

EVENTUAL HYPERBOLIC DIMENSION OF ENTIRE FUNCTIONS AND POINCARÉ FUNCTIONS OF POLYNOMIALS

ALEXANDRE DEZOTTI AND LASSE REMPE-GILLEN

ABSTRACT. Let $P: \mathbb{C} \rightarrow \mathbb{C}$ be an entire function. A *Poincaré function* $L: \mathbb{C} \rightarrow \mathbb{C}$ of P is the entire extension of a linearising coordinate near a repelling fixed point of P . We propose such Poincaré functions as a rich and natural class of dynamical systems from the point of view of measurable dynamics, showing that the measurable dynamics of P influences that of L . More precisely, the *hyperbolic dimension* $\dim_{\text{hyp}}(P)$ is a lower bound for $\dim_{\text{hyp}}(L)$.

Our results allow us to describe a large collection of hyperbolic entire functions having full hyperbolic dimension, and hence no natural invariant measures. (The existence of such examples was only recently established, using very different and much less direct methods.) We also give a negative answer to a natural question concerning the behaviour of *eventual dimensions* under quasiconformal equivalence.

1. INTRODUCTION

This article is related to the *measurable dynamics* of transcendental entire functions of one complex variable. The goal of measurable dynamics is to understand the statistically typical behaviour of a dynamical system, which means finding natural invariant measures that describe the average behaviour of the system in question. In the case of functions of one real variable, the desired measures will often be absolutely continuous with respect to Lebesgue measure so that they represent “typical behaviour” in the usual sense; compare e.g. [Lyu00]. For *complex* one-dimensional systems, this approach must be modified. Indeed, the locus where the interesting dynamics of a complex polynomial f takes place, known as the *Julia set* $J(f)$, frequently has zero Lebesgue measure and even Hausdorff dimension less than two. Sullivan [Sul82, Sul83] proposed the solution of finding natural geometric measures known as *conformal measures* to replace Lebesgue measure, and then constructing invariant measures that are absolutely continuous with respect to these.

In the following decades, a very clear picture of this theory has emerged for rational functions. In order to provide meaningful information about the dynamics, a conformal measure should be supported on the locus of non-uniform hyperbolicity, called the *radial Julia set* $J_r(f)$ (Definition 2.1). *Topological Collet-Eckmann maps* (Definition 2.6) provide a large class of functions for which $\dim(J_r(f)) = \dim(J(f))$ [Prz98], and for which the desired conformal measures and invariant measures exist [PRL07]. (Here \dim denotes Hausdorff dimension.) Indeed, if one requires additional mixing properties for the invariant measure,

The second author was supported by a Philip Leverhulme Prize.

then their existence becomes equivalent to the topological Collet-Eckmann condition; it is plausible that such maps have full measure in the space of rational functions or polynomials of a given degree. Remarkably, they include all *hyperbolic* functions.

More recently, the dynamics of transcendental entire functions $f: \mathbb{C} \rightarrow \mathbb{C}$ has received considerable attention; significant difficulties arise from the non-compactness of the phase space and the nature of the transcendental singularity at ∞ . Early results concerning the measurable dynamics of such a function were mainly negative and highlighted differences to the rational case, such as the existence of hyperbolic examples whose Julia sets have empty interior and positive measure [McM87] or the fact that the complex exponential map is not recurrent [Lyu86, Ree86] and indeed not ergodic [Lyu87]. These examples suggested that the measurable theory as developed in the rational case breaks down completely for transcendental entire functions. However, subsequently it was realised that the radial Julia set of such a function often has strictly smaller dimension than the full Julia set – a phenomenon unknown in rational dynamics until very recently, see below. Stallard [Sta99, Theorem C] was the first to notice that this may occur; in a breakthrough Urbański and Zdunik later observed that this holds for every hyperbolic exponential map f [UZ03] and, crucially, constructed conformal and invariant measures supported on $J_r(f)$. (See Definition 2.8 for the definition of hyperbolic entire functions.) This raised the question of whether there might not be a general theory of measurable transcendental dynamics after all, in analogy to in the rational case. This problem is not only interesting in its own right, but receives additional relevance through recently announced results of Avila and Lyubich [AL15], who gave the first examples of rational maps for which the radial Julia set has smaller Hausdorff dimension than the full Julia set. The similarity between this result and the above-mentioned transcendental phenomena suggests that a good understanding of measurable dynamics for transcendental functions will also lead to insights into the most intricate aspects of the polynomial theory.

Since the seminal work of Urbański and Zdunik, there has been much work on generalising and extending their results. We refer to the survey [KU08] for background and results, and mention here only work by Mayer and Urbański [MU08], who treat a large class of hyperbolic transcendental meromorphic functions satisfying a certain strong regularity condition (see below – this condition is satisfied, in particular, for many maps given by explicit formulae, such as exponential and trigonometric functions). It is natural to ask whether these results might extend to *all* hyperbolic entire functions, or at least those having finite order of growth. It turns out that this is not the case: there is a finite-order hyperbolic entire function f for which $J_r(f)$ has Hausdorff dimension 2 and therefore cannot support a conformal measure [Rem14].

Despite much progress, the area suffered somewhat from a lack of good examples beyond the explicit families covered by [MU08]. (We remark that the function in [Rem14] is constructed in a very non-explicit and rather artificial manner.) We propose to address this issue by studying the geometric properties of *Poincaré functions* of polynomials. In particular, we show (Corollary 1.3) that there is a large collection of hyperbolic entire functions f of finite order satisfying $\dim(J_r(f)) = 2$. This provides an alternative (and

simpler) proof of the main result of [Rem14]. We also answer a natural question arising from work of Stallard and the second author [RS10] (Corollary 1.7).

Poincaré functions. Let f be an entire function, and let $\xi_0 \in \mathbb{C}$ be a fixed point of f . Suppose that this fixed point is repelling; i.e., its *multiplier* $\rho = f'(\xi_0)$ has modulus greater than 1. Then, by a classical theorem of Koenigs (see e.g. [Bea91, Section 6.3]), there exists a conformal map L , defined near zero, such that $L(0) = \xi_0$ and L satisfies

$$(1.1) \quad L(\rho z) = f(L(z)).$$

Using the functional equation (1.1), this linearising function L extends to an entire function $L: \mathbb{C} \rightarrow \mathbb{C}$ satisfying (1.1) everywhere, which is called a *Poincaré function* for f .

The functional relation (1.1) leads to close connections between the dynamics of f and the function-theoretic properties of L . (As pointed out by Eremenko and Sodin[ES90], this idea can be found already in the work of Julia, Fatou and Lattès). Furthermore, the function L can be effectively computed using the functional relation; hence it is amenable to computer experiments. As far as we are aware, the idea of studying the *dynamics* of Poincaré functions – as examples that are tangible yet quite different from the usually studied families – was first proposed by Epstein in the 1990s; see [MBP12] for another instance of this approach. As we shall show, the measurable dynamics of the function f leaves an imprint on the dynamics of L *near infinity*, making these maps excellent test cases for the above-mentioned questions. (Such a connection was conjectured in [ER15, Section 7].) To make the preceding statement precise, we shall change our point of view slightly, from non-uniformly to *uniformly* hyperbolic behaviour.

Hyperbolic sets and (eventual) hyperbolic dimension. A *hyperbolic set* $K \subset \mathbb{C}$ of an entire function f is a compact and forward-invariant set on which the function is uniformly expanding; i.e., $\left| (f^n)'(z) \right| \geq \lambda$ for some $n \geq 1$, some $\lambda > 1$, and all $z \in K$. The *hyperbolic dimension* of f measures the size of such uniformly hyperbolic behaviour in geometric terms:

$$\dim_{\text{hyp}}(f) := \sup \{ \dim(K) : K \text{ is a hyperbolic set for } f \}.$$

It is known that always $\dim_{\text{hyp}}(f) = \dim(J_r(f))$ [Rem09a]; hence this quantity plays an important role in measurable dynamics. As mentioned above, the difficulty in studying transcendental functions (when compared with rational ones) arises from the presence of a transcendental singularity at infinity. The following notion measures the “limiting” properties of hyperbolic dimension as one approaches this singularity.

1.1. Definition (Eventual hyperbolic dimension).

Let f be an entire function. The *eventual hyperbolic dimension* of the mapping f , denoted by $\text{edim}_{\text{hyp}}(f)$, is

$$\text{edim}_{\text{hyp}}(f) := \lim_{R \rightarrow \infty} \sup \left\{ \dim(K) : K \text{ is a hyperbolic set for } f \text{ with } \inf_{z \in K} |z| \geq R \right\}.$$

We will mainly be interested in the case of entire functions f that belong to the *Eremenko-Lyubich class* \mathcal{B} of functions having a bounded set of critical and asymptotic values. In this case Barański, Karpińska and Zdunik [BKZ09] proved that always $\dim_{\text{hyp}}(f) > 1$; their proof shows also that $\text{edim}_{\text{hyp}}(f) \geq 1$.

The notion of eventual hyperbolic dimension plays a role – albeit implicitly – already in the work of Urbański and Zdunik on exponential maps. Indeed, a key step in their construction of conformal measures is to show that that certain measures do not give much mass to points close to the essential singularity. Their arguments imply that, for exponential maps, $\text{edim}_{\text{hyp}}(f) = 1$. In essence, this is what allows one to disregard behaviour near infinity. More generally, any class \mathcal{B} entire function covered by the results of Mayer and Urbański also satisfies $\text{edim}_{\text{hyp}}(f) = 1$ (see Proposition 3.6).

Measurable dynamics of linearisers. The following result relates the eventual hyperbolic dimension of a Poincaré function to the hyperbolic dimension of the original function.

1.2. Theorem (Linearisers with $\dim_{\text{hyp}} < 2$).

Let $P: \mathbb{C} \rightarrow \mathbb{C}$ be a polynomial of degree at least 2 with connected Julia set satisfying the topological Collet-Eckmann condition, and let L be a Poincaré function associated to a repelling fixed point of P . Then

$$1 \leq \dim_{\text{hyp}}(P) = \text{edim}_{\text{hyp}}(L) < 2.$$

In particular, if P is not conformally conjugate to a power map or a Chebyshev polynomial¹, then $\text{edim}_{\text{hyp}}(L) > 1$.

If additionally L is of disjoint type; that is, hyperbolic with connected Fatou set, then $\text{edim}_{\text{hyp}}(L) \leq \dim_{\text{hyp}}(L) < 2$.

Observe that the part on the dimension being greater than 1 follows from the known fact that $\dim_{\text{hyp}}(P) > 1$ for such polynomials [Zdu90]. This provides many examples of functions having eventual hyperbolic dimension strictly greater than one. Some of these functions do not satisfy the conditions of [MU08], and provide an interesting class of examples for the further study of measurable transcendental dynamics.

In general we still have some relation between the eventual hyperbolic dimension of a linearizer and the polynomial. If $P: \mathbb{C} \rightarrow \mathbb{C}$ is a polynomial of degree at least 2 and L is a Poincaré function associated to a repelling fixed point of P then

$$(1.2) \quad \text{edim}_{\text{hyp}}(L) \geq \dim_{\text{hyp}}(P).$$

This is a consequence of Theorem 1.4 below.

On the other hand, by Shishikura's famous results [Shi98], there is a residual subset of the boundary of the Mandelbrot set where all maps have hyperbolic dimension two. Indeed, by [McM00, Section 12], this is true in any non-trivial bifurcation locus of polynomials (and rational maps). By linearising such functions, we hence obtain many examples of entire functions having hyperbolic dimension two.

¹If P is conformally conjugate to a power map or a Chebyshev polynomial, then L agrees with the exponential or the cosine function, respectively, up to pre- and post-composition with affine maps.

1.3. Corollary (Linearisers with $\dim_{\text{hyp}}(L) = 2$).

There exists a residual subset R of the boundary of the Mandelbrot set with the following property. For any $c \in R$, the Poincaré function L associated to a repelling periodic point of the polynomial $z \mapsto z^2 + c$ belongs to class \mathcal{B} , has finite order and $\dim_{\text{hyp}}(L) = \text{edim}_{\text{hyp}}(L) = 2$.

While the set R is residual, there are no known *explicit* examples with hyperbolic dimension equal to two. However, using classical Wiman-Valiron theory, we show also that one inequality (1.2) holds more generally for linearisers of any entire function (polynomial or transcendental, and regardless of the connectivity of the Julia set):

1.4. Theorem (Eventual hyperbolic dimension of linearisers).

Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be a nonlinear entire function, and let L be a Poincaré function associated to a repelling fixed point of f . Then

$$\text{edim}_{\text{hyp}}(L) \geq \dim_{\text{hyp}}(f).$$

In particular, we obtain the following completely explicit example of a hyperbolic entire function having hyperbolic dimension two (albeit of infinite order and extremely rapid growth).

1.5. Corollary (Exponential lineariser).

Consider the function $f: \mathbb{C} \rightarrow \mathbb{C}; z \mapsto 2\pi i e^z$, and let L be a Poincaré function of f at the fixed point $2\pi i$. Then $\text{edim}_{\text{hyp}}(L) = 2$.

If furthermore L is normalised such that $|L'(0)| < \frac{1}{20}$, then L is hyperbolic with connected Fatou set.

Remark. Observe that the function L even belongs to the *Speiser class* of transcendental entire functions whose set of critical and asymptotic values is finite.

In order to prove the above theorems, we introduce a useful quantity, called the *vanishing exponent* of a function $f \in \mathcal{B}$. This exponent always provides an upper bound for the eventual hyperbolic dimension of f ; see Section 3 and, in particular, Lemma 3.1 and Lemma 3.3.

