$ -surgery on a resolution diagram.

Remark 25 It is interesting to observe that if the degree of a vertex is 3, then the lines of the associated framing are defined by the edges. Namely, let p_i be a vertex of G of degree 3. Assume that p_i is adjacent to edges e_{ij} , e_{ik} , and e_{is} . Then we have $e_{ik} \subset \ell_{jis}$, $e_{ij} \subset \ell_{kis}$, and $e_{is} \subset \ell_{jik}$:



Another remark concerns a simple equation on the lines of the associated framing for vertices of degree 4. Let p_i be a vertex of G of degree 4. Assume p_i is adjacent to edges e_{ij} , e_{ik} , e_{is} , and e_{it} . Then $\ell_{jik} = \ell_{sit}$:



Similarly we get $\ell_{jit} = \ell_{kis}$ and $\ell_{jis} = \ell_{kit}$.

6.1.3 Construction of geometric equations for framed cycles in general position

Let $C((p_1, \dots, p_k), (\ell_1, \dots, \ell_k))$ be a framed cycle in general position. Let us iteratively apply $k-3$ times the projection operation defined in Subsection 3.3 replacing the first and the second vertices by a new one, i.e.,

$$\underbrace{\omega_1 \circ \dots \circ \omega_1}_{k-3 \text{ times}}(C(P, L)).$$

Each time the number of vertices in the cycle decreases by one. We end up with the framed cycle $((p, p_{k-1}, p_k), (\ell, \ell_{k-1}, \ell_k))$ in general position.

Remark 26 Note that ℓ is the result of the composition of 2-point and 2-line operations on points p_1, \dots, p_k and lines ℓ_1, \dots, ℓ_k arising in the above composition of projection operations. In fact, one can fix a different choice of the sequence of projection operations. There are $\frac{k!}{3!}$ different possibilities to do so. They all lead to equivalent geometric equations.

Definition 42 The geometric equation

$$\ell \cap \ell_{k-1} \cap \ell_k = \text{true}$$

is the *geometric equation associated with C* . (Here ℓ is constructed as above.)

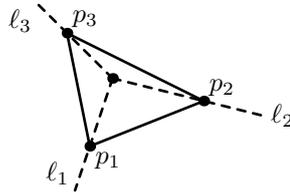
Proposition 14 Let $C(P, L)$ be a framed cycle in general position. Then $C(P, L)$ satisfies the monodromy cycle condition if and only if it satisfies any geometric equation defined by this cycle.

Proof The statement for triangular cycles follows directly from Lemma 1. The statement for framed cycles with 4 and more vertices is reduced to the triangular statement by projection operations. Equivalence of the initial and the reduced statements follows from Proposition 5 and the definition of the projection operation in terms of 2-point and 2-line operations (see Subsection 3.3). \square

Example 7 If a framed cycle has three vertices then its associated geometric equation is equivalent to

$$\ell_1 \cap \ell_2 \cap \ell_3 = \text{true}.$$

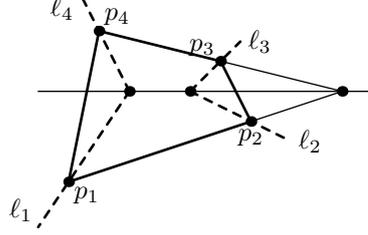
The corresponding sketch is as follows:



In the case of four cycles we have:

$$(\ell_1 \cap \ell_4, \ell_2 \cap \ell_3, p_1 p_2 \cap p_3 p_4) = \text{true},$$

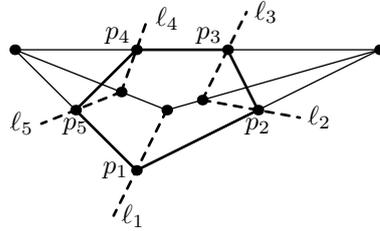
the sketch is as follows:



Finally, for five vertex cycles we have:

$$(\ell_2 \cap \ell_3, p_1 p_2 \cap p_3 p_4) \cap \ell_1 \cap (\ell_4 \cap \ell_5, p_1 p_3 \cap p_3 p_4) = \text{true},$$

the sketch is as follows:



Notice that if the number of vertices is greater than 3, then there are several different choices of equivalent geometric equations.

