

ON DISTRIBUTIONAL AND ASYMPTOTIC RESULTS FOR EXPONENTIAL FUNCTIONALS IN RENEWAL-REWARD PROCESSES DESCRIBING INSURANCE RISK MODELS *

J. AKAHORI[†], C. CONSTANTINESCU[‡], Y. IMAMURA[§], AND HH. PHAM[¶]

Abstract.

Inspired by the double-debt problem in Japan where the mortgagor has to pay the remaining loan even if their house was destroyed by a catastrophic event, we model the lender's cash flow, by an exponential functional of a renewal-reward process. We propose an insurance add-on to the loan repayments and analyse the asymptotic behavior of the distribution of the first hitting time, which represents the probability of full repayment. We show that the finite-time probability of full loan repayment converges exponentially fast to the infinite-time one. In a few concrete scenarios, we calculate the exact form of the infinite-time probability and the corresponding premiums.

Key words. exponential functional, renewal-reward process, ruin probability, double-debt problem, Fredholm integro-differential equation, stochastic fixed point equation

AMS subject classifications. 68Q25, 68R10, 68U05

1. Introduction. We develop a mathematical model for mortgage loans so that we can estimate/measure the risks of the lenders. The model is inspired by a framework designed to hedge the so-called “double-debt problem”, introduced by Ohgaki [24]. For analyzing the risks, we employ methods from risk theory, culminating in an analysis of solutions integro-differential equations with boundary conditions.

The Motivation. After the 2011 Great East Japan Earthquake, a lot of people who lost their houses are still kept under the due of their mortgage loan, which made their recovery rather difficult. It is commonly referred to as the double-debt problem. As Japan is exposed to the risk of further big earthquakes, Ohgaki [24] proposed a practical framework within the regime of the Japanese financial system, where the mortgage loan is combined with a marketized earthquake insurance, like a CAT bond. This paper mathematically formalizes Ohgaki's [24] proposed scheme. The proposed model will be in continuous time as the mortgage payments of individual mortgagors are not likely to be paid at the same time in a given payment period. However, in order to marketize this insurance-mortgage-security, we need to know the risk exposure.

The Model. Specifically, we consider an initial loan u given to a cohort of borrowers (as mortgages) that are paying it back continuously at a constant rate $c > 0$. Although the loans are fixed term, as mentioned before, the assumption of continuous payments is reasonable, due to the cohort effect, meaning that new customers are coming in as others are leaving the programme. Furthermore, we

*

Funding: This research was partially supported by the European Union's Seventh Framework Programme for research, technological development and demonstration under grant agreement no 318984 - RARE.

[†]Ritsumeikan University, Kusatsu, Japan (akahori@se.ritsumei.ac.jp,).

[‡]University of Liverpool, Liverpool, UK (C.Constantinescu@liverpool.ac.uk).

[§]Kanazawa University, Kanazawa, Japan (imamura@se.kanazawa-u.ac.jp).

[¶] International University, Vietnam National University Hochiminh city, Vietnam (ppha@hcmiu.edu.vn).

consider that disasters occur at random times T_i and after each disaster i only a ratio e^{-X_i} of borrowers are left to repay the loan. Thus, as time passes and disasters occur we have fewer and fewer borrowers paying back the loan (at the same rate c). We define as default/ruin the event that the (cash contribution) "process U_t never reaching u ", meaning that the borrowers will never fully repay the loan. Thus, the probability of default, $\phi(u)$, describes the probability of the process U_t never reaching level u , while the survival probability, $\psi(u)$, defines the probability of the first crossing of the level u , or *probability of full loan repayment*.

Let (T_i, X_i) , $i = 1, 2, \dots$ be a marked point process, with $T_0 = 0$. In our model, the cash flow process of the mortgage loan at time t is given by

$$(1.1) \quad U_t = c \sum_{n=0}^{\infty} 1_{[T_n, T_{n+1})}(t) \left\{ (t - T_n) e^{-\sum_{i=1}^n X_i} + \sum_{i=1}^n (T_i - T_{i-1}) e^{-\sum_{k=1}^{i-1} X_k} \right\},$$

where c is a positive constant, T_i is the occurrence time of the i -th disaster and X_i is the rate of the borrowers who survived the i -th disaster, with the requirement $X_i > 0$ (it cannot be zero). Here we denote $\sum_{i=n}^m \dots = 0$ for $m < n$. The process can be understood as an *exponential functional* of the renewal reward process

$$U_t = c \int_0^t e^{R_s} ds$$

where R is the renewal-reward process associated with (T_i, X_i) , $i = 1, 2, \dots$, that is,

$$R_t := - \sum_{j=1}^{J_t} X_j, \quad J_t = \sum_{k=1}^{\infty} 1_{\{T_k \leq t\}}.$$

Figure 1 is the graphical representation of the repayment process. Each red dot

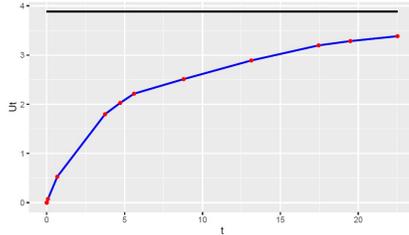


FIG. 1. A possible sample path.

represents a disaster hit. After every hit, the rate of repayment decreases since the affected mortgagors are forced to leave the pool. The repayment rate therefore slows down as disasters occur. Our aim is to make sure the path reaches the black line (the total amount lent u), in other words the full amount is repaid, within a finite time. Note that the model was inspired by the Japanese double-debt problem, but could account for any risk processes that temporises the effect of claims/losses.

Let τ_u be the first hitting time of $u > 0$, meaning the time the process U (which starts at zero) reaches u for the first time, which means the time the loan has been paid back in full. We shall study the probability of full loan repayment ψ , before a given time t ,

$$\psi(u, t) := P(\tau_u < t),$$

and its corresponding probability of loan default in finite time horizon

$$\phi(u, t) := P(\tau_u \geq t) = 1 - \psi(u, t),$$

for $u > 0$ and $t > 0$.

The Proposal. The solution proposed in this paper is to include a small insurance premium in the contracted mortgage payments and in return the mortgagor would be relieved of the outstanding mortgage principal in the event of property destruction by a natural disaster. With every disaster hit, it is expected that properties will be destroyed and mortgagors will subsequently default. The premium will be set to ensure that the total amount of all *combined* principals is completely paid off within a finite time-scale. In our model, the premium rate for each mortgagor is going to be the same indifferent of the size of their loan or region of residence.

The combined premiums from all of the mortgagors would be enough to cover the defaulted loans due to natural disasters. If a natural disaster occurs and causes damages of greater amount than the current reserve, the remainder of damages costs would be paid off gradually as the premiums continue to be collected from the mortgagors not affected by the disaster. The insurance premium is calculated and set to ensure that the total amount of combined principals could be paid off in a finite time.

The Results. We derive the probability of full loan repayment, ψ , as a solution of an integral equation. This expression could be further used for the numerical evaluation of the default probability, but in this paper, for specific examples, we propose to use a calibration of the parameters of the infinite-time full-loan-repayment-probability

$$\psi^\infty(u) := \lim_{t \rightarrow \infty} \psi(u, t) = P(\tau_u < \infty),$$

and its counter-part, the probability of loan default

$$(1.2) \quad \phi^\infty(u) := P(\tau_u = \infty) = 1 - \psi^\infty(u),$$

for all $u > 0$. This strategy of approximation can be justified by Theorem 3.1 which shows that the difference, between the finite and infinite time probabilities, decays exponentially fast, as $t \rightarrow \infty$, namely for any $u > 0$, there exists $C(u) > 0$ such that

$$0 \leq \psi^\infty(u) - \psi(u, t) \leq C(u)e^{-\xi t}, \quad t \geq 0,$$

for some $\xi > 0$. The numerical experiments in section 5.1 show that under a practical parameter set, the difference is negligible.

Furthermore, under the infinite horizon setting, as in [10], we will discuss the risk parameters for which the ruin probability stays within a range established by say a supervisory authority. We will present the cases of Poisson, non-Poisson and randomized arrivals, all the while the ratio of clients remaining from the initial cohort is exponentially distributed. Concretely, we are considering models with Poisson arrivals with parameter λ , followed by models with non-Poisson arrivals, as in Gamma(2, λ) inter-arrival times, to account for some memory in the process. We study the probability of default or full repayment, by means of integro-differential equations and Laplace transforms, as in e.g. [3, 4, 2]. Conjointly, we discuss the case of random Poisson parameter λ , equivalently to the distribution of inter-arrival time W being an

exponential random variable with random parameter Λ , to account for the clustering effect of the earthquake events. Thus one could estimate Λ and dynamically adjust the risks involved, as in [10].

Connections to exponential functionals of Lévy processes. The default (or survival) probability in our model can be interpreted as the distribution of an exponential functional of a stochastic process. Exponential functionals of Lévy processes abound in both finance and insurance mathematics literature. In finance, the distribution of the exponential functional of a process is key in computing the Asian option price, and has attracted a lot of attention, featuring different approaches such as [8], [25], and [27]. Some further references on applications of exponentials of Brownian motion can be found in [28]. In insurance, [16] used the exponential functional of a Lévy process in the analysis of an insurer liabilities when its variable annuity guarantees benefits on an exponential maturity of counterpart. Moreover, *perpetuities*, that can be seen as exponential functionals of a renewal-reward process, arise in a diverse range of fields, see [26] and [14] for a variety of references and examples of applications, including insurance [22, 23] and economics [11]. In [13], the distribution of a perpetuity presents applications in risk theory and pensions.

Connections to stochastic perpetuities. We note that our cash contribution process U is closely related to a *stochastic perpetuity*, which is the present value of a stochastically discounted series of independent, identically distributed (i.i.d.) cashflows, that is,

$$(1.3) \quad D = \sum_{k \geq 1} \left(\prod_{i=1}^k d_i \right) C_k,$$

where the cashflows, (C_i) , and the stochastic discount rates, (d_i) , are mutually independent, identically distributed sequences. The full-repayment probability in *infinite time* (1.2) is understood as the distribution function of a stochastic perpetuity. When considering cash flows C_k , arriving at T_k , $k = 1, 2, \dots$, and discounted at a rate δ , its present value is given by $\sum_{k \geq 1} e^{-\delta T_k} C_k$, which is identified with D of (1.3) if we assume $T_i - T_{i-1}$, $i = 1, 2, \dots$ are i.i.d., by setting $d_i = e^{-\delta(T_i - T_{i-1})}$. In [13] it is shown to exhibit a Gamma distribution for Poisson inter-arrivals of claims of exponentially distributed intensity (ex. 5.1.2.), result that we can retrieve with our approach. Similar models are interpreted as risk models with stochastic returns on investments (see e.g. [22, 23, 9]).

The main difference between our model and a stochastic perpetuity (or a stochastic interest risk model) consists in the fact that we are dealing with a stochastic process, whereas the perpetuity models are random variables. They are infinite sums of random variable, discounted (stochastic discounting) at fixed times, while our model is, at each time t , a finite sum of random variables which can be seen as “discounted at random times”.

Connections to stochastic fixed point equations. The integral equation we derive for the infinite horizon full repayment probability can be seen as a *stochastic fixed point equation*. As we will be looking at the tail, the structure of corresponding stochastic equation (1.1) resonates with the *stochastic fixed point equations* described in [17] for the first time, [9] for insurance applications and amply analysed in the book of Mikosch [20]. For such stochastic model of perpetuity-type, for various dependence

structures, it is shown the tails of the distribution are regularly varying both in the univariate and multivariate cases. Here references [18, 17, 7, 9, 23, 22]. More recent literature could be found in [21, 12].

Thus, in studying "default probabilities in infinite horizon", we are lead to stochastic fixed point equations, which are equal in distribution to the ones for stochastic perpetuity. Moreover, by choosing specific distributions for the random variables involved, the time "of discounting", or when an event occurs that would reduce the number of mortgagors, we can obtain explicit solutions of the tail distribution/ loan repayment. Stochastic fixed point equations literature abounds in asymptotic results.

The Structure of the Paper. The rest of the paper is organized as follows. In section 2 we give a general equation for the loan default probability ϕ in a loan model driven by a general marked point process (T_i, X_i) . The finite time and infinite time default probabilities are expressed via Fredholm equations, with solutions to be analyzed asymptotically or numerically, if not readily available in closed forms. We present an approximation of the finite-time probability from its infinite-time counterpart, in 3. In section 4, based on the infinite-horizon equation, we consider the case of X exponentially distributed and derive the ruin probability when W are exponentially and Erlang distributed, respectively, under both independent and conditionally independent scenarios. Finally, we calculate the ruin probability in the case of randomized arrival times when W is exponentially distributed with random parameter Λ . One can then numerically calculate the insurance premium to be incorporated in the mortgage plan, such that the probability of default stays within a small range, as in Section 5. We conclude in Section 6.

Note: In this paper, for keeping the context clear for the reader, the classical notation for ruin probability ψ will be referred to as the *probability of loan repayment*, or *probability of full loan repayment*, while the classical probability of non-ruin ϕ will be referred to as *probability of default*.

2. Ruin probability in a general loan model. Earthquakes are considered rare, extreme events in most parts of the world, but could not be considered as such in Japan. The incidence of earthquakes has increased after the 2011 one. Looking at earthquake data, one can see that the arrivals of earthquakes can be described by one Poisson distribution before 2011, and a different parameter Poisson distribution after 2011. However, being still perceived as an extreme, rare event, in addition to cultural reasons, at the moment only a minor part of home-owners in Japan have the relevant insurance cover. In order to evaluate the risks associated with a marketize earthquake insurance, we calculate the probability of default of a lender, based on a model that can be seen as an *exponentiation of the Cramér-Lundberg model*, which is the classical model in collective non-life insurance (see e.g. [11], [20], [5]).

Let $W_i = T_i - T_{i-1}$, $i = 1, \dots$, $T_0 = 0$. Firstly, assume that (W_i, X_i) , $i = 1, \dots$, are independent and identically distributed with (W, X) . For a bounded measurable function h , we set

$$(2.1) \quad \mathcal{K}h(u, t) := \mathbf{E}[I_{\{W < t \wedge \frac{u}{c}\}} h(e^X(u - cW), t - W)].$$

Then clearly \mathcal{K} defines a linear transformation on $L^\infty(\mathbf{R}_+^2)$.

THEOREM 2.1. *The finite-time probability ϕ satisfies the Fredholm type equation*

$$(2.2) \quad \phi(u, t) = \phi_0(u, t) + \mathcal{K}\phi(u, t),$$

where $\phi_0(u, t) = \mathbf{P}(W > t)I_{\{t \leq \frac{u}{c}\}}$, with $(u, t) \in (0, \infty)^2$.

Proof. Since U is an increasing process, we have that $\phi(u, t) = \mathbf{P}(U_t \leq u)$. Since

$$\phi(u, t) = \mathbf{P}(U_t \leq u, T_1 \geq t) + \mathbf{P}(U_t \leq u, T_1 < t),$$

and since for $T_1 \geq t$, $U_t = ct$, then

$$(2.3) \quad \phi(u, t) = \phi_0(u, t) + \mathbf{P}(U_t \leq u, T_1 < t).$$

When $T_1 < t$, we have that

$$U_t = cT_1 + ce^{-X_1} \sum_{n=1}^{\infty} 1_{[T_n, T_{n+1})}(t) \left\{ (t - T_n) e^{-\sum_{i=2}^n X_i} + \sum_{i=2}^n (T_i - T_{i-1}) e^{-\sum_{k=2}^{i-1} X_k} \right\}.$$

Noting that (T_i, X_i) is a marked point process,

$$U'_{t-T_1} = c \sum_{n=1}^{\infty} 1_{[T_n, T_{n+1})}(t) \left\{ (t - T_n) e^{-\sum_{i=2}^n X_i} + \sum_{i=2}^n (T_i - T_{i-1}) e^{-\sum_{k=2}^{i-1} X_k} \right\}$$

has the same distribution as U_{t-T_1} , and it's independent from T_1 and X_1 . Consequently we have that

$$(2.4) \quad \begin{aligned} \mathbf{P}(U_t \leq u, T_1 < t) &= \mathbf{P}(cT_1 + e^{-X_1} U'_{t-T_1} \leq u, T_1 < t) \\ &= \mathcal{K}\phi(u, t). \end{aligned}$$

Combining (2.3) with (2.4), we obtain (1). \square

COROLLARY 2.2. *The finite-time probability of full loan repayment is expressed by a Fredholm type equation*

$$(2.5) \quad \psi(u, t) = \psi_0(u, t) + \mathcal{K}\psi(u, t),$$

where $\psi_0(u, t) := \mathbf{P}(W \geq \frac{u}{c})I_{\{t > \frac{u}{c}\}}$, with $(u, t) \in (0, \infty)^2$.

