The Geometrical Origin of Dark Energy

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Abstract

The geometrical formulation of the quantum Hamilton-Jacobi theory shows that the quantum potential is never vanishing, so that it plays the rôle of intrinsic energy. Such a key property selects the Wheeler-DeWitt (WDW) quantum potential $Q[g_{jk}]$ as the natural candidate for the dark energy. This leads to the WDW Hamilton-Jacobi equation with a vanishing kinetic term, and with the identification

$$\Lambda = -\frac{\kappa^2}{\sqrt{g}} Q[g_{jk}] \ .$$

This shows that the cosmological constant is a quantum correction of the Einstein tensor, reminiscent of the von Weizsäcker correction to the kinetic term of the Thomas-Fermi theory. The quantum potential also defines the Madelung pressure tensor. The geometrical origin of the vacuum energy density, a strictly non-perturbative phenomenon, provides strong evidence that it is due to a graviton condensate. Time independence of the regularizied WDW equation suggests that the ratio between the Planck length and the Hubble radius may be a time constant, providing an infrared/ultraviolet duality. Such a duality is related to the local to global geometry theorems for constant curvatures, suggesting a key rôle of Thurston geometry and higher dimensional uniformization theory. This shows that understanding Universe's geometry is crucial for a formulation of Quantum Gravity.

1 Introduction

In spite of the tremendous efforts, understanding the origin of the cosmological constant [1][2][3] is still an open question. In this paper we show that the cosmological constant is naturally interpreted in terms of the quantum potential associated to the spatial metric tensor. The starting point concerns the geometrical formulation of the Quantum Hamilton-Jacobi Equation (QHJE), suggested by the $x - \psi$ duality observed in [4] and introduced in [5] (see [6] for a short review). In the following we call such a formulation, which differs with respect to the Bohmian one, *Geometrical Quantum Hamilton-Jacobi* (GQHJ) theory. Such a theory reproduces the main results of Quantum Mechanics (QM), including energy quantization and tunneling, without using any probabilistic interpretation of the wave function, which is one of the problems in formulating a consistent theory of quantum gravity.

Another consequence of the GQHJ theory is that if space is compact, then there is no notion of particle trajectory [7]. It follows that the GQHJ theory reproduces the results of QM following a geometrical approach without the axiomatic interpretation of the wave function as probability amplitude.

The idea underlying the geometrical derivation of the QHJE is that, like General Relativity (GR), even QM has a geometrical interpretation. This is done by imposing the existence of point transformations connecting different states, which, in turn, leads to a cocycle condition that uniquely fixes the structure of the QHJE. In such a formulation, it has been shown that the quantum Hamilton characteristic function S is non-trivial even in the case of the free particle with vanishing energy. Such a result is deeply related to the solution of Einstein's paradox, discussed later, and concerning the classical limit of bound states in the de Broglie-Bohm theory.

In the present paper we are interested in the fact that, unlike in the de Broglie-Bohm theory, even in the case of a free particle with vanishing energy, the quantum potential is non-trivial [5]. It is just such a property that led in [8] to the proposal that there is a deep relation between QM and gravity. In particular, it was emphasized that the characteristic property of the quantum potential is its universal nature, which is, like gravity, a property possessed by all forms of matter. Subsequently, the deep relation between gravity and QM was also stressed by Susskind in his GR=QM paper [9] and where it is emphasized that where there is quantum mechanics there is also gravity. An explicit relation between quantum mechanics and gravity arises in the case of the free particle with vanishing energy, whose quantum potential includes the Planck length $\ell_P = \sqrt{\hbar G/c^3}$ [8]

$$Q(x) = \frac{\hbar^2}{4m}\{S, x\} = -\frac{\hbar^2}{2m}\frac{\ell_P^2}{(x^2 + \ell_P^2)^2} , \tag{1.1}$$

where $\{f, x\} = f'''/f' - \frac{3}{2}(f''/f')^2$ is the Schwarzian derivative of f. Such a result follows by requiring that, in the case of a free particle of energy E, the QHJE consistently reproduces both the $\hbar \to 0$ and $E \to 0$ limits. On the other hand, since in the problem there are no scales, one is forced to use universal constants. It turns out that the Planck length is the only candidate satisfying the limit conditions, a result related to the invariance of the quantum potential under Möbius transformations of S. Since $E = 0$ corresponds to the ground state, such a non-trivial Q can be considered as an intrinsic energy.

The GQHJ theory includes another relation between QM and geometry of the universe. Namely, compactness of space would imply that the energy spectra are quantized [7]. The essential reason is that solutions of the Schrödinger equation should satisfy gluing conditions, so implying a quantized spectra, even in the case of the free particle [7]. This is also connected to the problem of definition of time. To see this, note that while in classical mechanics we have the equivalence between the definition of trajectory given by $p = \vec{\nabla} S$ and the one following by Jacobi theorem, that is

$$p = \vec{\nabla} S \quad \longleftrightarrow \quad t - t_0 = \frac{\partial S}{\partial E} \ . \tag{1.2}$$

However, at the quantum level the two definitions do not coincide. As shown in [10][7], trajectories, if any, should be defined by the Jacobi theorem. On the other hand, since a compact universe implies a quantized energy spectra, it follows that in this case the derivative of S with respect to E is ill-defined.[1] We then have

$$\text{Compact Universe} \longrightarrow \{E_n\} \longrightarrow \frac{\partial S}{\partial E} \text{ is ill-defined} \longrightarrow \text{no notion of trajectories} \ . \tag{1.3}$$

This leads to a possible relation between the problem of time in GR and the fact that time is not an observable in QM. It should be stressed that in Quantum Field Theory (QFT), even particle's spatial position is represented by parameters, so that, like time, even spatial coordinates are not observables.

