SINGULAR ORBITS AND BAKER DOMAINS

LASSE REMPE

Abstract. We show that there is a transcendental meromorphic function with an invariant Baker domain U such that every singular value of f is a super-attracting periodic point. This answers a question of Bergweiler from 1993. We also show that U can be chosen to contain arbitrarily large round annuli, centred at zero, of definite modulus. This answers a question of Mihaljević and the author from 2013, and complements recent work of Barański et al concerning this question.

I long for the Person from Porlock
To bring my thoughts to an end,
I am becoming impatient to see him
I think of him as a friend.

Stevie Smith

1. Introduction

Let $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function of degree at least 2. The *Fatou set* $F(f)$ consists of those points $z \in \mathbb{C}$ near which the iterates of f form a normal family in the sense of Montel. In other words, these are the points at which the dynamics generated by f is stable under small perturbations. A connected component of $F(f)$ is called a *Fatou component*; such a component is *invariant* if $f(U) \subset U$. An invariant Fatou component may be an immediate basin of attraction of an attracting or parabolic fixed point, or a simply or doubly connected domain on which f is conjugate to an irrational rotation.

It was shown by Fatou [Fat20, §30–31] that there is a close relationship between invariant Fatou components and the critical values of f. Indeed, every attracting or parabolic basin must contain a critical value, and the boundary of any rotation domain is contained in the *postcritical set* $\mathcal{P}(f)$, i.e., the closure of the union of all critical orbits of f. See Lemmas 8.5, 15.7 and Theorems 10.15, 11.17 of [Mil06].

These relationships carry over to the case of a transcendental meromorphic function $f\colon \mathbb{C} \to \hat{\mathbb{C}}$, with the set of critical values replaced by the set $\operatorname{sing}(f^{-1})$ of critical or asymptotic values of f, and the postcritical set by the *postsingular set* defined analogously. In this setting, there is another possible type of invariant Fatou component: An *invariant Baker domain*, in which the iterates converge to the essential singularity at ∞. Bergweiler [Ber93, Question 4] asked whether there is a relation between $\operatorname{sing}(f^{-1})$ and the boundary of such a Baker domain. He also asked a more precise version of this question [Ber93, Question 5]:

1.1. Question. Is it possible that a meromorphic function f has Baker domains if the forward orbit of z is bounded for all $z \in \operatorname{sing}(f^{-1})$?

2020 *Mathematics Subject Classification.* Primary 37F10, secondary 30D05, 37F31.

We give an affirmative answer.

1.2. Theorem (Baker domains and super-attracting points)**.** *There is a transcendental meromorphic function* $f\colon \mathbb{C} \to \hat{\mathbb{C}}$ *with a Baker domain such that every point of* $\operatorname{sing}(f^{-1})$ *is a super-attracting periodic point of period* 2*.*

Regarding [Ber93, Question 4], Bergweiler [Ber95, Theorem 3] obtained the following answer when f is a transcendental *entire* function: If U contains no critical or asymptotic values of f, then there exists a sequence $(p_n)_{n=0}^{\infty}$ of postsingular points $p_n \in \mathcal{P}(f)$ of f such that

(a) $|p_n| \to \infty$;
(b) $\frac{\operatorname{dist}(p_n,W)}{|p_n|} \to 0$; and
(c) $\lim_{n\to\infty}|p_{n+1}|/|p_n| = 1$.

On the other hand [Ber95, Theorem 1], there is an entire function with a Baker domain U such that $\operatorname{dist}(\mathcal{P}(f), \partial U) > 0$, where dist denotes Euclidean distance.

For Baker domains of general transcendental *meromorphic* functions, it was shown in [MR13, Theorem 1.5] that an analogue of [Ber95, Theorem 3] holds, without the hypothesis that U contains no critical or asymptotic values, where condition (c) is replaced by the weaker

$$\limsup_{n\to\infty} \frac{|p_{n+1}|}{|p_n|} < \infty. \tag{1.1}$$

The article [MR13] also poses the following question.