Proof. By theorem 2.1, for $\psi = 1 - \phi$. \square

Let \mathcal{A}_p be the (completion of the) set of all functions h on \mathbf{R}_+^2 such that

$$\|h\|_{\mathcal{A}_p} := \sup_{t>0} \int_0^{\infty} |h(u, t)|^p du < \infty.$$

LEMMA 2.3. *Suppose that X is non-trivial. Then \mathcal{K} defined by (2.1) is a linear operator on \mathcal{A}_p . Moreover, the operator norm $\|\mathcal{K}\|_{\mathcal{L}(\mathcal{A}_p)} = \sup\{\|\mathcal{K}h\|_{\mathcal{A}_p} : \|h\|_{\mathcal{A}_p} = 1, h \in \mathcal{A}_p\}$ is strictly less than 1, for any $1 \leq p < \infty$.*

Proof. For $h \in \mathcal{A}_p$, by using Jensen's inequality and Minkowski's inequalities, we have that

$$\begin{aligned} \|\mathcal{K}h\|_{\mathcal{A}_p} &\leq \mathbf{E} \left[\sup_{t>0} \int_0^{\infty} |I_{\{W < t \wedge \frac{u}{c}\}} h(e^X(u - cW), t - W)|^p du \right] \\ &\leq \mathbf{E} \left[e^{-X} \sup_{t>0} \int_0^{\infty} |h(Z, t)|^p dZ \right] = \|h\|_{\mathcal{A}_p} \mathbf{E}[e^{-X}]. \end{aligned}$$

Since X is non-trivial, we completed the proof. \square

LEMMA 2.4. *Suppose that the first moment of W is finite. Then $\psi_0 \in \cap_{p \geq 1} \mathcal{A}_p$.*

Proof. Let $p \geq 1$. We note that

$$\|\psi_0\|_{\mathcal{A}_p} = \sup_{t>0} \int_0^\infty \psi_0(u, t)^p du dt = \sup_{t>0} \int_0^{ct} \left(\mathbf{P}(W \geq \frac{u}{c}) \right)^p du.$$

Having an increasing function,

$$\sup_{t>0} \int_0^{ct} \left(\mathbf{P}(W \geq \frac{u}{c}) \right)^p du = \int_0^\infty \left(\mathbf{P}(W \geq \frac{u}{c}) \right)^p du \leq \int_0^\infty \mathbf{P}(W \geq \frac{u}{c}) du = c\mathbf{E}[W],$$

with the inequality given by the fact that the probability ranges in $[0,1]$. Hence we conclude $\psi_0 \in \mathcal{A}_p$, by the boundedness of the first moment of W . \square

THEOREM 2.5. *Under the assumptions of Lemmas 2.3 and 2.4, we have that*

$$\psi = \sum_{n=0}^{\infty} \mathcal{K}^n \psi_0 \in L(\cap_{p \geq 1} \mathcal{A}_p),$$

and moreover $\phi = 1 - \sum_{n=0}^{\infty} \mathcal{K}^n \psi_0$.

Proof. Since the operator norm of \mathcal{K} is strictly less than 1, by Lemma 2.3, the Neumann series $\sum_{n=0}^{\infty} \mathcal{K}^n$ is convergent in $L(\cap_{p \geq 1} \mathcal{A}_p)$ and thus defines (describes) the inverse operator of $1 - \mathcal{K}$. \square

Remark 2.6. All the results in this section are valid for possibly negative X , as long as $\mathbf{E}[e^{-X}] < 1$. Although such situations are not realistic in our mortgage loan modelling, it might be applicable to different contexts. Furthermore, since the results cover the exponential functionals of compound Poisson processes, this may contribute to the literature on exponential functionals of Lévy processes (see e.g. [6]).

3. From finite-time to infinite-time ruin probability. In this section we will work on infinite horizon ruin probabilities as limit of finite-time ruin probabilities when the time goes to infinity. By continuity of probability measure, it holds that

$$\psi^\infty(u) = \lim_{t \rightarrow \infty} \psi(u, t) = \mathbf{P}(\tau_u < \infty), u > 0.$$

We assume that the assumptions of Lemmas 2.3 and 2.4 are satisfied, that is, $\mathbf{P}(X > 0) > 0$ and $\mathbf{E}[W] < \infty$. Moreover, we assume that $\mathbf{P}(W > 0) > 0$ and that the joint density function of W and X exists, and we denote it by $f_{W,X}(w, x)$. Let L_p be the collection of functions which satisfy $\int_0^\infty |g(u)|^p du < \infty$, for $g \in L_p$. By identifying functions $g \in L_p$, via $g \in \mathcal{A}_p$ with $g(u, t) = g(u)$ ($u, t > 0$), L_p is a subset of \mathcal{A}_p . We define an operator \mathcal{K}_∞ on L_p by

$$\mathcal{K}_\infty g(u) := \mathbf{E} [1_{\{cW < u\}} g(e^X(u - cW))], \quad u > 0.$$

Clearly, $\mathcal{K}_\infty g(u) = \lim_{t \rightarrow \infty} (\mathcal{K}g)(u, t)$, and therefore, by Lemma 2.3, the operator norm of \mathcal{K}_∞ is strictly less than one, and it leads to

$$(3.1) \quad \psi^\infty = \sum_{n=0}^{\infty} \mathcal{K}_\infty^n \psi_0^\infty \in \cap_{p \geq 1} L_p,$$

where $\psi_0^\infty(u) := \mathbf{P}(W > \frac{u}{c})$. We note that for any $t > \frac{u}{c}$, $\psi_0(u, t)$ is equal to $\psi_0^\infty(u)$.

3.1. Convergence rate of ruin probability from finite-horizon to infinite-horizon. The finite-time horizon probability of default is extremely relevant for the mortgage markets that deals fixed-term loans. Having such a fast convergence to the infinite-time probability, we can actually use the infinite-time probability (which we can more often calculate explicitly) as an approximation for the finite-time one. In this section, we show that the finite-time ruin probability converges to the infinite-horizon one exponentially fast.

THEOREM 3.1. *We assume that there exists $\theta_0 > 0$ such that the moment generating function $M_W(\theta) = \mathbf{E}[e^{\theta W}]$ exists for any $\theta < \theta_0$. Then, we have the following estimates.*

(i) *There exists $\xi > 0$ and $C > 0$ such that for any $t > 0$,*

$$(3.2) \quad \int_0^\infty |\psi^\infty(u) - \psi(u, t)| du \leq C e^{-\xi t}.$$

(ii) *For any $u > 0$, there exists $C(u) > 0$ such that for any $t > 0$,*

$$0 \leq \psi^\infty(u) - \psi(u, t) \leq C(u) e^{-\xi' t},$$

for some $\xi' > 0$.

Proof. We first show that (i) implies (ii). By (i), we can choose $\xi > 0$ and $C > 0$ such that (3.2) holds true. Let $\xi' \in (0, \xi)$. For each $n \in \mathbf{N}$, set

$$A_n := \{u > 0; \psi^\infty(u) - \psi(u, n) \geq C e^{-\xi' n}\}.$$

Then the Lebesgue measure of the event A_n is bounded as

$$\text{Leb}(A_n) \leq \frac{1}{C e^{-\xi' n}} \int_0^\infty |\psi^\infty(u) - \psi(u, n)| du \leq \frac{C e^{-\xi n}}{C e^{-\xi' n}} = e^{-(\xi - \xi') n}.$$

Therefore we have that $\sum_{n=0}^\infty \text{Leb}(A_n) < \infty$. Hence, by Borel-Cantelli lemma, $\text{Leb}(\limsup_{n \rightarrow \infty} A_n) = 0$, which leads to

$$\psi^\infty(u) - \psi(u, t) < C(u) e^{-\xi' t}, \text{ Leb-a.s. } u,$$

for some $C(u) > 0$. Since $\psi^\infty(u)$ and $\psi(u, t)$ are continuous in u , it holds for any u .

Next we prove (i). Since

$$\psi^\infty(u) - \psi(u, t) = \sum_{n=0}^\infty (\mathcal{K}_\infty^n \psi_0^\infty(u) - \mathcal{K}^n \psi_0(u, t)),$$

by Theorem 2.5 and (3.1), it suffices to show that there exist constants $C' > 0$, $a \in (0, 1)$, and $\xi > 0$ such that

$$(3.3) \quad \int_0^\infty |\mathcal{K}_\infty^n \psi_0^\infty(u) - \mathcal{K}^n \psi_0(u, t)| du \leq C' a^n e^{-\xi t}.$$

To establish (3.3), we rely on the following recursive relation:

$$(3.4) \quad \begin{aligned} & \int_0^\infty |\mathcal{K}_\infty^n \psi_0^\infty(u) - \mathcal{K}^n \psi_0(u, t)| du \\ & \leq \mathbf{E}[e^{-X}] \mathbf{E}[I_{\{W < t\}} \int_0^\infty |\mathcal{K}_\infty^{n-1} \psi_0^\infty(u) - \mathcal{K}^{n-1} \psi_0(u, t - W)| du] \\ & \quad + c \mathbf{E}[e^{-X}]^n \mathbf{P}(W > t) \mathbf{E}[W]. \end{aligned}$$

A proof of (3.4) will appear in Appendix.

Using (3.4) recursively,

$$\begin{aligned}
(3.5) \quad & \int_0^\infty |\mathcal{K}_\infty^n \psi_0^\infty(u) - \mathcal{K}^n \psi_0(u, t)| du \\
& \leq \mathbf{E}[e^{-X}] \mathbf{E}[I_{\{W_1 < t\}}] \\
& \quad \times \left(\mathbf{E}[e^{-X}] \mathbf{E}[I_{\{W_2 < t - W_1\}}] \int_0^\infty |\mathcal{K}_\infty^{n-2} \psi_0^\infty(u) - \mathcal{K}^{n-2} \psi_0(u, t - W_1 - W_2)| du | W_1] \right. \\
& \quad \left. + c \mathbf{E}[e^{-X}]^{n-1} \mathbf{E}[W] \mathbf{E}[I_{\{W_1 < t\}}] \mathbf{P}(W_2 > t - W_1 | W_1) \right] + c \mathbf{E}[e^{-X}]^n \mathbf{P}(W_1 > t) \mathbf{E}[W] \\
& = \mathbf{E}[e^{-X}]^2 \mathbf{E}[I_{\{S_2 < t\}}] \int_0^\infty |\mathcal{K}_\infty^{n-2} \psi_0^\infty(u) - \mathcal{K}^{n-2} \psi_0(u, t - S_2)| du \\
& \quad + c \mathbf{E}[e^{-X}]^n \mathbf{P}(S_2 > t) \mathbf{E}[W] \\
& \leq \mathbf{E}[e^{-X}]^n \mathbf{E}[I_{\{S_n < t\}}] \int_0^\infty |\psi_0^\infty(u) - \psi_0(u, t - S_n)| du + c \mathbf{E}[e^{-X}]^n \mathbf{E}[W] \mathbf{P}(S_n > t),
\end{aligned}$$

where $S_n = \sum_{k=1}^n W_k$. The L^1 - norm of the difference between the finite and the infinite ruin probabilities in the right-most of (3.5) is calculated as

$$(3.6) \quad \int_0^\infty |\psi_0^\infty(u) - \psi_0(u, t)| du = \int_{ct}^\infty \mathbf{P}(W > \frac{u}{c}) du = c \mathbf{E}[I_{\{W > t\}}(W - t)],$$

for $t > 0$. Therefore by substituting (3.6) to (3.5), we have that

$$\begin{aligned}
& \text{(the right hand side of (3.5))} \\
(3.7) \quad & = c \mathbf{E}[e^{-X}]^n \mathbf{E}[I_{\{S_n < t\}}] \mathbf{E}[I_{\{W_{n+1} > t - S_n\}}(W_{n+1} - (t - S_n)) | S_n] \\
& \quad + c \mathbf{E}[e^{-X}]^n \mathbf{E}[W] \mathbf{P}(S_n > t) \\
& = c \mathbf{E}[e^{-X}]^n \mathbf{E}[I_{\{S_{n+1} > t\}}(W_{n+1} - ((t - S_n) \vee 0))].
\end{aligned}$$

The claim (3.3) is fulfilled once we establish

$$(3.8) \quad \mathbf{E}[I_{\{S_{n+1} > t\}}(W_{n+1} - ((t - S_n) \vee 0))] \leq e^{-\xi t} C'' (a')^n$$

with some $\xi, C'' > 0$ and $a' < 1/\mathbf{E}[e^{-X}]$. To see this, we note that for any $\theta \in (0, \theta_0)$,

$$(3.9) \quad \mathbf{E}[I_{\{S_{n+1} > t\}}(W_{n+1} - ((t - S_n) \vee 0))] \leq \mathbf{E}[I_{\{S_{n+1} > t\}} W_{n+1}] \leq \mathbf{E}[W_{n+1} e^{\theta(S_{n+1} - t)}],$$

allowing for infinite value in the rightmost. Since W_{n+1} and S_n are independent, the right hand side of (3.9) is decomposed as follows:

$$\mathbf{E}[W_{n+1} e^{\theta(W_{n+1} + S_n - t)}] = e^{-\theta t} \mathbf{E}[W_{n+1} e^{\theta W_{n+1}}] \mathbf{E}[e^{\theta S_n}] = e^{-\theta t} \mathbf{E}[W e^{\theta W}] M_W^n(\theta).$$

Since X is non-trivially positive, we can choose $\xi \in (0, \theta_0)$ such that

$$M_W(\xi) := \mathbf{E}[e^{\xi W}] < \frac{1}{\mathbf{E}[e^{-X}]}.$$

Thus, we established (3.8), which implies (3.3), and hence the proof is complete. \square

3.2. Exponential decay rate for exponential time arrival. Firstly, we consider that the arrivals of disasters W are exponentially distributed, with parameter λ , meaning the moment generating function exists for all $\theta < \lambda$, $M_W(\theta) = \frac{\lambda}{\lambda - \theta}$, so Theorem 3.1 applies, moreover the constants c and C of (3.2) can be determined explicitly.

PROPOSITION 3.2. *For W exponentially distributed with parameter $\lambda > 0$,*

$$\int_0^\infty |\psi^\infty(u) - \psi(u, t)| du \leq \frac{1}{\alpha} e^{-\lambda \alpha t}, \quad t \geq 0,$$

where $\alpha = 1 - \mathbf{E}[e^{-X}]$.

Proof. The partial sum $S_n = \sum_{k=1}^n W_k$ is Erlang distributed with parameters n and λ , and its tail distribution function is given by

$$\mathbf{P}(S_n > z) = e^{-\lambda z} \sum_{k=0}^{n-1} \frac{(\lambda z)^k}{k!}.$$

For $n \geq 0$, by tower property of expectations, conditioned by S_n , we have that

$$\begin{aligned} (3.10) \quad \mathbf{E}[I_{\{S_{n+1} > t\}}(W_{n+1} - ((t - S_n) \vee 0))] &= \mathbf{E}\left[\int_{t-S_n}^\infty (w - ((t - S_n) \vee 0)) \lambda e^{-\lambda w} dw\right] \\ &= \mathbf{E}[I_{\{S_n < t\}} e^{-\lambda(t-S_n)}] + \mathbf{P}(S_n > t). \end{aligned}$$

By (7), the first term of (3.10) is rewritten as

$$(3.11) \quad e^{-\lambda t} \mathbf{E}[I_{\{S_n < t\}} e^{\lambda S_n}] = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

By the combination of (3.7), (3.10) and (3.11), we obtain that

$$(3.12) \quad \int_0^\infty |\mathcal{K}_\infty^n \psi_0^\infty(u) - \mathcal{K}^n \psi_0(u, t)| du \leq \mathbf{E}[e^{-X}]^n e^{-\lambda t} \sum_{k=0}^n \frac{(\lambda t)^k}{k!}, \quad \forall n \geq 0.$$

Summing up (3.12) for $n \geq 0$, we get the decay rate

$$\int_0^\infty |\psi^\infty(u) - \psi(u, t)| du \leq \sum_{n=0}^\infty \mathbf{E}[e^{-X}]^n e^{-\lambda t} \sum_{k=0}^n \frac{(\lambda t)^k}{k!} = \frac{\exp(-\lambda t(1 - \mathbf{E}[e^{-X}]))}{1 - \mathbf{E}[e^{-X}]}. \quad \square$$

4. Explicit ruin probabilities in loan models. We will set the distribution parameters for earthquake frequency and severity to values that minimize the probability of a cohort not repaying the loans, that we will call ruin/default. In real life, historical data should be used to determine these parameters and further set the insurance premium which minimizes this probability of ruin. We will work on infinite horizon default/full repayment probabilities, for which the equation (2.2) reduces to

$$(4.1) \quad \phi^\infty(u) = \phi_0^\infty(u) + \mathcal{K}_\infty \phi^\infty(u),$$

with $\phi_0^\infty(u) = \lim_{t \rightarrow \infty} \mathbf{P}(W > t) I_{\{t \leq \frac{u}{c}\}} = 0$ and the equation for ψ^∞ is

$$(4.2) \quad \psi^\infty(u) = \psi_0^\infty(u) + \mathcal{K}_\infty \psi^\infty(u),$$

where $\psi_0^\infty(u) = \lim_{t \rightarrow \infty} \mathbf{P}(W \geq \frac{u}{c}) I_{\{t > \frac{u}{c}\}} = \mathbf{P}(W \geq \frac{u}{c})$.

In the following, we consider that the proportion e^{-X} of borrowers left in the loan programme is described by an exponentially distributed random variable X , with parameter α . In the remainder, for ease of notation, we will use $\phi(u)$ for $\phi^\infty(u)$ and the equivalent for ψ .

4.1. Memory-less Arrivals. We will start with Poisson arrivals, meaning that we wait an exponential amount of time between events. We also assume that the arrivals of disasters W are independent of the effect on the bank's remaining proportion of borrowers X .

