# Strategy Complexity of Parity Objectives in Countable MDPs 

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#### Abstract

We study countably infinite MDPs with parity objectives. Unlike in finite MDPs, optimal strategies need not exist, and may require infinite memory if they do. We provide a complete picture of the exact strategy complexity of $\varepsilon$-optimal strategies (and optimal strategies, where they exist) for all subclasses of parity objectives in the Mostowski hierarchy. Either MD-strategies, Markov strategies, or 1-bit Markov strategies are necessary and sufficient, depending on the number of colors, the branching degree of the MDP, and whether one considers $\varepsilon$-optimal or optimal strategies. In particular, 1-bit Markov strategies are necessary and sufficient for $\varepsilon$-optimal (resp. optimal) strategies for general parity objectives.


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## 1 Introduction

Background. Markov decision processes (MDPs) are a standard model for dynamic systems that exhibit both stochastic and controlled behavior [17]. MDPs play a prominent role in numerous domains, including artificial intelligence and machine learning [20, 19], control theory [4, 1], operations research and finance [5, 18, and formal verification [7, 2].

An MDP is a directed graph where states are either random or controlled. Its observed behavior is described by runs, which are infinite paths that are, in part, determined by the choices of a controller. If the current state is random then the next state is chosen according to a fixed probability distribution. Otherwise, if the current state is controlled, the controller can choose a distribution over all possible successor states. By fixing a strategy for the controller (and initial state), one obtains a probability space of runs of the MDP. The goal of the controller is to optimize the expected value of some objective function on the runs.

The type of strategy necessary to achieve an optimal (resp. $\varepsilon$-optimal) value for a given objective is called its strategy complexity. There are different types of strategies, depending on whether one can take the whole history of the run into account (history-dependent; $(\mathrm{H})$ ), or whether one is limited to a finite amount of memory (finite memory; (F)) or whether decisions are based only on the current state (memoryless; (M)). Moreover, the strategy type depends on whether the controller can randomize ( R ) or is limited to deterministic choices (D). The simplest type, MD, refers to memoryless deterministic strategies. Markov strategies are strategies that base their decisions only on the current state and the number of steps in the history of the run. Thus they do use infinite memory, but only in a very


Figure 1 These diagrams show the strategy complexity of $\varepsilon$-optimal strategies and optimal strategies (where they exist) for parity objectives. Depending on the position in the Mostowski hierarchy, either MD-strategies (green), deterministic Markov-strategies (blue) or deterministic 1-bit Markov strategies (red) are necessary and sufficient (and randomization does not help [12). If the MDPs are finitely branching then the Markov strategies can be replaced by MD-strategies (i.e., the blue parts turn green), but the deterministic 1-bit Markov part (red) remains unchanged.
restricted form by maintaining an unbounded step-counter. Slightly more general are 1-bit Markov strategies that use 1 bit of extra memory in addition to a step-counter.

Parity objectives. We study countably infinite MDPs with parity objectives. Parity conditions are widely used in temporal logic and formal verification, e.g., they can express $\omega$-regular languages and modal $\mu$-calculus [9]. Every state has a color, out of a finite set of colors encoded as natural numbers. A run is winning iff the highest color that is seen infinitely often is even. The controller wants to maximize the probability of winning runs. The Mostowski hierarchy [15] is a classification of parity conditions based on restricting the set of allowed colors. For instance, $\{1,2,3\}$-Parity objectives only use colors 1, 2, and 3 . This includes Büchi ( $\{1,2\}$-Parity) and co-Büchi objectives ( $\{0,1\}$-Parity), both of which further subsume reachability and safety objectives.

Related work. In finite MDPs, there always exist optimal MD-strategies for parity objectives. In fact, this holds even for finite turn-based 2-player stochastic parity games [6, 23]. Similarly, there always exist optimal MD-strategies in countably infinite non-stochastic turn-based 2-player parity games [22].

The picture is more complex for countably infinite MDPs. Optimal strategies need not exist (not even for reachability objectives [17, [16]), and $\varepsilon$-optimal strategies for Büchi objectives [10] and optimal strategies for parity objectives [13] require infinite memory.

The paper [13] gave a complete classification whether MD-strategies suffice or whether infinite memory is required for $\varepsilon$-optimal (resp. optimal) strategies for all subclasses of parity objectives in the Mostowski-hierarchy.

However, the mere fact that infinite memory is required for (a subclass of) parity does not establish the precise strategy complexity. E.g., are Markov strategies (or Markov strategies with finite extra memory) sufficient?

In [12] we showed that deterministic 1-bit Markov strategies are both necessary and sufficient for $\varepsilon$-optimal strategies for Büchi objectives. I.e., deterministic 1-bit Markov strategies
are sufficient, but neither randomized Markov strategies nor randomized finite-memory strategies are sufficient. This solved a 40-year old problem in gambling theory from [10, 11]. The same paper [12] showed that even for finitely branching MDPs with $\{1,2,3\}$-Parity objectives, optimal strategies (where they exist) need to be at least deterministic 1-bit Markov in general, i.e., neither randomized Markov nor randomized finite-memory strategies are sufficient.

While the lower bounds for $\varepsilon$-optimal strategies for Büchi objectives (resp. for optimal strategies for $\{1,2,3\}$-Parity objectives) carry over to general parity objectives, the upper bounds on the strategy complexity of $\varepsilon$-optimal (resp. optimal) parity remained open.
A basic upper bound and related conjecture. A basic upper bound on the complexity of $\varepsilon$-optimal strategies for parity can be obtained by using a combination of the results of [12] on Büchi objectives (1-bit Markov) and Lévy's zero-one law as follows. (However, note that the following argument does not work directly for optimal strategies.)

Informally speaking, Lévy's zero-one law implies that, for a tail objective (like parity) and any strategy, the level of attainment from the current state almost surely converges to either zero or one. I.e., the runs that always stay in states where the strategy attains something in $(0,1)$ is a null-set (cf. Appendix B). A consequence for parity is that almost all winning runs must eventually, with ever higher probability, commit to winning by some particular color. Thus, with minimal losses (e.g., $\varepsilon / 2$ ), after a sufficiently long finite prefix (depending on $\varepsilon$ ), one can switch to a strategy that aims to visit some particular color $x$ infinitely often. The latter objective is like a Büchi objective where the states of color $x$ are accepting and states of color $>x$ are considered losing sinks. By [12], an $\varepsilon / 2$-optimal strategy for such a Büchi objective can be chosen 1-bit Markov. However, one would also need to remember which color $x$ one is supposed to win by and stick to that color. The latter is critical, since strategies that switch focus between winning colors infinitely often (e.g., if they follow some local criteria based on the value of the current state wrt. various colors) can end up losing. Overall, the memory needed for such an $\varepsilon$-optimal strategy for parity is: $\left\lceil\log _{2}(c)\right\rceil$ bits for $c$ even colors to remember which color $x$ one is supposed to win by and Markov plus 1 bit for the Büchi strategy (see above), where the Markov step-counter also determines whether one still plays in the prefix. Thus Markov plus $\left(1+\left\lceil\log _{2}(c)\right\rceil\right)$ bits are sufficient. This argument would suggest that more memory is required for more colors. However, our result shows that this is not the case.

Our contributions. We show tight upper bounds on the strategy complexity of $\varepsilon$-optimal (resp. optimal) strategies for parity objectives: They can be chosen as deterministic 1-bit Markov, regardless of the number of colors. I.e., we provide matching upper bounds to the lower bounds from 12 .

In Section 3 we prove Theorem 1 An iterative plastering construction (i.e., fixing player choices on larger and larger subspaces) builds an $\varepsilon$-optimal 1-bit Markov strategy where the probability of never switching between winning even colors is $\geq 1-\varepsilon$. Its correctness relies heavily on Lévy's zero-one law. The number of iterations is finite and proportional to the number of even colors. It eliminates the need to remember the winning color $x$ and the $\left\lceil\log _{2}(c)\right\rceil$ part of the memory.

- Theorem 1. Consider an MDP $\mathcal{M}$, a parity objective and a finite set $S_{0}$ of initial states. For every $\varepsilon>0$ there exists a deterministic 1-bit Markov strategy that is $\varepsilon$-optimal from every state $s \in S_{0}$.

In Section 4 we prove Theorem 2. If an optimal strategy exists, then an optimal 1-bit Markov strategy can be constructed by the so-called sea urchin construction. It is a very
complex plastering construction with infinitely many iterations that uses the results of Theorem 1 and Lévy's zero-one law as building blocks. Its name comes from the shape of the subspace in which player choices get fixed: a growing finite body (around a start set $S_{0}$ ) with a finite, but increasing, number of spikes, where each spike is of infinite size; cf. Figure 4 E.g., if the initial states are almost surely winning then, at the stage with $i$ spikes, this strategy attains parity with some probability $\geq 1-2^{-i}$ already inside this subspace, and in the limit of $i \rightarrow \infty$ it attains parity almost surely. A further step even yields a single deterministic 1-bit Markov strategy that is optimal from every state that has an optimal strategy.

- Theorem 2. Consider an MDP $\mathcal{M}$ with a parity objective and let $S_{o p t}$ be the subset of states that have an optimal strategy.

There exists a deterministic 1-bit Markov strategy that is optimal from every $s \in S_{o p t}$.
In Theorem 1 and Theorem 2 the initial content of the 1-bit memory is irrelevant (cf. Lemma 9. Lemma 18 and Remark 8).

Moreover, we show in Section 5 and Section 6 that in certain subcases deterministic Markov strategies are necessary and sufficient (i.e., these require a Markov step-counter, but not the extra bit): optimal strategies for co-Büchi and $\{0,1,2\}$-Parity, and $\varepsilon$-optimal strategies for safety and co-Büchi. In the special case of finitely branching MDPs, these Markov strategies (but not the 1-bit Markov strategies) can be replaced by MD-strategies.

Together with the previously established lower bounds, this yields a complete picture of the exact strategy complexity of parity objectives at all levels of the Mostowski hierarchy, for countable MDPs. Figure 1 gives a complete overview.

## 2 Preliminaries

A probability distribution over a countable set $S$ is a function $f: S \rightarrow[0,1]$ with $\sum_{s \in S} f(s)=1$. We write $\mathcal{D}(S)$ for the set of all probability distributions over $S$.

We study Markov decision processes (MDPs) over countably infinite state spaces. Formally, an MDP $\mathcal{M}=\left(S, S_{\square}, S_{\bigcirc}, \longrightarrow, P\right)$ consists of a countable set $S$ of states, which is partitioned into a set $S_{\square}$ of controlled states and a set $S_{\bigcirc}$ of random states, a transition relation $\longrightarrow \subseteq$ $S \times S$, and a probability function $P: S_{\bigcirc} \rightarrow \mathcal{D}(S)$. We write $s \longrightarrow s^{\prime}$ if $\left(s, s^{\prime}\right) \in \longrightarrow$, and refer to $s^{\prime}$ as a successor of $s$. We assume that every state has at least one successor. The probability function $P$ assigns to each random state $s \in S_{\bigcirc}$ a probability distribution $P(s)$ over its set of successors. A sink is a subset $T \subseteq S$ closed under the $\longrightarrow$ relation. An MDP is acyclic if the underlying graph $(S, \longrightarrow)$ is acyclic. It is finitely branching if every state has finitely many successors and infinitely branching otherwise. An MDP without controlled states $\left(S_{\square}=\emptyset\right)$ is a Markov chain.

Strategies and Probability Measures. A run $\rho$ is an infinite sequence $s_{0} s_{1} \cdots$ of states such that $s_{i} \longrightarrow s_{i+1}$ for all $i \in \mathbb{N}$; write $\rho(i) \stackrel{\text { def }}{=} s_{i}$ for the $i$-th state along $\rho$. A partial run is a finite prefix of a run. We say that (partial) run $\rho$ visits $s$ if $s=\rho(i)$ for some $i$, and that $\rho$ starts in $s$ if $s=\rho(0)$.

A strategy is a function $\sigma: S^{*} S_{\square} \rightarrow \mathcal{D}(S)$ that assigns to partial runs $\rho s \in S^{*} S_{\square}$ a distribution over the successors of $s$. A (partial) run $s_{0} s_{1} \cdots$ is induced by strategy $\sigma$ if for all $i$ either $s_{i} \in S_{\square}$ and $\sigma\left(s_{0} s_{1} \cdots s_{i}\right)\left(s_{i+1}\right)>0$, or $s_{i} \in S_{\bigcirc}$ and $P\left(s_{i}\right)\left(s_{i+1}\right)>0$.

A strategy $\sigma$ and an initial state $s_{0} \in S$ induce a standard probability measure on sets of infinite plays. We write $\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}(\mathcal{R})$ for the probability of a measurable set $\mathcal{R} \subseteq s_{0} S^{\omega}$ of runs starting from $s_{0}$. As usual, it is first defined on the cylinders $s_{0} s_{1} \ldots s_{n} S^{\omega}$, where
$s_{1}, \ldots, s_{n} \in S$ : if $s_{0} s_{1} \ldots s_{n}$ is not a partial run induced by $\sigma$ then $\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(s_{0} s_{1} \ldots s_{n} S^{\omega}\right) \stackrel{\text { def }}{=} 0$. Otherwise, $\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(s_{0} s_{1} \ldots s_{n} S^{\omega}\right) \stackrel{\text { def }}{=} \prod_{i=0}^{n-1} \bar{\sigma}\left(s_{0} s_{1} \ldots s_{i}\right)\left(s_{i+1}\right)$, where $\bar{\sigma}$ is the map that extends $\sigma$ by $\bar{\sigma}(w s)=P(s)$ for all $w s \in S^{*} S_{\bigcirc}$. By Carathéodory's theorem [3], this extends uniquely to a probability measure $\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}$ on measurable subsets of $s_{0} S^{\omega}$. We will write $\mathbb{E}_{\mathcal{M}, s_{0}, \sigma}$ for the expectation w.r.t. $\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}$. We may drop the subscripts from notations, if it is understood.

Objectives. The objective of the player is determined by a predicate on infinite plays. We assume familiarity with the syntax and semantics of the temporal logic LTL [8]. Formulas are interpreted on the structure $(S, \longrightarrow)$. We use $\llbracket \varphi \rrbracket^{s} \subseteq s S^{\omega}$ to denote the set of runs starting from $s$ that satisfy the LTL formula $\varphi$, which is a measurable set [21]. We also write $\llbracket \varphi \rrbracket$ for $\bigcup_{s \in S} \llbracket \varphi \rrbracket^{s}$. Where it does not cause confusion we will identify $\varphi$ and $\llbracket \varphi \rrbracket$ and just write $\mathbb{P}_{\mathcal{M}, s, \sigma}(\varphi)$ instead of $\mathbb{P}_{\mathcal{M}, s, \sigma}\left(\llbracket \varphi \rrbracket^{s}\right)$.

Given a set $T \subseteq S$ of states, the reachability objective Reach $(T)$ is the set of runs that visit $T$ at least once; and the safety objective $\operatorname{Safety}(T)$ is the set of runs that never visit $T$.

Let $\mathcal{C} \subseteq \mathbb{N}$ be a finite set of colors. A color function $\operatorname{Col}: S \rightarrow \mathcal{C}$ assigns to each state $s$ its color $\operatorname{Col}(s)$. The parity objective, written as $\operatorname{Parity}(\mathrm{Col})$, is the set of infinite runs such that the largest color that occurs infinitely often along the run is even. To define this formally, let $\operatorname{even}(\mathcal{C})=\{i \in \mathcal{C} \mid i \equiv 0 \bmod 2\}$. For $\triangleright \in\{<, \leq,=, \geq,>\}, n \in \mathbb{N}$, and $Q \subseteq S$, let $[Q]$ Col $\triangleright n \stackrel{\text { def }}{=}\{s \in Q \mid \operatorname{Col}(s) \triangleright n\}$ be the set of states in $Q$ with color $\triangleright n$. Then

$$
\operatorname{Parity}(C o l) \stackrel{\text { def }}{=} \bigvee_{i \in \operatorname{even}(\mathcal{C})}\left(\mathrm{GF}[S]^{C o l=i} \wedge \mathrm{FG}[S]^{C o l \leq i}\right)
$$

The Mostowski hierarchy [15] classifies parity objectives by restricting the range of Col to a set of colors $\mathcal{C} \subseteq \mathbb{N}$. We write $\mathcal{C}$-Parity for such restricted parity objectives. In particular, the classical Büchi and co-Büchi objectives correspond to $\{1,2\}$-Parity and $\{0,1\}$-Parity, respectively. These two classes are incomparable but both subsume the reachability and safety objectives. Assuming that $T$ is a $\operatorname{sink}, \operatorname{Reach}(T)=\operatorname{Parity}(C o l)$ for the coloring with $\operatorname{Col}(s)=1 \Longleftrightarrow s \notin T$ and $\operatorname{Safety}(T)=\operatorname{Parity}(\operatorname{Col})$ for the coloring with $\operatorname{Col}(s)=1 \Longleftrightarrow s \in T$. Similarly, $\{0,1,2\}$-Parity and $\{1,2,3\}$-Parity are incomparable, but they both subsume (modulo renaming of colors) Büchi and co-Büchi objectives.

An objective $\varphi$ is called a tail objective (resp. suffix-closed) iff for every run $\rho^{\prime} \rho$ with some finite prefix $\rho^{\prime}$ we have $\rho^{\prime} \rho \in \varphi \Leftrightarrow \rho \in \varphi$ (resp. $\rho^{\prime} \rho \in \varphi \Rightarrow \rho \in \varphi$ ). In particular, Parity (Col) is tail for every coloring Col. Moreover, if $\varphi$ is suffix-closed then $\mathrm{F} \varphi$ is tail.

Strategy Classes. Strategies $\sigma: S^{*} S_{\square} \rightarrow \mathcal{D}(S)$ are in general randomized (R) in the sense that they take values in $\mathcal{D}(S)$. A strategy $\sigma$ is deterministic (D) if $\sigma(\rho)$ is a Dirac distribution for all partial runs $\rho \in S^{*} S_{\square}$.

We formalize the amount of memory needed to implement strategies. Let M be a countable set of memory modes. An update function is a function $u: \mathrm{M} \times S \rightarrow \mathcal{D}(\mathrm{M} \times S)$ that meets the following two conditions, for all modes $\mathrm{m} \in \mathrm{M}$ :

- for all controlled states $s \in S_{\square}$, the distribution $u((\mathrm{~m}, s))$ is over $\mathrm{M} \times\left\{s^{\prime} \mid s \longrightarrow s^{\prime}\right\}$.
- for all random states $s \in S_{\bigcirc}$, we have that $\sum_{m^{\prime} \in \mathrm{M}} u((\mathrm{~m}, s))\left(\mathrm{m}^{\prime}, s^{\prime}\right)=P(s)\left(s^{\prime}\right)$.

An update function $u$ together with an initial memory $\mathrm{m}_{0}$ induce a strategy $u\left[\mathrm{~m}_{0}\right]$ : $S^{*} S_{\square} \rightarrow \mathcal{D}(S)$ as follows. Consider the Markov chain with states set M $\times S$, transition relation $(\mathrm{M} \times S)^{2}$ and probability function $u$. Any partial run $\rho=s_{0} \cdots s_{i}$ in $\mathcal{M}$ gives rise to a set $H(\rho)=\left\{\left(\mathrm{m}_{0}, s_{0}\right) \cdots\left(\mathrm{m}_{i}, s_{i}\right) \mid \mathrm{m}_{0}, \ldots, \mathrm{~m}_{i} \in \mathrm{M}\right\}$ of partial runs in this Markov chain. Each $\rho s \in s_{0} S^{*} S_{\square}$ induces a probability distribution $\mu_{\rho s} \in \mathcal{D}(\mathrm{M})$, the probability of being in
state $(\mathrm{m}, s)$ conditioned on having taken some partial run from $H(\rho s)$. We define $u\left[\mathrm{~m}_{0}\right]$ such that $u\left[\mathrm{~m}_{0}\right](\rho s)\left(s^{\prime}\right) \stackrel{\text { def }}{=} \sum_{\mathrm{m}, \mathbf{m}^{\prime} \in \mathrm{M}} \mu_{\rho s}(\mathrm{~m}) u((\mathrm{~m}, s))\left(\mathrm{m}^{\prime}, s^{\prime}\right)$ for all $\rho s \in S^{*} S_{\square}$ and $s^{\prime} \in S$.

We say that a strategy $\sigma$ can be implemented with memory M (and initial memory $\mathrm{m}_{0}$ ) if there exists an update function $u$ such that $\sigma=u\left[\mathrm{~m}_{0}\right]$. In this case we may also write $\sigma\left[\mathrm{m}_{0}\right]$ to explicitly specify the initial memory mode $\mathrm{m}_{0}$. Based on this, we can define several classes of strategies:

- A strategy $\sigma$ is memoryless (M) (also called positional) if it can be implemented with a memory of size 1 . We may view M-strategies as functions $\sigma: S_{\square} \rightarrow \mathcal{D}(S)$.
- A strategy $\sigma$ is finite memory $(\mathrm{F})$ if there exists a finite memory M implementing $\sigma$. More specifically, a strategy is $k$-bit if it can be implemented with a memory of size $2^{k}$. Such a strategy is then determined by a function $u:\{0,1\}^{k} \times S \rightarrow \mathcal{D}\left(\{0,1\}^{k} \times S\right)$.
- A strategy $\sigma$ is Markov if it can be implemented with the natural numbers $\mathrm{M}=\mathbb{N}$ as the memory, initial memory mode $\mathrm{m}_{0}=0$ and a function $u$ such that the distribution $u(\mathrm{~m}, s)$ is over $\{\mathrm{m}+1\} \times S$ for all $\mathrm{m} \in \mathrm{M}$ and $s \in S$. Intuitively, such a strategy depends only on the current state and the number of steps taken so far.
- A strategy $\sigma$ is $k$-bit Markov if it can be implemented with memory $\mathrm{M}=\mathbb{N} \times\{0,1\}^{k}$, $\mathrm{m}_{0} \in\{0\} \times\{0,1\}^{k}$ and a function $u$ such that the distribution $u((n, b, s))$ is over $\{n+1\} \times\{0,1\}^{k} \times S$ for all $(n, b) \in \mathrm{M}$ and $s \in S$.

Deterministic 1-bit strategies are central in this paper; by this we mean strategies that are both deterministic and 1-bit.

Optimal and $\varepsilon$-optimal Strategies. Given an objective $\varphi$, the value of state $s$ in an MDP $\mathcal{M}$, denoted by $\operatorname{val}_{\mathcal{M}}(s)$, is the supremum probability of achieving $\varphi$. Formally, we have $\operatorname{val}_{\mathcal{M}}(s) \stackrel{\text { def }}{=} \sup _{\sigma \in \Sigma} \mathbb{P}_{\mathcal{M}, s, \sigma}(\varphi)$ where $\Sigma$ is the set of all strategies. For $\varepsilon \geq 0$ and state $s \in S$, we say that a strategy is $\varepsilon$-optimal from $s$ iff $\mathbb{P}_{\mathcal{M}, s, \sigma}(\varphi) \geq \operatorname{val}_{\mathcal{M}}(s)-\varepsilon$. A 0 -optimal strategy is called optimal. An optimal strategy is almost-surely winning if $\operatorname{val}_{\mathcal{M}}(s)=1$.

Considering an MD strategy as a function $\sigma: S_{\square} \rightarrow S$ and $\varepsilon \geq 0, \sigma$ is uniformly $\varepsilon$-optimal (resp. uniformly optimal) if it is $\varepsilon$-optimal (resp. optimal) from every $s \in S$.

Fixing and Safe Sets. Let $\sigma$ be an MD strategy. Given a set $S^{\prime} \subseteq S$ of states, write $\mathcal{M}\left[\sigma, S^{\prime}\right]$ for the MDP obtained from $\mathcal{M}$ by fixing the strategy $\sigma$ for all states in $S^{\prime}$, that is, $\mathcal{M}\left[\sigma, S^{\prime}\right] \stackrel{\text { def }}{=}\left(S, S_{\square} \backslash S^{\prime}, S_{\bigcirc} \cup S^{\prime}, \longrightarrow, P^{\prime}\right)$ where $P^{\prime}(s) \stackrel{\text { def }}{=} \sigma(s)$ for all $s \in S^{\prime}$.

For an objective $\varphi$ and a threshold $\beta \in[0,1]$, denote by $\operatorname{Safe}_{\mathcal{M}, \sigma, \varphi}(\beta)$ the set of all states $s$ starting from which $\sigma$ attains at least probability $\beta$; and denote by $\operatorname{Safe}_{\mathcal{M}, \varphi}(\beta)$ the set of states whose value for $\varphi$ is at least $\beta$. Formally,

$$
\begin{equation*}
\operatorname{Safe}_{\mathcal{M}, \sigma, \varphi}(\beta) \stackrel{\text { def }}{=}\left\{s \in S \mid \mathbb{P}_{\mathcal{M}, s, \sigma}(\varphi) \geq \beta\right\}, \quad \operatorname{Safe}_{\mathcal{M}, \varphi}(\beta) \stackrel{\text { def }}{=}\left\{s \in S \mid \operatorname{val}_{\mathcal{M}, \varphi}(s) \geq \beta\right\} \tag{1}
\end{equation*}
$$

## $3 \quad \varepsilon$-Optimal Strategies for Parity

In this section we prove Theorem 1, stating that $\varepsilon$-optimal strategies for parity objectives can be chosen 1-bit Markov. Given an MDP we convert it by three successive reductions to a structurally simpler MDP where strategies require less sophistication to achieve parity.

First reduction (Finitely Branching). This reduction converts an infinitely branching MDP $\mathcal{M}$ to a finitely branching one $\mathcal{M}^{\prime}$, with a clear bijection between the strategies in $\mathcal{M}$ and $\mathcal{M}^{\prime}$. The construction, first presented in our previous work [12], replaces each controlled state $s$, that has infinitely many successors $\left(s_{i}\right)_{i \in \mathbb{N}}$, with a "ladder" of controlled states $\left(q_{i}\right)_{i \in \mathbb{N}}$, where each $q_{i}$ has only two successors: $q_{i+1}$ and $s_{i}$. Roughly speaking, the controller
choice of successor $s_{n}$ at $s$ in $\mathcal{M}$, is simulated by a series of choices $q_{i+1}$ at $q_{i}, 0 \leq i<n$, followed by a choice of successor $s_{n}$ in state $q_{n}$ in $\mathcal{M}^{\prime}$, and vice versa.

To prevent scenarios when the controller in $\mathcal{M}^{\prime}$ stays on a ladder and never commits to a decision, we assign color 1 to all states $\left(q_{i}\right)_{i \geq 1}$ on the ladder ( $q_{0}$ inherits the color of $s$ ). Hence, a hesitant run on the ladder is losing for parity. So w.l.o.g. we can assume that the given $\mathcal{M}$ is finitely branching.

## - Lemma 3.

1. Suppose that for every finitely branching acyclic MDP with a finite set $S_{0}$ of initial states, and a parity objective, there exist $\varepsilon$-optimal deterministic 1-bit strategies from $S_{0}$.
Then even for every infinitely branching acyclic MDP with a finite set $S_{0}$ of initial states and a parity objective, there exist $\varepsilon$-optimal deterministic 1-bit strategies from $S_{0}$.
2. Suppose that for every finitely branching acyclic MDP with a parity objective, there exists a deterministic 1-bit strategy that is optimal from all states that have an optimal strategy. Then even for every infinitely branching acyclic MDP with a parity objective, there exists a deterministic 1-bit strategy that is optimal from all states that have an optimal strategy.

Second reduction (Acyclicity). A deterministic 1-bit Markov strategy can be seen as a function $\sigma: \mathbb{N} \times\{0,1\} \times S \rightarrow\{0,1\} \times S$, where $\sigma$ has access to an internal bit $b \in\{0,1\}$, which can be updated freely, and a step counter $k \in \mathbb{N}$, which increments by one in each step. Having $b$ and $k, \sigma$ produces a decision based on the current state of the MDP.

Following [12], we encode the step-counter from strategies into MDPs s.t. the current state of the system uniquely determines the length of the path taken so far. This translation allows us to focus on acyclic MDPs.

- Lemma 4. Consider MDPs with a parity objective and $k \in \mathbb{N}$.

1. Suppose that for every acyclic MDP $\mathcal{M}^{\prime}$ and every finite set of initial states $S_{0}^{\prime}$ and $\varepsilon>0$, there exists a deterministic $k$-bit strategy that is $\varepsilon$-optimal from all states $s \in S_{0}^{\prime}$.
Then for every MDP $\mathcal{M}$ and every finite set of initial states $S_{0}$ and $\varepsilon>0$, there exists a deterministic $k$-bit Markov strategy that is $\varepsilon$-optimal from all states $s \in S_{0}$.
2. Suppose that for every acyclic $M D P \mathcal{M}^{\prime}$ and $\varepsilon>0$, there exists a deterministic $k$-bit strategy that is $\varepsilon$-optimal from all states. Then for every $M D P \mathcal{M}$ and $\varepsilon>0$, there exists a deterministic $k$-bit Markov strategy that is $\varepsilon$-optimal from all states.
3. Suppose that for every acyclic $M D P \mathcal{M}^{\prime}$, where $S_{o p t}^{\prime}$ is the subset of states that have an optimal strategy, there exists a deterministic $k$-bit strategy that is optimal from all states $s \in S_{o p t}^{\prime}$. Then for every $M D P \mathcal{M}$, where $S_{\text {opt }}$ is the subset of states that have an optimal strategy, there exists a deterministic $k$-bit Markov strategy that is optimal from all states $s \in S_{o p t}$.
By Lemma 4, the sufficiency of deterministic 1-bit strategies in acyclic MDPs implies the sufficiency of deterministic 1-bit Markov strategies in general MDPs. Thus to prove Theorem 1, it suffices to prove the following:

- Theorem 5. Consider an acyclic MDP $\mathcal{M}$, a parity objective and a finite set $S_{0}$ of states. For every $\varepsilon>0$ there exists a deterministic 1-bit strategy that is $\varepsilon$-optimal from every $s \in S_{0}$.

Third reduction (Layered MDP). This reduction is in the same spirit of the previous one, in which the bit $b \in\{0,1\}$ is transferred from strategies to MDPs. Given an MDP $\mathcal{M}$, the corresponding layered MDP $\mathcal{L}(\mathcal{M})$ has two copies of each state $s \in S$ and each transition $t \in$ $\longrightarrow_{1}$ of $\mathcal{M}$, one augmented with bit 0 and another with bit $1:(s, i)$ and $(t, j)$ with $i, j \in\{0,1\}$. The states $(s, i)$ are random if $s \in S_{\bigcirc}$ and controlled if $s \in S_{\square}$. All the $(t, j)$ are controlled.

If there is a transition $t=(a, b)$ from state $a$ to $b$ in $\mathcal{M}$, there will be two transitions from $(a, i)$ to $(t, i)$, and four transitions from $(t, i)$ to $(b, j)$ in $\mathcal{L}(\mathcal{M})$; see Figure 2

A 1-bit deterministic strategy in $\mathcal{M}$ at a state $a$ picks a single successor $b$ and may flip the bit from $i$ to $j$; this is simulated in $\mathcal{L}(\mathcal{M})$ with an MD strategy $\sigma$ within two consecutive steps: $\sigma$ first chooses the transition $t=(a, b)$ by $\sigma(a, i)=(t, i)$ and then updates the bit by $\sigma(t, i)=(b, j)$ thereby moving from layer $i$ to layer $j$. The controlled states $(t, i)$ are essential for a correct simulation, since otherwise the controller cannot freely flip the bit (switch between layers) after it observes the successor chosen randomly at a random state.