1.6. Definition (Vanishing exponent).

Let $f \in \mathcal{B}$. Then the *vanishing exponent* $\vartheta(f)$ is defined as

$$\vartheta(f) := \inf \left\{ t \geq 0 : \limsup_{w \rightarrow \infty} \sum_{z \in f^{-1}(w)} \left(\frac{|w|}{|z| |f'(z)|} \right)^t = 0 \right\} \in [0, +\infty].$$

Eventual hyperbolic dimension and quasiconformal equivalence. Two entire functions f and g are called *quasiconformally equivalent* if there are quasiconformal homeomorphisms $\varphi, \psi: \mathbb{C} \rightarrow \mathbb{C}$ such that

$$\psi \circ f = g \circ \varphi,$$

and *affinely equivalent* if φ and ψ can be chosen to be affine. Quasiconformal equivalence classes form the natural parameter spaces of transcendental entire functions.

It was proved in [RS10] that, for any function $f \in \mathcal{B}$, the dimension of the *escaping set* (those points converging to infinity under iteration) does not change under affine equivalence, and the question was raised [RS10, Question 1.7] whether or not this remains true under quasiconformal equivalence. The same article also introduced a notion similar to our concept of eventual hyperbolic dimension: the *eventual dimension*

$$\text{edim}(f) = \inf_{R>0} \dim\{z \in J(f) : |f^n(z)| \geq R \text{ for all } n \geq 0\}.$$

This quantity is also invariant under affine equivalence; the same argument shows that the eventual hyperbolic dimension from Definition 1.1 is invariant by affine equivalence for functions in \mathcal{B} . In fact, it can be shown that, for $f \in \mathcal{B}$, always $\text{edim}_{\text{hyp}}(f) \leq \text{edim}(f) = \dim(I(f))$.

This raises the natural question of whether the eventual hyperbolic dimension remains constant inside quasiconformal classes. We can deduce from Theorem 1.2 that this is not the case.

1.7. Corollary (Eventual hyperbolic dimension may change).

There exist functions $f, g \in \mathcal{B}$ of finite positive order such that f and g are quasiconformally equivalent, but such that

$$1 < \text{edim}_{\text{hyp}}(f) < \text{edim}_{\text{hyp}}(g).$$

Furthermore, f is quasiconformally conjugate to $z \mapsto e^z - 2$ on a neighbourhood of its Julia set, and likewise for g .

Independent work of Mayer and Urbański. While this article was being completed, and after our results were first announced, Volker Mayer and Mariusz Urbański informed us of a new preprint [MU19] also treating the measurable dynamics and thermodynamic formalism of hyperbolic entire functions, including Poincaré functions of polynomials. In particular, they give an alternative proof of our Theorem 1.2, and show that the final inequality can be replaced by $\text{edim}_{\text{hyp}}(L) < \dim_{\text{hyp}}(L) < 2$ if P is hyperbolic [MU19, Theorems 1.6 and 7.3]. Their results also imply an alternative characterisation of our vanishing exponent; see Remark 3.2, item (d).

Acknowledgements. We thank Dave Sixsmith for interesting discussions and feedback on our manuscript. We are also grateful to Volker Mayer and Mariusz Urbański for making us aware of their recent preprints [May17, MU19].

2. NOTATION AND PRELIMINARIES

As usual, \mathbb{N} denotes the set of non-negative integers and \mathbb{C} denotes the complex plane. The (Euclidean) disc of radius R around a point $z \in \mathbb{C}$ is denoted $D(z, R)$. If f is a non-constant, nonlinear entire function, then $J(f)$ and $F(f)$ denote its Julia and Fatou

sets. The *singular set* $S(f)$ of f is the closure of the set of finite critical and asymptotic values of f ; compare [RS17, Section 2]. The *postsingular set* of f is

$$\text{PS}(f) := \overline{\bigcup_{s \in S(f)} \{f^n(s) : n \geq 0\}}.$$

An important subset of $J(f)$ is provided by its *radial points* (also known as *conical points*) [Prz99, McM00, Rem09a], as mentioned in the introduction.

2.1. Definition (Radial Julia set).

Let f be a rational or transcendental meromorphic function. The *radial Julia set* of f , denoted $J_r(f)$, consists of all points $z \in J(f)$ for which there are a positive number r and an increasing sequence $(n_k)_{k \in \mathbb{N}}$ of natural numbers with the following property: for all $k \in \mathbb{N}$, the inverse branch of f^{n_k} that sends $f^{n_k}(z)$ to z extends to the spherical disc of radius r around $f^{n_k}(z)$.

Let f be a nonlinear entire function. *Partition functions*, also known as the *Poincaré sequence* (see [Prz99]) will play an important role in some of our arguments. They measure, in a certain sense, the size of the preimage of a small disc around a point w under the n -th iterate of f .

2.2. Definition (Partition functions).

Let f again be rational or transcendental meromorphic. Let σ be a conformal metric on \mathbb{C} , let $t > 0$ and $w \in \mathbb{C}$. Then the *Poincaré sequence* of f is the sequence of *partition functions*

$$Z^\sigma(t, f^n, w) := \sum_{z \in f^{-n}(w)} \frac{1}{\|Df^n(z)\|_\sigma^t},$$

where $\|Df^n(z)\|_\sigma$ denotes the norm of the derivative of f^n at z .

By a *conformal metric* σ we mean here a form that can be written as $\rho_\sigma(z)|dz|$ with $\rho_\sigma(z) > 0$ for all $z \in \mathbb{C}$; the norm of the derivative of a holomorphic function φ with respect to σ is

$$\|D\varphi(z)\|_\sigma := |\varphi'(z)| \cdot \frac{\rho_\sigma(f(z))}{\rho_\sigma(z)}.$$

The metric used will be one of the Euclidean or spherical metric or the *cylindrical metric* $|dz|/|z|$ (defined on $\mathbb{C} \setminus \{0\}$), or its one-sided version

$$\rho_{\text{oc}}(z) = \frac{1}{\max\{|z|, 1\}}.$$

We shall write Z^{eucl} , Z^{cyl} , $\|Df(z)\|_{\text{oc}}$ etc. to indicate which metric is used. For polynomials, the choice of metric in the definition of partition functions is usually irrelevant, as long as the metric is defined in a neighbourhood of the Julia set, which is compact. For transcendental entire functions, we shall usually use the (one-sided) cylindrical metric.

2.3. Proposition (Continuity of partition functions).

Let $\sigma : \mathbb{C} \rightarrow \mathbb{R}_{>0}$ be the bounded density of a conformal metric on \mathbb{C} . Assume that

$$(2.1) \quad \limsup_{\delta \rightarrow 0} \left\{ \left| \frac{\sigma(z')}{\sigma(z)} - 1 \right| : z, z' \in \mathbb{C}, |z - z'| \leq \delta \right\} = 0.$$

Let $f \in \mathcal{B}$. Let $w \in \mathbb{C} \setminus S(f)$ be such that

$$(2.2) \quad \inf \left\{ |f'(z)| : z \in f^{-1}(w) \right\} > 0.$$

Then for all $t \geq 0$ the partition function

$$\xi \mapsto Z^\sigma(t, f, \xi) = \sum_{\zeta \in f^{-1}(\xi)} \|Df(\zeta)\|_\sigma^t$$

is continuous on a neighborhood of w .

2.4. Remarks.

- (a) The condition (2.1) is satisfied by the one-sided cylindrical metric.
- (b) If $|w|$ is large enough then condition (2.2) is automatically satisfied; see (2.6) below.

Proof. Let

$$r := \min \left\{ \text{dist}(w, S(f)), 1 \right\} > 0.$$

Then for all $z \in f^{-1}(w)$, the map f has a well defined holomorphic inverse branch f_z^{-1} defined on the disk $D(w, r)$ such that $f_z^{-1}(w) = z$. Let $u \in]0, 1[$. It follows from the distortion theorem that there exists $\kappa = \kappa(u) > 1$ such that $\kappa(u) \rightarrow 1$ as $u \rightarrow 0$ and for all $\xi \in D(w, ur)$

$$(2.3) \quad \kappa^{-1} \leq \left| \frac{(f_z^{-1})'(\xi)}{(f_z^{-1})'(w)} \right| \leq \kappa$$

and

$$\kappa^{-1} \frac{|\xi - w|}{r} \leq \frac{|f_z^{-1}(\xi) - f_z^{-1}(w)|}{|(f_z^{-1})'(w)|} \leq \kappa \frac{|\xi - w|}{r}.$$

Using (2.2) and the above we can find $\varepsilon = \varepsilon(u) > 0$ small enough so that if $\xi \in D(w, \varepsilon)$ then for all $z \in f^{-1}(w)$

$$\left| \frac{\sigma(f_z^{-1}(\xi))}{\sigma(z)} - 1 \right| \leq u.$$

Moreover we can also assume that

$$\left| \frac{\sigma(\xi)}{\sigma(w)} - 1 \right| \leq u.$$

From the above and (2.3) it follows that there exists $C = C(u) > 1$ such that $C(u) \rightarrow 1$ as $u \rightarrow 0$ and for all $\xi \in D(w, \varepsilon)$ and all $z \in f^{-1}(w)$

$$C^{-1} \leq \frac{\|Df_z^{-1}(\xi)\|_\sigma}{\|Df_z^{-1}(w)\|_\sigma} \leq C$$

Since $|\xi - w| < \frac{r}{2}$ it follows that any inverse branches of f defined near ξ is also well defined at w , hence $f^{-1}(\xi) = \{f_z^{-1}(\xi) : z \in f^{-1}(w)\}$. Thus

$$C^{-t} \leq \frac{Z^\sigma(t, f, \xi)}{Z^\sigma(t, f, w)} \leq C^t.$$

Consequently $Z^\sigma(t, f, w) < \infty$ if and only if $Z^\sigma(t, f, \xi) < \infty$. We assume now that $Z^\sigma(t, f, w) < \infty$. Since $C(u)^{\pm t} \rightarrow 1$ as $u \rightarrow 0$ it follows that $Z^\sigma(t, f, \xi) \rightarrow Z^\sigma(t, f, w)$ as $\xi \rightarrow w$. ■

By definition, for every $t < \dim_{\text{hyp}} f$ (with f an entire function) there is a hyperbolic set of dimension at least t . This implies easily that the partition functions of f with exponent t grow exponentially. We shall require a sharper, but more technical statement, given by Lemma 2.5 below. It states roughly that, if the definition of the partition function is modified to count only preimages near a given point in the Julia set, the growth remains exponential. As we are not aware of a reference, we include a proof in Appendix A for completeness.

2.5. Lemma (Local growth of partition functions).

Let f be a non-constant, non-linear entire function, and let U be an open set intersecting the Julia set of f . Let $t < \dim_{\text{hyp}} f$.

Then there exists a Jordan domain $D \subset U$, positive constants C, β, a and b , a sequence $(I_p)_p$ of positive integers, a sequence of finite families of simply connected domains $(D_i^p)_{1 \leq i \leq I_p}$, $D_i^p \subset D$ and a sequence of integers $(\nu_p)_{p \geq 1}$ converging to ∞ such that $|\nu_p - ap| \leq b$ for all p , and such that the following holds.

For all $p \geq 1$ and all $1 \leq i \leq I_p$, $f^{\nu_p} : D_i^p \rightarrow D$ is a conformal isomorphism. For any $w \in D$ and $1 \leq i \leq I_p$, let z_i denote the unique preimage of w by f^{ν_p} in D_i^p . Then

$$(2.4) \quad \sum_{i=1}^{I_p} \left| (f^{\nu_p})'(z_i) \right|^{-t} \geq C e^{\beta \nu_p}.$$

Pressure and topological Collet-Eckmann maps. If f is a polynomial², then the *topological pressure* of f is defined as

$$\mathcal{P}: (0, \infty) \rightarrow \mathbb{R};$$

$$(2.5) \quad \mathcal{P}(t) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{z \in f^{-n}(w)} \|Df^n(z)\|_{\text{sph}}^{-t} = \limsup_{n \rightarrow \infty} \frac{1}{n} \log Z^{\text{sph}}(t, f^n, w).$$

This function is defined and independent of w provided that w does not belong to a certain set $E \subset \mathbb{C} \setminus \text{PS}(f)$ of zero Hausdorff dimension; see [Prz99]. There are several other (equivalent) definitions of the pressure function, see [Prz99, PRLS04]. A key property of the pressure function is that the hyperbolic dimension of f coincides with the smallest zero of \mathcal{P} ; see [DU91, Prz93, Prz99], and also [PRLS04, PU10]. Furthermore, $\mathcal{P}(t)$ is convex in t .

Remark. The formula (2.5) can also be used to define the pressure for a large class of transcendental entire or meromorphic functions in \mathcal{B} including the Speiser class, see [BKZ12]. Here $\mathcal{P}(t)$ may be infinite. Again, the hyperbolic dimension is given by the infimum of the set of t for which the pressure is not positive. However, we will only use the pressure functions of polynomials.

It is possible to characterise the *topological Collet-Eckmann (TCE) condition* using the values of the pressure function; this is the most convenient definition of TCE for our purposes. For many other equivalent definitions of the TCE condition, see [PRLS03].

2.6. Definition (Topological Collet-Eckmann maps).

A polynomial satisfies the *topological Collet-Eckmann condition* if its pressure $\mathcal{P}(t)$ is negative for large values of t .

Since the pressure function is convex in t , for TCE maps there is a unique zero of \mathcal{P} , which coincides with the hyperbolic dimension. We will also use the following result.