6.1.4 Geometric equations for framed cycles associated to quantizations

Finally, combining together the above two constructions we have the following descriptive definition.

Definition 43 Assume we are given by the following data.

- Let $G(P)$ be a framework.
- Let $G_{\mathcal{T}}(\mathcal{L}, P)$ be a generic quantization for $G(P)$.
- Let C be a simple cycle of $G(P)$ that does not contain all vertices of P .
- Consider a geometric equation for the cycle $C(G, \mathcal{T}, P, \mathcal{L})$ as in Definition 42.
- Consider the expressions for the framed lines of the cycle $C(G, \mathcal{T}, P, \mathcal{L})$ in terms of points of P and lines of $\mathcal{L}(E(G_{\mathcal{T}}))$ in Definition 41.

By Subsection 6.1.1 the lines of $\mathcal{L}(E(G_{\mathcal{T}}))$ are identified with the non-fixed lines of $\Xi_G(P)$. So we end up with a geometric equation on the points and the lines of $\Xi_G(P)$. We say that this equation is the *geometric equation on $\Xi_G(P)$* for the framed cycle $C(G, \mathcal{T}, P, \mathcal{L})$ in the quantization $G_{\mathcal{T}}(\mathcal{L}, P)$.

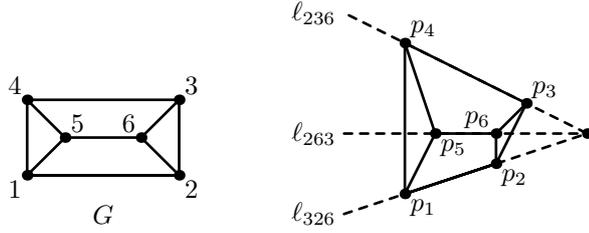


Fig. 12 The graph G (on the left) and the consistency condition for the cycle $p_2p_3p_6$ (on the right).

Remark 27 In fact the geometric equation on $\Xi_G(P)$ depends neither on the choice of configuration P nor on the choice of lines in the configuration. Although it is defined using quantizations, it is a characteristic of the graph G itself.

Let us prove the following property for geometric equations on $\Xi_G(P)$. It will be used in the proof of our main result (Theorem 1).

Corollary 2 *Let $G(P)$ be a framework in general position, let $G_{\mathcal{T}}(\mathcal{L}, P)$ be a generic quantization for $G(P)$, and let C be a simple cycle of $G(P)$ that does not contain all the vertices of P . Then the quantization $G_{\mathcal{T}}(\mathcal{L}, P)$ is consistent at C if and only if the quantization satisfies the geometric equation on $\Xi_G(P)$ for the cycle $C(G, \mathcal{T}, P, \mathcal{L})$ with non-fixed lines identified with the lines associated to the quantization.*

Proof By Definition 39 the cycle C is consistent if and only if $C(G, \mathcal{T}, P, \mathcal{L})$ satisfies the monodromy cycle condition. Since $G(P)$ is a framework in general position and $G_{\mathcal{T}}(\mathcal{L}, P)$ is its generic quantization, the framed cycle $C(G, \mathcal{T}, P, \mathcal{L})$ is in general position. By Proposition 14 the cycle $C(G, \mathcal{T}, P, \mathcal{L})$ satisfies the monodromy cycle condition if and only if it satisfies the geometric equation defined by this cycle. Hence, by Definition 43 the last is equivalent to the fact that the cycle $C(G, \mathcal{T}, P, \mathcal{L})$ satisfies the geometric equation on $\Xi_G(P)$. \square

Example 8 Let us consider a simple example related to the Desargues configuration as in Example 1 (see Figure 12, on the left). Consider the following graph and a cycle $p_2p_3p_6$ in it. Here the cycle $p_2p_3p_6$ is consistent if

$$\ell_{326} \cap \ell_{263} \cap \ell_{236} = \text{true},$$

see Figure 12 (on the right). According to Remark 25 the line $\ell_{326} = p_1p_2$; $\ell_{263} = p_5p_6$; and $\ell_{236} = p_3p_4$. Hence the above condition coincides with

$$p_1p_2 \cap p_5p_6 \cap p_3p_4 = \text{true}.$$

6.2 Proof of Theorem 1

By Theorem 3 a framework $G(P)$ in general position admits a non-parallelizable tensegrity if and only if there exists a consistent generic quantization of the framework. By Definition 40 a quantization is consistent if and only if it is consistent at each simple cycle that does not contain all vertices of G . By Corollary 2 a generic quantization is consistent at a simple cycle if and only if the associated geometric equation on $\Xi_G(P)$ for this cycle is satisfied. This concludes the proof. \square

Remark 28 Quantization of a graph is an auxiliary tool to get the geometric equations. Once the geometric equations are constructed, the quantization is no longer needed.

