SINGULAR ORBITS AND BAKER DOMAINS

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ABSTRACT. We show that there is a transcendental meromorphic function with an invariant Baker domain U such that every singular value of f is a super-attracting periodic point. This answers a question of Bergweiler from 1993. We also show that U can be chosen to contain arbitrarily large round annuli, centred at zero, of definite modulus. This answers a question of Mihaljević and the author from 2013, and complements recent work of Barański et al concerning this question.

> I long for the Person from Porlock
> To bring my thoughts to an end,
> I am becoming impatient to see him
> I think of him as a friend.

> <div style="text-align: right">Stevie Smith</div>

1. INTRODUCTION

Let $f \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function of degree at least 2. The *Fatou set* $F(f)$ consists of those points $z \in \mathbb{C}$ near which the iterates of f form a normal family in the sense of Montel. In other words, these are the points at which the dynamics generated by f is stable under small perturbations. A connected component of $F(f)$ is called a *Fatou component*; such a component is *invariant* if $f(U) \subset U$. An invariant Fatou component may be an immediate basin of attraction of an attracting or parabolic fixed point, or a simply or doubly connected domain on which f is conjugate to an irrational rotation.

It was shown by Fatou [Fat20, §30–31] that there is a close relationship between invariant Fatou components and the critical values of f. Indeed, every attracting or parabolic basin must contain a critical value, and the boundary of any rotation domain is contained in the *postcritical set* $\mathcal{P}(f)$, i.e., the closure of the union of all critical orbits of f. See Lemmas 8.5, 15.7 and Theorems 10.15, 11.17 of [Mil06].

These relationships carry over to the case of a transcendental meromorphic function $f \colon \mathbb{C} \to \hat{\mathbb{C}}$, with the set of critical values replaced by the set $\operatorname{sing}(f^{-1})$ of critical or asymptotic values of f, and the postcritical set by the *postsingular set* defined analogously. In this setting, there is another possible type of invariant Fatou component: An *invariant Baker domain*, in which the iterates converge to the essential singularity at ∞. Bergweiler [Ber93, Question 4] asked whether there is a relation between $\operatorname{sing}(f^{-1})$ and the boundary of such a Baker domain. He also asked a more precise version of this question [Ber93, Question 5]:

1.1. Question. Is it possible that a meromorphic function f has Baker domains if the forward orbit of z is bounded for all $z \in \operatorname{sing}(f^{-1})$?

We give an affirmative answer.

1.2. Theorem (Baker domains and super-attracting points)**.** *There is a transcendental meromorphic function* $f\colon \mathbb{C} \to \hat{\mathbb{C}}$ *with a Baker domain such that every point of* $\mathrm{sing}(f^{-1})$ *is a super-attracting periodic point of period* 2.

Regarding [Ber93, Question 4], Bergweiler [Ber95, Theorem 3] obtained the following answer: If U contains no critical or asymptotic values of f, then there exists a sequence $(p_n)_{n=0}^{\infty}$ of postsingular points $p_n \in \mathcal{P}(f)$ of f such that

(a) $|p_n| \to \infty$;
(b) $\frac{\mathrm{dist}(p_n, W)}{|p_n|} \to 0$; and
(c) $\lim |p_{n+1}|/|p_n| = 1$.

On the other hand [Ber95, Theorem 1], there is an entire function with a Baker domain U such that $\mathrm{dist}(\mathcal{P}(f), \partial U) > 0$, where dist denotes Euclidean distance.

For Baker domains of general transcendental meromorphic functions, it was shown in [MR13, Theorem 1.5] that an analogue of [Ber95, Theorem 3] holds, without the hypothesis that U contains no critical or asymptotic values, where condition (c) is replaced by the weaker

$$(1.1) \qquad\qquad \limsup |p_{n+1}|/|p_n| < \infty.$$

The article [MR13] also poses the following question.

1.3. Question ([MR13, Remark on p. 1603])**.** Can (1.1) be replaced by (c)?

In [BFJK20, Theorem A], it is shown that the answer is positive if $\mathbb{C} \setminus U$ has an unbounded connected component. Here we show that the answer to Question 1.3 is negative in general.

1.4. Theorem (Annuli of definite modulus)**.** *Let* $\rho > 0$. *Then the function in Theorems 1.2 can be chosen such that there is a sequence* $R_j \to \infty$ *with*

$$(1.2) \qquad\qquad \{z \in \mathbb{C} \colon R_j < |z| < \rho R_j\} \subset U$$

for all j.