- Definition 6 (Layered MDP). Given an $M D P \mathcal{M}=\left(S, S_{\square}, S_{\bigcirc}, \longrightarrow_{1}, P_{1}\right)$ with coloring $C o l_{1}: S \rightarrow \mathcal{C}$, we define the corresponding layered $\operatorname{MDP} \mathcal{L}(\mathcal{M})=\left(L, L_{\square}, L_{\bigcirc}, \longrightarrow_{2}, P_{2}\right)$ with coloring $\mathrm{Col}_{2}: L \rightarrow \mathcal{C}$ as follows.
- $L \stackrel{\text { def }}{=}\left(S \cup \longrightarrow_{1}\right) \times\{0,1\}$ where the set of controlled states is $L_{\square} \stackrel{\text { def }}{=}\left(S_{\square} \cup \longrightarrow_{1}\right) \times\{0,1\}$.
- For all $t \in \longrightarrow_{1}$ such that $t=\left(s, s^{\prime}\right)$ and for all $i, j \in\{0,1\}$, we have:

1. $(s, i) \longrightarrow_{2}(t, i)$ and $(t, i) \longrightarrow_{2}\left(s^{\prime}, j\right)$,
2. $P(s, i)((t, i)) \stackrel{\text { def }}{=} P(s)\left(s^{\prime}\right)$ iff $s \in S_{\bigcirc}$, and
3. $\mathrm{Col}_{2}((s, i)) \stackrel{\text { def }}{=} \operatorname{Col}_{1}(s)$ and $\operatorname{Col}_{2}((t, i)) \stackrel{\text { def }}{=} \operatorname{Col}_{1}\left(s^{\prime}\right)$.

The layered MDP of an acyclic MDP is acyclic. For $q \in S \cup \longrightarrow_{1}$, we refer to the copies of $q$ in layer 0 and layer 1 as siblings: $(q, 0)$ and $(q, 1)$. A set $B \subseteq L$ is closed if for each state $(q, i) \in B$ its sibling is also in $B$. Denote by $C l(B)$ the minimal closed superset of $B$.

- Lemma 7. Consider an acyclic $M D P \mathcal{M}=\left(S, S_{\square}, S_{\bigcirc}, \longrightarrow, P\right)$ with a parity objective $\varphi=$ Parity $(C o l)$ and let $\mathcal{L}(\mathcal{M})$ be the corresponding layered MDP.

For every deterministic 1-bit strategy $u\left[\mathrm{~m}_{0}\right]$ in $\mathcal{M}$ there is a corresponding $M D$ strategy $\tau$ in $\mathcal{L}(\mathcal{M})$, and vice-versa, such that for every $s_{0} \in S, \mathbb{P}_{\mathcal{L}(\mathcal{M}),\left(s_{0}, \mathrm{~m}_{0}\right), \tau}(\varphi)=\mathbb{P}_{\mathcal{M}, s_{0}, u\left[\mathrm{~m}_{0}\right]}(\varphi)$.

- Remark 8. We note that in a layered system $\mathcal{L}(\mathcal{M})$, any two siblings have the same value w.r.t. a parity objective $\varphi$. Moreover, any state $s$ in $\mathcal{M}$ has an optimal strategy iff $(s, 0) \in \mathcal{L}(\mathcal{M})$ has an optimal strategy iff its sibling $(s, 1)$ has an optimal strategy.

Suppose $\tau$ is an MD strategy in $\mathcal{L}(\mathcal{M})$ that is optimal for all states that have an optimal strategy. Let $u$ be the update function of a corresponding 1-bit strategy in $\mathcal{M}$, derived as described in Lemma 7. Then for every state $s$ in $\mathcal{M}$ that has an optimal strategy we have $\mathbb{P}_{\mathcal{M}, s, u[0]}(\varphi)=\mathbb{P}_{\mathcal{L}(\mathcal{M}),(s, 0), \tau}(\varphi)=\mathbb{P}_{\mathcal{L}(\mathcal{M}),(s, 1), \tau}(\varphi)=\mathbb{P}_{\mathcal{M}, s, u[1]}(\varphi)$. That is, both $u[0]$ and $u[1]$ are optimal from $s$, so the initial memory mode is irrelevant.

To prove Theorem 5, given an acyclic MDP, a set of initial states $S_{0}$ and $\varepsilon>0$, we consider the layered $\operatorname{MDP} \mathcal{L}(\mathcal{M})$ and set $L_{0}=S_{0} \times\{0\}$ of initial states. In the following lemma, we prove that there exists a single MD strategy that is $\varepsilon$-optimal starting from every state $\ell_{0} \in L_{0}$ in $\mathcal{L}(\mathcal{M})$. This and Lemma 7 will directly lead to Theorem 5

- Lemma 9. Consider an acyclic MDP $\mathcal{M}$ and parity objective $\varphi=\operatorname{Parity}(\operatorname{Col})$. Let $\mathcal{L}(\mathcal{M})$ be the layered $M D P$ of $\mathcal{M}$ and Col. For all finite sets $L_{0}$ of states in $\mathcal{L}(\mathcal{M})$ and all $\varepsilon>0$ there exists a single $M D$ strategy that is $\varepsilon$-optimal for $\varphi$ from every state $\ell_{0} \in L_{0}$.

In the rest of this section, we prove Lemma 9 We fix a layered $\operatorname{MDP} \mathcal{L}(\mathcal{M})$ (or simply $\mathcal{L}$ ) obtained from a given acyclic and finitely branching MDP $\mathcal{M}$ and a coloring $\mathrm{Col}: S \rightarrow \mathcal{C}$, where the set of states is $L$ and the finite set of initial states is $L_{0} \subseteq L$. Let $\varphi$ be the resulting parity objective in $\mathcal{L}$.

Recall that $\operatorname{even}(\mathcal{C})=2 \mathbb{N} \cap \mathcal{C}$ denotes the set of even colors. We denote by $e_{\text {max }}$ the largest even color in $\operatorname{even}(\mathcal{C})$ and assume w.l.o.g., that $\operatorname{even}(\mathcal{C})$ contains all even numbers


Figure 2 An $\operatorname{MDP} \mathcal{M}$ (in grey) and the corresponding layered $\operatorname{MDP} \mathcal{L}(\mathcal{M})$ with states of layer 0 and 1 in red and blue, respectively. Here, $t=(a, b), t^{\prime}=(b, c)$ and $t^{\prime \prime}=(b, d)$ are transitions of $\mathcal{M}$.
from 2 to $e_{\text {max }}$ inclusive. We have:

$$
\begin{aligned}
& \varphi \stackrel{\text { def }}{=} \bigvee_{e \in \operatorname{even}(\mathcal{C})}\left(\mathrm{GF}[L]^{C o l=e} \wedge \mathrm{FG}[L]^{C o l \leq e}\right) \\
&= \bigvee_{e \in \operatorname{even}(\mathcal{C})}\left(\mathrm{FGF}[L]^{C o l=e} \wedge \mathrm{FG}[L]^{C o l \leq e}\right) \quad \text { since } \mathrm{GF}[L]^{C o l=e} \text { is a tail objective } \\
&= \bigvee^{C \in \operatorname{even}(\mathcal{C})} \mathrm{F}\left(\mathrm{GF}[L]^{C o l=e} \wedge \mathrm{G}[L]^{C o l \leq e}\right) \quad \text { since } \mathrm{FG} A \wedge \mathrm{FG} B=\mathrm{F}(\mathrm{G} A \wedge \mathrm{G} B) \\
&= \bigvee^{C \in \operatorname{even}(\mathcal{C})} \mathrm{F} \varphi_{e},
\end{aligned}
$$

where $\varphi_{e} \stackrel{\text { def }}{=}\left(\mathrm{GF}[L]^{\text {Col }=e} \wedge \mathrm{G}[L]^{\text {Col } \leq e}\right)$. Indeed, $\varphi_{e}$ is the set of runs that win through color $e$ (i.e., by visiting color $e$ infinitely often and never visiting larger colors). Since the $\mathrm{F} \varphi_{e}$ are disjoint, for all states $\ell$ and strategies $\sigma$, we have:

$$
\begin{equation*}
\mathbb{P}_{\mathcal{L}, \ell, \sigma}(\varphi)=\sum_{e \in \operatorname{even}(\mathcal{C})} \mathbb{P}_{\mathcal{L}, \ell, \sigma}\left(\mathrm{F} \varphi_{e}\right) . \tag{2}
\end{equation*}
$$

Fix $\varepsilon>0$ and define $\gamma \stackrel{\text { def }}{=} \frac{\varepsilon}{e_{\text {max }}+2}$. To construct an MD strategy $\hat{\sigma}$ that is $\varepsilon$-optimal starting from every state in $L_{0}$ we have an iterative procedure. In each iteration, we define $\hat{\sigma}$ at states in some carefully chosen region; and continuing in this fashion, we gradually fix all choices of $\hat{\sigma}$. In an iteration, in order to fix "good" choices in the "right" region we need to carefully observe the behavior of finitely many $\frac{\gamma}{2}$-optimal strategies $\sigma_{\ell_{0}}$, one for each $\ell_{0} \in L_{0}$, which must respect the choices already fixed in previous iterations. We thus view these strategies $\sigma_{\ell_{0}}$ to be $\frac{\gamma}{2}$-optimal not in $\mathcal{L}$ but in another layered MDP that is derived from $\mathcal{L}$ after fixing the choices of partially defined $\hat{\sigma}$.

In more detail, the proof consists of exactly $\frac{e_{\text {max }}}{2}+1$ iterations: one iteration for each even color $e$ and a final "reach" iteration. Starting from color 2 and $\mathcal{L}_{0} \stackrel{\text { def }}{=} \mathcal{L}$, in the iteration $e \in\left\{2, \cdots, e_{\max }\right\}$, we obtain a layered $\operatorname{MDP} \mathcal{L}_{e}$ from $\mathcal{L}_{e-2}$ by fixing a single choice for each controlled state in a set $f x_{e}$. Roughly speaking, a run that falls in the set $f x_{e}$ is likely going to win through $\varphi_{e}$ (win through color $e$ ). We identify a certain subspace of $f x_{e}$, referred to as core $_{e}$, such that the following crucial fact holds: Once core $e_{e}$ is visited the run remains in fix $e_{e}$ with probability at least $1-\gamma$. At the final iteration, we fix the choices of all remaining states to maximize the probability of falling into the union of core $e_{e}$ sets. As
mentioned, the majority of such runs that visit core $_{e}$, for some color $e$, will stay in $f x_{e}$ forever and thus win parity through color $e$. After all the iterations, all choices of all controlled states are fixed, and this prescribes the MD strategy $\hat{\sigma}$ from $L_{0}$ in $\mathcal{L}$.

In order to define the sets $f x_{e}$ we heavily use Lévy's zero-one law and follow an inductive transformation on objectives. Lévy's zero-one states that, for a given set of (infinite) runs of a Markov chain, if we gradually observe a random run of the chain, we will become more and more certain whether the random run belongs to that set. This law has a strong implications for tail objectives. It asserts that on almost all runs $s_{0} s_{1} s_{2} \cdots$ the limit of the value of $s_{i}$ w.r.t. a tail objective tends to either 0 or 1 .

In each iteration $e \in\left\{2, \cdots, e_{\max }\right\}$, we transform an objective $\psi_{e-2}$ to a next objective $\psi_{e}$ where $\psi_{0} \stackrel{\text { def }}{=} \varphi$ is the parity objective and the result of the last transformation is $\psi_{e_{\max }}=$ $\bigvee_{e \in \text { even(C) }} \mathrm{F}$ core $_{e}$. We will also move from the $\operatorname{MDP} \mathcal{L}_{e-2}$ to $\mathcal{L}_{e}$ after the fixings so as to maintain the following invariant: For all $\ell_{0} \in L_{0}$, the value of $\ell_{0}$ for $\psi_{e}$ in $\mathcal{L}_{e}$ is almost as high as its value for $\varphi$ in $\mathcal{L}$, that is

$$
\begin{equation*}
\operatorname{val}_{\mathcal{L}_{e}, \psi_{e}}\left(\ell_{0}\right) \geq \operatorname{val}_{\mathcal{L}, \varphi}\left(\ell_{0}\right)-e \cdot \gamma \tag{3}
\end{equation*}
$$

Recall that $\varphi=\bigvee_{e \in \operatorname{even}(\mathcal{C})} \mathrm{F} \varphi_{e}$. Let $\mathrm{Fix}_{0} \stackrel{\text { def }}{=} \emptyset$ and write $\mathrm{Fix}_{e} \stackrel{\text { def }}{=} \bigcup_{e^{\prime} \leq e} C l\left(f i x_{e^{\prime}}\right)$ for $e \in$ $\left\{2,4, \cdots, e_{\max }\right\}$. We define:

$$
\begin{equation*}
\psi_{0} \stackrel{\text { def }}{=} \bigvee_{e^{\prime}>0} \mathrm{~F} \varphi_{e^{\prime}} \wedge \mathrm{G} \neg \mathrm{Fix}_{0}=\varphi \quad \psi_{e} \stackrel{\text { def }}{=} \bigvee_{e^{\prime} \leq e} \mathrm{Fcore}_{e^{\prime}} \vee \bigvee_{e^{\prime}>e}\left(\mathrm{~F} \varphi_{e^{\prime}} \wedge \mathrm{G} \neg \mathrm{Fix}_{e}\right) \tag{4}
\end{equation*}
$$

At each transformation, we examine the disjunct $\chi_{e} \stackrel{\text { def }}{=} \mathrm{F} \varphi_{e} \wedge \mathrm{G} \neg \mathrm{Fix}_{e-2}$ in $\psi_{e-2}$. The set of runs satisfying this objective $\chi_{e}$ not only win through color $e$ but also avoid the previously fixed regions. Roughly speaking, the aim is to transform $\chi_{e}$ to Fcore $e$, to move from $\psi_{e-2}$ to $\psi_{e}$. We apply Lévy's zero-one law to deduce that the runs satisfying the $\chi_{e}$ are likely to enter a region that has a high value for a slightly simpler objective, namely

$$
\begin{equation*}
\theta_{e} \stackrel{\text { def }}{=} \varphi_{e} \wedge \mathrm{G} \neg \mathrm{Fix}_{e-2} \tag{5}
\end{equation*}
$$

To do so, we observe in $\mathcal{L}_{e-2}$ the behavior of several arbitrary $\frac{\gamma}{2}$-optimal strategies $\sigma_{\ell_{0}}$ for $\psi_{e-2}$, one for each $\ell_{0} \in L_{0}$. Then, for each $\sigma_{\ell_{0}}$, we apply Lévy's zero-one law separately; this provides that there exists a finite set $R_{e}$ of states that have a high value for $\theta_{e}$, and is reached by one of the $\sigma_{\ell_{0}}$ with probability as high as the probability of satisfying the disjunct $\chi_{e}$. Now we use our previous results [12] on the strategy complexity of Büchi objectives and prove the existence of an MD strategy $\tau_{e}$ that is almost optimal for $\theta_{e}$ (error less than $\gamma$ ), starting from every state in $R_{e}$. We define sets fix $e_{e}$ and core $e_{e}$ to be the set of states from which $\tau_{e}$ attains a high probability for $\theta_{e}$ in $\mathcal{L}_{e-2}$; see Figure 3 Define $\beta \stackrel{\text { def }}{=} 1-\gamma$ and $\alpha \stackrel{\text { def }}{=} 1-\gamma^{2}$, and

$$
\begin{equation*}
f i x_{e} \stackrel{\text { def }}{=} \operatorname{Safe}_{\mathcal{L}_{e-2}, \tau_{e}, \theta_{e}}(\beta) \quad \operatorname{core}_{e} \stackrel{\text { def }}{=} \operatorname{Safe}_{\mathcal{L}_{e-2}, \tau_{e}, \theta_{e}}(\alpha) \tag{6}
\end{equation*}
$$

We fix the strategy $\tau_{e}$ in the $f i x_{e}$-region to derive the $\operatorname{MDP} \mathcal{L}_{e}$ from $\mathcal{L}_{e-2}$. Formally,

$$
\begin{equation*}
\mathcal{L}_{e} \stackrel{\text { def }}{=} \mathcal{L}_{e-2}\left[\tau_{e}, f x_{e}\right] . \tag{7}
\end{equation*}
$$

Iteration $e \in\left\{2, \cdots, e_{\max }\right\}$ : For all states $\ell_{0} \in L_{0}$, let $\sigma_{\ell_{0}}$ be a general (not necessarily MD) $\frac{\gamma}{2}$-optimal strategy w.r.t. $\psi_{e-2}$ in the layered MDP $\mathcal{L}_{e-2}$. Consider the Markov chain $\mathcal{C}_{\ell_{0}}$ induced by $\mathcal{L}_{e-2}$, the fixed initial state $\ell_{0}$ and strategy $\sigma_{\ell_{0}}$.


Figure 3 The construction for Lemma 9. In the first iteration, for color 2, we fix the MD strategy $\tau_{2}$ in the $f i x_{2}$-region. In the second iteration, for color 4 , we fix $\tau_{4}$ in $f i x_{4}$, and so on for all even colors. Everywhere else we fix an $\gamma$-optimal reachability strategy towards $\bigcup_{e=2}^{e_{\text {max }}}$ core $_{e}$ (in green).

By definition (Equation 5), $\theta_{e}$ is suffix-closed and $\mathrm{F} \theta_{e}$ is tail. The strategy $\sigma_{\ell_{0}}$ attains $\mathrm{F} \theta_{e}$ with probability at least as large as it achieves disjunct $\chi_{e}$ in $\psi_{e-2}$. We apply Lévy's zero-one law to deduce that the winning runs of $\mathrm{F} \theta_{e}$ likely reach a finite set $R_{e}$ of states that have a high value for $\theta_{e}$. In other words, most runs that eventually win through color $e$, while eventually avoiding Fix ${ }_{e-2}$, will reach $R_{e}$ within a bounded number of steps.

- Lemma 10. Let $s_{0} \in S$ and $\mathcal{E}$ be a suffix-closed objective. For all $\varepsilon, \varepsilon^{\prime}>0$, there exist $n$ and a finite set $F \subseteq \operatorname{Safe} \mathcal{E}_{\mathcal{E}}(1-\varepsilon)$ such that $\mathbb{P}_{s_{0}}\left(\mathrm{FE} \wedge \mathrm{F}^{\leq n} F\right) \geq \mathbb{P}_{s_{0}}(\mathrm{FE})-\varepsilon^{\prime}$.

By Lemma 10, there exist $n_{\ell_{0}}$ and a finite set $R_{\ell_{0}} \subseteq \operatorname{Safe}_{\mathcal{L}_{e-2}, \theta_{e}}(\alpha)$ such that

$$
\begin{equation*}
\mathbb{P}_{\mathcal{L}_{e-2}, \ell_{0}, \sigma_{\ell_{0}}}\left(\mathrm{~F} \theta_{e} \wedge \mathrm{~F}^{\leq n \ell_{0}} R_{\ell_{0}}\right) \geq \mathbb{P}_{\mathcal{L}_{e-2}, \ell_{0}, \sigma_{\ell_{0}}}\left(\mathrm{~F} \theta_{e}\right)-\frac{\gamma}{2} \tag{8}
\end{equation*}
$$

Define $n_{e} \stackrel{\text { def }}{=} \max _{\ell_{0} \in L_{0}}\left(n_{\ell_{0}}\right)$ and $R \stackrel{\text { def }}{=} \bigcup_{\ell_{0} \in L_{0}} R_{\ell_{0}}$. Write $R_{e} \stackrel{\text { def }}{=}\{(s, 0) \mid \exists b \cdot(s, b) \in R\}$ for the projection of $R_{e}$ on the layer 0 .

- Remark 11. Suppose $\mathcal{E}^{\prime} \subseteq \mathcal{E}$ and $\varepsilon>0$ are such that $\mathbb{P}\left(\mathcal{E}^{\prime}\right) \geq \mathbb{P}(\mathcal{E})-\varepsilon$. Then, for any $\mathcal{R}$, we have $\mathbb{P}\left(\mathcal{E}^{\prime} \cap \mathcal{R}\right) \geq \mathbb{P}(\mathcal{E} \cap \mathcal{R})-\varepsilon$.

Proof. We have:

$$
\mathbb{P}\left(\mathcal{E}^{\prime} \cap \mathcal{R}\right)=\mathbb{P}\left(\mathcal{E}^{\prime}\right)-\mathbb{P}\left(\mathcal{E}^{\prime} \backslash \mathcal{R}\right) \geq \mathbb{P}(\mathcal{E})-\varepsilon-\mathbb{P}\left(\mathcal{E}^{\prime} \backslash \mathcal{R}\right) \geq \mathbb{P}(\mathcal{E})-\varepsilon-\mathbb{P}(\mathcal{E} \backslash \mathcal{R})=\mathbb{P}(\mathcal{E} \cap \mathcal{R})-\varepsilon
$$

We apply Remark 11 to Equation (8) to get

$$
\mathbb{P}_{\mathcal{L}_{e-2}, \ell_{0}, \sigma_{\ell_{0}}}\left(\mathrm{~F} \theta_{e} \wedge \mathrm{G} \neg \mathrm{Fix}_{e-2} \wedge \mathrm{~F} C l\left(R_{e}\right)\right) \geq \mathbb{P}_{\mathcal{L}_{e-2}, \ell_{0}, \sigma_{\ell_{0}}}\left(\mathrm{~F} \theta_{e} \wedge \mathrm{G} \neg \mathrm{Fix}_{e-2}\right)-\frac{\gamma}{2}
$$

Since FG $\neg \mathrm{Fix}_{e-2} \wedge \mathrm{G} \neg \mathrm{Fix}_{e-2}=\mathrm{G} \neg \mathrm{Fix}_{e-2}$ and $\chi_{e}=\mathrm{F} \varphi_{e} \wedge \mathrm{G}_{\mathrm{Fix}}^{e-2}$,

$$
\begin{equation*}
\mathbb{P}_{\mathcal{L}_{e-2}, \ell_{0}, \sigma_{\ell_{0}}}\left(\chi_{e} \wedge \mathrm{~F} C l\left(R_{e}\right)\right) \geq \mathbb{P}_{\mathcal{L}_{e-2}, \ell_{0}, \sigma_{\ell_{0}}}\left(\chi_{e}\right)-\frac{\gamma}{2} \tag{9}
\end{equation*}
$$

We think of $\mathrm{GF}[S]^{C o l=e}$ as a Büchi condition on a slightly modified MDP. This allows us to apply the following theorem from [12] about the strategy complexity of Büchi objectives.

- Theorem 12 (Theorem 5 in [12]). For every acyclic countable MDP M, a Büchi objective $\varphi$, finite set $I$ of initial states and $\varepsilon>0$, there exists a deterministic 1-bit strategy that is $\varepsilon$ optimal from every $s \in I$.

Using Theorem 12 we prove the following.
$\triangleright$ Claim 13. In MDP $\mathcal{L}_{e-2}$, there is an MD strategy $\tau_{e}$, that is $(\alpha-\beta)$-optimal for $\theta_{e}$ from $R_{e}$.

Notice that $\tau_{e}$ is used to define regions core $_{e} \subseteq f i x_{e}$; see Equation (6) and Figure 3. Since $\operatorname{val}_{\mathcal{L}_{e-2}, \theta_{e}}(\ell)=\operatorname{val}_{\mathcal{L}_{e-2}, \theta_{e}}\left(\ell^{\prime}\right)$ holds for all siblings $\ell$ and $\ell^{\prime}$, all states in $R_{e}$ have value $\geq \alpha$ w.r.t. $\theta_{e}$. We have chosen $\tau_{e}$ to be $(\alpha-\beta)$-optimal, which implies $\mathbb{P}_{\mathcal{L}_{e-2}, \ell, \tau_{e}}\left(\theta_{e}\right) \geq \beta$ for all $\ell \in R_{e}$. This shows that $R_{e} \subseteq f i x_{e}$. Strategy $\tau_{e}$ is also used to obtain $\mathcal{L}_{e}$ from $\mathcal{L}_{e-2}$ : for all controlled states $\ell \in f i x_{e}$, the successor is fixed to be $\tau_{e}(\ell)$ in $\mathcal{L}_{e}$, see Equation (7).

Invariant (3): Given a state $\ell_{0} \in L_{0}$, this invariant states that, for all colors $e, \operatorname{val} \mathcal{L}_{\mathcal{L}_{e}, \psi_{e}}\left(\ell_{0}\right) \geq$ $\operatorname{val}_{\mathcal{L}, \varphi}\left(\ell_{0}\right)-e \cdot \gamma$ holds. Recall that $\psi_{0}=\varphi$ and $\mathcal{L}_{0}=\mathcal{L}$. To prove the invariant, by an induction on even colors $e$, it suffices to prove the following:

$$
\operatorname{val}_{\mathcal{L}_{e}, \psi_{e}}\left(\ell_{0}\right) \geq \operatorname{val}_{\mathcal{L}_{e-2}, \psi_{e-2}}\left(\ell_{0}\right)-2 \gamma
$$

We construct a strategy $\pi$ for $\psi_{e}$ in $\mathcal{L}_{e}$ such that $\mathbb{P}_{\mathcal{L}_{e}, \ell_{0}, \pi}\left(\psi_{e}\right) \geq \operatorname{val}_{\mathcal{L}_{e-2}, \psi_{e-2}}\left(\ell_{0}\right)-2 \gamma$. Intuitively speaking, $\pi$ enforces that most runs that win through colors $e^{\prime}$, with $e^{\prime} \leq e$, eventually reach the core $_{e^{\prime}}$-region and most remaining winning runs always avoid the $\mathrm{Fix}_{e^{-}}$ region.

The strategy $\pi$ is defined by combining $\sigma_{\ell_{0}}$ and $\tau_{e}$; recall that the strategy $\sigma_{\ell_{0}}$ is $\frac{\gamma}{2}$-optimal w.r.t. $\psi_{e-2}$ starting from $\ell_{0}$ in $\mathcal{L}_{e-2}$. We define $\pi$ such that it starts by following $\sigma_{\ell_{0}}$. If it ever enters $C l\left(f i x_{e}\right)$ then we ensure that it enters $f i x_{e}$ as well (in at most one more step). Then $\pi$ continues by playing as $\tau_{e}$ does forever.

The following claim concludes the proof of Invariant 3.
$\triangleright$ Claim 14. $\quad \mathbb{P}_{\mathcal{L}_{e}, \ell_{0}, \pi}\left(\psi_{e}\right) \geq \operatorname{val}_{\mathcal{L}_{e-2}, \psi_{e-2}}\left(\ell_{0}\right)-2 \gamma$.
We summarize the main steps in the proof of Claim 14 here. We first prove the claim that if $\pi$ ever enters $C l\left(f i x_{e}\right)$ then it is possible to define it in such a way that it actually enters $f i x_{e}$.

Comparing $\psi_{e}$ with $\psi_{e-2}$, one notices that two significant terms in the symmetric difference of these two objectives are $\chi_{e}$ and ccore $_{e}$. Roughly speaking, we use Equation (9) to move from $\chi_{e}$ to $\mathrm{F} C l\left(f x_{e}\right)$. Then we move from $\mathrm{F} C l\left(f i x_{e}\right)$ to F core $e_{e}$ by proving that $\mathbb{P}_{\mathcal{L}_{e}, \ell_{0}, \pi}\left(\mathrm{~F}\right.$ core $\left._{e}\right)$ is almost as high as $\mathbb{P}_{\mathcal{L}_{e-2}, \ell_{0}, \pi}\left(\mathrm{FCl}\left(f i x_{e}\right)\right)$, modulo small errors. To derive the latter, we rely on two facts: another application of Lévy's zero-one law that guarantees $\mathbb{P}_{\mathcal{L}_{e}, \ell_{0}, \pi}\left(\theta_{e} \wedge \mathrm{~F}\right.$ core $\left.e_{e}\right)$ is equal to $\mathbb{P}_{\mathcal{L}_{e}, \ell_{0}, \pi}\left(\theta_{e}\right)$; and the fact that, as soon as $\pi$ visits the first state $\ell \in f i x_{e}$, it switches to $\tau_{e}$ forever, and thus attains $\theta_{e}$ with probability at least $\beta$.

Reach iteration: After all $\frac{e_{\max }}{2}$-iterations for even colors and the fixing, by Invariant (3), for all $\ell_{0} \in L_{0}$, we have:

$$
\begin{equation*}
\operatorname{val}_{\mathcal{L}_{e_{\max }}, \psi_{e_{\max }}}\left(\ell_{0}\right) \geq \operatorname{val}_{\mathcal{L}, \varphi}\left(\ell_{0}\right)-e_{\max } \gamma \tag{10}
\end{equation*}
$$

Recall that $\psi_{e_{\max }}=\bigvee_{e \in \operatorname{even}(\mathcal{C})} \mathrm{F}_{\text {core }}^{e}$. At this last iteration, we fix the choice of all remaining states in $\mathcal{L}_{e_{\max }}$ such that the probability of $\psi_{e_{\max }}$ is maximized. Recall that there are uniformly $\varepsilon$-optimal MD strategies for reachability objectives [16]. Hence, there is a single MD strategy $\tau_{\text {reach }}$ in $\mathcal{L}_{e_{\max }}$ that is uniformly $\gamma$-optimal w.r.t. $\psi_{e_{\max }}$; in particular, $\tau_{\text {reach }}$ is $\gamma$-optimal from every state $\ell_{0} \in L_{0}$.

Let $\mathcal{L}^{\prime} \stackrel{\text { def }}{=} \mathcal{L}_{e_{\text {max }}}\left[\tau_{\text {reach }}, L\right]$. Let $\hat{\sigma}$ be the MD strategy in $\mathcal{L}$ that plays from $L_{0}$ as prescribed by all the fixings in $\mathcal{L}^{\prime}$. Since all choices in all the $f i x_{e}$-region are resolved according to $\tau_{e}$, $e \in\left\{2, \cdots, e_{\max }\right\}$, we can apply Lévy's zero-one law another time.


Figure 4 Initial segment of the sea urchin construction. $\mathcal{L}_{i}$ is the result of fixing $\tau_{i}$ inside $\mathrm{BETA}_{i}$ and then $\rho_{i}$ inside the $k_{i}$-bubble (the set of states reachable from the initial state(s) in $\leq k_{i}$ steps). Drawn here for $i=1,2,3,4$.

- Lemma 15. Let $0<\beta_{1}<\beta_{2} \leq 1$ and $\mathcal{E}$ a tail objective. For $s \in \operatorname{Safe}_{\mathcal{E}}\left(\beta_{2}\right)$, the following holds: $\mathbb{P}_{s}\left(\operatorname{GSaf} e_{\mathcal{E}}\left(\beta_{1}\right)\right) \geq \frac{\beta_{2}-\beta_{1}}{1-\beta_{1}}$.

By Lemma 15, for all states $\ell \in$ core $_{e}$,

$$
\begin{equation*}
\mathbb{P}_{\mathcal{L}_{e_{\max }}, \ell, \tau_{e}}\left(\mathrm{G} f i x_{e}\right) \geq \frac{\alpha-\beta}{1-\beta} \geq 1-\gamma \tag{11}
\end{equation*}
$$

States in $f i x_{e}$ have a high value for $\theta_{e}$ and thus also for $\mathrm{F} \varphi_{e}$.

- Lemma 16. Let $0<\beta<1$ and $\mathcal{E}$ a tail objective. For all states $s \in \operatorname{Safe}_{\mathcal{E}}(\beta)$ :

1. $\mathbb{P}_{s}\left(\operatorname{FGSafe} e_{\mathcal{E}}(\beta) \backslash \mathcal{E}\right)=0$; and
2. $\mathbb{P}_{s}\left(\mathcal{E} \backslash \operatorname{FGSafe} \mathcal{E}_{\mathcal{E}}(\beta)\right)=0$.