2.7. Theorem (Hausdorff dimensions of polynomial Julia sets).

Let P be a polynomial with connected Julia set, and suppose that P is not conformally conjugate to a Chebyshev polynomial or a power map. Then

$$\dim(J(P)) > 1.$$

If, moreover, P satisfies the topological Collet-Eckmann condition, then

$$1 < \dim(J(P)) < 2.$$

Proof. The first inequality is well-known. Indeed, by [Zdu90, Theorem 2], the Hausdorff dimension of the Julia set of P is greater than the Hausdorff dimension of the equilibrium measure of P . The equilibrium measure is given by the harmonic measure on $J(P)$, as

²In this article, we consider only the iteration of polynomials and entire functions, and hence restrict our discussion to this case. However, anything stated for polynomials in this subsection also holds for rational functions with non-empty Fatou set, without modifications.

viewed from ∞ [Bro65, Section 16]. If $J(P)$ is connected, then the basin of infinity is simply-connected, and the dimension of the harmonic measure on the boundary of a simply-connected domain is 1 [Mak85]; see also [GM05, Section VI.5].

The strict upper bound for TCE maps follows from [PRLS03, Theorem 4.3]. ■

The Eremenko-Lyubich class. Recall that the *Eremenko-Lyubich class* \mathcal{B} is defined as

$$\mathcal{B} := \{f: \mathbb{C} \rightarrow \mathbb{C} \text{ transcendental entire: } S(f) \text{ is bounded}\}.$$

Suppose that $f \in \mathcal{B}$ and $R > 1$ is such that $S(f) \subset D(0, R)$. Set $W := \mathbb{C} \setminus \overline{D(0, R/2)}$. Then every component T of $f^{-1}(W)$ is simply-connected and $f: T \rightarrow W$ is a universal covering map. These components are called *logarithmic tracts* of f (over ∞).

Suppose furthermore that R was chosen such that additionally $|f(0)| < R$. A well-known estimate due to Eremenko and Lyubich [EL92, Lemma 1] states that

$$(2.6) \quad \|Df(z)\|_{\text{cyl}} \geq \frac{1}{4} \log \left| \frac{f(z)}{R} \right|$$

whenever $|f(z)| > R$. In particular, the cylindrical derivative of $\|Df(z)\|_{\text{cyl}}$ at z tends to infinity as $f(z) \rightarrow \infty$.

Hyperbolic and disjoint-type entire functions.

2.8. Definition (Hyperbolic functions).

A transcendental entire function f is said to be *hyperbolic* if $f \in \mathcal{B}$ and $S(f)$ is a subset the union of the basins of attraction of the attracting periodic cycles of f .

Equivalently, a transcendental entire function f is hyperbolic if $\text{PS}(f)$ is a compact subset of the Fatou set. Compare [RS17] for a discussion of and background on the definition of hyperbolicity for transcendental functions.

An entire transcendental function is said to be of *disjoint type* if it is hyperbolic with connected Fatou set. We remark that any function $f \in \mathcal{B}$ can be rescaled into a disjoint-type function; more precisely, λf is of disjoint type provided that $|\lambda|$ is small enough; see e.g. [BK07, Example p.392].

3. VANISHING EXPONENT AND EVENTUAL HYPERBOLIC DIMENSION

In this section we investigate the eventual hyperbolic dimension for functions in the Eremenko-Lyubich class, using the vanishing exponent $\vartheta(f)$ defined in Definition 1.6. We begin by connecting this exponent with the eventual hyperbolic dimension. We then show that both exponents are at least 1, and equal 1 for some functions in the Mayer-Urbański class of functions satisfying the balanced growth condition (Proposition 3.6). Finally we turn to the behaviour of both quantities under affine and quasiconformal equivalence.

Vanishing exponent. Note that the series in the vanishing exponent from Definition 1.6 is precisely the cylindrical partition function $Z^{\text{cyl}}(t, f, w)$. We begin by relating this exponent to the behaviour of the partition functions of the disjoint type functions λf , as λ tends to zero.

3.1. Lemma.

Let $f \in \mathcal{B}$ and $t \geq 0$. Then the two following statements are equivalent.

- (a) $Z^{\text{cyl}}(t, f, w) \rightarrow 0$ as $w \rightarrow \infty$,
- (b) $Z^{\text{cyl}}(t, \lambda f, w) \rightarrow 0$ as $\lambda \rightarrow 0$ uniformly in w , for $|w| > 1$.

Proof. As the cylindrical metric is invariant under linear rescalings, it follows that $Z^{\text{cyl}}(t, \lambda f, w) = Z^{\text{cyl}}(t, f, \frac{w}{\lambda})$ for all $\lambda \in \mathbb{C}^*$ and all w . Hence, the two properties are equivalent. \blacksquare

3.2. Remark (Vanishing exponents).

- (a) The *vanishing exponent* is precisely the infimum of the values of t for which the properties of Lemma 3.1 hold.
- (b) The assumption $f \in \mathcal{B}$ is not used in the proof. However, an entire function $f \notin \mathcal{B}$ cannot satisfy the statements in Lemma 3.1 for any $t \in \mathbb{R}$; compare [Six14].
- (c) Note that $Z^{\text{cyl}}(t, f, w) = Z^{\text{oc}}(t, f, w)$ for all sufficiently large w . Indeed, if w is sufficiently large, then all preimages of w have modulus at least 1, and hence the density of the two metrics agree at these preimages. Similarly, for $|w| > 1$ and all sufficiently small λ , $Z^{\text{cyl}}(t, \lambda f, w) = Z^{\text{oc}}(t, \lambda f, w)$.

So we may replace the cylindrical metric by the one-sided cylindrical metric in Lemma 3.1, and also in the definition of the vanishing exponent.

- (d) In [MU19, Theorem 4.1], the authors prove the following important statement: if $Z^{\text{cyl}}(\tilde{t}, f, w) < \infty$ for sufficiently large w , then a holds for $t > \tilde{t}$. Hence $\vartheta(f)$ is also the critical exponent for finiteness of the partition function.

Moreover, the authors introduce another exponent, $\Theta(f)$, related to an “integral mean spectrum”. For certain functions – those which have “negative spectrum” in the sense of [MU19] – this exponent agrees with our vanishing exponent $\vartheta(f)$.

3.3. Lemma (Vanishing exponent and eventual hyperbolic dimension).

Let $f \in \mathcal{B}$. Then

$$\limsup_{\lambda \rightarrow 0} \dim_{\text{hyp}}(\lambda f) \leq \vartheta(f) \quad \text{and} \quad \text{edim}_{\text{hyp}}(f) \leq \vartheta(f).$$

Proof. Let $t > \vartheta(f)$. We shall show that, if $|\lambda|$ is small enough, the hyperbolic dimension of $f_\lambda := \lambda f$ is smaller than t .

If $|\lambda|$ is sufficiently small, then $Z^{\text{cyl}}(t, f_\lambda, w) \leq 1/2$ whenever $|w| \geq 1$. Furthermore, again for λ small enough, the singular set of f_λ is a subset of the disc $D(0, \frac{1}{2})$ and $f_\lambda(\overline{D(0, 1)}) \subset D(0, \frac{1}{2})$. As a consequence, the forward orbits of the elements of the singular set of f_λ belong to $D(0, \frac{1}{2})$ and the Julia set lies in the complement of the closed unit disc.

Fix such a λ and let X be a hyperbolic set for f_λ , say $X \subset D(0, R)$ with $R > 1$. Cover X by a finite family \mathcal{Q} of closed topological discs $Q \subset \overline{D(0, R)} \setminus D(0, 1)$. For such Q and $n \geq 0$, let $\mathcal{U}_n(Q)$ denote the set of connected components of $f_\lambda^{-n}(Q)$ intersecting X . Then $\mathcal{U}_n := \bigcup_{Q \in \mathcal{Q}} \mathcal{U}_n(Q)$ is a cover of X for all $n \geq 0$. We shall use these covers to estimate the Hausdorff dimension of X from above.

For each $Q \in \mathcal{Q}$, there is a slightly larger topological disc $\tilde{Q} \supset Q$ disjoint from $D(0, 1/2)$, and thus not intersecting $\text{PS}(f)$. Hence, if $n \geq 1$ and U is a connected component of $f_\lambda^{-n}(Q)$, then f_λ^n is univalent on the component \tilde{U} of $f_\lambda^{-n}(\tilde{Q})$ containing U . Observe that \tilde{U} does not intersect $D(0, 1)$ by choice of λ . Applying Koebe's distortion theorem to a branch φ of $\log((f_\lambda^n|_{\tilde{Q}})^{-1})$, there is a constant $C_0 > 1$ (depending only on \mathcal{Q}) such that

$$\frac{\|Df_\lambda^n(\zeta)\|_{\text{cyl}}}{\|Df_\lambda^n(z)\|_{\text{cyl}}} = \frac{|\varphi'(f_\lambda^n(z))| \cdot |f_\lambda^n(z)|}{|\varphi'(f_\lambda^n(\zeta))| \cdot |f_\lambda^n(\zeta)|} \leq R \cdot \frac{|\varphi'(f_\lambda^n(z))|}{|\varphi'(f_\lambda^n(\zeta))|} \leq C_0$$

for all $z, \zeta \in U$. In particular, there is a constant C_1 depending only on \mathcal{Q} such that

$$\text{diam}_{\text{cyl}}(U) \leq \frac{C_1}{\|Df_\lambda^n(z)\|_{\text{cyl}}}.$$

Let $Q \in \mathcal{Q}$ and fix $\zeta_Q \in Q$. For every $n \geq 1$ and all $U \in \mathcal{U}_n(Q)$, let ζ_U be the unique element of $f_\lambda^{-n}(\zeta_Q) \cap U$. If V_1, \dots, V_m are the components of $f_\lambda^{-1}(U)$ that intersect X , then

$$\begin{aligned} \sum_{i=1}^m \|Df_\lambda^{n+1}(\zeta_{V_i})\|_{\text{cyl}}^{-t} &= \|Df_\lambda^n(\zeta_U)\|_{\text{cyl}}^{-t} \cdot \sum_{i=1}^m \|Df_\lambda(\zeta_{V_i})\|_{\text{cyl}}^{-t} \\ &\leq \|Df_\lambda^n(\zeta_U)\|_{\text{cyl}}^{-t} \cdot Z^{\text{cyl}}(t, f_\lambda, \zeta_U) \leq \frac{1}{2} \cdot \|Df_\lambda^n(\zeta_U)\|_{\text{cyl}}^{-t}. \end{aligned}$$

By induction,

$$\sum_{U \in \mathcal{U}_n(Q)} \|Df_\lambda^n(\zeta_U)\|_{\text{cyl}}^{-t} \leq 2^{-n}.$$

Hence

$$\begin{aligned} \sum_{U \in \mathcal{U}_n} \text{diam}_{\text{cyl}}(U)^t &= \sum_{Q \in \mathcal{Q}} \sum_{U \in \mathcal{U}_n(Q)} \text{diam}_{\text{cyl}}(U)^t \\ &\leq C_1^t \cdot \sum_{Q \in \mathcal{Q}} \sum_{U \in \mathcal{U}_n(Q)} \|Df_\lambda^n(\zeta_U)\|_{\text{cyl}}^{-t} \leq C_1^t \cdot \#\mathcal{Q} \cdot 2^{-n} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Thus $\dim(X) \leq t$, and the first claim of the lemma is proved.

The proof of the second claim is very similar. Choose $t > \vartheta(f)$ and let $R_0 > 0$ large enough so that $Z^{\text{cyl}}(t, f, w) \leq 1/2$ for $|w| > R_0$.

Let X be a hyperbolic set of f which lies outside the disc of radius $R_0 + 1$. We now continue as above, and cover X by a finite union \mathcal{Q} of closed topological discs Q , for each of which there is a larger disc \tilde{Q} that does not intersect $D(0, R_0)$ and is contained in a neighbourhood of X on which the map f is expanding. The latter assumption ensures that, while \tilde{Q} may intersect the postsingular set, any component of $f^{-n}(\tilde{Q})$ that intersects X lies

outside $D(0, R_0)$ and is mapped univalently to \tilde{Q} . The remainder of the proof proceeds as above, and we conclude that $\dim X \leq t$. This proves the second claim. ■

3.4. Corollary (Non-maximal hyperbolic dimension).

If $\vartheta(f) < 2$ and if f is of disjoint type, then $\dim_{\text{hyp}}(f) < 2$.

Proof. By Lemma 3.3, there is $\lambda < 1$ such that $\dim_{\text{hyp}}(f_\lambda) < 2$. Since both f and f_λ are of disjoint type, it follows from Theorem 3.1 of [Rem09b] that these maps are quasiconformally conjugate on neighborhoods of their respective Julia sets.

Let $t_0 < 2$ be an upper bound for the dimension of the hyperbolic sets of f_λ . The hyperbolic sets of f are the images of the hyperbolic sets of f_λ under the above-mentioned quasiconformal conjugacy ψ . By [GV73, Theorem 12], the dimensions of these images are bounded by a constant $t_1 < 2$ depending only on t_0 and the quasiconstant of ψ . ■

Remark. Suppose that f is a disjoint-type entire function and $\vartheta(f) < 2$. It is mentioned in [MU19, Section 8] that various results from the theory of thermodynamic formalism, as developed in [MU08, MU10] will carry over for $t > \vartheta(f)$. In particular, if $t_0 := \dim_{\text{hyp}}(f) > 2$, then there is a probability measure that is conformal (with respect to the cylinder metric) and supported on the radial Julia set. In particular, it follows that $t_0 < 2$, giving an alternative proof of Corollary 3.4.