1.3. Question ([MR13, Remark on p. 1603])**.** Can (1.1) be replaced by (c)?

In [BFJK20, Theorem A], it is shown that the answer is positive if $\mathbb{C} \setminus U$ has an unbounded connected component. Here we show that the answer to Question 1.3 is negative in general.

1.4. Theorem (Annuli of definite modulus)**.** *Let* $\rho > 0$*. Then the function in Theorem 1.2 can be chosen such that there is a sequence* $R_j \to \infty$ *with*

$$\{z \in \mathbb{C}\colon R_j < |z| < \rho R_j\} \subset U \tag{1.2}$$

for all j*.*

The constructions in our paper are inspired by those of Rippon and Stallard [RS06, Theorem 1.2] and Barański et al [BFJK15, Theorem C]. Both of these construct meromorphic functions with multiply-connected Baker domains, obtained from an affine map by inserting poles at a sequence of points tending to infinity. We shall use quasiconformal surgery instead of explicit formulae, which allows us to obtain the required control of the singular orbits of f.

Notation. $\mathbb{C}$, $\hat{\mathbb{C}}$ and $\mathbb{D}$ denote the complex plane, Riemann sphere and unit disc, respectively. The Euclidean disc of radius r around $z \in \mathbb{C}$ is denoted by $D(z, r)$.

Acknowledgements. I am grateful to Bogusława Karpińska, whose excellent presentation about [BFJK20] at the 2020 online conference *On geometric complexity of Julia sets - II* inspired this work. I also thank her and Walter Bergweiler for interesting discussions following the talk, and impan and the Banach Centre in Warsaw for hosting the above conference. Finally, I am grateful to the referee for their thoughtful comments, which improved the paper.

2. Quasiconformal surgery

The function in Theorem 1.2 is constructed by a "cut and paste" quasiconformal surgery (see [BF14, Chapter 7]), starting from the linear map

$$f_0(z) := \mu z, \tag{2.1}$$

where $\mu > 1$. Consider the closed and connected set

$$U_0 := \{z \in \mathbb{C}\colon \operatorname{Re} z \geq 0\} \cup \bigcup_{j\geq 0} \left\{ z \in \mathbb{C}\colon \mu^j \leq |z| \leq \mu^{j+\frac{1}{2}} \right\},$$

which is forward-invariant under f_0. Also let D_0 be a closed disc

$$D_0 = \overline{D(\zeta_0, r_0)} \subset \{z \in \mathbb{C}\colon \operatorname{Re} z < 0 \text{ and } \sqrt{\mu} < |z| < \mu\} \subset \mathbb{C} \setminus U_0;$$

If we define

$$D_j := f_0^j(D_0) = \overline{D(\mu^j \cdot \zeta_0, \mu^j \cdot r_0)} =: \overline{D(\zeta_j, r_j)},$$

then all D_j are disjoint from U_0, and from each other.

We construct our functions from f_0 by a quasiconformal surgery that inserts poles in the discs D_j. The following makes this procedure precise.

2.1. Proposition (Quasiconformal surgery of f_0)**.** *Let $\mu > 1$, and let f_0, U_0 and $(D_j)_{j=0}^{\infty}$ be defined as above. Let $K > 1$, and let*

$$h_j\colon D_j \to \hat{\mathbb{C}}$$

be a sequence of K-quasiregular functions such that each h_j extends continuously to ∂D_j with $h_j = f_0$ on ∂D_j. Then

$$F\colon \mathbb{C} \to \hat{\mathbb{C}}; \quad z \mapsto \begin{cases} h_j(z) & \text{if } z \in D_j \\ f_0(z) & \text{if } z \notin \bigcup_j D_j. \end{cases} \tag{2.2}$$

is K-quasiregular, with no finite asymptotic values, and every critical value of F is a critical value of some h_j.

Suppose furthermore that, for infinitely many j, the map h_j is not a homeomorphism $h_j\colon D_j \to D_{j+1}$, and that for every j there is an open set X_j with the following properties.

(a) The dilatation of h_j is supported on X_j.
(b) F is conformal on $F^n(X_j)$ for all $n \geq 1$.