PROPOSITION 4.1. *If X is exponentially distributed with parameter $\alpha > 0$ and W is exponentially distributed with parameter $\lambda > 0$, and independent of X , then the probability of default ϕ is given by*

$$(4.3) \quad \phi(u) = \frac{1}{\Gamma(\alpha + 1)} \int_0^{\frac{\lambda u}{c}} z^\alpha \exp(-z) dz, \quad u \geq 0.$$

Proof. Here we give a direct proof using only differentiation. From (4.1) and using the joint density of (W, X) , $f_{W,X}(w, x) = \lambda \alpha e^{-\lambda w - \alpha x}$, we have that the probability of default ϕ satisfies

$$(4.4) \quad \phi(u) = \int_0^\infty \int_0^{\frac{u}{c}} \phi(e^x(u - cw)) \lambda \alpha e^{-\lambda w - \alpha x} dw dx.$$

By differentiating both sides of (4.4), twice, we obtain the following ordinary differential equation with non-constant coefficients

$$\phi''(u) = \left(-\frac{\lambda}{c} + \frac{\alpha}{u} \right) \phi'(u),$$

equipped with two boundary conditions, one at infinity and one at zero. Using the boundary condition $\phi(0) = 0$, the solution of the differential equation reads

$$\phi(u) = C \left(\frac{c}{\lambda} \right)^{\alpha+1} \int_0^{\frac{\lambda u}{c}} z^\alpha \exp(-z) dz.$$

Furthermore, from the infinity condition $\lim_{u \rightarrow \infty} \phi(u) = 1$, $C = \frac{1}{\Gamma(\alpha+1)} \left(\frac{\lambda}{c} \right)^{\alpha+1}$, completing the proof. \square

4.2. General Arrivals.

PROPOSITION 4.2. *If X is exponentially distributed with parameter $\alpha > 0$ and W is a random variable independent of X , with density a positive integrable function f_W , then the Laplace transform of the ruin probability satisfies*

$$(4.5) \quad \hat{\psi}(s) = s^{-1} (1 - \hat{f}_W(cs)) + \alpha s^{\alpha-1} \hat{f}_W(cs) \int_0^s u^{\alpha-1} (1 - \hat{f}_W(cu)) e^{\int_u^s \alpha v^{-1} \hat{f}_W(cv) dv} du,$$

where \hat{f} denotes the Laplace transform of f .

Proof. The joint density function of (W, X) is $f_{W,X}(w, x) = \alpha f_W(w) e^{-\alpha x}$. By taking Laplace transform for the both sides of (4.2), we have that, for $s > 0$,

$$\hat{\psi}(s) = \frac{1}{s}(1 - \hat{f}_W(cs)) + \int_0^\infty e^{-su} \int_0^\infty \int_0^{\frac{u}{c}} \psi(e^x(u - cw)) \alpha f_W(w) e^{-\alpha x} dw dx du.$$

By changing the order of the integrals of the second term, which is possible since they are all positive, we obtain that

$$s^\alpha \hat{\psi}(s) = s^{\alpha-1}(1 - \hat{f}_W(cs)) + \alpha s^{-1} \hat{f}_W(cs) \int_0^s x^\alpha \hat{\psi}(x) dx.$$

By denoting $G(s) = \int_0^s x^\alpha \hat{\psi}(x) dx$, we have a separable ordinary differentiable equation in $G(s)$,

$$G'(s) = s^{\alpha-1}(1 - \hat{f}_W(cs)) + \alpha s^{-1} \hat{f}_W(cs) G(s),$$

which leads to

$$G(s) = \left(G(t) + \int_t^s u^{\alpha-1} (1 - \hat{f}_W(cu)) e^{-\int_t^u \alpha v^{-1} \hat{f}_W(cv) dv} du \right) e^{\int_t^s \alpha v^{-1} \hat{f}_W(cv) dv},$$

for $0 < t < s$. Since $\lim_{t \rightarrow 0} G(t) = 0$, by taking $t \rightarrow 0$ for the both sides, we obtain that

$$G(s) = \int_0^s u^{\alpha-1} (1 - \hat{f}_W(cu)) e^{\int_u^s \alpha v^{-1} \hat{f}_W(cv) dv} du,$$

and thus we conclude that (4.5) is verified. \square

Remark 4.3. When W is exponential with parameter $\lambda > 0$,

$$\hat{F}_W(cs) = \int_0^\infty e^{-csx} e^{-\lambda x} dx = \frac{1}{\lambda + cs}$$

and thus

$$\hat{\psi}(s) = \frac{1}{s} - \frac{c}{\lambda + cs} + c\alpha s^\alpha (\lambda + cs)^{\alpha-1} \int_0^s \left(\frac{u}{\lambda + cu} \right)^\alpha \left(\frac{1}{u} - \frac{c}{\lambda + cu} \right) du,$$

which after Laplace inversion leads to the results of Proposition 4.1.

4.3. Arrivals with Memory. When the inter-arrival times W are Erlang distributed, with shape parameter k and rate parameter λ , we can obtain again explicit results for the ruing probability. Recall the density function of an Erlang distribution is given by

$$f(w; k, \lambda) = \frac{\lambda^k w^{k-1} e^{-\lambda w}}{(k-1)!}, \quad x > 0, \quad k \in \mathbf{N}, \quad \lambda > 0, \quad w > 0.$$

PROPOSITION 4.4. *If X is exponentially distributed with parameter $\alpha > 0$ and W is Erlang distributed with shape parameter 2 and rate parameter λ , then ϕ is given by*

$$\phi(u) = Cg(u), \quad u \geq 0$$

where

$$g(u) = \int_0^{\frac{\lambda u}{c}} \frac{1}{\sqrt{y}} e^{-y} \cosh\left(2\sqrt{(\alpha+2)y}\right) \left(\int_0^{\frac{\lambda u}{c}-y} e^{-x} x^{\alpha+\frac{1}{2}} dx \right) dy, \quad C = \lim_{u \rightarrow \infty} \frac{1}{g(u)}.$$

Proof. Here $f_W(w)$ is $f(w; 2, \lambda)$. The Laplace transform of f_W is given by

$$\hat{f}_W(s) = \int_0^\infty e^{-sw} f_W(w) dw = \frac{\lambda^2}{(s+\lambda)^2},$$

and its derivative is

$$(\hat{f}_W)'(s) = -\frac{2\lambda^2}{(s+\lambda)^3}.$$

By Proposition 4.2, we have that, for some constant C ,

$$\hat{\phi}(s) = CF(s), \quad s > 0,$$

where $F(s) = \frac{1}{s(s+\frac{\lambda}{c})^{2+\alpha}} e^{-\frac{\alpha\lambda}{cs+\lambda}}$ $s > 0$. We claim that

(4.6)

$$\begin{aligned} & \mathcal{L}^{-1}(F)(t) \\ &= \frac{1}{\Gamma(\alpha + \frac{3}{2})\sqrt{\pi}} \left(\frac{c}{\lambda}\right)^{\alpha+2} \int_0^{\frac{\lambda t}{c}} \frac{1}{\sqrt{y}} e^{-y} \cosh\left(2\sqrt{(\alpha+2)y}\right) \left(\int_0^{\frac{\lambda t}{c}-y} e^{-x} x^{\alpha+\frac{1}{2}} dx \right) dy. \end{aligned}$$

This can be seen in the following way. We set $F_1(s) = s^{-1}(s+\frac{\lambda}{c})^{-\alpha-\frac{3}{2}}$ and $F_2(s) = s^{-\frac{1}{2}} e^{-\frac{(\alpha+2)\lambda}{cs}}$. Then the inverse Laplace transform of F_1 and F_2 are given by

$$\mathcal{L}^{-1}(F_1)(t) = \frac{1}{\Gamma(\alpha + \frac{3}{2})} \left(\frac{c}{\lambda}\right)^{\alpha+\frac{3}{2}} \int_0^{\frac{\lambda t}{c}} e^{-x} x^{\alpha+\frac{1}{2}} dx,$$

and

$$\mathcal{L}^{-1}(F_2)(t) = \frac{1}{\sqrt{\pi t}} \cosh\left(2\sqrt{\frac{(\alpha+2)\lambda t}{c}}\right),$$

(see e.g. Chapter 5.5 (32) in [15]). Therefore we obtain (4.6). Hence we conclude that

$$\phi(s) = C\mathcal{L}^{-1}(F)(s).$$

where $\mathcal{L}^{-1}(F)$ is our g . Since $\lim_{u \rightarrow \infty} \phi(u) = 1$, $C = \lim_{u \rightarrow \infty} \frac{1}{g(u)}$. \square

4.4. Randomized Arrival Times. We assume that W is exponentially distributed with random parameter Λ . For any given $\Lambda = \lambda$, W is exponentially distributed with random parameter λ . This special dependence structure is referred to as conditional independence. As in [1], for every given $\Lambda = \lambda$, we calculate the ruin probability, say $\psi_\lambda(u)$, and then integrate over all the possible values of Λ , with distribution function F_Λ , leading to

$$\psi(u) = \int_0^\infty \psi_\lambda(u) dF_\Lambda(\lambda).$$

From (4.3),

$$(4.7) \quad P(\tau_u = \infty | \Lambda = \lambda) =: \phi_\lambda(u) := \frac{1}{\Gamma(\alpha + 1)} \int_0^{\frac{\lambda u}{c}} z^\alpha \exp(-z) dz,$$

for each $\lambda(> 0)$, and then the probability of loan repayment satisfies

$$\phi(u) = \int_0^\infty \phi_\lambda(u) dF_\Lambda(\lambda),$$

where $F_\Lambda(\lambda)$ is the distribution function of Λ . When Λ is Erlang distributed with parameter $(k, \frac{1}{\theta})$ with $k, \theta > 0$, namely,

$$(4.8) \quad dF_\Lambda(\lambda) = \frac{1}{\Gamma(k)\theta^k} \lambda^{k-1} e^{-\frac{\lambda}{\theta}} d\lambda,$$

we can derive explicitly the probability of default.

THEOREM 4.5. *The probability of default under (4.7) and (4.8) is expressed by*

$$(4.9) \quad \phi(u) = \frac{\Gamma(k + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(k)} B_{\frac{u\theta}{c+u\theta}}(\alpha + 1, k),$$

where $B_x(u, v) = \int_0^x y^{u-1} (1-y)^{v-1} dy$ is the incomplete beta function.

Proof. One has that

$$\phi(u) = \int_0^\infty \phi_\lambda(u) \frac{1}{\Gamma(k)\theta^k} \lambda^{k-1} e^{-\frac{\lambda}{\theta}} d\lambda.$$

Differentiating both sides, we have that

$$\phi'(u) = \frac{\Gamma(k + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(k)} \left(\frac{\theta}{c}\right) \left(\frac{c}{c + u\theta}\right)^{k+1} \left(1 - \frac{c}{c + u\theta}\right)^\alpha.$$

Therefore for some constant C , the probability of default is given by

$$\phi(u) = \frac{\Gamma(k + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(k)} \left(\frac{\theta}{c}\right) \int_0^u \left(\frac{c}{c + x\theta}\right)^{k+1} \left(1 - \frac{c}{c + x\theta}\right)^\alpha dx + C.$$

Since $\phi_\lambda(0) = 0$ for each $\lambda > 0$, and $\phi(0) = 0$, we have that $C = 0$. By a further change of variables, we conclude that the probability of default is expressed as in (4.9). \square

Remark 4.6. We note that the order of $\phi(u)$ is $\mathcal{O}(1/u^k)$.

5. Numerical experiments. In this section we present firstly two algorithms for the calculating the probability of default (ruin), apply them to some concrete numerical examples/chosen parameters, then analyse the corresponding premium rates.

5.1. Simulation for finite-time ruin versus infinite-time one. We present two algorithms. Algorithm 2 is faster and less variant than the Algorithm 1. However, Algorithm 2 can only be used for the case that inter-arrival is exponential distributed (or conditional exponential distributed for randomized arrival times model).

Algorithm 1. First, an algorithm which simulates the default state up to a given finite horizon time t by simulating scenarios of cash flows:

$$U_t = c \sum_{n=1}^{\infty} \left[1_{[T_n, T_{n+1})(t)} (t - T_n) e^{-\sum_{i=1}^n X_i} + \sum_{i=1}^n (T_i - T_{i-1}) e^{-\sum_{i=1}^{i-1} X_i} \right].$$

- Initiate $k = 0$, $T(0) = 0$, $X(0) = 0$ as time and effect at time 0.
- While $T(k) < t$
 - $k = k + 1$,
 - Simulate (W, X) given a specific distribution,
 - Inter-arrival time $W(k) = W$,
 - Arrival time $T(k) = T(k - 1) + W(k)$,
 - Effect $X(k) = X$.
- Calculate the cash flow at $t: U_t = (t - T_{k-1}) e^{-\sum_{i=1}^{k-1} X(i)} + \sum_{i=1}^{k-1} W(i) e^{-\sum_{i=1}^{i-1} X(i)}$.
- Determine default state $I = 1_{\{U_t < u\}}$.

Repeat the procedure N times (with N large). The sample average of I is an estimation for the default probability $\phi(u, t)$.

Algorithm 2. In the memoryless arrival model, the number of arrivals up to time $(T_n)_{n \geq 0}$ is a homogeneous Poisson process with parameter λ . The number of arrivals up to t , $N_t = \sum_{n=1}^{\infty} 1_{\{T_n \leq t\}}$ is also Poisson distributed with parameter λ . Given $N_t = n$, arrival times T_1, \dots, T_n is the ordered statistics of n i.i.d U_1, \dots, U_n with uniform distribution on $[0, t]$. So we can get the algorithm as follows:

- Simulate a random number n from Poisson distribution with parameter λt .
- Simulate n random number U_1, \dots, U_n from n i.i.d uniform distribution $U([0, t])$.
- Sort U_1, \dots, U_n to get arrival time T_1, \dots, T_n .
- Simulate effect X_1, \dots, X_n given a specific distribution.
- Calculate cash flow at $t: U_t = (t - T_{k-1}) e^{-\sum_{i=1}^{k-1} X(i)} + \sum_{i=1}^{k-1} W(i) e^{-\sum_{i=1}^{i-1} X(i)}$.
- Determine the default state $I = 1_{\{U_t < u\}}$.

Repeat the procedure N times (with N large). The expectation of I is an estimation for default probability $\phi(u, t)$.

5.2. Risk Adjusted Premium Rates. As in [10], imposing a solvency level on $\psi(u)$ we could derive c . Comparing the resulting c with the mortgage repayment rate, we could find out the amount that would be considered as premium. We are conducting a sensitivity analysis to see how each variable impacts the results. The aim is to find the values of the parameters which minimise the associated risks, but still keep the mortgage attractive, i.e. keep the monthly payments relatively low. In real world the parameters α and λ would be estimated using historical data, but for our analysis purposes, we will start from some given values.

Memoryless Arrivals. Let's assume X follows an exponential distribution with parameter α , and W follows an exponential distribution with a parameter λ . X and W are independent of each other. Then, according to Proposition 4.1, for any $u \geq \theta$, the probability of default is an incomplete Gamma function, (4.3). In our analysis, we focus on the repayment rate (with insurance premium included), as a percentage of the loan, namely c/u . When we impose a fixed solvency target, ϵ , namely $\phi(u) \leq \epsilon$, from $\frac{1}{\Gamma(\alpha+1)} \int_0^{\frac{\lambda u}{c}} z^\alpha e^{-z} dz \leq \epsilon$ we obtain that $\frac{c}{u} \geq \frac{\lambda}{\Gamma_\alpha^{-1}(\epsilon)}$, where Γ_α^{-1} is the inverse function of regularized incomplete Gamma with parameter α . Table 1 presents the minimum values of $\frac{c}{u} = \frac{\lambda}{\Gamma_\alpha^{-1}(\epsilon)}$, for $\epsilon = 0.00001$, when varying the parameters α and λ . Moreover, one can simulate histograms of the cash flow at a specific times. For

TABLE 1
 $\frac{c}{u}$ for different values of the parameters λ and α

$\lambda \backslash \alpha$	1	2	3	4	5	6
1	0.0850603	0.0717969	0.0628385	0.0562366	0.0511059	0.0469712
2	0.170121	0.143594	0.125677	0.112473	0.102212	0.0939424
3	0.255181	0.215391	0.188515	0.16871	0.153318	0.140914
4	0.340241	0.287188	0.251354	0.224946	0.204424	0.187885
5	0.425301	0.358985	0.314192	0.281183	0.25553	0.234856

instance, for $\lambda = 0.5$ and $\alpha = 20$, the histograms at $T = 100$ (Figure 2 and Figure 3), present very small differences, which are caused by the different estimates of the probability of default.

