On the Curve Complexity of 3-Colored Point-set Embeddings[⋆,⋆⋆]

Emilio Di Giacomo[a,*], Leszek Gąsieniec[b], Giuseppe Liotta[a], Alfredo Navarra[a]

[a] *Università degli Studi di Perugia, Italy*
[b] *University of Liverpool, UK*

Abstract

We establish new results on the curve complexity of k-colored point-set embeddings when $k = 3$. We show that there exist 3-colored caterpillars with only three independent edges whose 3-colored point-set embeddings may require $\Omega(n^{\frac{1}{3}})$ bends on $\Omega(n^{\frac{2}{3}})$ edges. This settles an open problem by Badent et al. [5] about the curve complexity of point set embeddings of k-colored trees and it extends a lower bound by Pach and Wenger [35] to the case that the graph only has $O(1)$ independent edges. Concerning upper bounds, we prove that any 3-colored path admits a 3-colored point-set embedding with curve complexity at most 4. In addition, we introduce a variant of the k-colored simultaneous embeddability problem and study its relationship with the k-colored point-set embeddability problem.

1. Introduction

A pioneering paper by Pach and Wenger [35] studies the problem of computing a planar drawing of a graph G with a given mapping between the vertices of G and the points that represent them in the drawing. They show that, for any given point set and for any given mapping, a planar graph with n vertices admits a planar drawing such that the *curve complexity*, i.e. the maximum number of bends along an edge, is $O(n)$. Furthermore, they prove that the bound on the curve complexity is tight when G has $\Omega(n)$ independent edges. This implies that the curve complexity of a planar drawing with vertices at fixed locations may be $\Omega(n)$ even for structurally very simple graphs such as paths or matchings, for which the number of independent edges is linear in n.

These results have motivated the study of a relaxed version of the problem, where the function that associates vertices of the graph to points of the plane is not a bijection. An instance of the *k-colored point set embeddability problem* receives as input an n-vertex planar graph G such that every vertex is given one of k distinct colors and a set S of n distinct points such that each point is given one of the k distinct colors. Furthermore, G and S are *compatible*, i.e, the number of points of S having a certain color i is the same as the number of vertices of G having color i. The goal is to compute a *k-colored point set*

⋆Preliminary results concerning this paper appeared in [14].

⋆⋆The work has been supported in part by the European project "Geospatial based Environment for Optimisation Systems Addressing Fire Emergencies" (GEO-SAFE), contract no. H2020-691161, by the Network Sciences and Technologies (NeST) initiative at University of Liverpool, by the project: "Algoritmi e sistemi di analisi visuale di reti complesse e di grandi dimensioni" - Ricerca di Base 2018, Dipartimento di Ingegneria dell'Università degli Studi di Perugia", and by the project "Algorithms and Graphs Drawing" - Ricerca di Base 2017, Dipartimento di Matematica e Informatica dell'Università degli Studi di Perugia

*Corresponding author

Email addresses: emilio.digiacomo@unipg.it (Emilio Di Giacomo), L.A.Gasieniec@liverpool.ac.uk (Leszek Gąsieniec), giuseppe.liotta@unipg.it (Giuseppe

embedding of G on S, that is a planar drawing of G where every vertex of a specific color is represented by a point of the same color.

When $k = n$ the k-colored point set embeddability problem coincides with the problem of computing a drawing with vertices at fixed locations and thus the lower bound by Pach and Wenger on the curve complexity holds. On the other hand, when $k = 1$ Kaufmann and Wiese [31] prove that every planar graph admits a 1-colored point set embedding onto any point set with curve complexity at most two. For $k = 2$, outerplanar graphs always admit a 2-colored point set embedding with curve complexity at most five [12] which reduces to at most one when the input is a 2-colored path [17]. However there are 2-colored planar graphs for which a 2-colored point set embedding may require $\Omega(n)$ bends on $\Omega(n)$ edges [5]. This result extends the lower bound of Pach and Wenger [35] to a much more relaxed set of constraints on the location of the vertices; different from the result by Pach and Wenger, the lower bound in [5] uses 2-connected graphs instead of (not necessarily connected) planar graphs. For structurally simpler graphs, such as trees or paths, the function that relates the curve complexity to the number of colors is not yet fully understood. Specifically, Badent et al. [5] ask to establish bounds on the curve complexity of k-colored trees for small values of $k \geq 3$. In this paper we prove the following theorem.

Theorem 1. *For sufficiently large n, a 3-colored point-set embedding of a 3-colored caterpillar may require $\Omega(n^{\frac{2}{3}})$ edges each having $\Omega(n^{\frac{1}{3}})$ bends.*

The caterpillars used to prove the lower bound of Theorem 1 have at most three independent edges. It may be worth remarking that the argument by Pach and Wenger [35] applies only to graphs with a linear number of independent edges. Hence, our lower bound extends the one by Pach and Wenger about the curve complexity of planar drawings with vertices at fixed locations also to those graphs for which the number of independent edges does not grow with n. Theorem 1 naturally raises the question about the curve complexity of 3-colored point set embeddings of paths. Another result of our paper is the following theorem that extends the upper bound on the curve complexity of colored point set embeddings of paths proved in [17] to more than two colors.

Theorem 2. *Every 3-colored path admits a 3-colored point-set embedding with curve complexity at most 4 on any compatible 3-colored point set.*

The proof of Theorem 2 is based on a new technique to compute a 2-colored point set embedding of a 2-colored path. This technique is used as a building block in an algorithm that guarantees curve complexity at most four for every 3-colored path.

In addition to the results above, this paper initiates the study of the *h-planar k-colored simultaneous embeddability problem*. Let G_1 and G_2 be two k-colored planar graphs that, for each color class, have the same number of vertices in the class. Let S be a k-colored set of points such that both G_1 and G_2 have a k-colored point set embedding on S that we call Γ_1 and Γ_2, respectively. The pair $\langle \Gamma_1, \Gamma_2 \rangle$ is a *k-colored simultaneous embedding* of G_1 and G_2. Inspired by recent papers [4, 6, 11, 21, 23, 24] that study the crossings between the edges of the two drawings that form a simultaneous embedding (see also next section), we consider the additional constraint that $\Gamma_1 \cup \Gamma_2$ be *h-planar*. A drawing is h-planar if h is the smallest number such that every edge is crossed by at most h other edges. When $\Gamma_1 \cup \Gamma_2$ is h-planar, we say that $\langle \Gamma_1, \Gamma_2 \rangle$ is an h-planar k-colored simultaneous embedding of G_1 and G_2. We apply and extend the ideas behind the proofs of Theorems 1 and 2 and use existing literature to establish upper and lower bounds on the value of h for various graph pairs. Namely, we show that there exists a 2-colored path G_1 and a 2-colored planar graph G_2 with n vertices such that for any 2-colored simultaneous embedding $\langle \Gamma_1, \Gamma_2 \rangle$ of G_1 and G_2 we have that $\Gamma_1 \cup \Gamma_2$ is $\Omega(n)$-planar. For three colors, there exists a 3-colored path G_1 and a 3-colored caterpillar G_2 with n vertices such that any 3-colored simultaneous embedding $\langle \Gamma_1, \Gamma_2 \rangle$ of G_1 and G_2 is $\Omega(n^{\frac{1}{3}})$-planar. On the positive side, if G_1 and G_2 are both 3-colored cycles or if they are both 4-colored cycles and one of them is 2-separable (i.e. it can be split into two 2-colored paths by removing two edges), we show that there always exists a simultaneous embedding $\langle \Gamma_1, \Gamma_2 \rangle$ of G_1 and G_2 such that $\Gamma_1 \cup \Gamma_2$ is a 2-planar graph and every edge has at most four bends.

Finally, it may be worth remarking that the results of a preliminary version of this paper [14] have found application in the context of robot motion planning [33].

1.1. Related Work

The problem of computing a drawing of a planar graph with fixed vertex positions was first studied by Halton [25] who proved that every n-colored planar graph admits an n-colored point-set embedding on any n-colored set of points; however Halton does not give a bound on the curve complexity of the computed drawings. Because of the $\Omega(n)$ lower bound by Pach and Wenger on the curve complexity of the k-colored point-set embeddings when $k = n$, several papers have focused on small values of k (typically $k \leq 3$) to see whether better bounds on the curve complexity can be achieved in this setting (see, e.g., [5, 16, 17, 18, 20**?**]). As mentioned above, the proof of the upper bound of Theorem 2 is based on the use of a new technique to compute a 2-colored point-set embedding of a 2-colored path with curve complexity four. It is worth recalling that an algorithm to compute a 2-colored point-set embedding of a 2-colored path with curve complexity one, is described in [17]. Although the algorithm in the present paper is worse than the one in [17] in terms of curve complexity, it can be effectively used as a building block for an algorithm to compute a 3-colored point-set embedding of a 3-colored path with curve complexity 4. The study of 2-colored point-set embeddings with straight-line edges (i.e., with curve complexity 0) has a long tradition in the computational geometry field, where various results have been published for paths [1, 2, 28], trees [8, 27, 26, 29, 34, 36], and cycles [30].

Since simultaneous embeddings with straight-line edges may not exist even for structurally simple graph families such as paths and trees [3, 22], many papers have studied simultaneous embeddings with bends along the edges. While we refer the interested reader to the extensive survey of Bläsius et al. [7], we only recall here that every two planar graphs have a simultaneous embedding with at most two bends per edge [15]. Argyriou et al. [4], Bekos et al. [6], and Grilli [23] study simultaneous embeddings such that any edge crossing of $\Gamma_1 \cup \Gamma_2$ occurs at right angles. Chan et al. [11], Grilli et al. [24], and Frati et al. [21] prove upper bounds on the curve complexity and on the number of crossings between edge pairs of $\Gamma_1 \cup \Gamma_2$. These papers bound the number of crossings between edge pairs of $\Gamma_1 \cup \Gamma_2$ but do not bound the number of crossings per edge. In this paper we study the case that $\Gamma_1 \cup \Gamma_2$ be h-planar, which implies that neither two edges can cross more than once, nor the number of crossings per edge can be larger than h. It may be worth recalling that h-planar drawings are among the most extensively studied topics in the research area of graph drawing beyond planarity; see, e.g. [19, 32] for references and surveys on this topic. Finally, we recall that k-colored simultaneous embeddings were first introduced in a paper by Brandes et al. [9] as a generalization of simultaneous embeddings in the "without-mapping scenario" (that is when $k = 1$) and in the "with-mapping scenario" (that is when $k = n$) first discussed in the paper by Brass et al. [10]. Different from our setting, Brandes et al. concentrate on straight-line drawings and do not take into account the edge crossings of $\Gamma_1 \cup \Gamma_2$.

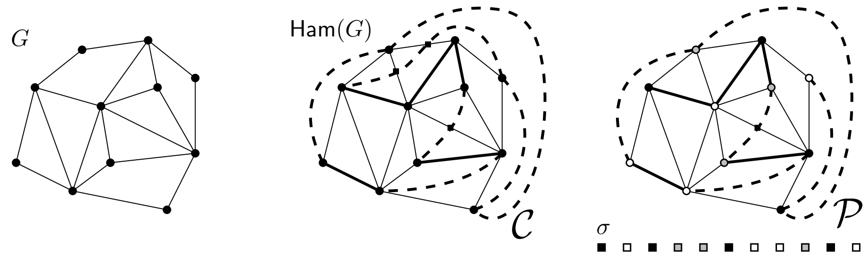
1.2. Paper Organization

The rest of the paper is organized as follows. Preliminaries can be found in Section 2. The lower bound on the curve complexity of 3-colored point-set embedding of caterpillars is proved in Section 3, while the upper bound for 3-colored paths is in Section 4. The results about h-planar k-colored simultaneous embeddings can be found in Section 5. Conclusion and open problems are in Section 6.

2. Preliminaries

Let $G = (V, E)$ be a graph. A *k-coloring* of G is a partition $\{V_0, V_1, \dots, V_{k-1}\}$ of V. The integers $0, 1, \dots, k-1$ are called *colors* and G is called a *k-colored graph*. For each vertex $v \in V_i$ we denote by $col(v)$ the color i of v.

Let S be a set of distinct points in the plane. For any point $p \in S$, we denote by $x(p)$ and $y(p)$ the x- and y-coordinates of p, respectively, and by $CH(S)$ the convex hull of S. Throughout the paper we always assume that the points of S have different x-coordinates (if not we can rotate the plane so to achieve this condition). A *k-coloring* of S is a partition $\{S_0, S_1, \dots, S_{k-1}\}$ of S. A set of points S with a k-coloring is called a *k-colored point set*. For each point $p \in S_i$, $col(p)$ denotes the color i of p. A k-colored point set S is *compatible with* a k-colored graph G if $|V_i| = |S_i|$ for every i. A *k-colored point-set embedding* of a planar graph G on a compatible k-colored point set S is a planar drawing of G such that: (i) every vertex v is mapped to a distinct point p of S with $col(p) = col(v)$, (ii) each edge e of G is drawn as a polyline λ. A point shared by any two consecutive (maximal) segments of λ is called a *bend* of



(a) (b) (c)

Figure 1: (a) A k-colored planar graph. (b) A Hamiltonian augmentation $\mathsf{Ham}(G)$ of G. The dashed edges are dummy edges; the little squares at the intersections of the dummy edges with the real ones are division vertices. (c) A k-colored Haimltonian path consistent with a k-colored sequence σ.

e. The maximum number of bends along an edge is the *curve complexity* of the k-colored point-set embedding. A planar drawing of G such that every vertex v is mapped to a distinct point p of S with $col(p) = col(v)$ and the edges are not required to be drawn as polylines (i.e, they are drawn as arbitrary simple Jordan arcs) will be called *k-colored topological point-set embedding* of G on S. A *k-colored sequence* σ is a sequence of (possibly repeated) colors c_0, c_1, $\ldots$, c_{n-1} such that $0 \leq c_j \leq k-1$ $(0 \leq j \leq n-1)$. We say that σ is *compatible with* a k-colored graph G if color i occurs $|V_i|$ times in σ. Let S be a k-colored point set. Let $p_0, \ldots, p_{n-1}$ be the points of S with $x(p_0) < \ldots < x(p_{n-1})$. The k-colored sequence $col(p_0), \ldots, col(p_{n-1})$ is called the *k-colored sequence induced by S*, and is denoted as $seq(S)$. A set of points S is *one-sided convex* if the points are in convex position and the two points with minimum and maximum x-coordinate are consecutive along $CH(S)$. In a k-colored one-sided convex point set, the sequence of colors encountered clockwise along $CH(S)$, starting from the point with minimum x-coordinate, coincides with $seq(S)$.

A *Hamiltonian cycle* of a graph G is a simple cycle that contains all vertices of G. A graph G that admits a Hamiltonian cycle is said to be *Hamiltonian*. A planar graph G is *sub-Hamiltonian* if either G is Hamiltonian or G can be augmented with dummy edges (but without dummy vertices) to a graph $\mathsf{aug}(G)$ that is Hamiltonian and planar. A *subdivision* of a graph G is a graph obtained from G by replacing each edge by a path with at least one edge. Internal vertices on such a path are called *division vertices*. Every planar graph has a subdivision that is sub-Hamiltonian [13, 31]. Let G be a planar graph, and let $\mathsf{sub}(G)$ be a sub-Hamiltonian subdivision of G. The graph $\mathsf{aug}(\mathsf{sub}(G))$ is called a *Hamiltonian augmentation of* G and will be denoted as $\mathsf{Ham}(G)$. Figures 1(a) and 1(b) shows a planar graph G and a Hamiltonian augmentation $\mathsf{Ham}(G)$ of G. Let $\mathcal{C}$ be the Hamiltonian cycle of a Hamiltonian augmentation $\mathsf{Ham}(G)$ of G. Let e be an edge of $\mathcal{C}$, let $\mathcal{P} = \mathcal{C} \setminus e$ be the Hamiltonian path obtained by removing e from $\mathcal{C}$, and let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of G in the order in which they appear along $\mathcal{P}$. Finally, let $\sigma = c_0, c_1, \ldots, c_{n-1}$ be a k-colored sequence. Path $\mathcal{P}$ is a *k-colored Hamiltonian path consistent with σ* if $\mathrm{col}(v_i) = c_i$ $(0 \leq i \leq n-1)$. Figure 1(c) shows a 3-colored Hamiltonian path consistent with the 3-colored sequence σ shown in the same picture for a 3-colored version of the graph G of Fig. 1(a). Let d be a division vertex used to split an edge (u, v) of G to obtain $\mathsf{sub}(G)$, and let (a, d) and (d, b) be the two edges incident to d that are part of (u, v) (notice that each of a and b may coincide with u or v or it can be a division vertex itself). If d is encountered between a and b when walking along $\mathcal{P}$, we say that d is a *flat division vertex* (notice that there can be vertices between a and d and between d and b). A cycle $\mathcal{C}$ is a *k-colored Hamiltonian cycle consistent with σ* if there exists an edge $e \in \mathcal{C}$ such that $\mathcal{P} = \mathcal{C} \setminus e$ is a k-colored Hamiltonian path consistent with σ. Notice that there can be different edges in $\mathcal{C}$ whose removal gives rise to k-colored paths consistent with σ. If this is the case let $\mathcal{P} = \mathcal{C} \setminus e$ the one of these paths for which the number of flat division vertices is minimum. The cycle $\mathcal{C}$ has at most δ_{f} flat division vertices per edge if $\mathcal{P}$ has at most δ_{f} flat division vertices per edge. A Hamiltonian augmentation $\mathsf{Ham}(G)$ that has a k-colored Hamiltonian cycle consistent with a k-colored sequence σ is called a *k-colored Hamiltonian augmentation of G consistent with σ*; also, we say that $\mathsf{Ham}(G)$ has at most δ_{f} flat division vertices per edge. The following theorem has been proved in [5].

Theorem 3. [5] *Let G be a k-colored planar graph, and let S be a k-colored point set compatible with G. If G has a k-colored Hamiltonian augmentation consistent with $seq(S)$ such that there are at most δ_{f} flat division vertices per edge and at most δ_{nf} non-flat division vertices, then G admits a k-colored point-set embedding on S with curve complexity at most $2\delta_{\mathrm{nf}} + \delta_{\mathrm{f}} + 1$.*

3. Lower Bound on the Curve Complexity of 3-colored Caterpillars

In this section, we establish that a 3-colored point-set embedding of a caterpillar may require $\Omega(n^{\frac{2}{3}})$ edges each having $\Omega(n^{\frac{1}{3}})$ bends. We actually prove a stronger result, i.e. that a forest of three stars may require $\Omega(n^{\frac{2}{3}})$ edges each having $\Omega(n^{\frac{1}{3}})$ bends. Our proof exploits a previous result about biconnected outerplanar graphs proved in [12]. We start by proving some technical lemmas that state useful properties of point-set embeddings on one-sided convex point-sets.