The constructions in our paper are inspired by those of Rippon and Stallard [RS06, Theorem 1.2] and Barański et al [BFJK15, Theorem C]. Both of these construct meromorphic functions with multiply-connected Baker domains, obtained from an affine map by inserting poles at a sequence of points tending to infinity. We shall use quasiconformal surgery instead of explicit formulae, which allows us to obtain the required control of the singular orbits of f.

Notation. $\mathbb{C}$, $\hat{\mathbb{C}}$ and $\mathbb{D}$ denote the complex plane, Riemann sphere and unit disc, respectively. The Euclidean disc of radius r around $z \in \mathbb{C}$ is denoted by $D(z, r)$.

2. Quasiconformal surgery

The function in Theorem 1.2 is constructed by a "cut and paste" quasiconformal surgery (see [PLB$^+$14]), starting from the linear map

$$f_0(z) := \mu z,$$

where $\mu > 1$.

Set

$$U_0 := \{z \in \mathbb{C} \colon \operatorname{Re} z \geq 0\} \cup \bigcup_{j \geq 0} \left\{z \in \mathbb{C} \colon \mu^j \leq |z| \leq \mu^{j+\frac{1}{2}}\right\}.$$

Observe that $f_0(U_0) \subset U_0$. Also let D_0 be a closed disc

$$D_0 = \overline{D(\zeta_0, r_0)} \subset \{z \in \mathbb{C} \colon \operatorname{Re} z < 0 \text{ and } \sqrt{\mu} < |z| < \mu\} \subset \mathbb{C} \setminus U_0.$$

Set

$$D_j := f_0^j(D_0) = \overline{D(\mu^j \cdot \zeta_0, \mu^j \cdot r_0)} =: \overline{D(\zeta_j, r_j)}.$$

Then all D_j are disjoint from U_0, and from each other.

We construct our functions from f_0 by a quasiconformal surgery that inserts poles in the discs D_j. The following makes this procedure precise.

2.1. Proposition (Quasiconformal surgery of f_0). *Let $\mu > 1$, and let f_0, U_0 and $(D_j)_{j=0}^{\infty}$ be defined as above. Let $K > 1$, and let*

$$h_j \colon D_j \to \hat{\mathbb{C}}$$

be a sequence of K-quasiregular functions such that each h_j extends continuously to ∂D_j with $h_j = f_0$ on ∂D_j. Then

$$(2.1) \qquad\qquad F \colon \mathbb{C} \to \hat{\mathbb{C}}; \quad z \mapsto \begin{cases} h_j(z) & \text{if } z \in D_j \\ f_0(z) & \text{if } z \notin \bigcup_j D_j. \end{cases}$$

is K-quasiregular, with no finite asymptotic values, and every critical value of F is a critical value of some h_j.

Suppose furthermore that, for infinitely many j, the map h_j is not a homeomorphism $h_j \colon D_j \to D_{j+1}$, and that for every j there is an open set X_j with the following properties.

(a) The dilatation of h_j is supported on X_j.

(b) F is conformal on $F^n(X_j)$ for all $n \geq 1$.

Then there is a quasiconformal homeomorphism $\psi \colon \mathbb{C} \to \mathbb{C}$ with $\psi(0) = 0$ such that

$$(2.2) \qquad\qquad f := \psi \circ F \circ \psi^{-1}$$

is a transcendental meromorphic function. Moreover, ψ is conformal on U_0, and $\psi(U_0)$ is contained in an invariant Baker domain U of f.

Proof. The claim that F is K-quasiregular follows from Royden's glueing lemma [Ber74, Lemma 2]. Any curve to infinity must intersect U_0, and therefore F is unbounded on any such curve. In particular, F has no finite asymptotic values. As $F = f_0$ on U_0, every critical value of F must come from one of the h_j.

Now suppose that the additional conditions hold. Since h_j maps ∂D_j to ∂D_{j+1} in one-to-one fashion, if h_j is not a homeomorphism, then h_j must have at least one pole in D_j. As this happens for infinitely many j, we see that F has infinitely many poles. Let

$\omega_0 := F_0^*(0)$ be the complex dilatation of F. By (b), this dilatation is forward-invariant on

$$X := \bigcup_{j,n \geq 0} F^n(X_j).$$

As F is meromorphic outside of X, we may extend ω_0 by pull-back to the grand orbit of X. Extending the resulting differential by the standard complex structure, we obtain an F-invariant measurable Beltrami differential ω on all of $\hat{\mathbb{C}}$. Observe that $\omega \equiv 0$ on U_0.