Then there is a quasiconformal homeomorphism $\psi\colon \mathbb{C} \to \mathbb{C}$ with $\psi(0) = 0$ such that

$$f := \psi \circ F \circ \psi^{-1} \tag{2.3}$$

is a transcendental meromorphic function. Moreover, ψ is conformal on U_0, and $\psi(U_0)$ is contained in an invariant Baker domain U of f.

Proof. The claim that F is K-quasiregular follows from Royden's glueing lemma [Ber74, Lemma 2], or the quasiconformal removability of quasiarcs [BF14, Theorem 1.19]. Any curve to infinity must intersect U_0, and therefore F is unbounded on any such curve. In particular, F has no finite asymptotic values. As $F = f_0$ on U_0, every critical value of F must come from one of the h_j.

Now suppose that the additional conditions hold. Since h_j maps ∂D_j to ∂D_{j+1} in one-to-one fashion, if h_j is not a homeomorphism, then h_j must have at least one pole in D_j. As this happens for infinitely many j, we see that F has infinitely many poles. Let $\omega_0 := F_0^*(0)$ be the complex dilatation of F. By (b), the restriction of ω_0 to

$$X := \bigcup_{j,n\geq 0} F^n(X_j)$$

is forward-invariant under F. As F is meromorphic outside of X, we may extend $\omega_0|_X$ by pull-back to the grand orbit of X. Extending the resulting differential by the standard complex structure, we obtain an F-invariant measurable Beltrami differential ω on all of $\hat{\mathbb{C}}$. Observe that $\omega \equiv 0$ on U_0.

By the measurable Riemann mapping theorem, there is a quasiconformal homeomorphism $\psi\colon \mathbb{C} \to \hat{\mathbb{C}}$ that solves the Beltrami equation for ω. Then ψ is conformal on U_0. Moreover, the map f defined by (2.3) is meromorphic; it is transcendental since F has infinitely many poles. Let $z \in \psi(U_0)$; say $z = \psi(\zeta)$ with $\zeta \in U_0$. Then

$$\begin{aligned} f_0(z) &= \psi(F(\psi^{-1}(z))) = \psi(F(\zeta)) \in \psi(U_0), \quad \text{and} \\ f_0^n(z) &= \psi(F^n(\psi^{-1}(z))) = \psi(F^n(\zeta)) = \psi(f_0^n(\zeta)) \to \infty. \end{aligned}$$

So $\psi(U_0)$ is contained in a Baker domain of f, as claimed. □

2.2. Proposition (Large annuli in U)**.** *For every $\rho > 1$, there is $\mu > 1$ with the following property. Let $f_0(z) = \mu \cdot z$, and let f be obtained from f_0 as in Proposition 2.1. Then f satisfies* (1.2) *for a sequence $R_j \to \infty$.*

Proof. Recall that U_0 contains arbitrarily large round annuli of modulus at least $M := (\log \mu)/2$. If $M \geq \Lambda(\rho)$, where $\Lambda(\rho)$ is the modulus of the Teichmüller annulus [Ahl73, §4–11], then any annulus that separates 0 from ∞ and has modulus at least M contains a round annulus of modulus $\log \rho$ centred at zero. Hence we can take $\mu = e^{2\Lambda(\rho)}$. □

3. Proof of Theorem 1.2

To prove Theorem 1.2, we construct a suitable sequence of functions h_j to use in Proposition 2.1. The idea is that each h_j has critical values in D_{j+1}, which are then mapped back to the original critical points by h_{j+1}. We begin with the following, where f_0 is again given by (2.1).

3.1. Proposition. *Let $\Delta = D(\zeta, r) \subset \mathbb{C}$ be a round disc, and let $K > 1$, $0 < \eta < r$, and $0 < \vartheta < \mu \cdot \eta$. For every $a \in \mathbb{C} \setminus f_0(\Delta)$, there is a K-quasiregular map*

$$g\colon \Delta \to \hat{\mathbb{C}}$$

such that

(1) g extends continuously to $\partial\Delta$, where it agrees with f_0.

