Controlling Lead Times and Minor Ordering Costs in the Joint Replenishment Problem with Stochastic Demands under the Class of Cyclic Policies

Abstract

In this paper, we consider the periodic review joint replenishment problem (JRP) under the class of cyclic policies. For each item, the demand in the protection interval is assumed stochastic. Moreover, a fraction of shortage is lost, while the other quota is backordered. We further suppose that lead times and minor ordering costs are controllable. These decision variables should not be neglected in the task of optimizing inventory systems, as their control may lead to substantial benefits. The problem concerns determining the cyclic replenishment policy, the lead times, and the minor ordering costs in order to minimize the long-run expected total cost per time unit. We established a number of properties of the cost function, which permit us to derive two heuristic algorithms. A lower bound on the minimum cost is obtained, which helps us evaluate the effectiveness of the proposed heuristics. Numerical experiments have shown that the first algorithm is more effective, although computationally more onerous. The second heuristic seems to be able to reach slightly worse solutions, but in a considerably less time.

Keywords: Inventory; Joint replenishment problem; Stochastic; Optimization; Heuristics; Lower bound

1. Introduction

In practical contexts, the coordination of replenishments among several items often leads to economic benefits. Examples may include the case of several products that are ordered from the same supplier or that are processed on the same piece of equipment (Nilsson & Silver, 2008). The problem related to the optimization of coordinated inventory replenishment policies among several items is typically referred to the joint replenishment problem (JRP).

The cost structure of the JRP is characterized by two ordering (or setup) cost components, i.e., a major ordering cost and a minor ordering cost. The major component is independent of the variety of items ordered and is borne at each replenishment cycle. The minor component depends on the variety of items ordered within a given ordering cycle. Because of the major cost component, group replenishments may lead to substantial cost savings, which potentially grows as the major ordering cost increases (Khouja & Goyal, 2008). Hence, in the JRP it is possible to exploit economies of scale (Kiesmüller, 2010).

The JRP has been widely studied in literature. The reader is referred to Khouja and Goyal (2008) for a review of papers published in the period 1989–2005. Since then more papers have been published. These works can be classified into two main groups, depending on whether the demand is supposed to be stochastic or not: deterministic JRP and stochastic JRP.

With regard to the first group, we can cite, for example, the following contributions. Porras and Dekker (2006) derived an exact solution method to approach the basic JRP problem with a minimum order quantity constraint. Under the assumption that shortages are not allowed, Hoque (2006) developed an exact algorithm for the problem with limited storage and transport capacities, and a budget constraint. Bayindir et al. (2006) considered a production environment with variable production costs, in the hypothesis that shortages are fully backordered. Robinson et al. (2007) presented several alternative heuristics to solve the JRP with dynamic demand. Olsen (2008) extended the basic model to consider interdependent minor ordering costs, and then solved the problem with an evolutionary algorithm. Moon et al. (2008) integrated the basic formulation with the supplier selection problem, taking into account quantity discounts. Narayanan and Robinson (2010a) presented two heuristic algorithms to solve the capacitated, dynamic lot-sizing JRP. Tsao and Sheen (2012) studied a two-actor, multi-item supply chain with a credit period and weight freight cost discounts. Zhang et al. (2012) developed a JRP model with complete backordering, where demands of some minor items are correlated with that of a major item. Amaya et al. (2013) presented a new heuristic approach based on linear programming to solve the JRP with resource constraints. Tsao and Teng (2013) developed two heuristic algorithms to solve the JRP with trade credit. Wang et al. (2012a; 2012b; 2013) used artificial intelligence techniques to approach the deterministic JRP. Paul et al. (2014) studied a JRP model that takes into account a random percentage of defective units in each replenishment and price discounts. Konur and Schaefer (2016) extended the basic problem by adding a second objective function that is related to environmental performance. Mohammaditabar and Ghodsypour (2016) proposed an integrated joint replenishment and supplier selection problem with a capacity constraint, which is approached using the direct grouping strategy.

In the stochastic JRP group, we can include the following works. Tsai et al. (2009) presented an approach to cluster items according to the correlation between their demands. Items are then managed using the can-order policy. Kiesmüller (2010) compared two different continuous review policies assuming that each item demand follows a compound renewal process, and that the total amount of products to be ordered is constrained. Narayanan and Robinson (2010b) carried out a study to evaluate the performance of nine lot-sizing heuristics in a dynamic rolling horizon system, where demands are Gaussian. Tanrikulu et al. (2010) developed a continuous review policy that takes into account a limited transportation capacity, assuming that each item demand follows a Poisson process. Feng et al. (2015) analysed a discounted cost model in which demands are correlated. Qu et al. (2015) approached the location-inventory problem under two different strategies, i.e., coordinated and independent replenishments, under the assumption that shortages are fully backordered. Finally, Braglia et al. (2016a; 2016b; 2017) studied various extensions of the JRP to take into account different aspects, such as backorders-lost sales mixtures, controllable lead time, investments to reduce the major ordering cost, or adopting distribution-free approach.

In classical production/inventory models, such as the economic production quantity (EPQ) and the economic order quantity (EOQ) models, the lead time and ordering/setup cost are considered as constants, i.e., as "givens". However, Silver (1992) recommended to treat the "givens" as decision variables, rather than as fixed parameters, so as to represent more realistic problem formulations. He also presented two numerical examples to show the benefits of changing a "given". We will explore this idea in the context of stochastic JRP.

In many practical circumstances, lead time can be shortened at the expense of an additional cost. In other words, lead time can be controllable. The extra cost of reducing lead time can be ascribed, for example, to administrative, transportation and supplier's speed-up costs (Huang, 2001). The just-in-time (JIT) philosophy suggests that, if lead time is reduced, several benefits can be achieved, such as lower investment in inventory, better product quality, less scrap, reduced storage space requirements, higher flexibility, increased productivity, and improved competitive position of the company (Hariga, 2000; Ouyang et al., 2002; Chuang et al., 2004; Lin, 2009; Glock, 2012). The concept of controllable lead time has been widely endorsed in the inventory management literature. The reader is referred to, e.g., Ouyang et al. (1996); Huang (2001); Lin (2009); Glock (2012); Sarkar and Moon (2014).

Further actions, other than shortening lead time, can be tackled to reach JIT goals. One of these initiatives is concerned with the setup/ordering cost reduction, which can be achieved in practice by means of various activities, such as procedural changes, specialized equipments acquisition and workers training (Leschke, 1996; Chuang et al., 2004). As observed in literature, decreasing the setup/ordering cost permits to improve quality and flexibility, lower investment in inventory, and increase effective capacity (Porteus, 1986; Nasri et al., 1990; Leschke & Weiss, 1997; Chuang et al., 2004). Setup cost control has been a topic of interest for many researchers in the field of production/inventory management. The reader is referred, for example, to Chuang et al. (2004); Lin (2009); Shu and Zhou (2014); Sarkar and Moon (2014); Sarkar et al. (2015).

The mixture of backorders and lost sales is an issue that production/inventory models should not neglect. This is particularly true when demand is stochastic. In fact, as Wang and Tang (2014) observed, a backorders-lost sales mixture can model different purchasing behaviours of customers when facing a stockout: some of them may wait until demand is satisfied (such demands are backordered); while others may be impatient and not willing to wait (such demands are lost). The importance of this aspect is shown by the number of researchers that have included a mixture of backorders and lost sales in their production/inventory models (see, e.g., Ouyang et al. (1996); Lodree (2007); Lin

(2009); Sarkar and Moon (2014); Castellano (2016)).

From the above literature review on stochastic JRP, we can see that little work has been done to consider lead time shortening, setup/ordering cost reduction, and backorders-lost sales mixtures simultaneously in the context of JRP. We therefore propose a novel periodic review stochastic JRP model that includes all these features. That is, in addition to the standard decision variables of the JRP (i.e., basic cycle time along with review period and target level of each item), the proposed model takes into account backorders-lost sales mixtures, controllable lead times, and investments to reduce the minor ordering costs. This optimization problem is approached under the class of cyclic policies. The objective is to determine the cyclic replenishment policy, the length of lead times and the minor ordering costs that minimize the long-run expected total cost per time unit. Due to the number of decisions and the nature of features considered at the same time, the problem under consideration is particularly challenging.

It is well-known that the simpler deterministic JRP is NP-hard (Arkin et al., 1989) and that obtaining the optimal policy may be computationally prohibitive for large problems (Khouja & Goyal, 2008). Being an extension of the classical deterministic JRP, we can deduce that our problem is at least equally complex to the deterministic JRP. Consequently, developing an efficient heuristic optimization method is recommended for the problem we propose (Silver, 2004).

To optimize the considered inventory system, we will propose two alternative heuristics. A lower bound on the minimum cost is derived and a procedure for its calculation is provided. Extensive numerical experiments are carried out to compare the heuristic solutions with the developed lower bound.

To summarize, the contributions of the study are as follows:

1. A periodic review stochastic JRP model with backorder-lost sales mixtures, controllable lead times and controllable minor ordering costs is formulated;
2. Two heuristic solution methods to approach the optimization problem under the class of cyclic policies are developed;
3. A lower bound on the minimum cost along with its calculation method is provided;
4. Extensive numerical experiments are conducted to evaluate the effectiveness of the proposed heuristics.

To the best of our knowledge, the aspects included into the present work have never been previously considered in literature. See Table 1 for a comparison between our study and others.

TABLE 1 HERE

The rest of the paper is organized as follows: Section 2 introduces the model and formalizes the problem; Section 3 proposes the first heuristic algorithm;

Section 4 presents the second solution approach; Section 5 is devoted to the development of a lower bound on the minimum cost; Section 6 presents numerical experiments; finally, Section 7 concludes the paper and indicates further research.

2. Model formulation and problem definition

The mathematical model uses the following notation:

Decision variables:

T	Basic cycle time, i.e., time interval between orders (time units).
L_n	Length of lead time of item n (time units).
z_n	Safety factor of item n.
R_n	Target level of item n. An equivalent decision variable to z_n.
k_n	Integer multiplier of item n.
a_n	Minor ordering cost of item n (money/order).

Parameters:

A	Major ordering cost (money/order).
N	Number of items, i.e., $n = 1, 2, ..., N$.
h_n	Unit holding cost rate of item n (money/quantity unit/time unit).
ρ_n	Fixed penalty cost per unit shortage of item n (money/quantity unit).
π_n	Marginal profit per unit of item n (money/quantity unit).
β_n	Fraction of shortage (i.e., the demand during the stockout period) of item n that will be lost.
σ_n	Standard deviation of the demand rate of item n (quantity unit/time unit).
D_n	Average demand rate of item n (quantity unit/time unit).

Random variables:

X_n	Demand of item n within its protection interval, i.e., within the period $k_nT + L_n$.

Functions and operators:

$f(\cdot)$	Standard normal probability density function (p.d.f.).
$F(\cdot)$	Standard normal cumulative distribution function (c.d.f.).
$G(\cdot)$	Standard normal loss function.
$E[\cdot]$	Mathematical expectation.
x^+	Maximum between 0 and x, i.e., $x^+ \equiv \max\{0, x\}$.
$\lVert\cdot\rVert$	Euclidean norm.

Sets:

$\mathbb{R}$	Real numbers.
$\mathbb{R}_+$	Positive real numbers.
$\mathbb{N}$	Natural numbers.

We make the following assumptions:

1. The random variables $X_1, X_2, ..., X_N$ are independent.
2. Inventory of item n is reviewed every k_nT time units. A sufficient quantity is ordered up to the target level R_n, and the order lot arrives after L_n time units. For each item, there is no more than a single order outstanding.
3. The target level of item n is given by $R_n = D_n (k_nT + L_n) + z_n\sigma_n\sqrt{k_nT + L_n}$,

for $n = 1, 2, ..., N$. The first addendum is the average demand during the protection interval, while the second one is the safety stock.

4. For each n, the random variable X_n is Gaussian with mean and standard deviation given by $D_n (k_n T + L_n)$ and $\sigma_n \sqrt{k_n T + L_n}$, respectively.
5. For each n, shortages are allowed and partially backordered with ratio $1 - \beta_n$. The fraction of shortage with ratio β_n is lost.
6. The time horizon is infinite.