Let G be a k-colored graph, let S be a k-colored one-sided convex point set compatible with G, and let Γ be a k-colored point-set embedding of G on S. Let e be an edge of Γ, let p be an intersection point between e and $CH(S)$, and let s be the segment of e that contains p. The point p can be one of the following (i) an end-vertex of e; or (ii) an endpoint of s that is not an end-vertex of e; (iii) an internal point of s and s is not a side of $CH(S)$; (iv) an internal point of s and s coincide with a side of $CH(S)$. In case (ii) we say that p is a *touching point of e*, while in case (iii) we say that p is a *hull crossing of E*. Let C be the set of touching points and hull crossings of e. The points of C divide e in $|C|+1$ pieces, which we call *sub-edges*. An end-vertex of a sub-edge that is not a vertex will be called *sub-vertex*. Thus a sub-vertex is either a touching point or a hull crossing. Each sub-edge is either completely inside $CH(S)$, or completely outside $CH(S)$, or it is completely contained in a side of $CH(S)$. Drawing Γ is said to be *canonical* if (i) no edge has a touching point; (ii) each sub-edge that is inside $CH(S)$ or belongs to a side of $CH(S)$ is drawn without bends; and (iii) each sub-edge that is outside $CH(S)$ is drawn with two bends. The following lemmas establish some useful relations between the number of bends along an edge and the number of its hull crossings.

Lemma 1. *Let G be a k-colored graph, let S be a k-colored one-sided convex point set compatible with G, and let Γ be a canonical k-colored point-set embedding on S. If an edge e has c hull crossings in Γ, then e has at most $2c+1$ bends. If e is drawn with b bends, then e crosses $CH(S)$ at least $\frac{b-1}{2}$ times.*

PROOF. Since Γ is canonical every sub-vertex of e is a hull crossing and therefore if e has c hull crossings it consists of $c+1$ sub-edges. These sub-edges are alternately inside and outside $CH(S)$, thus at most $\lceil \frac{c}{2} \rceil$ of them are outside $CH(S)$ and have two bend each; since each of the c crossing can be an additional bend, the number of bends along e is at most $2\lceil \frac{c}{2} \rceil + c = 2c+1$.

Assume now that e is drawn with b bends and let c' be the number of hull crossings of e. By the argument above, we have that $b \leq 2c'+1$, which implies $c' > \frac{b-1}{2}$. □

Lemma 2. *Let G be a k-colored graph, let S be a k-colored one-sided convex point set compatible with G, and let Γ be a k-colored point-set embedding on S. If the number of sub-vertices of an edge e in Γ is c, then e has at least $\lfloor \frac{c+1}{4} \rfloor$ bends in Γ.*

PROOF. If a sub-vertex is a touching point of e, then it is a bend of e. Hence, if e has at least $\lfloor \frac{c}{2} \rfloor$ touching points, then it has at least $\lfloor \frac{c}{2} \rfloor \geq \lfloor \frac{c+1}{4} \rfloor$ bends. If the number of touching points is less than $\lfloor \frac{c}{2} \rfloor$ we can slightly perturb the drawing in order to remove them. More precisely, let p be one such point. If the two sub-edges incident to p are both inside (resp. outside) $CH(S)$, we can slightly perturb the drawing of these two sub-edges so that their meeting point is no longer on $CH(S)$ but it is inside (resp. outside) $CH(S)$; if the two sub-edges incident to p are one inside (resp. outside) $CH(S)$ and the other coincide with a side of $CH(S)$, we can slightly perturb the drawing of these two sub-edges so that their meeting point is inside (resp. outside) $CH(S)$; finally if the two sub-edges incident to p coincide with two consecutive sides of $CH(S)$, we can slightly perturb their drawing so that their meeting point is inside $CH(S)$. Notice that no new hull crossing is created by these perturbations. Once all touching points have been removed, the sub-vertices of e are all hull crossings and they are at least $\lfloor \frac{c}{2} \rfloor$. These points define at least $\lfloor \frac{c}{2} \rfloor + 1$ sub-edges that are alternately inside and outside $CH(S)$; thus at least $\left\lfloor \frac{\lfloor \frac{c+1}{2} \rfloor}{2} \right\rfloor = \lfloor \frac{c+1}{4} \rfloor$ of them are outside $CH(S)$ and must have at least one bend each. □

The next lemma shows that every k-colored point-set embedding on any compatible k-colored one-sided convex point set can always be transformed into a canonical one.

Lemma 3. *Let G be a k-colored graph, and let S be a k-colored one-sided convex point set compatible with G. If G has a k-colored topological point-set embedding on S, then G admits a canonical k-colored point-set embedding on S such that each edge has the same set of sub-vertices in both drawings.*

PROOF. Let Γ be a k-colored topological point-set embedding on S. We now prove that G admits a canonical k-colored point-set embedding Γ' on S such that Γ' has the same planar embedding as Γ and the same set of sub-vertices as Γ.

The sub-edges that are either inside $CH(S)$ or are sides of $CH(S)$ can be drawn straight-line without crossings. Namely, suppose for a contradiction, that two of these sub-edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ cross each other when drawn as straight-line segments. Let $\prec$ denote the order of the vertices and sub-vertices along $CH(S)$ starting from a chosen reference point and assume without loss of generality that $u_1 \prec v_1$, $u_2 \prec v_2$, and $u_1 \prec u_2$. In order to have a crossing between e_1 and e_2 it must be $u_1 \prec u_2 \prec v_1 \prec v_2$. Since the two sub-edges e_1 and e_2 have the same end-points in Γ and are completely inside $CH(S)$ in Γ, then they would cross also in Γ, which contradicts the fact that Γ is planar.

We now prove that the sub-edges that are outside $CH(S)$ can be drawn with two bends each with the same end-points they have in Γ and preserving the planar embedding of Γ. Let $S' = \{p_1, p_2, \ldots, p_h\}$ be the set of points of $CH(S)$ representing vertices and sub-vertices that belong to the boundary of the external face of Γ. Choose any point p_c inside $CH(S')$ and for each vertex and sub-vertex v of Γ denote by ℓ_v the ray starting at p_c and passing through v. Let (u, v) be a sub-edge that is outside $CH(S)$; the two rays ℓ_u and ℓ_w define two wedges one of which does not contain any point of S' distinct from u and v. We denote such a wedge by $W_{(u,v)}$. By the choice of p_c the angle spanned by this wedge is at most π (see Fig. 2(a) for an illustration).

We now prove by induction on the number m of sub-edges that are outside $CH(S)$ that there exists a planar drawing of these sub-edges such that each of them has two bends and for each point w of S' the ray ℓ_w is not crossed by any sub-edge outside $CH(S)$. If $m = 1$, let (u, v) be the only sub-edge outside $CH(S)$ and let γ be a circle centered at p_c and enclosing $CH(S)$. We consider two rays, ℓ'_u starting from u and ℓ'_v starting from v, that are both inside $W_{(u,v)}$ and outside $CH(S)$. By choosing ℓ'_u sufficiently close to ℓ_u and ℓ'_v sufficiently close to ℓ_v we can always guarantee this property (see also Fig. 2(b)). Since the angle of the wedge $W_{(u,v)}$ is at most π, it is possible to choose two points, p_u along ℓ'_u and p_v along ℓ'_v, that can be connected by a segment that is completely outside γ. The polyline connecting the points u, p_u, p_v, and v is a drawing of the sub-edge (u, v) that has two bends and is completely outside $CH(S)$ except for u and v. Since any point w of S' distinct from u and w is not inside $W_{(u,v)}$,

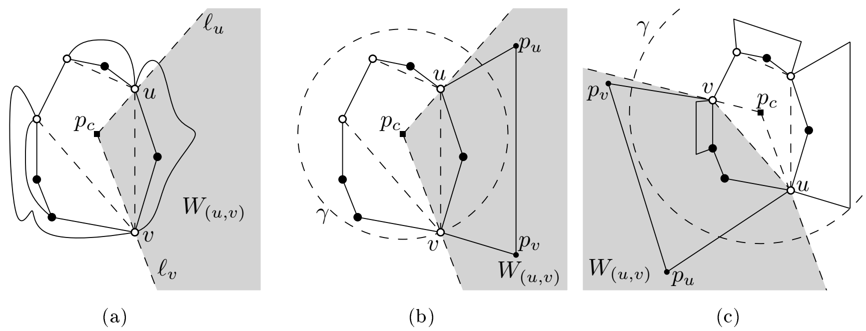


Figure 2: Proof of Lemma 3. (a) Choice of point p_c and illustration of the wedge $W_{(u,v)}$; the white points are the vertices and sub-vertices of S'. (b) Case base of the proof: addition of the sub-edge (u,v) inside $W_{(u,v)}$; (c) Inductive case of the proof: addition of the sub-edge (u,v) inside $W_{(u,v)}$.

the ray ℓ_w is not crossed by (u,v), which is the only sub-edge drawn outside $CH(S)$.

Assume now that $m > 1$ and remove from Γ a sub-edge (u,v) that belongs to the boundary of the external face. By induction, there exists a drawing Γ^- of all sub-edges outside $CH(S)$ except (u,v) so that each sub-edge has only two bends and so that for each vertex and sub-vertex w that belong to the boundary of the external face, the ray ℓ_w is not crossed by any edge outside $CH(S)$. We now add (u,v) to Γ^-. Analogously to the base case, let γ be a circle centered at p_c and enclosing Γ^-. Since ℓ_u and ℓ_v are not crossed by any edge outside $CH(S)$ we can choose two rays ℓ'_u starting from u and ℓ'_v starting from v, that are both inside $W_{(u,v)}$ and that do not intersect Γ^- except at u and v (see also Fig. 2(c)). Also in this case this property can be guaranteed by choosing ℓ'_u sufficiently close to ℓ_u and ℓ'_v sufficiently close to ℓ_v. Again we can choose two points p_u along ℓ'_u and p_v along ℓ'_v outside γ that can be connected with a segment. This result in a drawing of (u,v) with two bends that intersects Γ^- only at u and v. Also, for each point w of S' distinct from u and v, the ray ℓ_w is not crossed outside $CH(S)$ by any edge of Γ^- by induction. Since w does not belong to $W_{(u,v)}$, ℓ_w is not intersected by (u,v). □

We now recall the result about biconnected outerplanar graphs proved in [12] that we use to prove our lower bound. An *alternating point set* S_n is a 3-colored one-sided convex point set such that: (i) S_n has n points for each color 0, 1, and 2, and (ii) when going along the convex hull $CH(S_n)$ of S_n in clockwise order, the sequence of colors encountered is $0, 1, 2, 0, 1, 2, \ldots$. Each set of consecutive points colored $0, 1, 2$ is called a *triplet*.

A 3-*fan*, denoted as G_n, is a 3-colored outerplanar graph with $3n$ vertices ($n \geq 2$) and defined as follows (refer to Fig. 3 for an illustration with $n = 12$). G_n consists of a simple cycle formed by n vertices of color 0, followed (in clockwise order) by n vertices of color 1, followed by n vertices of color 2. The vertex of color i adjacent in the cycle to a vertex of color $i-1$ (indices taken modulo 3)

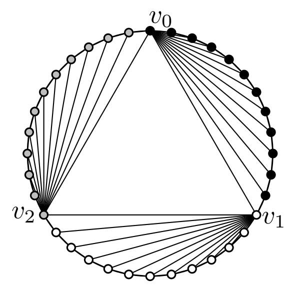


Figure 3: The 3-fan G_n for $n = 12$.

is denoted as v_i. Also, in G_n every vertex colored i is adjacent to v_i ($i = 0, 1, 2$) and vertices v_0, v_1, v_2 form a 3-cycle of G_n. The following theorem has been proved in [12].

Theorem 4. [12] *Let h be a positive integer, let G_n be a 3-fan for $n \geq 79h^3$, and let S_n be an alternating point set. In every 3-colored point-set embedding of G_n on S_n there is an edge with more than h bends.*

The forest of stars that we use to establish our lower bound is called a 3-*sky* and is denoted by F_n. It consists of three stars T_0, T_1, T_2 such that: (i) each T_i ($i = 0, 1, 2$) has n vertices ($n \geq 2$); (ii) all the vertices of each T_i ($i = 0, 1, 2$) have the same color i.

Let Γ_n be a point-set embedding of F_n on S_n. An *uncrossed triplet* of Γ_n is a triplet p_i, p_{i+1}, p_{i+2} of points of S_n such that, when moving along $CH(S_n)$ in clockwise order, no edge of Γ_n crosses $CH(S_n)$ between p_i and p_{i+1} and between p_{i+1} and p_{i+2}. A triplet is *crossed k times* if the total number of times that $CH(S_n)$ is crossed by some edges between p_i and p_{i+1} and between p_{i+1} and p_{i+2} is k. A *leaf triplet* of Γ_n is a triplet of S_n whose points represent leaves of F_n. Analogously, a *root triplet* is a triplet of S_n whose points represent the three roots of F_n.

A high-level description for the proof of our lower bound is as follows. Suppose that there exists a point-set embedding Γ_n of F_n on S_n with few bends per edge (i.e. with $o(n^{\frac{1}{3}})$ bends per edge). This implies that each edge crosses $CH(S_n)$ few times (because each segment of the polyline representing an edge can cross $CH(S_n)$ at most twice). We first prove (Lemma 4) that if this is the case, and if Γ_n has an uncrossed leaf triplet, then G_n admits a point-set embedding on S_n where each edge crosses $CH(S_n)$ few times. By Lemma 3, this implies that G_n admits a point-set embedding on S_n with few bends per edge, which contradicts Theorem 4. We then use Lemmas 5 and 6 to prove that if Γ_n does not have an uncrossed leaf triplet, then we can transform γ_n to obtain a point-set embedding of a smaller 3-sky that has an uncrossed leaf triplet. The crucial point is that this transformation keeps the number of crossings with $CH(S_n)$ small and does not remove too many vertices. The argument above implies that in any point-set embedding Γ_n of F_n on S_n there is at least one

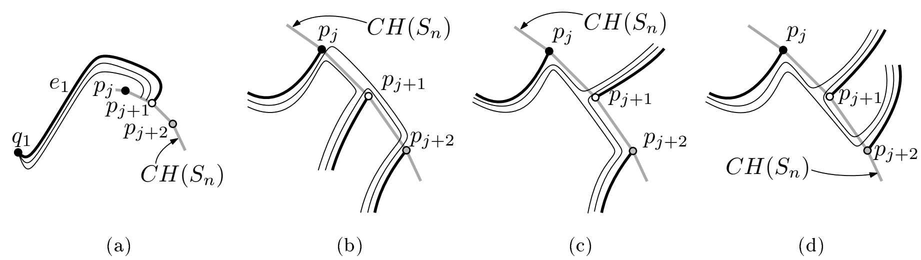


Figure 4: Insertion of a cycle connecting q_0, q_1 and q_2. (a) Drawing of the curves following the edge e_1. (b), (c), and (d) Connection of the six curves to form a cycle.

edge with many crossings (i.e. with $\Omega(n^{\frac{1}{3}})$ bends per edge). Lemmas 7 and 8 show how to repeatedly apply the argument to prove that the number of edges with many bends is $\Omega(n^{\frac{2}{3}})$.

Lemma 4. *Let F_n be a 3-sky, let S_n be an alternating point set, and let Γ_n be a 3-colored topological point-set embedding of F_n on S_n. If Γ_n has an uncrossed leaf triplet and each edge of Γ_n has at most b hull crossings, then the 3-fan G_n has a 3-colored topological point-set embedding on S_n such that each edge has at most $3b + 4$ hull crossings.*

PROOF. We show how to use Γ_n to construct a topological point-set embedding of the 3-fan G_n on S_n with at most $3b + 2$ hull crossings per edge.

Let p_j, p_{j+1}, p_{j+2} be an uncrossed leaf triplet. Recall that the points of a triplet are colored 0, 1 and 2 in this order. Every point of the triplet represents a leaf of a different star (because they have different color). Denote by q_i the point of Γ_n representing the root of T_i ($i = 0, 1, 2$) and denote by e_i the edge connecting q_i to p_{j+i}. The idea is to connect the three points q_0, q_1, q_2 with a 3-cycle that does not cross any existing edge. For each edge e_i we draw two curves that from q_i run very close to e_i until they reach $CH(S_n)$. The two curves are drawn on the same side of e_i such that they are consecutive in the circular order of the edges around q_i (see Fig. 4(a) for an illustration). These two curves do not intersect any existing edges and cross $CH(S_n)$ the same number of times as e_i. The six drawn curves are now suitably connected to realize a cycle C connecting q_0, q_1, q_2. Depending on which side the various curves reach $CH(S_n)$, the connections are different. In all cases, however, we can connect two curves to form a single edge by crossing $CH(S_n)$ at most two additional times and without violating planarity (see Fig. 4(b), 4(c), and 4(d)). Thus, we have added to Γ_n three edges e'_0, e'_1, and e'_2. Each e'_i connects q_i to q_{i+1} (indices taken modulo 3) and crosses $CH(S_n)$ at most $2b + 2$ times. Also, since the two curves that follow an edge e_i are both drawn on the same side of e_i, the cycle C does not have any vertices inside (see also Fig. 5(a) for an example).