By Lemma 16 . 2, we satisfy $\mathrm{F} \varphi_{e}$ almost surely:

$$
\begin{equation*}
\mathbb{P}_{\mathcal{L}_{e_{\max }}, \ell, \tau_{e}}\left(\mathrm{~F} \varphi_{e} \mid \mathrm{G} f i x_{e}\right)=1 \tag{12}
\end{equation*}
$$

Using Equations (10) and 11, we prove that following.
$\triangleright$ Claim 17. The MD strategy $\hat{\sigma}$ is $\varepsilon$-optimal for parity objective $\varphi$, from every state $\ell_{0} \in L_{0}$. This concludes the proof of Lemma 9

## 4 Optimal Strategies for Parity

In this section we show Theorem 2, i.e., that optimal strategies for parity, where they exist, can be chosen deterministic 1-bit Markov.

First we show the main technical result of this section.

- Lemma 18. Let $\mathcal{L}(\mathcal{M})$ be the layered MDP obtained from an acyclic and finitely branching MDP $\mathcal{M}$ and a coloring Col such that all states are almost surely winning for $\varphi=$ $\operatorname{Parity}(C o l)$ (i.e., every state $s$ has a strategy $\sigma_{s}$ such that $\mathbb{P}_{\mathcal{L}(\mathcal{M}), s, \sigma_{s}}(\varphi)=1$ ).

For every initial state $s_{0}$ there exists an $M D$ strategy $\sigma$ that almost surely wins, i.e., $\mathbb{P}_{\mathcal{L}(\mathcal{M}), s_{0}, \sigma}(\varphi)=1$.

Proof sketch. The full version of this rather complex proof can be found in Appendix E,
For some intuition consider Figure 4. The sea urchin construction is a plastering construction with infinitely many iterations where MD strategies are fixed in larger and larger subspaces. Its name comes from the shape of the subspace in which player choices are fixed up-to iteration $i$ : A growing finite body of states that are reachable from the initial state $s_{0}$ within $\leq k_{i}$ steps, plus $i$ different spikes of infinite size. Each spike is composed of nested subsets ALPHA $_{i} \subseteq \mathrm{BETA}_{i}$ (and $\subseteq$ GAMMA $_{i}$, which is used only in the correctness argument) that correspond to different levels of attainment of certain $\varepsilon$-optimal MD strategies $\tau_{i}$, obtained from Lemma 9 Strategy $\tau_{i}$ is then fixed in $\mathrm{BETA}_{i}$ (and thus in $\mathrm{ALPHA}_{i}$ ). Other MD strategies $\rho_{i}$ are fixed elsewhere in the finite body, up-to horizon $k_{i}$. Using Lévy's zero-one law, we prove that, once inside $\mathrm{ALPHA}_{i}$, there is a high chance of never leaving the $i$-th spike $\mathrm{BETA}_{i}$. Moreover, almost all runs that stay in the $i$-th spike satisfy parity. Finally, the strategies $\rho_{i}$ ensure that at least $1 / 2$ (by probability mass) of the runs from $s_{0}$ that don't stay in one of the first $i$ spikes will eventually stay in the $(i+1)$-th spike and satisfy parity there. Thus, at the stage with $i$ spikes, the fixed MD strategy attains parity with some probability $\geq 1-2^{-i}$ already inside this fixed subspace. In the limit of $i \rightarrow \infty$, the resulting MD strategy attains parity almost surely.

- Definition 19. For a tail objective $\varphi$ and an $M D P \mathcal{M}=\left(S, S_{\square}, S_{\circ}, \longrightarrow, P\right)$, we define the conditioned version of $\mathcal{M}$ w.r.t. $\varphi$ to be the $M D P \mathcal{M}_{*}=\left(S_{*}, S_{* \square}, S_{* \bigcirc}, \longrightarrow_{*}, P_{*}\right)$ with $S_{*}=\left\{s \in S \mid \exists \sigma . \mathbb{P}_{\mathcal{M}, s, \sigma}(\varphi)=\operatorname{val}_{\mathcal{M}}(s)>0\right\}$ and $S_{* \square}=S_{*} \cap S_{\square}$ and $S_{* \bigcirc}=S_{*} \cap S_{\bigcirc}$ and

$$
\longrightarrow_{*}=\left\{(s, t) \in S_{*} \times S_{*} \mid s \longrightarrow t \text { and if } s \in S_{* \square} \text { then } \operatorname{val}_{\mathcal{M}}(s)=\operatorname{val}_{\mathcal{M}}(t)\right\}
$$

and $P_{*}: S_{* \bigcirc} \rightarrow \mathcal{D}\left(S_{*}\right)$ so that $P_{*}(s)(t)=P(s)(t) \cdot \frac{\operatorname{val}_{\mathcal{M}}(t)}{\operatorname{val}_{\mathcal{M}}(s)}$ for all $s \in S_{* \bigcirc}$ and $t \in S_{*}$ with $s \longrightarrow * t$.

See Appendix Cfor a proof that $P_{*}(s)$ is a probability distribution for all $s \in S_{* \bigcirc}$ and therefore that the conditioned MDP $\mathcal{M}_{*}$ is well-defined. The name stems from a useful property (cf. Lemma 33 2) that for all strategies that are optimal for $\varphi$ in $\mathcal{M}$, the probability in $\mathcal{M}_{*}$ of any event is the same as that of its probability in $\mathcal{M}$ conditioned under $\varphi$.

The following theorem is a very slight generalization of [13, Theorem 5] (cf. Appendix E). It gives a sufficient condition under which we can conclude the existence of MD optimal strategies from the existence of MD almost-sure winning strategies.

- Theorem 20. Let $\varphi$ be a tail objective. Let $\mathcal{M}=\left(S, S_{\square}, S_{\bigcirc}, \longrightarrow, P\right)$ be an MDP and $\mathcal{M}_{*}=\left(S_{*}, S_{* \square}, S_{* ○}, \longrightarrow_{*}, P_{*}\right)$ its conditioned version wrt. $\varphi$. Then:

1. For all $s \in S_{*}$ there exists a strategy $\sigma$ with $\mathbb{P}_{\mathcal{M}_{*}, s, \sigma}(\varphi)=1$.
2. Suppose that for every $s \in S_{*}$ there exists an MD strategy $\sigma^{\prime \prime}$ with $\mathbb{P}_{\mathcal{M}_{*}, s, \sigma^{\prime \prime}}(\varphi)=1$. Then there is an MD strategy $\sigma^{\prime}$ such that for all $s \in S$ :

$$
\left(\exists \sigma \in \Sigma \cdot \mathbb{P}_{\mathcal{M}, s, \sigma}(\varphi)=\operatorname{val}_{\mathcal{M}}(s)\right) \Longrightarrow \mathbb{P}_{\mathcal{M}, s, \sigma^{\prime}}(\varphi)=\operatorname{val}_{\mathcal{M}}(s)
$$

- Theorem 21. Consider an acyclic MDP $\mathcal{M}$ and a parity objective.

There exists a deterministic 1-bit strategy that is optimal from all states that have an optimal strategy.

Proof. Consider the corresponding layered system $\mathcal{L}(\mathcal{M})$ (cf. Definition 6), which is also acyclic. Let $S_{\text {opt }}$ be the subset of states that have an optimal strategy in $\mathcal{M}$. Thus all states in $S_{\text {opt }} \times\{0,1\}$ have an optimal strategy in $\mathcal{L}(\mathcal{M})$ by Lemma 7 .

We now use Theorem 20 to obtain an MD strategy $\sigma^{\prime}$ in $\mathcal{L}(\mathcal{M})$ that is optimal for all states in $\mathcal{L}(\mathcal{M})$ that have an optimal strategy. First, the parity objective is tail. Second, in
$\mathcal{L}(\mathcal{M})$, any two siblings have the same value w.r.t. parity by Remark 8. Therefore the changes from $\mathcal{L}(\mathcal{M})$ to its conditioned version $\mathcal{L}(\mathcal{M})_{*}$ (wrt. the parity objective) are symmetric in the two layers. Thus $\mathcal{L}(\mathcal{M})_{*}$ is also a layered acyclic MDP (i.e., there exists some acyclic $\operatorname{MDP} \mathcal{M}^{\prime}$ s.t. $\mathcal{L}(\mathcal{M})_{*}=\mathcal{L}\left(\mathcal{M}^{\prime}\right)$ ), and by Theorem 201 all states in $\mathcal{L}(\mathcal{M})_{*}$ are almost surely winning. Now we can apply Lemma 18 (generalized to infinitely branching acyclic layered MDPs by Lemma 3) to $\mathcal{L}(\mathcal{M})_{*}$ and obtain that for every state in $\mathcal{L}(\mathcal{M})_{*}$ there is an MD strategy that almost surely wins. By Theorem 202 there is an MD strategy $\sigma^{\prime}$ in $\mathcal{L}(\mathcal{M})$ that is optimal for all states that have an optimal strategy. In particular, $\sigma^{\prime}$ is optimal for the states in $S_{\text {opt }} \times\{0,1\}$ in $\mathcal{L}(\mathcal{M})$. By Lemma 7 this yields a deterministic 1-bit strategy in $\mathcal{M}$ that is optimal for all states in $S_{o p t}$.

In Theorem 21 the initial memory mode of the 1-bit strategy is irrelevant (recall Remark 8). Theorem 2 now follows directly from Theorem 21 and Lemma 4 (3).

## 5 Optimal Strategies for $\{0,1,2\}$-Parity

- Theorem 22. Let $\mathcal{M}=\left(S, S_{\square}, S_{\bigcirc}, \longrightarrow, P\right)$ be an $M D P, \varphi$ a $\{0,1,2\}$-Parity objective and $\mathcal{M}_{*}=\left(S_{*}, S_{* \square}, S_{* ○}, \longrightarrow_{*}, P_{*}\right)$ its conditioned version wrt. $\varphi$. Assume that in $\mathcal{M}_{*}$ for every safety objective (given by some target $T \subseteq S_{*}$ ) and $\varepsilon>0$ there exists a uniformly $\varepsilon$-optimal $M D$ strategy. Let $S_{\text {opt }}$ be the subset of states that have an optimal strategy for $\varphi$ in $\mathcal{M}$.

Then there exists an $M D$ strategy in $\mathcal{M}$ that is optimal for $\varphi$ from every state in $S_{o p t}$.
The above result generalizes [13, Theorem 16], which considers only finitely-branching MDPs and uses the fact that for every safety objective, an MD strategy exists that is uniformly optimal. This is not generally true for infinitely-branching acyclic MDPs [13]. To prove Theorem 22 we adjust the construction so that it only requires uniformly $\varepsilon$-optimal MD strategies for safety objectives (in the conditioned MDP $\mathcal{M}_{*}$ ).

In order to apply Theorem 22 to infinitely-branching acyclic MDPs, we now show that acyclicity guarantees the existence of uniformly $\varepsilon$-optimal MD strategies for safety objectives.

- Lemma 23. For every acyclic MDP with a safety objective and every $\varepsilon>0$ there exists an MD strategy that is uniformly $\varepsilon$-optimal.

While we defined $\varepsilon$-optimality wrt. additive errors (cf. Section 2), our proof of Lemma 23 shows that the claim holds even wrt. multiplicative errors (in the style of [16]).

- Theorem 24. Consider an MDP $\mathcal{M}$ with a $\{0,1,2\}$-Parity objective and let $S_{o p t}$ be the subset of states that have an optimal strategy.

1. If $\mathcal{M}$ is acyclic then there exists an $M D$ strategy that is optimal from every state in $S_{\text {opt }}$.
2. There exists a deterministic Markov strategy that is optimal from every state in $S_{\text {opt }}$.

Proof. Towards item 1, if $\mathcal{M}$ is acyclic then also its conditioned version $\mathcal{M}_{*}$ (wrt. $\{0,1,2\}$-Parity) is acyclic. Thus, by Lemma 23, in $\mathcal{M}_{*}$ for every $\varepsilon>0$ and every safety objective there is a uniformly $\varepsilon$-optimal MD strategy. The result now follows from Theorem 22

Item 2 follows from Item 1 and Lemma 4 (item 3 with $k=0$ ).

## $6 \quad \varepsilon$-Optimal Strategies for $\{0,1\}$-Parity (co-Büchi)

Theorem 25. Suppose that $\mathcal{M}=\left(S, S_{\square}, S_{\bigcirc}, \longrightarrow, P\right)$ is an MDP such that for every safety objective (given by some target $T \subseteq S$ ) and $\varepsilon>0$ there exists a uniformly $\varepsilon$-optimal $M D$ strategy.

Then for every co-Büchi objective (given by some coloring Col:S $\rightarrow\{0,1\}$ ) and $\varepsilon>0$ there exists a uniformly $\varepsilon$-optimal MD strategy.

The precondition of Theorem 25 is satisfied by many classes of MDPs. Indeed, we obtain the following.

- Corollary 26. Consider an MDP $\mathcal{M}$ and a co-Büchi objective.

1. If $\mathcal{M}$ is acyclic then, for every $\varepsilon>0$, there exists a uniformly $\varepsilon$-optimal MD strategy.
2. If $\mathcal{M}$ is finitely branching then, for every $\varepsilon>0$, there exists a uniformly $\varepsilon$-optimal $M D$ strategy.
3. For every $\varepsilon>0$ there exists a deterministic Markov strategy that, from every initial state $s$, attains at least $\operatorname{val}_{\mathcal{M}}(s)-\varepsilon$.

Proof. Towards (1), for acyclic MDPs, uniformly $\varepsilon$-optimal strategies for safety can be chosen MD by Lemma 23 Towards (2), for finitely branching MDPs there always exists even a uniformly optimal MD strategy for every safety objective. In both cases the claim then follows from Theorem 25 Claim (3) follows directly from (1) and Lemma 4 (item 2 with $k=0$ ).

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## A Reductions in Section 3 and related Lemmas

By the following lemma, the strategy complexity of general parity objectives does not depend on the branching degree of the MDPs. However, this does not hold for particular parity objectives with a restricted set of colors, since the construction introduces an extra color.

## - Lemma 3.

1. Suppose that for every finitely branching acyclic MDP with a finite set $S_{0}$ of initial states, and a parity objective, there exist $\varepsilon$-optimal deterministic 1-bit strategies from $S_{0}$.
Then even for every infinitely branching acyclic MDP with a finite set $S_{0}$ of initial states and a parity objective, there exist $\varepsilon$-optimal deterministic 1-bit strategies from $S_{0}$.
2. Suppose that for every finitely branching acyclic MDP with a parity objective, there exists a deterministic 1-bit strategy that is optimal from all states that have an optimal strategy. Then even for every infinitely branching acyclic MDP with a parity objective, there exists a deterministic 1-bit strategy that is optimal from all states that have an optimal strategy.

Proof. Towards item (1), we encode an infinitely branching acyclic MDP $\mathcal{M}$ into a finitely branching acyclic MDP $\mathcal{M}^{\prime}$. Every controlled state $x$ with infinite branching $x \rightarrow y_{i}$ for all $i \in \mathbb{N}$ is replaced by a gadget $x \rightarrow z_{1}, z_{i} \rightarrow z_{i+1}, z_{i} \rightarrow y_{i}$ for all $i \in \mathbb{N}$ with fresh controlled states $z_{i}$. Infinitely branching random states with $x \xrightarrow{p_{i}} y_{i}$ for all $i \in \mathbb{N}$ are replaced by a gadget $x \xrightarrow{1} z_{1}, z_{i} \xrightarrow{1-p_{i}^{\prime}} z_{i+1}, z_{i} \xrightarrow{p_{i}^{\prime}} y_{i}$ for all $i \in \mathbb{N}$, with fresh random states $z_{i}$ and suitably adjusted probabilities $p_{i}^{\prime}$ to ensure that the gadget is left at state $y_{i}$ with probability $p_{i}$, i.e., $p_{i}^{\prime}=p_{i} /\left(\prod_{j=1}^{i-1}\left(1-p_{j}^{\prime}\right)\right)$. The fresh states are labeled with an unfavorable color that is smaller than all other colors, e.g., -1 .

We take an $\varepsilon$-optimal deterministic 1-bit strategy $\sigma^{\prime}$ for parity from all states $s \in S_{0}$ in $\mathcal{M}^{\prime}$. We construct a 1-bit deterministic $\varepsilon$-optimal strategy $\sigma$ for $\mathcal{M}$ as follows. Consider some state $x$ that is infinitely branching in $\mathcal{M}$ and its associated gadget in $\mathcal{M}^{\prime}$. Whenever a run in $\mathcal{M}^{\prime}$ according to $\sigma^{\prime}$ reaches $x$ with some memory value $\alpha \in\{0,1\}$ there exist values $p_{i}$ for the probability that the gadget is left at state $y_{i}$. Let $p \stackrel{\text { def }}{=} 1-\sum_{i \in \mathbb{N}} p_{i}$ be the probability that the gadget is never left. (If $x$ is controlled then only one $p_{i}$ (or $p$ ) is nonzero, since $\sigma^{\prime}$ is deterministic. If $x$ is random then $p=0$.) Since $\sigma^{\prime}$ is deterministic, the memory updates are deterministic, and thus there are values $\alpha_{i}^{\prime} \in\{0,1\}$ such that whenever the gadget is left at state $y_{i}$ the memory will be $\alpha_{i}^{\prime}$. We now define the behavior of the 1-bit deterministic strategy $\sigma$ at state $x$ with memory $\alpha$ in $\mathcal{M}$.

If $x$ is controlled and $p \neq 1$ then $\sigma^{\prime \prime}$ picks the successor state $y_{i}$ where $p_{i}=1$ and sets the memory to $\alpha_{i}^{\prime}$. If $p=1$ then any run according to $\sigma^{\prime}$ that enters the gadget does not satisfy the objective. Thus $\sigma$ performs at least as well in $\mathcal{M}$ regardless of its choice, e.g., pick successor $y_{1}$ and $\alpha^{\prime}=\alpha$.

If $x$ is random then $p=0$ and the successor is chosen according to the defined distribution (which is the same in $\mathcal{M}$ and $\mathcal{M}^{\prime}$ ) and $\sigma$ can only update its memory. Whenever the successor $y_{i}$ is chosen, $\sigma$ updates the memory to $\alpha_{i}^{\prime}$.

In states that are not infinitely branching in $\mathcal{M}, \sigma$ does exactly the same in $\mathcal{M}$ as $\sigma^{\prime}$ in $\mathcal{M}^{\prime}$.

Since all states in the gadgets are labeled with color $-1, \sigma$ performs at least as well in $\mathcal{M}$ as $\sigma^{\prime}$ in $\mathcal{M}^{\prime}$ and is thus $\varepsilon$-optimal from every $s \in S_{0}$.

Towards item (2), the proof is almost identical, expect that we consider optimal strategies from initial states $s$ that have an optimal strategy.

In order to show the existence of Markov (resp. 1-bit Markov) strategies, it suffices to show the existence of memoryless (resp. 1-bit) strategies in an MDP that is made acyclic
by encoding a step counter in the state space. (Note that deterministic 0-bit strategies are MD strategies and 0-bit Markov strategies are Markov strategies.) This idea appears already in [12] and can be formally stated as follows.

- Lemma 4. Consider MDPs with a parity objective and $k \in \mathbb{N}$.

1. Suppose that for every acyclic MDP $\mathcal{M}^{\prime}$ and every finite set of initial states $S_{0}^{\prime}$ and $\varepsilon>0$, there exists a deterministic $k$-bit strategy that is $\varepsilon$-optimal from all states $s \in S_{0}^{\prime}$.
Then for every MDP $\mathcal{M}$ and every finite set of initial states $S_{0}$ and $\varepsilon>0$, there exists a deterministic $k$-bit Markov strategy that is $\varepsilon$-optimal from all states $s \in S_{0}$.
2. Suppose that for every acyclic $M D P \mathcal{M}^{\prime}$ and $\varepsilon>0$, there exists a deterministic $k$-bit strategy that is $\varepsilon$-optimal from all states. Then for every $M D P \mathcal{M}$ and $\varepsilon>0$, there exists a deterministic $k$-bit Markov strategy that is $\varepsilon$-optimal from all states.
3. Suppose that for every acyclic $M D P \mathcal{M}^{\prime}$, where $S_{o p t}^{\prime}$ is the subset of states that have an optimal strategy, there exists a deterministic $k$-bit strategy that is optimal from all states $s \in S_{o p t}^{\prime}$. Then for every MDP $\mathcal{M}$, where $S_{\text {opt }}$ is the subset of states that have an optimal strategy, there exists a deterministic $k$-bit Markov strategy that is optimal from all states $s \in S_{o p t}$.

Proof. The construction is similar for all three items.
Consider an MDP $\mathcal{M}=\left(S, S_{\square}, S_{\bigcirc}, \longrightarrow, P\right)$ with sets of initial states $S_{0}$ (finite), $S$ and $S_{o p t}$, respectively.

We transform it into an acyclic MDP $\mathcal{M}^{\prime}$ by encoding a step-counter into the states, i.e., $\mathcal{M}^{\prime}=\left(S^{\prime}, S_{\square}^{\prime}, S_{\bigcirc}^{\prime}, \longrightarrow \longrightarrow^{\prime}, P^{\prime}\right)$ where $S^{\prime} \stackrel{\text { def }}{=} S \times \mathbb{N}, S_{\square}^{\prime} \stackrel{\text { def }}{=} S_{\square} \times \mathbb{N}, S_{\bigcirc}^{\prime} \stackrel{\text { def }}{=} S_{\bigcirc} \times \mathbb{N}, \operatorname{Col}((s, n)) \stackrel{\text { def }}{=}$ $\operatorname{Col}(s),(s, n) \longrightarrow{ }^{\prime}\left(s^{\prime}, n+1\right)$ iff $s \longrightarrow s^{\prime}$ and $P^{\prime}((s, n))\left(\left(s^{\prime}, n+1\right)\right) \stackrel{\text { def }}{=} P(s)\left(s^{\prime}\right)$.

For every deterministic $k$-bit strategy $\sigma^{\prime}$ in $\mathcal{M}^{\prime}$ there is a corresponding deterministic $k$-bit Markov strategy $\sigma$ in $\mathcal{M}$, and vice-versa. At any state $s, \sigma$ in memory mode m and step-counter $n$ plays exactly like $\sigma^{\prime}$ in memory mode m at state $(s, n)$.

It follows from the definition of the colorings that $\sigma$ (with memory mode $m$ ) attains the same from any initial state $s$ as $\sigma^{\prime}$ (with memory mode m ) attains from $(s, 0)$. Moreover, every state $s$ has the same value as its corresponding state $(s, 0)$.

1. In $\mathcal{M}^{\prime}$ we consider the set of initial states $S_{0}^{\prime} \stackrel{\text { def }}{=} S_{0} \times\{0\}$, which is finite since $S_{0}$ is finite. By our assumption, for every $\varepsilon>0$, there exists a deterministic $k$-bit strategy $\sigma^{\prime}$ in $\mathcal{M}^{\prime}$ that is $\varepsilon$-optimal from all states $s \in S_{0}^{\prime}$. Thus $\sigma$ is $\varepsilon$-optimal from all states $s \in S_{0}$.
2. Like above, except that the set of initial states $S_{0} \stackrel{\text { def }}{=} S$ is not finite. Since $\sigma^{\prime}$ is assumed to be $\varepsilon$-optimal from all states in $\mathcal{M}^{\prime}$, in particular it is $\varepsilon$-optimal from all states in $S_{0}^{\prime} \stackrel{\text { def }}{=} S \times\{0\}$. Thus $\sigma$ is $\varepsilon$-optimal from all states $s \in S$.
3. Here the set of initial states is $S_{o p t}$. Every state $s \in S_{o p t}$ has the same value as its corresponding state $(s, 0) \in S_{o p t} \times\{0\}$ and the corresponding strategies $\sigma$ and $\sigma^{\prime}$ attain the same from $s$ and $(s, 0)$, respectively. Therefore $S_{o p t} \times\{0\} \subseteq S_{o p t}^{\prime}$. Since the strategy $\sigma^{\prime}$ is assumed to be optimal from all states $s \in S_{o p t}^{\prime}$, it is optimal from all states in $S_{o p t} \times\{0\}$, and thus $\sigma$ is optimal from all states in $S_{o p t}$.

For ease of presentation, we will, instead of showing the existence of 1-bit strategies in an acyclic MDP $\mathcal{M}$, show the existence of MD strategies in the corresponding layered MDP $\mathcal{L}(\mathcal{M})$, which encodes the two memory modes into the states by having two copies of $\mathcal{M}$ (called layers 0 and 1). The transitions and probability functions, as well as whether a state is randomized, and its (parity) color, are lifted naturally.

The next lemma shows the correspondence between deterministic 1-bit strategies in $\mathcal{M}$ and MD strategies in $\mathcal{L}(\mathcal{M})$.

- Lemma 7. Consider an acyclic $M D P \mathcal{M}=\left(S, S_{\square}, S_{\bigcirc}, \longrightarrow, P\right)$ with a parity objective $\varphi=$ Parity $(C o l)$ and let $\mathcal{L}(\mathcal{M})$ be the corresponding layered MDP.

For every deterministic 1 -bit strategy $u\left[\mathrm{~m}_{0}\right]$ in $\mathcal{M}$ there is a corresponding $M D$ strategy $\tau$ in $\mathcal{L}(\mathcal{M})$, and vice-versa, such that for every $s_{0} \in S, \mathbb{P}_{\mathcal{L}(\mathcal{M}),\left(s_{0}, m_{0}\right), \tau}(\varphi)=\mathbb{P}_{\mathcal{M}, s_{0}, u\left[m_{0}\right]}(\varphi)$.

Proof. For the " $\mathcal{M} \Longrightarrow \mathcal{L}(\mathcal{M})$ " direction, given $u\left[m_{0}\right]$, we define the MD strategy $\tau$ to play in $\mathcal{L}(\mathcal{M})$ as follows. For $b, b^{\prime} \in\{0,1\}$,

- for a controlled state $s \in S_{\square}$, if $u\left[m_{0}\right](b, s)=\left(b^{\prime}, s^{\prime}\right)$ meaning that $u\left[m_{0}\right]$ chooses $s^{\prime}$ at $s$, by taking a transition $t=\left(s, s^{\prime}\right)$, and updates the bit to $b^{\prime}$, we define $\tau((s, b)) \stackrel{\text { def }}{=}(t, b)$ and $\tau((t, b)) \stackrel{\text { def }}{=}\left(s^{\prime}, b^{\prime}\right)$;
- for a random state $s \in S_{\bigcirc}$, if $u\left[m_{0}\right]$ updates the memory bit to $b^{\prime}$ in case the random successor resolves to $s^{\prime}$, by taking a transition $t=\left(s, s^{\prime}\right)$, we define $\tau((t, b)) \stackrel{\text { def }}{=}\left(s^{\prime}, b^{\prime}\right)$.

Similarly, for the " $\mathcal{M} \Longleftarrow \mathcal{L}(\mathcal{M})$ " direction, given $\tau$ in $\mathcal{L}(\mathcal{M})$, we define an update function $u$, such that for all initial bit $m_{0} \in\{0,1\}$ the deterministic 1-bit strategy $u\left[m_{0}\right]$ in $\mathcal{M}$ plays from any state $s \in S$ as $\tau$ plays in $\mathcal{L}(\mathcal{M})$ from $\left(s, m_{0}\right)$. The construction is as follows. For all $b, b^{\prime} \in\{0,1\}$ and all transitions $t=\left(s, s^{\prime}\right)$,

- if $s \in S_{\square}$, and if $\tau((s, b))=(t, b)$ and $\tau((t, b))=\left(s^{\prime}, b^{\prime}\right)$, we define $u(b, s) \stackrel{\text { def }}{=}\left(b^{\prime}, s^{\prime}\right)$;
- if $s \in S_{\bigcirc}$, and if $\tau((t, b))=\left(s^{\prime}, b^{\prime}\right)$, we define $u(b, s)\left(b^{\prime}, s^{\prime}\right) \stackrel{\text { def }}{=} P(s)\left(s^{\prime}\right)$.

Denote by $\mathcal{C}^{\tau}$ the Markov chain obtained from $\mathcal{L}(\mathcal{M})$ after fixing $\tau$, and by $\mathcal{C}^{u\left[m_{0}\right]}$ the Markov chain obtained from $\mathcal{M}$ after fixing $u\left[m_{0}\right]$. Observe there is a clear bijection between the runs in the Markov chains $\mathcal{C}^{\tau}$ and $\mathcal{C}^{u\left[m_{0}\right]}$. Since the parity colors are lifted accordingly, we conclude that $\mathbb{P}_{\mathcal{L}(\mathcal{M}),\left(s_{0}, m_{0}\right), \tau}(\varphi)=\mathbb{P}_{\mathcal{M}, s_{0}, u\left[m_{0}\right]}(\varphi)$, as required.

## B Lévy's zero-one law

We fix a finitely branching Markov chain $\mathcal{C}$ with state space $S$. We use the probability measure $\mathbb{P}_{s}$ when starting in a state s.

For an event $\mathcal{E} \in \mathcal{F}$, the indicator function $\mathbf{1}_{\mathcal{E}}: S^{\omega} \rightarrow\{0,1\}$ is defined by

$$
\mathbf{1}_{\mathcal{E}}(\rho)= \begin{cases}1 & \text { if } \rho \in \mathcal{E} \\ 0 & \text { otherwise }\end{cases}
$$

Below we recall Lévy's zero-one law; we state this result for a specific family of sub $\sigma$ algebras that is used throughout our proofs. Consider the simplest sequence of sub $\sigma$-algebras $\left(\mathcal{F}_{i}\right)_{i \in \mathbb{N}}$ of $\mathcal{F}$ where each $\mathcal{F}_{i}$ is the $\sigma$-algebra generated by all events that depend only on the length- $i$ prefixes. Formally, for all $i \in \mathbb{N}$, define the sub $\sigma$-algebra

$$
\mathcal{F}_{i}=\left\{A \cdot S^{\omega} \subseteq S^{\omega} \mid A \subseteq S^{i}\right\}
$$

Observe that $\mathcal{F}_{1} \subset \mathcal{F}_{2} \subset \cdots \subset \mathcal{F}_{\infty}$ where $\mathcal{F}_{\infty}=\mathcal{F}$ is the smallest $\sigma$-algebra containing all the $\mathcal{F}_{i}$. The sub $\sigma$-algebra $\mathcal{F}_{i}, i \in \mathbb{N}$, introduces an equivalence class $\sim_{i}$ on $S^{\omega}$ where $\rho \sim_{i} \rho^{\prime}$ if and only if for all $\mathcal{E} \in \mathcal{F}_{i}$, the condition $\rho \in \mathcal{E} \Leftrightarrow \rho^{\prime} \in \mathcal{E}$ is met. Given a run $\rho$, denote by $[\rho]_{\sim_{i}}$ the equivalence class of $\rho$. By definition of the $\mathcal{F}_{i}$, if $\rho \in s_{1} \cdots s_{i} S^{\omega}$ then $[\rho]_{\sim_{i}}=s_{1} \cdots s_{i} S^{\omega}$.

Given a state $s$, define the random variable $\mathbb{P}_{s}\left(\mathcal{E} \mid \mathcal{F}_{i}\right): S^{\omega} \rightarrow[0,1] \cup\{\perp\}$ such that, for all runs $\rho \in s_{1} \cdots s_{i} S^{\omega}$,

$$
\mathbb{P}_{s}\left(\mathcal{E} \mid \mathcal{F}_{i}\right)(\rho)= \begin{cases}\mathbb{P}_{s}\left(\mathcal{E} \mid[\rho]_{\sim_{i}}\right) & \text { if } \mathbb{P}_{s}\left(s_{1} \cdots s_{i}\right) \neq 0  \tag{13}\\ \perp(\text { read as undefined }) & \text { otherwise }\end{cases}
$$

By Lévy's zero-one law for all events $\mathcal{E} \subseteq \mathcal{F}_{\infty}$ we have that

$$
\lim _{i \rightarrow \infty} \mathbb{P}_{s}\left(\mathcal{E} \mid \mathcal{F}_{i}\right)=\mathbf{1}_{\mathcal{E}}
$$

holds $\mathbb{P}$-almost-surely.

