On the risk of credibility premium rules

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Abstract

A discrete-time risk process is considered where the full distribution of the claim size X is not completely known to the insurance company. Rather, it assumes that the distribution of X given $Z=\zeta$ is F_ζ where Z is some structural random variable for which a prior is available. The main emphasis of the paper is the unconditional ruin probability $\psi(u)$ in this setting where the premium is either updated according to incoming information about the claim distribution or computed by the expected value principle. This is in turn studied via the conditional ruin probability $\psi_\zeta(u)$, for which large deviations estimates are available. Rigorous proofs are given only for the case of the F_ζ forming a scale parameter family, including the classical case of gamma claims with a gamma prior. However, the analysis readily suggests what should be the behaviour of $\psi(u)$ in different models for the claims.

Keywords: Conjugate prior, credibility premiums, gamma distribution, large deviations, ruin probability, scale parameter family.

1 Introduction

Consider a discrete time claims surplus process of the form

$$S_n=\sum_{k=1}^{n}\bigl(X_k-\Pi_{k-1}\bigr), \tag{1.1}$$

where X_k is the total claim amount encountered in period k and Π_{k-1} the premium charged at the beginning of that period. In the standard setting where the X_k are i.i.d. with common distribution F and premium are calculated by the expected value premium principle $\Pi_{k-1}=(1+\delta)\mathbb{E}X_1$ where $\delta>0$ is the safety loading, the asymptotics of the ruin probability,

$$\psi(u)=\mathbb{P}(S_n>u \text{ for some } n),$$

as function of the initial reserve u is well understood for both light- and heavy-tailed F, see Theorem 3.1 below. This paper is concerned with the same problem under more general assumptions. One of our initial motivations was a new line of business with no statistics available on the claims.

One of these extensions deals with dependent X_k. More precisely, the setting is that $X_1, X_2, \ldots$ are i.i.d. with distribution F_ζ given $Z = \zeta$ for some structural random variable Z; in Bayesian terms, the distribution of Z is the prior. This introduces dependence among claims, which we refer to as a Bayesian structure. The other one is credibility updated premiums,

$$\Pi_n = (1+\delta)\Bigl(\frac{n}{n+d}\overline{X}_n + \frac{d}{n+d}H_0\Bigr), \tag{1.2}$$

where $\overline{X}_n = (X_1 + \cdots + X_n)/n$ and d, H_0 are constants. A basic connection between these two extensions is that if the prior distribution of Z is conjugate, then the Bayes premium has the form (1.2), see further Section 2 below.

We are thus facing $4 = 2 \times 2$ different situations: claims can be i.i.d. (IID) or with the Bayesian structure (BAYES), and premiums can calculated based on the classical expected-value-principle (EVP) or of credibility type (CRED) as in (1.2).

The study of credibility in insurance can be traced back to Mowbray (1914) and Whitney (1918). For a historical and practical overview of the development afterwards, we refer to Goulet (1998). In the following we concentrate on the much smaller part of the credibility literature, namely the part that intersects with ruin theory.

The earliest of such references may have been Bühlmann (1972), where a Bayesian component in the Cramér-Lundberg model was introduced; claims arrive according to a Poisson process with a parameter which is randomized. In this setting, assuming the prior to be conjugate, the ruin probability was studied when the premium is credibility adjusted continuously according to claim history, similar to the present paper. In extension hereof, with no assumption on the prior, Dubey (1977) studied the ruin probability in the same model under three different experience rating premiums: the Bayesian premium, the claim statistics premium, and the credibility premium. Both of these papers consider a model that builds on what we call a Bayesian structure and others call a mixing model. Others who studied the impact of mixing on the ruin probability (in a constant premium setting) are Albrecher et al. (2011) and Dutang et al. (2013).

Another early paper dealing with aspects of similar problems is Asmussen (1999), who considered the Cramér-Lundberg model in the standard i.i.d. setting, i.e. without Bayesian assumptions. The premium rule was a suitable continuous-time version of (1.2) with $d = 0$, and it was shown that this lead to a considerable reduction of the ruin probability in the light-tailed case; the heavy-tailed case was left open.