It is worth mentioning that the GQHJ theory has been inspired by uniformization theory, with the Schrödinger equation playing the analogous rôle of the uniformizing equation. In particular, the ratio of two linearly independent solutions of the Schrödinger equation, plays the analogous rôle of the uniformizing map. The basic duality, that is the Möbius symmetry, which extends to the QHJE in higher dimension [11], is the defining property of the Schwarzian derivative. Such a duality, that relates small and large scales, and acts like the map between different fundamental domains, is at the heart of the proof of the energy quantization [5]. The above connection between compactness of space, discrete spectra and the analogies with uniformization theory, suggests that higher dimensional uniformization theory is related to the geometry of the universe. This would imply that Thurston's geometry [12] is the appropriate framework to describe the Universe. In this context, the 3-torus plays a central rôle.

Besides (1.1), also (1.3) provides a relation between small and large scales. In particular, as in the case of a particle in a ring of radius R, that gives $E_n = n^2\hbar^2/(2mR^2)$, $n \in \mathbb{Z}$, an analogous relation shows that the energy spacing depends on the parameters defining the compact geometry of space.

We saw that the GQHJ theory indicates that quantum mechanics and general relativity are deeply related. In particular, in the GQHJ theory, time is not a well-defined observable. On the other hand, in the quantum gravity equation par excellence, that is the WheelerDeWitt (WDW) equation [13][14], there is no time variable at all.

The above analysis suggests considering the rôle of the WDW quantum potential. In the case of quantum gravity, the quantum potential represents an intrinsic energy density.

[1] An alternative to the ill-defined derivative $\partial_E S$ is to consider finite differences in the $E - S$ plane. One may easily check that this leads to a heuristic uncertainty relation between E and t.

The WDW quantum potential in the vacuum is then of interest. Actually, in complete analogy with the GQHJ theory and, in particular, with (1.1), the natural interpretation is that such an intrinsic energy density of the vacuum is just the one of dark energy, that is

$$\Lambda = -\frac{\kappa^2}{\sqrt{\bar{g}}} Q[g_{jk}] \ , \tag{1.4}$$

where $\bar{g} = \det g_{ij}$. Let us stress that the novelty here is not that the WDW quantum potential represents an energy density of the vacuum, this follows by dimensional analysis. The novelty is that, according to the GQHJ theory, there is strong evidence that it never vanishes. We then have that the cosmological constant is a quantum correction to the Einstein tensor. This is reminiscent of the von Weizsäcker correction to the kinetic term of the Thomas-Fermi theory [15]. It is worth mentioning that also the Madelung pressure tensor is defined in terms of the quantum potential.

Since (1.4) refers to the vacuum, it follows that there are not dynamical degree of freedoms, so that $S = 0$. This means that (1.4) coincides with the WDW equation in the vacuum. A consequence of our investigation is that since the metric tensor is the only field involved in (1.4), it follows that dark energy is naturally identified with a graviton condensate. In a quite different context, the rôle of the (Bohmian) quantum potential in cosmology, suggesting that the vacuum is a graviton condensate, has been in proposed in [16].

We will argue that, as suggested by Feng's volume average regularization [17], and by the minisuperspace approximation, a regularized WDW equation would need, besides the Planck length, the addition of an infrared scale, that we identify with the Hubble radius $R_H = c/H_0 = 1.36 \cdot 10^{26} m$. Time independence of the regularized WDW equation would then imply that, like R_H, even the Planck length is time-dependent. In particular, time independence of the WDW wave-functional, suggests that the ratio

$$\mathcal{K} = \frac{\ell_P}{R_H} = 5.96 \cdot 10^{-61} \ , \tag{1.5}$$

is a space-time constant. This provides an exact infrared/ultraviolet duality.

The paper is organized as follows. In sect. 2 we shortly review the derivation of the WDW Hamilton-Jacobi equation. Sect. 3 illustrates the main points of the GQHJ theory formulated in [5], focusing on its geometrical origin and on the solution of Einstein's paradox, which in turn is related to the non-triviality of the QHJE for the free particle with $E = 0$. In sect. 4 we show that, contrary to the de Broglie-Bohm formulation, the quantum potential is non-trivial even in the case of the WDW Hamilton-Jacobi equation with ${}^3R = 0$ and vanishing cosmological constant. In sect. 5 we show that the cosmological constant is naturally interpreted in terms of the WDW quantum potential in the vacuum. We then derive the wave-functional in the minisuperspace approximation. Sect. 6 is devoted to the infrared/ultravilet duality in the context of the regularized WDW equation, related to the local to global geometry theorems concerning manifolds of constant curvature. It turns out that the global geometry is strongly constrained in case the local one has constant curvature. This is just the geometrical counterpart of the fact that large scale physics seems constrained by the physics at small scales. Another manifestation of the connection between QM and GR. Finally, we argue that time independence of the regularized WDW equation would imply that $\mathcal{K}$ is a space-time constant.