The obtained drawing is not yet a topological point-set embedding of G_n because the cycle C' connecting all the vertices is missing. Each of the edges that are still missing connects either two leaves of the same color, or one leaf of

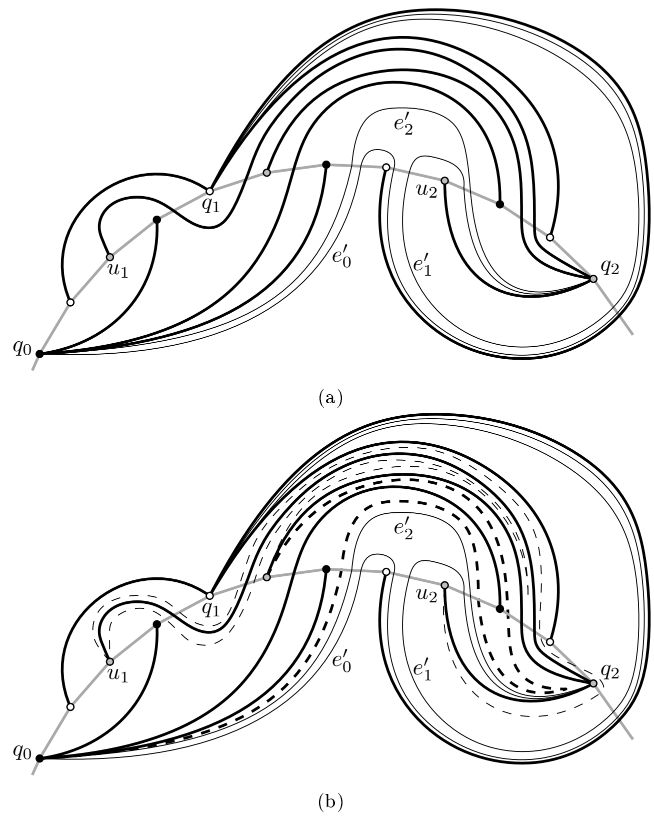


Figure 5: (a) A topological point-set embedding of the 3-sky F_4 (thick edges) after the addition of the cycle connecting q_0, q_1 and q_2 (thin edges). (b) Addition of the edges connecting the leaves adjacent to q_2 (thin dashed edges) and of the edge connecting q_1 to a leaf with color 2 (thick dashed edge).

color i to vertex v_{i+1} (indices taken modulo 3). Consider any two leaves u_1 and u_2 of color i whose edges are consecutive in the circular order around q_i; denote by $\overline{e}_j$ the edge (v_i, u_j) for $j = 1, 2$. We add the edge (u_1, u_2) as follows. Starting from u_1 we draw a curve following the edge $\overline{e}_1$ until we arrive very close to q_i and then we follow $\overline{e}_2$ until we reach u_2 (see the dashed edges in Fig. 5(b)). The added edges do not cross any existing edges and cross $CH(S_n)$ a number of times equal to the number of hull crossings of $\overline{e}_1$ plus the number of hull crossings of $\overline{e}_2$ plus, possibly, two additional times in case it has to go around q_i (see for example the edge connecting the leaves u_1 and u_2 in Fig. 5(b)). We repeat the same procedure for every pair of leaves whose edges are consecutive in the circular order around q_i (there are $n-2$ such pairs). Each of the added edges does not cross any other existing edge and crosses $CH(S_n)$ at most $2b+2$ times. Furthermore the added edges connect all the leaves of color i in a path π_i. It remains to add the edges that connect a leaf of color i to the vertex v_{i+1}. Notice that the edge (v_i, v_{i+1}) and one of the two edges connecting v_i to the end-vertices of π_i are consecutive in the circular order of the edges around q_i; denote by e''_i this edge. Starting from q_{i+1} we draw a curve following the edge e'_i (i.e., the edges that connects v_i to v_{i+1}) until we arrive very close to q_i and then we follow e''_i until we reach the leaf of e''_i (see the bold dashed edge in Fig. 5(b)). The constructed curve connects q_{i+1} to a leaf of color i and does not cross any existing edge. It crosses $CH(S_n)$ at most the number of hull crossings of e'_{i+1} (that is $2b+2$), plus the number of hull crossings of e''_{i+1} (that is c), plus, possibly, two additional times in case it has to go around q_i. Thus the total number of crossing of $CH(S_n)$ is at most $3b+4$. □

The next two lemmas explain how to obtain a 3-colored topological point-set embedding that satisfies Lemma 4.

Lemma 5. *Let F_n be a 3-sky, let S_n be an alternating point set, and let Γ_n be a 3-colored topological point-set embedding of F_n on S_n with a root triplet. If Γ_n has a leaf triplet τ that is crossed c times ($c < n$) and each edge of Γ_n has at most b hull crossings, then there exists a 3-sky $F_{n'}$ which is a subgraph of F_n and an alternating point set $S_{n'}$ which is a subset of S_n such that: (i) $n' \geq n - c$; (ii) there exists a 3-colored topological point-set embedding $\Gamma_{n'}$ of $F_{n'}$ on $S_{n'}$ such that each edge of $\Gamma_{n'}$ has at most $b+1$ hull crossings; (iii) τ is an uncrossed leaf triplet of $\Gamma_{n'}$.*

Proof. The idea is to make τ uncrossed by removing all edges that cross it. In order to keep the set of points alternating, for each removed edge e we will remove the whole triplet that contains the point representing the leaf of e. Suppose first that none of the triplets to be removed coincides with τ (i.e., no edge that crosses τ has an endpoint in τ). In this case, we remove all edges that cross τ and all the triplets containing their leaves. Notice that, since Γ_n has a root triplet, no root is removed. Since we always remove triplets, the final drawing has the same number of points for each color. Thus, such a drawing is a 3-colored topological point-set embedding $\Gamma_{n'}$ of a 3-sky $F_{n'}$ on an alternating point set $S_{n'}$ with an uncrossed triplet τ. The number of triplets removed is at

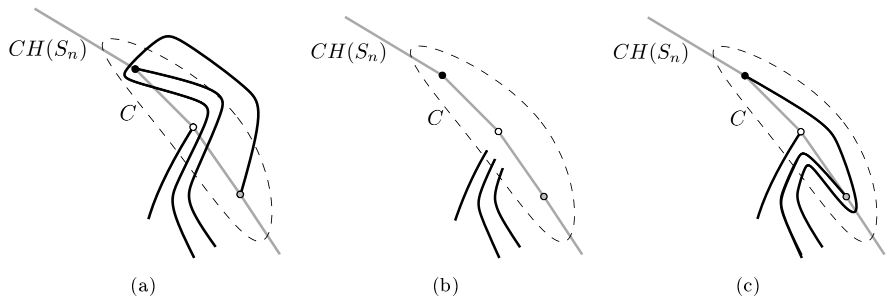


Figure 6: Re-drawing of the edges incident to a leaf triplet to make it uncrossed.

most c, and therefore $n' \geq n - c$. Finally, since we have only removed edges, the number of hull crossings of each edge remains at most b.

Suppose now that at least one edge that crosses τ has an endpoint in τ. Denote by $\overline{E}$ the set of these edges. We first remove all edges that cross τ but are not in $\overline{E}$. Then we redraw the edges of $\overline{E}$ so that they do not cross τ. To this end, consider a closed curve C around τ that only intersects edges that are incident to or cross τ. See Fig. 6 for an illustration.

We cut the three edges incident to the points of τ (these three edges include those in $\overline{E}$, possibly coinciding with them) at the point where they cross C for the first time when walking towards the leaves. We then connect the three points on the curve C to the points of τ. One can observe that this can always be done by adding to each edge at most one crossing of $CH(S_n)$. Also in this case the resulting drawing is a 3-colored topological point-set embedding $\Gamma_{n'}$ of a 3-sky $F_{n'}$ on an alternating point set $S_{n'}$ with an uncrossed triplet τ. Again, the number of triplets removed is at most c, and therefore $n' \geq n - c$. □

Lemma 6. *Let F_n be a 3-sky, let S_n be an alternating point set, and let Γ_n be a 3-colored topological point-set embedding of F_n on S_n. There exist an alternating point set $S_{n'} \subseteq S_n$ and a 3-colored topological point-set embedding $\Gamma_{n'}$ of $F_{n'}$ on $S_{n'}$ such that: (i) $n' \geq \frac{n}{3}$; (ii) if an edge has b hull crossings in Γ_n then it has at most $b+2$ hull crossings in $\Gamma_{n'}$; (iii) $\Gamma_{n'}$ has a root triplet.*

PROOF. If Γ_n has a root triplet, then $\Gamma_{n'}$ coincides with Γ_n and the statement holds. Suppose then that Γ_n does not have a root triplet. Consider the triplets that contain the roots. They are at most three. Denote them as τ_1, τ_2 and τ_3. Denote by $T_{i,i+1}$ (for $i = 0, 1, 2$, indices taken modulo 3) the set of triplets that are encountered between τ_i and τ_{i+1} when moving clockwise along $CH(S_n)$. The number of triplets in $T_{0,1}$, $T_{1,2}$, and $T_{2,0}$ is $n - 3$ and thus at least one of these three sets has $\frac{n-3}{3}$ triplets. We remove all the triplets of the other two sets. Furthermore we remove the points of the triplets containing the roots that do not represent the roots. This removes at most six extra vertices. The three roots are now consecutive. If they form a triplet, i.e., they are colored 0, 1, 2 in the clockwise order, then the drawing obtained after the removal is the

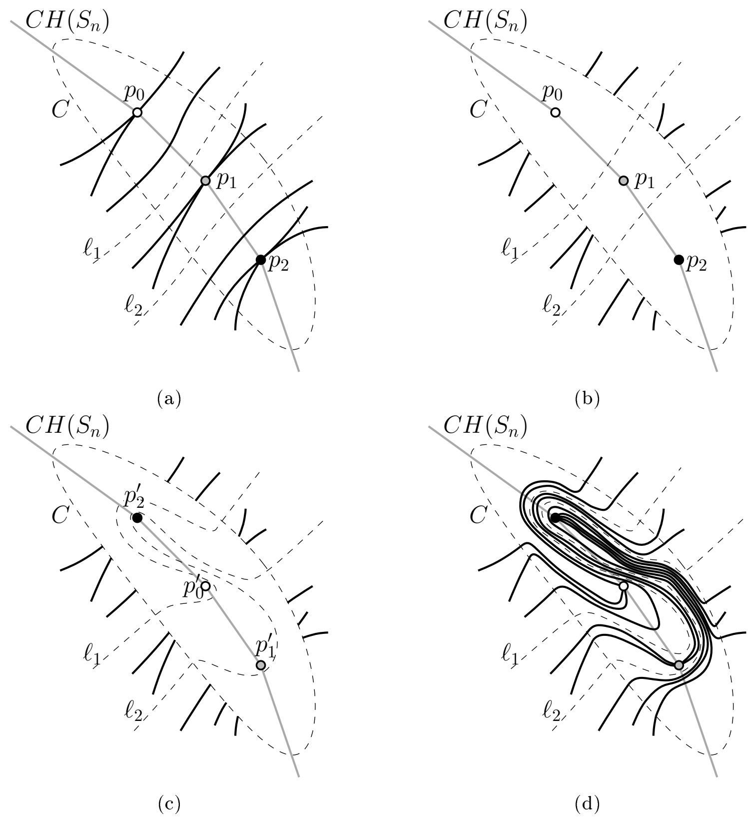


Figure 7: Re-drawing of the edges incident to and crossing the root triplet, in order to change the sequence of colors.

desired $\Gamma_{n'}$. In $\Gamma_{n'}$, there are at least $\frac{n-3}{3}$ triplets plus the root triplet, hence $n' \geq \frac{n-3}{3} + 1 = \frac{n}{3}$.

If the three roots do not form a triplet, we locally modify the drawing to transform them into a triplet. To this aim, we consider a closed curve C around the three roots that is crossed only by the edges incident to the three roots and by the edges that cross $CH(S_n)$ between them. The idea is to reassign colors to the points representing the three roots so that they form a triplet and then to reroute the edges crossing C so that each edge is reconnected to the point that, after the recoloring, has the color of the edge's root. This can be done by creating for each edge at most two additional hull crossings. Let p_0, p_1, and p_2 be the points representing the three roots in the clockwise order along $CH(S_n)$ and denote by c_0, c_1, and c_2 the colors of p_0, p_1, and p_2, respectively. Since p_0, p_1, and p_2 do not form a triplet, for at least two of them the color c_i is not i. We use two curves ℓ_1 and ℓ_2 to subdivide the region enclosed by C into three sub-regions: the first one containing p_0, a portion of each edge incident to it, and a portion of each edge that crosses $CH(S_n)$ between p_0 and

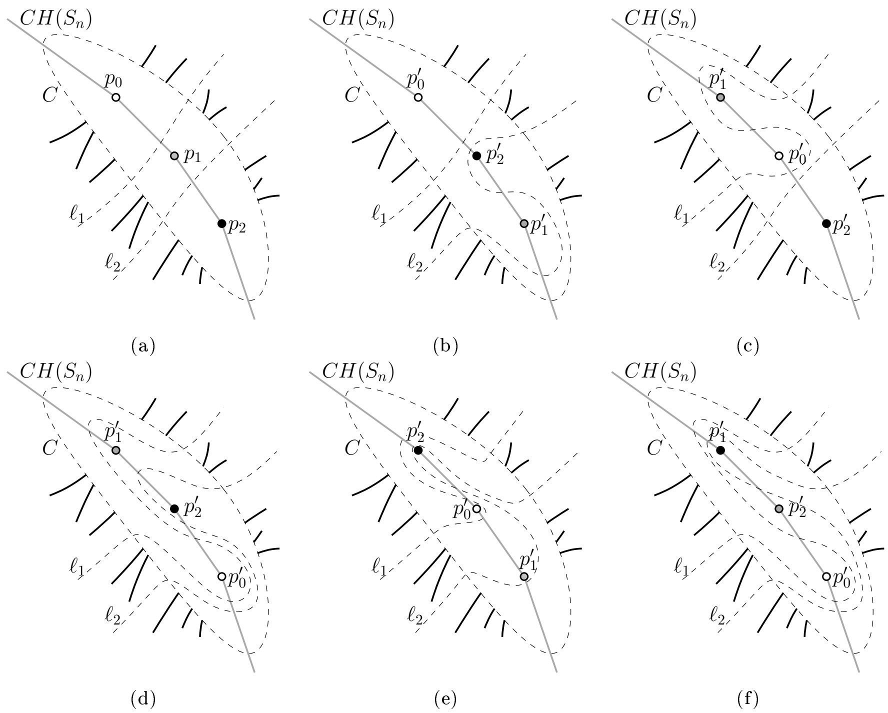


Figure 8: Transformation of lines ℓ_1 and ℓ_2 for all possible orders of p'_0, p'_1 and p'_2.

p_1; the second one containing p_1 and a portion for each edge incident to it; the third one containing p_2, a portion of each edge incident to it, and a portion of each edge that crosses $CH(S_n)$ between p_1 and p_2 (see Fig. 7(a)). We cut all the edges crossing C at the points where they intersect C (see Fig. 7(b)) and reassign colors to p_0, p_1, and p_2 so that each p_i is colored i (for $i = 0, 1, 2$). Denote by p'_i the point that, after the recoloring, has color c_i. We now make an homotopic transformation of ℓ_1 and ℓ_2 such that p'_0 is to the left of ℓ_1, p'_2 is to the right of ℓ_2, and p'_1 is between ℓ_1 and ℓ_2 (see Fig. 7(c)). This transformation of ℓ_1 (resp. ℓ_2) can create at most two additional crossings between ℓ_1 (resp. ℓ_2) and $CH(S_n)$ (Fig. 8 shows the transformation for all possible orders of p'_0, p'_1 and p'_2). The edges that were incident to p_0 must now be reconnected to p'_0; all these edges have been cut at points that are to the left of ℓ_1. Since p'_0 is also to the left of ℓ_1, these edges can be connected to p'_0 following the profile of ℓ_1 without creating any crossing. Analogously, the edges that were incident to p_1 have been cut at points between ℓ_1 and ℓ_2 and therefore can be reconnected to p'_1 following the profile of either ℓ_1 or ℓ_2 without creating any crossing. The edges that were incident to p_2 can be reinserted analogously to the right of ℓ_2. Finally, the edges that crossed $CH(S_n)$ between p_0 and p_1 can be rerouted to the left of ℓ_1 and those that crossed $CH(S_n)$ between p_1 and p_2 can be rerouted to the right of ℓ_2 (see Fig. 7(d)). After the rerouting, each of the rerouted edges has at most two additional hull crossings and we are in the same situation as in the previous case. The resulting drawing is the desired $\Gamma_{n'}$ with $n' \geq \frac{n}{3}$. □

We are now ready to provide our first main result about the lower bound on the 3-colored point-set embedding.

Lemma 7. *Let $h > 15$ be a positive integer. For $i = 1, 2, \ldots, h^2$ let F_{n_i} be the 3-sky for $n_i = 173563h^3 + 7h \cdot i$ and let S_{n_i} be an alternating point set. In every 3-colored point-set embedding of F_{n_i} on S_{n_i} that has a root triplet there exist at least i edges with more than h bends each.*

PROOF. We first prove that in any 3-colored point-set embedding Γ_{n_i} of F_{n_i} on S_{n_i} with a root triplet there exists one edge with more than h bends. Suppose for a contradiction that Γ_{n_i} has curve complexity h. This implies that each edge of Γ_{n_i} crosses $CH(S_{n_i})$ at most $2h$ times. Namely, each edge consists of at most $h+1$ segments and each of these segments can have at most two intersections with $CH(S_{n_i})$; on the other hand, two of these intersections are the two end-vertices of the edge. Since each edge of Γ_{n_i} crosses $CH(S_{n_i})$ at most $2h$ times and there are $3(n_i - 1)$ edges in total, there are at most $3 \cdot 2h(n_i - 1)$ crossings of $CH(S_{n_i})$ in total. The number of leaf triplets in Γ_{n_i} is $n_i - 1$. It follows that at least one leaf triplet τ is crossed at most $\frac{3 \cdot 2h(n_i-1)}{(n_i-1)} = 6h \leq 7h$ times. By Lemma 5 there exists a 3-colored point-set embedding $\Gamma_{n'}$ of a 3-sky $F_{n'}$ on an alternating point set $S_{n'}$ such that: (i) $n' \geq n_i - 7h$; (ii) each edge of $\Gamma_{n'}$ crosses $S_{n'}$ at most $2h+1$ times; (iii) τ is uncrossed. By Lemma 4, the 3-fan $G_{n'}$ has a 3-colored topological point-set embedding on $S_{n'}$ such that each edge crosses $CH(S_{n'})$ at most $6h+7$ times and by Lemma 3 a 3-colored point-set embedding with curve complexity at most $2(6h+7)+1 = 12h+15$. On the other hand, since $n_i = 173563h^3 + 7h \cdot i$, we have that $n' \geq n_i - 7h \geq= 173563h^3 + 7h \cdot i - 7h = 173563h^3 - 7h(i-1) \geq 173563h^3 = 79(13h)^3$ and by Theorem 4, in every 3-colored point-set embedding of $G_{n'}$ on $S_{n'}$ at least one edge has more than $13h$ bends. For $h > 15$, $13h$ is larger than $12h + 15$, which leads to a contradiction.