Next, we proceed to references closest to the present work. By means of Monte-Carlo simulations, Tsai and Parker (2004) studied the impact of credibility rating probability on the ruin probability in a discrete-time risk model. A more theoretical approach was considered in a similar framework in Trufin and Loisel (2013), a paper that we came across at a rather advanced stage of the present research. The motivation for the credibility premium in Trufin and Loisel (2013) is a portfolio of size say N selected as a mixture of the N_1 bad risks and N_2 good risks in the total population of size $N_0 = N_1 + N_2 \geq N$, so that the structural variable Z is a discrete random variable representing one of the $\binom{N}{N_0}$ possible combinations. Their primary focus was on finite horizon credibility and the impact on the asymptotic ruin probability of one particular portfolio was studied, corresponding to what we call the conditional ruin probability. The unconditional ruin is then simply a weighted sum over the conditional versions. In the present paper, we intend to work with a setting that captures continuous prior distributions on Z. In that case, integration of the conditional ruin formulas to get the unconditional ruin probability is non-trivial. Therefore, the study of the unconditional ruin for continuous priors is one of the major contributions of this paper. Some of our main results are only rigorously stated and proved in a scale-parameter set-up for the claims, introduced in the next section. A main classical of such is the gamma distribution. However, the analysis readily suggests what should be the behaviour of the unconditional ruin probability in different models for the claims.

The paper is organized as follows. Section 2 contains some preliminaries. Our main results are stated in Section 3, with the proofs given in Sections 4–7 which also include a number of examples. Finally, the Appendix contains the proof of a key auxiliary result and some details on the gamma-gamma case.

2 Bayesian set-up

Recall that Bayesian dependency structure implies that claims $X_1, X_2, \ldots$ are i.i.d. with distribution F_ζ given $Z = \zeta$ where Z is the structural random variable. We write $\mathbb{P}_\zeta$ for the conditional distribution $\mathbb{P}(\cdot \,|\, Z = \zeta)$ given $Z = \zeta$, $X_{(\zeta)}$ for a r.v. with distribution F_ζ, and

$$\mu(\zeta) = \mathbb{E}_\zeta X = \mathbb{E}X_{(\zeta)}, \;\; \sigma^2(\zeta) = \mathbb{V}\mathrm{ar}_\zeta X = \mathbb{V}\mathrm{ar} X_{(\zeta)}, \tag{2.1}$$

$$\mu = \mathbb{E}\mu(Z), \;\; \sigma^2 = \mathbb{E}\sigma^2(Z), \;\; \tau^2 = \mathbb{V}\mathrm{ar}\, \mu(Z)\,. \tag{2.2}$$

The collective premium is $H_0 = \mathbb{E}\mu(Z)$, the Bayes premium in period $n+1$ is $H_n^{\mathrm{Bayes}} = (1+\delta)\mathbb{E}[\mu(Z) \,|\, X_1, \ldots, X_n]$, and the classical credibility premium is given by (1.2) with $d = \sigma^2/\tau^2$. Note, however, that these particular choices of d and H_0 are not crucial for our results and we refer to (1.2) for any $0 \le d < 1$ and $H_0 > 0$ as the credibility premium. Let further $\kappa_\zeta(a) = \log \mathbb{E}_\zeta \mathrm{e}^{aX} = \log \mathbb{E}X_{(\zeta)}$ be the conditional cumulant function of X given $Z = \zeta$ and $a_\zeta^* = \sup\{a : \kappa_\zeta(a) < \infty\}$. The distribution F_ζ is said to be light-tailed if there is an $s > 0$ such that $\int_0^\infty \mathrm{e}^{sx} F_\zeta(\mathrm{d}x) < \infty$, and heavy-tailed if no such s exists and therefore $\int_0^\infty \mathrm{e}^{sx} F_\zeta(\mathrm{d}x) = \infty$ for all $s > 0$. In relation to a_ζ^*, then $a_\zeta^* = \infty$ if F_ζ is sufficiently light-tailed and on the contrary, one has $a_\zeta^* = 0$ if F_ζ is heavy-tailed.