It is a natural question to ask whether always $\dim_{\text{hyp}}(f) > \text{edim}_{\text{hyp}}(f)$ when f is of disjoint type with $\text{edim}_{\text{hyp}}(f) < 2$. This was recently answered in the negative by Mayer and Zdunik [MZ19], who show that there is a disjoint-type entire function f with $\text{edim}_{\text{hyp}}(f) = \dim_{\text{hyp}}(f) = \vartheta(f) < 2$. Moreover, there is no conformal (with respect to the cylinder or spherical metric) probability measure of exponent $\vartheta(f)$ that is supported on the radial Julia set.

A lower bound on the eventual hyperbolic dimension. We now give the details concerning the facts, stated in the introduction, that the eventual hyperbolic dimension of a function $f \in \mathcal{B}$ is at least 1, and that this lower bound is achieved by some of the functions studied by Mayer and Urbański.

3.5. Lemma (Eventual hyperbolic dimension in \mathcal{B}).

Let $f \in \mathcal{B}$, and let $R > 0$. Then

$$\sup_X \dim(X) > 1,$$

where the supremum is taken over all hyperbolic sets $X \subset \mathbb{C} \setminus D(0, R)$. In particular,

$$\vartheta(f) \geq \text{edim}_{\text{hyp}}(f) \geq 1.$$

Proof. We may assume that R is sufficiently large to ensure that $S(f) \subset D(0, R/2)$. Then every component of $f^{-1}(\mathbb{C} \setminus \overline{D(0, R)})$ is a logarithmic tract of f over infinity, and the first part follows from [BKZ09]. The second part is a direct consequence of the first, together with the definition of $\text{edim}_{\text{hyp}}(f)$, and Lemma 3.3. ■

Let us use the vanishing exponent to justify the claim in the introduction that entire functions $f \in \mathcal{B}$ covered by the results of [MU08] have eventual hyperbolic dimension 1.

It is worth noting that [MU08] treats more than just entire functions. Their results cover finite-order meromorphic functions. In our work we restrict to the entire case. Furthermore, [MU08] does not explicitly restrict to functions in \mathcal{B} , although we are not aware of any known examples of entire functions outside of \mathcal{B} to which their results apply. An entire function $f \in \mathcal{B}$ of finite positive order ρ has *balanced growth* in the sense of Mayer-Urbański if there exists $C > 1$ such that for all $z \in J(f)$,

$$(3.1) \quad C^{-1} \leq \frac{|f'(z)|}{(1 + |z|^{\rho-1})(1 + |f(z)|)} \leq C$$

(see [MU08, Lemma 3.1]).

3.6. Proposition (Eventual hyperbolic dimension and balanced growth).

Let f be a class \mathcal{B} entire function of finite order ρ . Suppose that f has balanced growth in the sense of Mayer-Urbański. Then $\text{edim}_{\text{hyp}}(f) = \vartheta(f) = 1 < \dim_{\text{hyp}}(f)$.

Proof. By Lemma 3.5, it only remains to show that $\vartheta(f) \leq 1$. The balanced growth condition (3.1) means precisely that $\|Df(z)\|_{\text{cyl}}$ is comparable to $\max(|z|, 1)^\rho$ for $z \in J(f)$, where ρ is the order of growth of f . So

$$(3.2) \quad Z^{\text{cyl}}(t, f, w) = \sum_{z \in f^{-1}(w)} \|Df(z)\|_{\text{cyl}}^{-t} \approx \sum_{z \in f^{-1}(w)} \frac{1}{|z|^{\rho t}},$$

for sufficiently large $w \in J(f)$. (Here \approx means that the two sides are uniformly comparable.)

Fix $\tilde{t} > t > 1$. By [MU08, Proposition 4.5], the second sum in (3.2) remains bounded as $w \rightarrow \infty$. Hence $\sum_{z \in f^{-1}(w)} 1/|z|^{\rho \tilde{t}}$ tends to zero as $w \rightarrow \infty$. We conclude that

$$(3.3) \quad Z^{\text{cyl}}(\tilde{t}, f, w) \rightarrow 0,$$

at least as $w \rightarrow \infty$ inside $J(f)$. Since $f \in \mathcal{B}$, every component of $J(f)$ is unbounded ([EL92, Proposition 2], [RG16, Corollary 3.11]). A simple application of Koebe's distortion theorem – which we provide below in Lemma 3.7 for completeness – shows that $Z^{\text{cyl}}(\tilde{t}, f, w)$ and $Z^{\text{cyl}}(\tilde{t}, f, \tilde{w})$ are comparable when $|w| = |\tilde{w}|$ and this modulus is sufficiently large. Therefore (3.3) also holds for the unrestricted limit $w \rightarrow \infty$, and $\vartheta(f) \leq \tilde{t}$. As $\tilde{t} > 1$ was arbitrary, the proof is complete. ■

3.7. Lemma (Distortion bounds on partition functions).

Let $f \in \mathcal{B}$ and $C > 1$. Then there is $R > 0$ such that

$$\frac{Z^{\text{cyl}}(t, f, w)}{Z^{\text{cyl}}(t, f, \tilde{w})} \leq C^t$$

whenever $t \geq 0$ and whenever $|w| = |\tilde{w}| \geq R$.

Proof. Let $R_0 > 0$ so large that $S(f) \subset D(0, R_0/2)$ and $|f(0)| < R_0$. Let $w, \tilde{w} \in \mathbb{C}$ with $|w| = |\tilde{w}| \geq R_0 + 4\pi$. Let γ be an arc of the circle $\partial D(0, |w|)$ connecting w to \tilde{w} . For each $z \in f^{-1}(w)$, let \tilde{z} be the element of $f^{-1}(\tilde{w})$ obtained from z by analytic continuation of

f^{-1} along γ . Observe that this defines a bijection between $f^{-1}(w)$ and $f^{-1}(\tilde{w})$. By the definition of the partition function, it is enough to show that

$$(3.4) \quad \frac{\|Df(z)\|_{\text{cyl}}}{\|Df(\tilde{z})\|_{\text{cyl}}} \leq C$$

when $|w|$ is sufficiently large.

This follows by applying Koebe's theorem to a *logarithmic transform* of f . More precisely, set $W := \mathbb{C} \setminus \overline{D(0, R_0)}$, let T be the connected component of $f^{-1}(W)$ containing z and \tilde{z} , and let V be a connected component of $f^{-1}(T)$. Recall that $0 \notin T$, so V is a simply-connected domain, and $f \circ \exp: V \rightarrow W$ is a universal covering. So there is a conformal isomorphism $\Phi: V \rightarrow H := \log W$ such that $\exp \circ \Phi = f \circ \exp$.

Let $\zeta, \tilde{\zeta} \in V$ with $\exp(\zeta) = z$, $\exp(\tilde{\zeta}) = \tilde{z}$. Then

$$\|Df(z)\|_{\text{cyl}} = |(\Phi^{-1})'(\zeta)|,$$

and similarly for $\tilde{\zeta}$. Also observe that $\text{Re } \Phi(\zeta) = \text{Re } \Phi(\tilde{\zeta}) = \log |w|$, and $|\text{Im } \Phi(\zeta) - \text{Im } \Phi(\tilde{\zeta})| < 2\pi \leq (|w| - R_0)/2$. By Koebe's distortion theorem, applied to the restriction of Φ^{-1} to the disc $D(\Phi(\zeta), |w| - R_0)$, we see that (3.4) holds with C replaced by a constant $C(|w|)$ that tends to 1 as $|w| \rightarrow \infty$. This completes the proof. \blacksquare

Affine and quasiconformal equivalence classes. Following [RS10], we show that, in \mathcal{B} , the eventual hyperbolic dimension is invariant under affine equivalence.

3.8. Theorem.

Let $f, g \in \mathcal{B}$ be affinely equivalent. Then $\text{edim}_{\text{hyp}}(f) = \text{edim}_{\text{hyp}}(g)$ and $\vartheta(f) = \vartheta(g)$.

Proof. The second claim is immediate from the definition. We show that $\text{edim}_{\text{hyp}}(g) \geq \text{edim}_{\text{hyp}}(f)$; the first claim then follows trivially.

Let $K > 1$. By [RS10, Corollary 2.2] (see also [Rem09b, Section 3]), there exist $R > 0$ and a K -quasiconformal map $\vartheta: \mathbb{C} \rightarrow \mathbb{C}$ such that

$$(3.5) \quad \vartheta(f(z)) = g(\vartheta(z))$$

for all $z \in J_R(f) = \{z \in J: |f^n(z)| \geq R \text{ for all } n \geq 0\}$.

Let $S > 0$ and suppose $R > 0$ is large enough, so that $|\vartheta(z)| \geq S$ for all $|z| \geq R$. From the conjugacy (3.5), it follows that if $X \subset J_R(f)$ is a hyperbolic subset for f , then $Y := \vartheta(X)$ is a compact subset of $J_S(g)$ which is forward invariant by g . By (2.6), Y is a hyperbolic set for g , assuming S was chosen sufficiently large.

From Theorem 8 of [GV73], one deduces the following inequality:

$$\sup \{ \dim(X): X \subset J_R(f) \text{ hyperbolic} \} \leq K \sup \{ \dim(Y): Y \subset J_S(g) \text{ hyperbolic} \}.$$

This implies that $\text{edim}_{\text{hyp}}(f) \leq K \text{edim}_{\text{hyp}}(g)$. Since K can be chosen arbitrarily close to 1, the result follows. \blacksquare

As stated in Corollary 1.7, the eventual hyperbolic dimension is *not* invariant under quasiconformal equivalence, and the proof will show that the same is true for the vanishing

exponent. However, the condition of having *full* eventual hyperbolic dimension is preserved under quasiconformal equivalence.

3.9. Proposition (Quasiconformal equivalence and eventual hyperbolic dimension).

Let $f, g \in \mathcal{B}$ be quasiconformally equivalent. If $\text{edim}_{\text{hyp}}(f) = 2$, then $\text{edim}_{\text{hyp}}(g) = 2$.

Proof. This follows again from [Rem09b]. We begin as in the proof of Theorem 3.8. The quasiconformal map ϑ still exists in this setting [Rem09b, Theorem 1.1], but it is no longer necessarily true that K can be chosen arbitrarily close to 1. Let S and R be as in the above proof, and let \mathcal{X} be the union of all hyperbolic sets of f contained in $J_R(f)$. Then $\mathcal{Y} := \vartheta(\mathcal{X}) \subset J_S(g)$ is a union of hyperbolic sets of g .

Since $\text{edim}_{\text{hyp}}(f) = 2$, we have $\dim(\mathcal{X}) = 2$ by definition. As quasiconformal maps preserve sets of Hausdorff dimension 2 [GV73, Corollary 13], it follows that $\dim(\mathcal{Y}) = 2$. Since S can be chosen arbitrarily large, we conclude that $\text{edim}_{\text{hyp}}(g) = 2$, as claimed. ■

Remark. It is natural to ask whether the condition $\vartheta(f) = 2$ is also preserved under quasiconformal equivalence. It seems plausible that this is the case, at least for functions with a finite set of singular values. Observe that this question is similar in spirit to [ER15, Proposition 4.2]; we leave it aside as it will not be required for the purposes of this paper.

4. POINCARÉ FUNCTIONS

In this section, we review some basic facts about the mapping properties of Poincaré functions. Recall that a Poincaré function, or lineariser, is an entire solution of the equation $L(\rho z) = f(L(z))$, with $L(0) = \xi_0$. Here f is an entire function with a repelling fixed point ξ_0 with multiplier ρ . If one fixes the value of $L'(0)$ to some non zero complex number, then the solution is unique. The unique Poincaré function such that $L'(0) = 1$ is called the *normalised Poincaré function* of f at ξ_0 .

4.1. Remark (Normalisation).

Let f, ξ_0, L be as above and let φ be a nonconstant affine map and $\lambda \in \mathbb{C}^*$. Then the mapping $f_1 := \varphi \circ f \circ \varphi^{-1}$ has a repelling fixed point at $\xi_1 := \varphi(\xi_0)$ with multiplier $\rho := f'(\xi_0)$. Then the function $L_1(z) := \varphi \circ L(\lambda z)$ is a Poincaré function for (f_1, ξ_1) . In particular, if the postsingular set of f (and hence of f_1) is bounded, then it follows from Theorem 3.8 that $\text{edim}_{\text{hyp}}(L_1) = \text{edim}_{\text{hyp}}(L)$ and $\vartheta(L_1) = \vartheta(L)$.

For example if f and ξ_0 are as above and L is a Poincaré function of f at ξ_0 , then

$$z \mapsto L\left(\frac{z}{L'(0)}\right) - \xi_0$$

is the normalised Poincaré function of $z \mapsto f(z + \xi_0) - \xi_0$ at 0. Therefore, in the following, we can usually assume that $\xi_0 = 0$ and that a given Poincaré function is normalised.

4.2. Remark (Quasiconformal equivalence of linearisers).

Similarly, if two entire functions are quasiconformally conjugate, then their respective Poincaré functions are quasiconformally equivalent, see [ER15, Proposition 3.2]. This will be important in Corollary 1.7.

Let L be a Poincaré function of an entire function f . Then we can use the defining functional relation to describe the preimage $L^{-1}(w)$ of a point $w \in \mathbb{C}$ in terms of the iterated preimages of w under f , as follows. (See Figure 1.)

4.3. Observation (Preimage structure of linearisers).

Let f be an entire function with a repelling fixed point ξ_0 of multiplier ρ , and let L be a Poincaré function of f at ξ_0 . Let $\Delta_0 = D(0, |\rho| r_0)$ be a small disc around 0 on which L is univalent, and set

$$\begin{aligned} A_0 &:= \{z \in \mathbb{C} : r_0 \leq |z| < r_0|\rho|\} \subset \Delta_0 \\ A_f &:= L(A_0). \end{aligned}$$

In particular A_0 is a fundamental domain for $z \mapsto \rho z$.