We now prove by induction on i that in every 3-colored point-set embedding of F_{n_i} on S_{n_i} with a root triplet there exist i edges with more than h bends. The statement is true for $i = 1$ by the argument above. Consider now any 3-colored point-set embedding Γ_{n_i} of F_{n_i} on S_{n_i} with a root triplet and assume that the statement is true for $i-1$. By the argument above Γ_{n_i} has at least one edge e crossed more than h times. We now remove this edge and the whole triplet τ that contains the point representing the leaf of e. Since Γ_{n_i} has a root triplet, the removal of τ does not remove any root. We then arbitrarily remove $7h-1$ leaf triplets. The resulting drawing is a 3-colored point-set embedding $\Gamma_{n_{i-1}}$ of $F_{n_{i-1}}$ on $S_{n_{i-1}}$ with a root triplet. By induction, it contains $i-1$ edges each having more than h bends. It follows that Γ_{n_i} has i edges each having more than h bends. □

The previous lemma shows that there exist instances requiring $\Omega(n^{\frac{1}{3}})$ bends on $\Omega(n^{\frac{2}{3}})$ edges (because $h \geq \sqrt[3]{\frac{n_i}{173563}}$ for $i = h^2$) if we only consider point-set embeddings that have a root triplet. In the next lemma we remove this restriction.

Lemma 8. *Let $h > 15$ be a positive integer, let F_n be the 3-sky for $n = 520689 \cdot h^3$, and let S_n be an alternating point set. In every 3-colored point-set embedding of F_n on S_n there exist at least h^2 edges with more than $\lfloor \frac{h-3}{8} \rfloor$ bends each.*

PROOF. Let Γ_n be any 3-colored point-set embedding of F_n on S_n. By Lemma 6 there exists a 3-colored point-set embedding $\Gamma_{n'}$ of the 3-sky $F_{n'}$ on an alternating point set $S_{n'}$ such that: (i) $n' \geq \frac{n}{3}$; (ii) if an edge has c hull crossings in Γ_n, then it crosses $CH(S_{n'})$ at most $c+2$ times in $\Gamma_{n'}$. By Lemma 3, $F_{n'}$ admits a canonical 3-colored point-set embedding $\overline{\Gamma_{n'}}$ on $S_{n'}$ such that each edge has the same hull crossings as in $\Gamma_{n'}$.

Since $n' \geq \frac{n}{3} = \frac{520689 \cdot h^3}{3} = 173563h^3$, Lemma 7 implies that $\overline{\Gamma_{n'}}$ has at least h^2 edges with at least h bends. By Lemma 1, each of these edges has at least $\frac{h-1}{2}$ hull crossings in $\overline{\Gamma_{n'}}$. Since $\overline{\Gamma_{n'}}$ and $\Gamma_{n'}$ have the same set of hull crossings, then these edges have at least $\frac{h-1}{2}$ hull crossings also in $\Gamma_{n'}$. This implies that the same edges have at least $\frac{h-1}{2} - 2 = \frac{h-5}{2}$ hull crossings in Γ_n. By Lemma 2 they have more than $\left\lfloor \frac{\frac{h-5}{2}+1}{4} \right\rfloor = \lfloor \frac{h-3}{8} \rfloor$ bends in Γ_n. $\square$

From the above lemma, by observing that $h \geq \sqrt[3]{\frac{n}{520689}}$, the next theorem can be stated.

Theorem 5. *For sufficiently large n, there exists a 3-colored forest F_n consisting of three monochromatic stars with n vertices and a 3-colored point set S_n in convex position compatible with F_n such that any 3-colored point-set embedding of F_n on S_n has $\Omega(n^{\frac{2}{3}})$ edges having $\Omega(n^{\frac{1}{3}})$ bends each.*

Since a caterpillar can be regarded as a set of stars whose roots are connected in a path, the lower bound of Theorem 5 also holds for caterpillars and thus Theorem 1 holds.

Theorem 1. *For sufficiently large n, a 3-colored point-set embedding of a 3-colored caterpillar may require $\Omega(n^{\frac{2}{3}})$ edges each having $\Omega(n^{\frac{1}{3}})$ bends.*

Theorem 1 answers an open problem posed in [5] about the curve complexity of k-colored point-set embeddings of trees for $k \geq 3$. Note that constant curve complexity for 2-colored outerplanar graphs has been proved in [12].

We conclude this section with some results derived from and/or related to Theorem 5. Firstly, Theorem 5 extends Theorem 4 since it implies that a 3-colored point set embedding of G_n may require $\Omega(n^{\frac{2}{3}})$ edges with $\Omega(n^{\frac{1}{3}})$ bends each. Moreover, Theorem 5 implies an analogous result for a k-colored forest of at least three stars for every $k \geq 3$. In particular, when $k = n$ we have the following result that extends the one by Pach and Wenger [35].

Corollary 1. *Let F be a forest of three n-vertex stars. Every planar drawing of F with vertices in one-sided convex position has $\Omega(n^{\frac{2}{3}})$ edges with $\Omega(n^{\frac{1}{3}})$ bends each.*

One may wonder whether the lower bound of Theorem 5 also holds when the number of colors or the number of stars is less than three. The following theorem shows that this is not the case.

Theorem 6. *Let F be a k-colored forest of h stars and S be a set of points compatible with F. If $\min\{k,h\}=2$ then F has a k-colored point-set embedding on S with curve complexity at most 2.*

PROOF. Suppose first that $k=2$. In this case we connect the roots of the stars and obtain a 2-colored caterpillar, which admits a 2-colored point-set embedding on any 2-colored set of points with curve complexity two [17]. Suppose now that $h=2$. Create a cycle C of n vertices colored in such a way that the sequence of colors along C is the same as in $seq(S)$. Let v be a vertex of C with the same color as the root of the first star. Connect this vertex to all the other vertices which it is not connected to, embedding all edges inside C (this can be easily done without creating any crossing). Analogously we can connect a vertex with the same color as the root of the second star to all the other vertices which it is not connected to, embedding all edges outside C. The obtained graph G is a k-colored Hamiltonian augmentation of the union of the two stars that is consistent with $seq(S)$ and has no division vertex. By Theorem 3, G admits a k-colored point-set embedding on S with curve complexity one. □

4. Upper Bound on the Curve Complexity of 3-colored Paths

In the light of Theorem 1, one may ask whether there exist subclasses of 3-colored caterpillars for which constant curve complexity can be guaranteed. In this section we first prove that this is the case for 3-colored paths and then we extend the result to 3-colored caterpillars whose leaves all have the same color. We start by introducing some tools that will be used throughout this section.

Let $G=(V,E)$ be a planar graph. A *topological book embedding* of G is a planar drawing such that all vertices of G are represented as points of a horizontal line ℓ, called the *spine*. Each of the two half-planes defined by ℓ is a *page*. Each edge of a topological book embedding is either in the top page, or completely in the bottom page, or it can be on both pages, in which case it crosses the spine. Each crossing between an edge and the spine is called a *spine crossing*. We assume that in a topological book embedding every edge is drawn as a sequence of one or more circular arcs, in particular semi-circles, such that no two consecutive arcs are in the same page[1]. Notice that, by using semi-circles, two arcs in the same page cross each other only if their end-points alternate along the spine. In the rest of the paper we shall often use the term arc to mean a circular arc. A spine crossing c is *flat* if the two end-points of the two arcs incident on c appear one before c and the other one after c along

[1]The more general concept of *p-page topological book embedding* exists, where each arc can be drawn on one among p different pages. For simplicity we use the term topological book embedding to mean 2-page topological book embedding.

ℓ. Let G be a k-colored graph, and let σ be a k-colored sequence compatible with G. A topological book embedding of G is *consistent* with σ if the sequence of vertex colors along the spine coincides with σ. The following lemma can be derived from Theorem 3.

Lemma 9. *Let G be a k-colored graph, and let S be a k-colored point set compatible with G. If G admits a topological book embedding consistent with $seq(S)$ and with at most χ_{f} flat spine crossing per edge and at most χ_{nf} non-flat spine crossing per edge, then G admits a point-set embedding on S with curve complexity at most $2\chi_{\mathrm{nf}} + \chi_{\mathrm{f}} + 1$.*

Proof. Replace each spine crossing with a dummy vertex on e and connect any two (real or dummy) vertices that are consecutive along the spine with an edge if they are not yet connected. The resulting graph is a k-colored Hamiltonian augmentation of G consistent with $seq(S)$ and with at most χ division vertices per edge and at most χ_{f} flat spine crossing per edge. By Theorem 3, G admits a k-colored point-set embedding on S with at most $2\chi_{\mathrm{nf}} + \chi_{\mathrm{f}} + 1$ bends per edge. □

According to Lemma 9, in order to prove that a 3-colored path P admits a 3-colored point-set embedding with constant curve complexity on every 3-colored set of points S, it is sufficient to prove that P has a topological book embedding consistent with $seq(S)$ with a constant number of spine crossings per edge. To this aim, we first remove the vertices and points of one color from P and S, obtaining a 2-colored path P' and a compatible 2-colored point set S'. Next, we construct a topological book embedding $\Gamma_{P'}$ of P' consistent with $seq(S')$ with at most two spine crossings per edge and with suitable properties. Then we use these properties to reinsert the third color and obtain a topological book embedding of P consistent with $seq(S)$. It is worth observing that a 2-colored path admits a topological book embedding without spine crossings [17]. However, the topological book embeddings produced by the algorithm in [17] do not have the properties that we need to reinsert the third color. In Section 4.1 we describe a new alternative algorithm for topological book embeddings of 2-colored paths which is then exploited in Section 4.2 to compute topological book embeddings of 3-colored paths. In this section we assume a path to be oriented from one end-vertex to the other, while its associated compatible sequence is ordered from left to right according to the definition.

4.1. Topological book embedding of a 2-colored path

Let P be a 2-colored path, and let $\sigma = seq(S)$ be the compatible sequence associated with P. Path P and sequence σ can be regarded as two binary strings of the same size where one color is represented by bit 0 and the other one by bit 1. We denote by $b_j(P)$ the j-th bit of P and by $b_j(\sigma)$ the j-th bit of σ. Path P and sequence σ are *balanced* if the number of zeros (ones, resp.) in P equals the number of zeros (ones, resp.) in σ. Path P and sequence σ are a *minimally balanced pair* if there does not exist a prefix of P and a corresponding prefix of σ that are balanced.

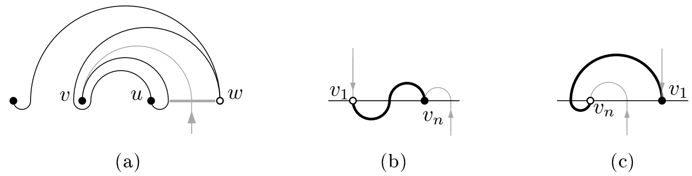


Figure 9: (a) Illustration of the hook visibility property. The bold segment is the access interval. The spine portion defined by u and w is the access spine portion. (b)–(c) Base cases for the proof of Lemma 11.

Lemma 10. *Let P and σ be a minimally balanced pair of length $k > 1$. Then, $b_1(P) \neq b_k(P)$, $b_k(P) = b_1(\sigma)$, and $b_1(P) = b_k(\sigma)$.*

PROOF. We first observe that $b_1(P) \neq b_1(\sigma)$ as otherwise $b_1(P)$ and $b_1(\sigma)$ would be balanced prefixes of P and of σ, respectively, thus contradicting the fact that P and σ are a minimally balanced pair. Also, $b_k(P) \neq b_k(\sigma)$ as otherwise the prefixes obtained by removing $b_k(P)$ from P and $b_k(\sigma)$ from σ would be balanced, again contradicting the fact that P and σ are a minimally balanced pair.

Assume now for a contradiction that $b_1(P) = b_k(P)$, which would also imply that $b_k(P) \neq b_1(\sigma)$ (since $b_1(P) \neq b_1(\sigma)$). Denote by Δ_l the number of zeros in the string $b_1(P)b_2(P)\dots b_l(P)$ minus the number of zeros in the string $b_1(\sigma)b_2(\sigma)\dots b_l(\sigma)$ (for $l = 1, 2, \dots, k$). Assume without loss of generality that $b_1(P) = 0$, then $b_1(\sigma) = 1$ and therefore $\Delta_1 = 1$. On the other hand, since P and σ are balanced, $\Delta_k = 0$. Since $b_k(P) \neq b_k(\sigma)$, $\Delta_{k-1} \neq 0$ and since $b_1(P) = b_k(P)$, we have that $b_k(P) = 0$, which implies $\Delta_{k-1} = -1$. The value of Δ_l changes by at most one unit (positively or negatively) when the index l is increased by one unit. Thus, $\Delta_1 = 1$ and $\Delta_{k-1} = -1$ imply that there must exist an index j, with $1 < j < k-1$, where $\Delta_j = 0$. But this would imply the existence of a prefix of P and a corresponding prefix of σ that are balanced – again a contradiction. Finally, note that $b_1(P) \neq b_1(\sigma)$, $b_k(P) \neq b_k(\sigma)$ and $b_1(P) \neq b_k(P)$ imply that $b_k(P) = b_1(\sigma)$ and $b_1(P) = b_k(\sigma)$. □

Let Γ be a topological book embedding with n vertices. The vertices of Γ divide the spine ℓ into $n + 1$ intervals, two of which (namely the rightmost and the leftmost) are unbounded. We call each of these intervals a *spine portion*. We consider a spine portion as an open interval, i.e., its endpoints are not part of the spine portion. Let p be a point of ℓ (possibly representing a vertex). We say that p is *visible from above (below)* if the vertical ray with origin at p and lying in the top (bottom) page does not intersect any edge of Γ. We say that a spine portion is *visible from above (below)* if it contains a sub-interval whose points are all visible from above (below). A vertex v of Γ is *hook visible* if there exists a point p of the spine that is not a vertex or a spine crossing such that p is visible from below and we can add the arc (v, p) in the top page of Γ without crossing any other edge of Γ. If p is to right (left) of v we say that v is *hook visible from*

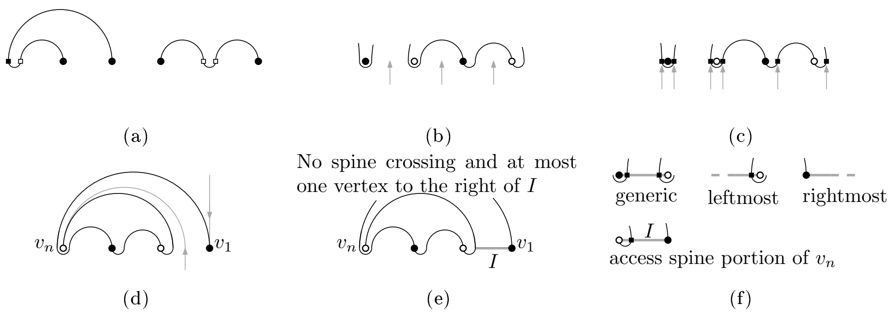


Figure 10: Illustration of the Properties 11.a–11.f. (a) Edges crossing the spine twice: the spine crossings are shown as small squares; the white squares are flat spine crossings; (b)–(c) Arrows show visibilities; squares are spine crossings; (d) v_1 and v_n are the first and the last vertex of P, respectively (arrows show visibilities); (e) I is the right access interval of the last vertex of P; (f) Illustration of the various types of spine portions; the small square are spine crossings .

the right (left) and p is a *right (left) access point* of v. An maximal interval of the spine consisting points that are all right (left) access points of v is called a *right (left) access interval* of v (or *access interval* for short). A spine portion that contains an access interval of a vertex v is called an *access spine portion* of v. For example, Fig. 9(a) depicts a 2-colored topological book embedding of a path; the right access interval for the hook visible vertex v is highlighted in bold; the gray curve describes the hook visibility property of v. The access spine portion is the spine portion defined by u and w. The next lemma states that a 2-colored path admits a 2-colored topological book embedding consistent with a compatible 2-colored sequence with some additional properties. These properties are useful to combine topological book embeddings of sub-paths, as we will do in the proof of the lemma, but also to use the computed topological book embedding as a basic tool to compute a topological book embedding of a 3-colored path. The additional properties are illustrated in Fig. 10

Lemma 11. *Let P be a 2-colored path, and let σ be a 2-colored sequence compatible with P. The path P admits a topological book embedding consistent with σ and with the following properties:*

11.a Every edge crosses the spine at least once and at most twice. If an edge crosses the spine twice, at least one of its spine crossings is flat.

11.b Every spine portion is visible from below.

11.c Every spine crossing is visible from below.

11.d The first vertex of P is visible from above; the last vertex of P is hook visible from the right;

11.e The last vertex of P has a single right access interval I and to the right of I there is no spine crossing and at most one vertex.

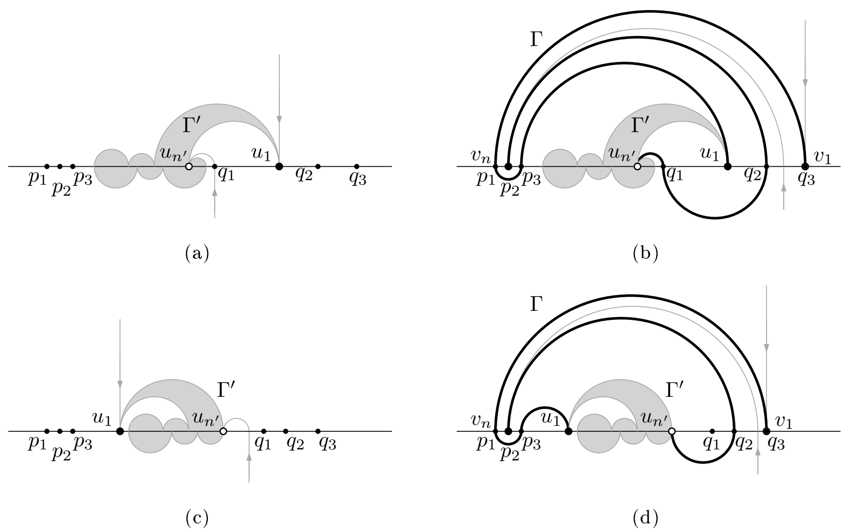


Figure 11: Case 1 of Lemma 11: (a) and (c) show the topological book embedding Γ' of P' and the choice of the points p_1, p_2, p_3, q_1, q_2, and q_3. In (a) u_1 is to the right of $u_{n'}$, while in (c) u_1 is to the left of $u_{n'}$. In both cases u_1 is accessible from above and $u_{n'}$ is hook visible from the right; (b) and (d) show the addition of the edges (v_1, u_1) and $(u_{n'}, v_n)$. In both cases v_1 is visible from above and v_n is hook visible from the right.