- Remark 27. Given a suffix-closed objective $\mathcal{E}$ and a run $\rho \in s_{1} \cdots s_{i} S^{\omega}$, if $\mathbb{P}_{s}\left(\mathcal{E} \mid \mathcal{F}_{i}\right)(\rho)$ is defined, then

$$
\begin{aligned}
\mathbb{P}_{s_{1}}\left(\mathcal{E} \mid \mathcal{F}_{i}\right)(\rho) & =\mathbb{P}_{s_{1}}\left(\mathcal{E} \mid[\rho]_{\sim_{i}}\right) \\
& =\mathbb{P}_{s_{1}}\left(\mathcal{E} \mid s_{1} \cdots s_{i} S^{\omega}\right) \\
& \leq \mathbb{P}_{s_{i}}\left(\mathcal{E} \mid s_{i} S^{\omega}\right) \\
& =\mathbb{P}_{s_{i}}(\mathcal{E}) .
\end{aligned}
$$

If $\mathcal{E}$ is tail then $\mathbb{P}_{s_{1}}\left(\mathcal{E} \mid \mathcal{F}_{i}\right)(\rho)=\mathbb{P}_{s_{i}}(\mathcal{E})$.
For the fixed Markov chain $\mathcal{C}$ and $\varepsilon>0$, we define $\operatorname{Safe}_{\mathcal{E}}(1-\varepsilon) \stackrel{\text { def }}{=}\left\{s \mid \mathbb{P}_{s}(\mathcal{E}) \geq 1-\varepsilon\right\}$.
Lemma 28. Let $s_{0} \in S$ and $\mathcal{E}$ be a suffix-closed objective and $\varepsilon>0$. Then $\mathbb{P}_{s_{0}}(F \mathcal{E} \wedge$ $\left.\neg \mathrm{FSafe}_{\mathcal{E}}(1-\varepsilon)\right)=0$.

Proof. Let $s_{0} \in S$. We have:

$$
\begin{aligned}
\llbracket \mathrm{G} \neg \operatorname{Safe}_{\mathcal{E}}(1-\varepsilon) \rrbracket^{s_{0}} & =\left\{s_{0} s_{1} \cdots \mid \forall i . \mathbb{P}_{s_{i}}(\mathcal{E})<1-\varepsilon\right\} \\
& \subseteq\left\{\rho \in s_{0} S^{\omega} \mid \forall i . \mathbb{P}_{s_{0}}\left(\mathcal{E} \mid \mathcal{F}_{i}\right)(\rho)<1-\varepsilon\right\} \quad \text { by Remark } 27 \\
& \subseteq\left\{\rho \in s_{0} S^{\omega} \mid \lim _{i \rightarrow \infty} \mathbb{P}_{s_{0}}\left(\mathcal{E} \mid \mathcal{F}_{i}\right)(\rho) \neq 1\right\}
\end{aligned}
$$

It follows

$$
\begin{equation*}
\mathbb{P}_{s_{0}}\left(\mathcal{E} \wedge G \neg \operatorname{Safe}_{\mathcal{E}}(1-\varepsilon)\right) \leq \mathbb{P}_{s_{0}}\left(\mathcal{E} \cap\left\{\rho \in s_{0} S^{\omega} \mid \lim _{i \rightarrow \infty} \mathbb{P}_{s_{0}}\left(\mathcal{E} \mid \mathcal{F}_{i}\right)(\rho) \neq 1\right\}\right)=0 \tag{14}
\end{equation*}
$$

by Lévy's zero-one law.
Let $s_{0} \in S$. We have:

$$
\begin{aligned}
& \mathbb{P}_{s_{0}}\left(F \mathcal{E} \wedge \neg \operatorname{FSafe}_{\mathcal{E}}(1-\varepsilon)\right) \\
& =\mathbb{P}_{s_{0}}\left(\mathrm{FE} \wedge \mathrm{G} \neg \operatorname{Safe}_{\mathcal{E}}(1-\varepsilon)\right) \\
& =\mathbb{P}_{s_{0}}\left(\bigcup_{s_{1} \cdots s_{i} \in S^{*}} s_{0} s_{1} \cdots s_{i-1}\left(\mathcal{E} \cap s_{i} S^{\omega}\right) \wedge G \neg \operatorname{Safe}_{\mathcal{E}}(1-\varepsilon)\right) \\
& \leq \sum_{s_{1} \cdots s_{i} \in S^{*}} \mathbb{P}_{s_{0}}\left(s_{0} s_{1} \cdots s_{i-1}\left(\mathcal{E} \cap s_{i} S^{\omega}\right) \wedge \mathrm{G} \neg \operatorname{Safe}_{\mathcal{E}}(1-\varepsilon)\right) \quad \text { union bound } \\
& \leq \sum_{s_{1} \cdots s_{i} \in S^{*}} \mathbb{P}_{s_{i}}\left(\left(\mathcal{E} \cap s_{i} S^{\omega}\right) \wedge G \neg \operatorname{Safe}_{\mathcal{E}}(1-\varepsilon)\right) \\
& =\sum_{s_{1} \cdots s_{i} \in S^{*}} \mathbb{P}_{s_{i}}\left(\mathcal{E} \wedge G \neg \operatorname{Safe}_{\mathcal{E}}(1-\varepsilon)\right) \\
& =0
\end{aligned}
$$

- Lemma 10. Let $s_{0} \in S$ and $\mathcal{E}$ be a suffix-closed objective. For all $\varepsilon, \varepsilon^{\prime}>0$, there exist $n$ and a finite set $F \subseteq \operatorname{Safe} e_{\mathcal{E}}(1-\varepsilon)$ such that $\mathbb{P}_{s_{0}}\left(\mathrm{FE} \wedge \mathrm{F}^{\leq n} F\right) \geq \mathbb{P}_{s_{0}}(\mathrm{FE})-\varepsilon^{\prime}$.

Proof. By Lemma 28 we have

$$
\mathbb{P}_{s_{0}}\left(\mathrm{FE} \wedge \mathrm{FSafe}_{\mathcal{E}}(1-\varepsilon)\right)=\mathbb{P}_{s_{0}}(\mathrm{FE})
$$

By continuity of measures it follows that there is $n$ such that

$$
\mathbb{P}_{s_{0}}\left(\mathrm{FE} \wedge \mathrm{~F}^{\leq n} \operatorname{Safe}_{\mathcal{E}}(1-\varepsilon)\right) \geq \mathbb{P}_{s_{0}}(\mathrm{~F} \mathcal{E})-\varepsilon^{\prime}
$$

Let bubble $e_{n}\left(s_{0}\right)$ be the set of states that can be reached from $s_{0}$ within at most $n$ steps. Since the Markov chain $\mathcal{C}$ is finitely branching, $F \stackrel{\text { def }}{=} \operatorname{Safe}_{\mathcal{E}}(1-\varepsilon) \cap$ bubble $_{n}\left(s_{0}\right)$ is a finite set. Then we have $\llbracket \mathrm{F} \leq n ~ F \rrbracket=\llbracket \mathrm{F} \leq{ }^{\leq} \operatorname{Safe}_{\mathcal{E}}(1-\varepsilon) \rrbracket$ and the statement of the lemma follows.

- Lemma 16. Let $0<\beta<1$ and $\mathcal{E}$ a tail objective. For all states $s \in \operatorname{Safe}_{\mathcal{E}}(\beta)$ :

1. $\mathbb{P}_{s}\left(\operatorname{FGSafe} e_{\mathcal{E}}(\beta) \backslash \mathcal{E}\right)=0$; and
2. $\mathbb{P}_{s}\left(\mathcal{E} \backslash \operatorname{FGSafe} e_{\mathcal{E}}(\beta)\right)=0$.

Proof. By Lévy's zero-one law,

$$
\begin{align*}
& \mathbb{P}_{s}\left(\left\{\rho \mid \lim _{i \rightarrow \infty} \mathbb{P}_{s}\left(\mathcal{E} \mid \mathcal{F}_{i}\right)(\rho)=\mathbf{1}_{\mathcal{E}}(\rho)\right\}\right)=1, \text { and } \\
& \mathbb{P}_{s}\left(\left\{\rho \mid \lim _{i \rightarrow \infty} \mathbb{P}_{s}\left(\mathcal{E} \mid \mathcal{F}_{i}\right)(\rho) \neq \mathbf{1}_{\mathcal{E}}(\rho)\right\}\right)=0 \tag{15}
\end{align*}
$$

On one hand Equation (15) implies that

$$
\begin{aligned}
& \mathbb{P}_{s}\left(\left\{\rho \mid \lim _{i \rightarrow \infty} \mathbb{P}_{s}\left(\mathcal{E} \mid \mathcal{F}_{i}\right)(\rho)=0 \wedge \mathbf{1}_{\mathcal{E}}(\rho)=0\right\}\right. \\
& \left.\cup\left\{\rho \mid \lim _{i \rightarrow \infty} \mathbb{P}_{s}\left(\mathcal{E} \mid \mathcal{F}_{i}\right)(\rho)=1 \wedge \mathbf{1}_{\mathcal{E}}(\rho)=1\right\}\right)=1 \\
\Rightarrow & \mathbb{P}_{s}\left(\left\{\rho \mid \lim _{i \rightarrow \infty} \mathbb{P}_{s}\left(\mathcal{E} \mid \mathcal{F}_{i}\right)(\rho)=0\right\} \cup\left\{\rho \mid \mathbf{1}_{\mathcal{E}}(\rho)=1\right\}\right)=1 \\
\Leftrightarrow & \mathbb{P}_{s}\left(\left\{\rho \mid \forall \varepsilon>0 \exists n \forall i \geq n \mathbb{P}_{s}\left(\mathcal{E} \mid \mathcal{F}_{i}\right)(\rho)<\varepsilon\right\} \cup \mathcal{E}\right)=1 \\
\Rightarrow & \mathbb{P}_{s}\left(\left\{\rho \mid \exists n \forall i \geq n \mathbb{P}_{s}\left(\mathcal{E} \mid \mathcal{F}_{i}\right)(\rho)<\beta\right\} \cup \mathcal{E}\right)=1 \\
\Rightarrow & \mathbb{P}_{s}\left(\mathrm{FG} \neg \operatorname{Safe} e_{\mathcal{E}}(\beta) \cup \mathcal{E}\right)=1 \quad \quad \text { by Remark } 27
\end{aligned}
$$

$$
\text { since } \llbracket \mathrm{FG} \neg \operatorname{Safe}_{\mathcal{E}}(\beta) \rrbracket \subseteq \llbracket \neg \mathrm{FG} \operatorname{Safe}_{\mathcal{E}}(\beta) \rrbracket
$$

$$
\Rightarrow \mathbb{P}_{s}\left(\neg \mathrm{FGSafe} e_{\mathcal{E}}(\beta) \cup \mathcal{E}\right)=1
$$

$$
\Leftrightarrow \mathbb{P}_{s}\left(\operatorname{FGSaf} e_{\mathcal{E}}(\beta) \cap \neg \mathcal{E}\right)=0
$$

$$
\Leftrightarrow \mathbb{P}_{s}\left(\operatorname{FGSafe}_{\mathcal{E}}(\beta) \backslash \mathcal{E}\right)=0
$$

On the other hand Equation implies that

$$
\begin{array}{rlr} 
& \mathbb{P}_{s}\left(\left\{\rho \mid \lim _{i \rightarrow \infty} \mathbb{P}_{s}\left(\mathcal{E} \mid \mathcal{F}_{i}\right)(\rho) \neq 1 \wedge \mathbf{1}_{\mathcal{E}}(\rho)=1\right\}\right)=0 & \\
\Rightarrow & \mathbb{P}_{s}\left(\left\{\rho \mid \lim _{i \rightarrow \infty} \mathbb{P}_{s}\left(\mathcal{E} \mid \mathcal{F}_{i}\right)(\rho) \neq 1\right\} \cap\left\{\rho \mid \mathbf{1}_{\mathcal{E}}(\rho)=1\right\}\right)=0 & \\
\Rightarrow & \mathbb{P}_{s}\left(\left\{\rho \mid \forall n \exists i \geq n \mathbb{P}_{s}\left(\mathcal{E} \mid \mathcal{F}_{i}\right)(\rho)<\beta\right\} \cap\left\{\rho \mid \mathbf{1}_{\mathcal{E}}(\rho)=1\right\}\right)=0 & \\
\Leftrightarrow & \mathbb{P}_{s}\left(\neg \operatorname{FGSafe} e_{\mathcal{E}}(\beta) \cap \mathcal{E}\right)=0 & \text { by Remark } 27 \\
\Leftrightarrow & \mathbb{P}_{s}\left(\mathcal{E} \backslash \operatorname{FGSafe}_{\mathcal{E}}(\beta)\right)=0 &
\end{array}
$$

Corollary 29. Let $0<\beta<1$ and $\mathcal{E}$ a tail objective. For all states $s \in \operatorname{Safe}_{\mathcal{E}}(\beta)$, we have $\mathbb{P}_{s}\left(\mathcal{E} \mid \operatorname{GSafe} e_{\mathcal{E}}(\beta)\right)=1$.

Proof. Since $\operatorname{GSafe} \mathcal{E}_{\mathcal{E}}(\beta)$ is contained in $\operatorname{FGSafe} \mathcal{E}_{\mathcal{E}}(\beta)$, Lemma 16 leads to $\mathbb{P}_{s}(\operatorname{GSafe} \mathcal{E}(\beta) \backslash \mathcal{E})=0$. Then,

$$
\begin{aligned}
& \mathbb{P}_{s}\left(\mathcal{E} \cap \operatorname{GSafe}_{\mathcal{E}}(\beta)\right)=\mathbb{P}_{s}\left(\operatorname{GSafe}_{\mathcal{E}}(\beta)\right)-\mathbb{P}_{s}\left(\operatorname{GSafe}_{\mathcal{E}}(\beta) \backslash \mathcal{E}\right) \\
& =\mathbb{P}_{s}\left(\mathrm{GSafe}_{\mathcal{E}}(\beta)\right) .
\end{aligned}
$$

By the above equality, we get that $\mathbb{P}_{s}\left(\mathcal{E} \mid \operatorname{GSafe}_{\mathcal{E}}(\beta)\right)=\frac{\mathbb{P}_{s}(\mathcal{E} \cap \operatorname{GSafe}(\beta))}{\mathbb{P}_{s}(\operatorname{GSafe\mathcal {E}}(\beta))}=1$.

- Corollary 30. Let $0<\beta_{1}, \beta_{2}<1$ and $\mathcal{E}$ a tail objective. For all states $s$ we have

$$
\mathbb{P}_{s}\left(\operatorname{FG} \operatorname{Safe}_{\mathcal{E}}\left(\beta_{1}\right) \backslash \mathrm{FG} \operatorname{Safe}_{\mathcal{E}}\left(\beta_{2}\right)\right)=0
$$

Proof. We have that

$$
\begin{align*}
\text { FG } \operatorname{Safe}_{\mathcal{E}}\left(\beta_{1}\right) \backslash \mathrm{FG} \operatorname{Safe}_{\mathcal{E}}\left(\beta_{2}\right) & \\
= & {\left[\mathcal{E} \cap\left(\mathrm{FG} \operatorname{Safe}_{\mathcal{E}}\left(\beta_{1}\right) \backslash \mathrm{FG} \operatorname{Safe}_{\mathcal{E}}\left(\beta_{2}\right)\right)\right] } \\
& \cup\left[\left(\mathrm{FG} \operatorname{Safe}_{\mathcal{E}}\left(\beta_{1}\right) \backslash \operatorname{FG} \operatorname{Safe}_{\mathcal{E}}\left(\beta_{2}\right)\right) \backslash \mathcal{E}\right]  \tag{16}\\
\subseteq & \left(\mathcal{E} \backslash \mathrm{FG} \operatorname{Safe}_{\mathcal{E}}\left(\beta_{2}\right)\right) \cup\left(\mathrm{FG} \operatorname{Safe}_{\mathcal{E}}\left(\beta_{1}\right) \backslash \mathcal{E}\right) .
\end{align*}
$$

Thus $\mathbb{P}_{s}\left(\operatorname{FG~Safe_{\mathcal {E}}}\left(\beta_{1}\right) \backslash \operatorname{FGSaf} e_{\mathcal{E}}\left(\beta_{2}\right)\right) \leq \mathbb{P}_{s}\left(\mathcal{E} \backslash \operatorname{FGSaf} e_{\mathcal{E}}\left(\beta_{2}\right)\right)+\mathbb{P}_{s}\left(\mathrm{FG} \operatorname{Safe} e_{\mathcal{E}}\left(\beta_{1}\right) \backslash \mathcal{E}\right)=0$, by Lemma 16.

- Lemma 15. Let $0<\beta_{1}<\beta_{2} \leq 1$ and $\mathcal{E}$ a tail objective. For $s \in \operatorname{Safe}_{\mathcal{E}}\left(\beta_{2}\right)$, the following holds: $\mathbb{P}_{s}\left(\operatorname{GSafe} \mathcal{E}_{\mathcal{E}}\left(\beta_{1}\right)\right) \geq \frac{\beta_{2}-\beta_{1}}{1-\beta_{1}}$.

Proof. Write $x$ for $\mathbb{P}_{s}\left(\operatorname{GSafe} \mathcal{E}\left(\beta_{1}\right)\right)$. We condition the probability of $\mathcal{E}$ under $\operatorname{G} \operatorname{Safe} \mathcal{E}\left(\beta_{1}\right)$. By the law of total probability, we have

$$
\beta_{2} \leq \mathbb{P}_{s}(\mathcal{E})=\mathbb{P}_{s}\left(\mathcal{E} \mid \mathrm{G} \operatorname{Safe}_{\mathcal{E}}\left(\beta_{1}\right)\right) \cdot x+\mathbb{P}_{s}\left(\mathcal{E} \mid \neg \mathrm{G} \operatorname{Safe}_{\mathcal{E}}\left(\beta_{1}\right)\right) \cdot(1-x)
$$

By Corollary 29, we have $\mathbb{P}_{s}\left(\mathcal{E} \mid \operatorname{GSafe} \mathcal{E}\left(\beta_{1}\right)\right)=1$. Hence we have $\beta_{2} \leq x+\beta_{1} \cdot(1-x)$; and $x \geq \frac{\beta_{2}-\beta_{1}}{1-\beta_{1}}$ follows.

## C The Conditioned MDP

In this section we adapt some results from [13].
We will need the following lemma, which is a variant of [14, Lemma 20]:

- Lemma 31. Let $\varphi$ be a tail objective. Let $\mathcal{M}=\left(S, S_{\square}, S_{\circ}, \longrightarrow, P\right)$ be an MDP, and $s_{0} \in S$, and $\sigma$ be a strategy with $\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}(\varphi)=\operatorname{val}_{\mathcal{M}}\left(s_{0}\right)$. Suppose that $s_{0} s_{1} \cdots s_{n}$ for some $n \geq 0$ is a partial run starting in $s_{0}$ and induced by $\sigma$. Then:

1. $\operatorname{val}_{\mathcal{M}}\left(s_{n}\right)=\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(\llbracket \varphi \rrbracket^{s_{0}} \mid s_{0} s_{1} \cdots s_{n} S^{\omega}\right)$.
2. If $s_{n} \in S_{\bigcirc}$ then $\operatorname{val}_{\mathcal{M}}\left(s_{n}\right)=\sum_{s_{n+1} \in S} P\left(s_{n}\right)\left(s_{n+1}\right) \cdot \operatorname{val}_{\mathcal{M}}\left(s_{n+1}\right)$.
3. If $s_{n} \in S_{\square}$ then $\operatorname{val}_{\mathcal{M}}\left(s_{n}\right)=\operatorname{val}_{\mathcal{M}}\left(s_{n+1}\right)$ for all $s_{n+1} \in \operatorname{supp}\left(\sigma\left(s_{0} s_{1} \cdots s_{n}\right)\right)$.

Proof. First we show $\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(\llbracket \varphi \rrbracket^{s_{0}} \mid s_{0} s_{1} \cdots s_{n} S^{\omega}\right) \leq \operatorname{val}_{\mathcal{M}}\left(s_{n}\right)$. Define a strategy $\sigma^{\prime}$ : $S^{*} S_{\square} \rightarrow \mathcal{D}(S)$ by $\sigma^{\prime}(w)=\sigma\left(s_{0} s_{1} \cdots s_{n-1} w\right)$ for all $w \in S^{*} S_{\square}$. Then we have $\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(\llbracket \varphi \rrbracket^{s_{0}} \mid\right.$ $\left.s_{0} s_{1} \cdots s_{n} S^{\omega}\right)=\mathbb{P}_{\mathcal{M}, s_{n}, \sigma^{\prime}}\left(\llbracket \varphi \rrbracket^{s_{n}}\right) \leq \operatorname{val}_{\mathcal{M}}\left(s_{n}\right)$.

Next we show $\operatorname{val}_{\mathcal{M}}\left(s_{n}\right) \leq \mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(\llbracket \varphi \rrbracket^{s_{0}} \mid s_{0} s_{1} \cdots s_{n} S^{\omega}\right)$. Towards a contradiction, suppose that $\operatorname{val}_{\mathcal{M}}\left(s_{n}\right)>\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(\llbracket \varphi \rrbracket^{s_{0}} \mid s_{0} s_{1} \cdots s_{n} S^{\omega}\right)$. Then, by the definition of $\operatorname{val}_{\mathcal{M}}\left(s_{n}\right)$,
there is a strategy $\sigma^{\prime}$ with $\mathbb{P}_{\mathcal{M}, s_{n}, \sigma^{\prime}}\left(\llbracket \varphi \rrbracket^{s_{n}}\right)>\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(\llbracket \varphi \rrbracket^{s_{0}} \mid s_{0} s_{1} \cdots s_{n} S^{\omega}\right)$. Define a strategy $\sigma^{\prime \prime}$ that plays according to $\sigma$; if and when partial run $s_{0} s_{1} \cdots s_{n}$ is played, then $\sigma^{\prime \prime}$ acts like $\sigma^{\prime}$ henceforth; otherwise $\sigma^{\prime \prime}$ continues with $\sigma$ forever. Using the tail property we get:

$$
\begin{array}{ll}
\mathbb{P}_{\mathcal{M}, s_{0}, \sigma^{\prime \prime}}\left(\llbracket \varphi \rrbracket^{s_{0}}\right) \\
=\mathbb{P}_{\mathcal{M}, s_{0}, \sigma^{\prime \prime}}\left(\llbracket \varphi \rrbracket^{s_{0}} \mid s_{0} s_{1} \cdots s_{n} S^{\omega}\right) \cdot \mathbb{P}_{\mathcal{M}, s_{0}, \sigma^{\prime \prime}}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right) & \\
+\mathbb{P}_{\mathcal{M}, s_{0}, \sigma^{\prime \prime}}\left(\llbracket \varphi \rrbracket^{s_{0}} \backslash s_{0} s_{1} \cdots s_{n} S^{\omega}\right) & \\
=\mathbb{P}_{\mathcal{M}, s_{n}, \sigma^{\prime}}\left(\llbracket \varphi \rrbracket^{s_{n}}\right) \cdot \mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right) & \text { def. of } \sigma^{\prime \prime} \\
\left.+\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(\llbracket \varphi \rrbracket^{\prime}\right]_{0} \backslash s_{0} s_{1} \cdots s_{n} S^{\omega}\right) & \text { def. of } \sigma^{\prime} \\
>\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(\llbracket \varphi \rrbracket^{s_{0}} \mid s_{0} s_{1} \cdots s_{n} S^{\omega}\right) \cdot \mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right) & \\
+\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(\llbracket \varphi \rrbracket^{s_{0}} \backslash s_{0} s_{1} \cdots s_{n} S^{\omega}\right) & \text { def. of } \sigma \\
=\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(\llbracket \varphi \rrbracket^{s_{0}}\right) & \\
=\operatorname{val}_{\mathcal{M}}\left(s_{0}\right) &
\end{array}
$$

This contradicts the definition of $\operatorname{val}_{\mathcal{M}}\left(s_{0}\right)$. Hence we have shown item 1 .
Towards items 2 and 3, we extend $\sigma: S^{*} S_{\square} \rightarrow \mathcal{D}(S)$ to $\sigma: S^{*} S \rightarrow \mathcal{D}(S)$ by defining $\sigma(w s)=P(s)$ for $w \in S^{*}$ and $s \in S_{\bigcirc}$. Then we have for all $s_{n+1} \in S$ :

$$
\begin{equation*}
\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} s_{n+1} S^{\omega}\right)=\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right) \cdot \sigma\left(s_{0} s_{1} \cdots s_{n}\right)\left(s_{n+1}\right) \tag{17}
\end{equation*}
$$

Further we have:

$$
\begin{aligned}
& \operatorname{val}_{\mathcal{M}}\left(s_{n}\right) \\
& \begin{array}{ll}
=\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(\llbracket \varphi \rrbracket^{s_{0}} \mid s_{0} s_{1} \cdots s_{n} S^{\omega}\right) & \text { by item } 1 \\
=\frac{\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(\llbracket \varphi \rrbracket^{s_{0}} \cap s_{0} s_{1} \cdots s_{n} S^{\omega}\right)}{\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right)} & \\
=\frac{\sum_{s_{n+1} \in S} \mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(\llbracket \varphi \rrbracket^{s_{0}} \cap s_{0} s_{1} \cdots s_{n} s_{n+1} S^{\omega}\right)}{\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right)} \\
=\frac{1}{\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right)} \cdot \sum_{s_{n+1} \in S} \mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} s_{n+1} S^{\omega}\right) \cdot \\
=\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(\llbracket \varphi \rrbracket^{s_{0}} \mid s_{0} s_{1} \cdots s_{n} s_{n+1} S^{\omega}\right) & \\
=\sum_{s_{n+1} \in S} \sigma\left(s_{0} s_{1} \cdots s_{n}\right)\left(s_{n+1}\right) \cdot \mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(\llbracket \varphi \rrbracket^{s_{0}} \mid s_{0} s_{1} \cdots s_{n} s_{n+1} S^{\omega}\right) & \text { by } 17) \\
=\sum_{s_{n+1} \in S} \sigma\left(s_{0} s_{1} \cdots s_{n}\right)\left(s_{n+1}\right) \cdot \operatorname{val}_{\mathcal{M}}\left(s_{n+1}\right) & \text { by item } 1
\end{array}
\end{aligned}
$$

Thus we have shown item 2 . Towards item 3 , suppose $s_{n} \in S_{\square}$. Then, by the tail property, $\operatorname{val}_{\mathcal{M}}\left(s_{n}\right) \geq \operatorname{val}_{\mathcal{M}}\left(s_{n+1}\right)$ for all $s_{n+1}$ with $s_{n} \longrightarrow s_{n+1}$. Since $\sigma\left(s_{0} s_{1} \cdots s_{n}\right)$ is a probability distribution, the equality chain above shows that $\operatorname{val}_{\mathcal{M}}\left(s_{n}\right)=\operatorname{val}_{\mathcal{M}}\left(s_{n+1}\right)$ for all $s_{n+1} \in$ $\operatorname{supp}\left(\sigma\left(s_{0} s_{1} \cdots s_{n}\right)\right)$. Thus we have shown item 3 .

- Lemma 32. The conditioned version $\mathcal{M}_{*}$ of $\mathcal{M}$ w.r.t. tail objective $\varphi$ (cf. Definition 19 is well defined.

Proof. By Lemma 312 we have that $P_{*}(s)$ is a probability distribution for all $s \in S_{* \bigcirc}$; hence the conditioned MDP $\mathcal{M}_{*}$ is well-defined.

The following lemma is a reformulation of [13, Lemma 6]:

Lemma 33. Let $\varphi$ be a tail objective. Let $\mathcal{M}=\left(S, S_{\square}, S_{\circ}, \longrightarrow, P\right)$ be an MDP, and let $\mathcal{M}_{*}=\left(S_{*}, S_{* \square}, S_{* \bigcirc}, \longrightarrow_{*}, P_{*}\right)$ be its conditioned version. Then:

1. For all $\sigma \in \Sigma_{\mathcal{M}_{*}}$ and all $n \geq 0$ and all $s_{0}, \ldots, s_{n} \in S_{*}$ with $s_{0} \longrightarrow_{*} s_{1} \longrightarrow_{*} \cdots \longrightarrow_{*} s_{n}$ :

$$
\mathbb{P}_{\mathcal{M}_{*}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right)=\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right) \cdot \frac{\operatorname{val}_{\mathcal{M}}\left(s_{n}\right)}{\operatorname{val}_{\mathcal{M}}\left(s_{0}\right)}
$$

2. For all $s_{0} \in S_{*}$ and all $\sigma \in \Sigma_{\mathcal{M}}$ with $\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}(\varphi)=\operatorname{val}_{\mathcal{M}}\left(s_{0}\right)>0$ and all measurable $\mathcal{R} \subseteq s_{0} S^{\omega}$ we have $\mathbb{P}_{\mathcal{M}_{*}, s_{0}, \sigma}(\mathcal{R})=\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(\mathcal{R} \mid \llbracket \varphi \rrbracket^{s_{0}}\right)$.

Proof. We prove item 1 by induction on $n$. For $n=0$ it is trivial. For the step, suppose that the equality in item 1 holds for some $n$. If $s_{n} \in S_{* \bigcirc}$ then we have:

$$
\begin{aligned}
& \mathbb{P}_{\mathcal{M}_{*}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} s_{n+1} S^{\omega}\right) \\
& =\mathbb{P}_{\mathcal{M}_{*}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right) \cdot P_{*}\left(s_{n}\right)\left(s_{n+1}\right) \\
& =\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right) \cdot \frac{\operatorname{val}_{\mathcal{M}}\left(s_{n}\right)}{\operatorname{val}_{\mathcal{M}}\left(s_{0}\right)} \cdot P_{*}\left(s_{n}\right)\left(s_{n+1}\right) \\
& =\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right) \cdot \frac{\operatorname{val}_{\mathcal{M}}\left(s_{n}\right)}{\operatorname{val}_{\mathcal{M}}\left(s_{0}\right)} \cdot P\left(s_{n}\right)\left(s_{n+1}\right) \cdot \frac{\operatorname{val}_{\mathcal{M}}\left(s_{n+1}\right)}{\operatorname{val}_{\mathcal{M}}\left(s_{n}\right)} \\
& =\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} s_{n+1} S^{\omega}\right) \cdot \frac{\operatorname{val}_{\mathcal{M}}\left(s_{n+1}\right)}{\operatorname{val}_{\mathcal{M}}\left(s_{0}\right)}
\end{aligned} \text { ind. hyp. } \quad \text { def. of } P_{*}
$$

Let now $s_{n} \in S_{* \square}$. If $\sigma\left(s_{0} s_{1} \ldots s_{n}\right)\left(s_{n+1}\right)=0$ then the inductive step is trivial. Otherwise we have:

$$
\begin{aligned}
& \mathbb{P}_{\mathcal{M}_{*}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} s_{n+1} S^{\omega}\right) \\
& =\mathbb{P}_{\mathcal{M}_{*}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right) \cdot \sigma\left(s_{0} s_{1} \ldots s_{n}\right)\left(s_{n+1}\right) \\
& =\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right) \cdot \frac{\operatorname{val}_{\mathcal{M}}\left(s_{n}\right)}{\operatorname{val}_{\mathcal{M}}\left(s_{0}\right)} \cdot \sigma\left(s_{0} s_{1} \ldots s_{n}\right)\left(s_{n+1}\right) \\
& =\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right) \cdot \frac{\operatorname{val}_{\mathcal{M}}\left(s_{n+1}\right)}{\operatorname{val}_{\mathcal{M}}\left(s_{0}\right)} \cdot \sigma\left(s_{0} s_{1} \ldots s_{n}\right)\left(s_{n+1}\right) \\
& =\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} s_{n+1} S^{\omega}\right) \cdot \frac{\operatorname{val}_{\mathcal{M}}\left(s_{n+1}\right)}{\operatorname{val}_{\mathcal{M}}\left(s_{0}\right)}
\end{aligned}
$$

This completes the inductive step, and we have proved item 1.
Towards item 2, let $s_{0} \in S_{*}$ and $\sigma \in \Sigma_{\mathcal{M}}$ such that $\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}(\varphi)=\operatorname{val}_{\mathcal{M}}\left(s_{0}\right)>0$. Observe that $\sigma$ can be applied also in the MDP $\mathcal{M}_{*}$. Indeed, for any $s \in S_{* \square}$, if $t$ is a possible successor state of $s$ under $\sigma$, then $\operatorname{val}_{\mathcal{M}}(s)=\operatorname{val}_{\mathcal{M}}(t)$ by Lemma 313 and thus $t \in S_{*}$.