By the measurable Riemann mapping theorem, there is a quasiconformal homeomorphism $\psi : \mathbb{C} \to \hat{\mathbb{C}}$ that solves the Beltrami equation for ω. Then ψ is conformal on U_0. Moreover, the map f defined by (2.2) is meromorphic; it is transcendental since F has infinitely many poles. Let $z \in \psi(U_0)$; say $z = \psi(\zeta)$ with $\zeta \in U_0$. Then

$$f_0(z) = \psi(F(\psi^{-1}(z))) = \psi(F(\zeta)) \in \psi(U_0), \quad \text{and}$$
$$f_0^n(z) = \psi(F^n(\psi^{-1}(z))) = \psi(F^n(\zeta)) = \psi(f_0^n(\zeta)) \to \infty.$$

So $\psi(U_0)$ is contained in a Baker domain of f, as claimed. ∎

2.2. Proposition (Large annuli in U). *For every $\rho > 1$, there is $\mu > 1$ with the following property. Let $f_0(z) = \mu \cdot z$, and let f be obtained from f_0 as in Proposition 2.1. Then f satisfies (1.2) for a sequence $R_j \to \infty$.*

Proof. Recall that U_0 contains arbitrarily large round annuli of modulus at least $M := (\log \mu)/2$. If $M \geq \Lambda(\rho)$, where $\Lambda(\rho)$ is the modulus of the Teichmüller annulus [Ahl73, §4–11], then any annulus that separates 0 from ∞ and has modulus at least M contains a round annulus of modulus $\log \rho$ centred at zero. Hence we can take $\mu = e^{2\Lambda(\rho)}$. ∎

3. Proof of Theorem 1.2

To prove Theorem 1.2, we construct a suitable sequence of functions h_j to use in Proposition 2.1. The idea is that each h_j has critical values in D_{j+1}, which are then mapped back to the original critical points by h_{j+1}. We begin with the following.

3.1. Proposition. *Let $\Delta = D(\zeta, r) \subset \mathbb{C}$ be a round disc, and let $K > 1$, $\eta < r$, and $\vartheta < \mu \cdot \eta$. For every $a \in \mathbb{C} \setminus f_0(\Delta)$, there is a K-quasiregular map*

$$g : \Delta \to \hat{\mathbb{C}}$$

such that

 (1) g extends continuously to $\partial \Delta$, where it agrees with f_0.
 (2) $g(\zeta) = a$.
 (3) g has exactly two critical points, c_1 and c_2, with $g(c_1) \neq g(c_2)$.
 (4) Each critical point c_j satisfies $g(c_j) \in D(f_0(\zeta), \vartheta)$.
 (5) g is meromorphic on $D(\zeta, \eta)$.
 (6) g is injective (i.e., quasiconformal) on $\Delta \setminus D(\zeta, \eta)$, and $\mu r > |g(z) - f_0(\zeta)| > \vartheta$ for $r > |z - \zeta| \geq \eta$.

Proof. Set

$$\alpha := \frac{a}{r\mu} - \frac{\zeta}{r} \neq 0.$$

Let $\varepsilon > 0$, and consider the map

$$\varphi = \varphi_{\alpha,\varepsilon} \colon \mathbb{D} \to \hat{\mathbb{C}}; \quad z \mapsto z + \frac{\varepsilon\alpha}{\alpha z + \varepsilon} = z + \frac{\varepsilon}{z - p},$$

where $p = -\varepsilon/\alpha$. Then $\varphi(0) = \alpha$ and $\varphi(p) = \infty$. By direct calculation, φ has two simple critical points at $p \pm \sqrt{\varepsilon}$, with critical values at $p \pm 2\sqrt{\varepsilon}$.

Now define $A \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the affine map taking $\mathbb{D}$ to Δ; i.e., $A(z) = r \cdot z + \zeta$. Define

$$g_\varepsilon \colon D(\zeta, \eta) \to \hat{\mathbb{C}}; \quad z \mapsto \mu \cdot A\big(\varphi_{\alpha,\varepsilon}\big(A^{-1}(z)\big)\big).$$

For all sufficiently small ε, g_ε satisfies (3), (4) and (5). Furthermore,

$$g_\varepsilon(\zeta) = \mu \cdot (r\alpha + \zeta) = a,$$

so g also satisfies (2).

