# ESCAPING SETS ARE NOT SIGMA-COMPACT

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Abstract. Let $f$ be a transcendental entire function. The *escaping set* $I(f)$ consists of those points that tend to infinity under iteration of $f$. We show that $I(f)$ is not $\sigma$-compact, resolving a question of Rippon from 2009.

## 1. Introduction

Let $f\colon \mathbb{C}\to\mathbb{C}$ be a transcendental entire function. The set

$$I(f) := \{z\in\mathbb{C}\colon f^n(z)\to\infty\}$$

is called the *escaping set* of $f$, where

$$f^n = \underbrace{f\circ\cdots\circ f}_{n \text{ times}}$$

denotes the $n$-th iterate of $f$. The escaping set was first studied by Eremenko [Ere89] and has been the subject of intensive research in recent years; see e.g. [Obe09, RRRS11, Ber18, RS19] and their references. By definition,

$$\begin{aligned} I(f) &= \{z\in\mathbb{C}\colon \forall M\geq 0\,\exists N\geq 0\colon |f^n(z)|\geq M \text{ for } n\geq N\}\\ &= \bigcap_{M\geq 0}\bigcup_{N\geq 0}\bigcap_{n=N}^{\infty} f^{-n}(\mathbb{C}\setminus D(0,M)),\end{aligned}$$

where $D(0,M)$ denotes the disc of radius $M$ around 0. So $I(f)$ is $F_{\sigma\delta}$, i.e., a countable intersection of countable unions of closed sets. Moreover, it is well-known that $I(f)$ cannot be a $G_\delta$ (a countable intersection of open sets); see Lemma 2.6 below.

This raises the question, posed by Rippon in 2009 [Obe09, Problem 8, p. 2960], whether $I(f)$ is ever $F_\sigma$. In other words, can $I(f)$ be $\sigma$-compact? (See also [Lip20b].) We answer this question in the negative.

Let $\mathrm{UO}(f)$ consist of all $z\in\mathbb{C}$ whose orbit $\{f^n(z)\colon n\geq 0\}$ is unbounded, and let $\mathrm{BU}(f)=\mathrm{UO}(f)\setminus I(f)$ denote the "bungee set"; see [OS16].

**1.1. Theorem.** *Let $f$ be a transcendental entire function. Then every $\sigma$-compact subset of $\mathrm{UO}(f)$ omits some point of $I(f)\cap J(f)$ and some point of $\mathrm{BU}(f)\cap J(f)$.*

*In particular, the sets $I(f)$, $\mathrm{UO}(f)$, $\mathrm{BU}(f)$ and their intersections with $J(f)$ are not $\sigma$-compact.*

**Acknowledgements.** I thank David Lipham and Phil Rippon for interesting discussions.

*Date* July 3, 2020.
2010 *Mathematics Subject Classification.* Primary 30D05, Secondary 37F10, 54D45.

## 2. Proof of the Theorem

We require a result on the existence of arbitrarily slowly escaping points, which follows from work of Rippon and Stallard [RS15]; see also [RS11, Theorem 1].

**2.1. Theorem.** *Let $f$ be a transcendental entire function, and let $R_0 \geq 0$. Then there exists $M_0 > R_0$ with the following property. If $(a_n)_{n=0}^{\infty}$ is a sequence with $a_n \to \infty$ and $a_n \geq M_0$ for all $n$, then there are $\zeta \in J(f) \cap I(f)$ and $\omega \in J(f) \cap \mathrm{BU}(f)$ with $|\zeta|, |\omega| \geq R_0$ such that $|f^n(\zeta)|, |f^n(\omega)| \leq a_n$ for all $n \geq 0$.*

*Proof.* The result is an easy consequence of [RS15, Theorem 1.2]. Indeed, in the proof of Corollary 1.3 (d) of that paper, the authors use this theorem to construct a point $\zeta \in I(f) \cap J(f)$ having an "annular itinerary" $(s_n)_{n=0}^{\infty}$ chosen such that $|f^n(\zeta)| \leq a_n$ for $n \geq N_1$ (using the notation in [RS15]). We may assume without loss of generality that $n_1 \geq 2$, which means that we can additionally ensure $s_n \geq 1$ for all $n$, and $s_n \leq n_1$ for $n < N_1$. Then we have $R \leq |f^n(\zeta)|$ for all $n$, and $|f^n(\zeta)| \leq M^{n_1}(R)$ for $n < N_1$. (Here $M(R) = M(R, f)$ is the maximum modulus function.) The point $\zeta$ has the desired properties if we choose $R \geq \max(R_0, R(f))$ in [RS15, Theorem 1.2] and set $M_0 := M^{n_1}(R)$. To obtain $\omega$, we instead use an unbounded sequence $(\tilde{s}_n)$ with $\tilde{s}_n \leq s_n$ such that $\tilde{s}_n \in \{1, 2\}$ for infinitely many $n$. ∎

