# An approach for evaluation of EVD and small failure probabilities of uncertain nonlinear structures under stochastic seismic excitations

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## Abstract

Efficient assessment of small first-passage failure probabilities of nonlinear structures with uncertain parameters under stochastic seismic excitations is an important but still challenging problem. In principle, the first-passage failure probabilities can be evaluated once the extreme value distribution (EVD) of studied structural response becomes available. With this in mind, this study presents a novel approach, termed as ***moment-generating function based mixture distribution*** (MGF-MD), for evaluation of the EVD. In this method, the MGF is firstly introduced to characterize the EVD, and the advantages of this characterization are highlighted. To calculate the MGF defined by a high-dimensional expectation integral, a low-discrepancy sampling technique, named Latinized partially stratified sampling (LPSS), is employed with a small sample size. Besides, the unbiasedness of the estimator is proven and the confidence interval is given. Then, a mixture of two generalized inverse Gaussian distributions (MTGIGD) with a closed-form MGF is proposed to approximate the EVD from the knowledge of its estimated MGF. The parameter estimation is conducted by matching the MGF of MTGIGD with seven values of the estimated one. Three numerical examples, including the EVD of random variables and reliability evaluation of two uncertain nonlinear structures subjected to fully non-stationary stochastic ground motions, are studied. Results indicate that the proposed approach can provide reasonable accuracy and efficiency and is applicable to very high-dimensional systems with small failure probabilities.

*Keywords:* Extreme value distribution, Small first-passage probability, Moment-generating function, Mixture distribution, Generalized inverse Gaussian distribution, Nonlinear structure, Stochastic seismic excitation

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**Abbreviations**

| | | | |
|---|---|---|---|
| C.O.V. | coefficient of variation | MCS | Monte Carlo simulation |
| CDF | cumulative distribution function | MD | mixture distribution |
| DOF | degree of freedom | MEM | maximum entropy method |
| DPIM | direct probability integral method | MGF | moment-generating function |
| EVD | extreme value distribution | MTGIGD | mixture of two generalized inverse Guassian distributions |
| FMNT | fourth-moment normal transformation | | |
| GEVD | generalized extreme value distribution | MTSND | mixture of two skew normal distributions |
| GIGD | generalized inverse Guassian distribution | PDEM | probability dnesity evoluation method |
| GPD | generalized Pareto distribution | PDF | probability density function |
| i.i.d. | independent and identically distributed | POE | probability of exceedance |
| IGD | inverse Gaussian distribution | SGLD | shifted generalized lognormal distribution |
| LPSS | Latinized partially stratified sampling | SS | subset simulation |
| LSS | Latinized stratified sampling | Std.D | standard deviation |

## 1. Introduction

Efficient assessment of reliability of engineering structures in the presence of various uncertainties is an important task not only in the entire design process, but also in the whole service life. In order to ensure the seismic safety, the dynamic reliability analysis of structures under stochastic seismic excitations is usually formulated as the well-known first-passage failure problem [1, 2]. That is, the failure occurs once the studied stochastic structural response exceeds the prescribed threshold during a given time interval. Many efforts have been devoted to evaluating the first-passage probability of stochastic dynamic systems, such as the out-crossing rate based methods [3, 4], stochastic averaging [5, 6], path integration [7, 8], etc. The first-passage problem nevertheless still remain one of the most difficult issues

in stochastic dynamics due to the existence of several challenges. First of all, uncertainties arising from both the earthquake loads and structural properties should be reasonably addressed simultaneously, which commonly results in high dimensions of the reliability problem to be solved. The performance functions of structures may also exhibit highly nonlinear behavior especially when subjected to strong earthquakes. Besides, as the structural systems become increasingly complex, the existence of multiple underlying failure modes with dependencies further increases the complexity of the problem. Apart from those challenges, the failure probabilities are typically expected to be very small, e.g., the order of magnitude $10^{-4}$ or smaller.

The extreme value distribution (EVD) of stochastic structural response has attracted considerable attention since the first-passage failure probabilities under different prescribed thresholds can be obtained equivalently through the associated EVD. Further, with the EVD, the time-dependent dynamic reliability problem can be conveniently converted into a time-invariant counterpart. In this setting, some well-known reliability analysis methods could be directly invoked to address the above-mentioned challenges, e.g., the Monte Carlo simulation (MCS) and Subset Simulation (SS) [9, 10] owing to their robustness to the dimension and complexity of the problem. However, the MCS is largely restricted by its computational cost, especially for a expensive-to-evaluate model with a low level of failure probability. The SS offers noticeably improved efficiency compared to the MCS, but is still far away from desirable for real-world applications. The present study particularly focuses on the evaluation of the entire EVD with efficiency and accuracy. Usually, two kinds of approaches have been developed for deriving the EVD of a stochastic structural response, i.e., the analytical approach and approximate approach. The analytical approaches are usually only applicable for some specific types of stochastic processes [11, 12, 13, 14], thus are infeasible for general engineering problems with the above-mentioned features/challenges. Alternatively, two major types of approximate approaches are available to capture the EVD: parametric approach and non-parametric approach. The basic idea of parametric approach is to assume that the extreme value samples coming from a population can be adequately modeled by a probability distribution with a fixed set of parameters. Based on the extreme value theory, the Gumbel distribution is used to model the main body and tail behavior of the EVD [15, 16], followed by the generalized extreme value distribution (GEVD) and generalized Pareto distribution (GPD) [17]. Among them, the GEVD could be more flexible, but still restricted by its limitations. Besides, some four-parameter distributions are also developed to recover the EVD from

the knowledge of its first-four moments, e.g., the fourth-moment normal transformation (FMNT) [18] and shifted generalized lognormal distribution (SGLD) [19]. Although these two models are generally versatile, the requirement of estimating the response moments with high accuracy and efficiency is still a challenging task. With the emergence of fractional moments, the maximum entropy method [20, 21, 22], kernel density maximum entropy method [23] and mixture distribution [24] are proposed. Despite these, other non-standard parametric methods can also be found in the literature [25, 26, 27]. For the non-parametric approach, assumptions on the distribution type of EVD are not requi. The probability density evolution method (PDEM) can be employed to evaluate the EVD by constructing a virtual stochastic process associated to the extreme value of the studied stochastic process [28, 29]. Recently, the direct probability integral method (DPIM) [30] is proposed for stochastic response analysis of static and dynamic structural systems based on the principle of probability conservation, which could also be used to capture the EVD of stochastic response process. The PDEM and DPIM are generally applicable to strongly nonlinear systems, which, however, are not easy to extend for problems with high-dimensional random inputs and small failure probabilities. In summary, the success of the parametric approach relies on assumptions of the distributional type and methods for parameter estimation. In other words, a good approximation can yield desirable results with less computational efforts. On the contrary, the non-parametric approach is distribution-free, but still suffers from several drawbacks.

The main objective of the present study is to develop an efficient approach for assessment of the EVD and small first-passage failure probabilities of nonlinear structures with uncertain parameters under stochastic seismic excitations. For this purpose, the moment-generating function (MGF) is firstly introduced to characterize the EVD, instead of the commonly-used integer moments and fractional moments. To calculate the MGF that is defined by a high-dimensional expectation integral, the latinized partially stratified sampling (LPSS) is employed with a uced number of samples due to its simultaneous variance uction associated with both main effects and variable interactions [31]. Then, a flexible mixture of two generalized inverse Gaussian distributions (MTGIGD) is proposed to recover the EVD from the knowledge of its estimated MGF, where a MGF-matching technique is developed for parameter estimation. Once the EVD is reconstructed in the entire distribution domain with reasonable accuracy, the first-passage failure probabilities under different thresholds can be obtained without difficulty.

The rest of this paper is arranged as follows. Section 2 outlines the EVD-based method for first-passage failure

probabilities evaluation of stochastic dynamic systems. The MGF is introduced in Section 3 to characterize the EVD, the advantages and numerical approximation of which are also elaborated. In Section 4, a mixture distribution is developed to recover the EVD from its estimated MGF. Three numerical examples are investigated to validate the proposed approach in Section 5. Some concluding remarks are available in Section 6.

## 2. First-passage probability: an EVD perspective

Without loss of generality, consider a generic stochastic dynamic structural system governed by the following state equation and initial condition:

$$\dot{\boldsymbol{X}} = \boldsymbol{A}(\boldsymbol{X}, \boldsymbol{\Theta}, t),\ \boldsymbol{X}(t)|_{t=0} = \boldsymbol{x}_0 \tag{1}$$

where $\boldsymbol{X} = (X_1, X_2, \ldots, X_d)^{\mathrm{T}}$ is a $d$-dimensional state vector; $\boldsymbol{A} = (A_1, A_1, \ldots, A_d)^{\mathrm{T}}$ is the state mapping; $\boldsymbol{\Theta} = (\Theta_1, \Theta_2, \ldots, \Theta_n)^{\mathrm{T}}$ denotes a $n$-dimensional random vector with known joint PDF $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$, which might arise from both the system properties and external excitations. In general, the system properties might originally occur as stochastic fields or directly as some random parameters. Besides, it is widely recognized that the random seismic excitations should be modelled as non-stationary stochastic processes or fields [32]. The stochastic fields or processes can usually be further decomposed to a set of standard random variables, e.g., by K-L expansion [33] and spectral representation method [34]. Therefore, a large number of random variables could be involved in a dynamic structural system with uncertain parameters subjected to non-stationary stochastic seismic excitations, leading to a high-dimensional system to be handled.

For a well-posed dynamical system, it is reasonable to assume that the solution to Eq. (1) exists uniquely as a function of $\boldsymbol{\Theta}$, which is expressed as:

$$\boldsymbol{X}(t) = \boldsymbol{H}(\boldsymbol{\Theta}, t),\ \dot{\boldsymbol{X}}(t) = \boldsymbol{h}(\boldsymbol{\Theta}, t) \tag{2}$$

where $\boldsymbol{H}$ and $\boldsymbol{h} = \partial \boldsymbol{H}/\partial t$ are the deterministic operators.