Set $\Delta_\eta := D(\zeta, \eta)$. As $\varepsilon \to 0$, $g_\varepsilon \to f_0$ uniformly on a neighbourhood of $\partial\Delta_\eta$. In particular, for sufficiently small ε, g_ε is conformal on a neighbourhood of $\partial\Delta_\eta$, and satisfies

$$\mu r > |g_\varepsilon(z) - f_0(\zeta)| > \vartheta$$

when $z \in \partial\Delta_\eta$. It follows that there is a quasiconformal homeomorphism G that maps the round annulus $\Delta \setminus \overline{\Delta_\eta}$ to the annulus bounded by $f_0(\partial\Delta)$ and $g_\varepsilon(\partial\Delta_\eta)$, and which agrees with f_0 on $\partial\Delta$ and with g_ε on $\partial\Delta_\eta$; see [Leh65]. We define g to agree with g_ε on D_η and with G outside; then g satisfies (1)–(6).

It is easy to see that the dilatation of G, and thus of g, tends to 1 as $\varepsilon \to 0$. So, for sufficiently small ε, g is indeed K-quasiconformal, as claimed. ∎

Now let f_0 and D_j be as above.

3.2. Proposition. *For any $K > 0$, there is a sequence of K-quasiregular maps*

$$h_j \colon D_j \to \hat{\mathbb{C}}$$

with the following properties for all j.

(a) *For every j, the map h_j has at least one critical point.*
(b) *If c is a critical point of h_j, then $h_j(c) \in D_{j+1}$ and $h_{j+1}(h_j(c)) = c$.*

Furthermore, for each j there are open sets $V_j \subset W_j \subset D_j$ such that

(c) *$D_j \setminus V_j$ is compact;*
(d) *h_j is injective on W_j;*
(e) *$h_j(z) = f_0(z)$ for $z \in V_j$;*
(f) *$h_j(W_j) \subset V_{j+1}$;*
(g) *the dilatation of h_j is supported on W_j.*

Proof. The sequence of functions is constructed inductively, in such a way as to ensure the following inductive hypotheses. For $j \geq 0$, let $\mathcal{C}_j \subset D_j$ be the set of critical points of h_j and $\Omega_{j+1} = h_j(\mathcal{C}_j) \subset \mathbb{C}$ the set of critical values. The construction will ensure the following inductive hypotheses:

(A) $\Omega_{j+1} \subset D_{j+1}$.
(B) $h_j \colon \mathcal{C}_j \to \Omega_{j+1}$ is injective, so there is $c_{j+1} \colon \Omega_{j+1} \to \mathcal{C}_j$ with $c_{j+1}(h_j(c)) = c$ for all $c \in \mathcal{C}_j$.
(C) $h_j(W_j) \subset D_{j+1}$ and $\overline{h_j(W_j)} \cap \Omega_{j+1} = \emptyset$.

To anchor the recursive construction, we also set $\Omega_0 := \{\zeta_0\}$ and $c(\zeta_0) := 0$. Note that this means that (A) holds also for $j = -1$.

Now suppose that, for $i < j$, h_i has been constructed satisfying these inductive hypotheses. To define h_j, let $\omega_j^1, \ldots, \omega_j^m$ be the elements of Ω_j. For each $i = 1, \ldots, m$, choose a small disc $\Delta_j^i = D(\omega_j^i, \rho_j^i) \Subset D_j$ around ω_j^i. If $j > 0$, these discs should be chosen such that the closures of the Δ_j^i do not intersect each other or $\overline{h_{j-1}(W_{j-1})}$. This is possible by (A) and (C). Set

$$(3.1) \qquad\qquad V_j := D_j \setminus \bigcup_{i=1}^{m} \overline{\Delta_j^i} \supset h_{j-1}(W_{j-1}).$$

For each $i = 1, \ldots, m$, apply Proposition 3.1 to $\Delta = \Delta_j^i$, $a = c(\omega_j^i)$, and $\vartheta, \eta = \eta_j^i$ arbitrary (subject to the conditions given in the proposition.). Let g_j^i denote the map obtained, define

$$(3.2) \qquad\qquad h_j(z) := \begin{cases} f_0(z) & \text{if } z \in V_j; \\ g_j^i(z) & \text{if } z \in \overline{\Delta_j^i} \end{cases}$$

and set

$$(3.3) \qquad\qquad W_j := D_j \setminus \bigcup_{i=1}^{m} \overline{D(\omega_j^i, \eta_j^i)} \supset V_j.$$

We claim that h_j satisfies (A)–(C). For the remainder of the proof, references to (1)–(5) shall mean the corresponding properties of the g_j^i established in Proposition 3.1.