It is well known (see, e.g., Diaconis and Ylvisaker (1979), Bernardo and Smith (2009) or Asmussen and Steffensen (2020, Sections II.2-3)) that if F_ζ belongs to a one-parameter natural exponential family with densities of the form

$$f_\zeta(x) = \exp\{\zeta x - \kappa(\zeta)\} \tag{2.3}$$

w.r.t. some reference measure $\lambda(\mathrm{d}x)$ and the prior is conjugate, then the Bayes and classical credibility premiums coincide. A trivial reparametrization shows that the same is true if

$$f_\zeta(x) = \exp\{-\zeta x - \kappa(\zeta)\}\,. \tag{2.4}$$

In order to have the relation 'increasing ζ means decreasing the risk due to lighter tails', we work with the latter parametrization in (2.4).

The same interpretation holds in the scale family class of distributions. This corresponds to $X_{(\zeta)} = X_{(1)}/\zeta$ so that $F_\zeta(x) = F(\zeta x)$ where $F(x) = F_1(x)$ is a distribution function independent of ζ. That is, ζ functions as a scale-parameter. Translated to the density, we have $f_\zeta(x) = \zeta f(\zeta x)$ where $f(x)$ is the density of $F(x)$.

In Section 6, we will consider examples involving distributions belonging to the natural exponential family but not to the scale family. More specifically, in these examples we consider distribution with lighter-than-exponential tails, and ζ serves as a tilting parameter. These may be of less direct actuarial interpretation, but serve to illustrate some mathematical aspects of the setting. Conversely, in Section 5 we will consider examples that are in the scale family but not in the natural exponential family. In the latter case, the credibility and Bayes premium no longer coincides.

Tsai and Parker (2004) argued as follows that a natural choice for the discrete time conditional claim size distribution is gamma. If the single claim size is exponential(ν) distributed and there is n claims during the considered time period, then aggregate claim during that period is gamma(n, ν). Hence, the first parameter (the shape parameter) represents claim frequency risk and the second (the rate or inverse scale parameter) represents claim severity risk during the time periods. However, unlike Tsai and Parker (2004) who deals with uncertain claim frequency risk, relating to the above, we assume that the shape parameter is known and model unknown claim severity. So our main example used for comparison, called the gamma-gamma case, is where F_ζ is gamma(α, ζ) and the conjugate prior on Z is gamma(α_0, β_0). Thus, we have the form (2.4) with

$$f_\zeta(x) = \frac{\zeta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \mathrm{e}^{-\zeta x}, \quad f_Z(\zeta) = \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \zeta^{\alpha_0-1} \mathrm{e}^{-\beta_0 \zeta}.$$

To ensure $\mathbb{V}\mathrm{ar} X < \infty$, we need to assume $\alpha_0 > 2$. Note that the gamma distribution belongs to both the natural exponential family and the scale family of distributions. A key fact for the ruin theory to be developed is that in the gamma-gamma case, the unconditional tail of X is Pareto-like,

$$\mathbb{P}(X > x) = \int_0^\infty \mathbb{P}_\zeta(X > x)\mathbb{P}(Z \in \mathrm{d}\zeta) \sim \frac{k}{x^{\alpha_0}}, \quad \text{as} \quad x \to \infty, \tag{2.5}$$

where k is a suitable constant. For this and further facts related to the gamma-gamma case, see Appendix A.

3 Ruin theory: statement of results

Using the standard notation, we write $f(u) \sim g(u)$ if $f(u)/g(u) \to 1$ as $u \to \infty$ and further write $f(u) \sim_{\log} g(u)$ if $\log f(u)/\log g(u) \to 1$ as $u \to \infty$. This convergence concept, common in large deviations theory, is weaker, but it captures the key features in many situations where exact asymptotics are out of the hand. We further write $\overline{F}(x) = 1 - F(x)$ to denote the tail function of the distribution F.

