Comments and Corrections to "New Results on the Fluctuating Two-Ray Model with Arbitrary Fading Parameters and Its Applications"

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***Abstract*—In [1], closed-form expressions were proposed for the distribution of the received signal-to-noise ratio in the fluctuating two-ray fading model with arbitrary fading parameters for millimetre wave bands. This correspondence provides some comments and corrections to the published closed-form expressions along with several novel analytical results.**

***Index Terms*—Fluctuating two-ray fading model, radio propagation, millimetre wave bands, wireless communications.**

I. INTRODUCTION

The analysis presented in [1] provided closed-form expressions for the Probability Density Function (PDF) and Cumulative Distribution Function (CDF) of the instantaneous Signal-to-Noise Ratio (SNR) in the Fluctuating Two-Ray (FTR) channel fading model with arbitrary fading parameters for millimetre wave bands. The interest of these results lies in their algebraic simplicity, which makes them significantly more analytically tractable than other earlier FTR channel fading models previously proposed in the literature. Unfortunately, due to the wrong integration expression in [2], the closed-form expressions provided in [1] for the PDF contain some errors that are propagated through the rest of the provided analytical results, including the expressions for the CDF, channel capacity and bit error rate. This correspondence provides corrections to these errors along with several additional observations on the correct evaluation of the analytical results. Moreover, some alternative novel analytical results are presented as well.

II. COMMENTS AND CORRECTIONS

The derivation of the PDF for the instantaneous SNR in the FTR fading model involves the following integral [1, eq. (22)]:

$$s_k = \int_0^\infty u^{j+m-1} e^{-(m+K)u} I_{2l-k}(-K\Delta u)\,du, \qquad (1)$$

where j,k,l are natural numbers (including zero), m,K,Δ are the FTR channel fading parameters (see [1] for details), and $I_\nu(x)$ is the modified Bessel function of the first kind [3, eq. (8.445)]. This integral is resolved in [1] by directly applying the second solution provided in [2, eq. (2.15.3.2)], which unfortunately leads to incorrect analytical results in [1]. The confusion arises from the fact that the second result for the integral in [2, eq. (2.15.3.2)] involves the associated Legendre function (or spherical function) of the first kind, $P_\nu^\mu(x)$, which has two different definitions depending on its argument:

$$P_\nu^\mu(x) = \begin{cases} A_\nu^\mu(x), & |x|<1 \\ B_\nu^\mu(x), & x>1 \end{cases} \qquad (2)$$

where $A_\nu^\mu(x)$ is given by [3, eq. (8.704)][4, eq. (14.3.1)] and $B_\nu^\mu(x)$ is given by [3, eq. (8.702)][4, eq. (14.3.6)], respectively. The relation between both expressions can be obtained from the value of the function $P_\nu^\mu(x)$ on the cut [4, eq. (14.23.1)]:

$$A_\nu^\mu(x) = e^{\pm\frac{\pi}{2}\mu i} B_\nu^\mu(x \pm i0), \qquad (3)$$

where $i=\sqrt{-1}$ is the imaginary number.

The integral of interest in [2, eq. (2.15.3.2)] is of the form $\int_0^\infty x^{\alpha-1} e^{-px} I_\nu(cx)\,dx$, which can be resolved by using the relation [3, eq. (8.406.1)]:

$$I_\nu(cx) = e^{-\frac{\pi}{2}\nu i} J_\nu(c\, e^{\frac{\pi}{2}i} x), \qquad (4)$$

where $J_\nu(x)$ is the (standard) Bessel function of the first kind [3, eq. (8.402)], along with the following result from the fourth solution in [3, eq. (6.621.1)]:

$$\int_0^\infty x^{\alpha-1} e^{-px} J_\nu(qx)\,dx$$
$$= \Gamma(\alpha+\nu)\,(p^2+q^2)^{-\frac{\alpha}{2}} P_{\alpha-1}^{-\nu}\left(\frac{p}{\sqrt{p^2+q^2}}\right) \qquad (5)$$
$$= \Gamma(\alpha+\nu)\,(p^2+q^2)^{-\frac{\alpha}{2}} A_{\alpha-1}^{-\nu}\left(\frac{p}{\sqrt{p^2+q^2}}\right), \qquad (6)$$

which is valid for $p>0$ and $q>0$. Notice that the condition $p>0$ and $q>0$ implies that the argument x of $P_\nu^\mu(x)$ in (5) is strictly lower than one. Thus, for the sake of clarity, $P_\nu^\mu(x)$ has been rewritten as $A_\nu^\mu(x)$ in (6).