Let $w \notin L(\Delta_0)$ and $n \geq 1$ and define

$$E_n := L^{-1}(w) \cap \{z \in \mathbb{C} : |\rho|^n \cdot r_0 \leq |z| < |\rho|^{n+1} \cdot r_0\}.$$

Observe that $L^{-1}(w) = \bigcup_{n \geq 1} E_n$ and

$$E_n = \rho^n \cdot (L|_{\Delta_0})^{-1}(f^{-n}(w) \cap A_f).$$

Finally, let $z \in E_n$ and set $\zeta := L(z/\rho^n) \in f^{-n}(w) \cap A_f$. Then

$$L'(z) = \frac{1}{\rho^n} L' \left(\frac{z}{\rho^n} \right) (f^n)'(\zeta).$$

Proof. The claims follow immediately from the functional relation (1.1). ■

We will need the following result (compare also [ER15, p. 581, Footnote 2]).

4.4. Proposition ([MBP12, Proposition 4.2 (ii)]).

Let f be an entire function and let L be a Poincaré function of f . Then $S(L) = \text{PS}(f)$.

5. A LOWER BOUND

The following result, together with Theorem 3.8, implies Theorem 1.4.

5.1. Theorem.

Let f be a non-constant entire function which is not affine. Let L be the normalised Poincaré function associated to a repelling fixed point of f . Then

$$\text{edim}_{\text{hyp}} L \geq \dim_{\text{hyp}} f.$$

In the proof of this theorem, we construct a sequence of closed round annuli \mathcal{A}_k , centered at 0 and with distance to 0 converging to ∞ , and satisfying the following: for any $t < \dim_{\text{hyp}} f$, the first partition function of $L|_{\mathcal{A}_k \cap L^{-1}(\mathcal{A}_k)}$ with exponent t will be bounded from below by a large constant as long as k is large enough.

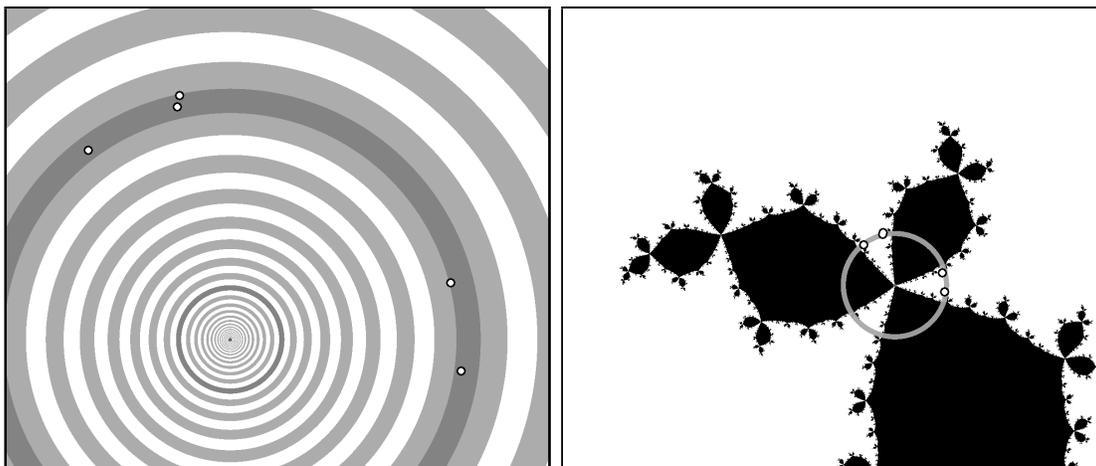


FIGURE 1. Illustration of Observation 4.3, for a lineariser L at a fixed point α of the Douady rabbit polynomial $P(z) = z^2 + c$, $c \approx 0.123 + 0.745i$. The domain of L is on the left, while the range (the dynamical plane of P) is on the right. An approximation of a fundamental annulus for the dynamics of P is shown in grey on the right. The five preimages of $w := 2 + 2i$ by P^{15} which belong to the annulus are represented by white discs with black borders. (Two of these preimages lie very close together.) The filled Julia set of P is shown in black in the background. On the left, A_0 is drawn as a small dark grey round annulus with its centre at 0. The different annuli $|\rho|^k A_0$ are differentiated by white and light grey colors. The annulus $|\rho|^{15} A_0$ is also represented in dark grey. By Observation 4.3, this annulus contains exactly five elements of $L^{-1}(2 + 2i)$, corresponding to the five points shown on the right; these are the elements of E_{15} .

From the Eremenko-Lyubich estimate (2.6) it follows that the set X of points whose orbits stay inside the annulus \mathcal{A}_k is a nonempty hyperbolic set for L for k large enough. Since the first partition function for this hyperbolic set is bounded from below by a constant which can be chosen as large as necessary (again with k large enough), the dimension of the set X is at least t . This implies $\text{edim}_{\text{hyp}} L \geq t$ for any $t < \dim_{\text{hyp}} f$, and hence the stated conclusion.

We start with the basic construction of the sequence $(\mathcal{A}_k)_k$. This is done in the following lemma.

5.2. Lemma.

Let f be a transcendental entire function or a polynomial of degree $d \geq 2$. Let $R_f > 0$, $m > 2$ and ρ a complex number such that $|\rho| > 1$. Let $M = 4 + 2m \log |\rho|$. Then there exists a sequence of natural numbers $n_k \rightarrow \infty$, and a sequence of bounded open subsets of the

complex plane S_k (with an explicit description given below) satisfying the following. Let

$$\mathcal{A}_k = A \left(|\rho|^{n_k} R_f, |\rho|^{n_k+m} R_f \right)$$

then,

- there is a sequence of complex numbers $\xi_k \rightarrow \infty$ and a nondecreasing sequence of natural numbers N_k such that

$$(5.1) \quad S_k = \left\{ z : \left| \log \left| \frac{z}{\xi_k} \right| \right| < \frac{M}{N_k}, \left| \arg \frac{z}{\xi_k} \right| < \frac{M}{N_k} \right\};$$

- For all $k \geq 0$, $f(S_k) \supset \mathcal{A}_k \cup S_{k+1}$;
- $\xi_{k+1} \in \mathcal{A}_k$.

Moreover Wiman-Valiron theory applies at ξ_k on S_k with central index N_k , that is,

- (a) $|f(\xi_k)| = M(r_k, f) := \max \left\{ |f(z)| : |z| = r_k \right\}$, with $r_k = |\xi_k|$;
 (b) For all $z \in S_k$

$$(5.2) \quad f(z) = \left(\frac{z}{\xi_k} \right)^{N_k} f(\xi_k) (1 + \varepsilon_0(z)),$$

and

$$(5.3) \quad f'(z) = \frac{N_k}{z} \left(\frac{z}{\xi_k} \right)^{N_k} f(\xi_k) (1 + \varepsilon_1(z)),$$

with $|\varepsilon_0(z)| + |\varepsilon_1(z)| \leq \varepsilon_k$ for all $z \in S_k$ and $\varepsilon_k \rightarrow 0$;

(c) We have the following estimate:

$$(5.4) \quad N_k \leq (\log M(r_k, f))^2.$$

Finally,

$$(5.5) \quad \frac{1}{|\rho|^2 R_f} M(r_k, f) \leq |\rho|^{n_k} \leq \frac{1}{|\rho| R_f} M(r_k, f).$$

Proof. We do a similar construction as the one that can be found in the early reference [Ere89].

For any $r \geq 0$, denote by $\xi(r)$ a definite arbitrary choice of a complex number such that $|\xi(r)| = r$ with $|f(\xi(r))| = M(r, f)$. Let $M > 1$. From Wiman-Valiron theory, we know that there exists an exceptional set $E \subset \mathbb{R}_+$ of finite logarithmic measure (that is $\text{lm } E := \int_E \frac{dx}{x} < \infty$), a nondecreasing function $r \mapsto N(r)$ from $\mathbb{R}_{>0} \setminus E$ to \mathbb{N} , and a positive function $r \mapsto \varepsilon(r)$ converging to 0 as $r \rightarrow \infty$, such that the following is satisfied for all $r \in \mathbb{R}_{>0} \setminus E$:

(a) For all $z \in D\left(\xi(r), \frac{2Mr}{N(r)}\right)$,

$$f(z) = \left(\frac{z}{\xi(r)}\right)^{N(r)} f(\xi(r)) (1 + \varepsilon_0(z)),$$

and

$$f'(z) = \frac{N(r)}{z} \left(\frac{z}{\xi(r)}\right)^{N(r)} f(\xi(r)) (1 + \varepsilon_1(z)),$$

with $|\varepsilon_0(z)| + |\varepsilon_1(z)| \leq \varepsilon(r)$ for all z in that disk.

(b) We have the following estimate:

$$N(r) \leq (\log M(r, f))^2.$$

In the case where f is a polynomial map of degree $d \geq 2$, the above is still true if the central index N is replaced by the constant equal to d and the exceptional set E is some bounded subset of $\mathbb{R}_{>0}$ depending on f .

Now recall that $M = 4 + 2m \log |\rho|$. With this, one can construct a sequence of points ξ_k and of sectors $S_k \subset D\left(\xi_k, \frac{2Mr_k}{N_k}\right)$ defined by equation (5.1) where $\xi_k = \xi(r_k)$, $r_k = |\xi_k| \notin E$, $N_k = N(r_k)$ and which is such that $f(S_k)$ contains both S_{k+1} and the round annulus $\mathcal{A}_k \ni \xi_{k+1}$, where $n_k \in \mathbb{N}$. For f transcendental the ratio $M/N(r)$ tends to 0 as $r \rightarrow \infty$ and thus we can assume that $e^{M/N(r)} \leq 1 + \frac{2M}{N(r)}$ for all r large enough. In the polynomial case $M/N(r)$ is still bounded and it is possible to find a $\kappa > 1$ large enough so that $e^{M/N(r)} \leq 1 + \frac{\kappa M}{N(r)}$ and work with $\frac{\kappa M}{N(r)}$ instead of $\frac{2M}{N(r)}$.

Choose first $r_0 \notin E$ and let $\xi_0 = \xi(r_0)$ and

$$S_0 = \left\{ z : \left| \log \left| \frac{z}{\xi_0} \right| \right| < \frac{M}{N_0}, \left| \arg \frac{z}{\xi_0} \right| < \frac{M}{N_0} \right\}.$$

Suppose now ξ_k , and thus S_k have already been defined. The mapping

$$\varphi_k : z \mapsto N_k (\log z - \log \xi_k) + \log f(\xi_k)$$

sends S_k univalently onto the square

$$Q_k = \{u : -M < \operatorname{Re}(u - \log f(\xi_k)) < M, -M < \operatorname{Im}(u - \log f(\xi_k)) < M\}.$$

Assuming r_0 large enough, $|\log f - \varphi_k| = |\log(1 + \varepsilon_0)| < 1$ is true for any $r \in [r_0, +\infty[\setminus E$. It then follows from Rouché's theorem that there exists an open subset $U_k \subset S_k$ which is sent by $\log f$ univalently onto the square

$$Q'_k = \{u : -M + 1 < \operatorname{Re}(u - \log f(\xi_k)) < M - 1, -M + 1 < \operatorname{Im}(u - \log f(\xi_k)) < M - 1\}.$$

In particular, the image of S_k by f contains the annulus

$$\mathcal{A}'_k = \left\{ z : e^{-M+1} \leq \left| \frac{z}{f(\xi_k)} \right| \leq e^{M-1} \right\}.$$

Then, by definition of M , the integer $n_k = \left\lfloor \frac{\log |f(\xi_k)/R_f|}{\log |\rho|} \right\rfloor - 1 \in \mathbb{N}$ is such that $\mathcal{A}_k \subset \mathcal{A}'_k$. Inequalities (5.5) follow from this choice. Note that for r_0 large enough, we will have $n_k \rightarrow \infty$. Choosing r_0 large enough, one has $\text{Im}(E \cap]r_0, +\infty]) < \text{Im}(\mathcal{A}_k \cap \mathbb{R}_{>0})$. Hence there is a $r_{k+1} \in (\mathcal{A}_k \cap \mathbb{R}_{>0}) \setminus E$. Also, one can suppose r_0 large enough so that $N(r) \geq 2$ for all $r \geq r_0$. As a consequence, one can find a $\xi_{k+1} \in \mathcal{A}_k$, with $|\xi_{k+1}| = r_{k+1}$ and $S_{k+1} \subset \mathcal{A}'_k$. \blacksquare

Now we can use Lemma 5.2 to prove Theorem 5.1.

Proof of Theorem 5.1. (See Figure 2). Let ρ be the multiplier of the repelling fixed point. Let $A_0 = A(R_f, |\rho|R_f)$ be the fundamental annulus from Observation 4.3 (i.e. $R_f = r_0/|\rho|$). Take $D \subset A_0$, $a, b, C, \beta > 0$ and the sequence $(\nu_p, (D_i^p)_{i=1, \dots, I_p})_p$ as in Lemma 2.5 and let $m = a + 2b + 2$. Lemma 5.2 gives the sequences of annuli \mathcal{A}_k , of natural numbers n_k and N_k , of sets S_k , and of points ξ_k .

The domain D intersects the Julia set of f , hence the family $(f^k)_{k \in \mathbb{N}}$ is not normal on D and for any bounded G there is k such that $G \subset f^k(D)$. It follows from the Ahlfors island theorem that there exists a subdomain $D_0 \subset D$, $j \in \{0, 1, 2\}$ and $k_0 \geq 1$ such that f^{k_0} send D_0 univalently onto S_j . If necessary, we remove the first terms of the sequence of S_n so that $j = 0$.