Several authors have pointed out that the lead time of an item can be supposed to be made of several components, such as setup time, process time, and queue time (Silver & Peterson, 1985; Liao & Shyu, 1991; Tersine, 2002). This observation makes it possible to assume that lead time be negotiable and controllable. That is, each component may be reduced with an additional charge. This approach to controlling lead time was originally proposed by Liao and Shyu (1991). They noted that the extra cost to shorten lead time may consist of the following three components:

1. Administrative costs (e.g., overtime payments, part-time employee wages for order preparation, etc.);
2. Transport costs (e.g., shipping time, freight charge for transiting items from the supplier, etc.); and
3. Supplier speed-up costs (e.g., setup costs, extra investments, etc.).

In particular, Liao & Shyu (1991) assumed that lead time can be decomposed into several components, each one having a different piecewise linear crashing cost function for lead time reduction. They further assumed that each component may be reduced to a given minimum duration. Over the years since its introduction, numerous scholars have endorsed this model for controlling lead time (see, e.g., Ouyang et al. (1996); Huang (2001); Chuang et al. (2004); Lin (2009); Glock (2012); Sarkar and Moon (2014); Soni and Patel (2015)).

In this paper, we assume that the lead time of each item can be controlled according to the same assumptions as in Liao & Shyu (1991). This is a licit approach as the above observations about lead time can clearly be adapted to any single- or multi-item inventory system, independently of the particular replenishment policy adopted. Moreover, we assume that the lead time of item n is different from, and independent of, that of item p, for $p \neq n$. Clearly, this permits us to generalize the case in which all items have the same lead time (which represents a special condition). Namely, our model can readily be reduced to the circumstance where all items are characterized by the same lead time. Note that two or more items may experience the same or different lead times depending on whether, for example, they are procured from the same or different suppliers. In the literature, it is possible to find papers on stochastic JRP where lead times are equal (see, e.g., Tsai et al. (2009); Kiesmüller (2010)) or not (see, e.g., Tanrikulu et al. (2010); Qu et al. (2015)).

In particular, it is supposed that the lead time L_n of item n is made up of M_n mutually independent, deterministic and constant components. The generic mth component has a minimum duration $b_{m,n}$, a normal duration $s_{m,n}$, and a crashing cost per time unit $c_{m,n}$, with $c_{1,n} \leq c_{2,n} \leq ... \leq c_{M_n,n}$. Components are crashed one at a time starting with the component of least $c_{m,n}$ and so on. If $L_{m,n}$ is the length of lead time with components $1, 2, ..., m$ crashed to their minimum durations, then $L_{m,n} = L_{0,n} - (s_{1,n} - b_{1,n}) - (s_{2,n} - b_{2,n}) - ... - (s_{m,n} - b_{m,n})$, where $L_{0,n} \equiv \sum_m s_{m,n}$. For $L_n \in [L_{m,n}, L_{m-1,n}]$, $m = 1, 2, ..., M_n$, the lead time crashing cost $U_n(L_n)$ relevant to item n can be expressed as follows:

$$\begin{aligned} U_n(L_n) \equiv\ & c_{m,n}(L_{m-1,n} - L_n) + c_{1,n}(s_{1,n} - b_{1,n}) \\ & + c_{2,n}(s_{2,n} - b_{2,n}) + ... + c_{m-1,n}(s_{m-1,n} - b_{m-1,n}). \end{aligned} \tag{1}$$

We can note that $U_n(L_n)$ is a piecewise-linear, decreasing function defined in the interval $[L_{M_n,n}, L_{0,n}]$, where it is also continuous and convex.

We further assume that the minor ordering cost a_n of item n, for $n = 1, 2, ..., N$, is controllable through a capital investment $I_n(a_n)$, which is a convex and strictly decreasing function of a_n. The investment $I_n(a_n)$ is required to reduce the minor ordering cost from the original level α_n to a target level a_n, with $0 < a_n \leq \alpha_n$. For example, $I_n(a_n)$ may be regarded as an investment of purchasing a more efficient vehicle, or an investment in new technology to facilitate the transport. The function $I_n(a_n)$ is the one-time investment cost whose benefits will extend to the long-term future. Hence, if τ_n is the annual fractional cost of capital investment (e.g., the interest rate), then $\tau_n I_n(a_n)$ is the annual cost of the investment. In this paper, we consider a logarithmic investment function:

$$I_n(a_n) = \frac{1}{\delta_n} \ln\left(\frac{\alpha_n}{a_n}\right), \ 0 < a_n \leq \alpha_n, \text{ for } n = 1, 2, ..., N, \tag{2}$$

where δ_n is the percentage decrease in a_n per money unit increase in investment. An early analytical treatment of investments to reduce setup cost is owed to Porteus (1985). The logarithmic investment function has been widely adopted in literature (see, e.g., Chuang et al. (2004); Lin (2009); Shu and Zhou (2014); Sarkar and Moon (2014); Sarkar et al. (2015)).

Under the above assumptions and notation, the long-run expected total cost per time unit relevant to the single and independent item n (whose review period is T_n) is expressed as follows:

$$C_n(a_n, T_n, z_n, L_n) = \tau_n I_n(a_n) + \mathcal{C}_n(a_n, T_n, z_n, L_n), \tag{3}$$

where the first term represents the investment cost to reduce the minor ordering cost for item n; letting $\bar{\pi}_n \equiv \rho_n + \pi_n \beta_n$,

$$\begin{aligned} \mathcal{C}_n(a_n, T_n, z_n, L_n) =\ & h_n\left(\frac{D_n T_n}{2} + z_n \sigma_n \sqrt{T_n + L_n} + \beta_n \sigma_n \sqrt{T_n + L_n} G(z_n)\right) \\ & + \frac{a_n}{T_n} + \frac{\bar{\pi}_n}{T_n} \sigma_n \sqrt{T_n + L_n} G(z_n) + \frac{U_n(L_n)}{T_n}. \end{aligned} \tag{4}$$

Equation (4) includes the inventory holding cost, the minor ordering cost, the shortage cost, and the lead time crashing cost. Equation (4) can be derived in a similar way to Annadurai and Uthayakumar (2010).

The cost function $C_n(a_n, T_n, z_n, L_n)$ (i.e., Eq. (3)) can readily be extended to the N-items case, i.e., to the JRP formulation. If we let $\mathbf{k} \equiv (k_1, k_2, ..., k_N)$, $\mathbf{z} \equiv (z_1, z_2, ..., z_N)$, $\mathbf{a} \equiv (a_1, a_2, ..., a_N)$, and $\mathbf{L} \equiv (L_1, L_2, ..., L_N)$, the long-run expected total cost per time unit for a family of N items under the JRP framework is

$$C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L}) = \frac{A}{T} + \sum_{n=1}^{N} C_n(T, a_n, k_n, z_n, L_n), \tag{5}$$

where the first term is the major ordering cost component and $C_n(T, a_n, k_n, z_n, L_n) \equiv \mathcal{C}_n(a_n, k_nT, z_n, L_n)$. Just as a clarification, we have defined T_n as the review period of item n when it is managed independently of the other items. When item n is instead managed in coordination with the others according to the JRP framework, its review period is k_nT, i.e., it is an integer multiple of a (common) base period T.

The objective is to find the cyclic replenishment policy, the length of lead times and the minor ordering costs that minimize Eq. (5). This problem can be formalized as follows:

$$\begin{array}{lll}
\text{(P)} & \min\limits_{(T,\mathbf{a},\mathbf{k},\mathbf{z},\mathbf{L})} & C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L}), \\
& \text{s.t.} & T > 0, \\
& & L_n \in [L_{M_n,n}, L_{0,n}] \ \forall\, n, \\
& & 0 < a_n \leq \alpha_n \ \forall\, n, \\
& & \mathbf{z} \in \mathbb{R}^N, \\
& & \mathbf{k} \in \mathbb{N}^N.
\end{array}$$

This problem has never been investigated before. Moreover, the global optimal solution to problem (P) is evidently difficult to obtain. We turn to develop two effective heuristic algorithms.

We conclude this section with a proposition that establishes some useful properties of the cost function, which can facilitate the development of heuristic algorithms.

Proposition 1. $C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L})$ *satisfies the following properties:*

1. *With fixed* $(T, \mathbf{a}, \mathbf{z}, \mathbf{k})$*,* $C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L})$ *is concave in* $\mathbf{L}$*.*
2. *With fixed* $(T, \mathbf{k}, \mathbf{z}, \mathbf{L})$*,* $C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L})$ *is convex in* $\mathbf{a}$*. Moreover, the only stationary point of* $C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L})$ *with respect to* a_n *is given by* $a_n = \bar{a}_n(k_nT) = \xi_n k_n T$*, for each* n*, where* $\xi_n \equiv \frac{\tau_n}{\delta_n}$*.*
3. *With fixed* $(\mathbf{a}, \mathbf{k}, \mathbf{L})$*,* $C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L})$ *is convex in* $(T, \mathbf{z})$*.*
4. *With fixed* $(\mathbf{k}, \mathbf{L})$ *and* $a_n = \bar{a}_n(k_nT)$*,* $C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L})$ *is convex in* $(T, \mathbf{z})$*.*

5. *With fixed* $(T, \mathbf{a}, \mathbf{k}, \mathbf{L})$ *and* $h_n(1-\beta_n) < \frac{\bar{\pi}_n}{k_n T}$ *for each* n, $C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L}) \to +\infty$ *as* $\|\mathbf{z}\| \to +\infty$.
6. *With fixed* $(T, \mathbf{a}, \mathbf{z}, \mathbf{L})$ *and relaxing the integrality constraint on* $\mathbf{k}$, $C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L}) \to +\infty$ *as* $\mathbf{k}$ *tends to the boundary of* $\mathbb{R}_+^N$.

Proof. Point No. 1. If we take the second-order partial derivative of $C_n(T, a_n, k_n, z_n, L_n)$ with respect to L_n, with $L_n \in [L_{m,n}, L_{m-1,n}]$, we have:

$$\frac{\partial^2}{\partial L_n^2} C_n(T, a_n, k_n, z_n, L_n) = -\frac{\sigma_n}{4(k_n T + L_n)^{3/2}} \left[h_n(z_n + \beta_n G(z_n)) + \frac{\bar{\pi}_n}{k_n T} G(z_n) \right] < 0,$$

which is valid for $m = 1, ..., M_n$ and for $n = 1, 2, ..., N$. Hence, the concavity of $C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L})$ in $\mathbf{L}$, with fixed $(T, \mathbf{a}, \mathbf{z}, \mathbf{k})$, is proved.

Point No. 2. Noting that $\frac{\partial^2}{\partial a_n^2} C_n(T, a_n, k_n, z_n, L_n) = \frac{\tau_n}{\delta_n} \frac{1}{a_n^2} > 0$, for each n, we can readily deduce that $C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L})$ is convex in $\mathbf{a}$, with fixed $(T, \mathbf{k}, \mathbf{z}, \mathbf{L})$. The only stationary point in a_n of $C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L})$ can be found solving the equation $\frac{\partial}{\partial a_n} C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L}) = 0$ with respect to a_n, which gives $a_n = \bar{a}_n(k_n T) = \xi_n k_n T$, for each n.

Point No. 3. Note that Annadurai and Uthayakumar (2010) proved that a function structurally identical to $\mathcal{C}_n(a_n, T_n, z_n, L_n)$ is convex in (T_n, z_n), for fixed (a_n, L_n). Since convexity is invariant under affine maps, we can affirm that $C_n(T, a_n, k_n, z_n, L_n)$ is convex in (T, z_n), for fixed (a_n, k_n, L_n). This is evidently true for each n. If we also observe that $\frac{A}{T}$ is convex in T, then the convexity of $C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L})$ in $(T, \mathbf{z})$, with fixed $(\mathbf{a}, \mathbf{k}, \mathbf{L})$, can readily be deduced.