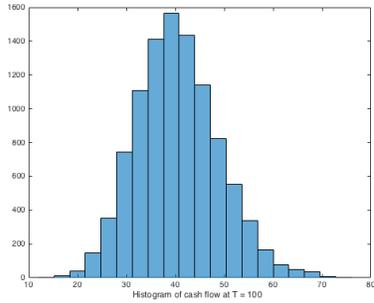


FIG. 2. Histogram for cash flow at a given time by Algorithm 1

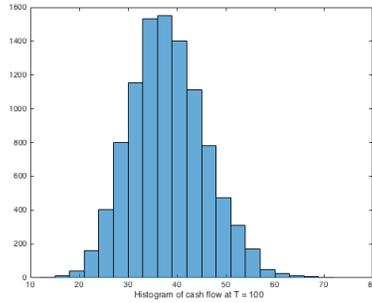


FIG. 3. Histogram for cash flow at a given time by Algorithm 2

Randomised Arrivals. Recall from Theorem 4.5, that for any $u \geq 0$, the probability of default is an incomplete Beta function, (4.9). We consider Λ , such that $E[\Lambda] = k\theta$ and $V[\Lambda] = k\theta^2$. When we impose a solvency target $\phi(u) \leq \epsilon$ and when $k\theta = 1$ ($\theta = 1/k$), then from Theorem 4.5, $\frac{c}{u} \geq \frac{(1 - B_{\alpha+1, k}^{-1}(\epsilon))}{kB_{\alpha+1, k}^{-1}(\epsilon)}$, where $B_{\alpha+1, k}^{-1}$ is the inverse function of regularised incomplete Beta with parameters $\alpha + 1$ and k . Table 2 describes the minimum values realized by c/u , for given parameters α and k , under a solvency target $\epsilon = 0.0001$. For inter-arrival times W exponentially distributed

TABLE 2
 $\frac{c}{u}$ for different values of the parameters k and α

$k \backslash \alpha$	1	2	3	4	5	6	7
1	99.	20.5443	9.	5.30957	3.64159	2.72759	2.16228
3	81.0935	14.9698	6.08913	3.41611	2.26003	1.64776	1.27925
5	76.9924	13.6923	5.42064	2.98014	1.94112	1.39791	1.07451
7	75.1637	13.1197	5.11958	2.78298	1.79635	1.28412	0.981004
9	74.1276	12.794	4.94774	2.67007	1.7132	1.21859	0.927026

with parameter $\lambda = 0.2$, X is exponentially distributed with parameter $\alpha = 20$, and $\frac{u}{c} = 50$, Figure 4 provides information about the convergence behavior of the default

probability, as the horizon time tends to infinity. After $N = 10^5$ simulations, one can see that the probability of default in finite time horizon converges exponentially to the one in infinite time horizon, namely, in this particular example, to 0.0035. (Figure 5).

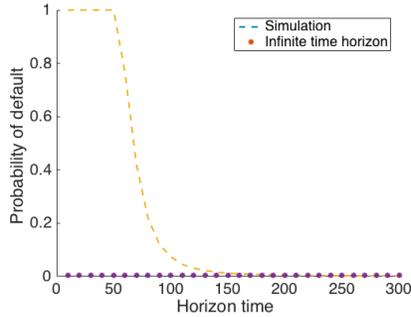


FIG. 4. A comparison of probability of default by simulation

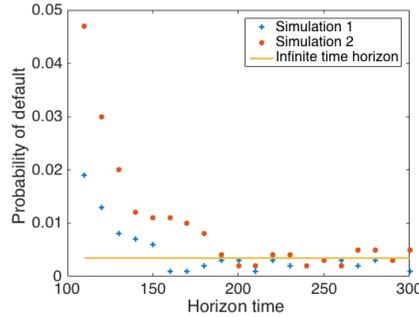


FIG. 5. Convergence of probability of default from finite horizon time to infinite horizon time

Comparison 1. For $\lambda > 0$, we assume that $k = 1$ and $\theta = \lambda$, meaning $E[\Lambda] = \lambda$ and $Var[\Lambda] = \lambda$. The graph 6 show the difference between the probability of loan repayment in the Memoryless Arrivals (MA) case versus the Randomize Arrivals (RA) case, when $\alpha = 20$, $c = 2$ and $u = 100$. Here the probabilities of default are given by (4.3) and (4.9), respectively.

Comparison 2. For $\lambda > 0$ and $v > 0$, we assume that $k = \frac{\lambda^2}{v}$ and $\theta = \frac{v}{\lambda}$, meaning $E[\Lambda] = \lambda$ and $Var[\Lambda] = v$. In other words, the variance of Λ is fixed. The graph 7 shows the difference of the default probabilities in the MA case versus the RA case, when $\alpha = 20$, $c = 2$ and $u = 100$.

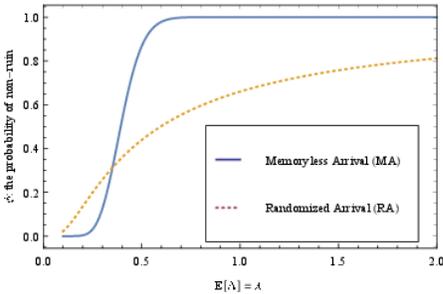


FIG. 6. Default probabilities

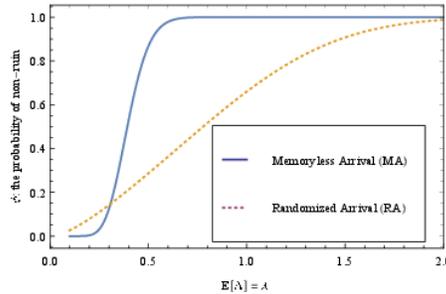


FIG. 7. Default probabilities

6. Conclusion. Although earthquakes are not rare events in Japan, for cultural reasons, only a small part of home-owners in Japan have the relevant insurance cover. Thus in the event of a disaster produced by an earthquake, the uninsured home-owners, roughly 70% of the market, could end up with a completely demolished property and an outstanding mortgage loan (e.g in 2009 only 23% had earthquake

insurance), the double-debt problem. The paper proposes a solution to this problem specific to Japanese mortgages, via an insurance mechanism incorporated in the mortgage repayment scheme. The premium of the insurance can be determined from the probability of default, by solvency control (i.e. less than a given/impose upper bound). By deriving the probability of default of the loan provider in infinite-time, one can then infer close approximations for any finite-time horizons. The modeling framework is that of exponential functional of renewal-reward processes and the methodology stems from mathematical risk theory. The set-up and the results could be translated in other financial or actuarial applications. For instance, as in [19], the exponential functional of a reward process can model the wealth of one person/ family (in danger of down-crossing the poverty line) with an insurance proposal to protect the vulnerable (close to the poverty line) from falling into *poverty traps* (which are absorbing states).

Acknowledgments. The authors would like to thank Professor Ohgaki for fruitful discussions during the preparation of this paper. Also, many thanks to BNP Paribas-Cardiff for their input regarding the practicality of the model.

REFERENCES

- [1] H. Albrecher, C. Constantinescu, and S. Loisel. Explicit ruin formulas for models with dependence among risks. *Insurance Math. Econom.*, 48(2):265–270, 2011.
- [2] H. Albrecher, C. Constantinescu, Z. Palmowski, G. Regensburger, and M. Rosenkranz. Exact and asymptotic results for insurance risk models with surplus-dependent premiums. *SIAM Journal on Applied Mathematics*, 73(1):47–66, 2013.
- [3] H. Albrecher, C. Constantinescu, G. Pirsic, G. Regensburger, and M. Rosenkranz. An algebraic operator approach to the analysis of Gerber-Shiu functions. *Insurance: Mathematics & Economics*, 46(1):42–51, 2010.
- [4] H. Albrecher, C. Constantinescu, and E. Thomann. Asymptotic results for renewal risk models with risky investments. *Stochastic Processes and their Applications*, 122(11):3767–3789, 2012.
- [5] S. Asmussen and H. Albrecher. *Ruin probabilities*, volume 14 of *Advanced Series on Statistical Science & Applied Probability*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, second edition, 2010.
- [6] Jean Bertoin and Marc Yor. Exponential functionals of Lévy processes. *Probab. Surv.*, 2:191–212, 2005.
- [7] D. Buraczewski and T. Damek, E. and Mikosch. *Stochastic models with power-law tails*. Springer Series in Operations Research and Financial Engineering. Springer, [Cham], 2016.
- [8] Philippe Carmona, Frédérique Petit, and Marc Yor. On the distribution and asymptotic results for exponential functionals of Lévy processes. *Exponential functionals and principal values related to Brownian motion*, pages 73–121, 1997.
- [9] J. F. Collamore. Random recurrence equations and ruin in a Markov-dependent stochastic economic environment. *Ann. Appl. Probab.*, 19(4):1404–1458, 2009.
- [10] C. Constantinescu, V. Maume-Deschamps, and R. Norberg. Risk processes with dependence and premium adjusted to solvency targets. *Eur. Actuar. J.*, 2(1):1–20, 2012.
- [11] A. Dassios and P. Embrechts. Martingales and insurance risk. *Comm. Statist. Stochastic Models*, 5(2):181–217, 1989.
- [12] A. Dassios, Yan Qu, and Jia Wei Lim. Exact simulation of generalised Vervaat perpetuities. *J. Appl. Probab.*, 56(1):57–75, 2019.
- [13] D. Dufresne. The distribution of a perpetuity, with applications to risk theory and pension funding. *Scand. Actuar. J.*, (1-2):39–79, 1990.
- [14] P. Embrechts and C. M. Goldie. Perpetuities and random equations. In *Asymptotic statistics (Prague, 1993)*, Contrib. Statist., pages 75–86. Physica, Heidelberg, 1994.
- [15] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. *Tables of integral transforms. Vol. I*. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. Based, in part, on notes left by Harry Bateman.
- [16] Runhuan Feng, Alexey Kuznetsov, and Fenghao Yang. Exponential functionals of Lévy processes

- and variable annuity guaranteed benefits. *Stochastic Processes and their Applications*, 129(2):604–625, 2019.
- [17] C. M. Goldie. Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Probab.*, 1(1):126–166, 1991.
- [18] H. Kesten. Random difference equations and renewal theory for products of random matrices. *Acta Math.*, 131:207–248, 1973.
- [19] Raimund M. Kovacevic and Georg Ch. Pflug. Does Insurance Help to Escape the Poverty Trap? A Ruin Theoretic Approach. *Journal of Risk & Insurance*, 78(4):1003–1028, December 2011.
- [20] T. Mikosch. *Non-life insurance mathematics*. Universitext. Springer-Verlag, Berlin, second edition, 2009. An introduction with the Poisson process.
- [21] T. Mikosch, M. Rezapour, and O. Wintenberger. Heavy tails for an alternative stochastic perpetuity model. *Stochastic Processes and their Applications*, 2018.
- [22] H. Nyrhinen. Finite and infinite time ruin probabilities in a stochastic economic environment. *Stochastic Process. Appl.*, 92(2):265–285, 2001.
- [23] H. Nyrhinen. Power estimates for ruin probabilities. *Adv. in Appl. Probab.*, 37(3):726–742, 2005.
- [24] H. Ohgaki. Shorai no nijusaimu mondai wo ikani kaihi suruka (in japanese), [how should we hedge the double debt problem in the future?]. *Ritsumeikan Hogaku*, (13-3):219–295, 2013.
- [25] J Vecer. A new pde approach for pricing arithmetic average asian options. *Journal of Computational Finance*, 4(4):105–113, 2001.
- [26] W. Vervaat. On a stochastic difference equation and a representation of non-negative infinitely divisible random variables. *Advances in Applied Probability*, 11(4):750–783, 1979.
- [27] Marc Yor. On some exponential functionals of brownian motion. *Advances in Applied Probability*, 24(3):509–531, 1992.
- [28] Marc Yor. *Exponential functionals of Brownian motion and related processes*. Springer Science & Business Media, 2001.

Appendix A. Proof of (3.4). The aim of this section is to prove (3.4), that is,

$$\begin{aligned} & \int_0^\infty |\mathcal{K}_\infty^n \psi_0^\infty(u) - \mathcal{K}^n \psi_0(u, t)| du \\ & \leq \mathbf{E}[e^{-X}] \mathbf{E}[I_{\{W < t\}}] \int_0^\infty |\mathcal{K}_\infty^{n-1} \psi_0^\infty(u) - \mathcal{K}^{n-1} \psi_0(u, t - W)| du \\ & \quad + c \mathbf{E}[e^{-X}]^n \mathbf{P}(W > t) \mathbf{E}[W]. \end{aligned}$$

To obtain (3.4), we need the following Lemmas.

LEMMA A.1. *For $h \in \mathcal{A}_1$, $t > 0$ and $n \in \mathbf{N} \cup \{0\}$, it holds that*

$$(A.1) \quad \int_0^\infty \mathcal{K}h(u, t) du = \mathbf{E}[e^{-X}] \mathbf{E}[I_{\{W < t\}}] \int_0^\infty h(z, t - W) dz.$$

Proof. By the definition of \mathcal{K} ,

$$(A.2) \quad \int_0^\infty \mathcal{K}h(u, t) du = \int_0^\infty du \int_0^{t \wedge \frac{u}{c}} dw \int_0^\infty dx h(e^x(u - cw), t - w) f_{W, X}(w, x).$$

Here we note that W and X are non-negative valued random variables. By change of the variables $(u, w, x) \mapsto (u, w, z = e^x(u - cw))$,

$$\begin{aligned} & \text{(the right hand side of (A.2))} \\ & = \int_0^t dw \int_{cw}^\infty du \int_{u-cw}^\infty dz h(z, t - w) f_{W, X}(w, \log \frac{z}{u - cw}) \frac{1}{z} \\ & = \int_0^t dw \int_0^\infty dz \int_{cw}^{z+cw} du h(z, t - w) f_{W, X}(w, \log \frac{z}{u - cw}) \frac{1}{z}. \end{aligned}$$

By changing variables $(w, z, u) \mapsto (w, z, v = \log \frac{z}{u - cw})$, we conclude equation (A.1). \square

LEMMA A.2. For $g \in L_1$ and $n \in \mathbf{N} \cup \{0\}$, it holds that

$$(A.3) \quad \int_0^\infty \mathcal{K}_\infty^n g(u) du = \mathbf{E}[e^{-X}]^n \int_0^\infty g(z) dz.$$

In particular, it holds that

$$\int_0^\infty \mathcal{K}_\infty^n \psi_0^\infty(u) du = c \mathbf{E}[e^{-X}]^n \mathbf{E}[W].$$

Proof. For $n = 0$, clearly (A.3) holds. For $n \geq 1$, we see that

$$(A.4) \quad \int_0^\infty \mathcal{K}_\infty^n g(u) du = \mathbf{E}[e^{-X}] \int_0^\infty \mathcal{K}_\infty^{n-1} g(u) du.$$

Hence we conclude (A.3) by induction. \square

Now we give a proof of (3.4). Let us consider a decomposition of $\mathcal{K}_\infty \psi_0^\infty(u)$. By the definition of \mathcal{K} and \mathcal{K}_∞ , for $n \in \mathbf{N}$, we see that

$$\begin{aligned} & \mathcal{K}_\infty^n \psi_0^\infty(u) \\ &= I_{\{t \leq \frac{u}{c}\}} \mathbf{E}[(I_{\{W \leq t\}} + I_{\{t \leq W \leq \frac{u}{c}\}}) \mathcal{K}_\infty^{n-1} \psi_0^\infty(e^X(u - cW))] \\ & \quad + I_{\{\frac{u}{c} < t\}} \mathbf{E}[I_{\{W \leq \frac{u}{c}\}} \mathcal{K}_\infty^{n-1} \psi_0^\infty(e^X(u - cW))] \\ &= \left(I_{\{\frac{u}{c} < t\}} \mathbf{E}[I_{\{W \leq \frac{u}{c}\}} \mathcal{K}_\infty^{n-1} \psi_0^\infty(e^X(u - cW))] + I_{\{\frac{u}{c} < t\}} \mathbf{E}[I_{\{W \leq \frac{u}{c}\}} \mathcal{K}_\infty^{n-1} \psi_0^\infty(e^X(u - cW))] \right) \\ & \quad + \mathbf{E}[I_{\{t \leq W \leq \frac{u}{c}\}} \mathcal{K}_\infty^{n-1} \psi_0^\infty(e^X(u - cW))] \\ &= \mathcal{K} \mathcal{K}_\infty^{n-1} \psi_0^\infty(u, t) + I_{\{t \leq \frac{u}{c}\}} \mathbf{E}[I_{\{t \leq W \leq \frac{u}{c}\}} \mathcal{K}_\infty^{n-1} \psi_0^\infty(e^X(u - cW))]. \end{aligned}$$

Here $\mathcal{K}_\infty^{n-1} \psi_0^\infty(e^X(u - cW))$ is identified with $\mathcal{K}_\infty^{n-1} \psi_0^\infty(e^X(u - cW), t - W)$. Since the operator \mathcal{K} is linear, we have that

$$(A.5) \quad \begin{aligned} & \mathcal{K}_\infty^n \psi_0^\infty(u) - \mathcal{K}^n \psi_0(u, t) \\ &= \mathcal{K}(\mathcal{K}_\infty^{n-1} \psi_0^\infty(u, t) - \mathcal{K}^{n-1} \psi_0(u, t)) + I_{\{t \leq \frac{u}{c}\}} \mathbf{E}[I_{\{t \leq W \leq \frac{u}{c}\}} \mathcal{K}_\infty^{n-1} \psi_0^\infty(e^X(u - cW))]. \end{aligned}$$

Therefore we obtain that

$$(A.6) \quad \begin{aligned} & \int_0^\infty |\mathcal{K}_\infty^n \psi_0^\infty(u) - \mathcal{K}^n \psi_0(u, t)| du \\ & \leq \int_0^\infty \mathcal{K} |\mathcal{K}_\infty^{n-1} \psi_0^\infty - \mathcal{K}^{n-1} \psi_0|(u, t) du \\ & \quad + \int_0^\infty I_{\{t \leq \frac{u}{c}\}} \mathbf{E}[I_{\{t \leq W \leq \frac{u}{c}\}} \mathcal{K}_\infty^{n-1} \psi_0^\infty(e^X(u - cW))] du. \end{aligned}$$

By Fubini's theorem, the second term of the right hand side in (A.5) is

$$(A.7) \quad \mathbf{E}[e^{-X}] \mathbf{P}(W > t) \int_0^\infty \mathcal{K}_\infty^{n-1} \psi_0^\infty(u) du.$$

Hence by Lemma A.1, Lemma A.2 and the combination of (A.6) and (A.7), we conclude (3.4).