If there are a set of system response quantities of interest $\boldsymbol{Q}(t) = (Q_1(t), Q_2(t), \ldots, Q_m(t))^{\mathrm{T}}$ for reliability analysis, such as the inter-story drifts and strains or stresses at some points, then $\boldsymbol{Q}$ can usually be calculated from its

connection with the state vectors. It is convenient to assume that the relationship takes the form:

$$\boldsymbol{Q}(t) = \boldsymbol{\Psi}\left[\boldsymbol{X}(t), \dot{\boldsymbol{X}}(t)\right] = \boldsymbol{W}(\boldsymbol{\Theta}, t) \tag{3}$$

where $\boldsymbol{\Psi}$ is a transfer operator; $\boldsymbol{W}$ denotes a mapping from $\boldsymbol{\Theta}$ to $\boldsymbol{Q}$. Alternatively, Eq. (3) can be expressed as its component form:

$$Q_l(t) = W_l(\boldsymbol{\Theta}, t),\ l = 1, 2, \ldots, m \tag{4}$$

For notational simplicity, the subscript $l$ will be omitted hereafter without inducing confusions.

In general, the first-passage reliability is defined as:

$$P_r = \Pr\{Q(t) \in \Omega_s, t \in [0, T]\} \tag{5}$$

where $\Pr\{\cdot\}$ is the probability operator; $\Omega_s$ is the safe domain; $T$ is the duration of time. For the case of symmetric double-boundary safe domain concerned in this study, the first-passage reliability can be further written as:

$$P_r = \Pr\{|Q(t)| < b, t \in [0, T]\} \tag{6}$$

where $b$ is the prescribed threshold. Since $Q(t)$ can be a general stochastic response process, evaluation of $P_r$ remains a non-trivial task, especially for a expensive-to-evaluate model.

Let $Z$ denote the maximum value of $|Q(t)|$, i.e.

$$Z = \max_{t\in[0,T]} |Q(t)| = \max_{t\in[0,T]} |W(\boldsymbol{\Theta}, t)| = G(\boldsymbol{\Theta}) \tag{7}$$

Note that $Z$ is conveniently assumed to be as a function of $\boldsymbol{\Theta}$ and should be a positive random variable in this form. Then, Eq. (6) is equivalent to:

$$P_r = \Pr\{Z = G(\boldsymbol{\Theta}) < b\} \tag{8}$$

That is to say, once the probability distribution of $Z$ is available, which is refer to as the extreme value distribution (EVD), $P_r$ can be straightforwardly and conveniently obtained. Denote the probability density function (PDF) and cumulative distribution function (CDF) of $Z$ as $f_Z(z)$ and $F_Z(z)$, respectively, and then the first-passage reliability is:

$$P_r = \int_{-\infty}^{b} f_Z(z)\mathrm{d}z = F_Z(b) \tag{9}$$

and the corresponding failure probability is

$$P_f = \int_b^{+\infty} f_Z(z)\mathrm{d}z = 1 - F_Z(b) \tag{10}$$

The problem of first-passage reliability is now equivalently transformed to evaluation of the EVD. Despite of the conceptual elegance, it is still not easy to directly derive the analytical PDF or CDF of the EVD of a general stochastic response process. To address this challenge, an efficient approach will be developed in the following sections to capture the EVD.

## 3. Moment-generating function of the EVD

To characterize the EVD, the statistical moments, such as the mean value and variance, are commonly employed in the literature. However, it is known that the probabilistic information from the low-order moments is not sufficient to accurately capture an unknown distribution. Additionally, the high-order moments, e.g., the skewness and kurtosis, containing more information regarding the tail behavior are conside for a more accurate characterization [18]. Nevertheless, high-order moments are invariably complicated to analyze and sampling variability of power moments increases with the order of moment. Recently, the fractional moments with low-order exponents are introduced primarily due to the fact that even a single low-order fractional moment can contain a large amount of information on numerous integer moments [24]. Even so, it is worth mentioning that either a finite sequence of integer moments or fractional moments can not uniquely determine a underlying probability distribution. Based on those considerations, the moment-generating function will be introduced in this work to characterize the EVD.

### *3.1. Definition of the moment-generating function*

For the positive random variable $Z$ defined in Eq. (7), its moment-generating function (MGF) can by expressed as:

$$M_Z(\tau) = \mathbb{E}\left[e^{\tau Z}\right] = \int_0^{+\infty} e^{\tau z} f_Z(z)\mathrm{d}z, \tau \in \mathbb{R} \tag{11}$$

where $\mathbb{E}[\cdot]$ denotes the expectation operator.

Consider the series expansion of $e^{\tau Z}$

$$e^{\tau Z} = 1 + \tau Z + \frac{\tau^2 Z^2}{2!} + \frac{\tau^3 Z^3}{3!} + \cdots + \frac{\tau^r Z^r}{r!} + \cdots \tag{12}$$

and hence

$$M_Z(\tau) = 1 + \tau\mathbb{E}\left[Z\right] + \frac{\tau^2\mathbb{E}\left[Z^2\right]}{2!} + \frac{\tau^3\mathbb{E}\left[Z^3\right]}{3!} + \cdots + \frac{\tau^r\mathbb{E}\left[Z^r\right]}{r!} + \cdots \tag{13}$$

where $\mathbb{E}\left[Z^r\right]$ is the $r$-th moment of $Z$. Eq. (13) implies that the MGF at any non-zero $\tau$ includes a bulk of probabilistic information on infinite number of integer moments. This feature is similar to that of fractional moments, while fractional moments only make sense with positive random variables.

Further, the $r$-th moment of $Z$ can be obtained from its MGF such that:

$$\mathbb{E}\left[Z^r\right] = M_Z^{(r)}(0) = \left.\frac{\mathrm{d}^r M_Z}{\mathrm{d}\tau^r}\right|_{\tau=0} \tag{14}$$

Eq. (14) implies that only the MGF near the origin already contains all the information for any order of integer moment. The MGF is so named because all moments of $Z$ can be computed from its derivative evaluated at $\tau = 0$. However, it is impossible to obtain any order integer moment from a single fractional moment (not the integer moment itself), even though it includes the information conceptually.

Besides, the MGF (if it exists) can uniquely determine the underlying distribution (see Section 30 of [35]). In other words, if two random variables have the same MGF, then they must follow the same distribution. Thus, the MGF characterization of the EVD is more suitable than traditional integer moments or fractional moments. For more properties concerning the MGF, please refer to, e.g., [36].

### *3.2. Numerical approximation of the moment-generating function*

For the EVD of a general stochastic response process, derivation of closed-form solution of its MGF is not trivial. Therefore, numerical evaluation is necessary for general applications. For our purpose, the MGF defined in Eq. (11) can be further expressed as:

$$M_Z(\tau) = \int_{\mathcal{D}_{\boldsymbol{\Theta}}} e^{\tau G(\boldsymbol{\theta})} p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} \tag{15}$$

where $\mathcal{D}_{\boldsymbol{\Theta}}$ is the distribution domain of $\boldsymbol{\Theta}$. As the dimension of $\boldsymbol{\Theta}$ could be very large, $M_Z(\tau)$ is generally a high-dimensional integral with a complicated integrand.

To evaluate such a integral, a recently developed low-discrepancy sampling technique, named Latinized partially

stratified sampling (LPSS) [31], will be employed. Eq. (15) can be estimated by the following estimator:

$$\hat{M}_Z(\tau) = \frac{1}{N}\sum_{i=1}^{N} e^{\tau G(\boldsymbol{\theta}^{(i)})} \tag{16}$$

where $N$ is the total number of samples; $\{\boldsymbol{\theta}^{(i)}\}_{i=1}^{N}$ are a set of samples of $\boldsymbol{\Theta}$. Since LPSS method is originally designed for generating uniformly distributed samples over the unit hypercube, i.e, $\mathcal{D}_{\boldsymbol{U}} = [0,1]^d$, a transformation $\boldsymbol{\theta}^{(i)} = T(\boldsymbol{u}^{(i)})$ should be applied to obtain the sample $\boldsymbol{\theta}^{(i)}$ from $\boldsymbol{u}^{(i)}$ ($\{\boldsymbol{u}^{(i)}\}_{i=1}^{N}$ are a set of samples generated by LPSS). A novel feature of the LPSS is that it can reduce the variance related to the main effects and variable interactions simultaneously, and hence a reduced set of samples could be sufficient to accurately evaluate the MGF. Readers may refer to [31] for technical details and [24, 37] for implementation issues of the LPSS method. The basic algorithm is presented in Algorithm 1. The Matlab code is readily available at: https://www.mathworks.com/matlabcentral/fileexchange/54847-latinized-partially-stratified-sampling.

**Algorithm 1** Latinized partially stratified sampling

1: **Input:** Specify the sample size $N$, number of dimensions $d$ and orthogonal subspaces $\mathcal{S}_j (j = 1, 2, \cdots, L)$, where $\mathcal{S}_{j_1} \perp \mathcal{S}_{j_2} (j_1 \neq j_2)$ and $\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \cdots \oplus \mathcal{S}_L = \mathcal{D}_{\boldsymbol{U}}$.

2: For each subspace $\mathcal{S}_j$, generate $N$ low-dimensional samples using Latinized stratified sampling (LSS) [31].

3: Randomly select a sample from each subspace (without replacement) and group the selected samples to produce a sample in $d$-dimensional space.

4: Repeat step 3 until $N$ samples are constructed in $d$-dimensional space.

5: **Output:** $N$ samples in $d$-dimensional unit hypercube.