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The integral in [2, eq. (2.15.3.2)] can be resolved as follows:

$$\int_0^\infty x^{\alpha-1}e^{-px}I_\nu(cx)\,dx$$
$$= e^{-\frac{\pi}{2}\nu i}\int_0^\infty x^{\alpha-1}e^{-px}J_\nu(c\,e^{\frac{\pi}{2}i}x)\,dx \qquad (7)$$
$$= e^{-\frac{\pi}{2}\nu i}\,\Gamma(\alpha+\nu)\,(p^2-c^2)^{-\frac{\alpha}{2}}A_{\alpha-1}^{-\nu}\left(\frac{p}{\sqrt{p^2-c^2}}\right) \qquad (8)$$
$$= \Gamma(\alpha+\nu)\,(p^2-c^2)^{-\frac{\alpha}{2}}B_{\alpha-1}^{-\nu}\left(\frac{p}{\sqrt{p^2-c^2}}\right), \qquad (9)$$

where (7) is obtained by using (4), (8) is obtained by using (6), and (9) is obtained by using (3). Notice that the second solution provided in [2, eq. (2.15.3.2)] is the same expression as in (8) but using the standard notation $P_\nu^\mu(x)$. This notation is certainly unfortunate and confusing since the argument of $P_\nu^\mu(x)$ in (8) is $x = p/\sqrt{p^2-c^2} > 1$ and this may mislead the reader to assume that $P_\nu^\mu(x)$ in [2, eq. (2.15.3.2)] should be evaluated using $B_\nu^\mu(x)$, when it actually needs to be evaluated using $A_\nu^\mu(x)$ as shown in (8). The analysis presented in [1] misinterprets $P_\nu^\mu(x)$ in [2, eq. (2.15.3.2)] as $B_\nu^\mu(x)$, which leads to incorrect analytical results. A more convenient notation for the solution of the integral in [2, eq. (2.15.3.2)] is the expression shown in (9), which is consistent with the fact that the argument of the associated Legendre function is greater than one and therefore removes any possible ambiguity.

Based on the discussion above, the solution to the integral in [2, eq. (2.15.3.2)] can be written unambiguously as:

$$\int_0^\infty x^{\alpha-1}e^{-px}I_\nu(cx)\,dx = \qquad (10)$$
$$\Gamma(\alpha+\nu)(p^2-c^2)^{-\frac{\alpha}{2}}P_{\alpha-1}^{-\nu}\left(\frac{p}{\sqrt{p^2-c^2}}\right),\quad c>0,$$

where $P_\nu^\mu(x)$ in (10) is given by [3, eq. (8.702)].

The result in (10) is valid for $c > 0$ only, i.e. positive arguments of $I_\nu(x)$. However, the integral of interest in (1) is evaluated over negative arguments of $I_\nu(x)$. From [3, eq. (8.445)], one can write $I_\nu(-cx) = -1^\nu I_\nu(cx)$, which yields the following result for negative arguments:

$$\int_0^\infty x^{\alpha-1}e^{-px}I_\nu(cx)\,dx = \qquad (11)$$
$$\Gamma(\alpha+\nu)(p^2-c^2)^{-\frac{\alpha}{2}}(-1)^\nu P_{\alpha-1}^{-\nu}\left(\frac{p}{\sqrt{p^2-c^2}}\right),\quad c<0,$$

where $P_\nu^\mu(x)$ in (11) is given by [3, eq. (8.702)].

Based on (11), the correct solution to (1) is obtained as:

$$s_k = \Gamma(j+m+2l-k)\left((m+K)^2-(K\Delta)^2\right)^{-\frac{j+m}{2}}$$
$$\times(-1)^{2l-k}\;P_{j+m-1}^{k-2l}\left(\frac{m+K}{\sqrt{(m+K)^2-(K\Delta)^2}}\right), \qquad (12)$$

which corrects the expression provided in [1, eq. (23)].

Consequently, [1, eq. (9)] needs to be corrected as follows:

$$d_j \triangleq \sum_{k=0}^{j}\binom{j}{k}\left(\frac{\Delta}{2}\right)^k\sum_{l=0}^{k}\binom{k}{l}\Gamma(j+m+2l-k) \qquad (13)$$
$$\times\left((m+K)^2-(K\Delta)^2\right)^{-\frac{j+m}{2}}$$
$$\times(-1)^{2l-k}\;P_{j+m-1}^{k-2l}\left(\frac{m+K}{\sqrt{(m+K)^2-(K\Delta)^2}}\right),$$

where $P_\nu^\mu(x)$ in (13) is given by [3, eq. (8.702)].