Point No. 4. We first observe that, for $a_n = \bar{a}_n(k_n T)$, $C_n(T, a_n, k_n, z_n, L_n)$ becomes:

$$C_n(T, k_n, z_n, L_n) = \xi_n \ln\left(\frac{\alpha_n}{\xi_n k_n T}\right) + \xi_n + \hat{C}_n(T, k_n, z_n, L_n),$$

where

$$\begin{aligned}\hat{C}_n(T, k_n, z_n, L_n) \quad &= \quad h_n\left(\frac{k_n T D_n}{2} + z_n \sigma_n \sqrt{k_n T + L_n} + \beta_n \sigma_n \sqrt{k_n T + L_n} G(z_n)\right) \\ &\quad + \frac{\bar{\pi}_n}{k_n T} \sigma_n \sqrt{k_n T + L_n} G(z_n) + \frac{U_n(L_n)}{k_n T}.\end{aligned}$$

Note that $\hat{C}_n(T, k_n, z_n, L_n)$ is convex in $(k_n T, z_n)$, with fixed L_n (this follows, e.g., from Annadurai and Uthayakumar (2010)). If we recall that convexity is invariant under affine maps, we can deduce that $\hat{C}_n(T, k_n, z_n, L_n)$ is convex in (T, z_n), with fixed (k_n, L_n). In addition, $\xi_n \ln\left(\frac{\alpha_n}{\xi_n k_n T}\right) + \xi_n$ is convex in T, for fixed k_n. Hence, $C_n(T, a_n, k_n, z_n, L_n)$ is convex in (T, z_n), with fixed (k_n, L_n) and $a_n = \bar{a}_n(k_n T)$. This is evidently true for each n. In conclusion, observing that $\frac{A}{T}$ is convex in T, we can conclude that $C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L})$ is convex in $(T, \mathbf{z})$, with fixed $(\mathbf{k}, \mathbf{L})$ and $a_n = \bar{a}_n(k_n T)$, for each n.

Finally, the limit properties given in the last two points are relatively easy to observe, and their proof is therefore omitted. □

Note that the assumption made at the fifth point of Proposition 1 is reasonable. In fact, over the inventory replenishment cycle, the unit holding cost can be assumed, in practice, smaller than the unit shortage cost.

3. The first heuristic algorithm

The properties of $C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L})$ given in Proposition 1 permit us to make the following useful observations to approach problem (P). First, we can note that the minimum of $C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L})$ in $\mathbf{L}$ is given by a vector $\bar{\mathbf{L}}$ whose nth component $\bar{L}_n$ belongs to $\mathscr{H}_n \equiv \{L_{m,n} : m = 0, 1, ..., M_n\}$. That is, the minimum of $C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L})$ in L_n, with $L_n \in [L_{m,n}, L_{m-1,n}]$, for each n, is located on one of the endpoints of the interval $[L_{m,n}, L_{m-1,n}]$. We can therefore write that

$$\min_{(T,\mathbf{a},\mathbf{k},\mathbf{z},\mathbf{L})} C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L}) = \min \left\{ \min_{(T,\mathbf{a},\mathbf{k},\mathbf{z},)} C\left(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \bar{\mathbf{L}}\right) : \bar{L}_n \in \mathscr{H}_n \, \forall\, n \right\}. \tag{6}$$

In accordance to Eq. (6), problem (P) is therefore reduced to solve the following sub-problem for each vector $\bar{\mathbf{L}}$:

$$\begin{array}{lll} (\mathrm{P}_1) & \displaystyle\min_{(T,\mathbf{a},\mathbf{k},\mathbf{z})} & C\left(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \bar{\mathbf{L}}\right), \\ & \text{s.t.} & T > 0, \\ & & 0 < a_n \leq \alpha_n \,\forall\, n, \\ & & \mathbf{z} \in \mathbb{R}^N, \\ & & \mathbf{k} \in \mathbb{N}^N. \end{array}$$

Note that the number of all vectors $\bar{\mathbf{L}}$ is given by $\prod_n (M_n + 1)$.

Although it is difficult to prove that $C\left(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \bar{\mathbf{L}}\right)$ is convex in $(T, \mathbf{a}, \mathbf{k}, \mathbf{z})$, Proposition 1 allows us to approach problem (P_1) according to the following relation:

$$\min_{(T,\mathbf{a},\mathbf{k},\mathbf{z})} C\left(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \bar{\mathbf{L}}\right) = \min \mathscr{U}, \tag{7}$$

where $\mathscr{U} \equiv \left\{ \min_{(T,\mathbf{a},\mathbf{z})} C\left(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \bar{\mathbf{L}}\right) : \mathbf{k} \in \mathbb{N}^N \right\}$. Evidently, it is practically impossible to evaluate $\min_{(T,\mathbf{a},\mathbf{z})} C\left(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \bar{\mathbf{L}}\right)$ *for each* $\mathbf{k} \in \mathbb{N}^N$ when N is large (we recall that $\bar{\mathbf{L}}$ is fixed in problem (P_1)). It therefore needs a heuristic procedure to explore the space of vectors $\mathbf{k}$. We here propose a method that aims at determining $\min_{(T,\mathbf{a},\mathbf{z})} C\left(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \bar{\mathbf{L}}\right)$ for vectors $\mathbf{k}$ by gradually increasing its norm, starting with the unitary vector, which is of the least norm. The process continues as long as $\min_{(T,\mathbf{a},\mathbf{z})} C\left(T,\mathbf{a},\mathbf{k},\mathbf{z},\bar{\mathbf{L}}\right)$ decreases. We note that the minimum of $C\left(T,\mathbf{a},\mathbf{k},\mathbf{z},\bar{\mathbf{L}}\right)$ in $(T,\mathbf{a},\mathbf{z})$, with fixed $\left(\mathbf{k},\bar{\mathbf{L}}\right)$, can be obtained by solving the first-order conditions for optimality.

By making use of the above observations, we can propose a heuristic algorithm below, which gives a heuristic solution $(T^*,\mathbf{a}^*,\mathbf{k}^*,\mathbf{z}^*,\mathbf{L}^*)$, and the cost $C^* \equiv C\left(T^*,\mathbf{a}^*,\mathbf{k}^*,\mathbf{z}^*,\mathbf{L}^*\right)$, to problem (P):

Algorithm 1.

Step 1. Set $\mathscr{C} \leftarrow \emptyset$.

Step 2. For each vector $\bar{\mathbf{L}}$ do Steps 2.1-2.3.

Step 2.1. Set $i = 1$, $\mathcal{C}_i = 1$, $\mathcal{C}_{i-1} = 0$.

Step 2.2. While $\mathcal{C}_i \neq \mathcal{C}_{i-1}$ AND stopping criterion is not satisfied, do Steps 2.2.1-2.2.4.

Step 2.2.1. Set $\mathscr{K} \leftarrow \{\mathbf{k} :\ k_n \in \{1,2,...,i\}\ \forall\, n\}$ and $\mathscr{J}_i \leftarrow \emptyset$.

Step 2.2.2. For each $\mathbf{k} \in \mathscr{K}$, do Steps 2.2.2.1-2.2.2.6.

Step 2.2.2.1. If stopping criterion is satisfied, then go to Step 2.2.3.

Step 2.2.2.2. Set $a_n \leftarrow \bar{a}_n\left(k_n T\right)$, for each n, and $\mathscr{D}_1 \leftarrow \{1,2,...,N\}$.

Step 2.2.2.3. Set $\mathscr{D}_2 \leftarrow \emptyset$, and $\left(\hat{T}_i,\hat{\mathbf{z}}_i\right) \leftarrow \arg\min_{(T,\mathbf{z})} C\left(T,\mathbf{a},\mathbf{k},\mathbf{z},\bar{\mathbf{L}}\right)$.

Step 2.2.2.4. If $\mathscr{D}_1 = \emptyset$, then set $\hat{\mathbf{a}}_i \equiv \left(\hat{a}_{1,i},\hat{a}_{2,i},...,\hat{a}_{N,i}\right)$ and go to Step 2.2.2.7; otherwise, go to Step 2.2.2.5.

Step 2.2.2.5. For each $n \in \mathscr{D}_1$, if $0 < \bar{a}_n\left(k_n\hat{T}_i\right) \leq \alpha_n$, then set $\hat{a}_{n,i} \leftarrow \bar{a}_n\left(k_n\hat{T}_i\right)$; otherwise, set $\hat{a}_{n,i} \leftarrow \alpha_n$ and $\mathscr{D}_2 \leftarrow \mathscr{D}_2 \cup \{n\}$.

Step 2.2.2.6. If $\mathscr{D}_2 = \emptyset$, then set $\hat{\mathbf{a}}_i \equiv \left(\hat{a}_{1,i},\hat{a}_{2,i},...,\hat{a}_{N,i}\right)$ and go to Step 2.2.2.7; otherwise, set $\mathscr{D}_1 \leftarrow \mathscr{D}_1 \setminus \mathscr{D}_2$, $a_n \leftarrow \bar{a}_n\left(k_n T\right)$, for each $n \in \mathscr{D}_1$, and $a_n \leftarrow \alpha_n$, for each $n \in \mathscr{D}_2$, then go to Step 2.2.2.3.

Step 2.2.2.7. Set $\mathscr{J}_i \leftarrow \mathscr{J}_i \cup \left\{C\left(\hat{T}_i,\hat{\mathbf{a}}_i,\mathbf{k},\hat{\mathbf{z}}_i,\bar{\mathbf{L}}\right)\right\}$.

Step 2.2.3. Set $\mathcal{C}_i \leftarrow \min \mathscr{J}_i$ and $\left(T_i^*,\mathbf{a}_i^*,\mathbf{k}_i^*,\mathbf{z}_i^*,\bar{\mathbf{L}}\right) \leftarrow \arg\min \mathscr{J}_i$.

Step 2.2.4. Set $i \leftarrow i + 1$.

Step 2.3. Set $\mathscr{C} \leftarrow \mathscr{C} \cup \left\{C\left(T_i^*,\mathbf{a}_i^*,\mathbf{k}_i^*,\mathbf{z}_i^*,\bar{\mathbf{L}}\right)\right\}$.

Step 3. Set $(T^*,\mathbf{a}^*,\mathbf{k}^*,\mathbf{z}^*,\mathbf{L}^*) \leftarrow \arg\min \mathscr{C}$, and $C^* \leftarrow \min \mathscr{C}$.

Steps 2.1–2.2 deal with problem (P_1). At Step 2.2.1, we define the subspace $\mathscr{K}$ of $\mathbb{N}^N$ where the search method aims to find a heuristic solution to $\min \mathscr{U}$. Note that, according to the heuristic procedure concerning the minimization over $\mathscr{U}$, subspace $\mathscr{K}$ is updated at each iteration, so as to include vectors $\mathbf{k}$ with gradually increasing norm. Algorithm 1 carries on with this sub-routine as long as it is possible to obtain increasingly smaller values of $\min_{(T,\mathbf{a},\mathbf{z})} C\left(T,\mathbf{a},\mathbf{k},\mathbf{z},\bar{\mathbf{L}}\right)$ (or until stopping criterion is satisfied). In this regard, at Step 2.2 Algorithm 1 checks whether the minimum costs obtained in two consecutive iterations coincide; if so, the heuristic to find $\min \mathscr{U}$, for fixed $\bar{\mathbf{L}}$, ends. Note that a stopping criterion is required since it is not possible to prove that the heuristic to find $\min \mathscr{U}$ will end in a finite time.

We then observe that Steps 2.2.2.2–2.2.2.6 are concerned with the minimization of $C\left(T,\mathbf{a},\mathbf{k},\mathbf{z},\bar{\mathbf{L}}\right)$ in $(T,\mathbf{a},\mathbf{z})$, for fixed $\left(\mathbf{k},\bar{\mathbf{L}}\right)$. Although this problem can be approached by solving the first-order conditions for optimality, it is required that the constraint $0 < a_n \leq \alpha_n$, for each n, be satisfied. To understand the logic behind the procedure at Steps 2.2.2.2–2.2.2.6, it may be useful to consider a simple example with two items. Let T^*, $\mathbf{a}^* \equiv (a_1^*, a_2^*)$, and $\mathbf{z}^* \equiv (z_1^*, z_2^*)$ be the solution to the problem of minimizing $C\left(T,\mathbf{a},\mathbf{k},\mathbf{z},\bar{\mathbf{L}}\right)$ in $(T,\mathbf{a},\mathbf{z})$, for fixed $\left(\mathbf{k},\bar{\mathbf{L}}\right)$. Let $\hat{T}$, $\hat{\mathbf{a}} \equiv (\hat{a}_1, \hat{a}_2)$, and $\hat{\mathbf{z}} \equiv (\hat{z}_1, \hat{z}_2)$ be the solution to the first-order conditions for optimality in $(T,\mathbf{a},\mathbf{z})$. Note that, thanks to Proposition 1, we can affirm that $C\left(T,\mathbf{a},\mathbf{k},\mathbf{z},\bar{\mathbf{L}}\right)$ admits only one stationary point $\left(\hat{T},\hat{\mathbf{a}},\hat{\mathbf{z}}\right)$. Once $\left(\hat{T},\hat{\mathbf{a}},\hat{\mathbf{z}}\right)$ is determined, the solution $(T^*,\mathbf{a}^*,\mathbf{z}^*)$ can be obtained according to the following cases:

1. If $\hat{a}_1 \leq \alpha_1$and $\hat{a}_2 \leq \alpha_2$, then we put $(T^*,\mathbf{a}^*,\mathbf{z}^*) = \left(\hat{T},\hat{\mathbf{a}},\hat{\mathbf{z}}\right)$.
2. If $\hat{a}_1 > \alpha_1$and $\hat{a}_2 \leq \alpha_2$, then we put $a_1 = \alpha_1$ and calculate the new $\hat{T}$, $\hat{a}_2$, and $\hat{\mathbf{z}}$, for $a_1 = \alpha_1$. If $\hat{a}_2 \leq \alpha_2$, then we put $T^* = \hat{T}$, $a_1^* = \alpha_1$, $a_2^* = \hat{a}_2$, and $\mathbf{z}^* = \hat{\mathbf{z}}$. Otherwise, we set $a_2 = \alpha_2$ and calculate the new $\hat{T}$ and $\hat{\mathbf{z}}$, for $a_1 = \alpha_1$ and $a_2 = \alpha_2$, which permit to attain the solution $T^* = \hat{T}$, $a_1^* = \alpha_1$, $a_2^* = \alpha_2$, and $\mathbf{z}^* = \hat{\mathbf{z}}$.
3. If $\hat{a}_1 \leq \alpha_1$and $\hat{a}_2 > \alpha_2$, then it is necessary to proceed similarly to the previous case.
4. If $\hat{a}_1 > \alpha_1$and $\hat{a}_2 > \alpha_2$, then we put $a_1 = \alpha_1$ and $a_2 = \alpha_2$, and calculate the new $\hat{T}$ and $\hat{\mathbf{z}}$, which gives the solution $T^* = \hat{T}$, $a_1^* = \alpha_1$, $a_2^* = \alpha_2$, and $\mathbf{z}^* = \hat{\mathbf{z}}$.