Hence, for all $w \in \mathcal{A}_k$, there exists $\omega \in D_0$, such that, $f^{k_0+j}(\omega) \in S_j$ for all $j = 0, \dots, k-1$ and $f^{k_0+k}(\omega) = w$.

For all k , there is a integer $p_k \geq 0$, such that

$$n_k + 1 - k - k_0 + b \leq ap_k \leq n_k - k - k_0 + m - 1 - b.$$

From above and Lemma 2.5, it follows that one can choose a sequence of p_k having the following properties. Let

$$\nu_k^+ := \nu_{p_k} + k + k_0,$$

then

$$n_k + 1 \leq \nu_k^+ \leq n_k + m - 1,$$

inequality (2.4) is satisfied and for all $i = 1, \dots, I_{p_k}$, $f_{|D_i^{p_k}}^{\nu_{p_k}} : D_i \rightarrow D$ is a conformal isomorphism.

Given k large choose an open set W_k such that $\mathcal{A}_k^{slit} := \mathcal{A}_k \setminus W_k$ is simply connected and $\rho^{\nu_k^+} L^{-1}(D) \subset \mathcal{A}_k^{slit}$. Choose $Y_k \subset S_0$ a univalent preimage of $\mathcal{A}_k^{slit} \cap \rho^{\nu_k^+} A_0$ by f^k . Consider the finite family of sets:

$$X_i^k = \rho^{\nu_k^+} L^{-1} \left(\left(f_{|D_i^{p_k}}^{\nu_{p_k}} \right)^{-1} \left(\left(f_{|D}^{k_0} \right)^{-1} (Y_k) \right) \right),$$

for $i = 1, \dots, I_{p_k}$.

Then X_i^k is a compact subset of $\mathcal{A}_k^{slit} \cap \rho^{\nu_k^+} A_0$ and $L : X_i^k \rightarrow \mathcal{A}_k^{slit} \cap \rho^{\nu_k^+} A_0$ is a conformal isomorphism.

We will show that the first partition function of the system of conformal isomorphisms $(L_{|X_i^k}, X_i^k)_{i=1, \dots, I_{p_k}}$ taken with respect to the cylindrical metric becomes arbitrary large as $k \rightarrow$

∞ . This would imply that the (cylindrical) pressure function for the IFS $(L|_{X_i^k}, X_i^k)_{i=1, \dots, I_k}$ is positive for k large.

Let $w \in \rho^{\nu_k^+} A_0$ and consider the first (modified) cylindrical partition function ³ of the system evaluated at w :

$$Z(w) := \sum_{i=1}^{I_{p_k}} \frac{|w|^t}{|z_i|^t \cdot |L'(z_i)|^t},$$

where $z_i = \left(L|_{X_i^k}\right)^{-1}(w)$.

For all $z_i \in X_i^k$, one has

$$L'(z_i) = \frac{1}{\rho^{\nu_k^+}} L' \left(\frac{z_i}{\rho^{\nu_k^+}} \right) f^{\nu_{p_k}'} \left(z_{IFS}^i \right) f^{k_0'} \left(z_{S_0}^i \right) f^{k'} \left(z_{WV}^i \right),$$

where

$$\begin{aligned} z_{IFS}^i &= L \left(\frac{z_i}{\rho^{\nu_k^+}} \right), \\ z_{S_0}^i &= f^{\nu_{p_k}} \left(z_{IFS}^i \right), \\ z_{WV}^i &= f^k \left(z_{S_0}^i \right). \end{aligned}$$

By construction $|z_i|$ is comparable to $|\rho^{\nu_k^+}|$. Since $\frac{z_i}{\rho^{\nu_k^+}} \in A_0$, the factor $\left| L' \left(\frac{z_i}{\rho^{\nu_k^+}} \right) \right|$ is uniformly bounded away from 0 and ∞ . Finally, the same is true for the derivative of $f|_D^{k_0}$. Hence $Z(w)$ is comparable to

$$\sum_{i=1}^{I_{p_k}} \frac{|w|^t}{\left| (f^{\nu_{p_k}})' \left(z_{IFS}^i \right) (f^k)' \left(z_{WV}^i \right) \right|}.$$

From Wiman-Valiron estimates (5.2) and (5.3), it follows that for any $y \in S_k$, one has $f'(y) = (1 + \varepsilon_2(y)) N_k \frac{f(y)}{y}$ for some bounded ε_2 . This implies,

$$\left| (f^k)' \left(z_{WV}^i \right) \right| \leq C_1^k N_0 N_1 \cdots N_{k-1} |w|,$$

³The modification on the metric can be ignored since, for k large, $|w|$ and the $|z_i|$ are all greater than 1.

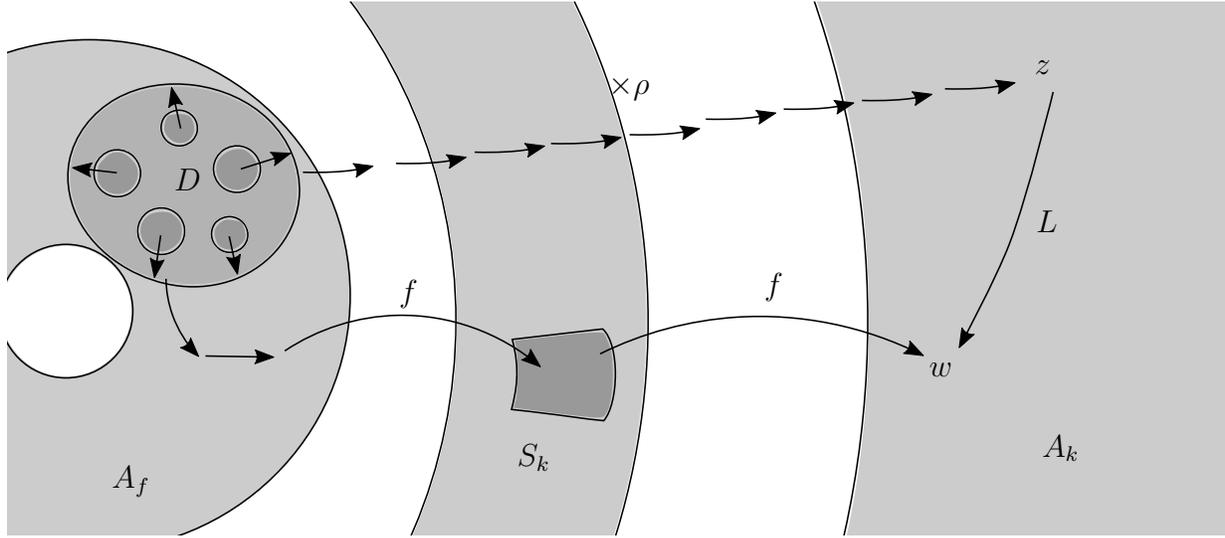


FIGURE 2. Illustration of the proof of theorem 5.1.

for some constant $C_1 > 0$. Consequently, for k large enough,

$$\begin{aligned} Z(w) &\geq C_2^k \frac{1}{N_0^t N_1^t \cdots N_{k-1}^t} \sum_{i=1}^{I_{P_k}} \left| f^{\nu_{P_k}'}(z_{IFS}^i) \right|^{-t} \\ &\geq C_3^k \frac{1}{N_0^t N_1^t \cdots N_{k-1}^t} e^{\beta n_k}, \end{aligned}$$

for suitable constants $C_i > 0$.

From (5.5) it follows that there exists $C_5 > 1$ such that

$$(5.6) \quad C_5^{-1} \log M(r_k, f) \leq n_k \leq C_5 \log M(r_k, f).$$

In particular from the above (5.4), and the fact that $n_k \rightarrow \infty$ it follows that there exists $C_6 > 0$ such that $N_j \leq C_6 n_{j+1}^2$ for all $j \leq k$. Hence for $j < k$,

$$N_0 N_1 \cdots N_{k-1} \leq C_6^k (n_1 \cdots n_k)^2.$$

Since the map f is entire it follows from (5.6) that the ratio $\frac{e^{\beta n_k}}{(n_1 \cdots n_k)^{2t}}$ grows to ∞ as $k \rightarrow \infty$ faster than any exponential growth in k . \blacksquare

6. LINEARISERS OF POLYNOMIALS

The following result, when combined with Theorem 5.1, will establish Theorem 1.2.

6.1. Lemma.

Let L be a Poincaré function for a topological Collet Eckmann polynomial P such that $J(P)$ is connected. Then

$$\text{edim}_{\text{hyp}}(L) \leq \vartheta(L) \leq \dim_{\text{hyp}}(P).$$

Proof. The first inequality is given by Lemma 3.3. Hence we must only prove the second.

Fix r_0 small so that the restriction of L to a neighborhood of the disk $D_0 = D(0, r_0)$ is univalent. Let $A_0 = D_0 \setminus D(0, r_0/|\rho|)$, $D_P = L(D_0)$, $A_P = D_P \setminus P^{-1}(D_P)$ and

$$\ell = L|_{D_0}^{-1}.$$

Note that, for $w \notin D_P$, the sets $P^{-n}(w) \cap A_P$ are empty for $n \leq \nu$, with some $\nu = \nu(w)$ such that $\nu \rightarrow \infty$ as $w \rightarrow \infty$. And more precisely $\nu(w)$ grows like $\frac{\log \log |w|}{\log d}$. From Observation 4.3, it follows that, for any $t > \dim_{\text{hyp}} P$ and w with $|w|$ large,

$$Z^{\text{oc}}(t, L, w) = \sum_{n \geq \nu} \sum_{\zeta \in P^{-n}(w) \cap A_P} \frac{1}{|uL'(u)|^t} \frac{|w|^t}{|P^{n'}(\zeta)|^t},$$

where $u = \ell(\zeta)$.

Since A_0 is fixed and L is close to the identity near 0, the factor $uL'(u)$ is bounded away from 0 and ∞ uniformly in $u \in A_0$. It follows that it is enough to verify the following claim.

Claim. Denote by $Z^{\text{eucl}}(t, P^n, w)$ the first partition function for P^n in the euclidian metric:

$$Z^{\text{eucl}}(t, P^n, w) = \sum_{z \in P^{-n}(w)} \frac{1}{|(P^n)'(z)|^t}.$$

Then, the series $\sum_{n \geq \nu} |w|^t Z^{\text{eucl}}(t, P^n, w)$ converges to 0 as $w \rightarrow \infty$.

Proof of the Claim. The polynomial P is topologically Collet-Eckmann and $t > \dim_{\text{hyp}} P$. Hence the topological pressure associated to P is negative at exponent t .

From that it follows that for any bounded set F compactly embedded in the basin of attraction of ∞ , there exists $C_F > 0$ and $\pi_t > 0$ such that for all $\zeta \in F$ and all $m \geq 0$,

$$(6.1) \quad \sum_{z \in P^{-m}(\zeta)} |P^{m'}(z)|^{-t} \leq C_F e^{-\pi_t m}.$$

Suppose F is an annulus which is a fundamental domain for the dynamics of P in the basin of attraction of ∞ .

Given $\varepsilon > 0$, we can choose F close enough to ∞ so that for w in the unbounded component U of the complement of F , there exists $k = k(w) \in \mathbb{N}$, with $\nu(w) \geq k(w) \rightarrow \infty$ as $w \rightarrow \infty$, and satisfying the following. The set $P^{-k-1}(w)$ is a subset of F and for all $\zeta \in P^{-k}(w)$, we have $|(P^k)'(\zeta)| \geq (1 - \varepsilon)^{k+1} d^k \frac{|w|}{|\zeta|}$. Indeed, $P'(\xi) = \left(1 + O\left(\frac{1}{\xi}\right)\right) d \xi^{d-1}$ for $\xi \in U \cup F$, which implies

$$\begin{aligned} P^{k'}(\zeta) &= P'(\zeta)P'(P(\zeta)) \cdots P'(P^{k-1}(\zeta)) \\ &= \left(1 + O\left(\frac{1}{\zeta}\right)\right) \left(1 + O\left(\frac{1}{\zeta^d}\right)\right) \cdots \left(1 + O\left(\frac{1}{\zeta^{d^{k-1}}}\right)\right) d^k \zeta^{d^k-1}. \end{aligned}$$

Now we suppose F fixed and consider $w \in U$,

$$\begin{aligned} \sum_{n \geq \nu} |w|^t Z^{\text{eucl}}(t, P^n, w) &\leq \sum_{n \geq \nu} \sum_{\zeta \in P^{-k}(w)} \frac{|\zeta|^t}{((1-\varepsilon)^t d^t)^k} \sum_{z \in P^{-(n-k)}(\zeta)} \left| P^{n-k'}(z) \right|^{-t} \\ &\leq \frac{C_F}{((1-\varepsilon)^t d^t)^k} \sum_{n \geq \nu} e^{-(n-k)\pi t} \sum_{\zeta \in P^{-k}(w)} |\zeta|^t, \end{aligned}$$

where we have applied (6.1) with $m = n - k \geq 0$. The set F is bounded and the set $P^{-k}(w)$ has at most d^k points, thus,

$$\sum_{n \geq \nu} |w|^t Z^{\text{eucl}}(t, P^n, w) \leq C_{F,t} \left(\frac{d}{(1-\varepsilon)^t d^t} \right)^k$$

for some constant $C_{F,t} > 0$ depending only on F and $t > \dim_{\text{hyp}} P$.