Regardless of the distribution of $Z$, we have

$$\mathbb{E}\left[\hat{M}_Z(\tau)\right] = \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} e^{\tau G(\boldsymbol{\theta}^{(i)})}\right] = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[e^{\tau G(\boldsymbol{\theta}^{(i)})}\right] = \frac{1}{N}\sum_{i=1}^{N} M_Z(\tau) = M_Z(\tau) \tag{17}$$

Thus, the MGF estimator in Eq. (16) is proven to be unbiased. Further, the estimator will approximately follow the normal distribution with mean $\hat{\mu}(\tau) = \hat{M}_Z(\tau)$ and variance given by:

$$\hat{\sigma}^2(\tau) = \frac{1}{N(N-1)}\sum_{i=1}^{N}\left[e^{\tau G(\boldsymbol{\theta}^{(i)})} - \hat{M}_Z(\tau)\right]^2 \tag{18}$$

Confidence interval can then be obtained accordingly. For example, the 95% confidence interval of the estimator is as follows:

$$[\hat{\mu}(\tau) - 1.96\hat{\sigma}(\tau), \hat{\mu}(\tau) + 1.96\hat{\sigma}(\tau)] \tag{19}$$

Eq. (16) can be used to obtain a point estimate of the MGF. In order to inform the accuracy of the point estimate, Eq. (19) gives a interval estimate of the MGF, which actually reflects the sampling variability and can be narrowed by increasing the sample size $N$.

## 4. Recovering the EVD from its moment-generating function by a mixture distribution

In this section, the problem of how to find the underlying EVD from the knowledge of its estimated MGF is studied. Although the MGF can uniquely determine a probability distribution, it is not easy to deduce the underlying distribution directly. Alternatively, we focus on offering an efficient and accurate parametric approach to approximate the EVD. For this purpose, a flexible mixture distribution will be developed on the basis of the generalized inverse Gaussian distribution, and then the parameter estimation based on the MGF is elaborated.

### *4.1. Generalized inverse Gaussian distribution*

The generalized inverse Gaussian distribution (GIGD) is a three-parameter family of continuous probability distributions, which has a wide variety of applications in many fields. A random variable has a GIGD if its PDF is given by [38]:

$$f_{\text{GIGD}}(z; \alpha, \beta, \lambda) = \frac{(\alpha/\beta)^{\lambda/2}}{2K_\lambda(\sqrt{\alpha\beta})} z^{(\lambda-1)} e^{-(\alpha x + \beta/z)/2}, z > 0 \tag{20}$$

where $\alpha > 0$, $\beta > 0$ and $\lambda \in \mathbb{R}$ are the three parameters; $K_\lambda$ is a modified Bessel function of the second kind with order $\lambda$.

The inverse Gaussian distribution (IGD) is a special case of GIGD when $\lambda = -1/2$. The special and limiting cases of the IGD also belong to the GIGD, such as the normal distribution and Lévy distribution. Besides, other special cases include the hyperbolic distribution for $\lambda = 0$, Gamma distribution for $\beta \to 0$ and inverse-gamma distribution for $\alpha \to 0$.

The closed-form solution of the MGF of GIGD exists and reads:

$$M_{\text{GIGD}}(\tau;\alpha,\beta,\lambda)=\left(\frac{\alpha}{\alpha-2\tau}\right)^{\frac{\lambda}{2}}\frac{K_{\lambda}(\sqrt{\beta(\alpha-2\tau)})}{K_{\lambda}(\sqrt{\alpha\beta})} \tag{21}$$

Based on the MGF, the mean and variance can be obtained as:

$$\mu_{\text{GIGD}}=\frac{\sqrt{\beta}K_{\lambda+1}(\sqrt{\alpha\beta})}{\sqrt{\alpha}K_{\lambda}(\sqrt{\alpha\beta})} \tag{22}$$

$$\sigma^2_{\text{GIGD}}=\left(\frac{\beta}{\alpha}\right)\left[\frac{K_{\lambda+2}(\sqrt{\alpha\beta})}{K_{\lambda}(\sqrt{\alpha\beta})}-\left(\frac{K_{\lambda+1}(\sqrt{\alpha\beta})}{K_{\lambda}(\sqrt{\alpha\beta})}\right)^2\right] \tag{23}$$

The reason why we specially select the GIGD as the basis of the proposed parametric distribution model is mostly based on the two following considerations. First of all, the GIGD is flexible in shapes and tail properties since several theoretical distributions with distinct features are included (as mentioned early). It is worthy mentioning that the IGD as a special case of the GIGD is the exact first-passage time distribution of the Brownian motion (Wiener process) [39]. Second, the MGF of the GIGD is readily available in a closed form (Eq. (21)), which enables the deviation of the MGF and parameter estimation of the proposed mixture distribution model (see the following two subsections).

### *4.2. A mixture of two generalized inverse Gaussian distributions*

Although the GIGD is more versatile than the IGD by adding an extra free parameter, its flexibility should be further enhanced to capture the EVD since more emphasis is placed on the tail, in addition to the main body of the distribution, especially for evaluation of small first-passage failure probabilities. Finite mixture models allow us to create new distributions with increased number of free parameters and hence flexibility in a simple, but efficient way [40]. However, identification of free parameters is an intractable task as the number of mixture components increases, since they usually cannot be derived explicitly. To this end, a mixture of two generalized inverse Gaussian distributions (MTGIGD) will be specially developed to capture the EVD in the present work. The PDF of the MTGIGD is given by:

$$\begin{aligned}f_{\text{MTGIGD}}(z;\Sigma)=&\varpi f_{\text{GIGD}}(z;\alpha^{(1)},\beta^{(1)},\lambda^{(1)})+(1-\varpi)f_{\text{GIGD}}(z;\alpha^{(2)},\beta^{(2)},\lambda^{(2)})\\=&\varpi\frac{(\alpha^{(1)}/\beta^{(1)})^{\lambda^{(1)}/2}}{2K_{\lambda^{(1)}}(\sqrt{\alpha^{(1)}\beta^{(1)}})}z^{(\lambda^{(1)}-1)}e^{-(\alpha^{(1)}x+\beta^{(1)}/z)/2}\\&+(1-\varpi)\frac{(\alpha^{(2)}/\beta^{(2)})^{\lambda^{(2)}/2}}{2K_{\lambda^{(2)}}(\sqrt{\alpha^{(2)}\beta^{(2)}})}z^{(\lambda^{(2)}-1)}e^{-(\alpha^{(2)}x+\beta^{(2)}/z)/2}\end{aligned} \tag{24}$$

where $0 < \varpi < 1$ and $1 - \varpi$ are the mixture weights of the first and second mixture components, respectively; $\Sigma = [\varpi, \alpha^{(1)}, \beta^{(1)}, \lambda^{(1)}, \alpha^{(2)}, \beta^{(2)}, \lambda^{(2)}]^{\mathrm{T}}$ is a set of seven free parameters to be determined; The superscript distinguishes those parameters from different mixture components.

The MGF of the MTGIGD can be derived as:

$$\begin{aligned}
M_{\text{MTGIGD}}(\tau;\Sigma) =& \int_0^{+\infty} e^{\tau z} f_{\text{MTGIGD}}(z;\Sigma)\mathrm{d}z \\
=& \int_0^{+\infty} e^{\tau z}\left(\varpi f_{\text{GIGD}}(z;\alpha^{(1)},\beta^{(1)},\lambda^{(1)}) + (1-\varpi) f_{\text{GIGD}}(z;\alpha^{(2)},\beta^{(2)},\lambda^{(2)})\right)\mathrm{d}z \\
=& \varpi \int_0^{+\infty} e^{\tau z} f_{\text{GIGD}}(z;\alpha^{(1)},\beta^{(1)},\lambda^{(1)})\mathrm{d}z + (1-\varpi)\int_0^{+\infty} e^{\tau z} f_{\text{GIGD}}(z;\alpha^{(2)},\beta^{(2)},\lambda^{(2)})\mathrm{d}z \\
=& \varpi M_{\text{GIGD}}(\tau;\alpha^{(1)},\beta^{(1)},\lambda^{(1)}) + (1-\varpi) M_{\text{GIGD}}(\tau;\alpha^{(2)},\beta^{(2)},\lambda^{(2)}) \\
=& \varpi\left(\frac{\alpha^{(1)}}{\alpha^{(1)}-2\tau}\right)^{\frac{\lambda^{(1)}}{2}} \frac{K_{\lambda^{(1)}}(\sqrt{\beta^{(1)}(\alpha^{(1)}-2\tau)})}{K_{\lambda^{(1)}}(\sqrt{\alpha^{(1)}\beta^{(1)}})} \\
&+ (1-\varpi)\left(\frac{\alpha^{(2)}}{\alpha^{(2)}-2\tau}\right)^{\frac{\lambda^{(2)}}{2}} \frac{K_{\lambda^{(2)}}(\sqrt{\beta^{(2)}(\alpha^{(2)}-2\tau)})}{K_{\lambda^{(2)}}(\sqrt{\alpha^{(2)}\beta^{(2)}})}
\end{aligned} \tag{25}$$

If we assume that the EVD follows the MTGIGD, i.e., $\hat{f}_Z(z) = f_{\text{MTGIGD}}(z;\Sigma)$, then how to estimate the unknown parameters collecting in $\Sigma$ could be another critical task.