The minimization problem at Step 2.2.2.3 can be approached by a standard constrained nonlinear minimization algorithm, or by a metaheuristic algorithm. Note that the first-order conditions for optimality in $(T,\mathbf{z})$ cannot be solved in closed form. Hence, only a numerical technique can be used.

With regard to the computational complexity of Algorithm 1, it is not possible to obtain an estimate or an upper bound. More precisely, while it is possible to observe that the minimization in $(T,\mathbf{a},\mathbf{z})$ of $C\left(T,\mathbf{a},\mathbf{k},\mathbf{z},\bar{\mathbf{L}}\right)$, with fixed $\left(\mathbf{k},\bar{\mathbf{L}}\right)$, may require, at most, $N+1$ iterations (i.e., instructions between Steps 2.2.2.2–2.2.2.6 may be repeated, at most, $N+1$ times), we cannot give an *a priori* estimate of the time (i.e., of the number of iterations) that the heuristic approaching the minimization over $\mathscr{U}$ would need. In fact, this heuristic is complex (mainly because of the combinatorial nature of the optimization in $\mathbf{k}$) and computational results are strongly dependent on parameter values, as extensive numerical experiments have shown later on.

However, note that we have included a stopping condition at Step 2.2. This permits us to assure that the heuristic approaching the minimization over $\mathscr{U}$ terminates within the time limit imposed by the stopping condition. It is possible to implement different stopping criteria. For example, (i) the maximum number of iterations performed to explore the space of vectors $\mathbf{k}$, or (ii) the maximum "extension" of vectors $\mathbf{k}$, where the "extension" of a vector can be evaluated according to its norm (e.g., Manhattan, Euclidean, or a different p-norm) or to the largest value that each component of the vector can take. Moreover, it is possible to observe that the heuristic solution in $\left(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \bar{\mathbf{L}}\right)$, with fixed $\bar{\mathbf{L}}$, must be determined for $\prod_n (M_n + 1)$ different vectors $\bar{\mathbf{L}}$. In Section 6, we will present the results of extensive numerical experiments aimed to evaluate the computational efficiency of Algorithm 1 over a quite large number of problems.

Numerical tests have shown that Algorithm 1 is effective. However, the same tests have pointed out that this algorithm may be computationally onerous for large problems. In the next section, we therefore present a computationally more efficient heuristic to approach problem (P).

4. The second heuristic algorithm

The second heuristic proposed to approach problem (P) is based on a simplified expression of $C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L})$, which is obtained by means of the passages below described. The first-order condition for optimality of $C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L})$ in z_n gives:

$$\frac{\partial}{\partial z_n} C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L}) = 0 \Leftrightarrow z_n = \bar{z}_n(k_n T), \text{ for } n = 1, 2, ..., N,$$

where

$$\bar{z}_n(k_n T) = F^{-1}\left(1 - \frac{h_n}{\beta_n h_n + \bar{\pi}_n \frac{1}{k_n T}}\right) = F^{-1}(1 - \lambda(k_n T)). \tag{8}$$

In Eq. (8), we have put $\lambda(k_n T) \equiv \frac{h_n}{\beta_n h_n + \bar{\pi}_n \frac{1}{k_n T}}$, and $F^{-1}(\cdot)$ represents the quantile function of the standard normal distribution. From Eq. (8), we have

$$1 - F(z_n) = \lambda(k_n T). \tag{9}$$

If we recall that $G(x) = f(x) - x(1 - F(x))$ (Zipkin, 2000), we can substitute $\lambda(k_n T)$ to $1 - F(z_n)$ (according to Eq. (9)) in $C_n(T, a_n, k_n, z_n, L_n)$, which becomes, after some algebraic manipulations,

$$\begin{aligned} C_n(T, a_n, k_n, L_n) &= \tau_n I_n(a_n) + \frac{1}{k_n T}(a_n + U_n(L_n)) + \frac{h_n D_n}{2} k_n T \\ &\quad + \sigma_n f(\bar{z}_n(k_n T)) \sqrt{k_n T + L_n}\left(\beta_n h_n + \bar{\pi}_n \frac{1}{k_n T}\right), \text{ for } n = 1, 2, ..., N. \end{aligned}$$

Equation (5) can finally be rewritten as follows:

$$C(T, \mathbf{a}, \mathbf{k}, \mathbf{L}) = \frac{A}{T} + \sum_{n=1}^{N} C_n(T, a_n, k_n, L_n). \tag{10}$$

The proposed second heuristic works on $C(T, \mathbf{a}, \mathbf{k}, \mathbf{L})$ and is inspired by the improved version of the original Silver's algorithm for the deterministic JRP (Kaspi & Rosenblatt, 1983). However, our heuristic algorithm approaches a more general and more complex problem than the original deterministic formulation: we deal with the stochastic JRP with lead times and minor ordering costs that are controllable. Below, we describe how the heuristic solution can be obtained. Finally, the second heuristic algorithm will be formalized.

Let us consider a fixed vector $\bar{\mathbf{L}}$. For each n, let ${}^*T_n$ be the minimum of $C_n(T, a_n, k_n, \bar{L}_n)$ in k_nT. Note that ${}^*T_n$ should be obtained by minimizing $C_n(T, a_n, k_n, \bar{L}_n)$ in (k_nT, a_n), which can easily be done according to the observations given in Section 3. The item with the smallest ${}^*T_n$ is the one that needs to be replenished most often; that is, its replenishment frequency is the highest. If we denote such item with index $n = 1$, then we can put $k_1 = {}^*k_1 \equiv 1$, and Eq. (10) can be rewritten as follows:

$$\begin{aligned} C\left(T, \mathbf{a}, \mathbf{k}, \bar{\mathbf{L}}\right) = & \frac{A}{T} + \tau_1 I_1(a_1) + \frac{1}{k_1 T}\left(a_1 + U_1\left(\bar{L}_1\right)\right) + \frac{h_1 D_1}{2} k_1 T \\ & + \sigma_1 f\left(\bar{z}_1(k_1 T)\right)\sqrt{k_1 T + \bar{L}_1}\left(\beta_1 h_1 + \frac{\bar{\pi}_1}{k_1 T}\right) \\ & + \sum_{n=2}^{N}\left[\tau_n I_n(a_n) + \frac{1}{k_n T}\left(a_n + U_n\left(\bar{L}_n\right)\right) + \frac{h_n D_n}{2} k_n T\right. \\ & \left. + \sigma_n f\left(\bar{z}_n(k_n T)\right)\sqrt{k_n T + \bar{L}_n}\left(\beta_n h_n + \frac{\bar{\pi}_n}{k_n T}\right)\right]. \end{aligned} \tag{11}$$

If we replace a_n with $\bar{a}_n(k_nT)$, for each n, then Eq. (11) becomes

$$\begin{aligned} C\left(T, \mathbf{k}, \bar{\mathbf{L}}\right) = & \frac{A}{T} + \xi_1 \ln\left(\frac{\alpha_1}{\xi_1 k_1 T}\right) + \xi_1 + \frac{U_1\left(\bar{L}_1\right)}{k_1 T} + \frac{h_1 D_1}{2} k_1 T \\ & + \sigma_1 f\left(\bar{z}_1(k_1 T)\right)\sqrt{k_1 T + \bar{L}_1}\left(\beta_1 h_1 + \frac{\bar{\pi}_1}{k_1 T}\right) \\ & + \sum_{n=2}^{N}\left[\xi_n \ln\left(\frac{\alpha_n}{\xi_n k_n T}\right) + \xi_n + \frac{U_n\left(\bar{L}_n\right)}{k_n T} + \frac{h_n D_n}{2} k_n T\right. \\ & \left. + \sigma_n f\left(\bar{z}_n(k_n T)\right)\sqrt{k_n T + L_n}\left(\beta_n h_n + \frac{\bar{\pi}_n}{k_n T}\right)\right]. \end{aligned} \tag{12}$$

We would observe that the following arguments concern the case in which $a_n = \bar{a}_n(k_nT)$, for each n, which is actually the initial stage of the heuristic. Note that the below passages are still valid in the case in which $a_i = \bar{a}_i(k_iT)$ and $a_j = \alpha_j$, for any $i, j \in \{1, 2, ..., N\}$ with $i \neq j$.

Let us relax the integrality constraint on k_n, with $n = 2, 3, ..., N$. If we now take the partial derivatives of $C(T, \mathbf{k}, \bar{\mathbf{L}})$ with respect to T and k_n, with $n = 2, 3, ..., N$, and then impose the first-order conditions for optimality, we obtain:

$$
\begin{aligned}
\frac{\partial}{\partial T} C\left(T, \mathbf{k}, \bar{\mathbf{L}}\right) \;=\;& -\frac{A}{T^2} - \frac{\xi_1}{T} - \frac{U_1\left(\bar{L}_1\right)}{k_1 T^2} + k_1 \frac{h_1 D_1}{2} \\
& + \sigma_1 \left\{ \left[\bar{z}_1\left(k_1 T\right) \frac{h_1 k_1 \bar{\pi}_1}{\left(\beta_1 h_1 k_1 T + \bar{\pi}_1\right)} \sqrt{k_1 T + \bar{L}_1} \right. \right. \\
& \left. + f\left(\bar{z}_1\left(k_1 T\right)\right) \frac{k_1}{2\sqrt{k_1 T + \bar{L}_1}} \right] \left(\beta_1 h_1 + \frac{\bar{\pi}_1}{k_1 T} \right) \\
& \left. - \frac{\bar{\pi}_1}{k_1 T^2} f\left(\bar{z}_1\left(k_1 T\right)\right) \sqrt{k_1 T + \bar{L}_1} \right\} \\
& + \sum_{n=2}^{N} \left\{ -\frac{\xi_n}{T} - \frac{U_n\left(\bar{L}_n\right)}{k_n T^2} + k_n \frac{h_n D_n}{2} \right. \\
& + \sigma_n \left\{ \left[\bar{z}_n\left(k_n T\right) \frac{h_n k_n \bar{\pi}_n}{\left(\beta_n h_n k_n T + \bar{\pi}_n\right)} \sqrt{k_n T + \bar{L}_n} \right. \right. \\
& \left. + f\left(\bar{z}_n\left(k_n T\right)\right) \frac{k_n}{2\sqrt{k_n T + \bar{L}_n}} \right] \left(\beta_n h_n + \frac{\bar{\pi}_n}{k_n T} \right) \\
& \left. \left. - \frac{\bar{\pi}_n}{k_n T^2} f\left(\bar{z}_n\left(k_n T\right)\right) \sqrt{k_n T + \bar{L}_n} \right\} \right\} = 0, \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial k_n} C\left(T, \mathbf{k}, \bar{\mathbf{L}}\right) \;=\;& -\frac{\xi_n}{k_n} - \frac{U_n\left(\bar{L}_n\right)}{k_n^2 T} + T \frac{h_n D_n}{2} \\
& + \sigma_n \left\{ \left[\bar{z}_n\left(k_n T\right) \frac{h_n T \bar{\pi}_n}{\left(\beta_n h_n k_n T + \bar{\pi}_n\right)} \sqrt{k_n T + \bar{L}_n} \right. \right. \\
& \left. + f\left(\bar{z}_n\left(k_n T\right)\right) \frac{T}{2\sqrt{k_n T + \bar{L}_n}} \right] \left(\beta_n h_n + \frac{\bar{\pi}_n}{k_n T} \right) \\
& \left. - \frac{\bar{\pi}_n}{k_n^2 T} f\left(\bar{z}_n\left(k_n T\right)\right) \sqrt{k_n T + \bar{L}_n} \right\} = 0, \qquad (14) \\
& \text{for } n = 2, 3, ..., N.
\end{aligned}
$$