The IID-EVP case is standard, and one has the following result where (i) covers the light-tailed case, (ii) the heavy-tailed one. See, e.g., Asmussen (2008, pp. 365–366, 295–298). Let the c.d.f.

$$F_I(x) = \frac{1}{\mu}\int_0^x \overline{F}(y)\,\mathrm{d}y\,.$$

denote the integrated tail distribution. We note that the evaluation of C in (i) requires the Wiener-Hopf factorization of $X - (1+\delta)\mu$ but this is not a concern here.

Theorem 3.1. *Consider the* IID-EVP *model, $X \sim F$ with c.g.f. $\kappa_X(s) = \log \mathbb{E}\mathrm{e}^{sX}$. Then:*

(i) *If a solution $R > 0$ of $\kappa_X(R) = (1+\delta)\mu R$ exists and satisfies $\kappa_X'(R) < \infty$, then there is a $C > 0$ such that $\psi(u) \sim C\mathrm{e}^{-Ru}$.*

(ii) *If both F and its integrated tail are subexponential, then*

$$\psi(u) \sim \frac{1}{\delta}\overline{F}_I(u). \tag{3.1}$$

In Theorem 3.1 above, it is assumed that the true mean of the i.i.d. claim amounts is known. More realistically, the insurance company would form some beliefs about the distribution of the claim amounts, say $\tilde{F}$, upon which it bases its best guess of the mean, $\tilde{\mu}$ (this would then be H_0). The collective premium of $(1+\delta)\tilde{\mu}$ then actually implies a true safety loading of $\tilde{\delta} = (1+\delta)\tilde{\mu}/\mu - 1$. Note in particular that the true safety loading is negative if $\tilde{\mu} < \mu/(1+\delta)$, and the company thus faces a risk that the premium fails to even fulfil the net premium condition; then ruin will be certain, in line with Theorem 3.3 below. The reasoning behind the prior beliefs in the i.i.d. case is not of interest in the present paper, which is why we consider μ (the true mean) rather than $\tilde{\mu}$ (the guess) in the IID-EVP model.

Theorem 3.2. *Consider the* IID-CRED *model, $X \sim F$ with c.g.f. $\kappa_X(s) = \log \mathbb{E}\mathrm{e}^{sX}$. Then:*

(i) *Let*

$$\phi_X(a) = \int_0^\infty \mathrm{e}^{-t}\kappa_X\Big(a\big[1-(1+\delta)t\big]\Big)\,\mathrm{d}t.$$

If a solution $R > 0$ of $\phi_X(R) = 0$ exists and satisfies $\phi'(R) < \infty$, then $\psi_X(u) \sim_{\log} \mathrm{e}^{-Ru}$.

(ii) *If F has a regularly varying tail, $\overline{F}(x) \sim L(x)x^{-\alpha}$ for $x \to \infty$ where $L(\cdot)$ is a slowly varying function and $\alpha > 0$, then again (3.1) holds with $\overline{F}_I(u) \sim (\alpha-1)L(u)u^{-(\alpha-1)}/\mu$.*

Comparing the IID-EVP and IID-CRED models, the adjustment coefficient obviously change character in the light-tailed case, affecting the asymptotic ruin probability. Nothing can be said in general, though, on which is preferred, as this depends on the specific claim distribution. However, in the heavy-tailed case, the asymptotic ruin probability remains just the same, if μ is known in the IID-EVP model, that is. Part (i) can be proved by minor modifications of arguments in Asmussen (1999) but is also a special case of Theorem 3.5 below (take the prior as degenerate). Part (ii) is proved in Section 4; by a minor modification it also applies to the Cramér-Lundberg model, thereby settling a problem left open in Asmussen (1999).