It is worth noting that the correct evaluation of (13) requires some specific considerations. For arguments greater than one (as it is here the case), the associated Legendre function of the first kind is given by [3, eq. (8.702)]:

$$P_\nu^\mu(x) = \left(\frac{x+1}{x-1}\right)^{\frac{\mu}{2}}\frac{{}_2F_1\left(-\nu,\nu+1;1-\mu;\frac{1-x}{2}\right)}{\Gamma(1-\mu)}, \qquad (14)$$

where ${}_2F_1(u,v;w;x)$ is the Gauss hypergeometric function [3, eqs. (9.14.1-2)]. In general, ${}_2F_1(u,v;w;x)$ is indeterminate when w is a non-positive integer ($w = 0,-1,-2,\ldots$), which occurs when μ is a natural number ($\mu = 1,2,3,\ldots$). In such case, and with the help of [3, eq. (9.101)], $P_\nu^\mu(x)$ can be evaluated as follows:

$$P_\nu^\mu(x) = \left(\frac{1-x}{2}\sqrt{\frac{x+1}{x-1}}\right)^\mu\frac{(-\nu)_\mu(\nu+1)_\mu}{\mu!}$$
$$\times\,{}_2F_1\left(\mu-\nu,\mu+\nu+1;1+\mu;\frac{1-x}{2}\right), \qquad (15)$$

where $(x)_n = \Gamma(x+n)/\Gamma(x) = x\,(x+1)\cdots(x+n-1)$ is the Pochhammer symbol. Thus, the evaluation of d_j should be based on (15) when μ is a natural number and (14) otherwise.

A further correction is required in Section III.B of [1]. By comparing the expressions in [1, eqs. (16)–(17)] with those shown in the proof in [1, eqs. (18)–(19)], it can be noticed that a factor $\alpha^\beta/2\Gamma(\beta)\Gamma(j+1)$ is missing in [1, eqs. (16)–(17)], which is required to obtain correct results for the bit error rate. Introducing the missing factor in [1, eq. (17)] yields:

$$B_G(j+1,2\sigma^2) \triangleq \frac{\Gamma(\beta+j+1)}{2\Gamma(\beta)\Gamma(j+1)}\frac{(2\alpha\sigma^2)^\beta}{(j+1)(1+2\alpha\sigma^2)^{\beta+j+1}}$$
$$= {}_2F_1\left(1,\beta+j+1;j+2;\frac{1}{1+2\alpha\sigma^2}\right), \qquad (16)$$

which provides the complete expression for $B_G(j+1,2\sigma^2)$.

III. New Analytical Results

The first solution in [2, eq. (2.15.3.2)] is valid for both positive and negative arguments of $I_\nu(x)$ and its direct application to the integral in (1) provides correct analytical results. Repeating the analysis presented in the previous section, a new closed-form expression for the PDF of the received SNR in the FTR fading channel can be obtained as follows:

$$f_\gamma(x) = \frac{m^m}{\Gamma(m)}\sum_{j=0}^{\infty}\frac{K^j a_j}{j!}f_G(x;j+1,2\sigma^2), \qquad (17)$$

where $f_G(x; j+1, 2\sigma^2)$ is given by [1, eq. (8)]:

$$f_G(x; j+1, 2\sigma^2) \triangleq \frac{x^j}{\Gamma(j+1)(2\sigma^2)^{j+1}} \exp\left(-\frac{x}{2\sigma^2}\right), \quad (18)$$

and the coefficient a_j is given by:

$$a_j \triangleq \sum_{k=0}^{j} \binom{j}{k} \sum_{l=0}^{k} \binom{k}{l} \Gamma(j+m+2l-k) \times (m+K)^{-(j+m+2l-k)} K^{2l-k} \left(\frac{\Delta}{2}\right)^{2l} \times (-1)^{2l-k} \, R_{j+m}^{k-2l}\left(\left[\frac{K\Delta}{m+K}\right]^2\right). \quad (19)$$

The function $R_\nu^\mu(x)$ in (19) is given by:

$$R_\nu^\mu(x) = \begin{cases} \left(\frac{\nu-\mu}{2}\right)_\mu \left(\frac{\nu-\mu+1}{2}\right)_\mu \frac{x^\mu}{\mu!} \times {}_2F_1\left(\frac{\nu+\mu}{2}, \frac{\nu+\mu+1}{2}; 1+\mu; x\right), \mu \in \mathbb{N}^+ \\ \frac{{}_2F_1\left(\frac{\nu-\mu}{2}, \frac{\nu-\mu+1}{2}; 1-\mu; x\right)}{\Gamma(1-\mu)}, \quad \text{otherwise} \end{cases} \quad (20)$$

The expressions for the CDF, channel capacity and bit error rate follow from (17) by replacing $f_G(x; j+1, 2\sigma^2)$ with $F_G(x; j+1, 2\sigma^2)$ [1, eq. (8)], $L_G(j+1, 2\sigma^2)$ [1, eq. (12)], and the corrected version of $B_G(j+1, 2\sigma^2)$ in (16), respectively. The main advantage of these results, which are based on the first solution in [2, eq. (2.15.3.2)], is that they are not ambiguous or prone to misinterpretations on their correct evaluation as it may be the case for the results obtained based on the second solution in [2, eq. (2.15.3.2)].