Let again $n \geq 0$ and $s_{0}, s_{1}, \ldots, s_{n} \in S$.

- Suppose $s_{0} s_{1} \cdots s_{n}$ is a partial run in $\mathcal{M}_{*}$ induced by $\sigma$. Then we have:

$$
\begin{array}{ll}
\mathbb{P}_{\mathcal{M}_{*}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right) \cdot \mathbb{P}_{\mathcal{M}, s_{0}, \sigma}(\varphi) & \text { item 1 } \\
=\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right) \cdot \frac{\operatorname{val}_{\mathcal{M}}\left(s_{n}\right)}{\operatorname{val}_{\mathcal{M}}\left(s_{0}\right)} \cdot \mathbb{P}_{\mathcal{M}, s_{0}, \sigma}(\varphi) & \text { assumption on } \sigma \\
=\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right) \cdot \operatorname{val}_{\mathcal{M}}\left(s_{n}\right) & \\
=\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right) \cdot \mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(\llbracket \varphi \rrbracket^{s_{0}} \mid s_{0} s_{1} \cdots s_{n} S^{\omega}\right) & \text { Lemma 31,1 } \\
=\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(\llbracket \varphi \rrbracket^{\left.s_{0} \cap s_{0} s_{1} \cdots s_{n} S^{\omega}\right)}\right. &
\end{array}
$$

- Suppose $s_{0} s_{1} \cdots s_{n}$ is not a partial run in $\mathcal{M}_{*}$ induced by $\sigma$. Hence $\mathbb{P}_{\mathcal{M}_{*}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right)=$ 0 . If $s_{0} s_{1} \cdots s_{n}$ is not a partial run in $\mathcal{M}$ induced by $\sigma$ then $\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right)=0$.

Otherwise, since $\sigma$ is optimal, there is $i \leq n$ with $\operatorname{val}_{\mathcal{M}}\left(s_{i}\right)=0$, hence $\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(\llbracket \varphi \rrbracket^{s_{0}} \cap\right.$ $\left.s_{0} s_{1} \cdots s_{n} S^{\omega}\right)$. In either case we have $\mathbb{P}_{\mathcal{M}_{*}, s_{0}, \sigma}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right) \cdot \mathbb{P}_{\mathcal{M}, s_{0}, \sigma}(\varphi)=0=$ $\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(\llbracket \varphi \rrbracket^{s_{0}} \cap s_{0} s_{1} \cdots s_{n} S^{\omega}\right)$.
In either case we have the equality $\mathbb{P}_{\mathcal{M}_{*}, s_{0}, \sigma}(\mathcal{R})=\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}\left(\mathcal{R} \mid \llbracket \varphi \rrbracket^{s_{0}}\right)$ for cylinders $\mathcal{R}=$ $s_{0} s_{1} \cdots s_{n} S^{\omega}$. Since probability measures extend uniquely from cylinders [3] , the equality holds for all measurable $\mathcal{R} \subseteq s_{0} S^{\omega}$. Thus we have shown item 2 .

The following lemma is [13, Lemma 7].

- Lemma 34. Let $\mathcal{M}=\left(S, S_{\square}, S_{\bigcirc}, \longrightarrow, P\right)$ be an $M D P$. Let $\varphi$ be an objective that is prefixindependent in $\{\mathcal{M}\}$. Suppose that for any $s \in S$ and any strategy $\sigma$ with $\mathbb{P}_{\mathcal{M}, s, \sigma}(\varphi)=1$ there exists an $M D$-strategy $\sigma^{\prime}$ with $\mathbb{P}_{\mathcal{M}, s, \sigma^{\prime}}(\varphi)=1$. Then there is an MD-strategy $\sigma^{\prime}$ such that for all $s \in S$ :

$$
\left(\exists \sigma \in \Sigma . \mathbb{P}_{\mathcal{M}, s, \sigma}(\varphi)=1\right) \quad \Longrightarrow \quad \mathbb{P}_{\mathcal{M}, s, \sigma^{\prime}}(\varphi)=1
$$

Proof. We can assume that all states are almost-surely winning, since in order to achieve an almost-sure winning objective, the player must forever remain in almost-surely winning states. So we need to define an MD-strategy $\sigma^{\prime}$ so that for all $s \in S$ we have $\mathbb{P}_{\mathcal{M}, s, \sigma^{\prime}}(\varphi)=1$.

Fix an arbitrary state $s_{1} \in S$. By assumption there is an MD-strategy $\sigma_{1}$ with $\mathbb{P}_{\mathcal{M}, s_{1}, \sigma_{1}}(\varphi)=1$. Let $U_{1} \subseteq S$ be the set of states that occur in plays that both start from $s_{1}$ and are induced by $\sigma_{1}$. We have $\mathbb{P}_{\mathcal{M}, s_{1}, \sigma_{1}}\left(\llbracket \varphi \rrbracket^{s_{1}} \cap U_{1}^{\omega}\right)=1$. In fact, for any $s \in U_{1}$ and any strategy $\sigma$ that agrees with $\sigma_{1}$ on $U_{1}$ we have $\mathbb{P}_{\mathcal{M}, s, \sigma}\left(\llbracket \varphi \rrbracket^{s} \cap U_{1}^{\omega}\right)=1$.

If $U_{1}=S$ we are done. Otherwise, consider the MDP $\mathcal{M}_{1}$ obtained from $\mathcal{M}$ by fixing $\sigma_{1}$ on $U_{1}$ (i.e., in $\mathcal{M}_{1}$ we can view the states in $U_{1}$ as random states). We argue that, in $\mathcal{M}_{1}$, for any state $s$ there is an MD-strategy $\sigma_{1}^{\prime}$ with $\mathbb{P}_{\mathcal{M}_{1}, s, \sigma_{1}^{\prime}}(\varphi)=1$. Indeed, let $s \in S$ be any state. Recall that there is an MD-strategy $\sigma$ with $\mathbb{P}_{\mathcal{M}, s, \sigma}(\varphi)=1$. Let $\sigma_{1}^{\prime}$ be the MD-strategy obtained by restricting $\sigma$ to the non- $U_{1}$ states (recall that the $U_{1}$ states are random states in $\mathcal{M}_{1}$ ). This strategy $\sigma_{1}^{\prime}$ almost surely generates a run that either satisfies $\varphi$ without ever entering $U_{1}$ or at some point enters $U_{1}$. In the latter case, $\varphi$ is satisfied almost surely: this follows from prefix-independence and the fact that $\sigma_{1}^{\prime}$ agrees with $\sigma_{1}$ on $U_{1}$. We conclude that $\mathbb{P}_{\mathcal{M}_{1}, s, \sigma_{1}^{\prime}}(\varphi)=1$.

Let $s_{2} \in S \backslash U_{1}$. We repeat the argument from above, with $s_{2}$ instead of $s_{1}$, and with $\mathcal{M}_{1}$ instead of $\mathcal{M}$. This yields an MD-strategy $\sigma_{2}$ and a set $U_{2} \ni s_{2}$ with $\mathbb{P}_{\mathcal{M}_{1}, s_{2}, \sigma_{2}}\left(\llbracket \varphi \rrbracket^{s_{2}} \cap U_{2}^{\omega}\right)=1$. In fact, for any $s \in U_{2}$ and any strategy $\sigma$ that agrees with $\sigma_{2}$ on $U_{2}$ and with $\sigma_{1}$ on $U_{1}$ we have $\mathbb{P}_{\mathcal{M}, s, \sigma}\left(\llbracket \varphi \rrbracket^{s} \cap U_{2}^{\omega}\right)=1$.

If $U_{1} \cup U_{2}=S$ we are done. Otherwise we continue in the same manner, and so forth. Since $S$ is countable, we can pick $s_{1}, s_{2}, \ldots$ to have $\bigcup_{i \geq 1} U_{i}=S$. Define an MD-strategy $\sigma^{\prime}$ such that for any $s \in S_{\square}$ we have $\sigma^{\prime}(s)=\sigma_{i}(s)$ for the smallest $i$ with $s \in U_{i}$. Thus, if $s \in U_{i}$, we have $\mathbb{P}_{\mathcal{M}, s, \sigma^{\prime}}(\varphi) \geq \mathbb{P}_{\mathcal{M}, s, \sigma^{\prime}}\left(\llbracket \varphi \rrbracket^{s} \cap U_{i}^{\omega}\right)=1$.

The following lemma is [14, Lemma 8].

- Lemma 35. Let $S$ be countable and $s \in S$. Call a set of the form $s w S^{\omega}$ for $w \in S^{*} a$ cylinder. Let $\mathbb{P}, \mathbb{P}^{\prime}$ be probability measures on $s S^{\omega}$ defined in the standard way, i.e., first on cylinders and then extended to all measurable sets $\mathcal{R} \subseteq s S^{\omega}$. Suppose there is $x \geq 0$ such that $x \cdot \mathbb{P}(\mathfrak{C}) \leq \mathbb{P}^{\prime}(\mathfrak{C})$ for all cylinders $\mathfrak{C}$. Then $x \cdot \mathbb{P}(\mathcal{R}) \leq \mathbb{P}^{\prime}(\mathcal{R})$ holds for all measurable $\mathcal{R} \subseteq s S^{\omega}$.

Proof. Let $\mathcal{C}=\left\{\mathfrak{C} \subseteq s S^{\omega} \mid \mathfrak{C}\right.$ cylinder $\}$ denote the class of cylinders. This class generates an algebra $\mathcal{C}_{*} \supseteq \mathcal{C}$, which is the closure of $\mathcal{C}$ under finite union and complement. The classes $\mathcal{C}$
and $\mathcal{C}_{*}$ generate the same $\sigma$-algebra $\sigma(\mathcal{C})$. The class $\mathcal{C}_{*}$ is the set of finite disjoint unions of cylinders [3, Section 2]. Hence $x \cdot \mathbb{P}(\mathcal{R}) \leq \mathbb{P}^{\prime}(\mathcal{R})$ for all $\mathcal{R} \in \mathcal{C}_{*}$.

Define

$$
\mathcal{Q}=\left\{\mathcal{R} \in \sigma(\mathcal{C}) \mid x \cdot \mathbb{P}(\mathcal{R}) \leq \mathbb{P}^{\prime}(\mathcal{R})\right\}
$$

We have $\mathcal{C} \subseteq \mathcal{C}_{*} \subseteq \mathcal{Q} \subseteq \sigma(\mathcal{C})$. We show that $\mathcal{Q}$ is a monotone class, i.e., if $\mathcal{R}_{1}, \mathcal{R}_{2}, \ldots \in \mathcal{Q}$, then $\mathcal{R}_{1} \subseteq \mathcal{R}_{2} \subseteq \cdots$ implies $\bigcup_{i} \mathcal{R}_{i} \in \mathcal{Q}$, and $\mathcal{R}_{1} \supseteq \mathcal{R}_{2} \supseteq \cdots$ implies $\bigcap_{i} \mathcal{R}_{i} \in \mathcal{Q}$. Suppose $\mathcal{R}_{1}, \mathcal{R}_{2}, \ldots \in \mathcal{Q}$ and $\mathcal{R}_{1} \subseteq \mathcal{R}_{2} \subseteq \cdots$. Then:

$$
\begin{aligned}
x \cdot \mathbb{P}\left(\bigcup_{i} \mathcal{R}_{i}\right) & =\sup _{i} x \cdot \mathbb{P}\left(\mathcal{R}_{i}\right) & & \text { measures are continuous from below } \\
& \leq \sup _{i} \mathbb{P}^{\prime}\left(\mathcal{R}_{i}\right) & & \text { definition of } \mathcal{Q} \\
& =\mathbb{P}^{\prime}\left(\bigcup_{i} \mathcal{R}_{i}\right) & & \text { measures are continuous from below }
\end{aligned}
$$

So $\bigcup_{i} \mathcal{R}_{i} \in \mathcal{Q}$. Using the fact that measures are continuous from above, one can similarly show that if $\mathcal{R}_{1}, \mathcal{R}_{2}, \ldots \in \mathcal{Q}$ and $\mathcal{R}_{1} \supseteq \mathcal{R}_{2} \supseteq \cdots$ then $\bigcap_{i} \mathcal{R}_{i} \in \mathcal{Q}$. Hence $\mathcal{Q}$ is a monotone class.

Now the monotone class theorem (see, e.g., [3, Theorem 3.4]) implies that $\sigma(\mathcal{C}) \subseteq \mathcal{Q}$, thus $\mathcal{Q}=\sigma(\mathcal{C})$. Hence $x \cdot \mathbb{P}(\mathcal{R}) \leq \mathbb{P}^{\prime}(\mathcal{R})$ for all $\mathcal{R} \in \sigma(\mathcal{C})$.

The following theorem is a variant of [13, Theorem 5].

- Theorem 20. Let $\varphi$ be a tail objective. Let $\mathcal{M}=\left(S, S_{\square}, S_{\bigcirc}, \longrightarrow, P\right)$ be an MDP and $\mathcal{M}_{*}=\left(S_{*}, S_{* \square}, S_{* ○}, \longrightarrow_{*}, P_{*}\right)$ its conditioned version wrt. $\varphi$. Then:

1. For all $s \in S_{*}$ there exists a strategy $\sigma$ with $\mathbb{P}_{\mathcal{M}_{*}, s, \sigma}(\varphi)=1$.
2. Suppose that for every $s \in S_{*}$ there exists an MD strategy $\sigma^{\prime \prime}$ with $\mathbb{P}_{\mathcal{M}_{*}, s, \sigma^{\prime \prime}}(\varphi)=1$. Then there is an $M D$ strategy $\sigma^{\prime}$ such that for all $s \in S$ :

$$
\left(\exists \sigma \in \Sigma \cdot \mathbb{P}_{\mathcal{M}, s, \sigma}(\varphi)=\operatorname{val}_{\mathcal{M}}(s)\right) \Longrightarrow \mathbb{P}_{\mathcal{M}, s, \sigma^{\prime}}(\varphi)=\operatorname{val}_{\mathcal{M}}(s)
$$

Proof. Towards item 1, let $s \in S_{*}$. By the definition of $S_{*}$, there is a strategy $\sigma$ with $\mathbb{P}_{\mathcal{M}, s, \sigma}(\varphi)=\operatorname{val}_{\mathcal{M}}(s)>0$. By Lemma 332, we have $\mathbb{P}_{\mathcal{M}_{*}, s, \sigma}(\varphi)=1$, as desired.

It remains to prove item 2. Suppose that for any $s \in S_{*}$ there exists an MD-strategy $\sigma^{\prime \prime}$ with $\mathbb{P}_{\mathcal{M}_{*}, s, \sigma^{\prime \prime}}(\varphi)=1$. By Lemma 34 it follows that there is an MD-strategy $\sigma^{\prime}$ with $\mathbb{P}_{\mathcal{M}_{*}, s, \sigma^{\prime}}(\varphi)=1$ for all $s \in S_{*}$. We show that this strategy $\sigma^{\prime}$ satisfies the property claimed in the statement of the theorem.

To this end, let $n \geq 0$ and $s_{0}, s_{1}, \ldots, s_{n} \in S$. If $s_{0} s_{1} \cdots s_{n}$ is a partial run in $\mathcal{M}_{*}$ then, by Lemma 33.1,

$$
\mathbb{P}_{\mathcal{M}_{*}, s_{0}, \sigma^{\prime}}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right)=\mathbb{P}_{\mathcal{M}, s_{0}, \sigma^{\prime}}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right) \cdot \frac{\operatorname{val}_{\mathcal{M}}\left(s_{n}\right)}{\operatorname{val}_{\mathcal{M}}\left(s_{0}\right)}
$$

and thus, as $\operatorname{val}_{\mathcal{M}}\left(s_{n}\right) \leq 1$,

$$
\operatorname{val}_{\mathcal{M}}\left(s_{0}\right) \cdot \mathbb{P}_{\mathcal{M}_{*}, s_{0}, \sigma^{\prime}}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right) \leq \mathbb{P}_{\mathcal{M}, s_{0}, \sigma^{\prime}}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right)
$$

If $s_{0} s_{1} \cdots s_{n}$ is not a partial run in $\mathcal{M}_{*}$ then $\mathbb{P}_{\mathcal{M}_{*}, s_{0}, \sigma^{\prime}}\left(s_{0} s_{1} \cdots s_{n} S^{\omega}\right)=0$ and the previous inequality holds as well. Therefore, by Lemma 35 we get for all measurable sets $\mathcal{R} \subseteq s_{0} S^{\omega}$ :

$$
\operatorname{val}_{\mathcal{M}}\left(s_{0}\right) \cdot \mathbb{P}_{\mathcal{M}_{*}, s_{0}, \sigma^{\prime}}(\mathcal{R}) \leq \mathbb{P}_{\mathcal{M}, s_{0}, \sigma^{\prime}}(\mathcal{R})
$$

In particular, since $\mathbb{P}_{\mathcal{M}_{*}, s_{0}, \sigma^{\prime}}(\varphi)=1$, we obtain $\operatorname{val}_{\mathcal{M}}\left(s_{0}\right) \leq \mathbb{P}_{\mathcal{M}, s_{0}, \sigma^{\prime}}(\varphi)$. The converse inequality $\mathbb{P}_{\mathcal{M}, s_{0}, \sigma^{\prime}}(\varphi) \leq \operatorname{val}_{\mathcal{M}}\left(s_{0}\right)$ holds by the definition of $\operatorname{val}_{\mathcal{M}}\left(s_{0}\right)$, hence we conclude $\mathbb{P}_{\mathcal{M}, s_{0}, \sigma^{\prime}}(\varphi)=\operatorname{val}_{\mathcal{M}}\left(s_{0}\right)$.

## D Missing proofs in Section 3

We first recall our results [12] on the strategy complexity of Büchi objectives:

- Theorem 12 (Theorem 5 in [12]). For every acyclic countable MDP M, a Büchi objective $\varphi$, finite set $I$ of initial states and $\varepsilon>0$, there exists a deterministic 1-bit strategy that is $\varepsilon$ optimal from every $s \in I$.

We will prove that
$\triangleright$ Claim 13. In MDP $\mathcal{L}_{e-2}$, there is an MD strategy $\tau_{e}$, that is $(\alpha-\beta)$-optimal for $\theta_{e}$ from $R_{e}$.

Proof. Consider the original MDP $\mathcal{M}$. Given a set $B \subseteq L$ in $\mathcal{L}$, we use $\operatorname{project}(B) \stackrel{\text { def }}{=}\{s \mid$ $(s, b) \in B, b \in\{0,1\}\}$ to project the set into $\mathcal{M}$.

We first slightly modify $\mathcal{M}$ to obtain $\mathcal{M}^{\prime}$. The modification guarantees that, for all states $s$ and runs $\rho$ of $\mathcal{M}^{\prime}$,

$$
s \rho \in \llbracket \mathrm{GF}[S]^{\text {Col=e }} \rrbracket \quad \text { if and only if } \quad s \rho \in \llbracket \mathrm{GF}[S]^{\text {Col=e }} \wedge \mathrm{G}[S]^{\text {Col } \leq e} \wedge \mathrm{G} \neg \operatorname{project}(\mathrm{Fix} e) \rrbracket .
$$

We redirect all out-going transitions of states $s^{\prime} \in \operatorname{project}\left(\right.$ Fix $\left._{e}\right)$ or $s^{\prime}$ with $\operatorname{Col}\left(s^{\prime}\right)>e$ to an infinite chain $q_{0} q_{1} q_{2} \cdots$ of controlled states where $\operatorname{Col}\left(q_{i}\right)=1$ and $q_{i} \longrightarrow q_{i+1}$. We also update the color of all states $s$ with $\operatorname{Col}(s)<e$ to 1 .

The objective GF[S] ${ }^{C o l=e}$ is a Büchi Objective in $\mathcal{M}^{\prime}$. By Theorem 12, given the finite set project $\left(R_{e}\right)$ of initial states, there exists a deterministic 1-bit strategy $\sigma$ in $\mathcal{M}^{\prime}$ that is $(\alpha-\beta)$-optimal w.r.t. $\mathrm{GF}[S]^{\text {Col }=e}$ for every state $s \in \operatorname{project}\left(R_{e}\right)$ (with the memory bit initially set to 0 ).

Since the fixed choices in $\mathcal{L}_{e-2}$ are only in the Fix ${ }_{e-2}$-region, strategy $\sigma$ can be translated in a natural way to a deterministic memoryless strategy $\sigma^{\prime}$ in $\mathcal{L}_{e-2}$ : For a state $s \in S_{\square}$ and $b \in\{0,1\}$, if $\sigma$ chooses the successor state $s^{\prime}$, by taking a transition $t=\left(s, s^{\prime}\right)$, and updates the bit to $b^{\prime}$, we define $\sigma^{\prime}((s, b)) \stackrel{\text { def }}{=}(t, b)$ and $\sigma^{\prime}((t, b))=\left(s^{\prime}, b^{\prime}\right)$. For a random state $s \in S_{\bigcirc}$ and $b \in\{0,1\}$, if the strategy $\sigma$ updates the memory bit to $b^{\prime}$ in case the random successor resolves to $s^{\prime}$, by taking a transition $t=\left(s, s^{\prime}\right)$, we define $\sigma^{\prime}((t, b)) \stackrel{\text { def }}{=}\left(s^{\prime}, b^{\prime}\right)$. Recall that the bit is initially set to 0 in $\sigma$. Consequently, the strategy $\sigma^{\prime}$ is $(\alpha-\beta)$-optimal for $\theta_{e}$ from every state $\ell \in R_{e}$ in the layered MDP $\mathcal{L}_{e-2}$.

We next prove the main technical claim in Section 3
$\triangleright$ Claim 14. $\quad \mathbb{P}_{\mathcal{L}_{e}, \ell_{0}, \pi}\left(\psi_{e}\right) \geq \operatorname{val}_{\mathcal{L}_{e-2}, \psi_{e-2}}\left(\ell_{0}\right)-2 \gamma$.
Proof. Recall the definition of $\pi$ : it starts by following $\sigma_{\ell_{0}}$. If it ever enters $C l\left(f i x_{e}\right)$ then we ensure that it enters $f x_{e}$ as well (in at most one more step). Then $\pi$ continues by playing as $\tau_{e}$ does forever.

Below we argue that if $\pi$ ever enters $C l\left(f x_{e}\right)$ then it is in fact possible to choose the layer in such a way that $\pi$ enters $f i x_{e}$ instead. Assume $\pi$ enters $C l\left(f i x_{e}\right)$ at $q$ after taking a transition from $p$ to $q$. Let $\bar{q} \in f i x_{e}$ be the sibling of $q$. By construction,

1. either $p \in \longrightarrow_{1} \times\{0,1\}$ is controlled: the controller switches the layer in $p$, by choosing $\bar{q}$ rather than $q$ and enters $f i x_{e}$;
2. or $q \in \longrightarrow_{1} \times\{0,1\}$ is controlled. By definition (6), the MD strategy $\tau_{e}$ attains a high value from state $\bar{q}$ for $\theta_{e}$. Hence, $\tau_{e}(\bar{q}) \in f i x_{e}$. Hence, the controller can switch the layer in $q$ by playing $\tau_{e}(\bar{q})$ and enters $f i x_{e}$.

For all $e^{\prime} \in\left\{2,4, \cdots, e_{\max }\right\}$ define

$$
\chi_{e^{\prime}} \stackrel{\text { def }}{=} \mathrm{F} \varphi_{e^{\prime}} \wedge \mathrm{G} \neg \mathrm{Fix}_{e-2} \quad \tilde{\chi}_{e^{\prime}} \stackrel{\text { def }}{=} \mathrm{F} \varphi_{e^{\prime}} \wedge \mathrm{G} \neg \mathrm{Fix}_{e}
$$

We define

$$
\begin{gather*}
\psi \stackrel{\text { def }}{=} \bigvee_{e^{\prime}<e} \mathrm{~F} \operatorname{core}_{e^{\prime}} \vee \bigvee_{e^{\prime}>e} \chi_{e^{\prime}} \\
\psi^{\prime} \stackrel{\text { def }}{=} \bigvee_{e^{\prime}<e} \mathrm{~F} \operatorname{core}_{e^{\prime}} \vee \bigvee_{e^{\prime}>e} \tilde{\chi}_{e^{\prime}} \tag{18}
\end{gather*}
$$

By definition of $\psi_{e-2}$ and $\psi_{e}$, see definition (4), we have $\psi_{e-2}=\psi \vee \chi_{e}$ and $\psi_{e}=\psi^{\prime} \vee \mathrm{F}$ core $e_{e}$. For brevity, further define $\rho \stackrel{\text { def }}{=} \mathrm{F} C l\left(f i x_{e}\right)$. Observe that $\llbracket \psi \wedge \neg \rho \rrbracket \subseteq \llbracket \psi^{\prime} \rrbracket$.

We first have that

$$
\begin{align*}
& \mathbb{P}_{\mathcal{L}_{e}, \ell_{0}, \pi}\left(\text { Fcore }_{e}\right) \\
& \geq \mathbb{P}_{\mathcal{L}_{e}, \ell_{0}, \pi}\left(\text { Fcore }_{e} \wedge \rho\right) \\
& \geq \mathbb{P}_{\mathcal{L}_{e}, \ell_{0}, \pi}\left(\neg C l\left(f i x_{e}\right) \text { until }\left(C l\left(f i x_{e}\right) \wedge \mathrm{F} \varphi_{e} \wedge \mathrm{Fcore}_{e} \wedge \mathrm{G} \neg \mathrm{Fix}_{e-2}\right)\right) \\
& =\sum_{\ell \in C l\left(f x_{e}\right)} \mathbb{P}_{\mathcal{L}_{e}, \ell_{0}, \pi}\left(\neg C l\left(f i x_{e}\right) \text { until } \ell\right) \cdot \mathbb{P}_{\mathcal{L}_{e}, \ell, \tau_{e}}\left(\theta_{e} \wedge \text { Fcore }_{e}\right) \\
& =\sum_{\ell \in C l\left(f x_{e}\right)} \mathbb{P}_{\mathcal{L}_{e}, \ell_{0}, \pi}\left(\neg C l\left(f i x_{e}\right) \text { until } \ell\right) \cdot \mathbb{P}_{\mathcal{L}_{e}, \ell, \tau_{e}}\left(\theta_{e}\right) \quad \text { by Lemma } 162 \\
& \geq \sum_{\ell \in C l\left(f i x_{e}\right)} \mathbb{P}_{\mathcal{L}_{e}, \ell_{0}, \pi}\left(\neg C l\left(f i x_{e}\right) \text { until } \ell\right) \cdot \beta \\
& =\mathbb{P}_{\mathcal{L}_{e}, \ell_{0}, \pi}(\rho) \cdot \beta \\
& =\mathbb{P}_{\mathcal{L}_{e-2}, \ell_{0}, \sigma_{\ell_{0}}}(\rho) \cdot \beta \\
& =\mathbb{P}_{\mathcal{L}_{e-2}, \ell_{0}, \sigma_{\ell_{0}}}(\rho) \cdot(1-\gamma) \\
& \geq \mathbb{P}_{\mathcal{L}_{e-2}, \ell_{0}, \sigma_{\ell_{0}}}(\rho)-\gamma \tag{19}
\end{align*}
$$

We use the law of total probability:

$$
\begin{equation*}
\mathbb{P}_{\mathcal{L}_{e-2}, \ell_{0}, \sigma_{\ell_{0}}}\left(\psi_{e-2}\right)=\mathbb{P}_{\mathcal{L}_{e-2}, \ell_{0}, \sigma_{\ell_{0}}}(\psi \wedge \neg \rho)+\mathbb{P}_{\mathcal{L}_{e-2}, \ell_{0}, \sigma_{\ell_{0}}}\left(\chi_{e}\right)+\mathbb{P}_{\mathcal{L}_{e-2}, \ell_{0}, \sigma_{\ell_{0}}}(\psi \wedge \rho) \tag{20}
\end{equation*}
$$

In one hand, since $\mathcal{L}_{e}$ and $\mathcal{L}_{e-2}$ only differ in the fix $x_{e}$-region, and since $\pi$ plays as $\sigma_{\ell_{0}}$ on all runs contained in $\neg \rho$ :

$$
\mathbb{P}_{\mathcal{L}_{e-2}, \ell_{0}, \sigma_{\ell_{0}}}(\psi \wedge \neg \rho)=\mathbb{P}_{\mathcal{L}_{e}, \ell_{0}, \pi}(\psi \wedge \neg \rho) \leq \mathbb{P}_{\mathcal{L}_{e}, \ell_{0}, \pi}\left(\psi^{\prime}\right)
$$

In the other hand, by Equation (9):

$$
\mathbb{P}_{\mathcal{L}_{e-2}, \ell_{0}, \sigma_{\ell_{0}}}\left(\chi_{e}\right) \leq \mathbb{P}_{\mathcal{L}_{e-2}, \ell_{0}, \sigma_{\ell_{0}}}\left(\chi_{e} \wedge \rho\right)+\frac{\gamma}{2}
$$

Applying the above to Equation 20 yields:

$$
\begin{array}{rlr}
\mathbb{P}_{\mathcal{L}_{e-2}, \ell_{0}, \sigma_{\ell_{0}}}\left(\psi_{e-2}\right) & \\
& \leq \mathbb{P}_{\mathcal{L}_{e}, \ell_{0}, \pi}\left(\psi^{\prime}\right)+\mathbb{P}_{\mathcal{L}_{e-2}, \ell_{0}, \sigma_{\ell}}\left(\chi_{e} \wedge \rho\right)+\mathbb{P}_{\mathcal{L}_{e-2}, \ell_{0}, \sigma_{0}}(\psi \wedge \rho)+\frac{\gamma}{2} & \\
& =\mathbb{P}_{\mathcal{L}_{e}, \ell_{0}, \pi}\left(\psi^{\prime}\right)+\mathbb{P}_{\mathcal{L}_{e-2}, \ell_{0}, \sigma_{\ell}}\left(\left(\chi_{e} \vee \psi\right) \wedge \rho\right)+\frac{\gamma}{2} & \text { since } \chi_{e} \text { and } \psi \text { are disjoint } \\
& \leq \mathbb{P}_{\mathcal{L}_{e}, \ell_{0}, \pi}\left(\psi^{\prime}\right)+\mathbb{P}_{\mathcal{L}_{e-2}, \ell_{0}, \sigma_{\ell_{0}}}(\rho)+\frac{\gamma}{2} & \\
& \leq \mathbb{P}_{\mathcal{L}_{e}, \ell_{0}, \pi}\left(\psi^{\prime}\right)+\mathbb{P}_{\mathcal{L}_{e-2}, \ell_{0}, \pi}\left(\mathrm{Fcore}_{e}\right)+\gamma+\frac{\gamma}{2} & \text { by Equation (19) } \\
& =\mathbb{P}_{\mathcal{L}_{e}, \ell_{0}, \pi}\left(\psi_{e}\right)+\frac{3 \gamma}{2} &
\end{array}
$$

To conclude the proof we recall that $\sigma_{\ell_{0}}$ is $\frac{\gamma}{2}$-optimal w.r.t $\psi_{e-2}$.
$\triangleright$ Claim 17. The MD strategy $\hat{\sigma}$ is $\varepsilon$-optimal for parity objective $\varphi$, from every state $\ell_{0} \in L_{0}$. Proof. For the MD strategy $\hat{\sigma}$, by the law of total probability, we have

$$
\mathbb{P}_{\mathcal{L}, \ell_{0}, \hat{\sigma}}(\varphi) \geq \sum_{e \in \operatorname{even}(\mathcal{C})} \mathbb{P}_{\mathcal{L}, \ell_{0}, \hat{\sigma}}\left(\mathrm{~F} \varphi_{e} \wedge \mathrm{~F} \text { core }_{e}\right)
$$

Let $\mathcal{C}^{\prime}$ be the set of even colors $e$ where $\mathbb{P}_{\mathcal{L}, \ell_{0}, \hat{\sigma}}\left(\mathrm{~F}_{\text {core }}^{e}\right)>0$. Then:

$$
\left.\left.\begin{array}{ll}
=\sum_{e \in \mathcal{C}^{\prime}} \mathbb{P}_{\mathcal{L}, \ell_{0}, \hat{\sigma}}\left(\mathrm{~F} \varphi_{e} \mid \mathrm{F} \operatorname{core}_{e}\right) \cdot \mathbb{P}_{\mathcal{L}, \ell_{0}, \hat{\sigma}}\left(\mathrm{~F} \text { core }_{e}\right) \\
\geq \sum_{e \in \mathcal{C}^{\prime}} \mathbb{P}_{\mathcal{L}, \ell_{0}, \hat{\sigma}}\left(\mathrm{Gfix} \mid \mathrm{F} \operatorname{core}_{e}\right) \cdot \mathbb{P}_{\mathcal{L}, \ell_{0}, \hat{\sigma}}\left(\mathrm{~F}_{\mathrm{core}}^{e}\right.
\end{array}\right) \quad \text { by Equation } 12\right\}
$$

since $\tau_{\text {reach }}$ is $\gamma$-optimal and by Equation 10,

$$
\begin{aligned}
& \geq(1-\gamma) \cdot\left(\operatorname{val}_{\mathcal{L}, \varphi}\left(\ell_{0}\right)-e_{\max } \gamma-\gamma\right) \\
& \geq \operatorname{val}_{\mathcal{L}, \varphi}\left(\ell_{0}\right)-\left(e_{\max }+2\right) \gamma
\end{aligned}
$$

Recall that $\varepsilon=\left(e_{\max }+2\right) \gamma$. Thus we have shown that the MD strategy $\hat{\sigma}$ is $\varepsilon$-optimal w.r.t. $\varphi$ from every state $\ell_{0} \in L_{0}$.