2 WDW Hamilton-Jacobi equation

In the ADM formulation space-time is foliated into a family of closed 3-dimensional hypersurfaces indexed by the time parameter. We choose the signature $(-,+,+,+)$. Denote by $g_{ij} = {}^4g_{ij}$ the metric tensor of the three dimensional spatial slices. Let $N = (-{}^4g^{00})^{-1/2}$ be the lapse and $N_k = {}^4g_{0k}$ the shift vector. We then have the standard 3+1 decomposition

$$ds^2 = (N_k N^k - N^2)c^2 dt^2 + 2N_k c dx^k dt + g_{jk} dx^j dx^k \ . \tag{2.1}$$

Set $\bar{g} = \det g_{ij}$ and $\kappa^2 = 8\pi G/c^4$. The Einstein-Hilbert Lagrangian density can be equivalently expressed in the form

$$\mathscr{L} = \frac{1}{2\kappa^2} N\sqrt{\bar{g}}({}^3R - 2\Lambda + K^{jk}K_{jk} - K^2) \ , \tag{2.2}$$

where ${}^3R$ is the intrinsic spatial scalar curvature, Λ the cosmological constant, K the trace of the extrinsic curvature

$$K_{jk} = \frac{1}{N}\Big(\frac{1}{2}g_{jk,0} - D_{(j}N_{k)}\Big) \ , \tag{2.3}$$

and D_j denotes the j component of the covariant derivative. Let π^0 and π^k be the momenta conjugate to N and N_k respectively. Since $\mathscr{L}$ is independent of both $\partial_{x_0}N$ and $\partial_{x_0}N_k$, we have the primary constraints $\pi^0 \approx 0$, $\pi^k \approx 0$. Time conservation of the primary constraints implies secondary constraints, given by the weak vanishing of the super-momentum,

$$\mathcal{H}_k = -2D_j\pi^j_k \approx 0 \ , \tag{2.4}$$

and of the super-Hamiltonian,

$$\mathcal{H} = 2\kappa^2 G_{ijkl}\pi^{ij}\pi^{kl} - \frac{1}{2\kappa^2}\sqrt{\bar{g}}({}^3R - 2\Lambda) \approx 0 \ , \tag{2.5}$$

where π_{jk} is the momentum canonically conjugated to g_{jk}, that is

$$\pi^{jk} = -\frac{1}{2\kappa^2}\sqrt{\bar{g}}(K^{jk} - g^{jk}K) \ , \tag{2.6}$$

and

$$G_{ijkl} = \frac{1}{2\sqrt{\bar{g}}}(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl}) \ , \tag{2.7}$$

is the DeWitt supermetric. The conservation in time of the secondary constraints do not imply further constraints.
By a Legendre transform one gets the Hamiltonian

$$H = \int d^3\mathbf{x}(N\mathcal{H} + N^k\mathcal{H}_k) \ , \tag{2.8}$$

showing that N and N^k are the Lagrange multipliers of $\mathcal{H}$ and $\mathcal{H}_k$ respectively.
The implementation of the primary constraints at the quantum level is obtained by setting

$$\hat{\pi}^0 = -i\hbar\frac{\delta}{\delta N} \ , \qquad \hat{\pi}^k = -i\hbar\frac{\delta}{\delta N_k} \ , \tag{2.9}$$

so that

$$- i\hbar\frac{\delta\Psi}{\delta N} = 0 \ , \qquad -i\hbar\frac{\delta\Psi}{\delta N_k} = 0 \ , \tag{2.10}$$

meaning that Ψ does not depend on any of the non-dynamical variables.
At the quantum level the conjugate momenta of a field ϕ would correspond to $-i\hbar\delta_\phi$, so that, since $[\delta^{(3)}] = L^{-3}$, we have $[\delta_\phi] = [\phi]^{-1}L^{-3}$. On the other hand, by (2.6) we have $[\pi_{ij}] = MT^{-2}$, which is different from the dimension of the canonical choice of $\hat{\pi}^{jk}$, namely $[-i\hbar\delta_{g_{jk}}] = ML^{-1}T^{-1}$. We then have

$$\hat{\pi}^{jk} = -i\hbar c\frac{\delta}{\delta g_{jk}} \ , \tag{2.11}$$

which also fixes the normalization of the classical relation

$$\pi^{jk} = c\frac{\delta S}{\delta g_{jk}} \ , \tag{2.12}$$

where S is the functional analogue of Hamilton's principal function. By (2.11), the supermomentum constraint reads

$$\hat{\mathcal{H}}_k\Psi = 2i\hbar c g_{ij}D_k\frac{\delta\Psi}{\delta g_{jk}} = 0 \ , \tag{2.13}$$

which is satisfied if Ψ is invariant under diffeomorphisms of the hypersurface.
The other secondary constraint, that is $\hat{\mathcal{H}}\Psi = 0$, is the WDW equation

$$\hbar c\Big[- 2\ell_P^2 G_{ijkl}\frac{\delta^2}{\delta g_{ij}\delta g_{kl}} - \frac{1}{2\ell_P^2}\sqrt{\bar{g}}({}^3R - 2\Lambda)\Big]\Psi[g_{ij}] = 0 \ , \tag{2.14}$$

where $\ell_P = \sqrt{8\pi\hbar G/c^3} = \kappa\sqrt{\hbar c}$ is the rationalized Planck length.
Let us now consider the key identity

$$\frac{1}{Ae^{\beta S}}\frac{\delta^2\left(Ae^{\beta S}\right)}{\delta g_{ij}\delta g_{kl}} = \beta^2\frac{\delta S}{\delta g_{ij}}\frac{\delta S}{\delta g_{kl}} + \frac{1}{A}\frac{\delta^2 A}{\delta g_{ij}\delta g_{kl}} + \frac{\beta}{A^2}\frac{\delta}{\delta g_{ij}}\left(A^2\frac{\delta S}{\delta g_{kl}}\right) \ , \tag{2.15}$$

which holds for any complex constant β. Setting $\beta = i/\hbar$ and

$$\Psi = Ae^{\frac{i}{\hbar}S} \ , \tag{2.16}$$

in (2.14) gives the WDW Hamilton-Jacobi equation, corresponding to the following quantum deformation of the Hamilton-Jacobi equation

$$2(c\kappa)^2 G_{ijkl}\frac{\delta S}{\delta g_{ij}}\frac{\delta S}{\delta g_{kl}} - \frac{1}{2\kappa^2}\sqrt{\bar{g}}({}^3R - 2\Lambda) - 2(c\kappa\hbar)^2\frac{1}{A}G_{ijkl}\frac{\delta^2 A}{\delta g_{ij}\delta g_{kl}} = 0 \ , \tag{2.17}$$

together with the continuity equation

$$G_{ijkl}\frac{\delta}{\delta g_{ij}}\Big(A^2\frac{\delta S}{\delta g_{kl}}\Big) = 0 \ . \tag{2.18}$$