6.3 Techniques to construct geometric equations defining tensegrities

In this subsection we give a summary of the algorithm to write down the system of geometric equations for the existence of non-parallelizable tensegrities.

Data. We start with a framework $G(P)$ in general position.

Step 1. Fix a collection of resolution diagrams \mathcal{T} and the corresponding configuration space $\Xi_G(P)$. Here we consider a family of quantizations $G_{\mathcal{T}}(\mathcal{L}, P)$ with fixed $G_{\mathcal{T}}$ and P , and with the lines of \mathcal{L} acting as parameters.

Step 2. Pick all simple cycles C_1, \dots, C_N in G that do not pass through all the points of G .

Step 3. Using the algorithm described in Subsection 6.1.2 we write all lines ℓ_{jik} in terms of compositions of Operations I–IV on the points of P and the lines of $\mathcal{L}(E(G))$. (See Definition 41.)

Step 4. Further, we use the construction of Subsection 6.1.3 to write down geometric equations for the framed cycles $C_i(G, \mathcal{T}, P, \mathcal{L})$ for $i = 1, \dots, N$ that does not contain all vertices of P . Recall that the lines of \mathcal{L} here play the role of variables.

Step 5. Combining together Steps 3 and 4 we get the geometric equations for the framed cycles $C_i(G, \mathcal{T}, P, \mathcal{L})$ (where $i = 1, \dots, N$) in terms of compositions of Operations I–IV on the points of P and the lines of $\mathcal{L}(E(G))$.

Step 6. Finally, we write down the geometric equations obtained in Step 5 in terms of the corresponding configuration space $\Xi_G(P)$ (following Subsection 6.1.1).

Output. As an output we get the system of geometric equations on the space $\Xi_G(P)$ for an arbitrary P (and hence on Ξ_G). By Theorem 1 this system is fulfilled if and only if there exists a non-parallelizable tensegrity at $G(P)$.

Remark 29 Since monodromies are multiplicative with respect to cycle addition, at Step 2 it is sufficient to pick only the simple cycles generating $H_1(G)$. In practice even less cycles are usually needed for the complete output.

7 Conjecture on systems of strong geometric equations for tensegrities

In Subsection 7.1 we define systems of strong geometric equations, show several examples and formulate a conjecture for tensegrities. Further, in Subsection 7.2 we say a few words regarding this conjecture for the case of graphs having less than 11 vertices.

7.1 Some examples and the conjecture formulation

First, we define strong geometric equations.

Definition 44 We say that a system of geometric equations (or a single geometric equation) on a configuration space $(P, L|R_S)$ is *strong* if it does not depend on lines of L and it does not include any point or line obtained by Operations III and IV.

In order to construct a strong geometric equation one uses only Operations I and II. Therefore, any strong geometric equation is represented by some Cayley algebra expression (see Remark 6).

Remark 30 Every strong geometric equation can be represented by some 3-point equation. It is always possible to do so, since every line in the equation is defined by a pair of already constructed points (including the points of P). So if $\ell = pq$ then

$$\ell \cap \ell_2 \cap \ell_3 = true \quad \Leftrightarrow \quad (p, q, \ell_2 \cap \ell_3) = true,$$

and

$$p_2 \in \ell = true \quad \Leftrightarrow \quad (p_2, p, q) = true.$$

We continue with several examples, showing difference between systems of geometric equations and systems of strong geometric equations.