## E Missing proofs in Section 4

- Definition 36 (Bubbles). Let $\mathcal{M}$ be an MDP with states $S, R \subseteq S, l \in \mathbb{N}$. The $l$-bubble around $R$ is the set

$$
\operatorname{bubble}(\mathcal{M}, R, l) \stackrel{\text { def }}{=}\left\{s \mid \exists s_{0} \in R . \exists \tau \cdot \mathbb{P}_{\mathcal{M}, s_{0}, \tau}\left(\mathrm{~F}^{\leq l} s\right)>0\right\}
$$

of states that can be reached from $R$ in at most $l$ steps. Any bubble around a closed set $R \subseteq L$ is closed.

Recall that for an MD strategy $\tau$, we write $\mathcal{M}[\tau, R]$ for the MDP obtained from $\mathcal{M}$ by fixing the strategy $\tau$ for all states in $R$. We will simply write $\mathcal{M}[\tau]$ for $\mathcal{M}[\tau, S]$, where $\tau$ is fixed everywhere, and $\mathcal{M}[\tau, R, l] \stackrel{\text { def }}{=} \mathcal{M}[\tau$, bubble $(\mathcal{M}, R, l)]$ that fixes $\tau$ in the $l$-bubble around $R$.

- Lemma 18. Let $\mathcal{L}(\mathcal{M})$ be the layered MDP obtained from an acyclic and finitely branching MDP $\mathcal{M}$ and a coloring Col such that all states are almost surely winning for $\varphi=$ $\operatorname{Parity}(C o l)$ (i.e., every state s has a strategy $\sigma_{s}$ such that $\mathbb{P}_{\mathcal{L}(\mathcal{M}), s, \sigma_{s}}(\varphi)=1$ ).

For every initial state $s_{0}$ there exists an MD strategy $\sigma$ that almost surely wins, i.e., $\mathbb{P}_{\mathcal{L}(\mathcal{M}), s_{0}, \sigma}(\varphi)=1$.
Proof. Directly from Lemma 37 (let $L_{0} \stackrel{\text { def }}{=} C l\left(\left\{s_{0}\right\}\right)$ ).

- Lemma 37. Let $\mathcal{L}(\mathcal{M})$ be the layered MDP obtained from an acyclic and finitely branching MDP $\mathcal{M}$ and a coloring Col such that all states are almost surely winning for $\varphi=$ $\operatorname{Parity}(C o l)$ (i.e., every state $s$ has a strategy $\sigma_{s}$ such that $\mathbb{P}_{\mathcal{L}(\mathcal{M}), s, \sigma_{s}}(\varphi)=1$ ).

For every finite closed set $L_{0}$ of initial states there exists an $M D$ strategy $\hat{\sigma}$ that almost surely wins from every state $s_{0} \in L_{0}$. That is, $\forall s_{0} \in L_{0} . \mathbb{P}_{\mathcal{L}(\mathcal{M}), s_{0}, \hat{\sigma}}(\varphi)=1$.
Proof. We iteratively produce an infinite sequence $\mathcal{L}_{0}, \mathcal{L}_{1}, \mathcal{L}_{2}, \ldots$ of layered MDPs. They have the same structure as $\mathcal{L}(\mathcal{M})$, but in each step from $\mathcal{L}_{i}$ to $\mathcal{L}_{i+1}$ the choices in some subset of states (reachable from $L_{0}$ ) are fixed. In the limit all choices from all controlled states reachable from $L_{0}$ are fixed. Hence this prescribes an MD strategy $\hat{\sigma}$ from $L_{0}$ in $\mathcal{L}(\mathcal{M})$. It is not sufficient that these fixings of MD strategies in subspaces are compatible with some strategy almost sure winning for $\varphi$, since progress (e.g., towards visiting a particular color) might only be made outside of the fixed subspace, and thus be delayed forever. Instead we prove the stronger property that $\hat{\sigma}$ ensures $\varphi$ with some probability $p_{i}\left(s_{0}\right)$ from $s_{0} \in L_{0}$ already in the fixed subspace of $\mathcal{L}_{i}$ alone, and that $\lim _{i \rightarrow \infty} p_{i}\left(s_{0}\right)=1$. This then implies that $\hat{\sigma}$ is almost surely winning for $\varphi$ in $\mathcal{L}(\mathcal{M})$.

## The sea urchin construction.

Its name comes from the shape of the subspace where strategies are fixed: a finite body $H_{i}$ out of which come finitely many spikes ( $\mathrm{BETA}_{i}$, where each spike is infinite). As the body grows, more spikes are added. Eventually the sea urchin covers the entire space; see Figure 4.

The construction uses some global thresholds $1>\alpha>\beta>\gamma>0$, to be determined later. Moreover, in each step from $\mathcal{L}_{i}$ to $\mathcal{L}_{i+1}$ we will define the following notions.

- Small error thresholds $\varepsilon_{j}>0$ for $\left.j \in\{0,1,2,3\}\right)$.
- Thresholds $l_{i}, k_{i} \in \mathbb{N}$ of a number of steps from $L_{0}$.
- Finite closed subsets of states $H_{i}$ where $H_{0} \stackrel{\text { def }}{=} \emptyset$ and $H_{i} \stackrel{\text { def }}{=} \operatorname{bubble}\left(\mathcal{L}, L_{0}, k_{i}\right)$ for $i>0$. ( $H_{i}$ is finite, because $\mathcal{L}$ is finitely branching.)
- Finite subsets $L_{i} \subseteq L$ as starting sets for certain modified objectives $\varphi_{i}$ (see below).
- MD strategies $\tau_{i}$ (for $i>0$ ) and subsets of states $\mathrm{ALPHA}_{i} \subseteq \mathrm{BETA}_{i} \subseteq \mathrm{GAMMA}_{i} \subseteq L$, where $\mathrm{ALPHA}_{i}$ (resp. $\mathrm{BETA}_{i}$, GAMMA $_{i}$ ) are the sets of states from which $\tau_{i}$ attains $\geq \alpha$ (resp. $\geq \beta, \geq \gamma$ ) for objective $\varphi_{i}$ (see below) in $\mathcal{L}_{i-1}\left(\right.$ and $\left.\mathcal{L}_{i}\right)$. Let ALPHA $_{0}=$ BETA $_{0}=$ GAMMA $_{0} \stackrel{\text { def }}{=} \emptyset$, ALPHA $_{i} \stackrel{\text { def }}{=} \operatorname{Safe}_{\mathcal{L}_{i-1}, \tau_{i}, \varphi_{i}}(\alpha)$ (and similar for BETA $_{i}$, GAMMA $_{i}$ ).
We write ALPHA $_{\leq i} \stackrel{\text { def }}{=} \bigcup_{j \leq i}$ ALPHA $_{j}$ and similar for BETA $_{\leq i}$, GAMMA $_{\leq i}$.
- Let $\mathrm{FIX}_{i} \stackrel{\text { def }}{=} \mathrm{BETA}_{\leq i} \cup H_{i}$. This is the subspace where choices are fixed in rounds up-to $i$.
- Modified objectives $\varphi_{i}$ with $\varphi_{0} \stackrel{\text { def }}{=} \varphi$ and $\varphi_{i+1} \stackrel{\text { def }}{=} \varphi \wedge \mathrm{G}\left(L \backslash C l\left(\mathrm{FIX}_{i}\right)\right)$. For $i>0$ the $\varphi_{i}$ are not strictly tail objectives, but they still enjoy the same properties as tail objectives wrt. the Levy zero-one law; cf. Remark 11
- In $\mathcal{L}_{i}$ the choices inside $\mathrm{FIX}_{i}$ are already fixed. Inside $\mathrm{BETA}_{i}$ the strategy $\tau_{i}$ is fixed, and inside $H_{i} \backslash \mathrm{BETA}_{\leq i}$ the choices are fixed according to another MD strategy $\rho_{i}$.
- It follows from the properties above that we have the invariant

$$
\begin{equation*}
\mathrm{GAMMA}_{i+1} \cap C l\left(\mathrm{FIX}_{i}\right)=\emptyset \tag{21}
\end{equation*}
$$

In particular, the sets $\mathrm{BETA}_{j}$ are disjoint for different $j$. However, for $j^{\prime}>j$, it is possible that GAMMA $j_{j}$ overlaps with BETA $_{j^{\prime}}$ (and GAMMA $j^{\prime}$ ).

## Base case.

We start with the $\operatorname{MDP} \mathcal{L}_{0} \stackrel{\text { def }}{=} \mathcal{L} \stackrel{\text { def }}{=} \mathcal{L}(\mathcal{M})$. By assumption, in $\mathcal{L}_{0}$ all states are almost surely winning for $\varphi_{0} \stackrel{\text { def }}{=} \varphi$ (the unrestricted parity objective). The invariant (21) is trivially satisfied for $i=0$, since $C l\left(\mathrm{FIX}_{0}\right)=\emptyset$.

## Step.

Now we define the step from $\mathcal{L}_{i}$ to $\mathcal{L}_{i+1}$ for $i \geq 0$. We assume that for all $j \leq i$ the MD strategies $\tau_{j}$ and the sets $\mathrm{ALPHA}_{j} \subseteq \mathrm{BETA}_{j} \subseteq \mathrm{GAMMA}_{j}$ and $H_{j}$ are already defined. Moreover, $\tau_{j}$ is fixed inside $\mathrm{BETA}_{j}$, and in $\mathcal{L}_{j}$ the strategy $\tau_{j}$ attains at least $\beta$ for objective $\varphi_{j}$ from each state $s \in \mathrm{BETA}_{j}$. Moreover, some other MD strategy is fixed in $H_{j} \backslash \mathrm{BETA}_{\leq j}$. (All this trivially holds for the base case $i=0$. For $i>0$ our construction will ensure these properties.)

We now consider $\mathcal{L}_{i}$. By $s_{0}$ we denote initial states in $L_{0}$. (General states are denoted by s.) We show that in $\mathcal{L}_{i}$, all initial states $s_{0} \in L_{0}$ are still almost surely winning for $\varphi$, as witnessed by a resetting strategy $\sigma$ defined below (where $\sigma$ is generally not MD , except inside the subspace $\mathrm{FIX}_{i}$ ). First we need a basic property of $\mathrm{ALPHA}_{j}, \mathrm{BETA}_{j}$.
$\triangleright$ Claim 38. Let $0 \leq j \leq i$ and $\sigma$ be an arbitrary strategy in $\mathcal{L}_{i}$. If $s \in$ ALPHA $_{j}$ then $\mathbb{P}_{\mathcal{L}_{i}, s, \sigma}\left(\mathrm{GBETA}_{j}\right) \geq \frac{\alpha-\beta}{1-\beta}$.

Proof. By Lemma 15, since $\sigma$ behaves just like $\tau_{j}$ in the relevant subspaces already fixed to $\tau_{j}$ in $\mathcal{L}_{i}$.

Recall that for every state $s \in L$ there exists an almost surely winning strategy $\sigma(s)$ for $\varphi$ in $\mathcal{L}$. The resetting strategy $\sigma$ in $\mathcal{L}_{i}$ starts in $L_{0}$ and behaves as specified in the three different modes $\mathrm{m}_{1}, \mathrm{~m}_{2}, \mathrm{~m}_{3}$ as follows. For all $j \leq i$ :

1. In $H_{j}$ it plays as prescribed by the fixing there, (starting in memory mode $\mathrm{m}_{1}$ ).
2. Whenever $\sigma$ enters a set $C l\left(\mathrm{BETA}_{j}\right) \backslash H_{i}$ then it switches to mode $\mathrm{m}_{2}$ and chooses the layer in such a way that it enters even $\mathrm{BETA}_{j}$ (in at most one more step) and continues playing $\tau_{j}$, as required by the fixing inside $\mathrm{BETA}_{j} . \square^{1}$ Inside $\mathrm{BETA}_{j}$, it plays $\tau_{j}$ that is fixed in $\mathrm{BETA}_{j}$. It continues to play $\tau_{j}$ even in $\mathrm{GAMMA}_{j} \backslash\left(\mathrm{BETA}_{\leq i} \cup H_{i}\right)$.
3. While playing in mode $\mathrm{m}_{2}$ (or $\mathrm{m}_{1}$ ), upon reaching an unfixed state $s$ outside of GAMMA ${ }_{j}$ (and outside of $H_{i}$ ), it goes to mode $\mathrm{m}_{3}$ and resets to an almost surely winning strategy $\sigma(s)$ for $\varphi$ in $\mathcal{L}$. It keeps playing $\sigma(s)$ until (and if) it reaches the fixed part $\mathrm{BETA}_{\leq i} \cup H_{i}$, whereupon it continues as before with mode $\mathrm{m}_{2}$.
[^0]We will see that, not only is the resetting strategy $\sigma$ almost surely winning for $\varphi$, but every time it re-enters $\mathrm{BETA}_{\leq i}$ it has a lower-bounded chance of eventually staying in $\mathrm{ALPHA}_{\leq i}$ forever.

We now classify the runs induced by the resetting strategy $\sigma$ (from some initial state $s_{0} \in L_{0}$ ) according to how often which modes $\mathrm{m}_{1}, \mathrm{~m}_{2}, \mathrm{~m}_{3}$ are used.

First we note that, since $\mathcal{L}_{i}$ is acyclic, under any strategy (and in particular $\sigma$ ), any run can visit any finite set (in particular $H_{i}$ ) only finitely often and therefore has an infinite suffix that is always outside $H_{i}$. Thus $\sigma$ is eventually always not in mode $\mathrm{m}_{1}$.

By our invariant 21, playing in $\mathrm{BETA}_{i}$ and $\mathrm{GAMMA}_{i}$ is not restricted by our previous fixings in $\mathrm{BETA}_{\leq i-1} \cup H_{i-1}$. Thus, when playing from $s \in \mathrm{BETA}_{i}$ in mode $\mathrm{m}_{2}$, we keep playing $\tau_{i}$ even in GAMMA ${ }_{i}$. Analogously to Claim 38, the chance of staying in the set $\mathrm{GAMMA}_{i}$ can be lower bounded.

$$
\begin{equation*}
\forall s \in \operatorname{BETA}_{i} \mathbb{P}_{\mathcal{L}_{i}, s, \sigma\left[m_{2}\right]}\left(\mathrm{G} \mathrm{GAMMA}_{i}\right) \geq \frac{\beta-\gamma}{1-\gamma}>0 \tag{22}
\end{equation*}
$$

This again follows from Lemma 15 observing that $\sigma$ behaves just like $\tau_{i}$ even inside GAMMA ${ }_{i}$ while staying in mode $\mathrm{m}_{2}$.

When playing from $s \in \mathrm{BETA}_{j}$ for some $j<i$, a similar property holds. If a run visits some state $s^{\prime} \in C l\left(\mathrm{BETA}_{j^{\prime}}\right)$, for some $j^{\prime}>j$, then we can assume that we have even $s^{\prime} \in \mathrm{BETA}_{j^{\prime}}$ by our assumption on $\sigma$ above, because outside of the fixed region the layer can be chosen freely. Then the strategy switches from $\tau_{j}$ to $\tau_{j^{\prime}}$ from $s^{\prime} \in \mathrm{BETA}_{j^{\prime}}$. Otherwise we keep playing $\tau_{j}$ while in $\mathrm{GAMMA}_{j}$, i.e., by Lemma 15 we get, for every $s \in \mathrm{BETA}_{j}$, that

$$
\begin{equation*}
\mathbb{P}_{\mathcal{L}_{i}, s, \sigma\left[\mathrm{~m}_{2}\right]}\left(\mathrm{G} \mathrm{GAMMA}_{j} \vee \mathrm{~F} \mathrm{BETA}>_{j}\right) \geq \frac{\beta-\gamma}{1-\gamma}>0 \tag{23}
\end{equation*}
$$

From $\sqrt[22]{ }$ and 23 , we obtain that the set of runs that infinitely often switch from mode $\mathrm{m}_{2}$ to $m_{3}$ are a null-set. Moreover, as shown above, every run has an infinite suffix where the mode is not $m_{1}$. It follows that, except for a null-set, all runs either have an infinite suffix in mode $\mathrm{m}_{2}$ or an infinite suffix in mode $\mathrm{m}_{3}$. Let $\mathcal{R}_{2}$ and $\mathcal{R}_{3}$ denote these subsets of runs, respectively. I.e., we have $\forall s_{0} \in L_{0}$

$$
\begin{equation*}
\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\mathcal{R}_{2} \cup \mathcal{R}_{3}\right)=1 \tag{24}
\end{equation*}
$$

In mode $\mathrm{m}_{3}$ the resetting strategy $\sigma$ plays an almost surely winning strategy for $\varphi$ outside of $C l\left(\mathrm{FIX}_{i}\right)$ that is not impeded by the fixings in $\mathcal{L}_{i}$, and $\varphi$ is a tail objective. Thus, for all $s_{0} \in L_{0}$,

$$
\begin{equation*}
\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\mathcal{R}_{3}\right)=\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\mathcal{R}_{3} \wedge \varphi \wedge \mathrm{FG}\left(L \backslash C l\left(\mathrm{FIX}_{i}\right)\right)\right) \tag{25}
\end{equation*}
$$

From the property that $\varphi$ is tail and the definition of $\varphi_{i+1}$ as $\varphi \wedge \mathrm{G}\left(L \backslash C l\left(\mathrm{FIX}_{i}\right)\right)$ we obtain that, for all $s_{0} \in L_{0}$,

$$
\begin{equation*}
\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\mathcal{R}_{3}\right)=\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\mathcal{R}_{3} \wedge \mathrm{~F} \varphi_{i+1}\right) \tag{26}
\end{equation*}
$$

In mode $\mathrm{m}_{2}$ the resetting strategy $\sigma$ plays some MD strategy $\tau_{j}$ in GAMMA ${ }_{j}$ (for some $j \leq i)$. Thus, for all $s_{0} \in L_{0}$,

$$
\begin{equation*}
\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\mathcal{R}_{2}\right)=\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\mathcal{R}_{2} \wedge \mathrm{FGGAMMA}_{\leq i}\right) \tag{27}
\end{equation*}
$$

Since in mode $\mathrm{m}_{2}$ the resetting strategy $\sigma$ plays some MD strategy $\tau_{j}$ with attainment $\geq \gamma$ (resp. $\geq \alpha$ ) in GAMMA (resp. ALPHA ${ }_{j}$ ), we can apply Levy's zero-one law (Corollary 30) and obtain even

$$
\begin{equation*}
\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\mathcal{R}_{2}\right)=\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\mathcal{R}_{2} \wedge \mathrm{FG} \mathrm{ALPHA}_{\leq i}\right) \tag{28}
\end{equation*}
$$

By (24), 26) and (28) we obtain

$$
\begin{equation*}
\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\mathrm{FG} \mathrm{ALPHA}_{\leq i} \vee \mathrm{~F} \varphi_{i+1}\right)=1 \tag{29}
\end{equation*}
$$

$\sigma$ plays like $\tau_{j}$ inside $\mathrm{ALPHA}_{j}$ which attains $\geq \alpha$ for $\varphi$. By using Levy's zero-one law (Lemma 16(1)) for safety sets at level $\alpha$, we obtain that $\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\varphi \wedge \mathrm{FGALPHA}_{\leq i}\right)=$ $\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\mathrm{FG} A L P H A_{\leq i}\right)$. Since $\mathrm{F} \varphi_{i+1} \subseteq \varphi$ it follows from Equation 29 that $\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}(\varphi)=1$, i.e., the resetting strategy $\sigma$ wins $\varphi$ almost surely.

For $s_{0} \in L_{0}$ let

$$
p_{i}\left(s_{0}\right) \stackrel{\text { def }}{=} \mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\varphi \wedge \mathrm{GIX}_{i}\right)
$$

be the attainment for $\varphi$ inside the fixed region $\mathrm{FIX}_{i}$ of $\mathcal{L}_{i}$.
Since $H_{i}$ is finite and $\mathcal{L}_{i}$ is acyclic, almost surely $H_{i}$ is eventually left forever. Moreover, the sets $\mathrm{ALPHA}_{j}$ are safety sets (at level $\alpha$ ) for $\varphi$. It follows from Levy's zero-one law (cf. Corollary 29) that

$$
\begin{equation*}
p_{i}\left(s_{0}\right)=\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\varphi \wedge \mathrm{GFIX}_{i}\right)=\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\mathrm{G} \mathrm{FIX}_{i}\right) \tag{30}
\end{equation*}
$$

Let's now consider only those runs from states $s_{0} \in L_{0}$ that do not satisfy $\mathrm{GFIX}_{i}$ (the rest satisfy $\varphi$ already inside the fixed part of $\mathcal{L}_{i}$ by (30). From (29) we obtain

$$
\begin{align*}
\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\left(\mathrm{FG} \mathrm{ALPHA}_{\leq i}\right.\right. & \left.\left.\vee \mathrm{F} \varphi_{i+1}\right) \wedge \neg \mathrm{G} \mathrm{FIX}_{i}\right) \\
& =\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\neg \mathrm{G} \mathrm{FIX}_{i}\right) \tag{31}
\end{align*}
$$

Using Lemma 10, we show the following claim.
$\triangleright$ Claim 39. For every $\varepsilon_{1}, \varepsilon_{2}>0$, there must exist a threshold $l_{i+1}$ and a finite set

$$
L_{i+1}^{\prime} \subseteq \operatorname{Safe}_{\mathcal{L}_{i},\left(\mathrm{GALPHA}_{\leq i} V \varphi_{i+1}\right)}\left(1-\varepsilon_{1}\right)
$$

such that, following $\sigma$ from any state $s_{0} \in L_{0}$, the chance of satisfying $\neg \mathrm{GIX}_{i}$ and within at most $l_{i+1}$ steps reaching a state $s$ in $L_{i+1}^{\prime}$ is at least $\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\neg \mathrm{GFX}_{i}\right)\left(1-\varepsilon_{2}\right)$.

$$
\begin{equation*}
\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\neg \mathrm{GFIX}_{i} \wedge \mathrm{~F}^{\leq l_{i+1}} L_{i+1}^{\prime}\right) \geq \mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\neg \mathrm{G} \mathrm{FIX}_{i}\right)\left(1-\varepsilon_{2}\right) \tag{32}
\end{equation*}
$$

Proof. For those $s_{0} \in L_{0}$ where $\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\neg \mathrm{GFIX}{ }_{i}\right)=0$ the claim holds trivially.
We now consider the remaining cases of those states $s_{0} \in L_{0}$ where $\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\neg \mathrm{G}\right.$ FIX $\left.{ }_{i}\right)>0$. Let

$$
\begin{equation*}
\delta \stackrel{\text { def }}{=} \varepsilon_{2} \cdot \min _{s_{0} \in L_{0}}\left\{\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\neg \mathrm{GFIX}_{i}\right)>0\right\} \tag{33}
\end{equation*}
$$

where $\delta>0$ since $L_{0}$ is finite. Let $\mathcal{E} \stackrel{\text { def }}{=} \mathrm{GALPHA}_{\leq i} \vee \varphi_{i+1}$. By (29) we have for every $s_{0} \in L_{0}$

$$
\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}(\mathrm{FE})=1
$$

We now consider the finitely many Markov chains $\mathcal{C}_{s_{0}}$ induced by playing $\sigma$ in $\mathcal{L}_{i}$ from the finitely many initial states $s_{0} \in L_{0}$. Thus we obtain for every $s_{0} \in L_{0}$

$$
\begin{equation*}
\mathbb{P}_{\mathcal{C}_{s_{0}}}(\mathrm{FE})=1 \tag{34}
\end{equation*}
$$

Since $\mathcal{E}$ is suffix-closed, we can apply Lemma 10 to each Markov chain $\mathcal{C}_{s_{0}}$. Thus there exist thresholds $l^{s_{0}}$ and finite sets

$$
L^{s_{0}} \subseteq \operatorname{Safe}_{\mathcal{C}_{s_{0}},(\mathcal{E})}\left(1-\varepsilon_{1}\right)
$$

such that

$$
\begin{equation*}
\mathbb{P}_{\mathcal{C}_{s_{0}}}\left(\mathrm{~F}^{\leq^{s_{0}}} L^{s_{0}}\right) \geq \mathbb{P}_{\mathcal{C}_{s_{0}}}(\mathrm{FE})-\delta=1-\delta \tag{35}
\end{equation*}
$$

where the last equality is due to 34 . Let now $L_{i+1}^{\prime} \stackrel{\text { def }}{=} \bigcup_{s_{0} \in L_{0}} L^{s_{0}}$ (which is finite, since it is a finite union of finite sets) and $l_{i+1} \stackrel{\text { def }}{=} \max _{s_{0} \in L_{0}} l^{s_{0}}$ (which is finite as the maximum of a finite set of numbers).

For every $s_{0} \in L_{0}$ we have

$$
\operatorname{Safe}_{\mathcal{C}_{s_{0}},(\mathcal{E})}\left(1-\varepsilon_{1}\right) \subseteq \operatorname{Safe}_{\mathcal{L}_{i},\left(\mathrm{GALPHA}_{\leq i} \vee \varphi_{i+1}\right)}\left(1-\varepsilon_{1}\right)
$$

since the required value for $\mathcal{E}=G \operatorname{ALPHA}_{\leq i} \vee \varphi_{i+1}$ is witnessed by the strategy $\sigma$, and thus

$$
L_{i+1}^{\prime} \subseteq \operatorname{Safe}_{\mathcal{L}_{i},\left(\mathrm{GALPHA}_{\leq i} \vee \varphi_{i+1}\right)}\left(1-\varepsilon_{1}\right)
$$

as required. From (35) we obtain that for all $s_{0} \in L_{0}$

$$
\begin{equation*}
\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\mathrm{~F}^{\leq l_{i+1}} L_{i+1}^{\prime}\right) \geq 1-\delta \tag{36}
\end{equation*}
$$

Now we are ready to show 32 . We have for all $s_{0} \in L_{0}$

$$
\begin{array}{rlr}
\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma} & \left(\neg \mathrm{GFIX}_{i} \wedge \mathrm{~F}^{\leq l_{i+1}} L_{i+1}^{\prime}\right) & \\
& =\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\mathrm{~F}^{\leq l_{i+1}} L_{i+1}^{\prime}\right)-\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\mathrm{G} \mathrm{FIX}_{i} \wedge \mathrm{~F}^{\leq l_{i+1}} L_{i+1}^{\prime}\right) & \text { law of total prob. } \\
& \geq \mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\mathrm{~F}^{\leq l_{i+1}} L_{i+1}^{\prime}\right)-\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\mathrm{GFIX}_{i}\right) & \\
& \geq 1-\delta-\left(1-\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\neg \mathrm{GFIX}_{i}\right)\right) & \text { by (36) } \\
& \geq 1-\varepsilon_{2} \cdot \mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\neg \mathrm{GIX}_{i}\right)-\left(1-\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\neg \mathrm{GFIX}_{i}\right)\right) & \text { by (33)} \\
& =\mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\neg \mathrm{GFXX}_{i}\right)\left(1-\varepsilon_{2}\right) &
\end{array}
$$

Notice that $L_{i+1} \cap \mathrm{FIX}_{i}=\emptyset$, because every state in $L_{i+1}$ must have a value $\geq 1-\varepsilon_{1}$ for $\varphi_{i+1}$. (In the special case of $i=0$ we have $\mathrm{FIX}_{0}=\emptyset$ and $\varphi_{1}=\varphi$ and thus $l_{1}=0$ and $L_{1}=L_{1}^{\prime}=L_{0}$.) Also recall that

$$
\begin{aligned}
L_{i+1}^{\prime} & \subseteq \text { Safe }_{\mathcal{L}_{i},\left(\mathrm{GALPHA}_{\leq i} \vee \varphi_{i+1}\right)}\left(1-\varepsilon_{1}\right) \\
& \subseteq \mathrm{ALPHA}_{\leq i} \cup \text { Safe }_{\mathcal{L}_{i}, \varphi_{i+1}}\left(1-\varepsilon_{1}\right) .
\end{aligned}
$$

We define $L_{i+1}$ as $L_{i+1} \stackrel{\text { def }}{=} L_{i+1}^{\prime} \backslash$ ALPHA $_{\leq i}$.
Since $\varphi_{i+1} \stackrel{\text { def }}{=} \varphi \wedge \mathrm{G}\left(L \backslash C l\left(\right.\right.$ FIX $\left.\left._{i}\right)\right)$ and $\varphi$ is a parity objective, we can, by Lemma 9 and Remark 11. pick an MD strategy $\tau_{i+1}$ that is $\varepsilon_{0}$-optimal for $\varphi_{i+1}$ from all states in $L_{i+1}$.