The last term in (2.17), that is

$$Q = -2(c\kappa\hbar)^2\frac{1}{A}G_{ijkl}\frac{\delta^2 A}{\delta g_{ij}\delta g_{kl}} \ , \tag{2.19}$$

is called the quantum potential.

3 QHJE and Einstein paradox

In this section we shortly discuss the main aspects of the *Geometrical Quantum Hamilton-Jacobi* (GQHJ) theory [5]. Let us start by showing why even in the case ${}^3R = 0$ there are non-trivial S and Q. To this end, it is useful to recall Einstein's paradox (see e.g. Ref. [18] pg. 243). This concerns the issue in Bohmian mechanics when considering the classical limit in the QHJE, in the case of a particle in an infinite potential well. More generally, the problem holds for all states described by a wave-function corresponding to Hamiltonian eigenstates of any one-dimensional bound state. In this case one can easily show that $\psi_E \in L^2(\mathbb{R})$ is proportional to a real function. Therefore, if one sets, as in Bohm theory, $\psi_E = Re^{\frac{i}{\hbar}S}$, then S is a constant. On the other hand, in the Bohmian formulation, $p = \partial_x S$ is identified with the mechanical momentum $m\dot{x}$, so that, quantum mechanically, one would have $p = 0$. Therefore, as in the case of the harmonic oscillator, a quantum particle at rest should start moving in the classical limit, where S and p are non-trivial. In other words, it is clear that it is not possible to get a non-trivial S as the $\hbar \to 0$ limit of a constant function.

The resolution of the paradox is that the quantum analogue of S is not necessarily the phase of the wave function. As we will show, this in fact also underlies the WKB approximation that even if one starts with the identification $\psi = \exp(iS_{WKB}/\hbar)$, with S_{WKB} complex, then real wave functions are identified with a linear combination of in and out waves. In our formulation, such a choice is not ad hoc, rather it follows from the request that the cocycle condition is always satisfied [5]. In particular, note that if $Re^{\frac{i}{\hbar}S}$ is a solution of the stationary Schrödinger equation, then, this is also the case of $Re^{-\frac{i}{\hbar}S}$. This is the key to introduce the so-called bipolar decomposition

$$\psi_E = R\Big(Ae^{\frac{i}{\hbar}S} + Be^{-\frac{i}{\hbar}S}\Big) \ . \tag{3.1}$$

As a result, in the case of a real ψ_E, the only constraint is just $|A| = |B|$ and one gets a non-trivial S with a well-defined classical limit. Such a solution of Einstein's paradox is a consequence of the GQHJ theory, that excludes in a natural way, and from the very beginning, the existence of states with a constant S [5]. The use of the bipolar decomposition was previously discussed by Floyd [10].

Later we will see that in the functional case of the WDW Hamilton-Jacobi equation, the corresponding S and the quantum potential assume a non-trivial rôle even when ${}^3R = 0$. This is just the functional analogue of basic properties of the quantum potential that we now discuss.

The main point that characterizes the non-trivial properties of the quantum potential is its connection with the Möbius invariance of the Schwarzian derivative $\{f, x\}$, that, in order to be well-defined, requires that $f \in C^2(\mathbb{R})$ and $\partial_x^2 f$ differentiable on $\mathbb{R}$. The continuity equation $\partial_x(R^2\partial_x S) = 0$ implies that R is proportional to $(\partial_x S)^{-1/2}$, so that the quantum potential can be expressed in terms of S only

$$Q = \frac{\hbar^2}{4m}\{S, x\} \ , \tag{3.2}$$

and the QHJE associated to a stationary Schrödinger equation reduces to the single equation

$$\frac{1}{2m}\Big(\frac{\partial S}{\partial x}\Big)^2 + V - E + Q = 0 \ . \tag{3.3}$$

Let us consider the basic identity

$$\left(\frac{\partial S}{\partial x}\right)^2 = \frac{\beta^2}{2}\left(\left\{\mathrm{e}^{\frac{2i}{\beta}S}, x\right\} - \{S, x\}\right) \ , \tag{3.4}$$

where β is a constant with the dimension of an action. Such an identity implies that the QHJE (3.5) can be also expressed in the form

$$\left\{\exp\left(\frac{2i}{\hbar}S\right), x\right\} = \frac{4m^2}{\hbar}(E - V) \ . \tag{3.5}$$

The solution of this non-linear differential equation is

$$\exp\left(\frac{2i}{\hbar}S\right) = \gamma\left[\frac{\psi^D}{\psi}\right] \ , \tag{3.6}$$

where ψ and ψ^D are two real linearly independent solutions of the stationary Schrödinger equation and $\gamma[f]$ is an arbitrary, generally complex, Möbius transformation of f

$$\gamma[f] = \frac{Af + B}{Cf + D} \ . \tag{3.7}$$

Thanks to the Möbius invariance of the Schwarzian derivative, one may consider a Möbius transformation of $\exp(2iS/\hbar)$, that we denote again by

$$\gamma\left[\exp\left(\frac{2i}{\hbar}S\right)\right] \ , \tag{3.8}$$

leaving $V - E$ invariant. On the other hand, since this corresponds to the transformation

$$S \longrightarrow \tilde{S} = \frac{\hbar}{2i}\log\gamma\left[\exp\left(\frac{2i}{\hbar}S\right)\right] \ , \tag{3.9}$$

we see that there is a non-trivial mixing between the kinetic term and the quantum potential in (3.3).