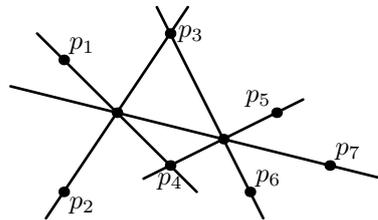
Example 9 Let us start with an example of a strong geometric equation. Let

$$(P, L|R_S) = ((p_1, p_2, \dots, p_7), () | \quad)$$

(here the list of lines L is empty). Consider the geometric equation:

$$(p_1p_4 \cap p_2p_3, p_3p_6 \cap p_4p_5, p_7) = true.$$

This geometric equation is a strong geometric equation. Here is a configuration of seven points satisfying this geometric equation:

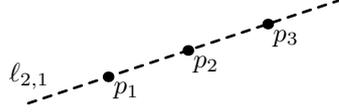


Example 10 In the second example we consider a system of geometric equations which, in fact, can be rewritten as a strong geometric equation. Let

$$(P, L|RS) = ((p_1, p_2, p_3), (\ell_{2,1}) | p_2 \in \ell_{2,1}).$$

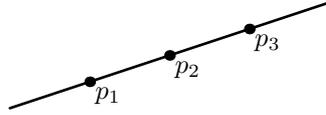
Then the system of geometric equations

$$\begin{cases} p_2 \in \ell_{2,1} = true \\ p_3 \in \ell_{2,1} = true. \end{cases}$$



(here the dashed line is the non-fixed line) is equivalent to the strong geometric equation

$$(p_1, p_2, p_3) = true.$$



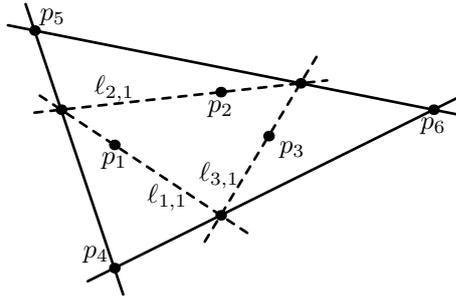
Example 11 In the last example here we consider a geometric equation which cannot be rewritten as a strong geometric condition. Let

$$(P, L|RS) = ((p_1, p_2, p_3, p_4, p_5, p_6), (\ell_{1,1}, \ell_{2,1}, \ell_{3,1}) | p_1 \in \ell_{1,1}, p_2 \in \ell_{2,1}, p_3 \in \ell_{3,1}).$$

The system of geometric equations

$$\begin{cases} \ell_{1,1} \cap \ell_{2,1} \cap p_4 p_5 = true \\ \ell_{2,1} \cap \ell_{3,1} \cap p_5 p_6 = true \\ \ell_{3,1} \cap \ell_{1,1} \cap p_6 p_4 = true \end{cases}$$

is not equivalent to any system of strong geometric equations on P . The reason for that is that certain initial generic point configurations P admit no solutions while the others generic ones admit two solutions. The set of configurations admitting solutions is not algebraic (it is semi-algebraic), while the solutions of any system of strong geometric equations is an algebraic set. Thus this system cannot be rewritten as a system of strongly geometric equations (we leave the details as an exercise for the reader). Below is the example of a configuration of nine points and three lines satisfying the above system of equations:



(here the dashed lines are the non-fixed lines).

Finally, let us formulate the conjecture on strong geometric descriptions of tensegrities (this is a slightly modified version of Conjecture 2 of [7]).

Conjecture 1 For every graph G there exists a system of strong geometric equations such that a framework $G(P)$ in general position admits a non-parallelizable tensegrity if and only if P satisfies this system of strong geometric equations.

If this conjecture is true then every condition is expressible in terms of Cayley algebra operations on the vertices of frameworks (see Remark 6). The important additional problem here is to give a constructive description of such systems.

7.2 System of strong geometric equations for graphs on small number of vertices

In this subsection we test Conjecture 1 on graphs with n vertices and $2n - 3$ edges for $n < 10$.

Remark 31 It is believed that all geometric conditions for an arbitrary graph can be reduced to the ones coming from graphs with n vertices and $2n - 3$ edges. In fact, the most interesting connected graphs with n vertices and $2n - 3$ edges are Laman graphs. Each subset of $m \geq 2$ vertices of such graph span at most $2m - 3$ edges. One of the important properties of such graphs is that they can be embedded as pseudo-triangulations (Theorem 1.1 of [12]).

Let us continue with the following two observations.

Observation A. If all vertices of a cycle are of degree 3 then the geometric equation defined by this cycle is a strong geometric equation.