The critical points and critical values of h_j are exactly those of the maps g_j^i. By (4) of Proposition 3.1, the critical values of g_j^i are in $f_0(\Delta_j^i) \subset f_0(D_i) = D_{i+1}$, establishing (A). Furthermore, by (3) of Proposition 3.1, g_j^i has exactly two different critical points, with different critical values belonging to $f_0(\Delta_j^i)$. Since different Δ_j^i are pairwise disjoint, and f_0 is injective, claim (B) follows. Finally, (C) is an immediate consequence of (4) and (6)

This completes the construction of the h_j. By (1) and Royden's glueing lemma, the map h_j is indeed a K-quasiregular map. It remains to establish (a)–(g).

(a) Each h_j^i has exactly two critical points, and $\Omega_0 \neq \emptyset$. It follows inductively that $\Omega_j \neq \emptyset$ for all j, and hence each h_j has at least one critical point.
(b) If c is a critical point of h_j, then $h_j(c) = \omega \in \Omega_{j+1} \subset D_{j+1}$ by (A). By (2), $h_{j+1}(h_j(c)) = c(\omega) = c$.
(c) By definition of V_j in (3.1), $D_j \setminus V_j$ is compact.
(d) $h_j|_{W_j}$ is injective by (3.3) and (6).
(e) By (3.2), $h_j = f_0$ on V_j.
(f) By (3.1), $V_j \supset h_j(W_j)$.
(g) By (5), the dilatation of h_j is supported on W_j.   ∎

Proof of Theorems 1.2 and 1.4. Choose μ as in Proposition 2.2, and let $(h_j)_{j=0}^{\infty}$ be as in Proposition 3.2. Let F be the quasiregular map defined by (2.1). Then $F^2(c) = c$ for every critical point c of F.

The dilatation of h_j is supported on W_j, and h_j is conformal on $\overline{V_j}$. Moreover, $F^n(X_j) \subset V_{n+j}$ for all n and j. Hence $X_j := W_j \setminus \overline{V_j}$ satisfies the conditions of Proposition 2.1. Let f be the entire function 2.2; then f has a Baker domain U and satisfies $f^2(c) = c$ for every critical point of f. This completes the proof of Theorem 1.2.

Furthermore, by Proposition 2.2, the function also satisfies 1.2 for a sequence $R_j \to \infty$, proving Theorem 1.4. ∎

References

[Ahl73]   Lars V. Ahlfors, *Conformal invariants: topics in geometric function theory*, McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973, McGraw-Hill Series in Higher Mathematics.

[Ber74]   Lipman Bers, *On moduli ofKleinian groups*, Russian Mathematical Surveys **29** (1974), no. 2, 88–102.

[Ber93]   Walter Bergweiler, *Iteration of meromorphic functions*, Bull. Amer. Math. Soc. (N.S.) **29** (1993), no. 2, 151–188.

[Ber95]   _____, *Invariant domains and singularities*, Math. Proc. Cambridge Philos. Soc. **117** (1995), no. 3, 525–532.

[BFJK15] Krzysztof Barański, Núria Fagella, Xavier Jarque, and Bogusława Karpińska, *Absorbing sets and Baker domains for holomorphic maps*, J. Lond. Math. Soc. (2) **92** (2015), no. 1, 144–162.

[BFJK20] _____, *Fatou components and singularities of meromorphic functions*, Proc. Roy. Soc. Edinburgh Sect. A **150** (2020), no. 2, 633–654.

[Fat20]   P. Fatou, *Sur les équations fonctionnelles (deuxième mémoire)*, Bulletin de la Société Mathématique de France **48** (1920), 33–94 (fr).

[Leh65]   Olli Lehto, *An extension theorem for quasiconformal mappings*, Proc. London Math. Soc. (3) **14a** (1965), 187–190.

[Mil06]   John Milnor, *Dynamics in one complex variable*, third ed., Annals of Mathematics Studies, vol. 160, Princeton University Press, Princeton, NJ, 2006.

[MR13]   Helena Mihaljević-Brandt and Lasse Rempe-Gillen, *Absence of wandering domains for some real entire functions with bounded singular sets*, Math. Ann. **357** (2013), no. 4, 1577–1604.

[PLB+14] Kevin M. Pilgrim, Tan Lei, Shaun Bullett, Bodil Branner, and Nria Fagella, *Cut and paste surgeries*, Cambridge Studies in Advanced Mathematics, p. 218306, Cambridge University Press, 2014.

[RS06]   P. J. Rippon and G. M. Stallard, *Singularities of meromorphic functions with Baker domains*, Mathematical Proceedings of the Cambridge Philosophical Society **141** (2006), no. 2, 371382.

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