As $t > 1$, it is possible to choose $\varepsilon > 0$ so that the right hand side tends to 0 as $w \rightarrow \infty$. △

■

Proof of Theorem 1.2. Suppose that L is a Poincaré function for a topological Collet-Eckmann polynomial P with connected Julia set. Then $\text{edim}_{\text{hyp}} L \geq \dim_{\text{hyp}} P$ by Theorem 5.1 and $\text{edim}_{\text{hyp}} L \leq \vartheta(L) \leq \dim_{\text{hyp}} P < 2$ by Lemma 6.1 and Theorem 2.7. Recall that $\text{edim}_{\text{hyp}} \geq 1$, so we have

$$1 \leq \text{edim}_{\text{hyp}}(L) = \vartheta(L) = \dim_{\text{hyp}}(P) < 2,$$

establishing the first claim of the theorem. If L is of disjoint type, then $\dim_{\text{hyp}}(L) < 2$ by Corollary 3.4.

Moreover, if P is not conformally conjugate to a Chebyshev polynomial or a power map, then $\dim_{\text{hyp}} P > 1$ by Theorem 2.7. ■

7. NON INVARIANCE INSIDE QUASICONFORMAL CLASSES

Proof of Corollary 1.7. For c in the main hyperbolic component H , let L_c be the normalised Poincaré function of the unique repelling fixed point of the polynomial P_c . Then, by Proposition 4.4, the map L_c has bounded postsingular set. Moreover its order ρ is given by the following:

$$\rho = \frac{\log 2}{\log |\lambda|}$$

where λ is the multiplier of the repelling fixed point (see [ES90]).

Since the polynomial P_c is hyperbolic, its hyperbolic dimension is equal to the dimension of its Julia set. Hence by Theorem 1.2,

$$\text{edim}_{\text{hyp}} L_c = \dim J(P_c).$$

The dimension of the Julia set is real analytic in $c \in H$ and nonconstant, see [Rue82]. Moreover, since the Julia set is a nondegenerate continuum, its dimension is bounded from below by 1. Meanwhile, all nonzero c inside this hyperbolic components are quasiconformally

conjugated to each other. Then, from Remark 4.2, it follows that the corresponding Poincaré functions are quasiconformally equivalent.

Now we fix some $c \neq 0$ in the main hyperbolic component and we use the notation L for L_c . It remains to show that for a choice of $\lambda > 0$ the function $f = \lambda L$ is quasiconformally conjugate to the map $E(z) := e^{-2}e^z$ (which is globally conjugated to $z \mapsto e^z - 2$) on a neighbourhood of the Julia sets.

Our aim is to apply Theorem 3.1 of [Rem09b]. We first need to recall a specific definition from this statement. If f and g are entire functions, they are said *quasiconformally equivalent near infinity* if there are quasiconformal functions $\varphi, \psi : \mathbb{C} \rightarrow \mathbb{C}$ such that $\varphi(f(z)) = g(\psi(z))$ whenever $|f(z)|$ and $|g(\psi(z))|$ are large enough.

Then, by [Rem09b], Theorem 3.1, if the functions f and g are of disjoint-type and quasiconformally equivalent near infinity, with the equivalence holding on disjoint-type tracts, then they are quasiconformally conjugated on a neighborhood of their respective Julia sets.

We will show that L and E are quasiconformally equivalent near infinity. Before that we need to check that the map L has only one tract and that this tract is a quasicircle.

Since the closure of the critical orbit is a subset of the interior of the filled Julia set $K(P_c)$ of the polynomial P_c , we can choose the tracts of L in a such way that their boundaries is the set $\Gamma = L^{-1}(J(P_c))$. Since the Julia set of P_c is a Jordan curve, the set Γ consists of simple (unbounded) curves. Moreover it follows from the invariance property of the Julia set that $\rho^n \Gamma = \Gamma$ for all $n \in \mathbb{Z}$.

Since L is univalent near 0 and since the Julia set of P_c is a Jordan curve, the intersection of Γ with a small disk $D = D(0, r_0)$ around 0 is made up of simple curves consisting of the points $z \in D$ such that $L(z) \in J(P_c)$. For $r_0 > 0$ small enough there is only one connected component Γ_0 of Γ that intersects D . Moreover this component is invariant by multiplication by ρ . This implies that there is only one tract. Indeed if Γ_1 is a connected component of Γ then for any $n \geq 0$ large enough $\rho^{-n} \Gamma_1$ intersects D hence Γ_0 thus Γ_1 intersects $\rho^n \Gamma_0 \subset \Gamma_0$.

The set $J(P_c)$ is a quasicircle. For any finite collection of points z_1, z_2, z_3, \dots in Γ one can find n such that $\rho^{-n} z_1, \rho^{-n} z_2, \rho^{-n} z_3, \dots$ all belong to $D \cap \Gamma$. It follows from that and from the geometric characterization of quasicircle (see e.g. [Ahl63]) that the curve $\Gamma \cup \{\infty\}$ is also a quasicircle. As a consequence the map L has only one tract over $\mathbb{C} \setminus K(P_c)$ and it is a quasidisk.

The Julia set $J(P_c)$ is bounded and bounded away from 0, hence the preimage of $\mathbb{C} \setminus K(P_c)$ by the map E is a (single) tract of E and a quasidisk. By Remark 2.7 of [Rem09b], it follows that L is quasiconformally equivalent to E on a tract, i.e. there are quasiconformal functions $\varphi, \psi : \mathbb{C} \rightarrow \mathbb{C}$ such that $\varphi(E(z)) = L(z)$ whenever $\operatorname{Re}(z)$ is large enough. ■

8. A HYPERBOLIC ENTIRE FUNCTION WITH HYPERBOLIC DIMENSION 2

Proof of Corollary 1.5. Let $f(z) = 2\pi i e^z$. The only finite singular value of f is 0 and it is sent onto the repelling fixed point $2\pi i$. Moreover, the two-dimensional Lebesgue measure of the escaping set of f is 0 (see for example [EL92], Theorem 7). As a consequence, for

almost all point z in \mathbb{C} , there exists $\delta > 0$ and a sequence of natural numbers $n_k \rightarrow \infty$ such that the spherical distance between $f^{n_k}(z)$ and the postsingular set $\{0, 2\pi i, \infty\}$ of f is at least δ^4 . By the main result of [Rem09a], it follows that the hyperbolic dimension of f is 2. Hence, it follows from Theorem 1.4 that $\text{edim}_{\text{hyp}} L = 2$.

Now, let $\lambda = L'(0)$ and let $r = \frac{\pi}{8|\lambda|}$. We show that if $|\lambda| < \frac{1}{20}$ then the image of the disk $D(0, r)$ by L is a compact subset of itself.

Indeed, the mapping f is univalent on the disk $D(2\pi i, \pi)$, and by the 1/4 Theorem,

$$f(D(2\pi i, \pi)) \supset D\left(2\pi i, \frac{2\pi \cdot \pi}{4}\right) \supset D(2\pi i, \pi).$$

Hence the lineariser L has an inverse defined on that disk. Hence by the 1/4 Theorem, L is univalent on $D(0, \pi/(4|\lambda|)) \subset (L|_{D(2\pi i, \pi)})^{-1}(D(2\pi i, \pi))$. By the Koebe Distortion Theorem, we have

$$L(D(0, r)) \subset D(2\pi i, 2r|\lambda|) = D(2\pi i, \pi/2).$$

The claim follows then from the assumption $|\lambda| < 1/20$.

According to Proposition 4.4, the singular set of L is the postsingular set of f . Hence it consists only of the two points 0 and $2\pi i$. As a consequence, the disk $D(0, r)$ is a subset of the Fatou set of L which absorbs all the singular orbits. Hence, the function L is hyperbolic and its Fatou set is connected. \blacksquare

A. EXISTENCE OF AN IFS

In this section we show that near every point of the Julia set of an entire function f , there is a finite conformal iterated function system, made of inverse branches of iterates of f , whose dimension is arbitrarily close to the hyperbolic dimension of f . Then we use this to bound the critical exponent of local pressures from below by the hyperbolic dimension.

This is used when estimating the eventual hyperbolic dimension of a Poincaré function from below.

A.1. Lemma.

Let f be an entire function and U an open set intersecting the Julia set of f . Then, for any $\varepsilon > 0$, there exists a finite conformal iterated function system defined on a subset of U made of inverse branches of iterates of f , such that the dimension of its limit set is at least $\dim_{\text{hyp}} f - \varepsilon$.

A.2. Remark.

This is also true in more generality for Ahlfors island maps.

Proof. Let $\chi = (D, (D_i, \varphi_i)_{i \in I})$ be a finite conformal iterated function whose limit set has dimension at least $\dim_{\text{hyp}} f - \varepsilon$, where, for all i , $D_i \subset D$ and $\varphi_i : D \rightarrow D_i$ is an inverse branch of some iterate f^{m_i} of f .

⁴In other words, almost all points belong to the radial Julia set of f .

Since D intersects the Julia set of f , there exists a natural number N such that $f^N(D) \cap U$ has nonempty interior and intersects $J(f)$. Let $V \subset D$ closed with nonempty interior such that $f|_V^N : V \rightarrow U' := f^N(V) \subset U$ is univalent and U' intersects the Julia set of f .

Since U' intersects the Julia set of f , there exists a natural number M such that $f^M(U') \supset D$. Then one can find, for all $i \in I$, closed sets with nonempty interiors $E_i \subset U'$ such that $f|_{E_i}^M : E_i \rightarrow f^M(E_i) \subset D_i$ is univalent.

Let k be any sufficiently large positive integer and consider the conformal iterated function system Y_k defined on U by the family of mappings $f|_{E_{i_0}}^{-M} \circ \varphi_{i_k} \circ \dots \circ \varphi_{i_1} \circ \varphi_{i_0} \circ f|_V^{-N}$, where $(i_0, i_1, \dots, i_k) \in I^{k+1}$. We claim that the dimension of its limit set is at least $\dim_{\text{hyp}} f - \varepsilon$ ⁵. Indeed, by using the pressure of the system X , we can find, for any $t < \dim_{\text{hyp}} f - \varepsilon$ and for any constant $C > 0$, a k_0 such that for all $k \geq k_0$, we have

$$\sum_{i \in I^k} \left\| (\varphi_{i_k} \circ \dots \circ \varphi_{i_1})' \right\|^t > C.$$

By a standard distortion argument, one can find k such that the first partition function of the system Y_k is greater than a given constant. It follows that the pressure of the system Y_k is positive for the exponent t . \blacksquare

Lemma 2.5 is a corollary of Lemma A.1.

Proof of Lemma 2.5. Let $\chi = (D, (D_i, \varphi_i)_{1 \leq i \leq I})$ be the iterated function system from Lemma A.1, with $\varphi_i = (f|_{D_i}^{m_i})^{-1}$. Denote \mathcal{P}_χ its pressure. By assumption, $\mathcal{P}_\chi(t) > 0$. Let $\varepsilon > 0$ be such that $\beta_0 = \mathcal{P}_\chi(t) - \varepsilon > 0$. Then, for all p large enough,

$$\sum_{|\alpha|=p} \|\varphi'_{\alpha_1}\|^t \cdot \dots \cdot \|\varphi'_{\alpha_p}\|^t \geq e^{\beta_0 p}.$$

Introduce, for $i = 1, \dots, I$, $\lambda_i = \|\varphi'_i\|^t$ and $\Lambda = \sum_{i=1}^I \lambda_i$. Note that

$$\Lambda^p = \sum_{|\alpha|=p} \lambda_{\alpha_1} \dots \lambda_{\alpha_p} = \sum_{|\alpha|=p} \|\varphi'_{\alpha_1}\|^t \cdot \dots \cdot \|\varphi'_{\alpha_p}\|^t.$$

Define $\varepsilon_i = \lambda_i / \Lambda$ and for each p choose a finite sequence of natural numbers $k_1(p), \dots, k_I(p)$ such that

$$(A.1) \quad \varepsilon_i p - 1 \leq k_i(p) < \varepsilon_i p + 1$$

and

$$\sum_{i=1}^I k_i(p) = p.$$

⁵It is indeed the same as the original system χ .

We will show that if

$$(A.2) \quad \nu_p = \sum_{i=1}^I k_i(p) m_i,$$

then the properties of the statement are satisfied for any p large.

Let's fix p and write k_i instead of $k_i(p)$. Consider the set S_p of multi-indexes α of length p such that the number of $0 \leq j \leq p$ such that $\alpha_j = i$ is precisely k_i for all $1 \leq i \leq I$. Then,

$$\frac{1}{\Lambda^p} \sum_{\alpha \in S_p} \lambda_{\alpha_1} \cdot \dots \cdot \lambda_{\alpha_p} = \frac{p!}{k_1! \cdot \dots \cdot k_I!} \cdot \frac{\lambda_1^{k_1} \cdot \dots \cdot \lambda_I^{k_I}}{\Lambda^p}.$$

Note that for $p \geq 1$ large enough, $k_i > 0$. From Stirling's approximation, it follows that for any p large enough,

$$\frac{p!}{k_1! \cdot \dots \cdot k_I!} \geq \left(\frac{1}{2\pi} \right)^{\frac{I-1}{2}} \cdot \sqrt{\frac{p}{k_1 \cdot \dots \cdot k_I}} \cdot \frac{p^p}{k_1^{k_1} \cdot \dots \cdot k_I^{k_I}}.$$

From equation (A.1), it follows that, for p large,

$$k_1 \cdot \dots \cdot k_I \leq 2p^I \varepsilon_1 \cdot \dots \cdot \varepsilon_I$$

and

$$\begin{aligned} k_1^{k_1} \cdot \dots \cdot k_I^{k_I} &\leq \left(\prod_i (\varepsilon_i p)^{k_i} \right) \prod_i \left(1 + \frac{1}{\varepsilon_i p} \right)^{k_i} \\ &\leq C_1 p^p \left(\varepsilon_1^{k_1} \cdot \dots \cdot \varepsilon_I^{k_I} \right) \end{aligned}$$

with some $C_1 > 0$ depending only on $(\varepsilon_i)_{i=1, \dots, I}$. Hence we have found a constant $C_2 > 0$ which depends only on $(I, \lambda_1, \dots, \lambda_I)$ and which is such that

$$\frac{p!}{k_1! \cdot \dots \cdot k_I!} \cdot \frac{\lambda_1^{k_1} \cdot \dots \cdot \lambda_I^{k_I}}{\Lambda^p} \geq C_2 \cdot \frac{1}{p^\delta} \cdot \frac{1}{\Lambda^p} \cdot \prod_{i=1}^I \left(\frac{\lambda_i}{\varepsilon_i} \right)^{k_i} = \frac{C_2}{p^\delta}.$$

with $\delta = (3I - 1)/2$.