If we now multiply Eq. (14) by $\frac{k_n}{T}$ and then substitute into Eq. (13), we get the equation

$$
\begin{aligned}
Y(T) \equiv\ & -\frac{A}{T^2}-\frac{\xi_1}{T}-\frac{U_1\left(\bar{L}_1\right)}{k_1T^2}+k_1\frac{h_1D_1}{2} \\
& +\sigma_1\left\{\left[\bar{z}_1\left(k_1T\right)\frac{h_1k_1\bar{\pi}_1}{\left(\beta_1h_1k_1T+\bar{\pi}_1\right)}\sqrt{k_1T+\bar{L}_1}\right.\right. \\
& \left.+f\left(\bar{z}_1\left(k_1T\right)\right)\frac{k_1}{2\sqrt{k_1T+\bar{L}_1}}\right]\left(\beta_1h_1+\frac{\bar{\pi}_1}{k_1T}\right) \\
& \left.-\frac{\bar{\pi}_1}{k_1T^2}f\left(\bar{z}_1\left(k_1T\right)\right)\sqrt{k_1T+\bar{L}_1}\right\}=0.
\end{aligned}
$$

We can note that $Y(T)$ coincides with the derivative with respect to T of $C_1\left(T,a_1,k_1,\bar{L}_1\right)$, with $a_1=\bar{a}_1\left(k_1T\right)$, plus the term $-\frac{A}{T^2}$. Recall that k_1 is a parameter, in this circumstance, and not a decision variable. Proposition 1 permits us to argue that the equation $Y(T)=0$ admits only one solution (in T), which will be denoted by $\tilde{T}$.

If we replace T with $\tilde{T}$ in Eq. (14) and then multiply it by $\frac{1}{\tilde{T}}$, we can see that the solution in $k_n\tilde{T}$ coincides with ${}^*T_n$, which means that $k_n\tilde{T}={}^*T_n$, for $n=2,3,...,N$. Since ${}^*T_n$, for each n, has been previously determined, we can put $k_n=\frac{{}^*T_n}{\tilde{T}}$ and $q_n=\lfloor k_n\rfloor$, for $n=2,3,...,N$. According to the unimodality of $C_n\left(T,a_n,k_n,\bar{L}_n\right)$ in $T_n\equiv k_nT$, we take q_n instead of q_n+1 if and only if $C_n\left(\tilde{T},a_n,q_n,\bar{L}_n\right)\leq C_n\left(\tilde{T},a_n,q_n+1,\bar{L}_n\right)$, with $a_n=\bar{a}_n\left(k_nT\right)$. In this case, we have ${}^*k_n\equiv q_n$, otherwise ${}^*k_n\equiv q_n+1$. We remind the reader that we have initially put $k_1={}^*k_1\equiv 1$.

The near-optimal value ${}^*T$ of T (for a given $\bar{\mathbf{L}}$, and for $a_n=\bar{a}_n\left(k_nT\right)$, with $n=2,3,...,N$, and $k_1={}^*k_1\equiv 1$) can now be evaluated by taking into consideration the integer values ${}^*k_n$, for $n=2,3,...,N$, just calculated. In particular, ${}^*T$ is obtained by solving Eq. (13) in T, once k_n has been replaced by ${}^*k_n$, for each n.

As ${}^*T$ has been determined, it is necessary to verify whether $0<\bar{a}_n\left({}^*k_n{}^*T\right)\leq\alpha_n$ or not, for each n. If this constraint is not satisfied by some item, we can denote the set $\mathscr{D}_2$ that contains all these items. According to the observations in Section 3, we set $a_n=\alpha_n$, for each $n\in\mathscr{D}_2$, and $a_n=\bar{a}_n\left(k_nT\right)$, for each $n\notin\mathscr{D}_2$. The solution procedure must then restart to calculate new ${}^*T$ and ${}^*k_n$, for $n=2,3,...,N$, keeping the value of ${}^*k_1$ unchanged. This sub-routine is repeated as long as, at the end of the pth run of the algorithm, $\bar{a}_n\left({}^*k_n{}^*T\right)>\alpha_n$ for any n that has not been included into $\mathscr{D}_2$ at the end of iteration $p-1$. Note that $\mathscr{D}_2$ changes in each iteration. This cycle is concluded in the iteration that ends with $\mathscr{D}_2=\emptyset$, giving in output the sub-optimal value ${}^*a_n$ of a_n, for each n, along with ${}^*T$ and ${}^*k_n$, for $n=2,3,...,N$, obtained in the last run.

If, at the end of the first iteration of the sub-routine, it turns out that $\mathscr{D}_2=\emptyset$, then that the constraint $0<\bar{a}_n\left({}^*k_n{}^*T\right)\leq\alpha_n$ is satisfied by all items. Thus, we can put ${}^*a_n\equiv\bar{a}_n\left({}^*k_n{}^*T\right)$, for each n, and further iterations are not required. In Figure 1, we summarize the decision process that should be followed to obtain the heuristic solution $\mathbf{s} \equiv ({}^*T, {}^*\mathbf{a}, {}^*k_2, {}^*k_3, ..., {}^*k_n)$, for fixed $\bar{\mathbf{L}}$ and for a given ${}^*k_1$, where ${}^*\mathbf{a} \equiv ({}^*a_1, {}^*a_2, ..., {}^*a_N)$.

FIGURE 1 HERE

We remind the reader that the above procedure has been initialized by imposing that the item with the highest replenishment frequency, denoted by $n = 1$, has unitary multiplier; that is, we have put ${}^*k_1 \equiv 1$. With this ${}^*k_1$ (and for a fixed $\bar{\mathbf{L}}$), we have then calculated the corresponding solution $\mathbf{s}^{(1)} \equiv \left({}^*T^{(1)}, {}^*\mathbf{a}^{(1)}, {}^*k_2^{(1)}, {}^*k_3^{(1)}, ..., {}^*k_n^{(1)}\right)$, where the superscript would emphasize that $\mathbf{s}^{(1)}$ is referred to ${}^*k_1 \equiv 1$. We should now investigate whether a greater value of ${}^*k_1$ may lead to a better solution $\mathbf{s}$ than $\mathbf{s}^{(1)}$, or not. To do this, once $\mathbf{s}^{(1)}$ has been determined, a new solution $\mathbf{s}^{(2)}$ (corresponding to ${}^*k_1 \equiv 2$) should be found, and we must check if the cost relevant to $\left(\mathbf{s}^{(2)}, {}^*k_1\right)$ is smaller than, or equal to, that corresponding to $\left(\mathbf{s}^{(1)}, {}^*k_1\right)$. If so, we proceed by searching for a new solution $\mathbf{s}^{(3)}$, whose cost should be compared with that of $\mathbf{s}^{(2)}$. This process ends when we obtain a solution $\mathbf{s}^{(r)}$ whose cost is greater than that of $\mathbf{s}^{(r-1)}$. When this occurs, we take $\left(\mathbf{s}^{(r-1)}, {}^*k_1\right)$, with ${}^*k_1 \equiv r - 1$, as heuristic solution to the problem of minimizing $C\left(T, \mathbf{a}, \mathbf{k}, \bar{\mathbf{L}}\right)$, with fixed $\bar{\mathbf{L}}$.

Finally, according to Eq. (6), the entire previously described procedure should be repeated for each vector $\bar{\mathbf{L}}$ to find the heuristic solution $(T^*, \mathbf{a}^*, \mathbf{k}^*, \mathbf{z}^*, \mathbf{L}^*)$, and the related cost C^*, to problem (P). The complete algorithm is formalized below:

Algorithm 2.

Step 1. Set $\mathscr{C} \leftarrow \emptyset$.

Step 2. For each vector $\bar{\mathbf{L}}$, do Steps 2.1-2.6.

Step 2.1. Initialize stopping condition.

Step 2.2. For each n, calculate ${}^*T_n$ as the minimum in k_nT of $C_n\left(T, a_n, k_n, \bar{L}_n\right)$.

Step 2.3. Let $n = 1$ be the index of the item with the smallest ${}^*T_n$. Set ${}^*k_1 = 1$ and ${}^*C = +\infty$.

Step 2.4. Set $\hat{a}_n \leftarrow \bar{a}_n(k_nT)$, for each n, $k_1 \leftarrow^* k_1$, and $\mathscr{D}_1 \leftarrow \{1, 2, ..., N\}$.

Step 2.5. Set $\mathscr{D}_2 \leftarrow \emptyset$ and $a_n \leftarrow \hat{a}_n$, for each n.

Step 2.6. Calculate $\tilde{T}$ as the solution in T of the equation $\frac{\partial}{\partial T}\left[\frac{A}{T} + C_1\left(T, a_1, k_1, \bar{L}_1\right)\right] = 0$.

Step 2.7. For $n = 2, 3, ..., N$, set $q_n \leftarrow \left\lfloor \frac{{}^*T_n}{\tilde{T}} \right\rfloor$.

Step 2.8. For $n = 2, 3, ..., N$, if $C_n\left(\tilde{T}, a_n, q_n, \bar{L}_n\right) \leq C_n\left(\tilde{T}, a_n, q_n + 1, \bar{L}_n\right)$, then set ${}^*k_n \leftarrow q_n$; otherwise, set ${}^*k_n \leftarrow q_n + 1$.

Step 2.9. Let ${}^*\mathbf{k} \equiv ({}^*k_1, {}^*k_2, ..., {}^*k_N)$. Calculate ${}^*T$ as the solution in T of the equation $\frac{\partial}{\partial T}C\left(T, \mathbf{a}, {}^*\mathbf{k}, \bar{\mathbf{L}}\right) = 0$.

Step 2.10. If $\mathscr{D}_1 = \emptyset$, then go to Step 2.13; otherwise, go to Step 2.11.

Step 2.11. For each $n \in \mathscr{D}_1$, if $0 < \bar{a}_n({}^*k_n \cdot {}^* T) \leq \alpha_n$, then set ${}^*a_n \leftarrow \bar{a}_n({}^*k_n \cdot {}^* T)$; otherwise, set ${}^*a_n \leftarrow \alpha_n$ and $\mathscr{D}_2 \leftarrow \mathscr{D}_2 \cup \{n\}$.

Step 2.12. If $\mathscr{D}_2 = \emptyset$, then go to Step 2.13; otherwise, set $\mathscr{D}_1 \leftarrow \mathscr{D}_1 \setminus \mathscr{D}_2$, $\hat{a}_n \leftarrow \bar{a}_n(k_nT)$, for each $n \in \mathscr{D}_1$, and $\hat{a}_n \leftarrow \alpha_n$, for each $n \in \mathscr{D}_2$, then go to Step 2.5.

Step 2.13. Let ${}^*\mathbf{a} \equiv ({}^*a_1, {}^*a_2, ..., {}^*a_N)$. If $C\left({}^*T, {}^*\mathbf{a}, {}^*\mathbf{k}, \bar{\mathbf{L}}\right) \leq {}^*C$, then set ${}^*\bar{T} \leftarrow {}^*T$, ${}^*\bar{\mathbf{k}} \leftarrow {}^*\mathbf{k}$, ${}^*\bar{\mathbf{a}} \leftarrow {}^*\mathbf{a}$.