In [5] the QHJE was derived by a slight modification of the way one gets the classical Hamilton-Jacobi equation. Namely, instead of looking for maps from (x, p) to (X, P), seen as independent variables, such that the new Hamiltonian is the trivial one, $\tilde{H} = 0$, we looked for transformations $x \rightarrow \tilde{x}$ such that $\tilde{V} - \tilde{E} = 0$, but with the transformation of p fixed by imposing that $S(x)$ transforms as a scalar function. That is

$$\tilde{S}(\tilde{x}) = S(x) \ , \tag{3.10}$$

holding for any pair of physical systems, including the one with $V - E = 0$.

A key consequence of (3.10) is that $S(x)$ can never be a constant. In particular, imposing that (3.10) holds even when the coordinate x refers to the state with $V - E = 0$, forces the introduction of an additional term in the classical Hamilton-Jacobi equation. Then, one considers three arbitrary states, denoted by A, B and C, and imposes the condition coming from the commutative diagram of maps

$$\begin{array}{ccccc} & & B & & \\ & \nearrow & & \searrow & \\ A & & \longrightarrow & & C \end{array}$$

Implementation of such a consistency condition is equivalent to a cocycle condition that fixes the additional term to be the quantum potential [5]. The outcome is just the QHJE. Another feature of the above formulation is that the quantum potential is never trivial even in the case $V - E = 0$. In particular, a careful analysis of the quantum potential for a free particle with vanishing energy shows that the $\hbar \to 0$ and $E \to 0$ limits in the case of the free particle of energy E, leads to the appearance of the Planck length in the expression for the quantum potential Q of a free particle with $E = 0$ (1.1). It should be stressed that the present formulation leads to a well-defined power expansion in $\hbar$ for S. This is different with respect to the WKB approximation since $S_{\rm WKB}$ is defined by

$$\psi = \exp\left(\frac{i}{\hbar}S_{\rm WKB}\right) , \tag{3.11}$$

so that, in general, $S_{\rm WKB}$ takes complex values. The GQHJ theory is also different with respect to the de Broglie-Bohm theory. Besides the case of real wave-functions illustrated above, also the quantum potential (1.1) turns out to be different. The difference also appears in the case of the free particle of energy E. Indeed, the solution of Eq.(3.3) with $V = 0$ is

$$S = \frac{\hbar}{2i}\log\left(\frac{Ae^{\frac{2i}{\hbar}\sqrt{2mE}x}+B}{Ce^{\frac{2i}{\hbar}\sqrt{2mE}x}+D}\right) . \tag{3.12}$$

Here the constants are chosen in such a way that $S \neq \pm\sqrt{2mE}x$. Such a choice, fixed by the consistency condition that the non-trivial S_0 is obtained from S in the $E \to 0$ limit, relates p-x duality, also called Legendre duality, and Möbius invariance of the Schwarzian derivative [5]. Another consistency condition comes from the classical limit. Since $S^{cl} = \pm\sqrt{2mE}x$, we have

$$\lim_{\hbar\longrightarrow 0}\log\left(\frac{Ae^{\frac{2i}{\hbar}\sqrt{2mE}x}+B}{Ce^{\frac{2i}{\hbar}\sqrt{2mE}x}+D}\right)^{\frac{\hbar}{2i}} = \pm\sqrt{2mE}x , \tag{3.13}$$

implying that the constants A, B, C and D depend on $\hbar$ [5].

The above analysis shows that S is the natural quantum analog of the classical action. In particular, the formulation solves Einstein's paradox and the power expansion of S in $\hbar$ is completely under control. Furthermore, it leads to a dependence of S on the fundamental constants, shedding light on the quantum origin of interactions. It also implies that if space is compact, then time parametrization cannot be defined [7]. The formulation, that follows from the simple geometrical principle (3.10), extends to arbitrary dimensions and to the relativistic case as well [11]. It reproduces, together with other features, such as energy quantization, the non existence of trajectories of the Copenhagen interpretation, without assuming any interpretation of the wave-function.

4 The WDW Hamilton-Jacobi equation with ${}^3R = 0$ and $\Lambda = 0$.

Let us go back to the WDW Hamilton-Jacobi equation by considering the case ${}^3R = 0$, $\Lambda = 0$, so that the WDW equation reduces to

$$G_{ijkl}\frac{\delta^2}{\delta g_{ij}\delta g_{kl}}\Psi = 0 \ . \tag{4.1}$$

Setting $\Psi = Ae^{\frac{i}{\hbar}S}$, the WDW Hamilton-Jacobi equation reads

$$G_{ijkl}\frac{\delta S}{\delta g_{ij}}\frac{\delta S}{\delta g_{kl}} - \frac{\hbar^2}{A}G_{ijkl}\frac{\delta^2 A}{\delta g_{ij}\delta g_{kl}} = 0 \ . \tag{4.2}$$