### *4.3. Moment-generating function based parameter estimation*

The MGF of the EVD is obtained in Section 3 and the closed-form expression of MGF of the MTGIGD exists. A natural idea for estimating parameters is to match the EVD's MGF with that of the MTGIGD, and the following equations can be established accordingly:

$$\begin{cases}
M_{\text{MTGIGD}}(\tau_1;\Sigma) = \hat{M}_Z(\tau_1) \\
M_{\text{MTGIGD}}(\tau_2;\Sigma) = \hat{M}_Z(\tau_2) \\
\qquad\qquad \cdots \\
M_{\text{MTGIGD}}(\tau_7;\Sigma) = \hat{M}_Z(\tau_7)
\end{cases} \tag{26}$$

where $\{\tau_j\}_{j=1}^7$ are seven discrete values of $\tau$. In fact, $\tau$ around zero is more concerned due to its high estimation accuracy and sufficient probabilistic information included. However, $\hat{M}_Z(\tau)$ could be sufficient large for $\tau > 0$, since $Z$ is a positive random variable in the context of this article. For this reason, the discrete values of $\tau$ are adopted as

$\{\tau_j\}_{j=1}^{7} = \{-0.1, -0.2, -0.3, -0.4, -0.5, -0.6, -0.7\}$ in this study. One can use the built-in function "fsolve" in Matlab to solve the above system of equations.

As a suggestion, the following estimates can be served as the initial values of the parameters as a guess:

$$\varpi = 0.4 \tag{27}$$

$$\alpha^{(1)} = \alpha^{(2)} = \frac{\hat{\mu}_Z}{\hat{\sigma}_Z^2} \tag{28}$$

$$\beta^{(1)} = \beta^{(2)} = \frac{\hat{\mu}_Z^3}{\hat{\sigma}_Z^2} \tag{29}$$

$$\lambda^{(1)} = \lambda^{(2)} = -\frac{1}{2} \tag{30}$$

where the parameters except for $\omega$ are estimated by the method of moments in the case that each mixture component is reduced to the IGD; $\hat{\mu}_Z$ and $\hat{\sigma}_Z^2$ are the estimated mean and variance of the EVD, which can be evaluated by:

$$\hat{\mu}_Z = \frac{1}{N}\sum_{i=1}^{N} G(\boldsymbol{\theta}^{(i)}) \tag{31}$$

$$\hat{\sigma}_Z^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(G(\boldsymbol{\theta}^{(i)}) - \hat{\mu}_Z\right)^2 \tag{32}$$

Note that no extra dynamic structural analysis is actually requi in this step.

Once the free parameters in the MTGIGD model are identified properly, one can obtain a point estimate of the EVD, and hence the first-passage failure probabilities under different prescribed thresholds. However, how to measure the accuracy of the results is a non-trivial task since both the sampling error containing in the MGF estimate and the model assumption error for functional form of the EVD should be addressed simultaneously. This is not attempted in this study, but we hope that our work will stimulate further studies.

The flowchart of the proposed approach for efficient assessment of EVD and first-passage failure probabilities of structures with uncertain parameters under stochastic seismic excitations is depicted in Fig. 1.

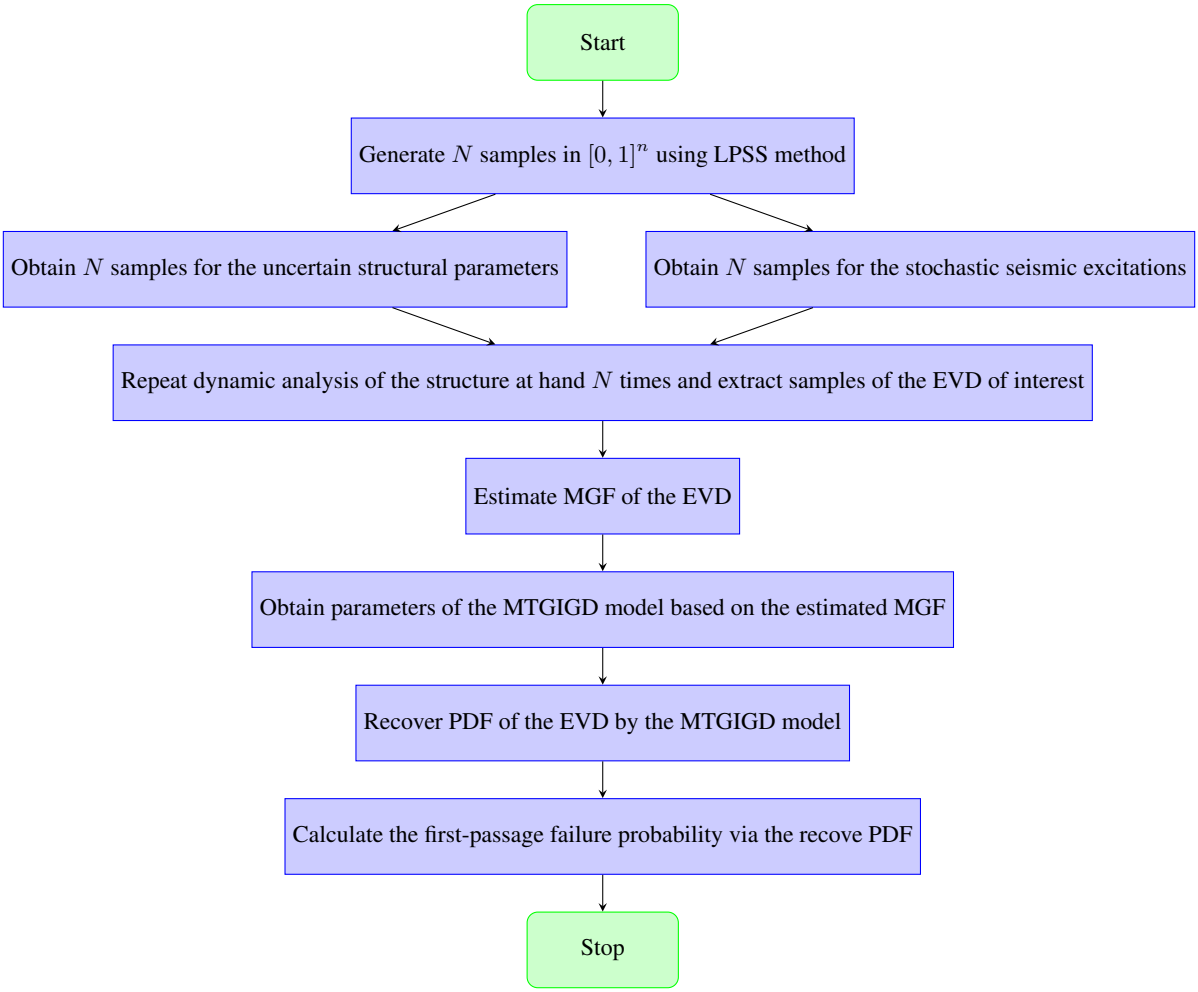



Figure 1: Flowchart of the proposed approach.

## 5. Numerical examples

To validate the proposed approach, three numerical examples will be presented in this section. In the first example, the EVD of random variables is investigated where the analytical solution of the EVD is available. The EVD approximated by the proposed method is verified through comparison with the analytical solution, indicating the accuracy of the proposed method. The EVD of the response of two nonlinear structures with uncertain parameters subjected to fully non-stationary stochastic seismic excitations is computed through the proposed approach, and the first-passage failure probabilities under different prescribed thresholds are obtained in example 2 and 3. In compassion with Monte Carlo simulation and other state-of-the-art methods, advantages of the proposed method are demonstrated.

### *5.1. Example 1: EVD of random variables*

Suppose that $\boldsymbol{X} = [X_1, X_2, \ldots, X_n]$ are i.i.d. standard normal random variables. Consider the maximum absolute value such that:

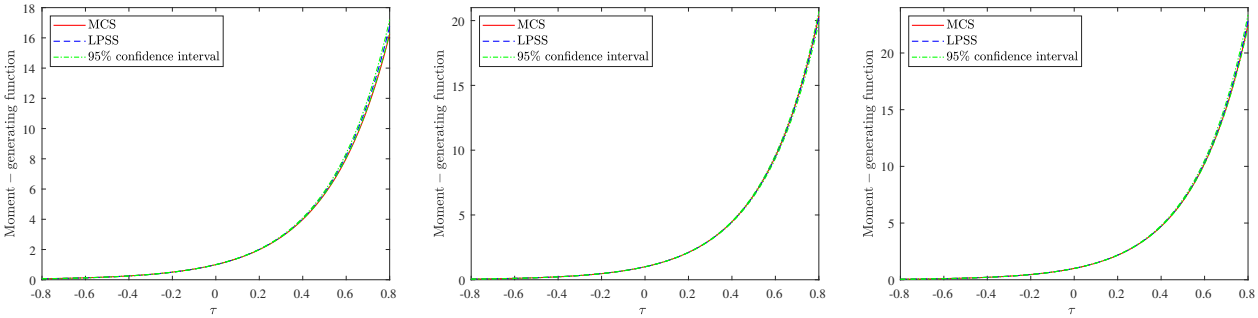
$$Z = \max(|X_1|, |X_2|, \ldots, |X_n|) \tag{33}$$

where $Y_i = |X_i|$ actually follows a folded standard normal distribution. Denote the PDF and CDF of $Y_i$ as $f_Y(y)$ and $F_Y(y)$, respectively. According to the probability theory, the closed-form solutions of the PDF and CDF of $Z$ are available, and given by:

$$f_Z(z) = n f_Y(z) F_Y^{n-1}(z), F_Z(z) = F_Y^n(z) \ (z \geq 0) \tag{34}$$

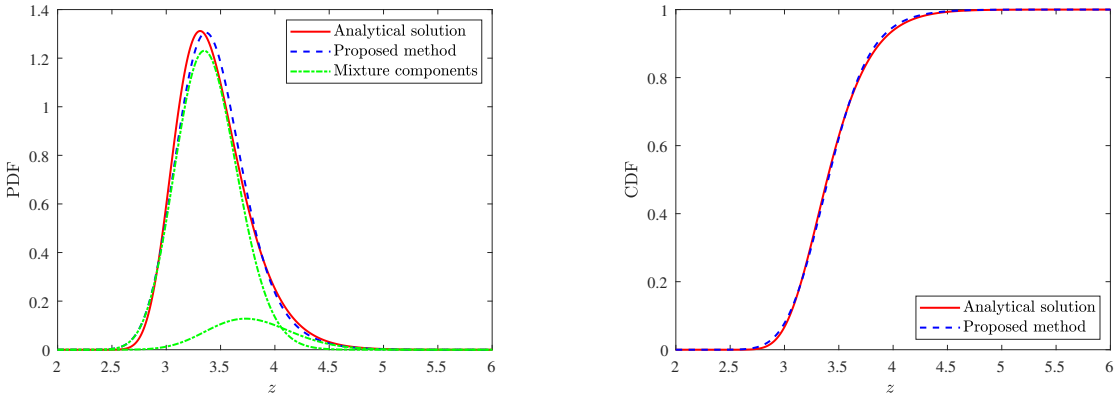
The proposed approach can also be employed to approximate the PDF and CDF of $Z$. Three cases, i.e., $n = 1000,\ 3000,\ 5000$, will be studied to verify the proposed method. First, the MGFs of $Z$ are estimated by the LPSS method with a small sample size of $N = 625$. The estimated MGFs are compared with those obtained by MCS (with $10^6$ runs), as shown in Fig. 2. It is observed that the confidence intervals are all very narrow, and contain the results given by MCS, indicating the accuracy of LPSS method. The MTGIGD is then used to recover the PDF and CDF of $Z$ by fitting seven values of the estimated MGF. The recove PDF and CDF of those three cases are compa with the analytical solutions in Figs. 3 - 5, respectively. Note that the two mixture components of the proposed MIGIND multiplied by their corresponding mixture weights are also given. As seen, the recovered PDF and CDF

agree well with those analytical solutions. This example demonstrates the accuracy of the proposed method, as well as the efficiency for such three high-dimensional cases.




(a) $n = 1000$ (b) $n = 3000$ (c) $n = 5000$

Figure 2: Moment-generating function of the EVD in Example 1.




(a) PDF (b) CDF

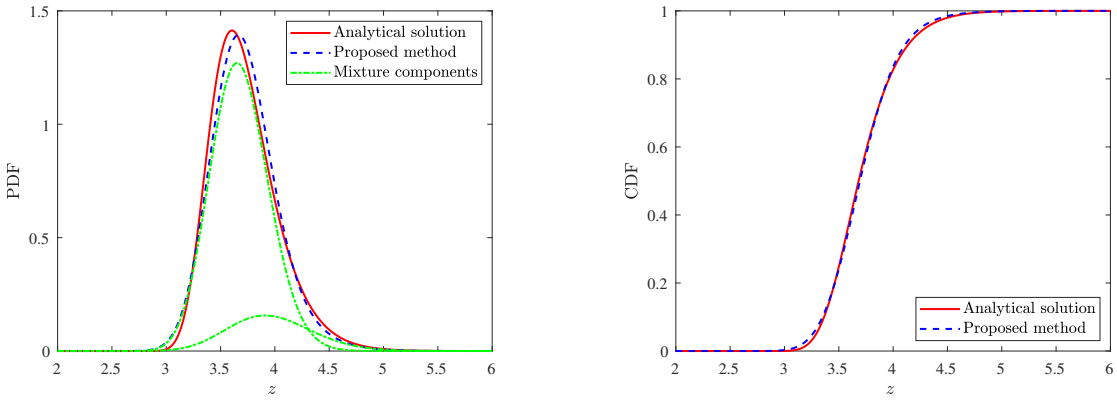
Figure 3: Comparison of PDF and CDF of the EVD in Example 1 ($n = 1000$).

### 5.2. *Example 2: EVD and first-passage failure probabilities of a nonlinear shear frame structure*

Consider a two-span ten-story nonlinear shear frame structure with uncertain parameters under stochastic seismic excitations, as shown in Fig. 6. The equation of motion of this structure is given by:

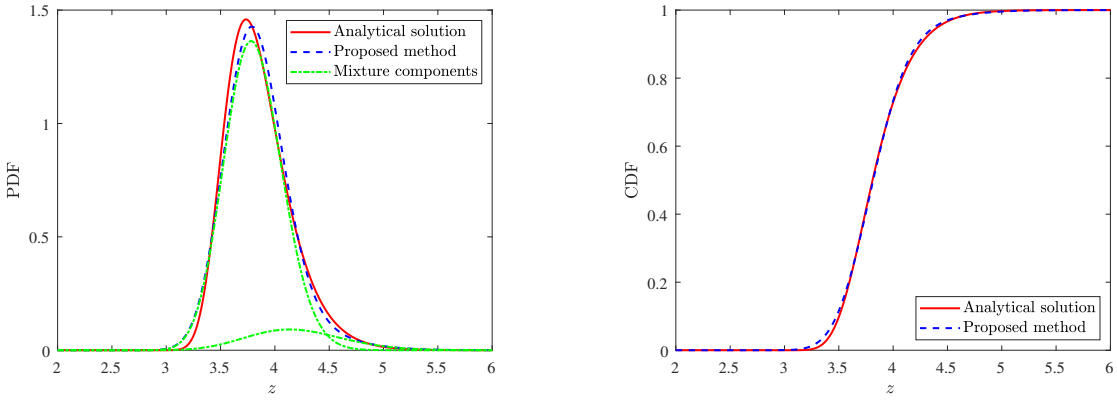
$$\boldsymbol{M}(\boldsymbol{\Theta})\ddot{\boldsymbol{X}}(t) + \boldsymbol{C}(\boldsymbol{\Theta})\dot{\boldsymbol{X}}(t) + \boldsymbol{K}(\boldsymbol{\Theta})\left[\varepsilon \boldsymbol{X}(t) + (1-\varepsilon)\boldsymbol{Y}(t)\right] = -\boldsymbol{M}(\boldsymbol{\Theta})I\ddot{U}_g(\boldsymbol{\Theta}, t) \tag{35}$$

where $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ are the mass, damping and initial stiffness matrices. Rayleigh damping is adopted such that $\boldsymbol{C} = a_1\boldsymbol{M} + a_2\boldsymbol{K}$, where $a_1$ and $a_2$ are obtained by assuming the damping ratio $\xi = 5\%$ for the first two modes. The




(a) PDF (b) CDF

Figure 4: Comparison of PDF and CDF of the EVD in Example 1 ($n = 3000$).




(a) PDF (b) CDF

Figure 5: Comparison of PDF and CDF of the EVD in Example 1 ($n = 5000$).

lumped mass and inter-story stiffness of each story are assumed to be independent random variables, the probabilistic information of which is summarized in Tabs. 1 and 2, respectively. That is, $d_1 = 20$ random variables are involved in the uncertain structural properties. $\ddot{\boldsymbol{X}}(t)$, $\dot{\boldsymbol{X}}(t)$ and $\boldsymbol{X}(t)$ are the acceleration, velocity and displacement vectors. The parameter $\varepsilon$, that controls the degree of hysteresis, is set to be 0.20. The term $\boldsymbol{Y}(t)$ is characterized by the Bouc–Wen model [41, 42]:

$$\dot{\boldsymbol{Y}}(t) = \rho\dot{\boldsymbol{X}}(t) - \eta|\dot{\boldsymbol{X}}(t)||\boldsymbol{Y}(t)|^{n_0-1}\boldsymbol{Y}(t) - \gamma\dot{\boldsymbol{X}}(t)|\boldsymbol{Y}(t)|^{n_0} \tag{36}$$

where $\rho$, $\eta$, $\gamma$ and $n_0$ are dimensionless quantities, which control the behaviour of the model. These parameters take the following values: $\rho = 1$, $n_0 = 3$ and $\eta = \gamma = 1/\left(2u_y^{n_0}\right)$, in which $u_y = 0.01$ m is the yielding displacement. $I$ is the loading inference vector. The seismic excitation $\ddot{U}_g(t)$ is modeled by a fully non-stationary stochastic process via the spectral representation method [43]:

$$\ddot{U}_g(t) = \sum_{i=1}^{d_2} \sqrt{2S_{\ddot{u}_g}(\omega_i, t)\Delta\omega}\,(A_i\cos(\omega_i t) + B_i\sin(\omega_i t)) \tag{37}$$

where $S_{\ddot{u}_g}(\omega, t)$ is the evolutionary power spectral density (EPSD) function, which is defined as:

$$S_{\ddot{u}_g}(\omega, t) = |f(\omega, t)|^2 G(\omega) \tag{38}$$

in which $f(\omega, t)$ is the modulation function of time and frequency, given by:

$$f(\omega, t) = \exp\left(-\delta_0\frac{\omega t}{\omega_a t_a}\right)\cdot\left[\frac{t}{c}\exp\left(1-\frac{t}{c}\right)\right]^k \tag{39}$$

and $G(\omega)$ is the one-sided PSD, assumed to follow the Clough- Penzien spectrum:

$$G(\omega) = \frac{\omega_g^4 + 4\zeta_g^2\omega_g^2\omega^2}{\left(\omega_g^2-\omega^2\right)^2 + \left(2\zeta_g\omega_g\omega\right)^2}\frac{\omega^4}{\left(\omega_f^2-\omega^2\right)^2 + \left(2\zeta_f\omega_f\omega\right)^2}S_0 \tag{40}$$

$\{A_i, B_i\}_{i=1}^{d_2}$ in Eq. (37) are $2d_2$ independent standard normal random variables. In this example, $d_2 = 800$ is adopted, i.e., a total of $d = d_1 + 2d_2 = 1620$ random variables are included in $\boldsymbol{\Theta}$. Other involved parameters are specified as: $\Delta t = 0.02$ s, $T = 30$ s; $\Delta\omega = 0.15$ rad/s, $\omega_{\mathrm{l}} = 0.15$ rad/s, $\omega_{\mathrm{u}} = 240$ rad/s; $\delta_0 = 0.15$, $c = 9.0$, $k = 2.0$; $\omega_g = 5\pi$ rad/s, $\zeta_g = \zeta_f = 0.60$, $\omega_f = 0.1\omega_g$, $S_0 = 54.6296$ cm$^2$/s$^3$.