Observation B. Suppose all vertices of a cycle are of degree 3 except for one which is of degree greater than 3. Then one of the non-fixed lines of the vertex of degree greater than 3 is defined by a composition of 2-point and 2-line operators on vertices of the cycle. This means that a geometric equation for the graph with n vertices has a reduction to a system of geometric equations on a certain graph with $n - 1$ vertices. In addition if the graph on $n - 1$ vertices is defined by a system of strong geometric equations, the original graph is defined by a system of strong geometric equations as well.

Lemma 2 Consider a connected graph G whose vertices are of degree ≥ 3 , suppose that m of them are of degree > 3 while the rest are of degree 3. Let G have k vertices of degree 3 forming a connected component (denote it by G_0). If $k + 2 > m$ then G has a simple cycle with at most one vertex of degree > 3 .

Proof Let G_0 be a maximal connected component satisfying the conditions of the lemma. If G_0 has a cycle then by Observation A the geometric equation defined by this cycle is a strong geometric equation. Suppose now G_0 is a tree. Then it has precisely $k + 2$ edges connecting G_0 with $G \setminus G_0$. Since G_0 is maximal, there is no vertices of degree 3 of $G \setminus G_0$ connected by an edge with G_0 . Since the number of vertices of degree ≥ 3 is less than k , there are two edges connecting G_0 with the same vertex of degree ≥ 3 . Therefore, we have one cycle C with the property that every vertex in C except one has degree 3 in G . So we are in position to use Observation B. This concludes the proof of the lemma. \square

In our paper [7] we have found all the system of geometric equations for graphs on n vertices and $2n - 3$ edges for $n = 6, 7, 8$. It is interesting to admit that Lemma 2 and Observations A and B are sufficient to write down systems of strong geometric equations for all the graphs considered before and also for the new case of graphs on 9 vertices (and the corresponding examples of papers [9,10,33]). Here is a way to do this.

A general remark. Suppose a graph G has a vertex of degree 1 or 2. Then G admits a non-parallelizable tensegrity only if the framework is not in general position. We skip such cases.

Case $n < 6$. If $n < 6$ then $\frac{3}{2}n > 2n - 3$. Hence we always have some vertex of degree 1 or 2.

Case $n = 6$. We have 9 edges here. All cycles have all vertices of degree 3. Hence according to Observation A we have strong geometric equations for them.

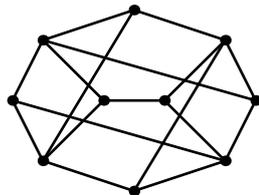
Case $n = 7$. We have 11 edges here. All cycles have all vertices of degree 3 except one which is of degree 4. Hence according to Observations A and B we have strong geometric equations for them (we either have a condition straight away or make a reduction to the case of $n = 6$).

Case $n = 8$. We have 13 edges here. Here we have at most two vertices of degree > 3 and at least 6 vertices of degree 3. Hence $k \geq 1$ and $m \leq 2$. Therefore by Lemma 2 we have either a strong geometric equation or a reduction to the case of $n = 7$.

Case $n = 9$. We have 15 edges here. If we have at most two vertices of degree > 3 then the situation is as in the case of $n = 8$. The only new case here is if we have three vertices of degree 4 and six vertices of degree 3. In this case there are at most 12 edges adjacent to vertices of degree 4. Hence there exists an edge that is adjacent to two vertices of degree 3. Hence we have a subtree with $k \geq 2$. Since $m = 3 < 4 \leq k + 2$ we can apply Lemma 2 and reduce this case to the case of $n = 8$.

We would like to conclude this paper with a small discussion on Conjecture 1. The main evidence that this conjecture is true was provided by numerous different examples confirming it. However it turns out that all the

examples examined satisfy the condition of Lemma 2. Meanwhile if the number of vertices is large, then the condition of Lemma 2 is not necessarily fulfilled. One might expect a large number of vertices of degree greater than 3, this can easily happen for graphs on n vertices and $2n - 3$ edges. The extremal example here is when we have $n - 6$ vertices of degree 4 and six vertices of degree 3. The case $n = 10$ already contains several non-equivalent (up to $H\Phi$ -surgeries) graphs for which the condition of Lemma 2 is not fulfilled. Here is an example of one of them:



For this graph we are no longer in position to apply Lemma 2, so this graph is a simplest candidate for a counterexample.

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