Note that in this case the formulation does not suffer the well known problem of the WDW equation, due to the presence of the order two functional derivative at the same point: such an operator is in general ill-defined since it may lead to $\delta^{(3)}(0)$-singularities. On the other hand, the wave functional $\Psi[g_{ij}]$ now depends linearly on g_{ij}, so that the action of the second-order functional derivative on $\Psi[g_{ij}]$ is well-defined. We then have

$$\Psi[g_{ij}] = A\mathrm{e}^{\frac{i}{\hbar}S} = \mathcal{T}g + C \ , \tag{4.3}$$

where

$$\mathcal{T}g := \int d^3\mathbf{x}\mathcal{T}^{jk}(\mathbf{x})g_{jk}(\mathbf{x}) \ , \tag{4.4}$$

with $\mathcal{T}_{jk}(\mathbf{x})$ an arbitrary complex tensor density field of weight 1 and C a complex constant. The general expression of S is

$$\exp\left(\frac{2i}{\hbar}S\right) = \frac{\mathcal{T}g + C}{\bar{\mathcal{T}}g + \bar{C}} \ , \tag{4.5}$$

and for A we have

$$A = |\mathcal{T}g + C| \ . \tag{4.6}$$

By (2.12) and (4.5), it follows that at the quantum level the momentum conjugate to g_{jk} is

$$\pi^{jk} = c\frac{\delta S}{\delta g_{jk}(\mathbf{x})} = \hbar c\,\mathrm{Im}\left(\frac{\mathcal{T}^{jk}(\mathbf{x})}{\mathcal{T}g + C}\right) \ , \tag{4.7}$$

so that the kinetic term in the WDW Hamilton-Jacobi equation reads

$$2(c\kappa)^2 G_{ijkl}(\mathbf{x})\frac{\delta S}{\delta g_{ij}(\mathbf{x})}\frac{\delta S}{\delta g_{kl}(\mathbf{x})} = \frac{2(c\kappa\hbar)^2}{\sqrt{\bar{g}}}\left(\frac{\mathcal{T}_{kl}(\mathbf{x})}{\mathcal{T}g + C}\right)\mathrm{Im}\left(\frac{\mathcal{T}^{kl}(\mathbf{x})}{\mathcal{T}g + C}\right) - \frac{1}{2}\Big[\mathrm{Im}\left(\frac{\mathrm{Tr}\,\mathcal{T}(\mathbf{x})}{\mathcal{T}g + C}\right)\Big]^2\Big\} \ . \tag{4.8}$$

Note that, by (4.2), this also corresponds to $-Q[g_{jk}]$. Furthermore, one may easily check that such an expression of $Q[g_{jk}]$ is just the functional analogue of the quantum potential of the free particle of vanishing energy (1.1).

5 Cosmological constant from the quantum potential

The discrepancy between the measured value of the cosmological constant and the theoretical prediction follows by considering Λ/κ^2 as a contribution to the effective vacuum energy density $\rho_{eff} = \rho + \Lambda/\kappa^2$, where $\langle T_{\mu\nu}\rangle = \rho g_{\mu\nu}$. Considering the QFT vacuum energy density as due to infinitely many zero-point energy of harmonic oscillators, we get (here $\hbar = c = 1$)

$$\rho = \int_0^{\Lambda_{UV}} \frac{4\pi k^2 dk}{(2\pi)^3} \frac{1}{2}\sqrt{k^2+m^2} \approx \frac{\Lambda_{UV}^4}{16\pi^2} \approx 10^{71}\mathrm{GeV}^4 \ , \tag{5.1}$$

where Λ_{UV} is the Planck mass. A result which is in complete disagreement with the estimation, based on experimental data, $\rho_{eff} \approx 10^{-47}\mathrm{GeV}^4$.

A problem with the above derivation is that it is based on the perturbative formulation of QFT. This corresponds to use the canonical commutation relations of the free theory that selects the vacuum of the free theory. On the other hand, the true vacuum of nontrivial QFT's is highly non-perturbative and is not unitarily equivalent to the free one. As a matter of fact, perturbation theory erroneously treats the quantum fields evolving as the free ones between point-like interaction events. From the physical point of view, the rôle of renormalization is to iteratively change the parameters of the theory, that then will depend on the physical scale. In other words, perturbation theory is a way to mimic the interacting theory by a free one, with the parameters becoming scale dependent.

It has been observed in [19] that the cutoff corresponding to the value of the cosmological constant may be related to an infrared/ultraviolet duality. In particular, the authors of [19], inspired by the Bekenstein bound $S \lesssim \pi M_P^2 L^2$ for the total entropy in a volume of size L^3, proposed the following relation between the infrared cutoff $1/L$ and Λ_{UV}

$$L^3\Lambda_{UV}^4 \lesssim L M_P^2 \ . \tag{5.2}$$

An estimation of the infrared scale of QFT can be derived by considering the precision tests of the electron's anomalous magnetic moment a_e. In this respect, as observed in [20], an estimate of the correction to the usual calculation imposed by the IR scale μ is

$$\delta a_e \approx \frac{\alpha}{\pi}\left(\frac{\mu}{m_e}\right) \approx 4 \cdot 10^{-9}\frac{\mu}{1\mathrm{eV}} \ . \tag{5.3}$$

Requiring that such an indeterminacy be smaller than the uncertainty of the theoretical prediction for a_e gives

$$\mu \leq 10^{-2}\,\mathrm{eV} \ , \tag{5.4}$$

which is the value corresponding to the cutoff that leads to the same order of magnitude of the experimental value of ρ.

The above analysis indicates that the cosmological constant is related to the infrared problem, a non-perturbative phenomenon concerning the structure of the vacuum which has physically measured consequences. For example, QED finite transition amplitudes are obtained by summing over states with infinitely many soft photons.

We saw that, unlike in Bohmian mechanics, the quantum potential is never vanishing [5]. This is the case even for the free particle of vanishing energy, implying that the quantum potential plays the rôle of particle intrinsic energy. Furthermore, Eq.(1.1) shows that the quantum potential includes the Planck length, which arises by consistency conditions in considering the $E \to 0$ and $\hbar \to 0$ limits [8]. This was one the reasons suggesting a strict relationship between QM and GR [8] (see also [9]). We then have the following result:

The WDW quantum potential in the vacuum corresponds to an intrinsic energy density.