Based on this strategy $\tau_{i+1}$ and parameters $\alpha>\beta>\gamma>0$, we define ALPHA $_{i+1} \subseteq$ $\mathrm{BETA}_{i+1} \subseteq \mathrm{GAMMA}_{i+1} \subseteq L$ to be the sets of states from which $\tau_{i+1}$ attains at least values $\alpha, \beta$ and $\gamma$, for $\varphi_{i+1}$, respectively. E.g.,

$$
\operatorname{BETA}_{i+1} \stackrel{\text { def }}{=} \text { Safe }_{\mathcal{L}_{i}, \tau_{i+1}, \varphi_{i+1}}(\beta)
$$

In particular, this definition satisfies our invariant 21, i.e., $\mathrm{GAMMA}_{i+1} \cap C l\left(\mathrm{FIX}_{i}\right)=\emptyset$, because a high attainment $\gamma$ for $\varphi_{i+1}=\varphi \wedge \mathrm{G}\left(L \backslash C l\left(\mathrm{FIX}_{i}\right)\right.$ requires that $C l\left(\mathrm{FIX}_{i}\right)$ is not visited.
W.l.o.g., by choosing $\varepsilon_{1}, \varepsilon_{0}$ sufficiently small, we can assume that $\alpha<\left(1-\varepsilon_{1}-\varepsilon_{0}\right)$, and therefore that $L_{i+1} \subseteq \mathrm{ALPHA}_{i+1} \subseteq \mathrm{BETA}_{i+1}$ (we only need $\subseteq \mathrm{BETA}_{i+1}$ ).

Let $\mathcal{L}_{i}^{\prime} \stackrel{\text { def }}{=} \mathcal{L}_{i}\left[\tau_{i+1}, \mathrm{BETA}_{i+1}\right]$. Note that in $\mathcal{L}_{i}^{\prime}$ the strategy $\sigma$ might not be able to reach $L_{i+1}$ with the same probability as in $\mathcal{L}_{i}$, because the choices in $\mathrm{BETA}_{i+1}$ are now fixed. However, a similar strategy $\sigma^{\prime}$ can reach $\mathrm{BETA}_{i+1}$ in $\mathcal{L}_{i}^{\prime}$ with at least the probability by which $\sigma$ reaches $L_{i+1}$ in $\mathcal{L}_{i}$. We now define a new resetting strategy $\sigma^{\prime}$ in $\mathcal{L}_{i}^{\prime}$. It behaves like the previous strategy $\sigma$ until (and if) it reaches $C l\left(\mathrm{BETA}_{i+1}\right)$. Without restriction we can assume that it reaches even $\mathrm{BETA}_{i+1}$ in this case (similar to the argument for $\sigma$ above). Then it plays like $\tau_{i+1}$ while in $\mathrm{GAMMA}_{i+1}$. This is possible, since $\mathrm{GAMMA}_{i+1} \cap C l\left(\mathrm{FIX}_{i}\right)=\emptyset$ by our invariant 21 . If and when it exits GAMMA ${ }_{i+1}$ at some state $s$ then it resets to some almost surely winning strategy $\sigma(s)$ for $\varphi$ in $\mathcal{L}$ until it reaches $\mathrm{BETA}_{i+1}$ (or another previously fixed part) again, etc.

From Claim 39 (Equation (32)) and the fact that $\sigma^{\prime}$ behaves like $\sigma$ until it reaches $\operatorname{BETA}_{i+1}$ we obtain that

$$
\begin{align*}
& \mathbb{P}_{\mathcal{L}_{i}^{\prime}, s_{0}, \sigma^{\prime}}\left(\neg \mathrm{GIX}_{i} \wedge \mathrm{~F}^{\leq l_{i+1}}\left(\mathrm{ALPHA}_{\leq i} \vee \mathrm{BETA}_{i+1}\right)\right) \\
& \geq \mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\neg \mathrm{GIX}_{i}\right)\left(1-\varepsilon_{2}\right)  \tag{37}\\
& =\mathbb{P}_{\mathcal{L}_{i}^{\prime}, s_{0}, \sigma^{\prime}}\left(\neg \mathrm{GFIX}_{i}\right)\left(1-\varepsilon_{2}\right),
\end{align*}
$$

where the last equality holds because $\mathcal{L}_{i}$ and $\mathcal{L}_{i}^{\prime}$ (resp. $\sigma$ and $\sigma^{\prime}$ ) coincide inside $\mathrm{FIX}_{i}$. Analogously to Claim [38, from any state in BETA $_{i+1}$, the chance of staying in the set GAMMA $_{i+1}$ can be lower-bounded.

$$
\begin{equation*}
\forall s \in \operatorname{BETA}_{i+1} \mathbb{P}_{\mathcal{L}_{i}^{\prime}, s, \sigma^{\prime}}\left(\operatorname{G~GAMMA}_{i+1}\right) \geq \frac{\beta-\gamma}{1-\gamma}>0 \tag{38}
\end{equation*}
$$

$\mathrm{GAMMA}_{i+1} \cap C l\left(\mathrm{FIX}_{i}\right)=\emptyset$ by (21) and $\sigma^{\prime}$ continues to play $\tau_{i+1}$ in GAMMA ${ }_{i+1}$. Since G GAMMA $_{i+1} \subseteq$ FG GAMMA ${ }_{i+1}$, we can apply Levy's zero-one law (Corollary 30) to 38) and obtain

$$
\begin{equation*}
\forall s \in \mathrm{BETA}_{i+1} \mathbb{P}_{\mathcal{L}_{i}^{\prime}, s, \sigma^{\prime}}\left(\mathrm{FG} \mathrm{ALPHA}_{i+1}\right) \geq \frac{\beta-\gamma}{1-\gamma}>0 \tag{39}
\end{equation*}
$$

By combining (37) with (39), we get

$$
\begin{aligned}
& \mathbb{P}_{\mathcal{L}_{i}^{\prime}, s_{0}, \sigma^{\prime}}\left(\neg \mathrm{GIX}_{i} \wedge \mathrm{~F}^{\leq l_{i+1}}\left(\mathrm{ALPHA}_{\leq i} \vee \mathrm{FG} \text { ALPHA }_{i+1}\right)\right) \\
& \geq\left(\mathbb{P}_{\mathcal{L}_{i}^{\prime}, s_{0}, \sigma^{\prime}}\left(\neg \mathrm{GFX}_{i}\right)\left(1-\varepsilon_{2}\right)\right) \frac{\beta-\gamma}{1-\gamma}
\end{aligned}
$$

By continuity of measures (recall that $\mathrm{F} X=\bigcup_{k \in \mathbb{N}} \mathrm{~F}^{k} X$ ), for every $\varepsilon_{3}>0$ there must exist a threshold $k_{i+1} \geq l_{i+1}$ of steps such that, for all $s_{0} \in L_{0}$,

$$
\begin{align*}
& \mathbb{P}_{\mathcal{L}_{i}^{\prime}, s_{0}, \sigma^{\prime}}\left(\neg \mathrm{GIX}_{i} \wedge \mathrm{~F}^{\leq k_{i+1}} \mathrm{ALPHA}_{\leq i+1}\right) \\
& \geq\left(\mathbb{P}_{\mathcal{L}_{i}^{\prime}, s_{0}, \sigma^{\prime}}\left(\neg \mathrm{GFIX}_{i}\right)\left(1-\varepsilon_{2}\right)\right) \frac{\beta-\gamma}{1-\gamma}\left(1-\varepsilon_{3}\right) \tag{40}
\end{align*}
$$

(Since $L_{0}$ is finite, we can have the same multiplicative error $\left(1-\varepsilon_{3}\right)$ for all $s_{0} \in L_{0}$.) (In the special case of $i=0$, we have $k_{1}=0$, since $L_{0}=L_{1} \subseteq$ ALPHA $_{1}$.) Once inside ALPHA ${ }_{\leq i+1}$, there is a bounded chance $\geq \frac{\alpha-\beta}{1-\beta}$ of staying inside BETA $_{\leq i+1}$ forever, by Claim 38 . Thus from (40) we get

$$
\begin{align*}
& \mathbb{P}_{\mathcal{L}_{i}^{\prime}, s_{0}, \sigma^{\prime}}\left(\neg \mathrm{GFXX}_{i} \wedge \mathrm{~F}^{\leq k_{i+1}} \mathrm{GBETA}_{\leq i+1}\right)  \tag{41}\\
& \geq\left(\left(\mathbb{P}_{\mathcal{L}_{i}^{\prime}, s_{0}, \sigma^{\prime}}\left(\neg \mathrm{GFX}_{i}\right)\left(1-\varepsilon_{2}\right)\right) \frac{\beta-\gamma}{1-\gamma}\left(1-\varepsilon_{3}\right)\right) \frac{\alpha-\beta}{1-\beta}
\end{align*}
$$

Consider the finite $k_{i+1}$-bubble $H_{i+1} \stackrel{\text { def }}{=}$ bubble $\left(\mathcal{L}, L_{0}, k_{i+1}\right)$ around $L_{0}$. Remember that in finite MDPs, there are uniformly optimal MD strategies for reachability objectives [16]. Consequently, since $H_{i+1}$ is finite, there exists an MD strategy $\rho_{i+1}$ in $\mathcal{L}_{i}^{\prime}$ that is optimal from $H_{i+1}$ for the objective of reaching ALPHA $_{\leq i+1}$ (from $L_{0}$ ) inside $H_{i+1}$ without leaving $H_{i+1}$. We fix $\rho_{i+1}$ inside $H_{i+1}$, and obtain our new MDP

$$
\mathcal{L}_{i+1} \stackrel{\text { def }}{=} \mathcal{L}_{i}^{\prime}\left[\rho_{i+1}, L_{0}, k_{i+1}\right]
$$

See Figure 4 for an illustration after round $i=3$. We define $\mathrm{FIX}_{i+1} \stackrel{\text { def }}{=} \mathrm{BETA}_{\leq i+1} \cup H_{i+1}$ as the region where the strategy is already fixed in $\mathcal{L}_{i+1}$. We now define an almost surely winning resetting strategy $\sigma^{\prime \prime}$ in $\mathcal{L}_{i+1}$, analogously as $\sigma$ previously in $\mathcal{L}_{i}$. Similarly as in Equation (30) for $\mathcal{L}_{i}$, we can derive the corresponding property for $\mathcal{L}_{i+1}$.

$$
\begin{equation*}
\mathbb{P}_{\mathcal{L}_{i+1}, s_{0}, \sigma^{\prime \prime}}\left(\varphi \wedge \mathrm{GFIX}_{i+1}\right)=\mathbb{P}_{\mathcal{L}_{i+1}, s_{0}, \sigma^{\prime \prime}}\left(\mathrm{GIX}_{i+1}\right) \tag{42}
\end{equation*}
$$

$\mathcal{L}_{i+1}, \mathcal{L}_{i}^{\prime}$ and $\mathcal{L}_{i}$ (resp. the strategies $\sigma^{\prime \prime}, \sigma^{\prime}$ and $\sigma$ ) coincide inside $\mathrm{FIX}_{i}$. Thus by 30) we have

$$
\begin{array}{rlr}
\mathbb{P}_{\mathcal{L}_{i+1}, s_{0}, \sigma^{\prime \prime}}\left(\mathrm{G} \mathrm{FIX}_{i}\right) & =\mathbb{P}_{\mathcal{L}_{i}^{\prime}, s_{0}, \sigma^{\prime}}\left(\mathrm{G} \mathrm{FIX}_{i}\right) \\
& = & \mathbb{P}_{\mathcal{L}_{i}, s_{0}, \sigma}\left(\mathrm{G} \mathrm{FIX}_{i}\right)  \tag{43}\\
& = & p_{i}\left(s_{0}\right)
\end{array}
$$

By the optimality of the reachability strategy $\rho_{i+1}$ that is fixed in $H_{i+1}$ and $\mathrm{FIX}_{i+1}=$ BETA $_{\leq i+1} \cup H_{i+1}$, we obtain from this and Equation 41) that

$$
\begin{align*}
& \mathbb{P}_{\mathcal{L}_{i+1}, s_{0}, \sigma^{\prime \prime}}\left(\neg \mathrm{GFIX}_{i} \wedge \mathrm{GFIX}_{i+1}\right) \\
& \geq\left(\left(1-p_{i}\left(s_{0}\right)\right)\left(1-\varepsilon_{2}\right) \frac{\beta-\gamma}{1-\gamma}\left(1-\varepsilon_{3}\right)\right) \frac{\alpha-\beta}{1-\beta} \tag{44}
\end{align*}
$$

The crucial question is how much $\sigma^{\prime \prime}$ attains for $\varphi$ in the fixed part alone, i.e., how large is $\mathbb{P}_{\mathcal{L}_{i+1}, s_{0}, \sigma^{\prime \prime}}\left(\varphi \wedge \mathrm{GFIX}_{i+1}\right)=p_{i+1}\left(s_{0}\right)$ ? For all $s_{0} \in L_{0}$ we have

$$
\begin{aligned}
& p_{i+1}\left(s_{0}\right) \\
& =\mathbb{P}_{\mathcal{L}_{i+1}, s_{0}, \sigma^{\prime \prime}}\left(\mathrm{GFIX}_{i+1}\right) \\
& =\mathbb{P}_{\mathcal{L}_{i+1}, s_{0}, \sigma^{\prime \prime}}\left(\mathrm{GFIX}_{i}\right)+\mathbb{P}_{\mathcal{L}_{i+1}, s_{0}, \sigma^{\prime \prime}}\left(\neg \mathrm{G} \mathrm{FIX}_{i} \wedge \mathrm{G} \mathrm{FIX}_{i+1}\right) \\
& \geq p_{i}\left(s_{0}\right)+\left(1-p_{i}\left(s_{0}\right)\right)\left(\left(1-\varepsilon_{2}\right) \frac{\beta-\gamma}{1-\gamma}\left(1-\varepsilon_{3}\right) \frac{\alpha-\beta}{1-\beta}\right)
\end{aligned}
$$

where the first equality is due to (42) and the last inequation is due to Equations (43) and (44).

We can suitably choose the parameters $\alpha, \beta, \gamma, \varepsilon_{2}, \varepsilon_{3}$ such that $\left(\left(1-\varepsilon_{2}\right) \frac{\beta-\gamma}{1-\gamma}\left(1-\varepsilon_{3}\right) \frac{\alpha-\beta}{1-\beta}\right)$ is arbitrarily close to 1 , and thus in particular $\geq 1 / 2$, and obtain that $p_{i+1}\left(s_{0}\right) \geq p_{i}\left(s_{0}\right)+$ $\left(1-p_{i}\left(s_{0}\right)\right) / 2$. Since $p_{0}\left(s_{0}\right)=0$, we get $1-p_{i}\left(s_{0}\right) \leq 2^{-i}$ and thus $\lim _{i \rightarrow \infty} p_{i}\left(s_{0}\right)=1$, as required.

Finally, let $\hat{\sigma}$ be the MD strategy in $\mathcal{L}$ that plays from $L_{0}$ as prescribed by all the fixings in $\bigcup_{i} \mathrm{FIX}_{i}$ in the systems $\mathcal{L}_{i}$. Then, for all $s_{0} \in S_{0}$ and every $i \in \mathbb{N}$, it holds that

$$
\mathbb{P}_{\mathcal{L}, s_{0}, \hat{\sigma}}(\varphi) \geq \mathbb{P}_{\mathcal{L}_{i}, s_{0}, \hat{\sigma}}\left(\varphi \wedge \mathrm{GFIX}_{i}\right)=p_{i}\left(s_{0}\right) \geq 1-2^{-i}
$$

Since this holds for every $i \in \mathbb{N}$ we get that $\mathbb{P}_{\mathcal{L}, s_{0}, \hat{\sigma}}(\varphi)=1$, i.e., the MD strategy $\hat{\sigma}$ wins $\varphi$ almost surely from every $s_{0} \in L_{0}$.

## F Optimal Strategies for $\{0,1,2\}$-Parity

- Theorem 22. Let $\mathcal{M}=\left(S, S_{\square}, S_{\bigcirc}, \longrightarrow, P\right)$ be an $M D P, \varphi$ a $\{0,1,2\}$-Parity objective and $\mathcal{M}_{*}=\left(S_{*}, S_{* \square}, S_{* ○}, \longrightarrow_{*}, P_{*}\right)$ its conditioned version wrt. $\varphi$. Assume that in $\mathcal{M}_{*}$ for every safety objective (given by some target $T \subseteq S_{*}$ ) and $\varepsilon>0$ there exists a uniformly $\varepsilon$-optimal $M D$ strategy. Let $S_{\text {opt }}$ be the subset of states that have an optimal strategy for $\varphi$ in $\mathcal{M}$.

Then there exists an $M D$ strategy in $\mathcal{M}$ that is optimal for $\varphi$ from every state in $S_{o p t}$.

In the rest of this section we prove Theorem 22. It generalizes [13, Theorem 16], which considers only finitely-branching MDPs and uses the fact that for every safety objective, an MD strategy exists that is uniformly optimal. This is not generally true for infinitelybranching acyclic MDPs [13]. To prove Theorem 22, we adjust the construction so that it only requires uniformly $\varepsilon$-optimal MD strategies for safety objectives (in the conditioned $\operatorname{MDP} \mathcal{M}_{*}$ ).

- Theorem 40 (from Theorem B in [16]). For every MDP $\mathcal{M}$ there exist uniform $\varepsilon$-optimal MD-strategies for reachability objectives.

The following simple lemma provides a scheme for proving almost-sure properties.

- Lemma 41 (Lem. 18 in [13]). Let $\mathbb{P}$ be a probability measure over the sample space $\Omega$. Let $\left(\mathcal{R}_{i}\right)_{i \in I}$ be a countable partition of $\Omega$ in measurable events. Let $\mathcal{E} \subseteq \Omega$ be a measurable event. Suppose $\mathbb{P}\left(\mathcal{R}_{i} \cap \mathcal{E}\right)=\mathbb{P}\left(\mathcal{R}_{i}\right)$ holds for all $i \in I$. Then $\mathbb{P}(\mathcal{E})=1$.

We need a few lemmas about safety objectives first. Recall the definition of safe sets (Equation (1)).

- Lemma 42. Let $\mathcal{M}=\left(S, S_{\square}, S_{\bigcirc}, \longrightarrow, P\right)$ be an $M D P, T \subseteq S$, $\sigma$ a strategy from state $s \in S$ and $\tau<1$. It holds that $\mathbb{P}_{\mathcal{M}, s, \sigma}\left(\mathrm{FG} \neg \operatorname{Safe}_{\mathcal{M}, \operatorname{Safety}(T)}(\tau) \wedge \mathrm{FG}(S \backslash T)\right)=0$.

Proof. For any $n \in \mathbb{N}$ define $Z_{n} \stackrel{\text { def }}{=}(S \backslash T)^{n}$. That is, $Z_{n} S^{\omega}$ is the event that the first $n$ visited states are outside $T$. For every state $s \notin \operatorname{Safe}_{\mathcal{M}, \operatorname{Safety}(T)}(\tau)$ and every strategy $\sigma$ from $s$ we have that $\lim _{n \rightarrow \infty} \mathbb{P}_{\mathcal{M}, s, \sigma}\left(Z_{n} S^{\omega}\right)<\tau<(1+\tau) / 2$ by Equation (1). Let $n(s) \in \mathbb{N}$ be the smallest number such that $\mathbb{P}_{\mathcal{M}, s, \sigma}\left(Z_{n(s)} S^{\omega}\right) \leq(1+\tau) / 2$. Let $L \subseteq S^{*}$ be the set of finite sequences $s_{0} s_{1} \cdots s_{n-1}$ such that $s_{0} \notin \operatorname{Safe}_{\mathcal{M}, \operatorname{Safety}(T)}(\tau)$ and $n=n\left(s_{0}\right)$ and $\forall i<n . s_{i} \in(S \backslash T) \backslash \operatorname{Safe}_{\mathcal{M}, \operatorname{Safety}(T)}(\tau)$.

We show for all $s \in S \backslash \operatorname{Safe}_{\mathcal{M}, \operatorname{Safety}(T)}(\tau)$ and all $k \in \mathbb{N}$ that $\mathbb{P}_{\mathcal{M}, s, \sigma}\left(L^{k} S^{\omega}\right) \leq\left(\frac{1+\tau}{2}\right)^{k}$. We proceed by induction on $k$. The case $k=0$ is trivial. For the induction step let $k \geq 0$.

$$
\begin{aligned}
\mathbb{P}_{\mathcal{M}, s, \sigma}\left(L^{k+1} S^{\omega}\right) & \leq \mathbb{P}_{\mathcal{M}, s, \sigma}\left(Z_{n(s)} L^{k} S^{\omega}\right) \\
& \leq \mathbb{P}_{\mathcal{M}, s, \sigma}\left(Z_{n(s)} S_{s^{\prime} \in S \backslash \text { Safe }_{\mathcal{M}, \text { Safety }(T)}(\tau)}^{\omega}\right) \cdot \sup _{\mathbb{P}_{\mathcal{M}}, s^{\prime}}\left(L^{k} S^{\omega}\right) \\
& \leq \mathbb{P}_{\mathcal{M}, s, \sigma}\left(Z_{n(s)} S^{\omega}\right) \cdot\left(\frac{1+\tau}{2}\right)^{k} \leq\left(\frac{1+\tau}{2}\right)^{k+1}
\end{aligned}
$$

where the first inequality uses that $L \cap\{s\} S^{*} \subseteq Z_{n(s)}$, the third uses the induction hypothesis, and the last the definition of $n(s)$. This completes the induction proof.

Write $\varphi \stackrel{\text { def }}{=} \mathrm{G} \neg \operatorname{Safe}_{\mathcal{M}, \text { Safety }(T)}(\tau) \wedge \mathrm{G}(S \backslash T)$. For all $s \in S$,

$$
\begin{array}{rlrl}
\mathbb{P}_{\mathcal{M}, s, \sigma}(\varphi) & =\mathbb{P}_{\mathcal{M}, s, \sigma}\left(L^{\omega}\right) & & \text { because } \llbracket \varphi \rrbracket= \\
& =\lim _{k \rightarrow \infty} \mathbb{P}_{\mathcal{M}, s, \sigma}\left(L^{k} S^{\omega}\right) & & \text { by continuity } \\
& \leq \lim _{k \rightarrow \infty}\left(\frac{1+\tau}{2}\right)^{k} & & \\
& =0 & & \text { as shown abov } \\
& & \text { because } \tau<1
\end{array}
$$

It follows that $\mathbb{P}_{\mathcal{M}, s, \sigma}\left(\mathrm{X}^{j} \varphi\right)=0$, for all $s \in S$ and all $j \in \mathbb{N}$ and therefore that

$$
\begin{aligned}
& \mathbb{P}_{\mathcal{M}, s, \sigma}\left(\mathrm{FG} S a f e_{\mathcal{M}, \text { Safety }(T)}(\tau) \wedge \mathrm{FG}(S \backslash T)\right) \\
& =\mathbb{P}_{\mathcal{M}, s, \sigma}(\mathrm{~F} \varphi) \\
& =\mathbb{P}_{\mathcal{M}, s, \sigma}\left(\bigcup_{j \in \mathbb{N}} \llbracket \mathrm{X}^{j} \varphi \rrbracket^{s}\right) \\
& \leq \sum_{j \in \mathbb{N}} \mathbb{P}_{\mathcal{M}, s, \sigma}\left(\mathrm{X}^{j} \varphi\right)=0
\end{aligned}
$$

Now we show that if an MDP admits uniformly $\varepsilon$-optimal strategies for all safety objectives, then optimal strategies for $\{0,1,2\}$-Parity (where they exist) can be chosen MD.

- Lemma 43. Let $\mathcal{M}=\left(S, S_{\square}, S_{\bigcirc}, \longrightarrow, P\right)$ be an MDP such that for every safety objective (given by some target set $T \subseteq S$ ) and $\varepsilon>0$ there exists a uniformly $\varepsilon$-optimal MD strategy. Let $s_{0} \in S$, Col $: S \rightarrow\{0,1,2\}, \varphi=\operatorname{Parity}(C o l)$, and $\sigma$ a strategy with $\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}(\varphi)=1$. Then there is an MD-strategy $\sigma^{\prime}$ with $\mathbb{P}_{\mathcal{M}, s_{0}, \sigma^{\prime}}(\varphi)=1$.

Proof. To achieve an almost-sure winning objective, the player must forever remain in states from which the objective can be achieved almost surely. So we can assume without loss of generality that all states are almost-sure winning, i.e., for all $s \in S$ we have $\mathbb{P}_{\mathcal{M}, s, \sigma}(\varphi)=1$ for some strategy $\sigma$. We will define an MD-strategy $\sigma^{\prime}$ with $\mathbb{P}_{\mathcal{M}, s, \sigma^{\prime}}(\varphi)=1$ for all $s \in S$.

Recall that $[S]^{C o l \neq 0} \subseteq S$ denotes the subset of states of color 1 or 2 . Let $T \stackrel{\text { def }}{=}[S]^{C o l \neq 0}$ and let $\sigma_{\varepsilon}$ be a uniformly $\varepsilon$-optimal MD strategy for $\operatorname{Safety}(T)$, whose existence is guaranteed by our assumption on $\mathcal{M}$. The precise $\varepsilon>0$ is immaterial, we only need that $\varepsilon<\frac{1}{3}$. The MD-strategy $\sigma^{\prime}$ will be based on special subsets (Equation (1)):

$$
\begin{equation*}
\operatorname{Safe}_{\mathcal{M}}(\tau) \stackrel{\text { def }}{=} \operatorname{Safe}_{\mathcal{M}, \sigma_{\varepsilon}, \operatorname{Safety}(T)}(\tau) \subseteq S \tag{45}
\end{equation*}
$$

We first define the MD-strategy $\sigma^{\prime}$ partially for the states in $\operatorname{Safe}_{\mathcal{M}}\left(\frac{1}{3}\right)$ and then extend the definition of $\sigma^{\prime}$ to all states. For the states in $\operatorname{Safe}_{\mathcal{M}}\left(\frac{1}{3}\right)$ define $\sigma^{\prime} \stackrel{\text { def }}{=} \sigma_{\varepsilon}$ (which is MD). Let $\mathcal{M}^{\prime}$ be the MDP obtained from $\mathcal{M}$ by restricting the transition relation as prescribed by the partial MD-strategy $\sigma^{\prime}$ in $S a f e_{\mathcal{M}}\left(\frac{1}{3}\right)$ (elsewhere the choices remain free). We define $\operatorname{Safe}_{\mathcal{M}^{\prime}}(\tau)$ for $\mathcal{M}^{\prime}$ as in Equation (45) for $\mathcal{M}$. Thus, for any $\tau \in[0,1]$, we have $\operatorname{Safe}_{\mathcal{M}}(\tau)=\operatorname{Safe}_{\mathcal{M}^{\prime}}(\tau)$. Indeed, since $\mathcal{M}^{\prime}$ restricts the options of the player, we have $\operatorname{Safe}_{\mathcal{M}}(\tau) \supseteq \operatorname{Safe}_{\mathcal{M}^{\prime}}(\tau)$. Conversely, let $s \in \operatorname{Safe}_{\mathcal{M}}(\tau)$. The strategy $\sigma_{\varepsilon}$ attains $\mathbb{P}_{\mathcal{M}, s, \sigma_{\varepsilon}}\left(\mathrm{G}[S]^{\text {Col=0 }}\right) \geq \tau$. Since $\sigma_{\varepsilon}$ can be applied in $\mathcal{M}^{\prime}$, and results in the same Markov chain as applying it in $\mathcal{M}$, we conclude $s \in \operatorname{Safe}_{\mathcal{M}^{\prime}}(\tau)$. This justifies to write $\operatorname{Safe}(\tau)$ for $\operatorname{Safe}_{\mathcal{M}}(\tau)=\operatorname{Safe}_{\mathcal{M}^{\prime}}(\tau)$ in the remainder of the proof.

Next we show that, also in $\mathcal{M}^{\prime}$, for all states $s \in S$ there exists a strategy $\sigma_{1}$ with $\mathbb{P}_{\mathcal{M}^{\prime}, s, \sigma_{1}}(\varphi)=1$. This strategy $\sigma_{1}$ is defined as follows. First play according to an almostsurely winning strategy $\sigma$ from the statement of the theorem. If and when the play visits Safe $\left(\frac{1}{3}\right)$, switch to the MD-strategy $\sigma_{\varepsilon}$. If and when the play then visits $[S]^{C o l \neq 0}$, switch back to an almost-surely winning strategy $\sigma$ from the statement of the theorem, and so forth.

We show that $\sigma_{1}$ attains $\mathbb{P}_{\mathcal{M}^{\prime}, s, \sigma_{1}}(\varphi)=1$. To this end we will use Lemma 41 . We partition the runs of $s S^{\omega}$ into three events $\mathcal{R}_{0}, \mathcal{R}_{1}, \mathcal{R}_{2}$ as follows:

- $\mathcal{R}_{0}$ contains the runs where $\sigma_{1}$ switches between $\sigma_{\varepsilon}$ and $\sigma$ infinitely often.
- $\mathcal{R}_{1}$ contains the runs where $\sigma_{1}$ eventually only plays according to $\sigma_{\varepsilon}$.
- $\mathcal{R}_{2}$ contains the runs where $\sigma_{1}$ eventually only plays according to $\sigma$.

Each time $\sigma_{1}$ switches to $\sigma_{\varepsilon}$, there is, by definition of $\operatorname{Safe}_{\mathcal{M}}\left(\frac{1}{3}\right)$, a probability of at least $\frac{1}{3}$ of never visiting a color- $\{1,2\}$ state again and thus of never again switching to $\sigma$. It follows that $\mathbb{P}_{\mathcal{M}^{\prime}, s, \sigma_{1}}\left(\mathcal{R}_{0}\right)=0$. By the definition of the switching behavior of $\sigma_{1}$, we have $\mathcal{R}_{1} \subseteq \llbracket \mathrm{FG}[S]^{\text {Col=0 }} \rrbracket \subseteq \llbracket \varphi \rrbracket$, and hence $\mathbb{P}_{\mathcal{M}^{\prime}, s, \sigma_{1}}\left(\mathcal{R}_{1} \cap \llbracket \varphi \rrbracket\right)=\mathbb{P}_{\mathcal{M}^{\prime}, s, \sigma_{1}}\left(\mathcal{R}_{1}\right)$. Since $\mathbb{P}_{\mathcal{M}, s, \sigma}(\varphi)=1$ and $\varphi$ is tail, we have that $\mathbb{P}_{\mathcal{M}^{\prime}, s, \sigma_{1}}\left(\mathcal{R}_{2} \cap \llbracket \varphi \rrbracket\right)=\mathbb{P}_{\mathcal{M}^{\prime}, s, \sigma_{1}}\left(\mathcal{R}_{2}\right)$. Using Lemma 41, we obtain $\mathbb{P}_{\mathcal{M}^{\prime}, s, \sigma_{1}}(\varphi)=1$.