11.f *Every spine portion has at most two spine crossings. The leftmost spine portion and the access spine portion of the last vertex of P have at most one spine crossing. The rightmost spine portion has no spine crossings.*

PROOF. We prove the statement by induction on the length n of P (and of σ). If $n = 1$ the statement trivially holds. If $n = 2$ we draw the unique edge e of P in one of two possible ways depending on the colors of the two vertices. If the first vertex of P has the same color as the first element of σ, we draw e with a spine crossing between the two points representing the vertices of P and in such a way that the first arc of e is in the bottom page; see Fig. 9(b). If the first vertex has a color different from the first element of σ, then we draw e with a spine crossing to the left of the leftmost vertex in such a way that the arc incident to the leftmost vertex is in the bottom page, see Fig. 9(c). In both cases properties 11.a–11.f hold. Suppose now that $n > 2$ and that the statement holds for every $k < n$. We distinguish between two cases.

Case 1: P and σ are a minimally balanced pair. By Lemma 10, the first vertex of P has the same color as the last element of σ, the last vertex of P has the same color as the first element of σ, and these two colors are different. It follows that by removing the first and the last elements from both P and σ, we obtain a new 2-colored path P' of length $n' = n-2$ and a new 2-colored sequence σ' compatible with P'. By our inductive hypothesis, P' admits a topological book embedding Γ' consistent with σ' such that Γ' satisfies properties 11.a–11.f (as schematically depicted in Fig. 11(a) and 11(c)). Denote by u_1 and $u_{n'}$ the first vertex and the last vertex of P', respectively. To construct a topological book embedding of P consistent with σ, we proceed as follows (refer also to Fig. 11(b) and 11(d)). Denote as p_1, p_2 and p_3 three points (in this order from left to right) to the left of all points of Γ'; denote as q_1, q_2, and q_3 (in this order from left to right) three points such that q_1 is in the access interval of $u_{n'}$, while q_2 and q_3 are to the right of all points of Γ' (notice that by property 11.d of Γ', $u_{n'}$ has a right access interval). Vertex v_n is mapped to p_2 and vertex v_1 is mapped to q_3. Edge (v_1, u_1) is drawn as the concatenation of three arcs: arc (v_1, p_1) in the top page, arc (p_1, p_3) in the bottom page, and arc (p_3, u_1) in the top page (Notice that by property 11.d of Γ', u_1 is visible from above). Edge $(u_{n'}, v_n)$ is drawn in one of two different ways. If $u_{n'}$ is not the rightmost vertex of Γ', then it is drawn as the concatenation of three arcs: arc $(u_{n'}, q_1)$ in the top page, arc (q_1, q_2) in the bottom page, and arc (q_2, v_n) in the top page (see Fig. 11(b)). If u'_n is the rightmost vertex of Γ' (and therefore by property 11.f there is no spine crossing to its right), then it is drawn as the concatenation of two arcs: arc $(u_{n'}, q_2)$ in the bottom page and arc (q_2, v_n) in the top page (see Fig. 11(d)). It is immediate to verify that we have constructed a topological book embedding Γ of P consistent with σ. Namely, the added edges are drawn so that they do not corss each other. Also, by the visibility properties of u_1 and $u_{n'}$, the added edges do not intersect any edge of Γ'.

We now prove that Γ satisfies properties 11.a–11.f. The two added edges have at least one and at most two spine crossing by construction. The spine crossing of (v_1, u_1) at p_3 is flat, since it is connected to p_1 (which is to its left) and to u_1 (which is to its right). If $(u_{n'}, v_n)$ is drawn with two spine crossings (first case above), its spine crossing at q_1 is flat, since it is connected to $u_{n'}$ (which is to its left) and to q_2 (which is to its right). Thus, property 11.a holds. As for property 11.b, it can only be violated because of the addition of some arc in the bottom page; however this can happen only if the interval of the added arc contains more than one vertex. We have added at most two such arcs: (p_1, p_3) and possibly (q_1, q_2). The interval of arc (p_1, p_3) only contains v_n by construction; the interval of arc (q_1, q_2) contains at most one vertex by property 11.d of Γ'. This observation together with the inductive hypothesis on Γ' implies that property 11.b holds for Γ. Since the at most two arcs added in the bottom page do not change the visibility property for the spine crossings of Γ' and the added spine crossings (p_1, p_3, q_2, and possibly q_1) are visible from below, property 11.c also holds for Γ. Concerning property 11.d, observe that v_1 is visible both from above and from below because it is the last point of Γ; on the other hand, v_n is hook visible from the right because all points between q_2 and v_1 are access points for it. These are the only access points due to the presence of the arcs added to represent (v_1, u_1) and $(u_{n'}, v_n)$. Moreover, to the right of these access points there is no spine crossing and only the vertex v_1. Thus, property 11.e holds. Concerning property 11.f, let w'_l and w'_r be the leftmost vertex and the rightmost vertex of Γ', respectively. There are three spine portions of Γ' that are modified by the addition of (v_1, u_1) and $(u_{n'}, v_n)$: the leftmost spine portion s'_l of Γ' to the left of w'_l, the rightmost spine portion s'_r of Γ' to the right of w'_r, and the spine portion s'_I containing the right access

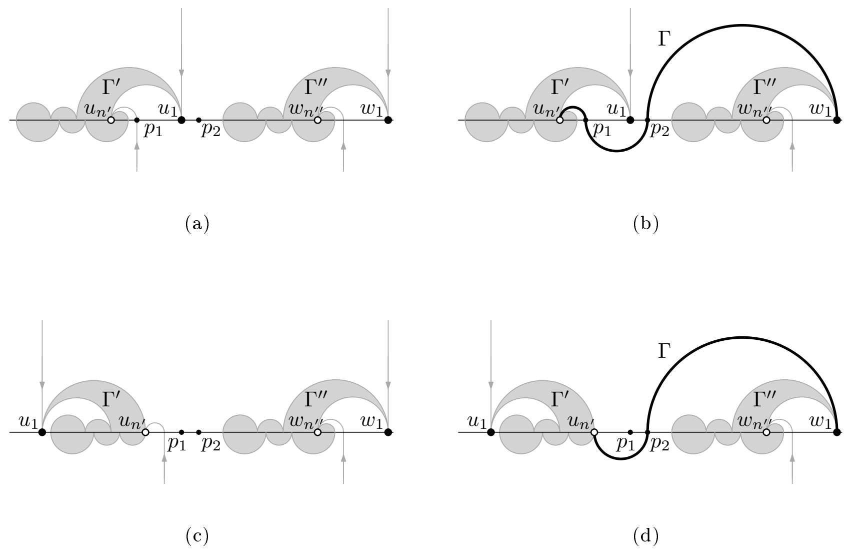


Figure 12: Case 2 of Lemma 11: (a) and (c) show the topological book embeddings Γ' of P' and Γ'' of P'' and the choice of the points p_1 and p_2. In (a) u_1 is to the right of $u_{n'}$, while in (c) u_1 is to the left of $u_{n'}$. In both cases u_1 and w_1 are accessible from above, while $u_{n'}$ and $w_{n''}$ are hook visible from the right; (b) and (d) show the addition of the edge $(u_{n'}, w_1)$. In both cases $u_1 = v_1$ is visible from above and $w_{n''} = v_n$ is hook visible from the right.

interval I of $u_{n'}$ (notice that s'_r and s'_I coincide if $u_{n'}$ is the rightmost vertex of Γ'). The spine portion s'_l is split in two by the addition of v_n: the leftmost spine portion s_l of Γ to the left of v_n and the spine portion s_1 between v_n and w'_l. Portion s_l contains only one spine crossing, that is p_1. As for s_1, by property 11.f of Γ' there is at most one spine crossing to the left of w'_l in Γ'; this and p_2 are the at most two spine crossings of s_1. The spine portion s'_r is also split in two by the addition of v_1: the rightmost spine portion s_r of Γ to the right of v_1 and the spine portion s_2 between w'_r and v_1. Section s_r has no spine crossing. Concerning s_2, by property 11.f of Γ' there is no spine crossing to the right of w'_r in Γ'; thus, s_2 only contains the spine crossing q_2. Consider now s'_I and assume that $s'_I \neq s'_r$ (otherwise there is nothing to prove). By property 11.f of Γ', the spine portion s'_I has at most one spine crossing in Γ'; the new spine crossing q_1 is the only one added inside s'_I and therefore s'_I has at most two spine crossings in Γ. Finally, observe that the spine portion s_I that contains the right access interval of v_n coincides with s_2 and we have already shown that it has at most one spine crossing.

Case 2: P and σ are not a minimally balanced pair. In this case there exists a prefix (i.e. a subpath) P' of P and a corresponding prefix σ' of σ that are balanced. P' is 2-colored path and σ' is a 2-colored sequence compatible with P' and their length is less than n. By inductive hypothesis, P' admits a topological book embedding Γ' consistent with σ' and satisfying properties 11.a–11.f. On the other hand, $P'' = P \setminus P'$ is also a 2-colored path and $\sigma'' = \sigma \setminus \sigma'$ is a 2-colored sequence consistent with P''. Thus, P'' also admits a topological book embedding Γ'' consistent with σ'' and satisfying properties 11.a–11.f.

Denote by u_1 and $u_{n'}$ the first and the last vertex of P', respectively, and by w_1 and $w_{n''}$ the first and the last vertex of P''. To construct a topological book embedding of P consistent with σ we proceed as follows (refer also to Fig. 12). Place Γ'' to the right of Γ' and denote as p_1 and p_2 two points (in this order from left to right) such that p_1 is in the access interval of $u_{n'}$, while p_2 is to the right of all points of Γ' and to the left of all the points of Γ''.

Edge $(u_{n'}, w_1)$ is drawn in one of two different ways. If $u_{n'}$ is not the rightmost vertex of Γ', then it is drawn as the concatenation of three arcs (see Fig. 12(b)): arc $(u_{n'}, p_1)$ in the top page, arc (p_1, p_2) in the bottom page, and arc (p_2, w_1) in the top page (Notice that by property 11.d of Γ'', w_1 is visible from above). If $u_{n'}$ is the rightmost vertex of Γ' (and therefore by property 11.f there is no spine crossing to its right), then it is drawn as the concatenation of two arcs (see Fig. 12(d)): $(u_{n'}, p_2)$ in the bottom page and (p_2, w_1) in the top page. It is immediate to verify that we have a constructed a topological book embedding Γ of P consistent with σ.

We now prove that Γ satisfies properties 11.a–11.f. The edge $(u_{n'}, w_1)$ added to create Γ has at least one and at most two spine crossings by construction. The two spine crossings are both flat. Thus, property 11.a holds. As for property 11.b, recall that the addition of $(u_{n'}, w_1)$ can cause the violation of property 11.b only if the interval of an arc added in the bottom page contains more than one vertex. We added in the bottom page either arc (p_1, p_2), whose interval only contains u_1 by construction, or $(u_{n'}, p_2)$, whose interval does not contain any vertex. This fact together with the inductive hypothesis on Γ' and Γ'' implies that property 11.b holds for Γ. Since the arc added in the bottom page does not change the visibility property for the spine crossings of Γ' and Γ'' and the added spine crossings (p_2 and possibly p_1) are visible from below, property 11.c also holds for Γ. About property 11.d, observe that v_1 coincides with u_1, which is visible from above in Γ' by property 11.d; since it is not covered from above by the edge $(u_{n'}, w_1)$, it remains visible from above also in Γ. Similarly, v_n coincides with $w_{n''}$, which is hook visible from the right in Γ'' by property 11.d; since its access interval is not covered from below by the edge $(u_{n'}, w_1)$, it remains hook visible from the right also in Γ. Since no vertex or spine crossing has been introduced to the right of the right access interval of $w_{n''} = v_n$, and since property 11.e holds for $w_{n''}$ in Γ'', property 11.e also holds for Γ. Concerning property 11.f, let w'_r be the rightmost vertex of Γ' and let w''_l be the leftmost vertex of Γ''. All the spine portions of Γ except the one between w'_r and w''_l, call it s_1, and the one containing the right access interval of $u_{n'}$, call it s'_I, have the same number of spine crossings that they have in Γ' or in Γ'', i.e. at most two. Notice that s'_I coincides with s_1 if $u_{n'}$ is the rightmost vertex of Γ'. Since by property 11.f of Γ' there is no spine crossing to the right of w'_r and by property 11.f of Γ'' there is at most one spine crossing to the left of w''_l, there is at most one spine crossing in s_1 different from p_2. Thus, s_1 has at most

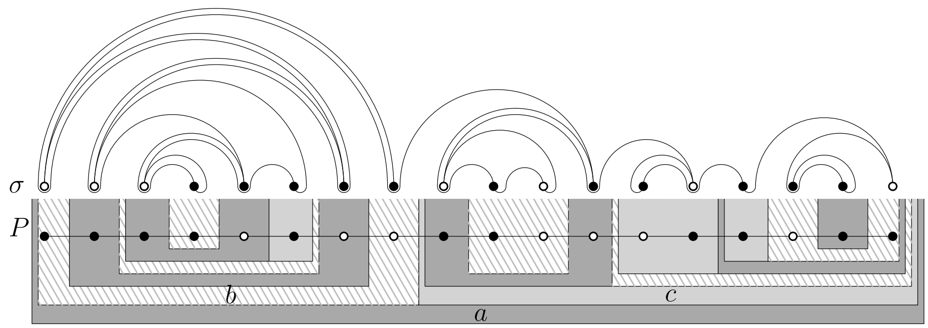


Figure 13: A topological book embedding of a 2-colored path consistent with a given 2-colored sequence constructed according to the algorithm described in the proof of Lemma 11.

two spine crossings. Consider now s'_I and assume that $s'_I \neq s_1$ (otherwise there is nothing to prove). By property 11.f of Γ', s'_I has at most one spine crossing in Γ'; the new spine crossing p_1 is the only one added inside s'_I and therefore s'_I has at most two spine crossings in Γ. Finally, since $v_n = w_{n''}$, the spine portion s_I that contains the right access interval of v_n coincides with the spine portion of Γ'' containing the right access interval of $w_{n''}$. No spine crossing has been added inside this portion and therefore s_I has at most one spine crossing by property 11.f of Γ''. □

Fig. 13 shows a topological book embedding of a 2-colored path P consistent with a given 2-colored sequence σ constructed according to the algorithm described in the proof of Lemma 11. To help the reader, colored regions are shown on top on the path in order to highlight the input of the various recursive calls. For example, the region labeled a represents the first call of the algorithm whose input is the whole path P and the whole sequence σ; since P and σ are not minimally balanced, Case 2 of Lemma 11 applies and we have two new recursive calls. The first call, represented by the region labeled b, takes as input the first eight vertices of P and the first eight elements of σ, which form the minimally balanced prefix of P and σ. The second call, represented by the region labeled c, takes as input the remaining part of P and σ.

4.2. Topological book embedding of a 3-colored path

In this section we exploit the algorithm of Lemma 11 to construct a topological book embedding of a 3-colored path. A vertex v of a topological book embedding Γ is *doubly hook visible* if there exist two points p_1 and p_2 of the spine that are not vertices or spine crossings such that p_2 is visible from above and we can add the arc (v, p_1) in the top page Γ and the arc (p_1, p_2) in the bottom page without crossing any other edge of Γ (see Fig. 14). If p_1 and p_2 are both to the right of v, we say that v is *doubly hook visible from the right.* The spine portions containing p_1 and p_2 are called the *access spine portions* of v. Notice that they both coincide with the rightmost spine portion if v is the rightmost vertex of Γ. The next lemma proves that every 3-colored path admits a topological book embedding consistent with any compatible 3-colored sequence and that has some additional properties. These properties will be useful in Section 5 to prove results about h-planar k-colored simultaneous embedding.

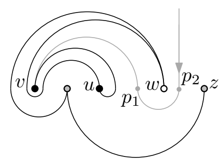


Figure 14: Illustration of the doubly hook visibility property. The two spine portions defined by u and w and by w and z are the access spine portions of v.

Lemma 12. *Let P be a 3-colored path and σ be a 3-colored sequence compatible with P. The path P admits a topological book embedding consistent with σ and with the following properties:*

12.a Every edge crosses the spine at most twice. If an edge crosses the spine twice, at least one of its spine crossings is flat.

12.b The first vertex of P is visible from above; the last vertex of P is doubly hook visible from the right;

12.c Every spine portion has at most two spine crossings. The leftmost spine portion and the two access spine portions of the last vertex of P have at most one spine crossing. The rightmost spine portion has no spine crossings.

Proof. Let c_0 and c_1 be the colors of the end-vertices of P (possibly $c_0 = c_1$), and let c_2 be a color distinct from c_0 and c_1. Let $w_1, w_2, \ldots, w_k$ be a maximal subpath of P colored c_2. Let u_1 be the vertex before w_1 along P and let u_2 be the vertex after w_k along P. Since the end-vertices of P have colors distinct from c_2, u_1 and u_2 always exist. We replace the subpath $u_1, w_1, w_2, \ldots, w_k, u_2$ with a *dummy edge* (u_1, u_2). We do this replacement for every maximal subpath colored c_2. Let P' be the resulting 2-colored path and let σ' be the 2-colored sequence obtained from σ by removing all elements of color c_2.