*Proof of Theorem 1.1.* If $D \subset \mathbb{C}$ is closed and $z \in \mathbb{C}$, we set

$$n_D(z) := \min\{n \geq 0 \colon f^n(z) \notin D\} \leq \infty,$$

with the convention that $n_D(z) = \infty$ if no such $n$ exists. Since $D$ is closed, and $f$ is continuous, $n_D$ depends upper semicontinuously on $z$. Note that $z \in \mathrm{UO}(f)$ if and only if $n_D(z) < \infty$ for every compact $D \subset \mathbb{C}$.

Now let $R_0 \geq 0$ be arbitrary, and choose $M_0$ as in Theorem 2.1. Let $X \subset \mathrm{UO}(f)$ be $\sigma$-compact; say $X = \bigcup_{j=0}^{\infty} K_j$ where each $K_j$ is compact. Define $M_j := M_0 + j$ and $D_j := \overline{D(0, M_j)}$. Then $n_{D_j}(z) < \infty$ for every $z \in K_j$.

Since $K_j$ is compact and $n_{D_j}$ is upper semicontinuous,

$$n_j := \max_{z \in K_j} n_{D_j}(z)$$

exists for every $j$. Set $N_0 := n_0$ and $N_{j+1} := \max(n_{j+1}, N_j + 1)$. For $n \geq 0$, define $j(n) := \min\{j \colon N_j \geq n\}$ and

$$a_n := M_{j(n)} = \min\{M_j \colon N_j \geq n\}.$$

Then $a_n \geq M_0$ for all $n$ and $\lim_{n\to\infty} a_n = \infty$. So by Theorem 2.1, there are $\zeta \in I(f) \cap J(f)$ and $\omega \in \mathrm{BU}(f) \cap J(f)$ such that $|f^n(z)| \leq a_n$ for $z \in \{\zeta, \omega\}$ and all $n \geq 0$.

Let $j \geq 0$. Then, for $n \leq n_j \leq N_j$, we have $|f^n(z)| \leq a_n \leq M_j$, and hence $f^n(z) \in D_j$. So $n_{D_j}(z) > n_j$, and $z \notin K_j$ by choice of $n_j$. Thus $z \notin X = \bigcup_{j=0}^{\infty} K_j$, as claimed. ∎

By the expanding property of the Julia set, we obtain the following, which answers a question of Lipham (personal communication).

**2.2. Corollary.** *Let $f$ be a transcendental entire function and let $Y$ be one of the sets $I(f) \cap J(f)$, $\mathrm{UO}(f) \cap J(f)$ and $\mathrm{BU}(f) \cap J(f)$. Then $Y$ is nowhere $\sigma$-compact. That is, if $X \subset Y$ is $\sigma$-compact, then $X$ is nowhere dense in $Y$.*

*Proof.* Let $X \subset \mathrm{UO}(f) \cap J(f)$ and assume that there is an open set $U \subset \mathbb{C}$ with $\emptyset \neq U \cap Y \subset X$. We will show that $X$ is not $\sigma$-compact.

Let $R_0$ and $M_0$ be as in Theorem 2.1, and set $L_0 := J(f) \cap \overline{D(0, M_0)} \setminus D(0, R_0)$. If $R_0$ is chosen sufficiently large, then $L_0$ contains no Fatou exceptional point of $f$. Hence there is some $n$ such that $f^n(U) \supset L_0$, and thus $f^n(X) \supset L_0 \cap Y$.