Step 2.14. If $C\left({}^*T, {}^*\mathbf{a}, {}^*\mathbf{k}, \bar{\mathbf{L}}\right) \leq {}^*C$ AND stopping condition is not satisfied, then set ${}^*C \leftarrow C\left({}^*T, {}^*\mathbf{a}, {}^*\mathbf{k}, \bar{\mathbf{L}}\right)$ and ${}^*k_1 \leftarrow^* k_1 + 1$, and go to Step 2.4; otherwise, go to Step 2.15.

Step 2.15. Set $\mathscr{C} \leftarrow \mathscr{C} \cup \left\{C\left({}^*\bar{T}, {}^*\bar{\mathbf{a}}, {}^*\bar{\mathbf{k}}, \bar{\mathbf{L}}\right)\right\}$.

Step 3. Set $(T^*, \mathbf{a}^*, \mathbf{k}^*, \mathbf{L}^*) \leftarrow \arg\min \mathscr{C}$, $C^* \leftarrow \min \mathscr{C}$, and $z_n^* \leftarrow \bar{z}_n(k_n^*T^*)$, for each n.

We can observe that:

- Steps 2.6–2.9 deal with the minimization in $(T, k_2, k_3, ..., k_N)$, with fixed $\left(\mathbf{a}, k_1, \bar{\mathbf{L}}\right)$.
- Steps 2.4–2.13 consider the minimization in $(T, \mathbf{a}, k_2, k_3, ..., k_N)$, with fixed $\left(k_1, \bar{\mathbf{L}}\right)$.
- Steps 2.1–2.14 find the heuristic solution in $(T, \mathbf{a}, \mathbf{k})$, with fixed $\bar{\mathbf{L}}$.

Similarly to Algorithm 1, it is not possible to give an estimate or an upper bound to the computational requirements of Algorithm 2. While the number of iterations to approach the minimization in $(T, \mathbf{a}, k_2, k_3, ..., k_N)$, with fixed $\left(k_1, \bar{\mathbf{L}}\right)$, is at most equal to $N+1$, the time (i.e., the number of iterations) needed to obtain the heuristic solution in $(T, \mathbf{a}, \mathbf{k})$, with fixed $\bar{\mathbf{L}}$, is strongly dependent on parameter values and nothing can *a priori* be said. Note that Algorithm 2 includes a stopping condition at Step 2.14, which is initialized at Step 2.1. This permits us to assure that the heuristic approaching the minimization in $(T, \mathbf{a}, \mathbf{k})$, with fixed $\bar{\mathbf{L}}$, terminates within the time limit imposed by the stopping criterion. Moreover, it is not possible to prove that the sub-routine concerned with the optimization in $(T, \mathbf{a}, \mathbf{k})$, with fixed $\bar{\mathbf{L}}$, will converge in a finite time. Although several stopping criteria may be adopted, a reasonable termination criterion is to specify the largest value that ${}^{*}k_1$ can take. We finally remind the reader that the heuristic solution in $\left(T, \mathbf{a}, \mathbf{k}, \bar{\mathbf{L}}\right)$, with fixed $\bar{\mathbf{L}}$, must be determined for $\prod_n (M_n + 1)$ different vectors $\bar{\mathbf{L}}$.

5. A lower bound on the minimum cost

The cost of the solution obtained from a heuristic algorithm is an upper bound to the minimum cost obtained solving problem (P) exactly. Since this minimum cost is unknown, a lower bound is, therefore, useful to evaluate the effectiveness of both heuristics. We will provide a lower bound according to the following procedure. We first allocate the joint (i.e., major) ordering cost A to each item in such a way that the sum of the allocated costs over items is less than or equal to the original cost A. The sum of $\sum_{n=1}^{N} C_n (T, a_n, k_n, z_n, L_n)$ plus the single-item costs derived from partitioning A among the items gives a lower bound for the total system cost, i.e., for $C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L})$. This lower-bound cost function has the property to be separable in items so that it can be minimized one item at a time. Note that this joint-cost allocation scheme, as originally proposed by Atkins and Iyogun (1987; 1988), has also been adopted, for example, by Viswanathan (1997) in the case of the JRP under a continuous-review policy with full backlogging.

Let $\phi_n \geq 0$, for each n, such that $\sum_{n=1}^{N} \phi_n = 1$. If ϕ_n is the fraction of the joint ordering cost A that is allocated to item n, then it is possible to prove that, for a given $\boldsymbol{\phi} \equiv (\phi_1, \phi_2, ..., \phi_N)$, the cost function

$$
\begin{aligned}
J(\boldsymbol{\phi}) &= \min_{(\mathbf{a}, T_1, T_2, \ldots, T_n, \mathbf{z}, \mathbf{L})} \sum_{n=1}^{N} \left[\frac{\phi_n A}{T_n} + C_n (a_n, T_n, z_n, L_n) \right] \\
&= \sum_{n=1}^{N} \min_{(a_n, T_n, z_n, L_n)} \left[\frac{\phi_n A}{T_n} + C_n (a_n, T_n, z_n, L_n) \right]
\end{aligned}
$$

is a lower bound on $\min\limits_{(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L})} C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L})$. Note that the second equality follows from the fact that $\sum_{n=1}^{N} \left[\frac{\phi_n A}{T_n} + C_n (a_n, T_n, z_n, L_n) \right]$ is separable in items. Evidently, the set $\left\{ J(\boldsymbol{\phi}) : \phi_n \geq 0, \ \sum_{n=1}^{N} \phi_n = 1 \right\}$ defines a class of lower bounds. The problem concerned with finding the best, or the highest, lower bound within the class of lower bounds can be stated as (Atkins & Iyogun, 1987; 1988; Iyogun & Atkins, 1993)

$$
\begin{aligned}
\text{(Q)} \qquad & \max_{\boldsymbol{\phi}} \quad J(\boldsymbol{\phi}) \\
& \text{s.t.} \quad \phi_n \geq 0,\ \forall\, n, \\
& \qquad\ \ \sum_{n=1}^{N} \phi_n = 1.
\end{aligned}
$$

We now prove that $J^* = \max_{\boldsymbol{\phi}} J(\boldsymbol{\phi})$ is the desired lower bound for $\min_{(T,\mathbf{a},\mathbf{k},\mathbf{z},\mathbf{L})} C(T,\mathbf{a},\mathbf{k},\mathbf{z},\mathbf{L})$ and then we give an algorithm to approach problem (Q) that is based on the procedure proposed by Atkins and Iyogun (1988).

Proposition 2. *J^* is a lower bound on $\min_{(T,\mathbf{a},\mathbf{k},\mathbf{z},\mathbf{L})} C(T,\mathbf{a},\mathbf{k},\mathbf{z},\mathbf{L})$.*

Proof. For any $\boldsymbol{\phi}$, with $\phi_n \geq 0$, for each n, such that $\sum_{n=1}^{N} \phi_n = 1$, we have

$$
\begin{aligned}
& \min_{(T,\mathbf{a},\mathbf{k},\mathbf{z},\mathbf{L})} C(T,\mathbf{a},\mathbf{k},\mathbf{z},\mathbf{L}) \\
= \ & \min_{(T,\mathbf{a},\mathbf{k},\mathbf{z},\mathbf{L})} \frac{A}{T} + \sum_{n=1}^{N} C_n(T,a_n,k_n,z_n,L_n) \\
= \ & \min_{(T,\mathbf{a},\mathbf{k},\mathbf{z},\mathbf{L})} \sum_{n=1}^{N} \phi_n \frac{A}{T} + \sum_{n=1}^{N} C_n(T,a_n,k_n,z_n,L_n) \\
\geq \ & \min_{(T,\mathbf{a},\mathbf{k},\mathbf{z},\mathbf{L})} \sum_{n=1}^{N} \frac{\phi_n}{k_n} \frac{A}{T} + \sum_{n=1}^{N} C_n(T,a_n,k_n,z_n,L_n).
\end{aligned}
$$

Since the above relations are valid for any vector $\boldsymbol{\phi}$, it is evidently true that

$$
\min_{(T,\mathbf{a},\mathbf{k},\mathbf{z},\mathbf{L})} C(T,\mathbf{a},\mathbf{k},\mathbf{z},\mathbf{L}) \geq \max_{\boldsymbol{\phi}} \left[\min_{(T,\mathbf{a},\mathbf{k},\mathbf{z},\mathbf{L})} \sum_{n=1}^{N} \frac{\phi_n}{k_n} \frac{A}{T} + \sum_{n=1}^{N} C_n(T,a_n,k_n,z_n,L_n) \right].
$$

If we also consider that, by relaxing the integrality constraint on k_n, for each n, we have

$$
\min_{(T,\mathbf{a},\mathbf{k},\mathbf{z},\mathbf{L})} \sum_{n=1}^{N} \frac{\phi_n}{k_n} \frac{A}{T} + \sum_{n=1}^{N} C_n(T,a_n,k_n,z_n,L_n) \geq \min_{(\mathbf{a},T_1,T_2,\ldots,T_n,\mathbf{z},\mathbf{L})} \sum_{n=1}^{N} \phi_n \frac{A}{T_n} + \sum_{n=1}^{N} C_n(a_n,T_n,z_n,L_n),
$$

then we can write

$$
\min_{(T,\mathbf{a},\mathbf{k},\mathbf{z},\mathbf{L})} C(T,\mathbf{a},\mathbf{k},\mathbf{z},\mathbf{L}) \geq \max_{\boldsymbol{\phi}} \left[\min_{(\mathbf{a},T_1,T_2,\ldots,T_n,\mathbf{z},\mathbf{L})} \sum_{n=1}^{N} \phi_n \frac{A}{T_n} + \sum_{n=1}^{N} C_n(a_n,T_n,z_n,L_n) \right] = J^*.
$$

This completes the proof. □

The following algorithm defines a procedure to approach problem (Q) so as to find the solution $\boldsymbol{\phi}^* \equiv (\phi_1^*, \phi_2^*, ..., \phi_N^*) = \arg\max_{\boldsymbol{\phi}} J(\boldsymbol{\phi})$:

Algorithm 3.

Step 1. *Set* $\phi_n = 0$ *and* $\phi_n^* \leftarrow \phi_n$.

Step 2. *Minimize* $C_n(a_n, T_n, z_n, L_n)$ *in* (a_n, T_n, z_n, L_n), *for each* n. *Let* T_n^* *be the minimum in* T_n, *for each* n.

Step 3. *Relabel items in nondecreasing order, i.e., let* $n = 1$ *be the item with the smallest* T_n^* *and* $n = N$ *be the item with the highest* T_n^*. *Set* $n^* \leftarrow n = 1$.

Step 4. *For each* $n = 1, 2, ..., n^*$, *increase the allocated fraction* $\phi_n A$ *and minimize* $\frac{\phi_n A}{T_n} + C_n(a_n, T_n, z_n, L_n)$ *in* (a_n, T_n, z_n, L_n). *Let* $\hat{T}_n$ *be the minimum in* T_n.

Step 5. *For each* $n = 1, 2, ..., n^*$, *if* $\hat{T}_n = T_{n^*+1}^*$, *then set* $\phi_n^* \leftarrow \phi_n$ *and go to Step 6; otherwise, go to Step 4.*

Step 6. *If* $\sum_{n=1}^{n^*} \phi_n = 1$, *then STOP; otherwise, set* $n^* \leftarrow n^* + 1$ *and go to Step 4.*

It is worth noting that problem (Q) can also be approached with a metaheuristic algorithm, such as the simulated annealing.

6. Numerical study

In this section, we evaluate the performance of Algorithm 1 and Algorithm 2 in terms of computational time and solution quality. We have carried out two different series of experiments. The first one considers smaller problems and the objective is to solve problem (P). The second one considers larger problems and the objective is to solve a sub-problem of problem (P). That is, in this second case, the purpose is the optimization of $C(T, \mathbf{a}, \mathbf{k}, \mathbf{z}, \mathbf{L})$ for a given vector $\mathbf{L}$. As the reader will note in this section, Algorithm 1 may become computationally onerous for large problems. Therefore, to ease the experimental comparison, which exploits a standard computing platform, it is practical reducing the problem complexity. Since the optimization over $\mathbf{L}$ is a standard combinatorial optimization problem approached with an exhaustive search method and the proposed heuristics do not differ in how the minimum in $\mathbf{L}$ is searched, we have, therefore, considered the simpler problem where $\mathbf{L}$ is a parameter, rather than a decision variable. The reader will also note that this choice does not change the relative performance between the heuristics that is observed in both series of experiments. Because of the time required to approach problem (P) for large problems, the number of instances in the first series of experiments is smaller than in the second one.