It is then natural to make the identification

$$Q[g_{jk}] = -\sqrt{g}\rho_{\text{vac}} \ , \tag{5.5}$$

$\rho_{\text{vac}} = \Lambda/\kappa^2$. Since in this case the only degrees of freedom are the ones associated to the metric tensor, the dark energy should correspond to a graviton condensate.
In this context, we stress that the vacuum energy is a purely quantum property and the absence of the kinetic term does not imply, as in the de Broglie-Bohm theory, the Einstein's paradox. The fact that the cosmological constant is a quantum correction to the Einstein tensor given in terms of the quantum potential, is reminiscent of the von Weizsäcker correction to the kinetic term of the Thomas-Fermi theory. Furthermore, we note that the quantum potential also defines the Madelung pressure tensor.
Now observe that the absence of propagating degrees of freedom implies that the quantum potential in (5.5) corresponds to the one of the WDW Hamilton-Jacobi equation without the kinetic term, that is

$$S = 0 \ . \tag{5.6}$$

Let us choose a metric with vanishing ${}^3R$. Eq.(5.6) implies a nice mechanism, namely by (2.17) it follows that in this case the continuity equation is trivially satisfied, so that Eq.(5.5), that by (5.6) is the full WDW Hamilton-Jacobi equation, coincides with the WDW equation (2.14) with $\Psi = A$. In this way the contribution to the WDW Hamilton-Jacobi equation comes only from the quantum potential. In other words, since by (5.6) Ψ takes real values, it follows by the definition of $Q[g_{ij}]$ in (2.19), that Eq.(5.5) is just the WDW equation in the vacuum

$$-2\ell_P^2 G_{ijkl}\frac{\delta^2}{\delta g_{ij}\delta g_{kl}}A = -\frac{\sqrt{g}}{\ell_P^2}\Lambda A \ . \tag{5.7}$$

We now adapt the analysis that led to Eq.(1.1), to the case of Eq.(5.7). The main difference is that now the problem includes both small and large scales.
To see how fundamental constants may appear in the present context, we derive an explicit solution of Eq.(5.7) in the case of the Friedmann-Lemaître-Robertson-Walker background. Let us then consider the line element

$$ds^2 = -N(t)^2c^2dt^2 + a^2(t)d\Sigma_k^2 \ , \tag{5.8}$$

where

$$d\Sigma_k^2 = \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \tag{5.9}$$

is the spatial line element of constant curvature k. In such an approximation the Hilbert-Einstein equation in the vacuum, with $k = 0$, reads

$$S_{HE} = \frac{V_0}{\kappa^2}\int dt\Big(-\frac{3a\dot{a}^2}{Nc} - Nca^3\Lambda\Big) \ , \tag{5.10}$$

where

$$V_0 = \int dr d\theta d\phi r^2 \sin\theta \ . \tag{5.11}$$

In such a minisuperspace approximation, the WDW equation (5.7) reads

$$\left(\frac{d^2}{da^2} + 12\frac{V_0^2\Lambda}{\ell_P^4}a^4\right)A = 0 \ , \tag{5.12}$$

whose solution is a linear combination of the Bessel functions of first and second kind

$$A(a) = \sqrt{a}\Big(\alpha J_{1/6}(Ca^3) + \beta Y_{1/6}(Ca^3)\Big) \ , \tag{5.13}$$

where

$$C = 2\frac{V_0}{\ell_P^2}\sqrt{\frac{\Lambda}{3}} \ . \tag{5.14}$$

In this approximation of the WDW equation, besides the Planck length, there is also another fundamental constant, Λ itself, and a natural choice, suggested by (5.14), is

$$V_0 = \Lambda^{-3/2} \ . \tag{5.15}$$

6 Infrared/ultraviolet duality and local to global geometry theorems

We saw that the WDW equation includes both large and small scales that can be interpreted as infrared and ultraviolet cutoffs, that should appear in a well-defined version of the WDW equation. It is clear that such an investigation should include a careful analysis of the involved local and global geometries.

A well-known problem with the WDW equation, is that due to the second-order functional derivative evaluated at the same point, it presents, in general, $\delta^{(3)}(\mathbf{x}=\mathbf{0})$-singularities. This is analogous to the normal ordering singularities in QFT, due to the joining of two legs of the same vertex; so giving the Feynman propagator evaluated at 0. Similarly, the infinite volume limit can be interpreted as the integral representation of the δ-distribution in momentum space at zero momentum. In other words, $\delta^{(3)}(\mathbf{p}=\mathbf{0})$ can be interpreted as the infinite volume limit of the space volume divided by $(2\pi)^3$. A related method is used, for example, in deriving the effective action for $\lambda\phi_4^4$ in Euclidean space to get the dependence of the coupling constant on the mass scale. In that case, the infrared regularization was done by supposing that the Euclidean space is S^4 rather than $\mathbb{R}^4$, and then considering S^4 as the surface of a five-dimensional sphere, so that one obtains a finite result and avoids such an infrared divergence.