This implies that

$$\begin{aligned} \sum_{\alpha \in S_p} \|\varphi'_{\alpha_1}\|^t \cdot \dots \cdot \|\varphi'_{\alpha_p}\|^t &= \frac{1}{\Lambda^p} \left(\sum_{\alpha \in S_p} \lambda_{\alpha_1} \cdot \dots \cdot \lambda_{\alpha_p} \right) \left(\sum_{|\alpha|=p} \|\varphi'_{\alpha_1}\|^t \cdot \dots \cdot \|\varphi'_{\alpha_p}\|^t \right) \\ &\geq \frac{C_2}{p^\delta} \cdot \sum_{|\alpha|=p} \|\varphi'_{\alpha_1}\|^t \cdot \dots \cdot \|\varphi'_{\alpha_p}\|^t \geq \frac{C_2 e^{\beta_0 p}}{p^\delta}. \end{aligned}$$

Then, by standard distortion estimates, and from the definition (A.2) of ν_p , there exists $C > 0$ such that for $w \in D$,

$$\sum_{z \in f^{-\nu_p}(w) \cap D} |f^{\nu_p'}(z)|^{-t} \geq C \sum_{\alpha \in S_p} \|\varphi'_{\alpha_1}\|^t \cdots \|\varphi'_{\alpha_p}\|^t \geq \frac{C'}{\nu_p^\delta} e^{\beta \nu_p},$$

with $\beta_1 = \beta_0/(2a)$ and $a = \sum_{i=1}^I \varepsilon_i m_i$.

For any $p \geq 1$, choose a bijection $\chi : \{1, \dots, I_p\} \rightarrow S_p$ with $I_p = \#S_p$, and define, for $1 \leq i \leq I_p$, $D_i^p := \varphi_{\chi(i)}(D)$. We will have equation (2.4) for $\beta > 0$ smaller than β_1 and an appropriate choice of $C > 0$. Finally, note that $ap - b \leq \nu_p < ap + b$ for $b = \sum_{i=1}^I m_i$. \blacksquare

REFERENCES

[Ahl63] Lars V. Ahlfors, *Quasiconformal reflections*, Acta Math. **109** (1963), 291–301. MR 0154978

[AL15] Artur Avila and Mikhail Lyubich, *Lebesgue measure of Feigenbaum Julia sets*, preprint arXiv:1504.02986, 2015.

[Bea91] Alan F. Beardon, *Iteration of rational functions*, Graduate Texts in Mathematics, vol. 132, Springer-Verlag, New York, 1991, Complex analytic dynamical systems.

[BK07] Krzysztof Barański and Bogusława Karpińska, *Coding trees and boundaries of attracting basins for some entire maps*, Nonlinearity **20** (2007), no. 2, 391–415. MR 2290468 (2007m:37105)

[BKZ09] Krzysztof Barański, Bogusława Karpińska, and Anna Zdunik, *Hyperbolic dimension of Julia sets of meromorphic maps with logarithmic tracts*, Int. Math. Res. Not. IMRN (2009), no. 4, 615–624. MR 2480096 (2009k:37109)

[BKZ12] ———, *Bowen’s formula for meromorphic functions*, Ergodic Theory Dynam. Systems **32** (2012), no. 4, 1165–1189. MR 2955309

[Bro65] Hans Brolin, *Invariant sets under iteration of rational functions*, Ark. Mat. **6** (1965), 103–144 (1965). MR 0194595 (33 #2805)

[DU91] Manfred Denker and Mariusz Urbański, *On Sullivan’s conformal measures for rational maps of the Riemann sphere*, Nonlinearity **4** (1991), no. 2, 365–384. MR 1107011 (92f:58097)

[EL92] Alexandre È. Eremenko and Mikail Yu Lyubich, *Dynamical properties of some classes of entire functions*, Ann. Inst. Fourier (Grenoble) **42** (1992), no. 4, 989–1020. MR 1196102 (93k:30034)

[ER15] Adam Epstein and Lasse Rempe-Gillen, *On invariance of order and the area property for finite-type entire functions.*, Ann. Acad. Sci. Fenn., Math. **40** (2015), no. 2, 573–599 (English).

[Ere89] Alexandre È. Eremenko, *On the iteration of entire functions*, Dynamical systems and ergodic theory (Warsaw, 1986), Banach Center Publ., vol. 23, PWN, Warsaw, 1989, pp. 339–345. MR 1102727 (92c:30027)

[ES90] A. È. Eremenko and M. L. Sodin, *Iterations of rational functions and the distribution of the values of Poincaré functions*, Teor. Funktsii Funktsional. Anal. i Prilozhen. (1990), no. 53, 18–25. MR 1077218 (92d:30016)

[GM05] John B. Garnett and Donald E. Marshall, *Harmonic measure*, New Mathematical Monographs, vol. 2, Cambridge University Press, Cambridge, 2005.

[GV73] F. W. Gehring and J. Väisälä, *Hausdorff dimension and quasiconformal mappings*, J. London Math. Soc. (2) **6** (1973), 504–512. MR 0324028 (48 #2380)

[KU08] Janina Kotus and Mariusz Urbański, *Fractal measures and ergodic theory of transcendental meromorphic functions*, Transcendental dynamics and complex analysis, London Math. Soc. Lecture Note Ser., vol. 348, Cambridge Univ. Press, Cambridge, 2008, pp. 251–316.

- [Lyu86] M. Yu. Lyubich, *Generic behavior of trajectories of the exponential function*, Uspekhi Mat. Nauk **41** (1986), no. 2(248), 199–200.
- [Lyu87] ———, *The measurable dynamics of the exponential*, Sibirsk. Mat. Zh. **28** (1987), no. 5, 111–127. MR 924986 (89d:58071)
- [Lyu00] Mikhail Lyubich, *The quadratic family as a qualitatively solvable model of chaos*, Notices Amer. Math. Soc. **47** (2000), no. 9, 1042–1052.
- [Mak85] N. G. Makarov, *On the distortion of boundary sets under conformal mappings*, Proc. London Math. Soc. (3) **51** (1985), no. 2, 369–384.
- [May17] Volker Mayer, *A lower bound of the hyperbolic dimension for meromorphic functions having a logarithmic hlder tract*, Preprint arXiv:1709.02188, 2017.
- [MBP12] Helena Mihaljević-Brandt and Jörn Peter, *Poincaré functions with spiders’ webs*, Proc. Amer. Math. Soc. **140** (2012), no. 9, 3193–3205. MR 2917092
- [McM87] Curtis T. McMullen, *Area and Hausdorff dimension of Julia sets of entire functions*, Trans. Amer. Math. Soc. **300** (1987), no. 1, 329–342. MR 871679 (88a:30057)
- [McM00] ———, *Hausdorff dimension and conformal dynamics. II. Geometrically finite rational maps*, Comment. Math. Helv. **75** (2000), no. 4, 535–593.
- [MU08] Volker Mayer and Mariusz Urbański, *Geometric thermodynamic formalism and real analyticity for meromorphic functions of finite order*, Ergodic Theory and Dynamical Systems **28** (2008), 915–946.
- [MU10] ———, *Thermodynamical formalism and multifractal analysis for meromorphic functions of finite order*, Mem. Amer. Math. Soc. **203** (2010), no. 954, vi+107. MR 2590263
- [MU19] Volker Mayer and Mariusz Urbanski, *Thermodynamic formalism and integral means spectrum of asymptotic tracts for transcendental entire functions*, Preprint arXiv:1709.05166, 2019.
- [MZ19] Volker Mayer and Anna Zdunik, *The failure of ruelle’s property for entire functions*, 2019.
- [PRL07] Feliks Przytycki and Juan Rivera-Letelier, *Statistical properties of topological Collet-Eckmann maps*, Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 1, 135–178. MR 2332354 (2008j:37093)
- [PRLS03] Feliks Przytycki, Juan Rivera-Letelier, and Stanislav Smirnov, *Equivalence and topological invariance of conditions for non-uniform hyperbolicity in the iteration of rational maps*, Inventiones mathematicae **151** (2003), no. 1, 29–63 (English).
- [PRLS04] Feliks Przytycki, Juan Rivera-Letelier, and Stanislav Smirnov, *Equality of pressures for rational functions*, Ergod. Th. Dynam. Sys. **24** (2004), no. 3, 891–914.
- [Prz93] Feliks Przytycki, *Lyapunov characteristic exponents are nonnegative*, Proceedings of the American Mathematical Society **119** (1993), no. 1, pp. 309–317 (English).
- [Prz98] ———, *Iterations of holomorphic Collet-Eckmann maps: conformal and invariant measures. Appendix: on non-renormalizable quadratic polynomials*, Trans. Amer. Math. Soc. **350** (1998), no. 2, 717–742. MR 1407501 (98d:58155)
- [Prz99] ———, *Conical limit set and Poincaré exponent for iterations of rational functions*, Trans. Amer. Math. Soc. **351** (1999), no. 5, 2081–2099. MR 1615954 (99h:58110)
- [PU10] Feliks Przytycki and Mariusz Urbanski, *Conformal fractals: Ergodic theory methods*, 1st ed., Cambridge University Press, New York, NY, USA, 2010.
- [Ree86] Mary Rees, *The exponential map is not recurrent*, Mathematische Zeitschrift **191** (1986), no. 4, 593–598 (English).
- [Rem09a] Lasse Rempe, *Hyperbolic dimension and radial Julia sets of transcendental functions*, Proc. Amer. Math. Soc. **137** (2009), no. 4, 1411–1420. MR 2465667 (2010h:37100)
- [Rem09b] ———, *Rigidity of escaping dynamics for transcendental entire functions*, Acta Math. **203** (2009), no. 2, 235–267. MR 2570071 (2011b:37084)
- [Rem14] Lasse Rempe-Gillen, *Hyperbolic entire functions with full hyperbolic dimension and approximation by Eremenko-Lyubich functions*, Proceedings of the London Mathematical Society **108** (2014), no. 5, 1193–1225.

- [RG16] Lasse Rempe-Gillen, *Arc-like continua, julia sets of entire functions, and eremenko's conjecture*, 2016.
- [RS10] Lasse Rempe and Gwyneth M. Stallard, *Hausdorff dimensions of escaping sets of transcendental entire functions*, Proc. Amer. Math. Soc. **138** (2010), no. 5, 1657–1665. MR 2587450 (2011a:37097)
- [RS17] Lasse Rempe-Gillen and Dave Sixsmith, *Hyperbolic entire functions and the Eremenko-Lyubich class: class \mathcal{B} or not class \mathcal{B} ?*, Math. Z. **286** (2017), no. 3-4, 783–800. MR 3671560
- [Rue82] David Ruelle, *Repellers for real analytic maps*, Ergod. Theor. Dynam. Syst. **2** (1982), 99–107.
- [Shi98] Mitsuhiro Shishikura, *The hausdorff dimension of the boundary of the mandelbrot set and julia sets*, Annals of Mathematics **147** (1998), no. 2, pp. 225–267.
- [Six14] D. J. Sixsmith, *A new characterisation of the Eremenko-Lyubich class*, J. Anal. Math. **123** (2014), 95–105. MR 3233575
- [Sta99] Gwyneth M. Stallard, *The Hausdorff dimension of Julia sets of hyperbolic meromorphic functions*, Math. Proc. Cambridge Philos. Soc. **127** (1999), no. 2, 271–288.
- [Sul82] Dennis Sullivan, *Discrete conformal groups and measurable dynamics*, Bull. Amer. Math. Soc. (N.S.) **6** (1982), no. 1, 57–73. MR 634434 (83c:58066)
- [Sul83] ———, *Conformal dynamical systems*, Geometric Dynamics (Jr. Palis, J., ed.), Lecture Notes in Mathematics, vol. 1007, Springer Berlin Heidelberg, 1983, pp. 725–752 (English).
- [UZ03] Mariusz Urbański and Anna Zdunik, *The finer geometry and dynamics of the hyperbolic exponential family*, Michigan Math. J. **51** (2003), no. 2, 227–250. MR 1992945 (2004d:37068)
- [Zdu90] Anna Zdunik, *Parabolic orbifolds and the dimension of the maximal measure for rational maps*, Invent. Math. **99** (1990), no. 3, 627–649. MR 1032883 (90m:58120)

DEPT. OF MATHEMATICAL SCIENCES, UNIVERSITY OF LIVERPOOL, LIVERPOOL L69 7ZL, UK.
ORCID: 0000-0002-6463-3304. MR AUTHOR ID: 906429.

E-mail address: adezotti@liverpool.ac.uk

DEPT. OF MATHEMATICAL SCIENCES, UNIVERSITY OF LIVERPOOL, LIVERPOOL L69 7ZL, UK.
ORCID: 0000-0001-8032-8580; MR AUTHOR ID: 738017.

E-mail address: l.rempe@liverpool.ac.uk