(2) $g(\zeta) = a$.
(3) *g has exactly two critical points,* c_1 *and* c_2*, with* $g(c_1) \neq g(c_2)$.
(4) *Each critical point* c_j *satisfies* $g(c_j) \in D(f_0(\zeta), \vartheta)$.
(5) *g is meromorphic on* $D(\zeta, \eta)$.
(6) *g is injective (and hence quasiconformal) on* $\Delta \setminus D(\zeta, \eta)$*, and* $\mu r > |g(z) - f_0(\zeta)| > \vartheta$ *for* $r > |z - \zeta| \geq \eta$.

Proof. Set

$$\alpha := \frac{a}{r\mu} - \frac{\zeta}{r} \neq 0.$$

Let $\varepsilon > 0$, and consider the map

$$\varphi = \varphi_{\alpha,\varepsilon} \colon \mathbb{D} \to \hat{\mathbb{C}}; \quad z \mapsto z + \frac{\varepsilon\alpha}{\alpha z + \varepsilon} = z + \frac{\varepsilon}{z - p},$$

where $p = -\varepsilon/\alpha$. Then $\varphi(0) = \alpha$ and $\varphi(p) = \infty$. By direct calculation, φ has two simple critical points at $p \pm \sqrt{\varepsilon}$, with critical values at $p \pm 2\sqrt{\varepsilon}$.

Now define $A \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ to be the affine map taking $\mathbb{D}$ to Δ; i.e., $A(z) = r \cdot z + \zeta$. Define

$$g_\varepsilon \colon D(\zeta, \eta) \to \hat{\mathbb{C}}; \quad z \mapsto \mu \cdot A\big(\varphi_{\alpha,\varepsilon}\big(A^{-1}(z)\big)\big).$$

For all sufficiently small ε, g_ε satisfies (3), (4) and (5). Furthermore,

$$g_\varepsilon(\zeta) = \mu \cdot (r\alpha + \zeta) = a,$$

so g also satisfies (2).

Set $\Delta_\eta := D(\zeta, \eta)$. As $\varepsilon \to 0$, $g_\varepsilon \to f_0$ uniformly on a neighbourhood of $\partial\Delta_\eta$. In particular, for sufficiently small ε, g_ε is conformal on a neighbourhood of $\partial\Delta_\eta$, and satisfies

$$\mu r > |g_\varepsilon(z) - f_0(\zeta)| > \vartheta$$

when $z \in \partial\Delta_\eta$. It follows that there is a quasiconformal homeomorphism G that maps the round annulus $\Delta \setminus \overline{\Delta_\eta}$ to the annulus bounded by $f_0(\partial\Delta)$ and $g_\varepsilon(\partial\Delta_\eta)$, and which agrees with f_0 on $\partial\Delta$ and with g_ε on $\partial\Delta_\eta$; see [Leh65]. We define g to agree with g_ε on D_η and with G outside; then g satisfies (1)–(6).

Finally, we claim that the dilatation of G, and thus of g, tends to 1 as $\varepsilon \to 0$. Indeed, this follows readily from the construction in [Leh65]. Alternatively, for small ε, we can define G directly by linear interpolation on each radius of $\Delta \setminus \overline{\Delta_\eta}$. An elementary calculation shows that the dilatation of G tends to 1. Hence, for sufficiently small ε, g is K-quasiconformal as claimed. ☐

For the remainder of the paper, let D_j denote the discs introduced in Section 2.

3.2. Proposition. *For any* $K > 1$*, there is a sequence of* K*-quasiregular maps*

$$h_j \colon D_j \to \hat{\mathbb{C}}$$

with the following properties for all j.

(a) For every j*, the map* h_j *has at least one critical point.*
(b) If c *is a critical point of* h_j*, then* $h_j(c) \in D_{j+1}$ *and* $h_{j+1}(h_j(c)) = c$.