IV. Validation

The analytical results obtained in this work are validated and corroborated with Monte Carlo simulations. Simulation results are obtained based on 10^7 realisations of the (random) complex baseband received signal amplitude V_r under the FTR fading channel according to [1, eq. (1)]. The empirical PDF of the instantaneous SNR γ is then estimated as the normalised weighted histogram of $\gamma = |V_r|^2 \cdot E_b/N_0$, where E_b/N_0 is the energy per bit to noise power spectral density ratio.

Fig. 1 compares the empirical PDF obtained by means of simulations with the analytical results presented in this work, including the analytical result of [1, eq. (6)] based on the corrected version of d_j in (13) as well as the new analytical result provided in (17). In both cases, the infinite series are truncated to a maximum of 70 terms. Results are shown for an average SNR $\bar{\gamma} = E_b/N_0 = 1$ and fading parameters $K = 15$, $\Delta = 0.5$ and $m \in \{1.5, 5, 20\}$. As it can be appreciated, there exists a perfect agreement between simulation and analytical results, thus corroborating the correctness and validity of the analytical results presented in this correspondence.

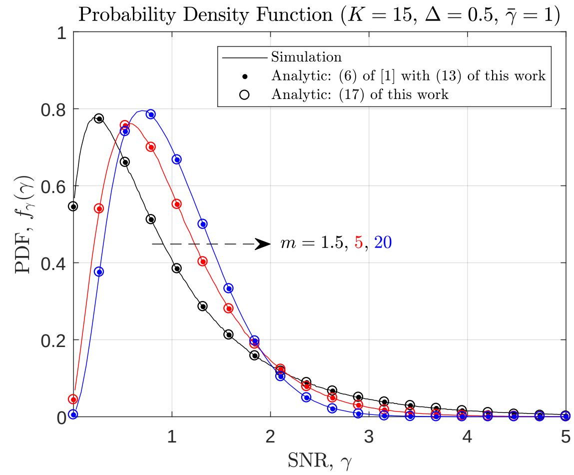

Fig. 1. PDF of the received SNR in the FTR fading channel.

V. Discussion

It is worth mentioning that the use of certain mathematical software packages that provide separate implementations for the functions $A_\nu^\mu(x)$ and $B_\nu^\mu(x)$ in (2) may coincidentally provide correct numerical results when such functions are mistakenly used to evaluate the analytical expressions in [1]. If the function $P_\nu^\mu(x)$ in [1, eq. (9)] is mistakenly evaluated as $A_\nu^\mu(x)$, then, according to (3), this is equivalent to evaluate $P_\nu^\mu(x)$ as $B_\nu^\mu(x)$ multiplied by a factor $e^{-\frac{\pi}{2}\mu i}$. The multiplication of this factor $e^{-\frac{\pi}{2}\mu i}$ (where $\mu = k - 2l$) with the term $e^{\frac{\pi}{2}(2l-k)i}$ in [1, eq. (9)] is equal to the term $e^{\pi(2l-k)i} = (-1)^{2l-k}$ that appears in the correct expression for d_j in (13), where $P_\nu^\mu(x)$ is evaluated as $B_\nu^\mu(x)$. Consequently, the numerical results obtained in this way will coincidentally be correct. The numerical results presented in [1] are correct even though some mathematical expressions contain errors.

Since [1] was published until the date, a number of studies have appeared in the literature making use of the incorrect form of d_j in [1, eq. (9)]. Notice that d_j plays the role of a multiplicative coefficient in the analytical results for the FTR fading channel and it does not alter their algebraic form. Moreover, the correct expression for d_j in (13) can be obtained by multiplying [1, eq. (9)] by the factor $e^{\frac{\pi}{2}(2l-k)i}$. Therefore, most (if not all) of the results published in the literature based on [1] may still be valid provided that the correct form of d_j in (13) is used or, wherever this is not feasible, possibly by multiplying the existing analytical results by the factor $e^{\frac{\pi}{2}(2l-k)i}$. However, a detailed analysis of the existing results in the literature derived from [1] was not carried out.

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