By Lemma 11, P' admits a topological book embedding Γ' consistent with σ' that satisfies properties 11.a–11.f. We add to Γ' a set Q of points colored c_2 to represent the removed vertices that will be added back. These points must be placed so that the sequence of colors along the spine coincides with σ. By property 11.b of Lemma 11, all these points can be placed so that they are visible from below. We now have to replace some edges of P' with paths of vertices colored c_2. Refer to Figure 15 for an illustration. Let (u_1, u_2) be an edge that has to be replaced by a path $\overline{P} = u_1, w_1, w_2, \ldots, w_k, u_2$. For each vertex w_i to be added ($i = 1, 2, \ldots, k$) we add an *image point* to the drawing. The image points are added as follows. By property 11.a of Lemma 11, the

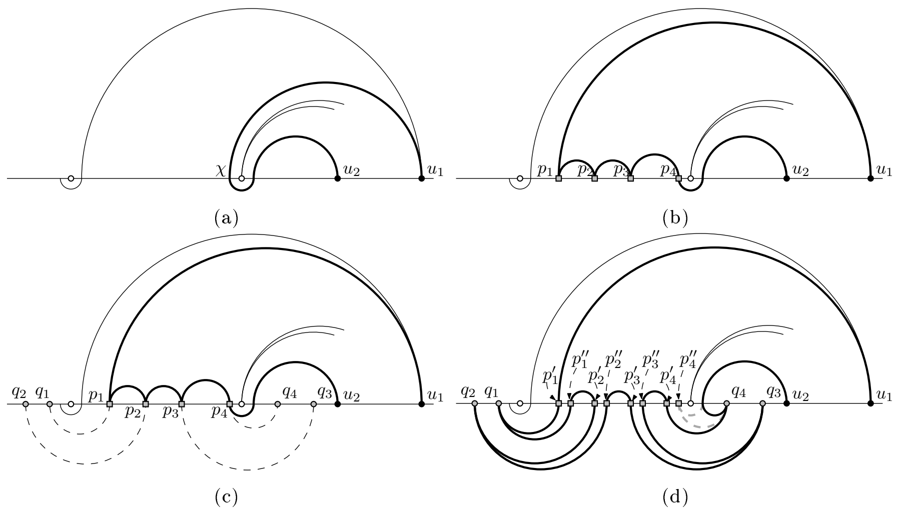


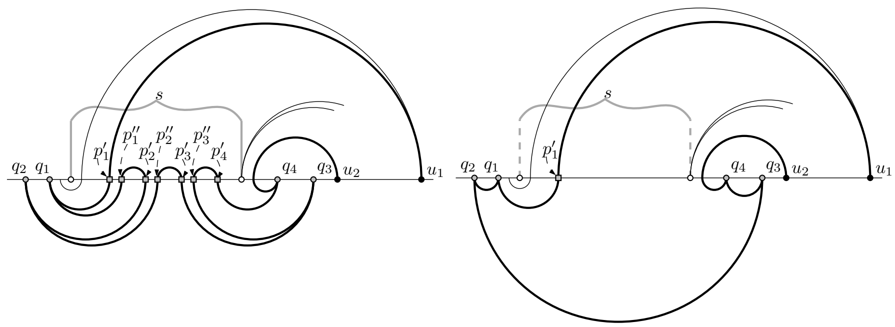
Figure 15: Addition of the vertices of color c_2 to create a topological book embedding of a 3-colored path.

edge (u_1, u_2) crosses the spine at least once. Let χ be the point where (u_1, u_2) crosses the spine for the first time when going from u_1 to u_2. By property 11.c of Lemma 11, χ is visible from below. This means that there is a segment s of ℓ with χ as an endpoint that is visibile from below. We place $k-1$ image points $p_1, p_2, \ldots, p_{k-1}$ inside this segment, while χ is the k-th image point p_k (it is the leftmost if s is to the left of χ, while it is the rightmost if s is to the right of χ). See Fig. 15(b) for an illustration. The first arc of the edge (u_1, u_2) is replaced by an arc connecting u_1 to p_1; each image point p_i is connected to the image point p_{i+1} $(i = 1, 2, k-1)$ by means of an arc in the top page; finally, the last image point p_k is already connected to u_2 by means of the remaining part of the original edge (u_1, u_2). Notice that the edge (u_1, p_1) does not cross the spine, and the same is true for any edge (p_i, p_{i+1}), while the edge (p_k, u_2) crosses the spine at most once (the original edge had at most two spine crossing one of which was at $\chi = p_k$). We have replaced the edge (u_1, u_2) with a path $\pi = \langle u_1, p_1, p_2, \ldots, p_k, u_2 \rangle$ with $k+1$ edges, as needed. However, the points representing the intermediate vertices of this path are not the points of the set Q. The idea then is to "connect" the image points to the points of Q. To this aim, we add matching edges in the bottom page between the image points and the points of Q. Since both the points of Q and the image points are visible from below, these matching edges do not cross any other existing edge. Moreover, by using a simple brackets matching algorithm, we can add the matching edges so that they do not cross each other. We finally use the matching edges to create the actual path that will represent $\overline{P}$. We "cut" the path π at each image point p_i and replace it with two consecutive points p'_i and p''_i. The edge (u_1, p_1) becomes an edge (u_1, p'_1), each edge (p_i, p_{i+1}) becomes an edge (p''_i, p'_{i+1}), and the edge (p_k, u_2) becomes an edge (p''_k, u_2). Denote by q_i the point of Q matched with p_i. For each q_i we add two edges (q_i, p'_i) and (q_i, p''_i) following the matching edge (p_i, q_i). In this way we have a path $u_1, p'_1, q_1, p''_1, p'_2, q_2, p''_2, \ldots, p'_k, q_k, p''_k, u_2$. The arc (q_k, p''_k) and the first arc of (p''_k, u_2) are both in the same page; thus we remove p''_k and connect q_k directly to the second arc of (p''_k, u_2). The points p'_i and p''_i $(i = 1, 2, \ldots, k)$ do not represent vertices but spine crossings, while the points q_i represent vertices of P. The resulting drawing is planar by construction and therefore it is a topological book embedding Γ of P.

Proof of property 12.a. Property 12.a holds for every edge of Γ that is also in Γ' by property 11.a of Lemma 11. Consider now the edges of a path $\overline{P} = u_1, w_1, w_2, \ldots, w_k, u_2$ that replace an edge (u_1, u_2) and recall that each vertex w_i is drawn at point q_i. The first edge from u_1 to q_1 has a spine crossing at p'_1; the edge from p_k to u_2 possibly has a spine crossing of the replaced edge (u_1, u_2); finally, each edge from q_i to q_{i+1} $(i = 1, 2, \ldots, k)$ has two spine crossings at p''_i and at p'_{i+1}. Moreover, at least one of the two spine crossings is flat. More precisely, if q_i and q_{i+1} are both to the left of p'_1, then p''_i is flat (see for example p''_1 in Fig. 15(d)); if q_i and q_{i+1} are both to the right of p_k, then p'_{i+1} is flat (see for example p'_4 in Fig. 15(d)); if q_i is to the left of p'_1 and q_{i+1} is to the right of p_k, then both p''_i and p'_{i+1} are flat (see for example p''_2 and p'_3 in Fig. 15(d)). Thus, property 12.a holds for all edges of Γ.

Proof of property 12.b. The first vertex v_1 of P is visible from above in Γ' by property 11.d of Lemma 11 (notice that by our choice of c_2, v_1 is the first vertex of P'). The only arcs added in the top page are those connecting p''_i to p'_{i+1}, which do not cover any existing vertex. Thus v_1 of P remains visible from above also in Γ. The last vertex v_n of P is hook visible from the right in Γ' by property 11.d of Lemma 11 (also in this case our choice of c_2 guarantees that v_n is the last vertex of P'). The hook visibility of v_n implies that for every point p_1 of the right access interval I of v_n and for every point p_2 of the rightmost spine portion s'_r (notice that the access spine portion of v_n and s'_r may coincide) we can add the two arcs (v_n, p_1) and (p_1, p_2) without creating any crossings between edges. By properties 11.e and 11.f of Γ', there is no spine crossing and at most one vertex between p_1 and p_2. This implies that no image point is added between p_1 and p_2; furthermore, if some point of the set Q is added inside I or s_r, it is always possible to choose a point p_1 of I and a point p_2 of s_r such that no point of Q is added between them. As a consequence, the two arcs (v_n, p_1) and (p_1, p_2) can be added to Γ without creating any crossings between edges. Moreover, p_2 is visible from above in Γ' because it belongs to s_r. Since no image point is added inside s_r (because no spine crossings belongs to s_r), none of the edges added in the top page defines an interval that contains p_2, which is therefore visible from above also in Γ. It follows that v_n is doubly hook visible and therefore property 12.b holds.

Proof of property 12.c. The construction of Γ shown so far may not satisfy this property. Let s be a spine portion of Γ; either s coincides with a spine portions s' of Γ' (if no point of Q is added inside s') or it is a sub-interval of s' obtained by splitting s' with points of Q. Spine portion s' will be called the *parent spine portion* of s. We now explain how to modify Γ to a new topological book embedding Γ'' that still satisfies properties 12.a and 12.b and such that



(a) (b)

Figure 16: Reduction of the spine crossings of a spine section.

each spine portion has no more spine crossings than its parent spine crossing.

Let s be a spine portion of Γ, and let s' be its parent spine portion. Assume first that no point of Q is added inside s', i.e., that $s = s'$. If s is not crossed by a dummy edge (u_1, u_2), then the number of spine crossings in s in Γ is the same as in Γ' (we create new spine crossings in Γ only when replacing dummy edges with paths). Suppose now that s' is crossed in Γ' by a dummy edge (u_1, u_2). Since (u_1, u_2) is dummy, it is replaced in Γ by a path $\overline{P} = u_1, w_1, w_2, \dots, w_k, u_2$. See, for example, Fig. 16(a). In Γ the spine crossing between (u_1, u_2) and s is replaced by $2k-1$ spine crossings (i.e, the points p'_i and p''_i). Thus, s has more than two spine crossings in Γ. We modify the drawing of each edge (w_i, w_{i+1}) of $\overline{P}$ so that s is crossed only once by the edges of $\overline{P}$ as shown, for example, in Fig. 16(b). Edge (w_i, w_{i+1}) consists of three arcs in Γ: arc (q_i, p''_i), arc (p''_i, p'_{i+1}), and arc (p'_{i+1}, q_{i+1}). Points p''_i and p'_{i+1} are two spine crossings and they are consecutive along the spine (i.e. there is no vertex or spine crossing between them). We replace the three arcs with a single arc (q_i, q_{i+1}) in the bottom page. We claim that no edge crossing is created. Replacing (q_i, p''_i), (p''_i, p'_{i+1}), and (p'_{i+1}, q_{i+1}) with (q_i, q_{i+1}) in the same page as (q_i, p''_i) and (p'_{i+1}, q_{i+1}) is topologically equivalent to moving arc (p''_i, p'_{i+1}) to the page opposite to the one where it was in Γ. This means that a crossing can be created only with arcs having an endpoint between p''_i and p'_{i+1}. Since p''_i and p'_{i+1} are consecutive, no such arc exists. When this transformation is applied to all edges (w_i, w_{i+1}) of $\overline{P}$, path $\overline{P}$ has only one remaining spine crossing inside s, namely the spine crossing at p'_1. When $\overline{P}$ was added to Γ' to obtain Γ, a spine crossing, denoted χ above, was removed. It follows that the number of spine crossings in s stays the same in Γ' and in Γ''.

Assume now that some points of Q are added inside s' and therefore s is one of the spine portions obtained by splitting s. At most two of these new spine portions inherit form s one spine crossing each. The same argument as above applies to these two spine portions. Observe that all other spine portions obtained by splitting s have no spine crossings. It follows that the number of spine crossings in s is at most the number of spine crossing in its parent spine

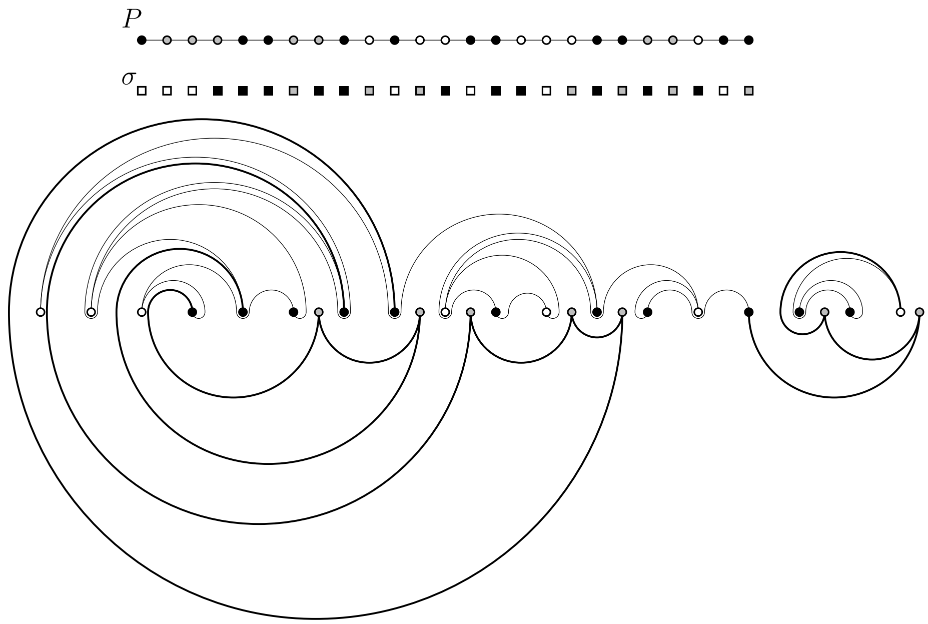


Figure 17: A topological book embedding of a 3-colored path consistent with a given 3-colored sequence constructed according to the algorithm described in the proof of Lemma 12.

portion s'.

From the argument above, both in the case that $s = s'$ and in the case that $s \subset s'$, the number of spine crossings in s is at most the one in s'. Thus, s has at most two spine crossings by property 11.f of Lemma 11. By the same property, if s is the leftmost spine portion it has at most one spine crossing and if it is the rightmost it has no spine crossing. Consider now the two access spine portions s_1 and s_2 of v_n and assume that they do not coincide with the rightmost spine portion (otherwise we have already seen that they do not contain any spine crossings). Assume s_1 to be on the left of s_2. As described in the proof of property 11.b, the parent spine portion of s_1 is the access spine portion of v_n in Γ'. From property 11.f of Lemma 11 it follows that s_1 has at most one crossing. The parent spine portion of s_2 is the rightmost spine portion of Γ' and therefore s_2 has at most one spine crossing (actually it has zero spine crossings). This concludes the proof that property 12.c holds for Γ''. We conclude by remarking that the transformation of Γ in Γ'' does not cause any violation of the properties 12.a and 12.b proved above, which therefore still hold for Γ''. $\square$

Fig. 17 shows a topological book embedding of a 3-colored path P consistent with a given 3-colored sequence σ constructed according to the algorithm described in the proof of Lemma 12. The book embedding is constructed starting from the one shown in Fig. 13, which is a topological book embedding of the path obtained from P by replacing sub-paths of one color (gray in the picture) with edges. We conclude this section by showing how Lemma 12 can be used to prove two theorems about 3-colored point-set embeddability. Other applications of Lemma 12 to 3-colored simultaneous embeddability will be given in Section 5.

Theorem 2. *Every* 3*-colored path admits a* 3*-colored point-set embedding with curve complexity at most* 4 *on any compatible* 3*-colored point set.*

Proof. Let P be a 3-colored path and let S be a 3-colored point set compatible with P. By property 12.a of Lemma 12, P admits a 3-colored topological book embedding consistent with $seq(S)$ with at most two spine crossings per edge and at most one flat spine crossing per edge. This together with Lemma 9 implies the statement. □

Theorem 2 can be extended to a subclass of 3-colored caterpillars.

Theorem 7. *Every* 3*-colored caterpillar with monochromatic leaves admits a* 3*-colored point-set embedding with curve complexity at most* 4 *on any compatible* 3*-colored point set.*

Proof. Let C be a 3-colored caterpillar with monochromatic leaves and let S be a compatible point set. As in the case of 3-colored paths, we first remove from C the vertices of one color. In particular, we remove the vertices with the color of the leaves; denote this color as c_2. Suppose first that the end-vertices of the backbone[2] P of C have a color distinct from c_2 (the case when the end-vertices of the backbone have color c_2 will be reduced to this case by suitably adding dummy vertices and points). The leaves are simply removed from C while the subpaths of P whose color is c_2 are replaced by dummy edges as in the case of paths (notice that under the assumption that no end-vertex of P is colored c_2 this replacement is always possible). Let P' be the resulting 2-colored path and let σ' be the 2-colored sequence obtained from $\sigma = seq(S)$ by removing all elements of color c_2.

By Lemma 11, P' admits a topological book embedding Γ' consistent with σ' that satisfies the properties 11.a–11.f. As in the proof of Lemma 12 we add to Γ' a set Q of points colored c_2 placed so that the sequence of colors along the spine coincides with σ. By property 11.b of Γ', all these points can be placed so that they are visible from below. We now have to remove the dummy edges of P' and re-insert the vertices colored c_2. The approach is very similar to the one used for paths. The difference is that each dummy edge has to be replaced with a subgraph that is a caterpillar instead of a path. Also, there can be leaves that must be attached to vertices of P' with color different from c_2.

We consider first the leaves that have to be attached to vertices with color different from c_2. Let u_1 be one such vertex and suppose that there are k leaves

[2] The path obtained from a caterpillar by removing all its leaves is usually called *spine*. To avoid confusion with the spine of the book embedding, we call it *backbone*.

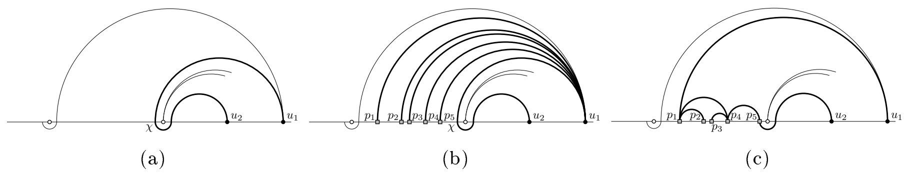


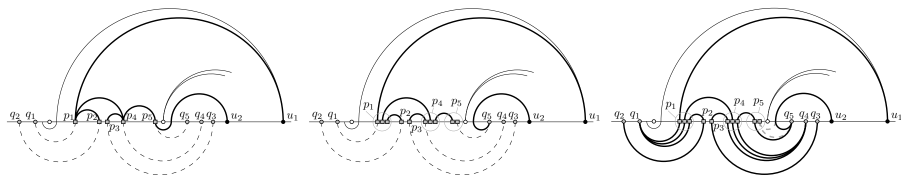
Figure 18: Addition of the image points. (a) An edge (u_1, u_2) that crosses the spine for the first time at χ; (b) $k = 5$ leaves have been attached to v; (c) the edge (u_1, u_2) has been replaced by a 1-page book embedding of a caterpillar with $k = 5$ vertices.

to be attached to u_1. Let (u_1, u_2) be the edge connecting u_1 to its predecessor or to its successor in P'. By property 11.a of Lemma 11, the edge (u_1, u_2) crosses the spine at least once. Let χ be the point where (u_1, u_2) crosses the spine for the first time when going from u_1 to u_2. By property 11.c of Lemma 11, χ is visible from below. This means that there is a segment s of ℓ with χ as an endpoint that is visible from below. We place k image points inside this segment and connect them to u_1. This can be done without creating any crossing since the added edges follow the profile of the arc from u_1 to χ (see Fig. 18(b)).