Furthermore, for each j *there are open sets* $V_j \subset W_j \subset D_j$ *such that*

(c) $D_j \setminus V_j$ *is compact;*

(d) h_j *is injective on* W_j*;*
(e) $h_j(z) = f_0(z)$ *for* $z \in V_j$*;*
(f) $h_j(W_j) \subset V_{j+1}$*;*
(g) *the dilatation of* h_j *is supported on* W_j*.*

Proof. The sequence of functions is constructed inductively, in such a way as to ensure the following inductive hypotheses. For $j \geq 0$, let $\mathcal{C}_j \subset D_j$ be the set of critical points of h_j and $\Omega_{j+1} = h_j(\mathcal{C}_j) \subset \mathbb{C}$ the set of critical values. The construction will ensure the following inductive hypotheses for $j \geq 0$:

(A) $\Omega_{j+1} \subset D_{j+1}$.
(B) $h_j \colon \mathcal{C}_j \to \Omega_{j+1}$ is bijective.
(C) $h_j(W_j) \subset D_{j+1}$ and $\overline{h_j(W_j)} \cap \Omega_{j+1} = \emptyset$.

To anchor the recursive construction, we also set $\Omega_0 := \{\zeta_0\}$. This means that (A) holds also for $j = -1$.

Now suppose that $j \geq 0$ is such that h_i has been constructed for $i < j$, satisfying the inductive hypotheses. To define h_j, let $\omega_j^1, \dots, \omega_j^m$ be the elements of Ω_j. Choose pairwise disjoint small discs $\Delta_j^i = D(\omega_j^i, \rho_j^i) \Subset D_j$ centred at the ω_j^i, for $j = 1, \dots, m$; this is possible by (A). If $j > 0$, then by (C) we may also ensure that the closures of the Δ_j^i do not intersect $\overline{h_{j-1}(W_{j-1})}$. Set

$$V_j := D_j \setminus \bigcup_{i=1}^{m} \overline{\Delta_j^i} \supset h_{j-1}(W_{j-1}). \tag{3.1}$$

For each $i = 1, \dots, m$, apply Proposition 3.1 to $\Delta = \Delta_j^i$, where $a = c_{j-1}^i \in \mathcal{C}_{j-1}$ is such that

$$h_{j-1}(c_{j-1}^i) = \omega_j^i, \tag{3.2}$$

if $j > 0$, and $c_{-1}^1 = 0$ otherwise. We may choose ϑ and $\eta = \eta_j^i$ arbitrary, subject to the conditions given in the proposition. Let g_j^i denote the maps obtained. We define

$$h_j(z) := \begin{cases} f_0(z) & \text{if } z \in V_j; \\ g_j^i(z) & \text{if } z \in \overline{\Delta_j^i} \end{cases} \qquad \text{and} \tag{3.3}$$

$$W_j := D_j \setminus \bigcup_{i=1}^{m} \overline{D(\omega_j^i, \eta_j^i)} \supset V_j. \tag{3.4}$$

We claim that h_j satisfies (A)–(C). For the remainder of the proof, references to (1)–(5) shall mean the corresponding properties of the g_j^i established in Proposition 3.1.

The critical points and critical values of h_j are exactly those of the maps g_j^i. By (4), the critical values of g_j^i are in $f_0(\Delta_j^i) \subset f_0(D_j) = D_{j+1}$, establishing (A). Furthermore, by (3), g_j^i has exactly two different critical points, with different critical values belonging to $f_0(\Delta_j^i)$. Since different Δ_j^i are pairwise disjoint, and f_0 is injective, claim (B) follows. Finally, (C) is an immediate consequence of (4) and (6).

This completes the construction of the h_j. By (1) and Royden's glueing lemma (or the quasiconformal removability of quasiarcs), the map h_j is indeed a K-quasiregular map. It remains to establish (a)–(g).

(a) Each h_j^i has exactly two critical values, and $\#\Omega_0 = 1$. It follows inductively that $\#\Omega_j = 2^j$ for all j, and in particular each h_j has at least one critical point.

(b) If $c \in \mathcal{C}_j$ is a critical point of h_j, then $c = c_j^i$ for some i, by (3.2) and (B). So by (2),

$$h_{j+1}(h_j(c)) = h_{j+1}(h_j(c_j^i)) = g_{j+1}^i(\omega_{j+1}^i) = c_j^i = c.$$

(c) By definition of V_j in (3.1), $D_j \setminus V_j$ is compact.