Experiments have been executed on a PC with an Intel® Core™ i3 processor at 3.10GHz and 4GB of RAM memory. Moreover, MATLAB® R2013b has been used as computing environment. In each problem, Algorithms 1 and 2 adopt the following stopping criterion:

- In Algorithm 1, the multiplier of each item cannot be larger than 30;
- In Algorithm 2, the multiplier of the item with the highest replenishment frequency cannot be larger than 30.

Both heuristics have been compared with a specifically developed benchmark algorithm, i.e., a hybrid genetic algorithm (HGA), which works as follows. The genetic algorithm (GA) included into Optimization Toolbox™ has been used to solve the following (sub-)problem for each vector $\bar{\mathbf{L}}$:

$$\begin{aligned} \min_{\mathbf{k}} \quad & C\left(\hat{T}, \hat{\mathbf{a}}, \mathbf{k}, \bar{\mathbf{L}}\right), \\ \text{s.t.} \quad & \mathbf{k} \in \mathbb{N}^N, \end{aligned}$$

where $\left(\hat{T}, \hat{\mathbf{a}}\right)$ is the vector $(T, \mathbf{a})$ that minimizes $C\left(T, \mathbf{a}, \mathbf{k}, \bar{\mathbf{L}}\right)$, for fixed $\left(\mathbf{k}, \bar{\mathbf{L}}\right)$. To find $\left(\hat{T}, \hat{\mathbf{a}}\right)$, we have used the same procedure described in Section 3 (see, e.g., the routine between Steps 2.2.2.2–2.2.2.6 of Algorithm 1). GA has been implemented with default parameter values, except those listed below that have been tuned with preliminary tests:

- Population size: $15 \cdot N$;
- Elite count: $\lfloor 0.2 \cdot \text{population size} \rfloor$;
- Crossover fraction: 0.6;
- Migration direction: both;
- Maximum generations number: $10 \cdot N$;
- Stall generations limit: $8 \cdot N$.

In the first series of experiments, we take into account three values of N: $N = 4, 5, 6$. The lead time of each item is supposed to be made of three components (i.e., $m = 1, 2, 3$ for each n). Parameter values belong to the ranges shown in Table 2. Table 3 shows the specific values that the parameters take. For these problems, we provide the selected parameter values and the obtained solutions so as to permit comparisons the reader may be interested to do.

Table 4 gives the results of this first round of experiments. We can first note that the greatest absolute percentage error (APE), evaluated with respect to the lower bound, is 1.9%, which has been achieved in problem P2 by Algorithm 2. In this problem, APE is 1.8% for Algorithm 1 and HGA. In the other problems, APE is always smaller than 1% for either heuristic algorithm. The best solutions have been obtained by Algorithm 1 and HGA. Algorithm 2 has found a slightly worse solution in problems P2 and P5. Since Algorithm 1 has always found the same solution as HGA (Algorithm 2 in problems P1, P3, and P4 only), it is reasonable that it has reached the optimum in each problem.

If we now consider the computational time, we can observe that Algorithm 2 is more efficient than both Algorithm 1 and HGA. Moreover, while Algorithm 2 seems to be less sensitive to variations in parameter values (for a given value of N), Algorithm 1 shows a notable variability in the computational time. In this regard, note that, for $N = 5$, the computational time of Algorithm 1 in problem P3 is two orders of magnitude greater than in problem P4.

In conclusion, we may affirm that Algorithm 2 seems promising from the practical perspective. In fact, the cost difference between the solutions obtained by Algorithm 1 and Algorithm 2 is negligible. Moreover, Algorithm 2 has turned out to be computationally more efficient than Algorithm 1.

TABLE 2 HERE

TABLE 3 HERE

TABLE 4 HERE

Similar conclusions can be drawn from the second series of experiments, whose results are given in Table 5. We remind the reader that these tests are concerned with solving a sub-problem of problem (P), where $\mathbf{L}$ is a parameter rather than a decision variable. Note that, for $N > 6$, Algorithm 1 has not been taken into consideration. In fact, its computational time grows fast as N increases, up to unpractical values in some cases. We can observe that, in general, HGA seems preferable to Algorithm 1 for several reasons: (1) HGA is able to converge, on average, more quickly (especially for $N > 5$); (2) the computational time of HGA is less variable, for fixed N; and (3) HGA and Algorithm 1 have always found the same solution in all generated problems. However, in terms of the computational requirements, Algorithm 2 has shown the best performance: if we observe, for example, the case with $N = 9$, the percentage of computational time reduction is, on average, more than 99.2%.

Now, consider the performance in terms of APE with respect to the lower bound. Although, in general, the efficiency is decreasing as N increases, the APE achieved by all heuristics is always below 5%. This permits us to say that their solutions appear to be fairly efficient. In conclusion, if we also note that, in terms of APE, Algorithm 2 differs from HGA (or from Algorithm 1) by 0.9% at most, we may conclude that Algorithm 2 seems practically preferable.

TABLE 5 HERE

7. Conclusions

This paper investigated the periodic review JRP under stochastic demands with backorders-lost sales mixtures. Lead times and minor ordering costs were assumed to be controllable. The objective was to find the cyclic replenishment policy, the length of lead times and the minor ordering costs that minimize the long-run expected total cost per time unit.

To approach this problem, we presented two heuristic algorithms. The performance of these algorithms was evaluated in comparison with a lower bound on the minimum cost and a specifically developed benchmark algorithm called hybrid genetic algorithm (HGA). Numerical experiments showed that the second heuristic (i.e., Algorithm 2) is practically preferable. In fact, it appeared, on average, computationally faster than the others, and this advantage is even more evident as the size of problems becomes larger. For example, if we consider the case with $N = 9$, the percentage of computational time reduction was, on average, more than 99.2%. Moreover, the APE of Algorithm 2, with respect to the lower bound, was smaller than 5% in all generated problems, and the performance with respect to HGA (or Algorithm 1) was, at most, 0.9% worse.

Future researches may be devoted to the following questions. Is it possible to extend the model presented in this paper, e.g., continuous lead time? How to analytically compare the performance of the proposed heuristics with the global optimal solution in terms of solution quality and computational efficiency?

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Figure 1. Decision process to find the heuristic solution for fixed $\left({}^{*}k_1, \bar{\mathbf{L}}\right)$.

Table 1

Author(s)	Demand		Lead time		Review policy		Major ordering cost		Minor ordering costs						
	Stochastic	*Deterministic*	*Controllable*	*Fixed*	*Periodic*	*Continuous*	*Controllable*	*Fixed*	*Controllable*	*Fixed*	*Backorders-lost sales mixtures*	*Cyclic replenishment policy*[1]	*Stockout costs*	*Distribution-free approach*	*Lower bound*
Kiesmüller (2010)	◆			◆		◆		◆		◆					
Tsao and Sheen (2012)		◆		◆	◆			◆		◆					
Braglia et al. (2016a)	◆		◆		◆		◆			◆	◆		◆		
Amaya et al. (2013)		◆		◆	◆			◆		◆		◆			
Tsao and Teng (2013)		◆		◆	◆			◆		◆		◆			
Wang et al. (2013)		◆		◆	◆			◆		◆		◆			
Paul et al. (2014)		◆		◆	◆			◆		◆					
Tsai et al. (2009)	◆			◆		◆		◆		◆			◆		
Braglia et al. (2016b)	◆		◆		◆			◆		◆	◆	◆	◆		
Konur and Schaefer (2016)		◆		◆	◆			◆		◆		◆			
Narayanan and Robinson (2010a)		◆		◆				◆		◆					
Tanrikulu et al. (2010)	◆			◆		◆		◆		◆			◆		
Qu et al. (2015)	◆			◆	◆			◆		◆		◆	◆		
Mohammaditabar and Ghodsypour (2016)		◆		◆	◆			◆		◆					
Yao and Huang (2014)		◆		◆	◆			◆		◆		◆			
Cheung et al. (2015)		◆		◆				◆		◆					
Brahimi et al. (2015)		◆		◆				◆		◆					
Bienkowski et al. (2015)		◆		◆				◆		◆					
Feng et al. (2015)	◆			◆		◆		◆		◆			◆		
Braglia et al. (2017)	◆		◆		◆		◆			◆				◆	
This paper	◆		◆		◆			◆	◆		◆	◆	◆		◆

[1] Only under periodic review.

Table 1. Comparison among the contributions of different authors.

Table 2

Parameters	Values
A	Random in [150, 250]
h_n	Random in [1, 25]
ρ_n	Random in [20, 70]
π_n	Random in [80, 150]
β_n	Random in [0.1, 0.9]
D_n	Random in [100, 2000]
σ_n/D_n	Random in [0.01, 0.40]
α_n	Random in [150, 250]
τ_n	0.1 (identical for each n)
$1/\delta_n$	Random in [4000, 7000]
$s_{m,n}$	Random in [17, 25]
$b_{m,n}$	Random in [7, 15]
$c_{1,n}$	Random in [0.1, 1.0]
$c_{2,n}$	Random in [1.6, 3.4]
$c_{3,n}$	Random in [4.2, 6.2]

Table 2. Values assigned to parameters in each problem.

Table 3

Problem	A	Item (n)	$1/\delta_n$	α_n	h_n	D_n	σ_n/D_n	ρ_n	π_n	β_n	$s_{1,n}$ (days)	$s_{2,n}$ (days)	$s_{3,n}$ (days)	$b_{1,n}$ (days)	$b_{2,n}$ (days)	$b_{3,n}$ (days)	$c_{1,n}$ (\$/day)	$c_{2,n}$ (\$/day)	$c_{3,n}$ (\$/day)
P1	232	1	4357	178	18	1271	0.13	24	109	0.9	22	25	19	11	15	8	0.3	1.9	5.8
		2	5495	205	1	999	0.22	40	83	0.7	20	25	21	14	13	10	0.5	3.1	4.4
		3	6880	246	7	768	0.07	33	144	0.6	24	20	25	13	11	14	0.6	2.8	5.4
		4	5021	247	2	1679	0.24	60	147	0.5	21	18	21	15	11	14	0.9	2.3	4.2
P2	241	1	5756	165	3	1212	0.11	42	114	0.5	20	19	21	15	7	7	0.5	1.9	5.1
		2	4671	248	21	1145	0.27	66	103	0.3	25	20	19	10	13	10	1.0	2.4	4.8
		3	6254	246	18	1843	0.28	29	143	0.7	24	22	21	13	7	11	0.4	2.5	4.5
		4	4765	199	8	643	0.30	33	106	0.3	21	19	22	8	7	10	0.7	1.8	4.6
P3	162	1	5518	230	24	1539	0.19	27	87	0.6	22	22	23	7	11	12	0.7	2.7	5.0
		2	6097	164	1	1532	0.04	26	135	0.2	22	23	20	13	7	12	0.6	2.0	4.4
		3	6673	192	11	823	0.10	64	107	0.4	18	18	20	11	14	9	0.7	2.3	5.1
		4	6878	242	10	1179	0.37	49	97	0.6	19	18	25	11	14	10	0.7	2.6	5.6
		5	5642	230	20	244	0.07	48	108	0.7	21	19	17	15	13	7	0.3	2.1	5.6
P4	213	1	4416	246	20	202	0.33	27	86	0.1	19	19	24	12	8	15	0.2	2.1	5.5
		2	4448	216	5	1109	0.22	63	89	0.8	24	20	25	12	12	8	1.0	2.7	4.3
		3	4772	153	13	1581	0.40	51	146	0.7	18	21	24	14	11	7	0.3	2.1	4.3
		4	6522	235	12	1875	0.04	37	147	0.5	19	17	17	14	15	10	0.1	3.1	4.8
		5	4763	244	17	346	0.18	46	120	0.4	18	19	19	12	12	8	0.6	3.4	5.3
P5	159	1	6443	218	18	1181	0.05	40	84	0.5	19	24	20	8	14	11	0.9	2.9	5.5
		2	4730	226	19	992	0.39	23	96	0.3	20	17	23	9	11	10	0.7	2.2	5.0
		3	6788	225	7	122	0.01	32	105	0.9	19	25	18	14	10	15	0.3	2.7	5.8
		4	5050	189	17	740	0.31	26	138	0.5	25	23	23	7	14	15	0.4	1.8	5.6
		5	4589	216	17	408	0.33	29	81	0.8	20	21	17	11	7	7	0.5	3.2	6.1
		6	4753	167	5	1609	0.35	32	83	0.7	18	22	22	8	8	13	1.0	3.2	5.3

Table 3. Values that parameters take in each problem in the first series of experiments.