ON DISTRIBUTIONAL AND ASYMPTOTIC
RESULTS FOR EXPONENTIAL
FUNCTIONALS OF RENEWAL-REWARD
PROCESSES DESCRIBING INSURANCE
RISK MODELS

J. AKAHORI^{1*}, C. CONSTANTINESCU², Y. IMAMURA³, and
HH. PHAM⁴

¹Ritsumeikan University, Kusatsu, Japan
(akahori@se.ritsumei.ac.jp).

²University of Liverpool, Liverpool, UK
(C.Constantinescu@liverpool.ac.uk).

³Kanazawa University, Kanazawa, Japan
(imamuray@se.kanazawa-u.ac.jp).

⁴International University, Vietnam National University Hochiminh
city, Vietnam (phha@hcmiu.edu.vn)..

Inspired by the double-debt problem in Japan where the mortgagor has to pay the remaining loan even if their house was destroyed by a catastrophic event, we model the lender's cash flow, by an exponential functional of a renewal-reward process. We propose an insurance add-on to the loan repayments and analyse the asymptotic behavior of the distribution of the first hitting time, which represents the probability of full repayment. We show that the finite-time probability of full loan repayment converges exponentially fast to the infinite-time one. In a few concrete scenarios, we calculate the exact form of the infinite-time probability and the corresponding premiums.

1 Introduction

We develop a mathematical model for mortgage loans so that we can estimate/measure the risks of the lenders. The model is inspired by a framework designed to hedge the so-called “double-debt problem”, introduced by Ohgaki [24]. For analyzing the risks, we employ methods from risk theory, culminating in an analysis of solutions integro-differential equations with boundary conditions.

The Motivation. After the 2011 Great East Japan Earthquake, a lot of people who lost their houses are still kept under the due of their mortgage loan, which made their recovery rather difficult. It is commonly referred to as the double-debt problem. As Japan is exposed to the risk of further big earthquakes, Ohgaki [24] proposed a practical framework within the regime of the Japanese financial system, where the mortgage loan is combined with a marketized earthquake insurance, like a CAT bond. This paper mathematically formalizes Ohgaki’s [24] proposed scheme. The proposed model will be in continuous time as the mortgage payments of individual mortgagors are not likely to be paid at the same time in a given payment period. However, in order to marketize this insurance-mortgage-security, we need to know the risk exposure.

The Model. Specifically, we consider an initial loan u given to a cohort of borrowers (as mortgages) that are paying it back continuously at a constant rate $c > 0$. Although the loans are fixed term, as mentioned before, the assumption of continuous payments is reasonable, due to the cohort effect, meaning that new customers are coming in as others are leaving the programme. Furthermore, we consider that disasters occur at random times T_i and after each disaster i only a ratio e^{-X_i} of borrowers are left to repay the loan. Thus, as time passes and disasters occur we have fewer and fewer borrowers paying back the loan (at the same rate c). We define as default/ruin the event that the (cash contribution) “process U_t never reaching u ”, meaning that the borrowers will never fully repay the loan. Thus, the probability of default, $\phi(u)$, describes the probability of the process U_t never reaching level u , while the survival probability, $\psi(u)$, defines the probability of the first crossing of the level u , or *probability of full loan repayment*.

Let (T_i, X_i) , $i = 1, 2 \dots$ be a marked point process, with $T_0 = 0$. In our

model, the cash flow process of the mortgage loan at time t is given by

$$U_t = c \sum_{n=0}^{\infty} 1_{[T_n, T_{n+1})}(t) \{ (t - T_n) e^{-\sum_{i=1}^n X_i} + \sum_{i=1}^n (T_i - T_{i-1}) e^{-\sum_{k=1}^{i-1} X_k} \}, \quad (1)$$

where c is a positive constant, T_i is the occurrence time of the i -th disaster and X_i is the rate of the borrowers who survived the i -th disaster, with the requirement $X_i > 0$ (it cannot be zero). Here we denote $\sum_{i=n}^m \dots = 0$ for $m < n$. The process can be understood as an *exponential functional* of the renewal reward process

$$U_t = c \int_0^t e^{R_s} ds$$

where R is the renewal-reward process associated with (T_i, X_i) , $i = 1, 2, \dots$, that is,

$$R_t := - \sum_{j=1}^{J_t} X_j, \quad J_t = \sum_{k=1}^{\infty} 1_{\{T_k \leq t\}}.$$

Figure 1 is the graphical representation of the repayment process. Each red

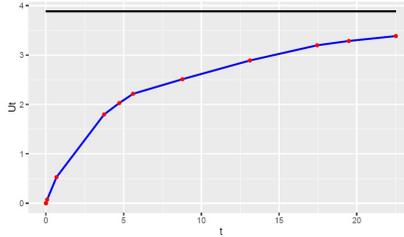


Figure 1: A possible sample path.

dot represents a disaster hit. After every hit, the rate of repayment decreases since the affected mortgagors are forced to leave the pool. The repayment rate therefore slows down as disasters occur. Our aim is to make sure the path reaches the black line (the total amount lended u), in other words the full amount is repaid, within a finite time. Note that the model was inspired by the Japanese double-debt problem, but could account for any risk processes that temporises the effect of claims/losses.

Let τ_u be the first hitting time of $u > 0$, meaning the time the process U (which starts at zero) reaches u for the first time, which means the time the loan has been paid back in full. We shall study the probability of full loan repayment

ψ , before a given time t ,

$$\psi(u, t) := P(\tau_u < t),$$

and its corresponding probability of loan default in finite time horizon

$$\phi(u, t) := P(\tau_u \geq t) = 1 - \psi(u, t),$$

for $u > 0$ and $t > 0$.

The Proposal. The solution proposed in this paper is to include a small insurance premium in the contracted mortgage payments and in return the mortgagor would be relieved of the outstanding mortgage principal in the event of property destruction by a natural disaster. With every disaster hit, it is expected that properties will be destroyed and mortgagors will subsequently default. The premium will be set to ensure that the total amount of all *combined* principals is completely paid off within a finite time-scale. In our model, the premium rate for each mortgagor is going to be the same indifferent of the size of their loan or region of residence.

The combined premiums from all of the mortgagors would be enough to cover the defaulted loans due to natural disasters. If a natural disaster occurs and causes damages of greater amount than the current reserve, the remainder of damages costs would be paid off gradually as the premiums continue to be collected from the mortgagors not affected by the disaster. The insurance premium is calculated and set to ensure that the total amount of combined principals could be paid off in a finite time.

The Results. We derive the probability of full loan repayment, ψ , as a solution of an integral equation. This expression could be further used for the numerical evaluation of the default probability, but in this paper, for specific examples, we propose to use a calibration of the parameters of the infinite-time full-loan-repayment-probability

$$\psi^\infty(u) := \lim_{t \rightarrow \infty} \psi(u, t) = P(\tau_u < \infty),$$

and its counter-part, the probability of loan default

$$\phi^\infty(u) := P(\tau_u = \infty) = 1 - \psi^\infty(u), \tag{2}$$

for all $u > 0$. This strategy of approximation can be justified by Theorem 3 which shows that the difference, between the finite and infinite time probabilities, decays exponentially fast, as $t \rightarrow \infty$, namely for any $u > 0$, there exists $C(u) > 0$ such that

$$0 \leq \psi^\infty(u) - \psi(u, t) \leq C(u)e^{-\xi t}, \quad t \geq 0,$$

for some $\xi > 0$. The numerical experiments in section 5.1 show that under a practical parameter set, the difference is negligible.

Furthermore, under the infinite horizon setting, as in [10], we will discuss the risk parameters for which the ruin probability stays within a range established by say a supervisory authority. We will present the cases of Poisson, non-Poisson and randomized arrivals, all the while the ratio of clients remaining from the initial cohort is exponentially distributed. Concretely, we are considering models with Poisson arrivals with parameter λ , followed by models with non-Poisson arrivals, as in Gamma(2, λ) inter-arrival times, to account for some memory in the process. We study the probability of default or full repayment, by means of integro-differential equations and Laplace transforms, as in e.g. [3, 4, 2]. Conjointly, we discuss the case of random Poisson parameter λ , equivalently to the distribution of inter-arrival time W being an exponential random variable with random parameter Λ , to account for the clustering effect of the earthquake events. Thus one could estimate Λ and dynamically adjust the risks involved, as in [10].

Connections to exponential functionals of Lévy processes. The default (or survival) probability in our model can be interpreted as the distribution of an exponential functional of a stochastic process. Exponential functionals of Lévy processes abound in both finance and insurance mathematics literature. In finance, the distribution of the exponential functional of a process is key in computing the Asian option price, and has attracted a lot of attention, featuring different approaches such as [8], [25], and [27]. Some further references on applications of exponentials of Brownian motion can be found in [28]. In insurance, [16] used the exponential functional of a Lévy process in the analysis of an insurer liabilities when its variable annuity guarantees benefits on an exponential maturity of counterpart. Moreover, *perpetuities*, that can be seen as exponential functionals of a renewal-reward process, arise in a diverse range of fields, see [26] and [14] for a variety of references and examples of applications,

including insurance [22, 23] and economics [11]. In [13], the distribution of a perpetuity presents applications in risk theory and pensions.

Connections to stochastic perpetuities. We note that our cash contribution process U is closely related to a *stochastic perpetuity*, which is the present value of a stochastically discounted series of independent, identically distributed (i.i.d.) cashflows, that is,

$$D = \sum_{k \geq 1} \left(\prod_{i=1}^k d_i \right) C_k, \quad (3)$$

where the cashflows, (C_i) , and the stochastic discount rates, (d_i) , are mutually independent, identically distributed sequences. The full-repayment probability in *infinite time* (2) is understood as the distribution function of a stochastic perpetuity. When considering cash flows C_k , arriving at T_k , $k = 1, 2, \dots$, and discounted at a rate δ , its present value is given by $\sum_{k \geq 1} e^{-\delta T_k} C_k$, which is identified with D of (3) if we assume $T_i - T_{i-1}$, $i = 1, 2, \dots$ are i.i.d., by setting $d_i = e^{-\delta(T_i - T_{i-1})}$. In [13] it is shown to exhibit a Gamma distribution for Poisson inter-arrivals of claims of exponentially distributed intensity (ex. 5.1.2.), result that we can retrieve with our approach. Similar models are interpreted as risk models with stochastic returns on investments (see e.g. [22, 23, 9]).

The main difference between our model and a stochastic perpetuity (or a stochastic interest risk model) consists in the fact that we are dealing with a stochastic process, whereas the perpetuity models are random variables. They are infinite sums of random variable, discounted (stochastic discounting) at fixed times, while our model is, at each time t , a finite sum of random variables which can be seen as “discounted at random times”.

Connections to stochastic fixed point equations. The integral equation we derive for the infinite horizon full repayment probability can be seen as a *stochastic fixed point equation*. As we will be looking at the tail, the structure of corresponding stochastic equation (1) resonates with the *stochastic fixed point equations* described in [17] for the first time, [9] for insurance applications and amply analysed in the book of Mikosch [20]. For such stochastic model of perpetuity-type, for various dependence structures, it is shown the tails of the distribution are regularly varying both in the univariate and multivariate cases. Here references [18, 17, 7, 9, 23, 22]. More recent literature could be found in

[21, 12].

Thus, in studying "default probabilities in infinite horizon", we are lead to stochastic fixed point equations, which are equal in distribution to the ones for stochastic perpetuity. Moreover, by choosing specific distributions for the random variables involved, the time "of discounting", or when an event occurs that would reduce the number of mortgagors, we can obtain explicit solutions of the tail distribution/ loan repayment. Stochastic fixed point equations literature abounds in asymptotic results.

The Structure of the Paper. The rest of the paper is organized as follows. In section 2 we give a general equation for the loan default probability ϕ in a loan model driven by a general marked point process (T_i, X_i) . The finite time and infinite time default probabilities are expressed via Fredholm equations, with solutions to be analyzed asymptotically or numerically, if not readily available in closed forms. We present an approximation of the finite-time probability from its infinite-time counterpart, in 3. In section 4, based on the infinite-horizon equation, we consider the case of X exponentially distributed and derive the ruin probability when W are exponentially and Erlang distributed, respectively, under both independent and conditionally independent scenarios. Finally, we calculate the ruin probability in the case of randomized arrival times when W is exponentially distributed with random parameter Λ . One can then numerically calculate the insurance premium to be incorporated in the mortgage plan, such that the probability of default stays within a small range, as in Section 5. We conclude in Section 6.

Note: In this paper, for keeping the context clear for the reader, the classical notation for ruin probability ψ will be referred to as the *probability of loan repayment*, or *probability of full loan repayment*, while the classical probability of non-ruin ϕ will be referred to as *probability of default*.

2 Ruin probability in a general loan model

Earthquakes are considered rare, extreme events in most parts of the world, but could not be considered as such in Japan. The incidence of earthquakes has increased after the 2011 one. Looking at earthquake data, one can see that the arrivals of earthquakes can be described by one Poisson distribution before

2011, and a different parameter Poisson distribution after 2011. However, being still perceived as an extreme, rare event, in addition to cultural reasons, at the moment only a minor part of home-owners in Japan have the relevant insurance cover. In order to evaluate the risks associated with a marketize earthquake insurance, we calculate the probability of default of a lender, based on a model that can be seen as an *exponentiation of the Cramér-Lundberg model*, which is the classical model in collective non-life insurance (see e.g. [11], [20], [5]).

Let $W_i = T_i - T_{i-1}$, $i = 1, \dots$, $T_0 = 0$. Firstly, assume that (W_i, X_i) , $i = 1, \dots$, are independent and identically distributed with (W, X) . For a bounded measurable function h , we set

$$\mathcal{K}h(u, t) := \mathbf{E}[I_{\{W < t \wedge \frac{u}{c}\}} h(e^X(u - cW), t - W)]. \quad (4)$$

Then clearly \mathcal{K} defines a linear transformation on $L^\infty(\mathbf{R}_+^2)$.

Theorem 1. *The finite-time probability ϕ satisfies the Fredholm type equation*

$$\phi(u, t) = \phi_0(u, t) + \mathcal{K}\phi(u, t), \quad (5)$$

where $\phi_0(u, t) = \mathbf{P}(W > t)I_{\{t \leq \frac{u}{c}\}}$, with $(u, t) \in (0, \infty)^2$.

Proof. Since U is an increasing process, we have that $\phi(u, t) = \mathbf{P}(U_t \leq u)$. Since

$$\phi(u, t) = \mathbf{P}(U_t \leq u, T_1 \geq t) + \mathbf{P}(U_t \leq u, T_1 < t),$$

and since for $T_1 \geq t$, $U_t = ct$, then

$$\phi(u, t) = \phi_0(u, t) + \mathbf{P}(U_t \leq u, T_1 < t). \quad (6)$$

When $T_1 < t$, we have that

$$U_t = cT_1 + ce^{-X_1} \sum_{n=1}^{\infty} 1_{[T_n, T_{n+1})}(t) \{ (t - T_n) e^{-\sum_{i=2}^n X_i} + \sum_{i=2}^n (T_i - T_{i-1}) e^{-\sum_{k=2}^{i-1} X_k} \}.$$

Noting that (T_i, X_i) is a marked point process,

$$U'_{t-T_1} = c \sum_{n=1}^{\infty} 1_{[T_n, T_{n+1})}(t) \{ (t - T_n) e^{-\sum_{i=2}^n X_i} + \sum_{i=2}^n (T_i - T_{i-1}) e^{-\sum_{k=2}^{i-1} X_k} \}$$

has the same distribution as U_{t-T_1} , and it's independent from T_1 and X_1 . Con-

sequently we have that

$$\begin{aligned}\mathbf{P}(U_t \leq u, T_1 < t) &= \mathbf{P}(cT_1 + e^{-X_1}U'_{t-T_1} \leq u, T_1 < t) \\ &= \mathcal{K}\phi(u, t).\end{aligned}\tag{7}$$

Combining (6) with (7), we obtain (2). \square

Corollary 1. *The finite-time probability of full loan repayment is expressed by a Fredholm type equation*

$$\psi(u, t) = \psi_0(u, t) + \mathcal{K}\psi(u, t),\tag{8}$$

where $\psi_0(u, t) := \mathbf{P}(W \geq \frac{u}{c})I_{\{t > \frac{u}{c}\}}$, with $(u, t) \in (0, \infty)^2$.