Next we show that for all $s \in S$ the strategy $\sigma_{1}$ defined above achieves $\mathbb{P}_{\mathcal{M}^{\prime}, s, \sigma_{1}}\left(\mathrm{~F} \operatorname{Safe}\left(\frac{2}{3}\right) \vee\right.$ $\left.\mathrm{F}[S]^{\mathrm{Col}=2}\right)=1$. To this end we will use Lemma 41 again. We partition the runs of $s S^{\omega}$ into three events $\mathcal{R}_{1}^{\prime}, \mathcal{R}_{2}^{\prime}, \mathcal{R}_{0}^{\prime}$ as follows:

- $\mathcal{R}_{1}^{\prime}=\llbracket \mathrm{FG}[S]^{\text {Col }=0} \rrbracket^{s}$
- $\mathcal{R}_{2}^{\prime}=\llbracket \mathrm{GF}[S]^{\mathrm{Col}=2} \rrbracket^{s}$
- $\mathcal{R}_{0}^{\prime}=s S^{\omega} \backslash \llbracket \varphi \rrbracket^{s}$

We have previously shown that $\mathbb{P}_{\mathcal{M}^{\prime}, s, \sigma_{1}}(\varphi)=1$ and hence that $\mathbb{P}_{\mathcal{M}^{\prime}, s, \sigma_{1}}\left(\mathcal{R}_{0}^{\prime}\right)=0$. We now invoke Lemma 42 with $\tau \stackrel{\text { def }}{=} \frac{2}{3}+\varepsilon<1$ and obtain that almost all runs in $\mathcal{R}_{1}^{\prime}$ satisfy $\operatorname{GFSafe}_{\mathcal{M}, \operatorname{Safety}(T)}(\tau)$. Since $\sigma_{\varepsilon}$ is uniformly $\varepsilon$-optimal for $\operatorname{Safety}(T)$ we have $\operatorname{Safe}_{\mathcal{M} \text {,Safety }(T)}(\tau) \subseteq \operatorname{Safe}(\tau-\varepsilon)=\operatorname{Safe}\left(\frac{2}{3}\right)$ and thus almost all runs in $\mathcal{R}_{1}^{\prime}$ satisfy GFSafe $\left(\frac{2}{3}\right)$. Since $\llbracket \mathrm{GF}$ Safe $\left(\frac{2}{3}\right) \rrbracket \subseteq \llbracket \mathrm{FSafe}\left(\frac{2}{3}\right) \rrbracket$, we observe that

$$
\mathbb{P}_{\mathcal{M}^{\prime}, s, \sigma_{1}}\left(\mathcal{R}_{1}^{\prime} \cap \llbracket \operatorname{FSafe}\left(\frac{2}{3}\right) \vee \mathrm{F}[S]^{\text {Col }=2} \rrbracket\right)=\mathbb{P}_{\mathcal{M}^{\prime}, s, \sigma_{1}}\left(\mathcal{R}_{1}^{\prime}\right)
$$

Since $\mathcal{R}_{2}^{\prime} \subseteq \llbracket \mathrm{F}[S]^{C o l=2} \rrbracket$, we also have that

$$
\mathbb{P}_{\mathcal{M}^{\prime}, s, \sigma_{1}}\left(\mathcal{R}_{2}^{\prime} \cap \llbracket \operatorname{FSafe}\left(\frac{2}{3}\right) \vee \mathrm{F}[S]^{C o l=2} \rrbracket\right)=\mathbb{P}_{\mathcal{M}^{\prime}, s, \sigma_{1}}\left(\mathcal{R}_{2}^{\prime}\right)
$$

By Lemma 41 we obtain $\mathbb{P}_{\mathcal{M}^{\prime}, s, \sigma_{1}}\left(\mathrm{FSafe}\left(\frac{2}{3}\right) \vee \mathrm{F}[S]^{\text {Col }=2}\right)=1$.
Writing $T^{\prime} \stackrel{\text { def }}{=} \operatorname{Safe}\left(\frac{2}{3}\right) \cup[S]^{\text {Col }=2}$ we have just shown that for all $s \in S$ there is a strategy $\sigma_{1}$ with $\mathbb{P}_{\mathcal{M}^{\prime}, s, \sigma_{1}}\left(\mathrm{~F} T^{\prime}\right)=1$. Since this holds for all $s \in S$, it follows from Theorem 40 that there is an MD-strategy $\hat{\sigma}$ for $\mathcal{M}^{\prime}$ with $\mathbb{P}_{\mathcal{M}^{\prime}, s, \hat{\sigma}}\left(\mathrm{~F} T^{\prime}\right)=1$ for all $s \in S$. We extend the (so far partially defined) strategy $\sigma^{\prime}$ by $\hat{\sigma}$. Thus we obtain a (fully defined) strategy $\sigma^{\prime}$ for $\mathcal{M}$ such that for all $s \in S$ we have $\mathbb{P}_{\mathcal{M}, s, \sigma^{\prime}}\left(\mathrm{F} T^{\prime}\right)=1$.

It remains to show that $\mathbb{P}_{\mathcal{M}, s, \sigma^{\prime}}(\varphi)=1$ holds for all $s \in S$. To this end we will use Lemma 41 again. We partition the runs of $s S^{\omega}$ into two events $\mathcal{R}_{1}^{\prime \prime}, \mathcal{R}_{2}^{\prime \prime}$ :

- $\mathcal{R}_{1}^{\prime \prime}=\llbracket \operatorname{GFSafe}\left(\frac{2}{3}\right) \rrbracket^{s}$, i.e., $\mathcal{R}_{1}^{\prime \prime}$ contains the runs that visit Safe $\left(\frac{2}{3}\right)$ infinitely often.
- $\mathcal{R}_{2}^{\prime \prime}=\llbracket \mathrm{FG} \neg \operatorname{Safe}\left(\frac{2}{3}\right) \rrbracket^{\text {s }}$, i.e., $\mathcal{R}_{2}^{\prime \prime}$ contains the runs that from some point on never visit $\operatorname{Safe}\left(\frac{2}{3}\right)$.
Recall that $\sigma^{\prime}$ plays like $\sigma_{\varepsilon}$ inside of $\operatorname{Safe}\left(\frac{1}{3}\right)$, that $\operatorname{Safe}\left(\frac{2}{3}\right) \subseteq \operatorname{Safe}\left(\frac{1}{3}\right)$, and that $\sigma_{\varepsilon}$ is an MD-strategy. Thus we can invoke Lemma 15 with $\beta_{2} \stackrel{\text { def }}{=} \frac{2}{3}$ and $\beta_{1} \xlongequal{\text { def }} \frac{1}{3}$ and conclude that every time a run (according to $\sigma^{\prime}$ ) enters $\operatorname{Safe}\left(\frac{2}{3}\right)$, the probability that the run remains in $\operatorname{Safe}\left(\frac{1}{3}\right)$ forever is at least $\frac{1}{2}$. It follows that almost all runs in $\mathcal{R}_{1}^{\prime \prime}$ eventually remain in $\operatorname{Safe}\left(\frac{1}{3}\right)$ forever. That is, $\mathbb{P}_{\mathcal{M}, s, \sigma^{\prime}}\left(\mathcal{R}_{1}^{\prime \prime} \cap \llbracket \operatorname{FGSafe}\left(\frac{1}{3}\right) \rrbracket\right)=\mathbb{P}_{\mathcal{M}, s, \sigma^{\prime}}\left(\mathcal{R}_{1}^{\prime \prime}\right)$. Since $\operatorname{Safe}\left(\frac{1}{3}\right) \subseteq[S]^{\text {Col=0 }}$, we have $\llbracket \mathrm{FGSafe}\left(\frac{1}{3}\right) \rrbracket \subseteq \llbracket \mathrm{FG}[S]^{\text {Col=0 }} \rrbracket \subseteq \llbracket \varphi \rrbracket$. Hence also $\mathbb{P}_{\mathcal{M}, s, \sigma^{\prime}}\left(\mathcal{R}_{1}^{\prime \prime} \cap \llbracket \varphi \rrbracket\right)=\mathbb{P}_{\mathcal{M}, s, \sigma^{\prime}}\left(\mathcal{R}_{1}^{\prime \prime}\right)$.

We have previously shown that $\mathbb{P}_{\mathcal{M}, s, \sigma^{\prime}}\left(\mathrm{F} T^{\prime}\right)=1$ holds for all $s \in S$. Hence also $\mathbb{P}_{\mathcal{M}, s, \sigma^{\prime}}\left(\mathrm{GF} T^{\prime}\right)=1$ holds for all $s \in S$. In particular, almost all runs in $\mathcal{R}_{2}^{\prime \prime}$ satisfy $\mathrm{GF} T^{\prime}$. By comparing the definitions of $\mathcal{R}_{2}^{\prime \prime}$ and $T^{\prime}$ we see that almost all runs in $\mathcal{R}_{2}^{\prime \prime}$ even satisfy $\operatorname{GF}[S]^{\text {Col=2 }}$. Since $\llbracket \mathrm{GF}[S]^{\mathrm{Col=}=2} \rrbracket \subseteq \llbracket \varphi \rrbracket$, we obtain $\mathbb{P}_{\mathcal{M}, s, \sigma^{\prime}}\left(\mathcal{R}_{2}^{\prime \prime} \cap \llbracket \varphi \rrbracket\right)=\mathbb{P}_{\mathcal{M}, s, \sigma^{\prime}}\left(\mathcal{R}_{2}^{\prime \prime}\right)$. A final application of Lemma 41 yields $\mathbb{P}_{\mathcal{M}, s, \sigma^{\prime}}(\varphi)=1$ for all $s \in S$.

We are ready to prove Theorem 22

Proof of Theorem [22. Let $\mathcal{M}=\left(S, S_{\square}, S_{\bigcirc}, \longrightarrow, P\right)$ be an MDP and $\varphi$ a $\{0,1,2\}$-Parity objective. Since $\varphi$ is tail, it is possible to define (see Definition 19) the conditioned version $\mathcal{M}_{*}=\left(S_{*}, S_{* \square}, S_{* \bigcirc}, \longrightarrow_{*}, P_{*}\right)$ of $\mathcal{M}$ wrt. $\varphi$. Assume that in $\mathcal{M}_{*}$ for every safety objective (given by some target $T \subseteq S_{*}$ ) and $\varepsilon>0$ there exists a uniformly $\varepsilon$-optimal MD strategy. Let $S_{\text {opt }}$ be the subset of states that have an optimal strategy for $\varphi$ in $\mathcal{M}$.

By Theorem 20, 1, all states in $\mathcal{M}_{*}$ are almost sure winning for $\varphi$. By our above condition about safety objectives in $\mathcal{M}_{*}$, we can apply Lemma 43 to $\mathcal{M}_{*}$ and obtain that for every state in $\mathcal{M}_{*}$ there is an MD strategy that is almost surely winning for $\varphi$. By Theorem 202, there is an MD strategy in $\mathcal{M}$ that is optimal for $\varphi$ from every state in $S_{o p t}$, as required.

In order to apply Theorem 22 to infinitely-branching acyclic MDPs, we now show that acyclicity guarantees the existence of uniformly $\varepsilon$-optimal MD strategies for safety objectives.

Lemma 23. For every acyclic MDP with a safety objective and every $\varepsilon>0$ there exists an MD strategy that is uniformly $\varepsilon$-optimal.

Proof. Let $\operatorname{Safety}(T)$ be the safety objective and shortly write $\operatorname{val}\left(s_{0}\right)=\operatorname{val}_{\mathcal{M}, \operatorname{Safety}(T)}\left(s_{0}\right)$ for the value of a state $s_{0}$ w.r.t. this objective. Assume w.l.o.g. that the target $T \subseteq S$ is a sink and let $\iota: S \rightarrow \mathbb{N}$ be an enumeration of the state space.

Let $\sigma$ be an MD-strategy that, at any state $s \in S_{\square}$, picks a successor $s^{\prime}$ such that

$$
\operatorname{val}\left(s^{\prime}\right) \geq \operatorname{val}(s)\left(1-\varepsilon 2^{-\iota(s)}\right)
$$

We show that $\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}(\operatorname{Safety}(T)) \geq \operatorname{val}\left(s_{0}\right)(1-\varepsilon)$ holds for every initial state $s_{0}$.
Let's write $\operatorname{Post}^{*}(s) \subseteq S$ for the set of states reachable from state $s \in S$ and define $L(s) \stackrel{\text { def }}{=} \prod_{s^{\prime} \in \text { Post }^{*}(s)}\left(1-\varepsilon 2^{-\iota\left(s^{\prime}\right)}\right)$. Further, let $\mathrm{val}_{s}(n)$ be the random variable denoting the value of the $n$th state of a random run that starts in $s$. In particular, val $l_{s}(0)=\operatorname{val}(s)$. An induction on $n$ using our choice of strategy gives, for every $s_{0} \in S$, that

$$
\begin{equation*}
\mathbb{E}\left(\operatorname{val}_{s_{0}}(n)\right) \geq \operatorname{val}\left(s_{0}\right) L\left(s_{0}\right) \tag{46}
\end{equation*}
$$

Indeed, this trivially holds for $n=0$. For the induction step there are two cases.
Case 1: $s_{0} \in S_{\square}$ and $\sigma\left(s_{0}\right)=s$. Then

$$
\begin{aligned}
& \mathbb{E}\left(\operatorname{val}_{s_{0}}(n+1)\right) \\
& =\mathbb{E}\left(\operatorname{val}_{s}(n)\right) \\
& \geq \operatorname{val}(s) L(s) \\
& \geq \operatorname{val}\left(s_{0}\right)(1-\varepsilon \\
& \geq \operatorname{val}\left(s_{0}\right) L\left(s_{0}\right)
\end{aligned}
$$

$$
\geq \operatorname{val}(s) L(s) \quad \text { ind. hyp. }
$$

$$
\geq \operatorname{val}\left(s_{0}\right)\left(1-\varepsilon 2^{-\iota\left(s_{0}\right)}\right) L(s) \quad \text { def. of } \sigma
$$

Case 2: $s_{0} \in S_{\bigcirc}$. Then

$$
\begin{aligned}
& \mathbb{E}\left(\operatorname{val}_{s_{0}}(n+1)\right) \\
& =\sum_{s_{0} \longrightarrow s} P\left(s_{0}\right)(s) \cdot \mathbb{E}\left(\operatorname{val}_{s}(n)\right) \\
& \geq \sum_{s_{0} \longrightarrow s} P\left(s_{0}\right)(s) \cdot \operatorname{val}(s) L(s) \quad \text { by ind. hyp. } \\
& \geq \sum_{s_{0} \longrightarrow s} P\left(s_{0}\right)(s) \cdot \operatorname{val}(s)\left(1-\varepsilon 2^{-\iota\left(s_{0}\right)}\right) L(s) \\
& \geq \sum_{s_{0} \longrightarrow s} P\left(s_{0}\right)(s) \cdot \operatorname{val}(s) L\left(s_{0}\right) \quad \text { acyclicity; def. of } L\left(s_{0}\right) \\
& =\operatorname{val}\left(s_{0}\right) L\left(s_{0}\right) .
\end{aligned}
$$

Together with the observation that $L\left(s_{0}\right)>(1-\varepsilon)$ for every $s_{0}$, we derive that

$$
\begin{equation*}
\liminf _{n \rightarrow \infty} \mathbb{E}\left(\operatorname{val}_{s_{0}}(n)\right) \geq \operatorname{val}\left(s_{0}\right)(1-\varepsilon) \tag{47}
\end{equation*}
$$

To show the claim, fix $s_{0} \in S$ and shortly write $\mathbb{P}$ for $\mathbb{P}_{\mathcal{M}, s_{0}, \sigma}$ here. Let $\left[\mathrm{X}^{n} \neg T\right]: S^{\omega} \rightarrow$ $\{0,1\}$ be the random variable that indicates that the $n$th state is not in the target set $T$. Note that $\left[\mathrm{X}^{n} \neg T\right] \geq \operatorname{val}_{s_{0}}(n)$ because target states have value 0 . We conclude that

$$
\begin{array}{ll}
\mathbb{P}(\operatorname{Safety}(T)) & \\
=\mathbb{P}\left(\bigcap_{i=0}^{\infty} \llbracket X^{i} \neg T \rrbracket\right) & \\
=\lim _{n \rightarrow \infty} \mathbb{P}\left(\bigcap_{i=0}^{n} \llbracket X^{i} \neg T \rrbracket\right) & \text { semantics of } \operatorname{Safety}(T)=\mathrm{G} \neg T \\
=\lim _{n \rightarrow \infty} \mathbb{P}\left(\llbracket X^{i} \neg T \rrbracket\right) & T \text { is a sink } \\
=\lim _{n \rightarrow \infty} \mathbb{E}\left(\left[\mathrm{X}^{n} \neg T\right]\right) & \text { definition of }\left[\mathrm{X}^{n} \neg T\right] \\
\geq \liminf _{n \rightarrow \infty} \mathbb{E}(\operatorname{val}(n)) & \text { as }\left[\mathrm{X}^{n} \neg T\right] \geq \operatorname{val}_{s_{0}}(n) \\
\geq \operatorname{val}\left(s_{0}\right)(1-\varepsilon) & \text { by Equation 47. }
\end{array}
$$

## G $\varepsilon$-Optimal Strategies for $\{0,1\}$-Parity

- Theorem 25. Suppose that $\mathcal{M}=\left(S, S_{\square}, S_{\bigcirc}, \longrightarrow, P\right)$ is an MDP such that for every safety objective (given by some target $T \subseteq S$ ) and $\varepsilon>0$ there exists a uniformly $\varepsilon$-optimal $M D$ strategy.

Then for every co-Büchi objective (given by some coloring Col :S $\rightarrow\{0,1\}$ ) and $\varepsilon>0$ there exists a uniformly $\varepsilon$-optimal MD strategy.

Proof. Let $\mathcal{M}=\left(S, S_{\square}, S_{\circ}, \longrightarrow, P\right)$ be an MDP such that for all safety objectives, uniformly $\varepsilon$-optimal strategies can be chosen MD. Let $\operatorname{Col}: S \rightarrow\{0,1\}$ be a coloring and $\varphi=$ $\operatorname{Parity}(\mathrm{Col})$ be the resulting co-Büchi objective.

We show that there exist uniformly $\varepsilon$-optimal MD-strategies for $\varphi$. I.e., for every $\varepsilon>0$ there is an MD-strategy $\sigma_{\varepsilon}$ with $\forall_{s_{0} \in S} \mathbb{P}_{\mathcal{M}, s_{0}, \sigma_{\varepsilon}}(\varphi) \geq \operatorname{val}_{\mathcal{M}}\left(s_{0}\right)-\varepsilon$.

To construct this MD-strategy strategy $\sigma_{\varepsilon}$, we first need several auxiliary notions.

Let $\varepsilon_{1}>0$ be a suitably small number (to be determined later) and $\tau_{1} \stackrel{\text { def }}{=} 1-\varepsilon_{1}>0$. Let $\tau_{2} \stackrel{\text { def }}{=} 1-\varepsilon_{1} / k+\lambda$ for a suitably large $k \geq 1$ (to be determined later) and let $\lambda<\varepsilon_{1} / k$ (e.g., $\left.\lambda \stackrel{\text { def }}{=} \varepsilon_{1} /(2 k)\right)$. Thus $\tau_{2}<1$.

Let $T \stackrel{\text { def }}{=} \operatorname{Col}^{-1}(\{1\})$ be the set of states with color 1 and $\operatorname{Safety}(T)$ the safety objective. We have that $\varphi=\mathrm{FG}(S \backslash T)$.

By our assumption on $\mathcal{M}$, there exists a uniformly $\lambda$-optimal MD-strategy $\sigma_{o p t-a v}$ for $\operatorname{Safety}(T)$. Let $S^{\prime} \stackrel{\text { def }}{=} \operatorname{Safe}_{\mathcal{M}, \sigma_{\text {opt-av }}, \operatorname{Safety}(T)}\left(\tau_{1}\right)$ be the set of states where $\sigma_{o p t-a v}$ achieves at least value $\tau_{1}$ for $\operatorname{Safety}(T)$ (refer to Equation (1) for the definition of safe sets). In particular $S^{\prime} \subseteq S \backslash T$, since $\tau_{1}>0$. From $\mathcal{M}$ we obtain a modified MDP $\mathcal{M}^{\prime}$ by fixing all player choices from states in $S^{\prime}$ according to $\sigma_{o p t-a v}$.

We show that the value w.r.t. objective $\varphi$ is only slightly smaller in $\mathcal{M}^{\prime}$, i.e.,

$$
\begin{equation*}
\operatorname{val}_{\mathcal{M}^{\prime}}\left(s_{0}\right) \geq \operatorname{val}_{\mathcal{M}}\left(s_{0}\right)-\varepsilon_{1} \quad \text { for every } s_{0} \in S \tag{48}
\end{equation*}
$$

Let $s_{0} \in S$. By definition of the value $\operatorname{val}_{\mathcal{M}}\left(s_{0}\right)$, for every $\delta>0$ there exists a strategy $\sigma_{\delta}$ in $\mathcal{M}$ from $s_{0}$ s.t. $\mathbb{P}_{\mathcal{M}, s_{0}, \sigma_{\delta}}(\varphi) \geq \operatorname{val}_{\mathcal{M}}\left(s_{0}\right)-\delta$. We define a strategy $\sigma_{\delta}^{\prime}$ in $\mathcal{M}^{\prime}$ from state $s_{0}$ as follows. First play like $\sigma_{\delta}$. If and when a state in $S^{\prime}$ is reached, then henceforth play like $\sigma_{\text {opt-av }}$. This is possible, since no moves from states outside $S^{\prime}$ have been fixed in $\mathcal{M}^{\prime}$, and all moves from states inside $S^{\prime}$ have been fixed according to $\sigma_{o p t-a v}$. Then we have:

$$
\begin{aligned}
& \mathbb{P}_{\mathcal{M}^{\prime}, s_{0}, \sigma_{\delta}^{\prime}}(\varphi) \\
& =\mathbb{P}_{\mathcal{M}, s_{0}, \sigma_{\delta}}(\varphi) \\
& \quad-\mathbb{P}_{\mathcal{M}, s_{0}, \sigma_{\delta}}\left(\mathrm{F} S^{\prime}\right) \cdot \mathbb{P}_{\mathcal{M}, s_{0}, \sigma_{\delta}}\left(\varphi \mid \mathrm{F} S^{\prime}\right) \\
& \quad+\mathbb{P}_{\mathcal{M}, s_{0}, \sigma_{\delta}}\left(\mathrm{F} S^{\prime}\right) \cdot \mathbb{P}_{\mathcal{M}^{\prime}, s_{0}, \sigma_{\delta}^{\prime}}\left(\varphi \mid \mathrm{F} S^{\prime}\right) \\
& \geq \mathbb{P}_{\mathcal{M}, s_{0}, \sigma_{\delta}}(\varphi) \\
& \quad-\mathbb{P}_{\mathcal{M}, s_{0}, \sigma_{\delta}}\left(\mathrm{F} S^{\prime}\right) \cdot \mathbb{P}_{\mathcal{M}, s_{0}, \sigma_{\delta}}\left(\varphi \mid \mathrm{F} S^{\prime}\right) \\
& \quad+\mathbb{P}_{\mathcal{M}, s_{0}, \sigma_{\delta}}\left(\mathrm{F} S^{\prime}\right) \cdot \tau_{1} \\
& \geq \operatorname{val}_{\mathcal{M}}\left(s_{0}\right)-\delta-\mathbb{P}_{\mathcal{M}, s_{0}, \sigma_{\delta}}\left(\mathrm{F} S^{\prime}\right)\left(1-\tau_{1}\right) \\
& \geq \operatorname{val}_{\mathcal{M}}\left(s_{0}\right)-\delta-\varepsilon_{1}
\end{aligned}
$$

Since this holds for every $\delta>0$ we have $\operatorname{val}_{\mathcal{M}^{\prime}}\left(s_{0}\right) \geq \operatorname{val}_{\mathcal{M}}\left(s_{0}\right)-\varepsilon_{1}$, thus 48.
Let $S^{\prime \prime} \stackrel{\text { def }}{=} S a f e_{\mathcal{M}, \sigma_{o p t-a v}, \operatorname{Safety}(T)}\left(\tau_{2}-\lambda\right)$ as by definition of safe sets in Equation 11. In particular, $S^{\prime \prime}=S a f e_{\mathcal{M}^{\prime}, \sigma_{\text {opt-av }}, \operatorname{Safety}(T)}\left(\tau_{2}-\lambda\right)$, since $\sigma_{\text {opt-av }}$ is a uniform MD-strategy that has been fixed on the subset $S^{\prime}$ in the step from $\mathcal{M}$ to $\mathcal{M}^{\prime}$.

Let $s_{0} \in S$ be an arbitrary state. By definition of the value w.r.t. $\varphi$ of $s_{0}$, for every $\varepsilon^{\prime}>0$ there exists a strategy $\sigma_{\varepsilon^{\prime}}$ from $s_{0}$ in $\mathcal{M}^{\prime}$ with $\mathbb{P}_{\mathcal{M}^{\prime}, s_{0}, \sigma_{\varepsilon^{\prime}}}(\varphi) \geq \operatorname{val}_{\mathcal{M}^{\prime}}\left(s_{0}\right)-\varepsilon^{\prime}$.

Since $\tau_{2}<1$, can we apply Lemma 42 and obtain $\mathbb{P}_{\mathcal{M}^{\prime}, s_{0}, \hat{\sigma}}\left(\operatorname{FSafe} \mathcal{M}_{\mathcal{M}^{\prime}, \operatorname{Safety}(T)}\left(\tau_{2}\right)\right) \geq$ $\mathbb{P}_{\mathcal{M}^{\prime}, s_{0}, \hat{\sigma}}(\varphi)$ for every strategy $\hat{\sigma}$ from $s_{0}$ and thus in particular for $\sigma_{\varepsilon^{\prime}}$.

Therefore, $\mathbb{P}_{\mathcal{M}^{\prime}, s_{0}, \sigma_{\varepsilon^{\prime}}}\left(\operatorname{FSafe}_{\mathcal{M}^{\prime}, \operatorname{Safety}(T)}\left(\tau_{2}\right)\right) \geq \operatorname{val}_{\mathcal{M}^{\prime}}\left(s_{0}\right)-\varepsilon^{\prime}$. Since this holds for every $\varepsilon^{\prime}>0$, in $\mathcal{M}^{\prime}$ the value of $s_{0}$ w.r.t. the reachability objective $\operatorname{FSaf} \operatorname{SM}_{\mathcal{M}^{\prime}, \operatorname{Safety}(T)}\left(\tau_{2}\right)$ is $\geq \operatorname{val}_{\mathcal{M}^{\prime}}\left(s_{0}\right)$ for every state $s_{0}$.

By Theorem 40 for every $\varepsilon_{2}>0$ there exists a uniformly $\varepsilon_{2}$-optimal MD-strategy $\sigma^{\prime}$ in $\mathcal{M}^{\prime}$ for this reachability objective. So we obtain

$$
\begin{equation*}
\mathbb{P}_{\mathcal{M}^{\prime}, s_{0}, \sigma^{\prime}}\left(\operatorname{FSafe}_{\mathcal{M}^{\prime}, \operatorname{Safety}(T)}\left(\tau_{2}\right)\right) \geq \operatorname{val}_{\mathcal{M}^{\prime}}\left(s_{0}\right)-\varepsilon_{2} \quad \text { for every state } s_{0} \tag{49}
\end{equation*}
$$

In particular, $\sigma^{\prime}$ must coincide with $\sigma_{\text {opt-av }}$ at all states in $S^{\prime}$, since in $\mathcal{M}^{\prime}$ these choices are already fixed.

Since $\sigma$ is a uniformly $\lambda$-optimal MD-strategy for $\operatorname{Safety}(T)$ in $\mathcal{M}$ and $\mathcal{M}^{\prime}$, we have $\operatorname{Safe}_{\mathcal{M}^{\prime}, \text { Safety }(T)}\left(\tau_{2}\right) \subseteq \operatorname{Safe}_{\mathcal{M}^{\prime}, \sigma, \operatorname{Safety}(T)}\left(\tau_{2}-\lambda\right)=S^{\prime \prime}$ and thus by 49 we get

$$
\begin{equation*}
\mathbb{P}_{\mathcal{M}^{\prime}, s_{0}, \sigma^{\prime}}\left(\mathrm{F} S^{\prime \prime}\right) \geq \mathbb{P}_{\mathcal{M}^{\prime}, s_{0}, \sigma^{\prime}}\left(\mathrm{FSafe}_{\mathcal{M}^{\prime}, \operatorname{Safety}(T)}\left(\tau_{2}\right)\right) \geq \operatorname{val}_{\mathcal{M}^{\prime}}\left(s_{0}\right)-\varepsilon_{2} \tag{50}
\end{equation*}
$$

We obtain the MD-strategy $\sigma_{\varepsilon}$ in $\mathcal{M}$ by combining the MD-strategies $\sigma^{\prime}$ and $\sigma_{\text {opt-av }}$. The strategy $\sigma_{\varepsilon}$ plays like $\sigma_{o p t-a v}$ at all states inside $S^{\prime}$ and like $\sigma^{\prime}$ at all states outside $S^{\prime}$ (i.e., at $\left.S \backslash S^{\prime}\right)$.

In order to show that $\sigma_{\varepsilon}$ has the required property $\mathbb{P}_{\mathcal{M}, s_{0}, \sigma_{\varepsilon}}(\varphi) \geq \operatorname{val}_{\mathcal{M}}\left(s_{0}\right)-\varepsilon$, we first estimate the probability that a play according to $\sigma_{\varepsilon}$ will never leave the set $S^{\prime}$ after having visited a state in $S^{\prime \prime}$.

Let $s \in S^{\prime \prime}$. Then, by applying Lemma 15 to the Markov chain obtained from applying the MD-strategy $\sigma_{\varepsilon}$ to $\mathcal{M}$, we obtain

$$
\begin{align*}
\mathbb{P}_{\mathcal{M}, s, \sigma_{\text {opt-av }}}\left(\mathrm{G} S^{\prime}\right) & \geq \frac{\left(\tau_{2}-\lambda\right)-\tau_{1}}{1-\tau_{1}} \\
& =\frac{\left(1-\varepsilon_{1} / k\right)-\left(1-\varepsilon_{1}\right)}{\varepsilon_{1}}  \tag{51}\\
& =1-\frac{1}{k} .
\end{align*}
$$

In particular we also have $\mathbb{P}_{\mathcal{M}, s, \sigma_{\varepsilon}}\left(\mathrm{G} S^{\prime}\right) \geq 1-\frac{1}{k}$, since $\sigma_{\varepsilon}$ coincides with $\sigma_{o p t-a v}$ inside the set $S^{\prime}$. Finally we obtain for every $s_{0} \in S$

$$
\begin{aligned}
\mathbb{P}_{\mathcal{M}, s_{0}, \sigma_{\varepsilon}}(\varphi) & =\mathbb{P}_{\mathcal{M}, s_{0}, \sigma_{\varepsilon}}(\mathrm{FG}(S \backslash T)) & & \text { by def. of } \varphi \\
& \geq \mathbb{P}_{\mathcal{M}, s_{0}, \sigma_{\varepsilon}}\left(\mathrm{F} S^{\prime \prime}\right) \cdot \mathbb{P}_{\mathcal{M}, s_{0}, \sigma_{\varepsilon}}\left(\mathrm{FG} S^{\prime} \mid \mathrm{F} S^{\prime \prime}\right) & & \text { since } S^{\prime} \subseteq S \backslash T \\
& \geq \mathbb{P}_{\mathcal{M}^{\prime}, s_{0}, \sigma^{\prime}}\left(\mathrm{F} S^{\prime \prime}\right) \cdot(1-1 / k) & & \text { by } 51 \\
& \geq\left(\operatorname{val}_{\mathcal{M}^{\prime}}\left(s_{0}\right)-\varepsilon_{2}\right) \cdot(1-1 / k) & & \text { by } 50 \\
& \geq\left(\operatorname{val}_{\mathcal{M}}\left(s_{0}\right)-\varepsilon_{1}-\varepsilon_{2}\right) \cdot(1-1 / k) & & \text { by } 48
\end{aligned}
$$

This holds for every $1>\varepsilon_{1}, \varepsilon_{2}>0$ and every $k \geq 1$, and moreover $\operatorname{val}_{\mathcal{M}}\left(s_{0}\right) \leq 1$. Thus we can set $\varepsilon_{1}=\varepsilon_{2} \stackrel{\text { def }}{=} \varepsilon / 4$ and $k \stackrel{\text { def }}{=} \frac{2}{\varepsilon}$ and obtain $\mathbb{P}_{\mathcal{M}, s_{0}, \sigma_{\varepsilon}}(\varphi) \geq \operatorname{val}_{\mathcal{M}}\left(s_{0}\right)-\varepsilon$ for every $s_{0} \in S$ as required.


[^0]:    ${ }^{1}$ By Definition 6. either the current state or the next state allows to switch between layers; cf. the proof of Claim 14
    ${ }^{2}$ Remember that 21 implies that the sets $\mathrm{BETA}_{j}$ are disjoint.