The outcome of such an analysis is that in general singularities may be removed by taking into account the physical scales. What is crucial is to preserve diffeomorphism invariance. In this respect, we recall that $[-i\hbar\delta_{g_{jk}}] = ML^{-1}T^{-1}$, while for the δ-distribution in configuration space we have $[\delta^{(3)}] = L^{-3}$. This means that, besides the Planck length, a well-defined regularized version of the WDW equation should also involve a large scale cutoff. An explicit example of such a mechanism is the one in the interesting paper by Feng, who proposed the volume average regularization [17]. Feng's regularization introduces a factor $1/V$, with V naturally identified with the space volume. In particular, Feng's regularized WDW equation has the structure

$$\hat{\mathcal{H}}[\ell_P, V, \Lambda; g_{ij}]\Psi[g_{ij}] = 0 \ , \tag{6.1}$$

see, for example, Eq.(2.24) of [17]. Feng's regularization is related to the standard heat kernel and point splitting regularizations [21]-[25]. In particular, it corresponds to averaging the displacement in the point splitting regularization.
The dual rôle of the δ-distributions in x and p spaces, shows that infrared and ultraviolet dualities are related to x-p duality, another manifestation of the dual property of the Fourier transform, which in fact is at the heart of the Heisenberg uncertainty relations.
Even if a well-defined version of the WDW functional differential equation is still unknown, it is clear that, as Eq.(5.12) shows, besides the Planck length it should also include a cosmological scale, making manifest an infrared/ultraviolet duality. A related issue concerns the Möbius symmetry of the Schwarzian derivative. In this respect, it was shown in [11] that even in the geometrical derivation of the quantum Hamilton-Jacobi equation in higher dimensions, there is an underlying global conformal symmetry, the generalization of the Möbius symmetry of the Schwarzian derivative. This is a crucial property, whose implementation requires a compact space, which in turn would imply that the energy spectra are quantized [7]. As a consequence, since by Jacobi theorem [10]

$$t - t_0 = \frac{\partial S}{\partial E} \ , \tag{6.2}$$

it follows that time-parametrization is ill-defined for discrete spectra, so that trajectories would never exist if space is compact [7]. The mentioned conformal transformation includes the space inversion relating large and small scales

$$x_k \to l^2 x_k / r^2 \ , \tag{6.3}$$

$r^2 = \sum_1^D x_k^2$, with l a length scale. This is another hint that an infrared/ultraviolet duality should appear in the cosmological context, and then in a well-defined version of the WDW equation. A similar situation arises in the uniformization theory by Klein, Koebe and Poincaré, where negatively curved Riemann surfaces have fundamental domains in their universal covering, e.g. the upper half-plane $\mathbb{H}$, which are related by Fuchsian transformations, that is discrete subgroups of[2] $\mathrm{SL}(2, \mathbb{R})$.
Finding an infrared/ultraviolet duality in the cosmological context could be used to consider the local to global theorems relating local and global geometries. In particular, according to Thurston [12], the global geometry is strongly constrained in case the local one has constant curvature. Interestingly, according to Bieberbach [26][27], all compact flat manifolds are finitely covered by tori, a result that in three dimension was previously obtained by Schoenflies [28]. The underlying idea is that the local structure of space provides information on its global structure, which includes the information on the topological structure and on points at large distances.

[2]This is in fact deeply related to the weak/strong duality transformations of the effective coupling constant $\tau \to -1/\tau$ of Seiberg-Witten theory, that, in the case of pure SU(2), posses a $\Gamma(2) \subset \mathrm{SL}(2, \mathbb{R})$ symmetry.

The discussed connection between compactness of space, discrete spectra and the analogies with uniformization theory, suggests that higher dimensional uniformization theory is right framework to investigate the geometry of the universe.
As explicitly seen in the minisuperspace approximation, it is clear that the solution of a well-defined version of the WDW equation should involve transcendental functions. As such, the dependence on the cosmological constant should be in the form of some dimensionless constant $\mathcal{K}$, that is

$$A[g_{ij}] = F[\mathcal{K}; g_{ij}] \ . \tag{6.4}$$

Let us stress that such a constant should be the same for any choice of the time slicing in the ADM foliation. So, in particular, $\mathcal{K}$ should be time-independent. Since the Planck length is naturally interpreted as ultraviolet cutoff, the constant $\mathcal{K}$ should have the form

$$\mathcal{K} = \frac{\ell_P}{L_U} \ , \tag{6.5}$$

with L_U a fundamental length describing the geometry of the Universe. The obvious candidate for L_U is the Hubble radius $R_H = c/H_0 = 1.36 \cdot 10^{26} m$, whose size is of the same order of the radius of the observable universe and that, besides Λ, is the only quantity which is spatially constant. We then have,

$$\mathcal{K} = \frac{\ell_P}{R_H} = 5.96 \cdot 10^{-61} \ . \tag{6.6}$$

Note that since A must depend on Λ, the space-time independence of $\mathcal{K}$ implies that

$$A[g_{ij}] = F[C\sqrt{\Lambda}; g_{ij}] \ , \tag{6.7}$$

with C, $[C] = L$, a space-time constant.
Eq.(6.6) implies that the Planck length is time-dependent. This is in agreement with the Dirac idea that fundamental constants are dynamical variables. On the other hand, the most natural candidate for time variation is just the Planck constant $\hbar$. The point is that the Einstein field equation contains Λ, c and G, and a possible time dependence of such constants would break diffeomorphism invariance. Therefore, preserving such an invariance means that only $\hbar$, that in fact appears only in considering the WDW equation, can change. On the other hand, Eq.(6.6) implies an infrared/ultraviolet duality, where the large scale is given by R_H, whose time dependence is the same of the scale representing the quantum regime, that is (the square root of) $\hbar$.
We stress that time variation of fundamental constants is a crucial and widely investigated subject [29][30][31]. In a different context, time dependence of the Planck constant has been investigated in the interesting paper [32].
We conclude by observing that very recently, in [33], it has been argued by a different perspective, that the formulation of the quantum Hamilton-Jacobi theory introduced in [5], could in fact be at the origin of the cosmological constant.

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