The LPSS method is firstly employed to generate $N = 625$ sample points in $[0, 1]^d$ $(d = 1620)$, serving as a basic point set. Then, these samples are transformed to the random-variate space of the vector $\boldsymbol{\Theta}$. Integrating the samples

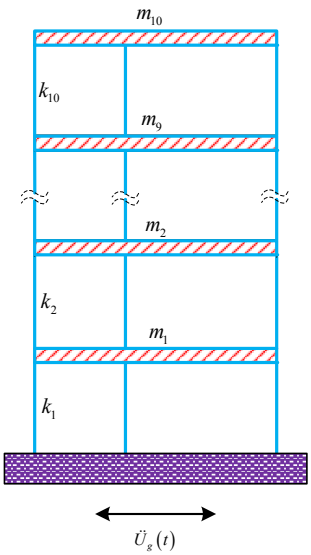



Figure 6: A two-bay ten-story shear frame structure subjected to stochastic seismic excitations.

Table 1: Probabilistic information of the lumped mass

| Lumped mass | $m_1$ | $m_2$ | $m_3$ | $m_4$ | $m_5$ | $m_6$ | $m_7$ | $m_8$ | $m_9$ | $m_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| Distribution | LN | LN | LN | LN | LN | LN | LN | LN | LN | LN |
| Mean ($\times 10^5$ kg) | 6.0 | 5.9 | 5.8 | 5.7 | 5.6 | 5.5 | 5.4 | 5.3 | 5.2 | 5.1 |
| C.O.V. | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 |

LN = Lognormal; C.O.V. = coefficient of variation

of $\{A_i, B_i\}_{i=1}^{d_2}$ with the spectral representation, one can have a set of 625 representative samples of the fully non-stationary stochastic seismic accelerations. Three representative samples of the seismic acceleration time histories are shown in Fig. 7, where the non-stationary behavior in time domain can be clearly observed. Besides, the mean and standard deviation (Std.D) of 625 seismic acceleration samples against their targets are depicted in Fig. 8. As seen, the simulated results accord well with the targets, indicating the applicability of the LPSS based spectral representation.

Once the representative samples of the random structural properties and seismic excitation are produced, the deterministic dynamic structural analysis is performed repeatedly to extract samples of the EVD of concern. The ode45 function in Matlab is used to solve Eq. (35). Fig. 9 shows a typical sample of the restoring force v.s. inter-story drift, indicating a strong hysteretic behavior of the random structure under the fully non-stationary seismic excitations.

Table 2: Probabilistic information of the initial inter-story stiffness

| Stiffness | $k_1$ | $k_2$ | $k_3$ | $k_4$ | $k_5$ | $k_6$ | $k_7$ | $k_8$ | $k_9$ | $k_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| Distribution | G | G | G | G | G | G | G | G | G | G |
| Mean ($\times 10^7$ N/m) | 2.50 | 2.45 | 2.40 | 2.35 | 2.30 | 2.25 | 2.20 | 2.15 | 2.10 | 2.05 |
| C.O.V. | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 |

G = Gaussian

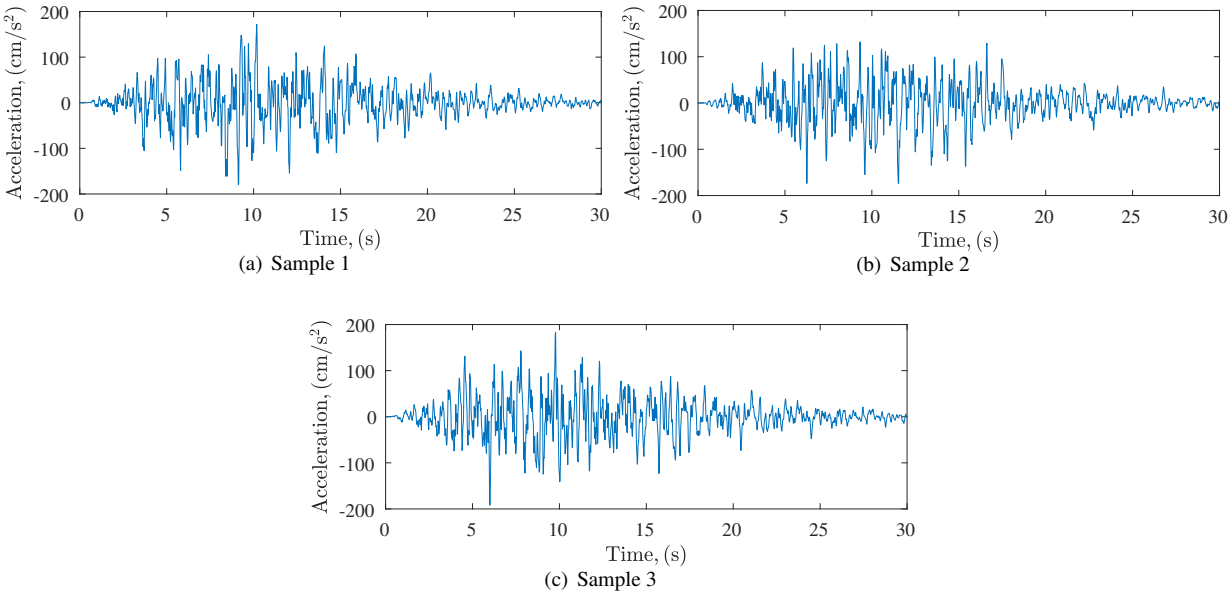
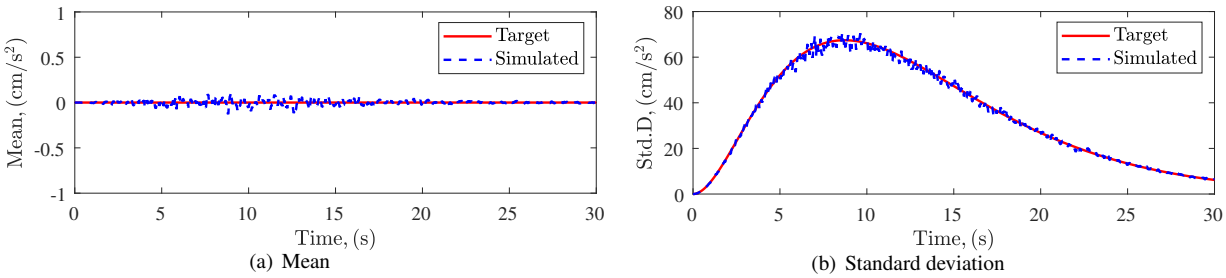



Figure 7: Representative samples of the fully non-stationary stochastic seismic acceleration.

In this study, we are especially interested in the EVDs of the 1-st, 5-th, and 10-th inter-story drift. The MGFs of interested EVDs computed by the LPSS method are depicted in Fig. 10, together with the reference results given by MCS (with $10^6$ runs). It can be seen that all the three MGFs are accurately estimated since the 95% confidence intervals are very narrow and contain the corresponding reference results.

Based on the knowledge of the estimated MGFs, the EVDs can be recovered by the MTGIGD model. Figs. 11 - 13 show the recovered PDF, CDF and probability of exceedance (POE) curves, where the results given by MCS (with $10^6$ runs) are also depicted for comparison. As seen, fairly good agreement can be observed between the results computed by the proposed method and those of MCS, indicating the accuracy of the proposed method for modeling not only the main body, but also the distribution tail of the EVD. From the POE curves, the first-passage failure probabilities under




(a) Mean (b) Standard deviation

Figure 8: Comparison of the simulated mean and standard deviation with their targets for the fully non-stationary stochastic seismic excitation.

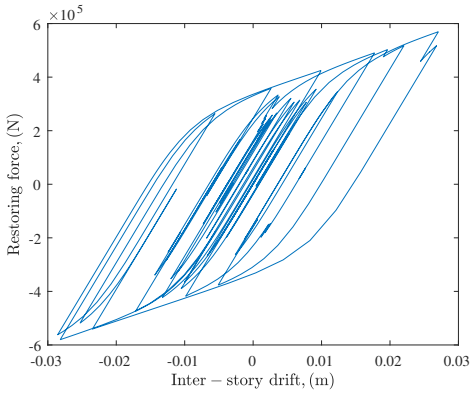
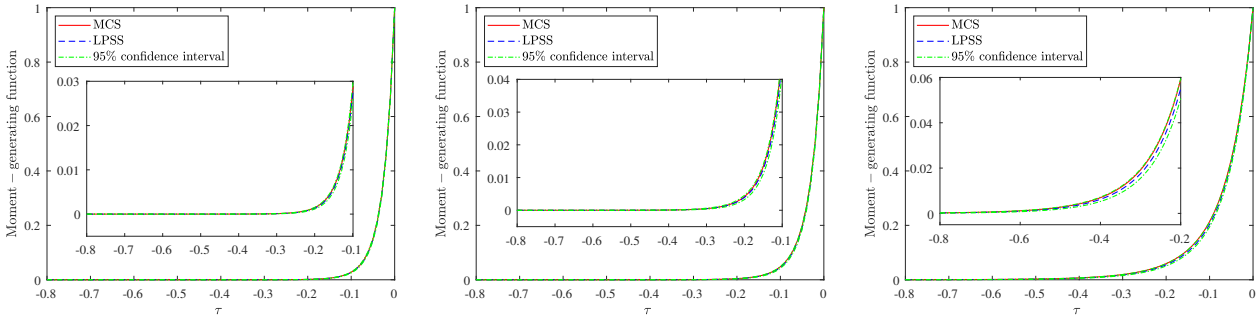