Proof. By theorem 1, for $\psi = 1 - \phi$. \square

Let \mathcal{A}_p be the (completion of the) set of all functions h on \mathbf{R}_+^2 such that

$$\|h\|_{\mathcal{A}_p} := \sup_{t>0} \int_0^\infty |h(u, t)|^p du < \infty.$$

Lemma 1. *Suppose that X is non-trivial. Then \mathcal{K} defined by (4) is a linear operator on \mathcal{A}_p . Moreover, the operator norm $\|\mathcal{K}\|_{\mathcal{L}(\mathcal{A}_p)} = \sup\{\|\mathcal{K}h\|_{\mathcal{A}_p} : \|h\|_{\mathcal{A}_p} = 1, h \in \mathcal{A}_p\}$ is strictly less than 1, for any $1 \leq p < \infty$.*

Proof. For $h \in \mathcal{A}_p$, by using Jensen's inequality and Minkowski's inequalities, we have that

$$\begin{aligned}\|\mathcal{K}h\|_{\mathcal{A}_p} &\leq \mathbf{E}[\sup_{t>0} \int_0^\infty |I_{\{W < t \wedge \frac{u}{c}\}} h(e^X(u - cW), t - W)|^p du] \\ &\leq \mathbf{E}[e^{-X} \sup_{t>0} \int_0^\infty |h(Z, t)|^p dZ] = \|h\|_{\mathcal{A}_p} \mathbf{E}[e^{-X}].\end{aligned}$$

Since X is non-trivial, we completed the proof. \square

Lemma 2. *Suppose that the first moment of W is finite. Then $\psi_0 \in \cap_{p \geq 1} \mathcal{A}_p$.*

Proof. Let $p \geq 1$. We note that

$$\|\psi_0\|_{\mathcal{A}_p} = \sup_{t>0} \int_0^\infty \psi_0(u, t)^p du dt = \sup_{t>0} \int_0^{ct} \left(\mathbf{P}(W \geq \frac{u}{c})\right)^p du.$$

Having an increasing function,

$$\sup_{t>0} \int_0^{ct} \left(\mathbf{P}(W \geq \frac{u}{c}) \right)^p du = \int_0^\infty \left(\mathbf{P}(W \geq \frac{u}{c}) \right)^p du \leq \int_0^\infty \mathbf{P}(W \geq \frac{u}{c}) du = c\mathbf{E}[W],$$

with the inequality given by the fact that the probability ranges in $[0,1]$. Hence we conclude $\psi_0 \in \mathcal{A}_p$, by the boundedness of the first moment of W . \square

Theorem 2. *Under the assumptions of Lemmas 1 and 2, we have that*

$$\psi = \sum_{n=0}^{\infty} \mathcal{K}^n \psi_0 \in L(\cap_{p \geq 1} \mathcal{A}_p),$$

and moreover $\phi = 1 - \sum_{n=0}^{\infty} \mathcal{K}^n \psi_0$.

Proof. Since the operator norm of \mathcal{K} is strictly less than 1, by Lemma 1, the Neumann series $\sum_{n=0}^{\infty} \mathcal{K}^n$ is convergent in $L(\cap_{p \geq 1} \mathcal{A}_p)$ and thus defines (describes) the inverse operator of $1 - \mathcal{K}$. \square

Remark 1. *All the results in this section are valid for possibly negative X , as long as $\mathbf{E}[e^{-X}] < 1$. Although such situations are not realistic in our mortgage loan modelling, it might be applicable to different contexts. Furthermore, since the results cover the exponential functionals of compound Poisson processes, this may contribute to the literature on exponential functionals of Lévy processes (see e.g. [6]).*

3 From finite-time to infinite-time ruin probability

In this section we will work on infinite horizon ruin probabilities as limit of finite-time ruin probabilities when the time goes to infinity. By continuity of probability measure, it holds that

$$\psi^\infty(u) = \lim_{t \rightarrow \infty} \psi(u, t) = \mathbf{P}(\tau_u < \infty), u > 0.$$

We assume that the assumptions of Lemmas 1 and 2 are satisfied, that is, $\mathbf{P}(X > 0) > 0$ and $\mathbf{E}[W] < \infty$. Moreover, we assume that $\mathbf{P}(W > 0) > 0$ and that the joint density function of W and X exists, and we denote it by $f_{W,X}(w, x)$. Let L_p be the collection of functions which satisfy $\int_0^\infty |g(u)|^p du < \infty$, for $g \in L_p$.

By identifying functions $g \in L_p$, via $g \in \mathcal{A}_p$ with $g(u, t) = g(u)$ ($u, t > 0$), L_p is a subset of \mathcal{A}_p . We define an operator \mathcal{K}_∞ on L_p by

$$\mathcal{K}_\infty g(u) := \mathbf{E} [1_{\{cW < u\}} g(e^X(u - cW))], \quad u > 0.$$

Clearly, $\mathcal{K}_\infty g(u) = \lim_{t \rightarrow \infty} (\mathcal{K}g)(u, t)$, and therefore, by Lemma 1, the operator norm of \mathcal{K}_∞ is strictly less than one, and it leads to

$$\psi^\infty = \sum_{n=0}^{\infty} \mathcal{K}_\infty^n \psi_0^\infty \in \cap_{p \geq 1} L_p, \quad (9)$$

where $\psi_0^\infty(u) := \mathbf{P}(W > \frac{u}{c})$. We note that for any $t > \frac{u}{c}$, $\psi_0(u, t)$ is equal to $\psi_0^\infty(u)$.

3.1 Convergence rate of ruin probability from finite-horizon to infinite-horizon

The finite-time horizon probability of default is extremely relevant for the mortgage markets that deals fixed-term loans. Having such a fast convergence to the infinite-time probability, we can actually use the infinite-time probability (which we can more often calculate explicitly) as an approximation for the finite-time one. In this section, we show that the finite-time ruin probability converges to the infinite-horizon one exponentially fast.

Theorem 3. *We assume that there exists $\theta_0 > 0$ such that the moment generating function $M_W(\theta) = \mathbf{E}[e^{\theta W}]$ exists for any $\theta < \theta_0$. Then, we have the following estimates.*

(i) *There exists $\xi > 0$ and $C > 0$ such that for any $t > 0$,*

$$\int_0^\infty |\psi^\infty(u) - \psi(u, t)| du \leq C e^{-\xi t}. \quad (10)$$

(ii) *For any $u > 0$, there exists $C(u) > 0$ such that for any $t > 0$,*

$$0 \leq \psi^\infty(u) - \psi(u, t) \leq C(u) e^{-\xi' t},$$

for some $\xi' > 0$.

Proof. We first show that (i) implies (ii). By (i), we can choose $\xi > 0$ and $C > 0$

such that (10) holds true. Let $\xi' \in (0, \xi)$. For each $n \in \mathbf{N}$, set

$$A_n := \{u > 0; \psi^\infty(u) - \psi(u, n) \geq C e^{-\xi' n}\}.$$

Then the Lebesgue measure of the event A_n is bounded as

$$\text{Leb}(A_n) \leq \frac{1}{C e^{-\xi' n}} \int_0^\infty |\psi^\infty(u) - \psi(u, t)| du \leq \frac{C e^{-\xi n}}{C e^{-\xi' n}} = e^{-(\xi - \xi')n}.$$

Therefore we have that $\sum_{n=0}^\infty \text{Leb}(A_n) < \infty$. Hence, by Borel-Cantelli lemma, $\text{Leb}(\limsup_{n \rightarrow \infty} A_n) = 0$, which leads to

$$\psi^\infty(u) - \psi(u, t) < C(u) e^{-\xi' t}, \text{ Leb-a.s. } u,$$

for some $C(u) > 0$. Since $\psi^\infty(u)$ and $\psi(u, t)$ are continuous in u , it holds for any u .

Next we prove (i). Since

$$\psi^\infty(u) - \psi(u, t) = \sum_{n=0}^\infty (\mathcal{K}_\infty^n \psi_0^\infty(u) - \mathcal{K}^n \psi_0(u, t)),$$

by Theorem 2 and (9), it suffices to show that there exist constants $C' > 0$, $a \in (0, 1)$, and $\xi > 0$ such that

$$\int_0^\infty |\mathcal{K}_\infty^n \psi_0^\infty(u) - \mathcal{K}^n \psi_0(u, t)| du \leq C' a^n e^{-\xi t}. \quad (11)$$

To establish (11), we rely on the following recursive relation:

$$\begin{aligned} & \int_0^\infty |\mathcal{K}_\infty^n \psi_0^\infty(u) - \mathcal{K}^n \psi_0(u, t)| du \\ & \leq \mathbf{E}[e^{-X}] \mathbf{E}[I_{\{W < t\}} \int_0^\infty |\mathcal{K}_\infty^{n-1} \psi_0^\infty(u) - \mathcal{K}^{n-1} \psi_0(u, t - W)| du] \\ & \quad + c \mathbf{E}[e^{-X}]^n \mathbf{P}(W > t) \mathbf{E}[W]. \end{aligned} \quad (12)$$

A proof of (12) will appear in Appendix.

Using (12) recursively,

$$\begin{aligned}
& \int_0^\infty |\mathcal{K}_\infty^n \psi_0^\infty(u) - \mathcal{K}^n \psi_0(u, t)| du \\
& \leq \mathbf{E}[e^{-X}] \mathbf{E}[I_{\{W_1 < t\}}] \\
& \quad \times \left(\mathbf{E}[e^{-X}] \mathbf{E}[I_{\{W_2 < t - W_1\}}] \int_0^\infty |\mathcal{K}_\infty^{n-2} \psi_0^\infty(u) - \mathcal{K}^{n-2} \psi_0(u, t - W_1 - W_2)| du | W_1] \right. \\
& \quad \left. + c \mathbf{E}[e^{-X}]^{n-1} \mathbf{E}[W] \mathbf{E}[I_{\{W_1 < t\}}] \mathbf{P}(W_2 > t - W_1 | W_1) \right] + c \mathbf{E}[e^{-X}]^n \mathbf{P}(W_1 > t) \mathbf{E}[W] \\
& = \mathbf{E}[e^{-X}]^2 \mathbf{E}[I_{\{S_2 < t\}}] \int_0^\infty |\mathcal{K}_\infty^{n-2} \psi_0^\infty(u) - \mathcal{K}^{n-2} \psi_0(u, t - S_2)| du \\
& \quad + c \mathbf{E}[e^{-X}]^n \mathbf{P}(S_2 > t) \mathbf{E}[W] \\
& \leq \mathbf{E}[e^{-X}]^n \mathbf{E}[I_{\{S_n < t\}}] \int_0^\infty |\psi_0^\infty(u) - \psi_0(u, t - S_n)| du + c \mathbf{E}[e^{-X}]^n \mathbf{E}[W] \mathbf{P}(S_n > t),
\end{aligned} \tag{13}$$

where $S_n = \sum_{k=1}^n W_k$. The L^1 - norm of the difference between the finite and the infinite ruin probabilities in the right-most of (13) is calculated as

$$\int_0^\infty |\psi_0^\infty(u) - \psi_0(u, t)| du = \int_{ct}^\infty \mathbf{P}(W > \frac{u}{c}) du = c \mathbf{E}[I_{\{W > t\}}(W - t)], \tag{14}$$

for $t > 0$. Therefore by substituting (14) to (13), we have that

$$\begin{aligned}
& \text{(the right hand side of (13))} \\
& = c \mathbf{E}[e^{-X}]^n \mathbf{E}[I_{\{S_n < t\}}] \mathbf{E}[I_{\{W_{n+1} > t - S_n\}}(W_{n+1} - (t - S_n)) | S_n] \\
& \quad + c \mathbf{E}[e^{-X}]^n \mathbf{E}[W] \mathbf{P}(S_n > t) \\
& = c \mathbf{E}[e^{-X}]^n \mathbf{E}[I_{\{S_{n+1} > t\}}(W_{n+1} - ((t - S_n) \vee 0))].
\end{aligned} \tag{15}$$

The claim (11) is fulfilled once we establish

$$\mathbf{E}[I_{\{S_{n+1} > t\}}(W_{n+1} - ((t - S_n) \vee 0))] \leq e^{-\xi t} C'' (a')^n \tag{16}$$

with some $\xi, C'' > 0$ and $a' < 1/\mathbf{E}[e^{-X}]$. To see this, we note that for any $\theta \in (0, \theta_0)$,

$$\mathbf{E}[I_{\{S_{n+1} > t\}}(W_{n+1} - ((t - S_n) \vee 0))] \leq \mathbf{E}[I_{\{S_{n+1} > t\}} W_{n+1}] \leq \mathbf{E}[W_{n+1} e^{\theta(S_{n+1} - t)}], \tag{17}$$

allowing for infinite value in the rightmost. Since W_{n+1} and S_n are independent,

the right hand side of (17) is decomposed as follows:

$$\mathbf{E}[W_{n+1}e^{\theta(W_{n+1}+S_n-t)}] = e^{-\theta t}\mathbf{E}[W_{n+1}e^{\theta W_{n+1}}]\mathbf{E}[e^{\theta S_n}] = e^{-\theta t}\mathbf{E}[We^{\theta W}]M_W^n(\theta).$$

Since X is non-trivially positive, we can choose $\xi \in (0, \theta_0)$ such that

$$M_W(\xi) := \mathbf{E}[e^{\xi W}] < \frac{1}{\mathbf{E}[e^{-X}]}.$$

Thus, we established (16), which implies (11), and hence the proof is complete. \square

3.2 Exponential decay rate for exponential time arrival

Firstly, we consider that the arrivals of disasters W are exponentially distributed, with parameter λ , meaning the moment generating function exists for all $\theta < \lambda$, $M_W(\theta) = \frac{\lambda}{\lambda - \theta}$, so Theorem 3 applies, moreover the constants c and C of (10) can be determined explicitly.

Proposition 1. *For W exponentially distributed with parameter $\lambda > 0$,*

$$\int_0^\infty |\psi^\infty(u) - \psi(u, t)| du \leq \frac{1}{\alpha} e^{-\lambda \alpha t}, \quad t \geq 0,$$

where $\alpha = 1 - \mathbf{E}[e^{-X}]$.

Proof. The partial sum $S_n = \sum_{k=1}^n W_k$ is Erlang distributed with parameters n and λ , and its tail distribution function is given by

$$\mathbf{P}(S_n > z) = e^{-\lambda z} \sum_{k=0}^{n-1} \frac{(\lambda z)^k}{k!}.$$

For $n \geq 0$, by tower property of expectations, conditioned by S_n , we have that

$$\begin{aligned} \mathbf{E}[I_{\{S_{n+1} > t\}}(W_{n+1} - ((t - S_n) \vee 0))] &= \mathbf{E}\left[\int_{t-S_n}^\infty (w - ((t - S_n) \vee 0))\lambda e^{-\lambda w} dw\right] \\ &= \mathbf{E}[I_{\{S_n < t\}}e^{-\lambda(t-S_n)}] + \mathbf{P}(S_n > t). \end{aligned} \tag{18}$$

By (3.2), the first term of (18) is rewritten as

$$e^{-\lambda t}\mathbf{E}[I_{\{S_n < t\}}e^{\lambda S_n}] = e^{-\lambda t} \frac{(\lambda t)^n}{n!}. \tag{19}$$

By the combination of (15), (18) and (19), we obtain that

$$\int_0^\infty |\mathcal{K}_\infty^n \psi_0^\infty(u) - \mathcal{K}^n \psi_0(u, t)| du \leq \mathbf{E}[e^{-X}]^n e^{-\lambda t} \sum_{k=0}^n \frac{(\lambda t)^k}{k!}, \quad \forall n \geq 0. \quad (20)$$

Summing up (20) for $n \geq 0$, we get the decay rate

$$\int_0^\infty |\psi^\infty(u) - \psi(u, t)| du \leq \sum_{n=0}^\infty \mathbf{E}[e^{-X}]^n e^{-\lambda t} \sum_{k=0}^n \frac{(\lambda t)^k}{k!} = \frac{\exp(-\lambda t(1 - \mathbf{E}[e^{-X}]))}{1 - \mathbf{E}[e^{-X}]}.$$

□

4 Explicit ruin probabilities in loan models

We will set the distribution parameters for earthquake frequency and severity to values that minimize the probability of a cohort not repaying the loans, that we will call ruin/default. In real life, historical data should be used to determine these parameters and further set the insurance premium which minimizes this probability of ruin. We will work on infinite horizon default/full repayment probabilities, for which the equation (5) reduces to

$$\phi^\infty(u) = \phi_0^\infty(u) + \mathcal{K}_\infty \phi^\infty(u), \quad (21)$$

with $\phi_0^\infty(u) = \lim_{t \rightarrow \infty} \mathbf{P}(W > t) I_{\{t \leq \frac{u}{c}\}} = 0$ and the equation for ψ^∞ is

$$\psi^\infty(u) = \psi_0^\infty(u) + \mathcal{K}_\infty \psi^\infty(u), \quad (22)$$

where $\psi_0^\infty(u) = \lim_{t \rightarrow \infty} \mathbf{P}(W \geq \frac{u}{c}) I_{\{t > \frac{u}{c}\}} = \mathbf{P}(W \geq \frac{u}{c})$.

In the following, we consider that the proportion e^{-X} of borrowers left in the loan programme is described by an exponentially distributed random variable X , with parameter α . In the remainder, for ease of notation, we will use $\phi(u)$ for $\phi^\infty(u)$ and the equivalent for ψ .

4.1 Memory-less Arrivals

We will start with Poisson arrivals, meaning that we wait an exponential amount of time between events. We also assume that the arrivals of disasters W are

independent of the effect on the bank's remaining proportion of borrowers X .