Table 4

Prob.	Lower bound	Item - *n*	ϕ_n^*	Algorithm 2 k_n	L_n	a_n	T	*Cost*	*Comp. time (sec.)*	Algorithm 1 k_n	L_n	a_n	T	*Cost*	*Comp. time (sec.)*	HGA k_n	L_n	a_n	T	*Cost*	*Comp. time (sec.)*
P1	13586	*1*	1	1	34	58.1				1	34	58.1				1	34	58.1			
		2	0	4	37	205.0				4	37	205.0				4	37	205.0			
		3	0	2	38	183.5				2	38	183.5				2	38	183.5			
		4	0	1	40	67.0				1	40	67.0				1	40	67.0			
							0.1334	13610 (< 1%)	17				0.1334	13610 (< 1%)	2595				0.1334	13610 (< 1%)	14554
P2	29106	*1*	0.0397	2	29	95.7				3	29	140.2				3	29	140.2			
		2	0.7441	1	33	38.8				1	33	37.9				1	33	37.9			
		3	0.2016	1	31	52.0				1	31	50.8				1	31	50.8			
		4	0.0146	1	25	39.6				1	25	37.7				1	25	37.7			
							0.0831	29652 (1.9%)	9				0.0812	29628 (1.8%)	500				0.0812	29628 (1.8%)	9152
P3	24346	*1*	1	1	30	47.8				1	30	47.8				1	30	47.8			
		2	0	5	32	164.0				5	32	164.0				5	32	164.0			
		3	0	1	34	57.8				1	34	57.8				1	34	57.8			
		4	0	1	35	59.6				1	35	59.6				1	35	59.6			
		5	0	2	35	97.8				2	35	97.8				2	35	97.8			
							0.0866	24467 (< 1%)	72				0.0866	24467 (< 1%)	79312				0.0866	24467 (< 1%)	10018
P4	25678	*1*	0.0002	2	35	82.1				2	35	82.1				2	35	82.1			
		2	0	1	32	41.3				1	32	41.3				1	32	41.3			
		3	0.9511	1	32	44.3				1	32	44.3				1	32	44.3			
		4	0.0472	1	39	60.6				1	39	60.6				1	39	60.6			
		5	0.0015	1	32	44.3				1	32	44.3				1	32	44.3			
							0.0929	25775 (< 1%)	37				0.0929	25775 (< 1%)	2188				0.0929	25775 (< 1%)	22883
P5	29378	*1*	0.2349	1	33	54.0				1	33	52.0				1	33	52.0			
		2	0.3494	1	30	39.7				1	30	38.2				1	30	38.2			
		3	0	6	39	225.0				9	39	225.0				9	39	225.0			
		4	0.0298	1	36	42.3				1	36	40.8				1	36	40.8			
		5	0	1	25	38.5				1	25	37.1				1	25	37.1			
		6	0.3859	1	29	39.8				1	29	38.4				1	29	38.4			
							0.0838	29564 (< 1%)	312				0.0808	29520 (< 1%)	$> 10^6$				0.0808	29520 (< 1%)	58518

Table 4. Results of the comparative analysis in the first series of experiments. Within brackets, the APE with respect to the lower bound is shown.

Table 5

No. of items	Lower bound	Computational time (sec.) Algorithm 1	Algorithm 2	HGA	Cost of solution Algorithm 1	Algorithm 2	HGA
	14524	< 1	< 1	12	14837 (2.1%)	14837 (2.1%)	14837 (2.1%)
	14842	< 1	< 1	14	15079 (1.6%)	15079 (1.6%)	15079 (1.6%)
	17759	< 1	< 1	11	17790 (< 1%)	17790 (< 1%)	17790 (< 1%)
	15925	7	< 1	18	15937 (< 1%)	15940 (< 1%)	15937 (< 1%)
4	15120	< 1	< 1	13	15167 (< 1%)	15167 (< 1%)	15167 (< 1%)
	10267	137	< 1	19	10290 (< 1%)	10290 (< 1%)	10290 (< 1%)
	11357	10	< 1	19	11400 (< 1%)	11400 (< 1%)	11400 (< 1%)
	11212	27	< 1	18	11262 (< 1%)	11271 (< 1%)	11262 (< 1%)
	8950	3	< 1	17	9178 (2.5%)	9178 (2.5%)	9178 (2.5%)
	10215	< 1	< 1	37	10268 (< 1%)	10268 (< 1%)	10268 (< 1%)
	9923	16	< 1	33	10006 (< 1%)	10054 (1.3%)	10006 (< 1%)
	13624	14	< 1	31	13648 (< 1%)	13723 (< 1%)	13648 (< 1%)
	15109	48	< 1	33	15502 (2.6%)	15502 (2.6%)	15502 (2.6%)
5	10104	4	< 1	29	10541 (4.3%)	10541 (4.3%)	10541 (4.3%)
	10215	< 1	< 1	33	10268 (< 1%)	10268 (< 1%)	10268 (< 1%)
	9923	13	< 1	29	10006 (< 1%)	10054 (1.3%)	10006 (< 1%)
	13624	12	< 1	26	13648 (< 1%)	13723 (< 1%)	13648 (< 1%)
	10464	3	< 1	21	10613 (1.4%)	10649 (1.8%)	10613 (1.4%)
	12626	66	< 1	41	13172 (4.3%)	13181 (4.4%)	13172 (4.3%)
	27988	451	< 1	43	28907 (3.3%)	29165 (4.2%)	28907 (3.3%)
	13230	68	< 1	49	13356 (< 1%)	13356 (< 1%)	13356 (< 1%)
	20761	7	< 1	34	20923 (< 1%)	20923 (< 1%)	20923 (< 1%)
6	19597	7	< 1	31	19848 (1.3%)	19856 (1.3%)	19848 (1.3%)
	10502	17332	< 1	46	10557 (< 1%)	10613 (1.1%)	10557 (< 1%)
	21075	< 1	< 1	37	21969 (4.2%)	21969 (4.2%)	21969 (4.2%)
	15483	85	< 1	53	15822 (2.2%)	15867 (2.5%)	15822 (2.2%)
	27723	< 1	< 1	32	27907 (< 1%)	27907 (< 1%)	27907 (< 1%)
	27468	-	< 1	59	-	28008 (2.0%)	27905 (1.6%)
	25408	-	< 1	73	-	25729 (1.3%)	25599 (< 1%)
	13986	-	< 1	97	-	14048 (< 1%)	14016 (< 1%)
	24809	-	< 1	78	-	25866 (4.3%)	25819 (4.1%)
7	20677	-	< 1	71	-	20830 (< 1%)	20822 (< 1%)
	21255	-	< 1	85	-	21413 (< 1%)	21376 (< 1%)
	23939	-	< 1	61	-	24701 (3.2%)	24668 (3.0%)
	20867	-	< 1	69	-	21232 (1.7%)	21209 (1.6%)
	20894	-	< 1	66	-	21743 (4.0%)	21692 (3.8%)
	24950	-	< 1	74	-	25899 (3.8%)	25864 (3.7%)
	28454	-	< 1	95	-	29036 (2.0%)	29009 (2.0%)
	28154	-	< 1	113	-	29409 (4.4%)	29341 (4.2%)
	25616	-	< 1	80	-	25908 (1.1%)	25862 (1.0%)
8	24326	-	< 1	73	-	24828 (2.1%)	24753 (1.8%)
	23484	-	< 1	101	-	24119 (2.7%)	23998 (2.2%)
	18078	-	< 1	97	-	18163 (< 1%)	18163 (< 1%)
	17299	-	< 1	94	-	17348 (< 1%)	17348 (< 1%)
	26844	-	< 1	100	-	28050 (4.5%)	28007 (4.3%)
	22964	-	< 1	123	-	23607 (2.8%)	23559 (2.6%)
	18891	-	< 1	125	-	19177 (1.5%)	19177 (1.5%)
	19797	-	< 1	126	-	20632 (4.2%)	20601 (4.1%)
	28626	-	< 1	132	-	29709 (3.8%)	29590 (3.4%)
9	22736	-	< 1	140	-	23175 (1.9%)	23157 (1.8%)
	22591	-	< 1	141	-	23079 (2.2%)	22970 (1.7%)
	13258	-	< 1	133	-	13552 (2.2%)	13549 (2.2%)
	21933	-	< 1	126	-	22703 (3.5%)	22635 (3.2%)
	25922	-	< 1	136	-	26693 (3.0%)	26655 (2.8%)

Table 5. Results of the comparative analysis in the second series of experiments. Within brackets, the APE with respect to the lower bound is shown.

Figure 1

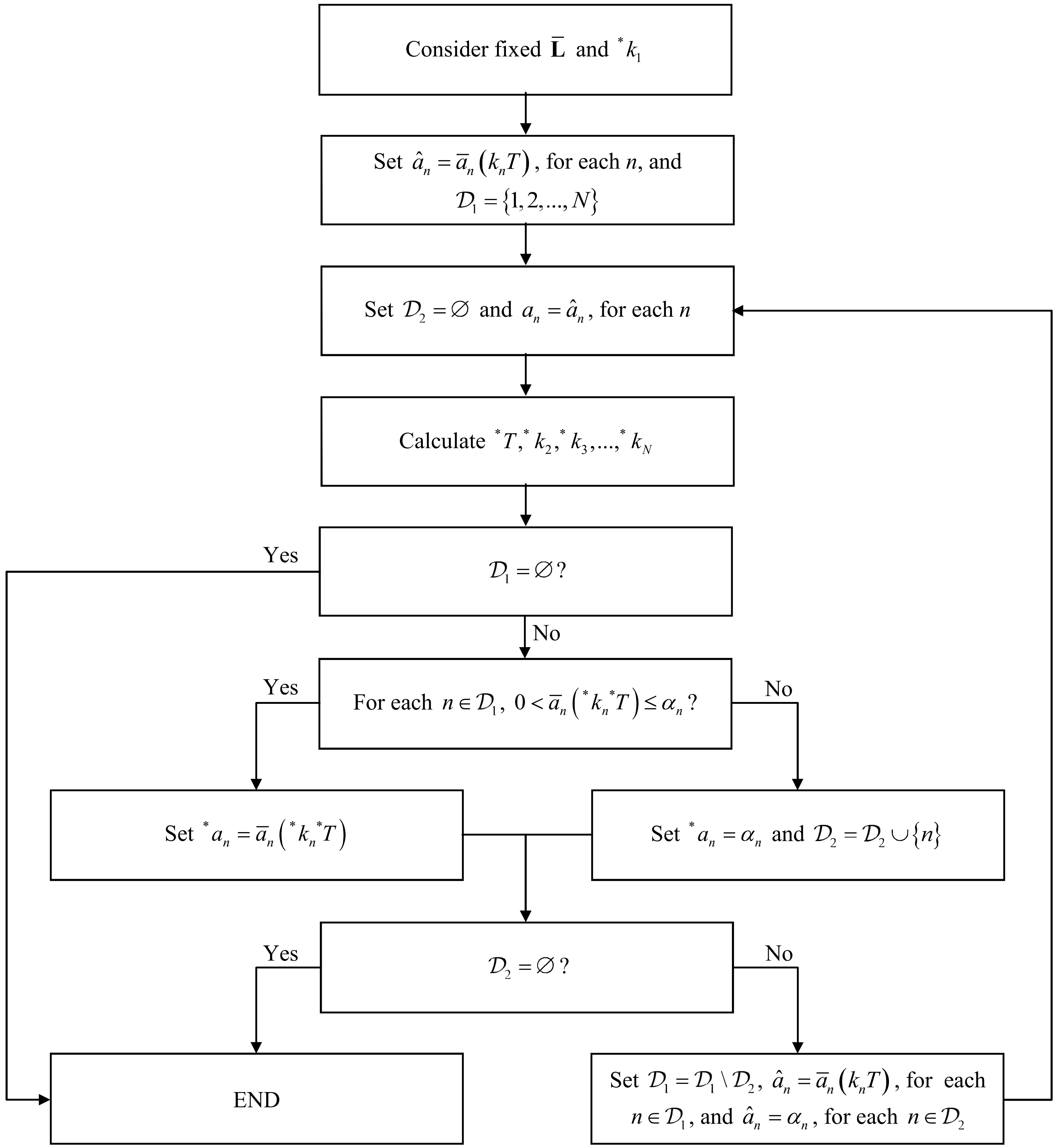

Figure 1. Decision process to find the heuristic solution for fixed $({}^*k_1, \bar{\mathbf{L}})$.