(d) $h_j|_{W_j}$ is injective by (3.4) and (6).

(e) By (3.3), $h_j = f_0$ on V_j.

(f) By (3.1), $V_{j+1} \supset h_j(W_j)$.

(g) By (5), the dilatation of h_j is supported on W_j. $\square$

Proof of Theorems 1.2 and 1.4. Choose μ as in Proposition 2.2, and let $(h_j)_{j=0}^{\infty}$ be as in Proposition 3.2. Let F be the quasiregular map defined by (2.2). Then $F^2(c) = c$ for every critical point c of F.

The dilatation of h_j is supported on W_j, and h_j is conformal on $\overline{V_j}$. Moreover, $F^n(X_j) \subset V_{n+j}$ for all n and j. Hence $X_j := W_j \setminus \overline{V_j}$ satisfies the conditions of Proposition 2.1. Let f be the meromorphic function (2.3); then f has a Baker domain U and satisfies $f^2(c) = c$ for every critical point of f. This completes the proof of Theorem 1.2.

Furthermore, by Proposition 2.2, the function also satisfies 1.2 for a sequence $R_j \to \infty$, proving Theorem 1.4. $\square$

References

[Ahl73] Lars V. Ahlfors, *Conformal invariants: topics in geometric function theory*, McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973, McGraw-Hill Series in Higher Mathematics.

[Ber74] Lipman Bers, *On moduli of Kleinian groups*, Russian Mathematical Surveys **29** (1974), no. 2, 88–102.

[Ber93] Walter Bergweiler, *Iteration of meromorphic functions*, Bull. Amer. Math. Soc. (N.S.) **29** (1993), no. 2, 151–188.

[Ber95] ———, *Invariant domains and singularities*, Math. Proc. Cambridge Philos. Soc. **117** (1995), no. 3, 525–532.

[BF14] Bodil Branner and Núria Fagella, *Quasiconformal surgery in holomorphic dynamics*, Cambridge Studies in Advanced Mathematics, vol. 141, Cambridge University Press, Cambridge, 2014, With contributions by Xavier Buff, Shaun Bullett, Adam L. Epstein, Peter Haïssinsky, Christian Henriksen, Carsten L. Petersen, Kevin M. Pilgrim, Tan Lei and Michael Yampolsky.

[BFJK15] Krzysztof Barański, Núria Fagella, Xavier Jarque, and Bogusława Karpińska, *Absorbing sets and Baker domains for holomorphic maps*, J. Lond. Math. Soc. (2) **92** (2015), no. 1, 144–162.

[BFJK20] ———, *Fatou components and singularities of meromorphic functions*, Proc. Roy. Soc. Edinburgh Sect. A **150** (2020), no. 2, 633–654.

[Fat20] P. Fatou, *Sur les équations fonctionnelles (deuxième mémoire)*, Bulletin de la Société Mathématique de France **48** (1920), 33–94 (fr).

[Leh65] Olli Lehto, *An extension theorem for quasiconformal mappings*, Proc. London Math. Soc. (3) **14a** (1965), 187–190.

[Mil06] John Milnor, *Dynamics in one complex variable*, third ed., Annals of Mathematics Studies, vol. 160, Princeton University Press, Princeton, NJ, 2006.

[MR13] Helena Mihaljević-Brandt and Lasse Rempe-Gillen, *Absence of wandering domains for some real entire functions with bounded singular sets*, Math. Ann. **357** (2013), no. 4, 1577–1604.

[RS06] P. J. Rippon and G. M. Stallard, *Singularities of meromorphic functions with Baker domains*, Mathematical Proceedings of the Cambridge Philosophical Society **141** (2006), no. 2, 371–382.

Dept. of Mathematical Sciences, University of Liverpool, Liverpool L69 7ZL, UK, https://orcid.org/0000-0001-8032-8580
Email address: `l.rempe@liverpool.ac.uk`