Proposition 2. *If X is exponentially distributed with parameter $\alpha > 0$ and W is exponentially distributed with parameter $\lambda > 0$, and independent of X , then the probability of default ϕ is given by*

$$\phi(u) = \frac{1}{\Gamma(\alpha + 1)} \int_0^{\frac{\lambda u}{c}} z^\alpha \exp(-z) dz, \quad u \geq 0. \quad (23)$$

Proof. Here we give a direct proof using only differentiation. From (21) and using the joint density of (W, X) , $f_{W,X}(w, x) = \lambda \alpha e^{-\lambda w - \alpha x}$, we have that the probability of default ϕ satisfies

$$\phi(u) = \int_0^\infty \int_0^{\frac{u}{c}} \phi(e^x(u - cw)) \lambda \alpha e^{-\lambda w - \alpha x} dw dx. \quad (24)$$

By differentiating both sides of (24), twice, we obtain the following ordinary differential equation with non-constant coefficients

$$\phi''(u) = \left(-\frac{\lambda}{c} + \frac{\alpha}{u} \right) \phi'(u),$$

equipped with two boundary conditions, one at infinity and one at zero. Using the boundary condition $\phi(0) = 0$, the solution of the differential equation reads

$$\phi(u) = C \left(\frac{c}{\lambda} \right)^{\alpha+1} \int_0^{\frac{\lambda u}{c}} z^\alpha \exp(-z) dz.$$

Furthermore, from the infinity condition $\lim_{u \rightarrow \infty} \phi(u) = 1$, $C = \frac{1}{\Gamma(\alpha+1)} \left(\frac{\lambda}{c} \right)^{\alpha+1}$, completing the proof. \square

4.2 General Arrivals

Proposition 3. *If X is exponentially distributed with parameter $\alpha > 0$ and W is a random variable independent of X , with density a positive integrable function f_W , then the Laplace transform of the ruin probability satisfies*

$$\hat{\psi}(s) = s^{-1}(1 - \hat{f}_W(cs)) + \alpha s^{\alpha-1} \hat{f}_W(cs) \int_0^s u^{\alpha-1} (1 - \hat{f}_W(cu)) e^{\int_u^s \alpha v^{-1} \hat{f}_W(cv) dv} du, \quad (25)$$

where \hat{f} denotes the Laplace transform of f .

Proof. The joint density function of (W, X) is $f_{W,X}(w, x) = \alpha f_W(w)e^{-\alpha x}$. By taking Laplace transform for the both sides of (22), we have that, for $s > 0$,

$$\hat{\psi}(s) = \frac{1}{s}(1 - \hat{f}_W(cs)) + \int_0^\infty e^{-su} \int_0^\infty \int_0^{\frac{u}{c}} \psi(e^x(u - cw)) \alpha f_W(w) e^{-\alpha x} dw dx du.$$

By changing the order of the integrals of the second term, which is possible since they are all positive, we obtain that

$$s^\alpha \hat{\psi}(s) = s^{\alpha-1}(1 - \hat{f}_W(cs)) + \alpha s^{-1} \hat{f}_W(cs) \int_0^s x^\alpha \hat{\psi}(x) dx.$$

By denoting $G(s) = \int_0^s x^\alpha \hat{\psi}(x) dx$, we have a separable ordinary differentiable equation in $G(s)$,

$$G'(s) = s^{\alpha-1}(1 - \hat{f}_W(cs)) + \alpha s^{-1} \hat{f}_W(cs) G(s),$$

which leads to

$$G(s) = \left(G(t) + \int_t^s u^{\alpha-1} (1 - \hat{f}_W(cu)) e^{-\int_t^u \alpha v^{-1} \hat{f}_W(cv) dv} du \right) e^{\int_t^s \alpha v^{-1} \hat{f}_W(cv) dv},$$

for $0 < t < s$. Since $\lim_{t \rightarrow 0} G(t) = 0$, by taking $t \rightarrow 0$ for the both sides, we obtain that

$$G(s) = \int_0^s u^{\alpha-1} (1 - \hat{f}_W(cu)) e^{\int_u^s \alpha v^{-1} \hat{f}_W(cv) dv} du,$$

and thus we conclude that (25) is verified. \square

Remark 2. When W is exponential with parameter $\lambda > 0$,

$$\hat{F}_W(cs) = \int_0^\infty e^{-csx} e^{-\lambda x} dx = \frac{1}{\lambda + cs}$$

and thus

$$\hat{\psi}(s) = \frac{1}{s} - \frac{c}{\lambda + cs} + c\alpha s^\alpha (\lambda + cs)^{\alpha-1} \int_0^s \left(\frac{u}{\lambda + cu} \right)^\alpha \left(\frac{1}{u} - \frac{c}{\lambda + cu} \right) du,$$

which after Laplace inversion leads to the results of Proposition 2.

4.3 Arrivals with Memory

When the inter-arrival times W are Erlang distributed, with shape parameter k and rate parameter λ , we can obtain again explicit results for the ruing probability. Recall the density function of an Erlang distribution is given by

$$f(w; k, \lambda) = \frac{\lambda^k w^{k-1} e^{-\lambda w}}{(k-1)!}, \quad x > 0, \quad k \in \mathbf{N}, \quad \lambda > 0, \quad w > 0.$$

Proposition 4. *If X is exponentially distributed with parameter $\alpha > 0$ and W is Erlang distributed with shape parameter 2 and rate parameter λ , then ϕ is given by*

$$\phi(u) = Cg(u), \quad u \geq 0$$

where

$$g(u) = \int_0^{\frac{\lambda u}{c}} \frac{1}{\sqrt{y}} e^{-y} \cosh\left(2\sqrt{(\alpha+2)y}\right) \left(\int_0^{\frac{\lambda u}{c}-y} e^{-x} x^{\alpha+\frac{1}{2}} dx \right) dy, \quad C = \lim_{u \rightarrow \infty} \frac{1}{g(u)}.$$

Proof. Here $f_W(w)$ is $f(w; 2, \lambda)$. The Laplace transform of f_W is given by

$$\hat{f}_W(s) = \int_0^\infty e^{-sw} f_W(w) dw = \frac{\lambda^2}{(s+\lambda)^2},$$

and its derivative is

$$(\hat{f}_W)'(s) = -\frac{2\lambda^2}{(s+\lambda)^3}.$$

By Proposition 3, we have that, for some constant C ,

$$\hat{\phi}(s) = CF(s), \quad s > 0,$$

where $F(s) = \frac{1}{s(s+\frac{\lambda}{c})^{2+\alpha}} e^{\frac{\alpha\lambda}{cs+\lambda}}$ $s > 0$. We claim that

$$\begin{aligned} & \mathcal{L}^{-1}(F)(t) \\ &= \frac{1}{\Gamma(\alpha + \frac{3}{2})\sqrt{\pi}} \left(\frac{c}{\lambda}\right)^{\alpha+2} \int_0^{\frac{\lambda t}{c}} \frac{1}{\sqrt{y}} e^{-y} \cosh\left(2\sqrt{(\alpha+2)y}\right) \left(\int_0^{\frac{\lambda t}{c}-y} e^{-x} x^{\alpha+\frac{1}{2}} dx \right) dy. \end{aligned} \tag{26}$$

This can be seen in the following way. We set $F_1(s) = s^{-1}(s + \frac{\lambda}{c})^{-\alpha - \frac{3}{2}}$ and $F_2(s) = s^{-\frac{1}{2}}e^{\frac{(\alpha+2)\lambda}{cs}}$. Then the inverse Laplace transform of F_1 and F_2 are given by

$$\mathcal{L}^{-1}(F_1)(t) = \frac{1}{\Gamma(\alpha + \frac{3}{2})} \left(\frac{c}{\lambda}\right)^{\alpha + \frac{3}{2}} \int_0^{\frac{\lambda t}{c}} e^{-x} x^{\alpha + \frac{1}{2}} dx,$$

and

$$\mathcal{L}^{-1}(F_2)(t) = \frac{1}{\sqrt{\pi t}} \cosh\left(2\sqrt{\frac{(\alpha+2)\lambda t}{c}}\right),$$

(see e.g. Chapter 5.5 (32) in [15]). Therefore we obtain (26). Hence we conclude that

$$\phi(s) = C\mathcal{L}^{-1}(F)(s).$$

where $\mathcal{L}^{-1}(F)$ is our g . Since $\lim_{u \rightarrow \infty} \phi(u) = 1$, $C = \lim_{u \rightarrow \infty} \frac{1}{g(u)}$. \square

4.4 Randomized Arrival Times

We assume that W is exponentially distributed with random parameter Λ . For any given $\Lambda = \lambda$, W is exponentially distributed with random parameter λ . This special dependence structure is referred to as conditional independence. As in [1], for every given $\Lambda = \lambda$, we calculate the ruin probability, say $\psi_\lambda(u)$, and then integrate over all the possible values of Λ , with distribution function F_Λ , leading to

$$\psi(u) = \int_0^\infty \psi_\lambda(u) dF_\Lambda(\lambda).$$

From (23),

$$P(\tau_u = \infty | \Lambda = \lambda) =: \phi_\lambda(u) := \frac{1}{\Gamma(\alpha + 1)} \int_0^{\frac{\lambda u}{c}} z^\alpha \exp(-z) dz, \quad (27)$$

for each $\lambda(> 0)$, and then the probability of loan repayment satisfies

$$\phi(u) = \int_0^\infty \phi_\lambda(u) dF_\Lambda(\lambda),$$

where $F_\Lambda(\lambda)$ is the distribution function of Λ . When Λ is Erlang distributed with parameter $(k, \frac{1}{\theta})$ with $k, \theta > 0$, namely,

$$dF_\Lambda(\lambda) = \frac{1}{\Gamma(k)\theta^k} \lambda^{k-1} e^{-\frac{\lambda}{\theta}} d\lambda, \quad (28)$$

we can derive explicitly the probability of default.

Theorem 4. *The probability of default under (27) and (28) is expressed by*

$$\phi(u) = \frac{\Gamma(k + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(k)} B_{\frac{u\theta}{c+u\theta}}(\alpha + 1, k), \quad (29)$$

where $B_x(u, v) = \int_0^x y^{u-1}(1-y)^{v-1} dy$ is the incomplete beta function.

Proof. One has that

$$\phi(u) = \int_0^\infty \phi_\lambda(u) \frac{1}{\Gamma(k)\theta^k} \lambda^{k-1} e^{-\frac{\lambda}{\theta}} d\lambda.$$

Differentiating both sides, we have that

$$\phi'(u) = \frac{\Gamma(k + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(k)} \left(\frac{\theta}{c}\right) \left(\frac{c}{c + u\theta}\right)^{k+1} \left(1 - \frac{c}{c + u\theta}\right)^\alpha.$$

Therefore for some constant C , the probability of default is given by

$$\phi(u) = \frac{\Gamma(k + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(k)} \left(\frac{\theta}{c}\right) \int_0^u \left(\frac{c}{c + x\theta}\right)^{k+1} \left(1 - \frac{c}{c + x\theta}\right)^\alpha dx + C.$$

Since $\phi_\lambda(0) = 0$ for each $\lambda > 0$, and $\phi(0) = 0$, we have that $C = 0$. By a further change of variables, we conclude that the probability of default is expressed as in (29). \square

Remark 3. *We note that the order of $\phi(u)$ is $\mathcal{O}(1/u^k)$.*

5 Numerical experiments

In this section we present firstly two algorithms for the calculating the probability of default (ruin), apply them to some concrete numerical examples/chosen parameters, then analyse the corresponding premium rates.

5.1 Simulation for finite-time ruin versus infinite-time one

We present two algorithms. Algorithm 2 is faster and less variant than the Algorithm 1. However, Algorithm 2 can only be used for the case that inter-arrival is exponential distributed (or conditional exponential distributed for randomized arrival times model).

Algorithm 1

First, an algorithm which simulates the default state up to a given finite horizon time t by simulating scenarios of cash flows:

$$U_t = c \sum_{n=1}^{\infty} \left[1_{[T_n, T_{n+1})}(t) (t - T_n) e^{-\sum_{i=1}^n X_i} + \sum_{i=1}^n (T_i - T_{i-1}) e^{-\sum_{i=1}^{i-1} X_i} \right].$$

- Initiate $k = 0$, $T(0) = 0$, $X(0) = 0$ as time and effect at time 0.
- While $T(k) < t$
 - $k = k + 1$,
 - Simulate (W, X) given a specific distribution,
 - Inter-arrival time $W(k) = W$,
 - Arrival time $T(k) = T(k - 1) + W(k)$,
 - Effect $X(k) = X$.
- Calculate the cash flow at $t: U_t = (t - T_{k-1}) e^{-\sum_{i=1}^{k-1} X(i)} + \sum_{i=1}^{k-1} W(i) e^{-\sum_{i=1}^{i-1} X(i)}$.
- Determine default state $I = 1_{\{U_t < u\}}$

Repeat the procedure N times (with N large). The sample average of I is an estimation for the default probability $\phi(u, t)$.

Algorithm 2

In the memoryless arrival model, the number of arrivals up to time $(T_n)_{n \geq 0}$ is a homogeneous Poisson process with parameter λ . The number of arrivals up to t , $N_t = \sum_{n=1}^{\infty} 1_{\{T_n \leq t\}}$ is also Poisson distributed with parameter λ . Given $N_t = n$, arrival times T_1, \dots, T_n is the ordered statistics of n i.i.d U_1, \dots, U_n with uniform distribution on $[0, t]$. So we can get the algorithm as follows:

- Simulate a random number n from Poisson distribution with parameter λt .
- Simulate n random number U_1, \dots, U_n from n i.i.d uniform distribution $U([0, t])$.
- Sort U_1, \dots, U_n to get arrival time T_1, \dots, T_n .
- Simulate effect X_1, \dots, X_n given a specific distribution.

- Calculate cash flow at t : $U_t = (t - T_{k-1})e^{-\sum_{i=1}^{k-1} X(i)} + \sum_{i=1}^{k-1} W(i)e^{-\sum_{i=1}^{i-1} X(i)}$.
- Determine the default state $I = 1_{\{U_t < u\}}$.

Repeat the procedure N times (with N large). The expectation of I is an estimation for default probability $\phi(u, t)$.

5.2 Risk Adjusted Premium Rates

As in [10], imposing a solvency level on $\psi(u)$ we could derive c . Comparing the resulting c with the mortgage repayment rate, we could find out the amount that would be considered as premium. We are conducting a sensitivity analysis to see how each variable impacts the results. The aim is to find the values of the parameters which minimise the associated risks, but still keep the mortgage attractive, i.e. keep the monthly payments relatively low. In real world the parameters α and λ would be estimated using historical data, but for our analysis purposes, we will start from some given values.

Memoryless Arrivals

Let's assume X follows an exponential distribution with parameter α , and W follows an exponential distribution with a parameter λ . X and W are independent of each other. Then, according to Proposition 2, for any $u \geq 0$, the probability of default is an incomplete Gamma function, (23). In our analysis, we focus on the repayment rate (with insurance premium included), as a percentage of the loan, namely c/u . When we impose a fixed solvency target, ϵ , namely $\phi(u) \leq \epsilon$, from $\frac{1}{\Gamma(\alpha+1)} \int_0^{\frac{\lambda u}{c}} z^\alpha e^{-z} dz \leq \epsilon$ we obtain that $\frac{c}{u} \geq \frac{\lambda}{\Gamma_\alpha^{-1}(\epsilon)}$, where Γ_α^{-1} is the inverse function of regularized incomplete Gamma with parameter α . Table 1 presents the minimum values of $\frac{c}{u} = \frac{\lambda}{\Gamma_\alpha^{-1}(\epsilon)}$, for $\epsilon = 0.00001$, when varying the parameters α and λ . Moreover, one can simulate histograms

Table 1: $\frac{c}{u}$ for different values of the parameters λ and α

$\lambda \backslash \alpha$	1	2	3	4	5	6
1	0.0850603	0.0717969	0.0628385	0.0562366	0.0511059	0.0469712
2	0.170121	0.143594	0.125677	0.112473	0.102212	0.0939424
3	0.255181	0.215391	0.188515	0.16871	0.153318	0.140914
4	0.340241	0.287188	0.251354	0.224946	0.204424	0.187885
5	0.425301	0.358985	0.314192	0.281183	0.25553	0.234856

of the cash flow at a specific times. For instance, for $\lambda = 0.5$ and $\alpha = 20$, the

histograms at $T = 100$ (Figure 2 and Figure 3), present very small differences, which are caused by the different estimates of the probability of default.