It remains to consider the Bayesian model of the claims. Here a result of Glynn and Whitt (1994) (see also Asmussen and Albrecher (2010, Section XIII.1)) will be extremely useful for analysing the conditional ruin probability $\psi_\zeta(u)$. It basically says that $\psi_\zeta(u) \sim_{\log} \mathrm{e}^{-R_\zeta u}$ provided one can show

$$\frac{1}{n}\log \mathbb{E}_\zeta \mathrm{e}^{aS_n} \to \phi_\zeta(a), \tag{3.2}$$

where ϕ_ζ is like the cumulant function of a r.v. with negative mean and $R_\zeta > 0$ solves

$$\phi_\zeta(R_\zeta) = 0. \tag{3.3}$$

In both models, we have complete results only for the scale family. First, we state the result in the BAYES-EVP model. Recall that $X_{(1)}$ corresponds to $\zeta = 1$.

Theorem 3.3. *Consider the* BAYES-EVP *model in the scale family set-up. Assume that there is a $B < \infty$ and $\varepsilon > 0$ such that $\mathbb{E}\big[\mathrm{e}^{\varepsilon(X_{(1)}-x)} \,\big|\, X_{(1)} > x\big] \le B$ for all x. Let $c = \mathbb{E}X_{(1)}/((1+\delta)H_0)$ and assume that $f_Z(\zeta)$ is differentiable on $[c,\infty)$ with $f_Z(c) \neq 0$. Then $\psi(u)$ can be written as $\psi_{\xi<c} + \psi_{\xi\ge c}(u)$ where $\psi_{\xi<c} \in (0,1)$ is constant and $\psi_{\xi\ge c}(u) \sim_{\log} u^{-1}$.*

Note that the assumption on $X_{(1)}$ in particular holds if the hazard rate is eventually bounded away from zero, i.e. if $\liminf f_1(x)/\overline{F}_1(x) > 0$. As the following corollary states, the gamma-gamma case satisfies the assumptions of Theorem 3.3.

Corollary 3.4. *The ruin probability in the* BAYES-EVP *model in the gamma-gamma case has the asymptotics of Theorem 3.3.*

In the BAYES-CRED model, we have the following intermediate result.

Theorem 3.5. *In the* BAYES-CRED *model, the relation (3.2) holds with*

$$\phi_\zeta(a) = \int_0^\infty \mathrm{e}^{-t}\kappa_\zeta\Big(a\big[1-(1+\delta)t\big]\Big)\,\mathrm{d}t. \tag{3.4}$$

The proof is given in Appendix B for the sake of completeness, but we acknowledge that it can be found already in Trufin and Loisel (2013). In Section 6, we provide some further results on the existence of the adjustment coefficient in this case. The next theorem is concerned with the unconditional ruin probability in the BAYES-CRED model.

Theorem 3.6. *Consider the* BAYES-CRED *model in the scale family set-up and assume that R_1 exists. Then*

$$\psi(u) \ \sim_{\log}\ \mathbb{E}\mathrm{e}^{-uR_1 Z} \ = \ \int_0^\infty \mathrm{e}^{-uR_1\zeta} f_Z(\zeta)\,\mathrm{d}\zeta.$$

For the gamma-gamma case, we can specify further:

Corollary 3.7. *If the prior density is gamma*(α_0, β_0) *or, more generally, satisfies* $f_Z(\zeta) \sim C\zeta^{\alpha_0 - 1}$ *as* $\zeta \downarrow 0$ *for some* $\alpha_0 > 1$*, then*

$$\psi(u) \sim_{\log} u^{-\alpha_0}.$$

This is to be compared with part (ii) of Theorem 3.2 and (2.5), which shows that the Bayesian structure of the claims improves the decay rate of $\psi(u)$ by one power of u compared to the i.i.d. case. One heuristic reason for this is the structure $X = V/Y$ of a Pareto X, where V, Y are independent exponential random variables: large values of X occur typically as a result of small values of Y rather than large values of V. In the i.i.d. case each X_n has its own Y_n, V_n, i.e. $X_n = V_n/Y_n$ where $(V_n)_{n\in\mathbb{N}}, (Y_n)_{n\in\mathbb{N}}$ are independent processes of i.i.d. exponential random variables. Hence, in this case there are many sources of risk, as there is an infinity of Y_n's that could be large (one for each X_n). Whereas in the Bayesian setting the X_n share the same Y, that is, $X_n = V_n/Y$, creating a dependency structure among the $(X_n)_{n\in\mathbb{N}}$, but fewer sources of risks for large values.