The proof of Theorem 1.1 shows that no $\sigma$-compact subset of $\mathrm{UO}(f) \cap J(f)$ contains $L_0 \cap Y$. (Note that we may take $a_0 = M_0$ in the proof to ensure that $|\zeta|, |\omega| \leq M_0$.) So $f^n(X)$ is not $\sigma$-compact. The image of a $\sigma$-compact set under a continuous function is again $\sigma$-compact, and the claim follows. ■

**2.3. Remark.** We may restate our proof of Theorem 1.1 in a very general setting, as follows.

Let $f\colon U \to V$ be continuous, where $U$ and $V$ are topological spaces and $U$ is $\sigma$-compact. Let $\mathrm{UO}(f)$ denote the set of $z \in U$ such that $f^n(z) \in U$ for all $n \geq 0$, but the orbit $\{f^n(z)\}$ is not contained in any compact subset of $U$.

Suppose that $X \subset \mathrm{UO}(f)$ is $\sigma$-compact and $D \subset U$ is compact. Then there is a non-decreasing sequence $(D_n)_{n=0}^{\infty}$ of compact subsets $D_n \subset U$ with $D \subset D_0$ and $\bigcup_{n=0}^{\infty} D_n = U$ such that $X$ contains no point $\zeta$ with $f^n(\zeta) \in D_n$ for all $n \geq 0$.

**2.4. Remark.** By Remark 2.3, the set of escaping points is not $\sigma$-compact in any setting where an analogue of Theorem 2.1 holds. This includes:

(a) transcendental meromorphic functions [RS11];
(b) transcendental self-maps of the punctured plane [MP18, Theorem 1.2];
(c) quasiregular self-maps $f\colon \mathbb{R}^d \to \mathbb{R}^d$ of transcendental type [Nic16];
(d) continuous functions $\varphi\colon [0, \infty)$ with $\varphi(t) \not\to \infty$ as $t \to \infty$, and such that $I(\varphi) \neq \emptyset$ [ORS19, Theorem 2.2].

**2.5. Remark.** A more general setting than both (a) and (b) in Remark 2.4 is provided by the *Ahlfors islands maps* of Epstein; see e.g. [RR12]. These are maps $f\colon W \to X$, where $X$ is a compact one-dimensional manifold, $W \subset X$ is open, and $f$ satisfies certain transcendence conditions near $\partial W$. We may define the escaping set $I(f)$ as the set of points $z \in W$ with $f^n(z) \in W$ for all $n$ and $\mathrm{dist}(f^n(z), \partial W) \to 0$. The same proof as in [RS11] should establish an analogue of Theorem 2.1 for transcendental Ahlfors islands maps, and hence show that $I(f)$ is not $\sigma$-compact in this general setting.

For completeness, we conclude by giving the simple proof that $I(f)$ is never a $G_\delta$ set; compare also [Lip20a, Corollary 3.2].

**2.6. Lemma.** *Let $f$ be a transcendental entire function. Then $I(f)$ and $I(f) \cap J(f)$ are not $G_\delta$ sets.*

*Proof.* $J(f)$ is closed, and hence $G_\delta$. The intersection of two $G_\delta$ sets is again $G_\delta$, so it is enough to prove the claim for $I(f) \cap J(f)$. By [Ere89, Theorem 2], $I(f) \cap J(f)$ is nonempty, and hence dense in $J(f)$ by Montel's theorem. By Baire's theorem, any two dense $G_\delta$ subsets of $J(f)$ must intersect. Hence it is enough to observe that $\mathrm{BU}(f) \cap J(f)$ contains a dense $G_\delta$ by Montel's theorem, namely the set of points whose orbits are dense in $J(f)$. (See [BD00, Lemma 1].) ■

**2.7. Remark.** Lipham has pointed out the following reformulation of Corollary 2.2: Any $G_\delta$ set $A \subset Y := J(f) \setminus I(f)$ is nowhere dense in $Y$. Otherwise, there are $\zeta \in J(f)$ and $\varepsilon > 0$ such that $\overline{D(\zeta,\varepsilon)} \cap Y \subset A$; but then $B := J(f) \cap \overline{D(\zeta,\varepsilon)} \setminus A$ is an $F_\sigma$ set with $D(\zeta,\varepsilon) \cap I(f) \cap J(f) \subset B \subset I(f) \cap J(f)$, which contradicts Corollary 2.2.

Since $Y$ has second category in $J(f)$, as seen in the proof of Lemma 2.6, we conclude that $Y$ is not *strongly $\sigma$-complete*; that is, it cannot be written as a countable union of *relatively closed* $G_\delta$ subsets. (On the other hand, $Y$ is clearly $G_{\delta\sigma}$ by definition.)

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