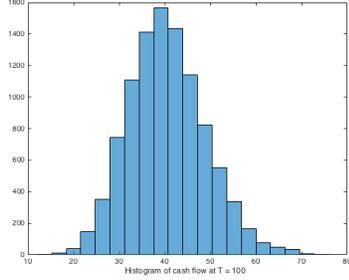


Figure 2: Histogram for cash flow at a given time by Algorithm 1

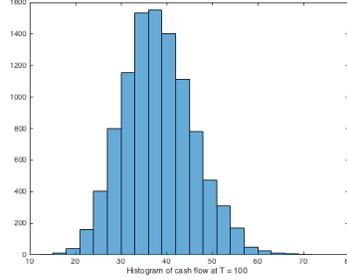


Figure 3: Histogram for cash flow at a given time by Algorithm 2

Randomised Arrivals

Recall from Theorem 4, that for any $u \geq 0$, the probability of default is an incomplete Beta function, (29). We consider Λ , such that $E[\Lambda] = k\theta$ and $V[\Lambda] = k\theta^2$. When we impose a solvency target $\phi(u) \leq \epsilon$ and when $k\theta = 1$ ($\theta = 1/k$), then from Theorem 4, $\frac{c}{u} \geq \frac{(1 - B_{\alpha+1,k}^{-1}(\epsilon))}{kB_{\alpha+1,k}^{-1}(\epsilon)}$, where $B_{\alpha+1,k}^{-1}$ is the inverse function of regularised incomplete Beta with parameters $\alpha + 1$ and k . Table 2 describes the minimum values realized by c/u , for given parameters α and k , under a solvency target $\epsilon = 0.0001$. For inter-arrival times W exponentially distributed

Table 2: $\frac{c}{u}$ for different values of the parameters k and α

$k \backslash \alpha$	1	2	3	4	5	6	7
1	99.	20.5443	9.	5.30957	3.64159	2.72759	2.16228
3	81.0935	14.9698	6.08913	3.41611	2.26003	1.64776	1.27925
5	76.9924	13.6923	5.42064	2.98014	1.94112	1.39791	1.07451
7	75.1637	13.1197	5.11958	2.78298	1.79635	1.28412	0.981004
9	74.1276	12.794	4.94774	2.67007	1.7132	1.21859	0.927026

with parameter $\lambda = 0.2$, X is exponentially distributed with parameter $\alpha = 20$, and $\frac{u}{c} = 50$, Figure 4 provides information about the convergence behavior of the default probability, as the horizon time tends to infinity. After $N = 10^5$ simulations, one can see that the probability of default in finite time horizon converges exponentially to the one in infinite time horizon, namely, in this particular example, to 0.0035. (Figure 5).

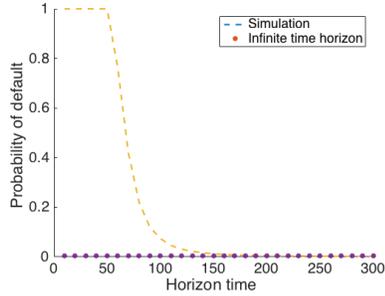


Figure 4: A comparison of probability of default by simulation

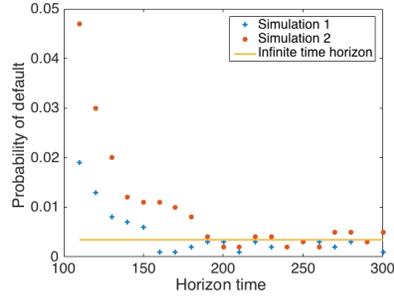


Figure 5: Convergence of probability of default from finite horizon time to infinite horizon time

Comparison 1

For $\lambda > 0$, we assume that $k = 1$ and $\theta = \lambda$, meaning $E[\Lambda] = \lambda$ and $Var[\Lambda] = \lambda$. The graph 6 show the difference between the probability of loan repayment in the Memoryless Arrivals (MA) case versus the Randomize Arrivals (RA) case, when $\alpha = 20$, $c = 2$ and $u = 100$. Here the probabilities of default are given by (23) and (29), respectively.

Comparison 2

For $\lambda > 0$ and $v > 0$, we assume that $k = \frac{\lambda^2}{v}$ and $\theta = \frac{v}{\lambda}$, meaning $E[\Lambda] = \lambda$ and $Var[\Lambda] = v$. In other words, the variance of Λ is fixed. The graph 7 shows the difference of the default probabilities in the MA case versus the RA case, when $\alpha = 20$, $c = 2$ and $u = 100$.

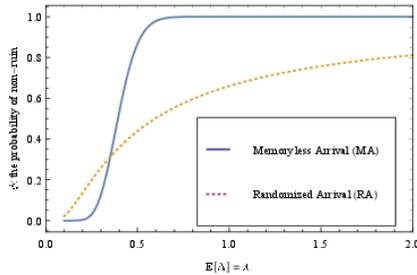


Figure 6: Default probabilities

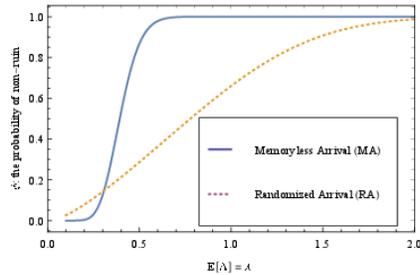


Figure 7: Default probabilities

6 Conclusion

Although earthquakes are not rare events in Japan, for cultural reasons, only a small part of home-owners in Japan have the relevant insurance cover. Thus in the event of a disaster produced by an earthquake, the uninsured home-owners, roughly 70% of the market, could end up with a completely demolished property and an outstanding mortgage loan (e.g in 2009 only 23% had earthquake insurance), the double-debt problem. The paper proposes a solution to this problem specific to Japanese mortgages, via an insurance mechanism incorporated in the mortgage repayment scheme. The premium of the insurance can be determined from the probability of default, by solvency control (i.e. less than a given/impose upper bound). By deriving the probability of default of the loan provider in infinite-time, one can then infer close approximations for any finite-time horizons. The modeling framework is that of exponential functional of renewal-reward processes and the methodology stems from mathematical risk theory. The set-up and the results could be translated in other financial or actuarial applications. For instance, as in [19], the exponential functional of a reward process can model the wealth of one person/ family (in danger of downcrossing the poverty line) with an insurance proposal to protect the vulnerable (close to the poverty line) from falling into *poverty traps* (which are absorbing states).

Acknowledgments

The authors would like to thank Professor Ohgaki for fruitful discussions during the preparation of this paper. Also, many thanks to BNP Paribas-Cardiff for their input regarding the practicality of the model.

References

- [1] H. Albrecher, C. Constantinescu, and S. Loisel. Explicit ruin formulas for models with dependence among risks. *Insurance Math. Econom.*, 48(2):265–270, 2011.
- [2] H. Albrecher, C. Constantinescu, Z. Palmowski, G. Regensburger, and M. Rosenkranz. Exact and asymptotic results for insurance risk models

- with surplus-dependent premiums. *SIAM Journal on Applied Mathematics*, 73(1):47–66, 2013.
- [3] H. Albrecher, C. Constantinescu, G. Pirsic, G. Regensburger, and M. Rosenkranz. An algebraic operator approach to the analysis of Gerber-Shiu functions. *Insurance: Mathematics & Economics*, 46(1):42–51, 2010.
- [4] H. Albrecher, C. Constantinescu, and E. Thomann. Asymptotic results for renewal risk models with risky investments. *Stochastic Processes and their Applications*, 122(11):3767–3789, 2012.
- [5] S. Asmussen and H. Albrecher. *Ruin probabilities*, volume 14 of *Advanced Series on Statistical Science & Applied Probability*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, second edition, 2010.
- [6] Jean Bertoin and Marc Yor. Exponential functionals of Lévy processes. *Probab. Surv.*, 2:191–212, 2005.
- [7] D. Buraczewski and T. Damek, E. and Mikosch. *Stochastic models with power-law tails*. Springer Series in Operations Research and Financial Engineering. Springer, [Cham], 2016.
- [8] Philippe Carmona, Frédérique Petit, and Marc Yor. On the distribution and asymptotic results for exponential functionals of lévy processes. *Exponential functionals and principal values related to Brownian motion*, pages 73–121, 1997.
- [9] J. F. Collamore. Random recurrence equations and ruin in a Markov-dependent stochastic economic environment. *Ann. Appl. Probab.*, 19(4):1404–1458, 2009.
- [10] C. Constantinescu, V. Maume-Deschamps, and R. Norberg. Risk processes with dependence and premium adjusted to solvency targets. *Eur. Actuar. J.*, 2(1):1–20, 2012.
- [11] A. Dassios and P. Embrechts. Martingales and insurance risk. *Comm. Statist. Stochastic Models*, 5(2):181–217, 1989.
- [12] A. Dassios, Yan Qu, and Jia Wei Lim. Exact simulation of generalised Vervaat perpetuities. *J. Appl. Probab.*, 56(1):57–75, 2019.
- [13] D. Dufresne. The distribution of a perpetuity, with applications to risk theory and pension funding. *Scand. Actuar. J.*, (1-2):39–79, 1990.

- [14] P. Embrechts and C. M. Goldie. Perpetuities and random equations. In *Asymptotic statistics (Prague, 1993)*, Contrib. Statist., pages 75–86. Physica, Heidelberg, 1994.
- [15] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. *Tables of integral transforms. Vol. I*. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. Based, in part, on notes left by Harry Bateman.
- [16] Runhuan Feng, Alexey Kuznetsov, and Fenghao Yang. Exponential functionals of lévy processes and variable annuity guaranteed benefits. *Stochastic Processes and their Applications*, 129(2):604–625, 2019.
- [17] C. M. Goldie. Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Probab.*, 1(1):126–166, 1991.
- [18] H. Kesten. Random difference equations and renewal theory for products of random matrices. *Acta Math.*, 131:207–248, 1973.
- [19] Raimund M. Kovacevic and Georg Ch. Pflug. Does Insurance Help to Escape the Poverty Trap?—A Ruin Theoretic Approach. *Journal of Risk & Insurance*, 78(4):1003–1028, December 2011.
- [20] T. Mikosch. *Non-life insurance mathematics*. Universitext. Springer-Verlag, Berlin, second edition, 2009. An introduction with the Poisson process.
- [21] T. Mikosch, M. Rezapour, and O. Wintenberger. Heavy tails for an alternative stochastic perpetuity model. *Stochastic Processes and their Applications*, 2018.
- [22] H. Nyrhinen. Finite and infinite time ruin probabilities in a stochastic economic environment. *Stochastic Process. Appl.*, 92(2):265–285, 2001.
- [23] H. Nyrhinen. Power estimates for ruin probabilities. *Adv. in Appl. Probab.*, 37(3):726–742, 2005.
- [24] H. Ohgaki. Shorai no nijusaimu mondai wo ikani kaihi suruka (in japanese), [how should we hedge the double debt problem in the future?]. *Ritsumeikan Hogaku*, (13-3):219–295, 2013.
- [25] J Vecer. A new pde approach for pricing arithmetic average asian options. *Journal of Computational Finance*, 4(4):105–113, 2001.

- [26] W. Vervaat. On a stochastic difference equation and a representation of non-negative infinitely divisible random variables. *Advances in Applied Probability*, 11(4):750–783, 1979.
- [27] Marc Yor. On some exponential functionals of brownian motion. *Advances in Applied Probability*, 24(3):509–531, 1992.
- [28] Marc Yor. *Exponential functionals of Brownian motion and related processes*. Springer Science & Business Media, 2001.

A Proof of (12)

The aim of this section is to prove (12), that is,

$$\begin{aligned} & \int_0^\infty |\mathcal{K}_\infty^n \psi_0^\infty(u) - \mathcal{K}^n \psi_0(u, t)| du \\ & \leq \mathbf{E}[e^{-X}] \mathbf{E}[I_{\{W < t\}} \int_0^\infty |\mathcal{K}_\infty^{n-1} \psi_0^\infty(u) - \mathcal{K}^{n-1} \psi_0(u, t - W)| du] \\ & \quad + c \mathbf{E}[e^{-X}]^n \mathbf{P}(W > t) \mathbf{E}[W]. \end{aligned}$$

To obtain (12), we need the following Lemmas.

Lemma 3. For $h \in \mathcal{A}_1$, $t > 0$ and $n \in \mathbf{N} \cup \{0\}$, it holds that

$$\int_0^\infty \mathcal{K}h(u, t) du = \mathbf{E}[e^{-X}] \mathbf{E}[I_{\{W < t\}} \int_0^\infty h(z, t - W) dz]. \quad (30)$$

Proof. By the definition of \mathcal{K} ,

$$\int_0^\infty \mathcal{K}h(u, t) du = \int_0^\infty du \int_0^{t \wedge \frac{u}{c}} dw \int_0^\infty dx h(e^x(u - cw), t - w) f_{W, X}(w, x). \quad (31)$$

Here we note that W and X are non-negative valued random variables. By change of the variables $(u, w, x) \mapsto (u, w, z = e^x(u - cw))$,

$$\begin{aligned} & \text{(the right hand side of (31))} \\ & = \int_0^t dw \int_{cw}^\infty du \int_{u-cw}^\infty dz h(z, t - w) f_{W, X}(w, \log \frac{z}{u - cw}) \frac{1}{z} \\ & = \int_0^t dw \int_0^\infty dz \int_{cw}^{z+cw} du h(z, t - w) f_{W, X}(w, \log \frac{z}{u - cw}) \frac{1}{z}. \end{aligned}$$

By changing variables $(w, z, u) \mapsto (w, z, v = \log \frac{z}{u-cw})$, we conclude equation (30). \square

Lemma 4. For $g \in L_1$ and $n \in \mathbf{N} \cup \{0\}$, it holds that

$$\int_0^\infty \mathcal{K}_\infty^n g(u) du = \mathbf{E}[e^{-X}]^n \int_0^\infty g(z) dz. \quad (32)$$

In particular, it holds that

$$\int_0^\infty \mathcal{K}_\infty^n \psi_0^\infty(u) du = c \mathbf{E}[e^{-X}]^n \mathbf{E}[W].$$

Proof. For $n = 0$, clearly (32) holds. For $n \geq 1$, we see that

$$\int_0^\infty \mathcal{K}_\infty^n g(u) du = \mathbf{E}[e^{-X}] \int_0^\infty \mathcal{K}_\infty^{n-1} g(u) du. \quad (33)$$

Hence we conclude (32) by induction. \square

Now we give a proof of (12). Let us consider a decomposition of $\mathcal{K}_\infty \psi_0^\infty(u)$. By the definition of \mathcal{K} and \mathcal{K}_∞ , for $n \in \mathbf{N}$, we see that

$$\begin{aligned} & \mathcal{K}_\infty^n \psi_0^\infty(u) \\ &= I_{\{t \leq \frac{u}{c}\}} \mathbf{E}[(I_{\{W \leq t\}} + I_{\{t \leq W \leq \frac{u}{c}\}}) \mathcal{K}_\infty^{n-1} \psi_0^\infty(e^X(u - cW))] \\ & \quad + I_{\{\frac{u}{c} < t\}} \mathbf{E}[I_{\{W \leq \frac{u}{c}\}} \mathcal{K}_\infty^{n-1} \psi_0^\infty(e^X(u - cW))] \\ &= \left(I_{\{\frac{u}{c} < t\}} \mathbf{E}[I_{\{W \leq \frac{u}{c}\}} \mathcal{K}_\infty^{n-1} \psi_0^\infty(e^X(u - cW))] + I_{\{t \leq \frac{u}{c}\}} \mathbf{E}[I_{\{W \leq \frac{u}{c}\}} \mathcal{K}_\infty^{n-1} \psi_0^\infty(e^X(u - cW))] \right) \\ & \quad + \mathbf{E}[I_{\{t \leq W \leq \frac{u}{c}\}} \mathcal{K}_\infty^{n-1} \psi_0^\infty(e^X(u - cW))] \\ &= \mathcal{K} \mathcal{K}_\infty^{n-1} \psi_0^\infty(u, t) + I_{\{t \leq \frac{u}{c}\}} \mathbf{E}[I_{\{t \leq W \leq \frac{u}{c}\}} \mathcal{K}_\infty^{n-1} \psi_0^\infty(e^X(u - cW))]. \end{aligned}$$

Here $\mathcal{K}_\infty^{n-1} \psi_0^\infty(e^X(u - cW))$ is identified with $\mathcal{K}_\infty^{n-1} \psi_0^\infty(e^X(u - cW), t - W)$.

Since the operator \mathcal{K} is linear, we have that

$$\begin{aligned} & \mathcal{K}_\infty^n \psi_0^\infty(u) - \mathcal{K}^n \psi_0(u, t) \\ &= \mathcal{K}(\mathcal{K}_\infty^{n-1} \psi_0^\infty(u, t) - \mathcal{K}^{n-1} \psi_0(u, t)) + I_{\{t \leq \frac{u}{c}\}} \mathbf{E}[I_{\{t \leq W \leq \frac{u}{c}\}} \mathcal{K}_\infty^{n-1} \psi_0^\infty(e^X(u - cW))]. \end{aligned} \quad (34)$$

Therefore we obtain that

$$\begin{aligned}
& \int_0^\infty |\mathcal{K}_\infty^n \psi_0^\infty(u) - \mathcal{K}^n \psi_0(u, t)| du \\
& \leq \int_0^\infty \mathcal{K} |\mathcal{K}_\infty^{n-1} \psi_0^\infty - \mathcal{K}^{n-1} \psi_0|(u, t) du \\
& \quad + \int_0^\infty I_{\{t \leq \frac{u}{c}\}} \mathbf{E}[I_{\{t \leq W \leq \frac{u}{c}\}} \mathcal{K}_\infty^{n-1} \psi_0^\infty(e^X(u - cW))] du.
\end{aligned} \tag{35}$$

By Fubini's theorem, the second term of the right hand side in (34) is

$$\mathbf{E}[e^{-X}] \mathbf{P}(W > t) \int_0^\infty \mathcal{K}_\infty^{n-1} \psi_0^\infty(u) du. \tag{36}$$

Hence by Lemma 3, Lemma 4 and the combination of (35) and (36), we conclude (12).