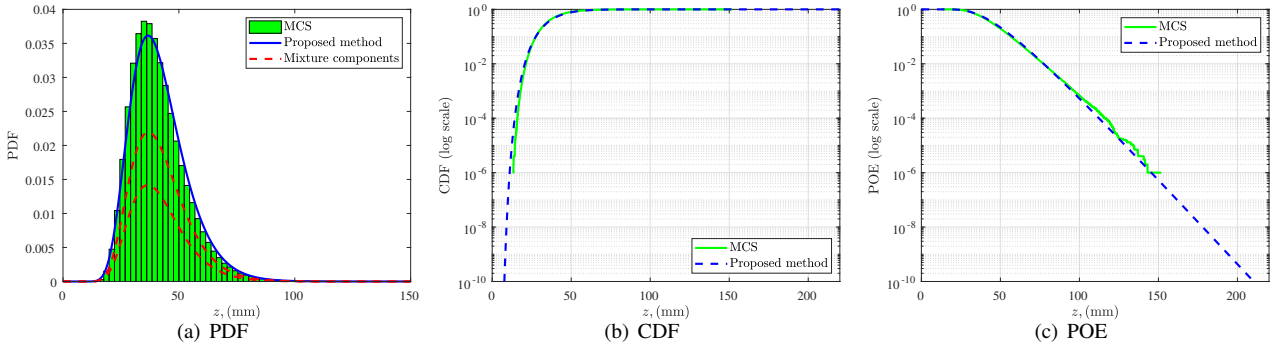

Figure 9: Typical hysteretic behavior for Example 3.

different thresholds can be simultaneously obtained. For example, the first-passage failure probabilities associated with the 1-st inter-story drift are listed in Tab. 3. In comparison with MCS, one can see that the proposed method can yield very accurate results, even for small failure probability levels less than $10^{-4}$. Besides, the proposed method is much more efficient than MCS, since only $N = 625$ samples are required in this study. Finally, the EVD of the 10-th inter-story drift is also approximated by the generalized extreme value distribution (GEVD) [17] with maximum likelihood estimation, shifted generalized lognormal distribution (SGLD) [44] with the method of moments (MOM) and maximum entropy method with four sample moments as constraint. Note that the same set of samples with the proposed method is employed in all these three methods. As shown in Fig. 14, large approximation errors appear, especially for the CDF and POE curves, raveling the limited flexibility of those parametric distribution models and/or inadequate parameter estimation methods.



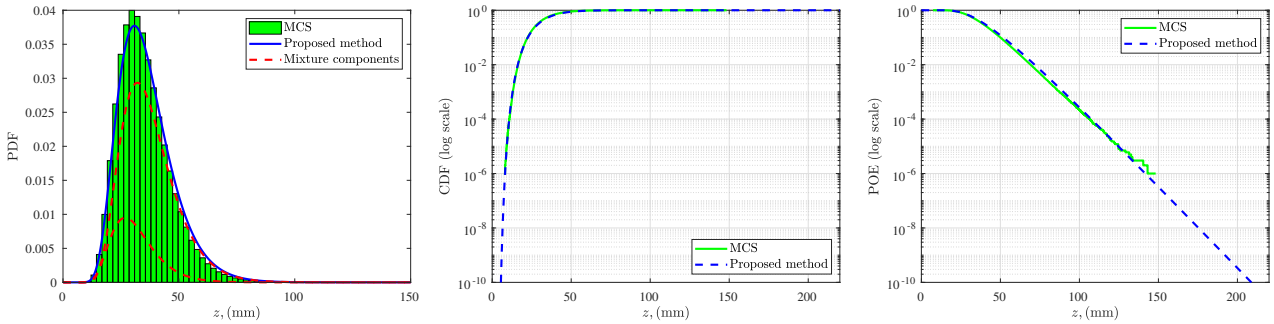

(a) 1-st inter-story drift (b) 5-th inter-story drift (c) 10-th inter-story drift

Figure 10: Moment-generating function of the EVD in Example 2.



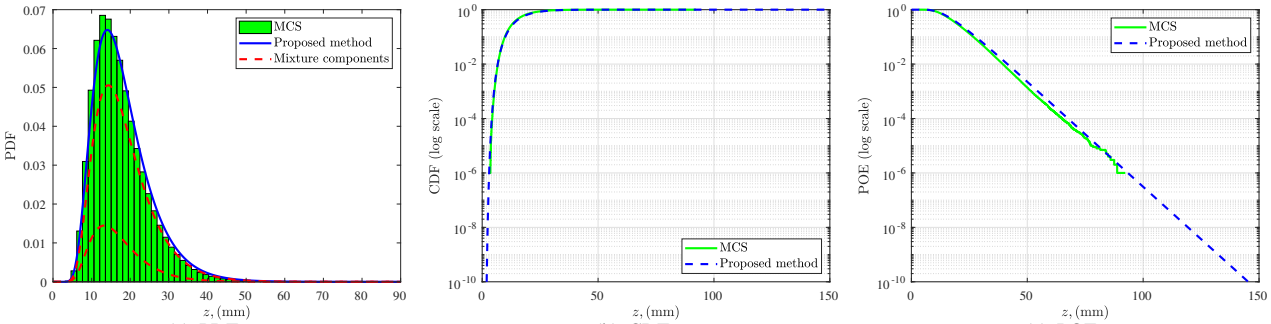

(a) PDF (b) CDF (c) POE

Figure 11: EVD of the 1-st inter-story drift.




(a) PDF (b) CDF (c) POE

Figure 12: EVD of the 5-th inter-story drift.

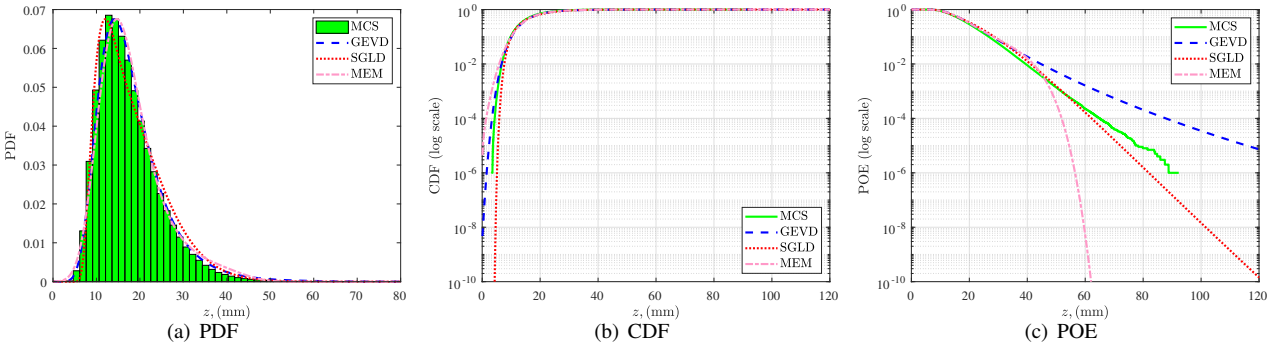



(a) PDF (b) CDF (c) POE

Figure 13: EVD of the 10-th inter-story drift.

Table 3: Comparison of the first-passage failure probabilities under different thresholds for example 2.

| Method | $N$ | Threshold (mm) | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 72 | 80 | 88 | 96 | 104 | 112 | 120 |
| MCS | $10^6$ | 0.0181 | 0.0073 | 0.0028 | $1.1020\times10^{-3}$ | $4.2000\times10^{-4}$ | $1.8000\times10^{-4}$ | $5.9000\times10^{-5}$ |
| Proposed | 625 | 0.0194 | 0.0072 | 0.0026 | $9.0353\times10^{-4}$ | $3.0960\times10^{-4}$ | $1.0450\times10^{-4}$ | $3.4841\times10^{-5}$ |




(a) PDF (b) CDF (c) POE

Figure 14: EVD of the 10-th inter-story drift by other methods.

### *5.3. Example 3: EVD and first-passage failure probabilities of a nonlinear bridge structure*

A simplified two-span bridge structure is taken as the last example, as shown in Fig. 15. The bridge structure is modelled in OpenSees, an open software framework for earthquake engineering simulation. Since the beam is expected to remain linear elastic under seismic excitation, six elastic beam-column elements are used. The pier is

modelled by four nonlinear beam-column elements, the cross section of which is discretized into 225 fibers. Constitutive law of the material adopts the bilinear model, as depicted in the figure. The Rayleigh damping is employed with assuming the damping ratio of 3% for the first two modes. Eight random variables are conside in the structural properties, whose information is listed in Tab. 4. For convenience, the same stochastic seismic excitations with example 2 are applied to this model. Therefore, a total of $d = 1608$ random variables are actually contained in this example. The EVD of displacement at the pile top is of particular concern for seismic reliability analysis.

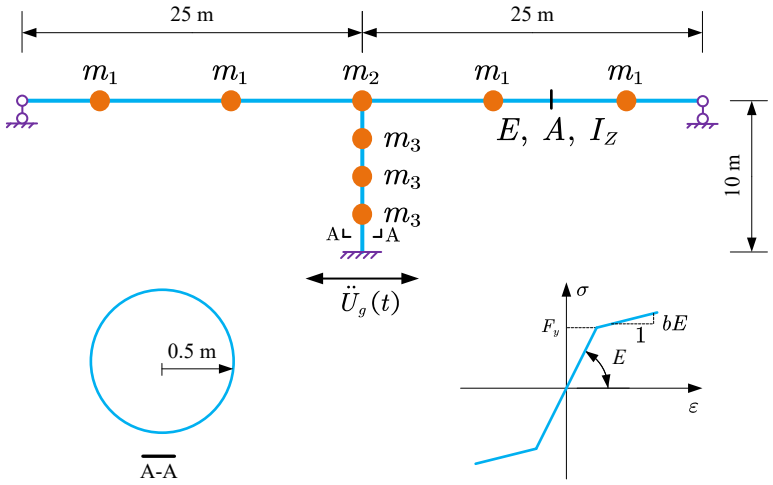



Figure 15: A two-span bridge structure subjected to stochastic seismic excitations.