Consider now an edge that has to be replaced by a caterpillar C' with k vertices (notice that C' contains u_1 and u_2 as leaves attached to the first and last vertex of the backbone of C'). Denote this edge as (u_1, u_2) and let χ and s be defined as in the previous case. We add k image points inside s and, as done in the proof of Lemma 12, the first arc of the edge (u_1, u_2) is replaced by an arc connecting u_1 to p_1, while the last image point p_k is already connected to u_2 by means of the remaining part of the original edge (u_1, u_2). Finally, let Γ'' be a 1-page book embedding of C' with the property that u_1 and u_2 are the first vertex and the last vertex of Γ'' (such an embedding always exists). We "embed" Γ'' on the image points, placing all the edges on the top page (see Fig. 18(c)).

Once all vertices colored c_2 have been added to the drawing using image points, we add the matching edges as done in the proof of Lemma 12 and use them to construct the final drawing. The idea is similar the one used in Lemma 12, the difference being that the vertices mapped to the image points can have degree larger than two (while in the case of Lemma 12 they all have degree two). Each image point p_i is replaced by a set of consecutive points in number equal to the degree of the vertex mapped to p_i; each of the edges incident to this vertex is connected to one of the points replacing p_i, paying attention to guarantee that the left-to-right order of these edges is preserved so to not create crossings (see Fig. 19(b)). Denote by q_i the point of Q matched with p_i; all the points replacing p_i are connected to q_i following the matching edge (see Fig. 19(c)). Also in this case the last point replacing the last image point p_k has two incident arcs that are in the same page and that are removed.

As for the case of paths, the final drawing is planar, each edge crosses the spine at most twice, and if an edge crosses the spine twice at least one of the two spine crossings is flat which, by Lemma 9, implies the statement for the case when the backbone of C has end-vertices whose color is not c_2



(a) (b) (c)

Figure 19: Addition of the matching edges and construction of the final drawing. (a) Each image point p_i is matched to a point q_i of Q; (b) Each image point p_i is replaced by a set of points in number equal to the number of incident edges. (c) Matching edges are used to construct the drawing.

If the backbone P of C has one or two end-vertices colored c_2, then we modify C by attaching to each end-vertex of the backbone colored c_2 a dummy leaf with a color c_1 different from c_2. The obtained caterpillar C^* has all leaves of the same color c_2 except for the dummy ones. We also add one or two dummy points to the point set S so that C^* and the new point set S^* are compatible. If we attached a dummy leaf to the beginning of the backbone of C then we add a dummy point with x-coordinate smaller than any other point; if we attached a dummy leaf to the end of the backbone of C then we add a dummy point with x-coordinate greater than any other point. With this choice the dummy points appear at the beginning and/or at the end of the sequence $\sigma^* = seq(S^*)$. Once these additions have been made we can remove the leaves colored c_2 as in the previous case, but we do not remove the dummy leave whose color is c_1. We are left with a "backbone" P^* (actually it is the backbone of C plus one or two dummy leaves at the end) whose end-vertices are not colored c_2. We can now proceed as in the previous case and compute a 3-colored point-set embedding Γ^* of C^* with curve complexity four. To obtain a 3-colored point-set embedding of C it is sufficient to remove the dummy leaves and the dummy points. This is possible because the dummy leaves are mapped to the dummy points. Namely, the algorithm of Lemma 11 is such that if the first vertex of a 2-colored path has the same color of the first element of the 2-colored sequence, then the first vertex is mapped to the first element of the sequence; analogously, if the last vertex of the path has the same color of the last element of the sequence, then the last vertex is mapped to the last element of the sequence. Based on our choice of the additional points this is our case and therefore the dummy leaves can be safely removed from Γ^*. □

5. *h*-planar *k*-colored Simultaneous Embeddings

Two k-colored planar graphs are *compatible* if they have the same number of vertices for each color. A *k-colored simultaneous embedding* of two compatible k-colored graphs G_1 and G_2 is a pair of k-colored point-set embeddings Γ_1 and Γ_2 on the same k-colored point set where Γ_i represents G_i ($i = 1, 2$). Such k-colored simultaneous embedding is *h-planar* if every edge of $\Gamma_1 \cup \Gamma_2$ is crossed at most h times and no two edges cross more than once.

We start by establishing lower bounds on the number h of crossings per edge in $\Gamma_1 \cup \Gamma_2$ when one of the two graphs is a structurally very simple graph. Namely, we exploit the following technical lemma.

Lemma 13. *Let h, k be any two positive integers and let G_1 and G_2 be a pair of compatible k-colored planar graphs such that G_2 is a cycle. If the pair G_1 and G_2 admits an h-planar k-colored simultaneous embedding then G_1 has a k-colored point-set embedding with curve complexity $2h+1$ on any compatible point set.*

PROOF. Since G_2 is a cycle, the h-planar k-colored simultaneous embedding $\Gamma_1 \cup \Gamma_2$ can be regarded as a k-colored Hamiltonian augmentation of G_1 with at most h division vertices per edge. In the worst case none of them is flat. Thus, Theorem 3 implies that G_1 has a k-colored point-set embedding with at most $2h+1$ bends per edge on any compatible point set. □

The following results can be proved using Lemma 13.

Theorem 8. *There exists a 2-colored planar triangulation G_1 and a compatible 2-colored cycle G_2 such that for every h-planar 2-colored simultaneous embedding of G_1 and G_2, we have $h \in \Omega(n)$.*

PROOF. It is known that there exists a 2-colored planar triangulation G and a compatible set of point S such that in every 2-colored point-set embedding of G on S, there exist $\Omega(n)$ edges each having $\Omega(n)$ bends [5]. Let G_1 coincide with G and let G_2 be a cycle. If G_1 and G_2 admitted an h-planar 2-colored simultaneous embedding with $h \in o(n)$, then by Lemma 13, G_1 would admit a 2-colored point-set embedding with curve complexity $o(n)$ on any compatible point set, including S – a contradiction. □

The next two theorems can be proved using the same argument as the one of Theorem 8 together with Theorem 1 (for the first one) and a result by Pach and Wenger [35] (for the second one) stating that for every n there exists an n-colored matching G and an n-colored point set such that, in every point-set embedding of G on S, there exist $\Omega(n)$ edges each having $\Omega(n)$ bends

Theorem 9. *There exists a 3-colored caterpillar G_1 and a compatible 3-colored cycle G_2 such that for every h-planar 3-colored simultaneous embedding of G_1 and G_2, we have $h \in \Omega(n^{\frac{1}{3}})$.*

Theorem 10. *There exists an n-colored matching G_1 and a compatible n-colored cycle G_2 such that for every h-planar n-colored simultaneous embedding of G_1 and G_2, we have $h \in \Omega(n)$.*

We now prove some upper bounds on the number of crossings per edge in $\Gamma_1 \cup \Gamma_2$ when both G_1 and G_2 are cycles.

Theorem 11. *A compatible pair of 3-colored cycles admits a 2-planar 3-colored simultaneous embedding with at most 4 bends per edge.*

Figure 20: Proof of Theorem 11: Addition of the edge (v_1, v_{n_1}).

PROOF. Let C_1 and C_2 be two 3-colored compatible cycles. For $i = 1, 2$, let P_i be the path obtained by arbitrarily removing one edge e_i from cycle C_i. Let σ be the sequence of colors of the vertices of P_2. By property 12.a of Lemma 12, P_1 admits a topological book embedding Γ'_1 consistent with σ such that every edge has at most two spine crossings and at most one flat spine crossing; moreover every spine portion has at most two spine crossings (property 12.c). We now explain how to add the edge e_1 to Γ'_1 so as to obtain a topological book embedding of C_1. By property 12.b of Lemma 12, the first vertex v_1 of P_1 is visible from above while the last vertex v_n is doubly hook visible from the right. Hence, we can add edge $e_1 = (v_1, v_n)$ to Γ'_1 without introducing any edge crossing (see Fig. 20). Namely, by the doubly hook visibility, v_n can be connected to a point visible from above by crossing the spine at most twice (if v_n is visible from above then no spine crossing is necessary). We thus obtain a topological book embedding Γ''_1 of C_1 compatible with σ such that each edge of C_1 has at most two spine crossings and at most one flat spine crossing, and each spine portion is crossed at most twice. Notice that by property 12.c the access spine portions of v_n have at most one spine crossing each in Γ'_1 and the addition of the edge (v_1, v_n) introduces one spine crossing in each access spine portion. Thus they have at most two spine crossings in Γ''_1.

Let S be a 3-colored point set lying on a horizontal line ℓ such that $seq(S) = \sigma$. By Lemma 9, C_1 admits a 3-colored point-set embedding on S with at most four bends per edge. In particular such a point-set embedding can be obtained as follows. Replace each semi-circle α representing an arc of an edge in Γ' with a 1-bend polyline whose endpoints are the endpoints of α and whose bend is the middle point of α. With this construction, the segments of each polyline are drawn with slope $+1$ or -1. This implies that two arcs in the same page and incident on a common endpoint overlap. Let (p, q) and (p, r) be two such arcs. Without loss of generality assume that p, q and r appear in this order along ℓ (refer also to Fig. 21). Consider the line ℓ' passing through p with slope $1-\varepsilon$, for suitably small $\varepsilon > 0$; let p' be the intersection point between ℓ' and the line with slope -1 passing through q. Re-drawing the arc (p, q) as the polyline connecting p, p', and q removes the overlap between (p, q) and (p, r) and does not create any crossing between them. On the other hand, by choosing ε small enough we can guarantee that (p, q) does not cross any other existing arc. By

(a) (b)

Figure 21: Proof of Theorem 11: Removal of edge overlaps.

applying this re-drawing to all pairs of overlapping arcs[3], we obtain a 3-colored point-set embedding Γ_1 of C_1 on S. The number of bends per edge is at most 4. Namely, since each edge has at most two spine crossings in Γ'_1, it consists of at most three arcs in Γ'_1. For each of these arcs there is one bend in Γ_1 and an additional bend can exist at each spine crossing, which would give an upper bound of five to the number of bends per edge in Γ_1. However one of the two spine crossings is flat and flat spine crossings do not create additional bends. Namely, let p be a flat spine crossing and let (p,q) and (p,r) be the two arcs that have p as a common endpoint. By definition of flat spine crossings, q is to the left of p and r is to the right of p (or vice versa) and the two arcs are in different pages. This implies that the two segments incident to p have the same slope and are not rotated to remove overlaps. Hence, no additional bend is created at p and the number of bends per edge in Γ_1 is at most four.

Since $seq(S) = \sigma$, connecting each point of S to the next one we obtain a drawing Γ'_2 of P_2 which, together with Γ_1, forms a 3-colored simultaneous embedding of C_1 and P_2. Since in Γ_1 each edge of C_1 crosses the spine (and therefore the edges of P_2 in Γ'_2) at most twice and each spine portion (and therefore each edge of P_2 in Γ'_2) is crossed at most twice, the simultaneous embedding of C_1 and P_2 is 2-planar. To obtain a simultaneous embedding of C_1 and C_2 it remains to add to Γ'_2 the edge e_2 removed to obtain P_2. Edge e_2 connects the first vertex u_1 of P_2 to its last vertex u_n, which are the leftmost vertex and the rightmost vertex in the drawing, respectively. Edge e_2 can be drawn with three bends: the first bend is a point p_1 of ℓ to the left of any other point in the drawing; the second bend is a point p_2 of ℓ to the right of any other point in the drawing; finally, the third bend is the intersection point p_3 of the line with slope $+1$ through p_1 and the line with slope -1 through p_2. The resulting drawing Γ_2 of C_2 is clearly planar. We now show that the added edge intersects Γ_1 in at most two points. The choice of the slopes of segments $\overline{p_1p_3}$ and $\overline{p_3p_2}$ guarantees that these two segments do not cross any edge of Γ_1. Segment $\overline{p_1v_1}$ is crossed by the edges of Γ_1 that cross the leftmost spine portion in Γ'_1 (notice that Γ_1 intersects line ℓ at the points representing the spine crossings of Γ''_1, which coincide with those of Γ'_1 except possibly for the two spine crossings added inside the access spine portions of v_n). By property 12.c of Lemma 12 there is only one edge that crosses the leftmost spine portion of Γ'_1. Concerning segment $\overline{u_np_2}$, in Γ'_1 there is no spine crossing to the right of

[3]If the two overlapping arcs have slope -1, the slope of the redrawn edge has to be $-1+\varepsilon$.

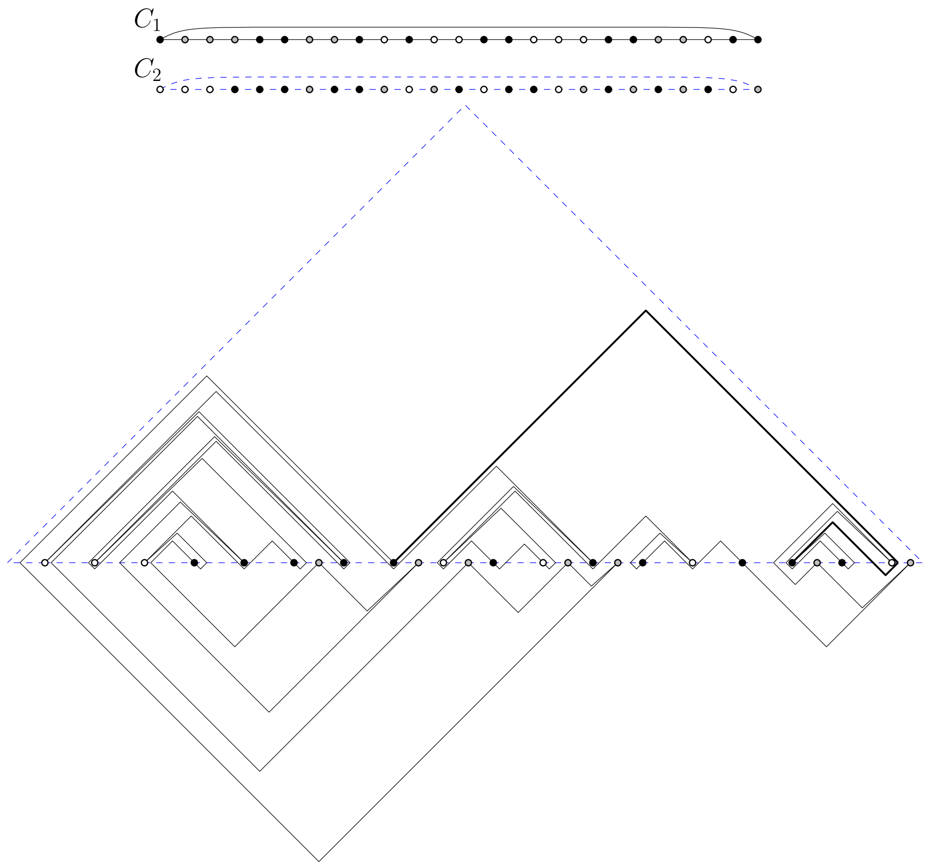


Figure 22: A 2-planar 3-colored simultaneous embedding of two 3-colored cycles with at most 4 bends per edge constructed according to the algorithm described in the proof of Theorem 11.

u_n by property 12.c of Lemma 12. When we create Γ_2', we possibly add a spine crossing inside the rightmost spine portion s_r of Γ_1', i.e. to the right of u_n (this happens if s_r is one of the access spine portions of v_n). Thus, segment $\overline{u_n p_2}$ has at most one spine crossing. It follows that edge e_2 crosses the edges of Γ_1 at most twice and therefore Γ_1 and Γ_2 form a 2-planar 3-colored simultaneous embedding of C_1 and C_2 with at most four bends per edge. □

Fig. 22 shows a 2-planar 3-colored simultaneous embedding of two 3-colored cycles with at most 4 bends per edge constructed according to the algorithm described in the proof of Theorem 11. The simultaneous embedding is constructed starting from the one shown in Fig. 17, which is a topological book embedding of the path obtained from C_1 by removing one edge (shown in bold in Fig. 22).

6. Concluding Remarks and Open Problems

In this paper we have established a lower bound on the curve complexity of k-colored point-set embeddings when $k = 3$ and the graph is acyclic. More precisely, Theorem 5 proves $\Omega(n^{\frac{1}{3}})$ bends on $\Omega(n^{\frac{2}{3}})$ edges. This settles an open problem by Badent et al. [5] about the curve complexity of point set embeddings of k-colored trees and it extends a lower bound by Pach and Wenger [35] to the case that the graph has only $O(1)$ independent edges. Since $O(n)$ bends are always sufficient to compute a point-set embedding of a 3-colored caterpillar [5, 35], we have the following.

Problem 1. *Establish whether the lower bound of Theorem 5 is tight.*

Motivated by the lower bound of Theorem 5, Theorem 7 describes a family of 3-colored caterpillars for which a constant number of bends per edge is always sufficient. Further exploring this topic is a natural question. Namely, we have the following.

Problem 2. *Characterize the* 3*-colored caterpillars which admit a* 3*-colored point-set embedding with constant curve complexity on any compatible set of points.*

In contrast to the case of caterpillars, 3-colored paths always have a 3-colored point-set embedding with constant curve complexity on any given set of points (see Theorem 2). On the other hand, there exist n-colored paths whose point set embeddings may require $\Omega(n)$ bends per edge.

Problem 3. *Do* 4*-colored paths always admit a point-set embedding with constant curve complexity on any compatible set of points?*

Problem 4. *Let k be a positive integer. Is there a function $f(k)$ such that every k-colored path has a k-colored point-set embedding with curve complexity $f(k)$ on any compatible set of points?*

The following theorem gives a partial answer to Problem 3. A 4-colored path is *separable* if, after removing one edge, it can be split into two 2-colored paths. A 4-colored point set is *separable* if there exists a vertical line that splits the point set into two 2-colored point sets.