Using the LPSS method with 625 samples, the MGF of the EVD is computed by repeatedly performing 625 deterministic dynamic structural analyses. For comparison purposes, the crude MCS is also carried out to obtain the MGF (with $10^6$ runs). As depicted in Fig. 16, it can be seen that the 95% confidence interval of the MGF is quite narrow and contains the result by MCS, indicating the accuracy of the estimated MGF.

The EVD can be recovered by the proposed mixture distribution, from the knowledge of the estimated MGF. In Fig. 17, one can observe that the PDF, CDF and POE in log scale are in good agreement with those reference results by crude MCS (with $10^6$ runs). Note that we only perform 625 finite-element based structural dynamic analyses in this study. Therefore, the performance of the proposed method for recovering the EVD is demonstrated once again in terms of accuracy and efficiency. From the recove EVD, the first-passage failure probabilities under different prescribed thresholds can be obtained. As listed in Tab. 5, the results obtained from the proposed method accord

Table 4: Random variables for Example 3

| Variable | Description | Distribution | Mean | C.O.V. |
|---|---|---|---|---|
| $m_1$ | Lumped mass | LN | $2.00\times10^5$ kg | 0.10 |
| $m_2$ | Lumped mass | LN | $2.05\times10^5$ kg | 0.10 |
| $m_3$ | Lumped mass | LN | $1.00\times10^4$ kg | 0.10 |
| $A$ | Cross-sectional area of the beam | G | $5.00\times10^5$ mm$^2$ | 0.05 |
| $I_Z$ | Moment of inertia of the beam | G | $5.00\times10^{12}$ mm$^4$ | 0.05 |
| $E$ | Young's modulus | G | $2.05\times10^5$ Mpa | 0.15 |
| $F_y$ | Yield stress | G | 400 Mpa | 0.15 |
| $b$ | Strain-hardening ratio | G | 0.02 | 0.10 |

well with those by MCS, even for a low level of failure probability. Further, two recently developed methods are also implemented in this example, i.e., MIGLD with fractional moments [24] and mixture of two skew normal distributions (MTSND) with Laplace transform [37]. For fair comparison, the same EVD samples used for the proposed method are adopted to calculate the fractional moments and Laplace transform. Different from the original selection, note that the Laplace transform is evaluated at a new set of points, i.e., $\{s_i\}_{i=1}^7 = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7\}$, so as to avoid very large numbers. The results by these two methods are compared with those by MCS in Fig. 18. Even though there are slight differences between the PDF curve and histogram, the MIGLD model can yield fairly good results concerning the CDF and PDF curves in log scale. On contrary, the MTSND model is unable to produce a comparative POE curve probably due to its limited capability for modelling an EVD. However, it would be difficult to identify which one is better among the MTGIGD and MIGLD for a general problem, and further research is required.

## 6. Conclusions and remarks

In this paper, an efficient approach is proposed for evaluation of the EVD and small first-passage failure probabilities of nonlinear structures with random parameters under stochastic seismic excitations. Unlike the traditional partial characterization, i.e., integer moments or fractional moments, the MGF is first introduced to characterize the

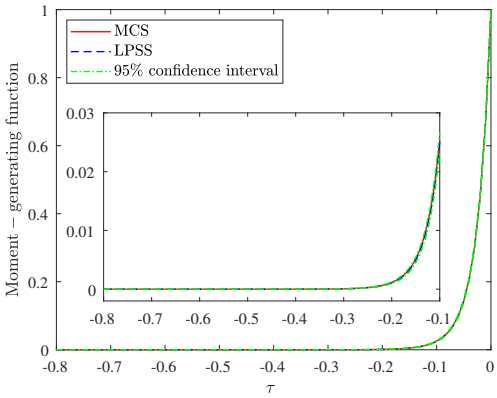
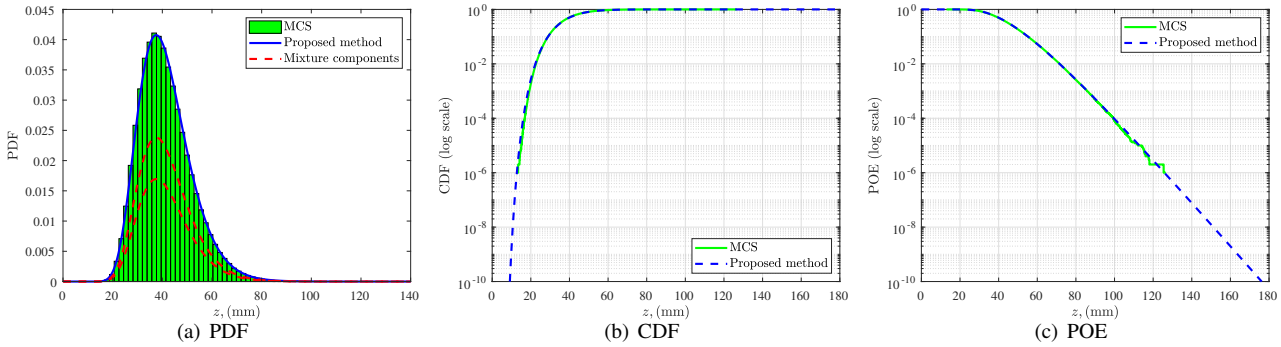
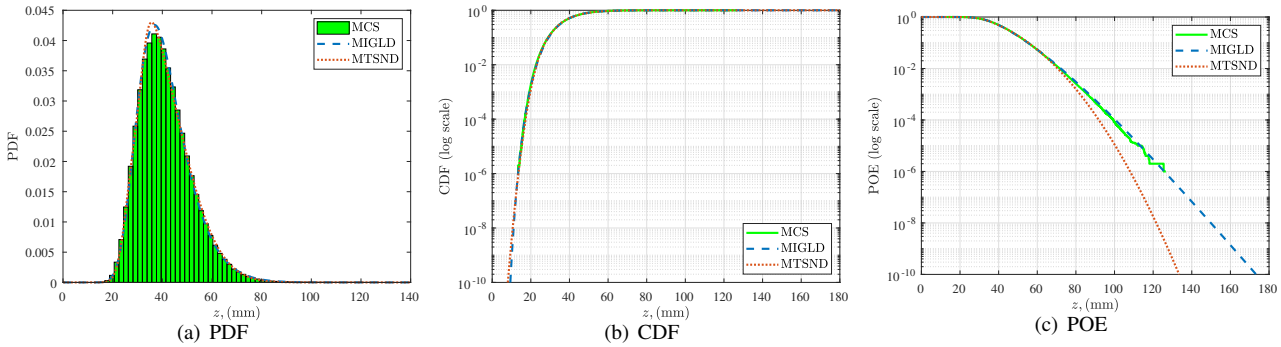



Figure 16: Moment-generating function of the EVD in Example 3.




(a) PDF (b) CDF (c) POE

Figure 17: EVD of displacement at the pile top.




(a) PDF (b) CDF (c) POE

Figure 18: EVD of displacement at the pile top by other methods.

EVD of a general stochastic response process, which is usually defined as a high-dimensional expectation integral. The advantages of MGF characterization are also demonstrated and discussed. Then, a recently developed variance-

Table 5: Comparison of the first-passage failure probabilities under different thresholds for Example 3.

| Method | $N$ | Threshold (mm) | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 60 | 68 | 76 | 84 | 92 | 100 | 108 |
| MCS | $10^6$ | 0.0535 | 0.0171 | 0.0049 | 0.0014 | $3.5200\times10^{-4}$ | $8.1000\times10^{-5}$ | $1.9000\times10^{-5}$ |
| Proposed | 625 | 0.0531 | 0.0170 | 0.0050 | 0.0014 | $3.6637\times10^{-4}$ | $9.3851\times10^{-5}$ | $2.3420\times10^{-5}$ |

reduction sampling technique, named LPSS, is employed to estimate the MGF with a small sample size. Besides, the unbiasedness of the estimator is proven and the confidence interval is given. To recover the EVD from the knowledge of its estimated MGF, a versatile MTGIGD model is proposed. The close-form solution of MGF of the MTGIGD is derived, and a novel parameter estimation technique based on the MGF is developed accordingly. Once the EVD is reconstructed, the first-passage failure probabilities with different thresholds can be simultaneously obtained. Three numerical examples are investigated to validate the proposed approach. The first example considers the EVD of random variables, where the close-form solution of the EVD is available. A nonlinear shear frame structure and a nonlinear bridge structure with random parameters subjected to fully non-stationary stochastic seismic excitations are studied as another two examples. Results indicate that the proposed method: (1) is able to take both the randomness from structural properties and external seismic excitations into consideration; (2) can be applied to very high-dimensional stochastic dynamic systems; (3) is capable of capturing the EVD with high efficiency, not only the main body, but also the distribution tail; (4) can yield very accurate estimation of the first-passage failure probabilities under different prescribed thresholds, even for a small level less than $10^{-4}$; (5) is applicable for evaluation of multiple EVDs in one single run.

Two main types of errors are included in the proposed approach. The first type is caused by the assumption on parametric form for the EVD, which results in biased estimates and cannot be eliminated. Another type of error relates to the fact that the MGF of EVD is evaluated by a sampling method. Since the estimator for the MGF is proven to be unbiased, the latter error can be reduced by increasing the sample size.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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