Theorem 12. *Let P be a* 4*-colored path with n vertices, and let S be a* 4*-colored point set compatible with P. If P is separable, it has a* 4*-colored point-set embedding on S with curve complexity at most* 4*. If S is separable, then P has a* 4*-colored point-set embedding on S with curve complexity at most* 5*.*

Proof. Suppose first that P is separable with respect to edge (u_1, u_2), and let P_1 and P_2 be the two 2-colored paths obtained by removing (u_1, u_2) from P. Orient each path P_i so that u_i is the first vertex, and let u_{n_i} be the last vertex of P_i $(i = 1, 2)$. Let $\sigma = seq(S)$, and let σ_i be the 2-colored sequence obtained from σ considering only the elements colored with the colors of P_i. By Lemma 11, P_i has a topological book embedding Γ_i consistent with σ_i such that properties 11.a–11.f hold. By mirroring Γ_2 vertically we obtain a topological book embedding Γ'_2 consistent with σ_2 such that: (i) every spine portion is visible from above; (ii) vertex u_2 is visible from below. Since each spine portion of Γ_1 is visible from below and each spine portion of Γ'_2 is visible from above, Γ_1 and Γ'_2 can be arranged together by intermixing their vertices according to the order defined by σ and without creating any intersection between the two. We now add back the edge (u_1, u_2) removed from P to obtain P_1 and P_2. Since u_1 is visible from above and u_2 is visible from below, we can connect u_1 and u_2 with an edge that crosses the spine once to the left of the leftmost point/spine crossing of $\Gamma_1 \cup \Gamma'_2$. We obtain a topological book embedding of P consistent with σ with at most two spine crossing per edge and at most one flat spine crossing per edge. By Lemma 9, P has a 4-colored point-set embedding on S with curve complexity at most 4.

Suppose now that S is separable. The idea is to exchange the roles of P and S. Namely, let P' be a path such that the sequence of colors along the path coincides with $seq(S)$, and let σ' be a sequence of colors coinciding with the sequence of colors of the vertices of P along the path. Clearly, σ' is compatible with P'. By the argument above, P' admits a topological book embedding Γ' consistent with σ' such that properties 11.a–11.f hold. Since the first vertex u_1 of P' is visible from above and the last vertex u_n is hook visible from below, we can connect these two vertices with an edge that crosses the spine at the access spine portion of u_n and at the rightmost spine portion (if these two spine portions coincide, u_n and u_1 are both visible from above and can be connected without spine crossings). We obtain a topological book embedding Γ'' of a cycle C consistent with σ' such that each spine portion has at most two spine crossings. Since the colors in σ' appear in the same order as in P and those in P' appear as those in $seq(S)$, the topological book embedding Γ'' defines a 4-colored Hamiltonian augmentation of P consistent with $seq(S)$ such that there are at most two division vertices per edge (we cannot say anything about how many of them are flat). By Theorem 3, P admits a 4-colored point-set embedding on S with curve complexity at most 5. □

Fig. 23(a) shows a 4-colored topological book embedding of a separable 4-colored path with at most two spine crossings per edge constructed according to the algorithm described in the proof of Theorem 12. The solid edges are the edges of the path P_1, the dashed edges are the edges of the path P_2, and the bold edge is the edge (u_1, u_2).

An additional contribution of this paper is the study of the h-planar k-colored simultaneous embeddability problem, which can be regarded as a variant of the k-colored simultaneous embeddability problem where it is required that the union of the two drawings is an h-planar graph. Theorems 8, 9, and 10 establish lower bounds on the value of h for which a simultaneous h-planar embedding of two k-colored planar graphs exists. Concerning upper bounds, Theorem 11 exploits the techniques behind Theorem 2 to prove that any two compatible 3-

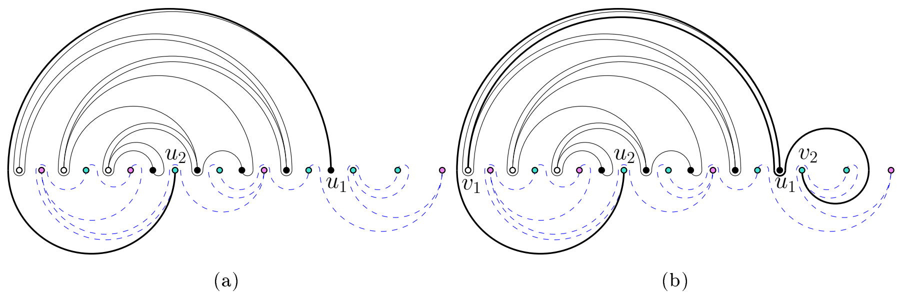


Figure 23: (a) A 4-colored topological book embedding of a separable 4-colored path with at most 2 spine crossings per edge constructed according to the algorithm described in the proof of Theorem 12. (b) A 4-colored topological book embedding of a separable 4-colored cycle with at most 3 spine crossings per edge constructed according to the algorithm described in the proof of Theorem 13.

colored cycles admit an 2-planar 3-colored simultaneous embedding. A natural question is therefore the following.

Problem 5. *Do any two compatible* 4*-colored cycles admit an* h*-planar* 4*-colored simultaneous embedding for constant* h *and with constant curve complexity?*

The following theorem is a preliminary result that sheds some light on Problem 5. A 4-colored cycle is *separable* if, after removing two edges, it can be split into two 2-colored paths.

Theorem 13. *A compatible pair of* 4*-colored cycles, one of which is separable, admits a* 3*-planar* 4*-colored simultaneous embedding with at most* 5 *bends per edge.*

PROOF. Let C_1 and C_2 be the two 4-colored cycles. Assume that C_1 is separable and let $P_{1,1}$ and $P_{1,2}$ be the two 2-colored paths obtained by removing two edges (u_1, u_2) and (v_1, v_2) from C_1. Let P_1 be the path obtained from C_1 by removing only the edge (u_1, u_2). Let P_2 be a 4-colored path obtained from C_2 by arbitrarily removing an edge, and let σ be the sequence of colors of the vertices of P_2. Path P_1 is separable and thus, as explained in the proof of Theorem 12, it admits a topological book embedding Γ consistent with σ with at most two spine crossing per edge and at most one flat spine crossing per edge. We now explain how to add the edge (v_1, v_2) so that it has at most three spine crossings and at most one flat spine crossing. Recall that Γ is obtained starting from two topological book embeddings Γ_1 and Γ_2 of $P_{1,1}$ and $P_{1,2}$, respectively, satisfying properties 11.a–11.f of Lemma 11; Γ_2 is then vertically mirrored, thus obtaining Γ_2', and finally the first vertex u_1 of $P_{1,1}$ in Γ_1 and the first vertex u_2 of $P_{1,2}$ in Γ_2' are connected by an edge that crosses the spine to the left of the leftmost point of $\Gamma_1 \cup \Gamma_2'$. By property 11.d, the last vertex v_1 of $P_{1,1}$ is hook visible from the right in Γ_1. Analogously, the last vertex v_2 of $P_{1,2}$ is hook visible from the right in Γ_2; this means that there exists a point p (actually an interval) that is

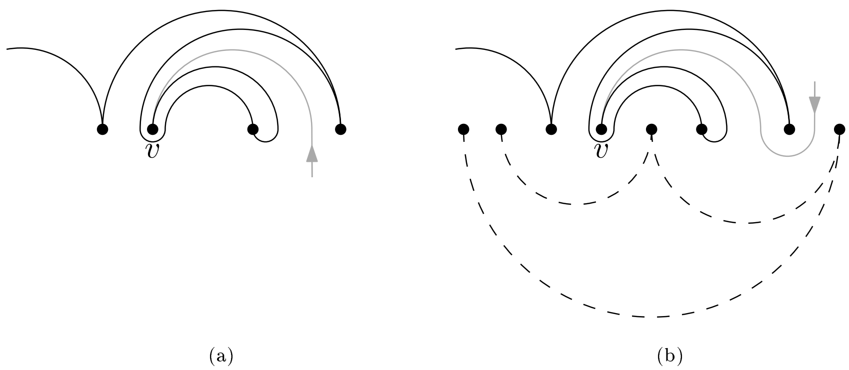


Figure 24: (a) A hook visible vertex v of; (b) Combination of Γ_1 and Γ'_2 (dashed): v becomes doubly hook visible.

visible from below and that can be connected to v_2 by means of an arc in the top page. It follows that in Γ'_2, point p is visible from above and can be connected to v_2 by means of an arc in the bottom page. Without loss of generality suppose that the last element of σ has one of the colors of $P_{1,2}$. In this case, when we combine Γ_1 and Γ'_2, the vertex v_1 is no longer hook visible because the visibility of v_1 is "blocked" by Γ'_2 (see Fig- 24 for an illustration). However, it becomes doubly hook visible (the situation is similar to that of Lemma 12 when some points of Q are added inside the rightmost spine portion). This means that there are two points q_1 and q_2 such that q_2 is visible from above and we can add arcs (v_1, q_1) in the top page and (q_1, q_2) in the bottom page without intersecting any existing edge. Since the point p of Γ'_2 remains visible from above after the combination of Γ_1 and Γ'_2, we can add the arc (q_2, p) without creating any edge crossing. We obtain a topological book embedding of C_1 consistent with σ where all edges except (v_1, v_2) have at most two spine crossings and at most one flat spine crossing, and edge (v_1, v_2) has at most three spine crossings and at most one flat spine crossing (the one at q_1).

With the same technique as the one described in the proof of Theorem 11, we can use the topological book embedding of C_1 to construct a 4-colored point-set embedding with at most five bends per edge on a point set S lying on a horizontal line ℓ such that $seq(S) = \sigma$. Connecting each point of S along ℓ to the next one, we obtain a drawing of P_2 and therefore a 4-colored simultaneous embedding of C_1 and P_2. Each edge of C_1 crosses the spine (and therefore the edges of P_2) at most three times. Also, each spine portion (and therefore each edge of P_2) is crossed at most twice. It follows that the simultaneous embedding of C_1 and P_2 is 3-planar. To obtain a simultaneous embedding of C_1 and C_2, it remains to add the edge connecting the first vertex of P_2 to its last vertex. This can be drawn with three bends and maintaining the 3-planarity of the drawing in the same way as explained in the proof of Theorem 11. □

Fig. 23(b) shows a 4-colored topological book embedding of a separable 4-colored cycle with at most 3 spine crossings per edge constructed according to the algorithm described in the proof of Theorem 13. The solid edges are the edges of the path $P_{1,1}$, the dashed edges are the edges of the path $P_{1,2}$, and the bold edges are the edges (u_1, u_2) and (v_1, v_2).

We conclude by mentioning that it would be interesting to study the optimization version of the problems studied in this paper. In particular, the problem of computing point-set embeddings where the total number of bends is minimum and the problem of computing h-planar k-colored simultaneous embeddings with the minimum number of bends and/or with the minimum value of h.

References

[1] M. Abellanas, J. Garcia-Lopez, G. Hernández-Peñver, M. Noy, and P. A. Ramos. Bipartite embeddings of trees in the plane. *Discrete Applied Mathematics*, 93(2-3):141–148, 1999.

[2] J. Akiyama and J. Urrutia. Simple alternating path problem. *Discrete Mathematics*, 84:101–103, 1990.

[3] P. Angelini, M. Geyer, M. Kaufmann, and D. Neuwirth. On a tree and a path with no geometric simultaneous embedding. *Journal of Graph Algorithms and Applications*, 16(1):37–83, 2012.

[4] E. N. Argyriou, M. A. Bekos, M. Kaufmann, and A. Symvonis. Geometric RAC simultaneous drawings of graphs. *Journal of Graph Algorithms and Applications*, 17(1):11–34, 2013.

[5] M. Badent, E. Di Giacomo, and G. Liotta. Drawing colored graphs on colored points. *Theoretical Computer Science*, 408(2-3):129–142, 2008.

[6] M. A. Bekos, T. C. van Dijk, P. Kindermann, and A. Wolff. Simultaneous drawing of planar graphs with right-angle crossings and few bends. *Journal of Graph Algorithms and Applications*, 20(1):133–158, 2016.

[7] T. Bläsius, S. G. Kobourov, and I. Rutter. Simultaneous embedding of planar graphs. In R. Tamassia, editor, *Handbook on Graph Drawing and Visualization.*, pages 349–381. Chapman and Hall/CRC, 2013.

[8] P. Bose, M. McAllister, and J. Snoeyink. Optimal algorithms to embed trees in a point set. *Journal of Graph Algorithms and Applications*, 2(1):1–15, 1997.

[9] U. Brandes, C. Erten, A. Estrella-Balderrama, J. J. Fowler, F. Frati, M. Geyer, C. Gutwenger, S. Hong, M. Kaufmann, S. G. Kobourov, G. Liotta, P. Mutzel, and A. Symvonis. Colored simultaneous geometric embeddings and universal pointsets. *Algorithmica*, 60(3):569–592, 2011.

[10] P. Braß, E. Cenek, C. A. Duncan, A. Efrat, C. Erten, D. Ismailescu, S. G. Kobourov, A. Lubiw, and J. S. B. Mitchell. On simultaneous planar graph embeddings. *Comput. Geom.*, 36(2):117–130, 2007.

[11] T. M. Chan, F. Frati, C. Gutwenger, A. Lubiw, P. Mutzel, and M. Schaefer. Drawing partially embedded and simultaneously planar graphs. In C. A. Duncan and A. Symvonis, editors, *Graph Drawing - 22nd International Symposium, (GD 2014)*, volume 8871 of *Lecture Notes in Computer Science*, pages 25–39. Springer, 2014.

[12] E. Di Giacomo, W. Didimo, G. Liotta, H. Meijer, F. Trotta, and S. K. Wismath. k-colored Point-set Embeddability of Outerplanar Graphs. *Journal of Graph Algorithms and Applications*, 12(1):29–49, 2008.

[13] E. Di Giacomo, W. Didimo, G. Liotta, and S. K. Wismath. Curve-constrained drawings of planar graphs. *Computational Geometry: Theory and Applications*, 30(1):1–23, 2005.

[14] E. Di Giacomo, L. Gąsieniec, G. Liotta, and A. Navarra. Colored point-set embeddings of acyclic graphs. In F. Frati and K. Ma, editors, *Graph Drawing and Network Visualization - 25th International Symposium, (GD 2017)*, volume 10692 of *Lecture Notes in Computer Science*, pages 413–425. Springer, 2017.

[15] E. Di Giacomo and G. Liotta. Simultaneous embedding of outerplanar graphs, paths, and cycles. *International Journal of Computational Geometry and Applications*, 17(2):139–160, 2007.

[16] E. Di Giacomo and G. Liotta. The Hamiltonian Augmentation Problem and Its Applications to Graph Drawing. In M. S. Rahman and S. Fujita, editors, *WALCOM: Algorithms and Computation, 4th International Workshop, (WALCOM 2010)*, volume 5942 of *Lecture Notes in Computer Science*, pages 35–46. Springer, 2010.

[17] E. Di Giacomo, G. Liotta, and F. Trotta. On Embedding a Graph on Two Sets of Points. *International Journal of Foundations of Computer Science*, 17(5):1071–1094, 2006.

[18] E. Di Giacomo, G. Liotta, and F. Trotta. Drawing Colored Graphs with Constrained Vertex Positions and Few Bends per Edge. *Algorithmica*, 57(4):796–818, 2010.

[19] W. Didimo, G. Liotta, and F. Montecchiani. A survey on graph drawing beyond planarity. *ACM Computing Surveys*, 52(1):4:1–4:37, 2019.

[20] F. Frati, M. Glisse, W. J. Lenhart, G. Liotta, T. Mchedlidze, and R. I. Nishat. Point-Set Embeddability of 2-Colored Trees. In W. Didimo and M. Patrignani, editors, *Graph Drawing - 20th International Symposium, (GD 2012(*, volume 7704 of *Lecture Notes in Computer Science*, pages 291–302. Springer, 2012.

[21] F. Frati, M. Hoffmann, and V. Kusters. Simultaneous embeddings with few bends and crossings. In E. Di Giacomo and A. Lubiw, editors, *Graph Drawing and Network Visualization - 23rd International Symposium, (GD 2015)*, volume 9411 of *Lecture Notes in Computer Science*, pages 166–179. Springer, 2015.

[22] M. Geyer, M. Kaufmann, and I. Vrto. Two trees which are self-intersecting when drawn simultaneously. *Discrete Mathematics*, 309(7):1909–1916, 2009.

[23] L. Grilli. On the NP-hardness of GRacSim drawing and k-SEFE problems. *Journal of Graph Algorithms and Applications*, 22(1):101–116, 2018.

[24] L. Grilli, S. Hong, J. Kratochvíl, and I. Rutter. Drawing simultaneously embedded graphs with few bends. In C. A. Duncan and A. Symvonis, editors, *Graph Drawing - 22nd International Symposium, (GD 2014)*, volume 8871 of *Lecture Notes in Computer Science*, pages 40–51. Springer, 2014.

[25] J. H. Halton. On the thickness of graphs of given degree. *Information Sciences*, 54:219–238, 1991.

[26] A. Kaneko and M. Kano. Straight-line embeddings of two rooted trees in the plane. *Discrete & Computational Geometry*, 21(4):603–613, 1999.

[27] A. Kaneko and M. Kano. Straight line embeddings of rooted star forests in the plane. *Discrete Applied Mathematics*, 101:167–175, 2000.

[28] A. Kaneko, M. Kano, and K. Suzuki. Path coverings of two sets of points in the plane. In J. Pach, editor, *Towards a Theory of Geometric Graphs*, volume 342 of *Contemporary Mathematics*. American Mathematical Society, 2004.

[29] A. Kaneko, M. Kano, and S. Tokunaga. Straight-line embeddings of three rooted trees in the plane. In *Canadian Conference on Computational Geometry, CCCG'98*, 1998.

[30] A. Kaneko, M. Kano, and K. Yoshimoto. Alternating hamilton cycles with minimum number of crossing in the plane. *International Journal of Computational Geometry & Application*, 10:73–78, 2000.

[31] M. Kaufmann and R. Wiese. Embedding vertices at points: Few bends suffice for planar graphs. *Journal of Graph Algorithms and Applications*, 6(1):115–129, 2002.

[32] S. G. Kobourov, G. Liotta, and F. Montecchiani. An annotated bibliography on 1-planarity. *Computer Science Review*, 25:49–67, 2017.

[33] A. Navarra and D. Cacciagrano. Energy saving and collision-free motion planning for oblivious robots. In *32nd International Conference on Advanced Information Networking and Applications Workshops, AINA 2018 workshops*, pages 591–596. IEEE Computer Society.

[34] J. Pach and J. Törőcsik. Layout of rooted tree. In W. T. Trotter, editor, *Planar Graphs (DIMACS Series in Discrete Mathematics and Theoretical Computer Science)*, volume 9, pages 131–137. American Mathematical Society, 1993.

[35] J. Pach and R. Wenger. Embedding Planar Graphs at Fixed Vertex Locations. *Graphs and Combinatorics*, 17(4):717–728, 2001.

[36] S. Tokunaga. On a straight-line embedding problem graphs. *Discrete Math.*, 150:371–378, 1996.