4 Proofs for the IID-CRED model, part (ii)

For simplicity of notation, write

$$S_n^X = X_1 + \cdots + X_n, \quad S_n^\Pi = \Pi_0 + \cdots + \Pi_{n-1},$$

so that

$$S_n = S_n^X - (1+\delta)S_n^\Pi = X_n + S_{n-1}^X - (1+\delta)S_n^\Pi.$$

Note that by the LLN, $S_n^X/n \to \mu$ and hence also $S_n^\Pi/n \to \mu$.

Proof of lower bound $\liminf_{u\to\infty} \dfrac{\psi(u)}{\overline{F}_I(u)/\delta} \geq 1.$

Define $\delta^* = (1+\delta)(1-\epsilon^*) - (1+\epsilon^*)$, where $\epsilon^* > 0$ is so small that $\delta^* > 0$. Let $\widetilde{A}_{n_0}^m$ be the event

$$\widetilde{A}_{n_0}^m = \big\{S_n^X < (1+\epsilon^*)n\mu \,, S_n^\Pi > (1-\epsilon^*)n\mu \text{ for all } n_0 \leq n < m\big\}.$$

Given $\epsilon > 0$, we can choose first n_0 such that $\mathbb{P}\widetilde{A}_{n_0}^\infty > 1 - \epsilon/2$ and next K such that $\mathbb{P}A_{n_0}^\infty > 1 - \epsilon$ where $A_{n_0}^m = \widetilde{A}_{n_0}^m \cap \{\sup_{n\leq n_0} S_n \leq K\}$. If $u \geq K$, ruin cannot occur before n_0 on $A_{n_0}^m$, and thus

$$\psi(u) \geq \sum_{n=n_0}^\infty \mathbb{P}(\tau(u) = n;\, A_{n_0}^{n-1}), \tag{4.1}$$

where $\tau(u) = \inf\{n \in \mathbb{N} | S_n^X > u + S_n^\Pi\}$ is the time of ruin, i.e. $\psi(u) = \mathbb{P}(\tau(u) < \infty)$. Now on $A_{n_0}^{n-1}$, we have $S_k < -\delta^* k\mu$ for $n_0 \leq k < n$. Thus ruin can not occur at such k, and hence the r.h.s. of (4.1) is the same as

$$\sum_{n=n_0}^\infty \mathbb{P}\big(X_n > u + (1+\delta)S_n^\Pi - S_{n-1}^X;\, A_{n_0}^{n-1}\big).$$

But $(1+\delta)S_n^\pi - S_{n-1}^X < \delta^* n\mu$ on $A_{n_0}^{n-1}$, and therefore this can in turn be bounded below by

$$\sum_{n=n_0}^{\infty} \mathbb{P}\big(X_n > u + \delta^* n\mu; A_{n_0}^{n-1}\big) = \mathbb{P}A_{n_0}^{n-1} \sum_{n=n_0}^{\infty} \mathbb{P}\big(X_n > u + \delta^* n\mu\big), \tag{4.2}$$

using the independence of X_n and $A_{n_0}^{n-1}$. Here the sum is bounded below by

$$\int_{n_0-1}^{\infty} \overline{F}(u + \delta^* t\mu)\,\mathrm{d}t = \frac{1}{\delta^*}\overline{F}_I(u + \delta^*(n_0-1)\mu) \sim \frac{1}{\delta^*}\overline{F}_I(u).$$

Using $A_{n_0}^{n-1} \supseteq A_{n_0}^{\infty}$, it therefore follows from (4.2) that

$$\liminf_{u\to\infty} \frac{\psi(u)}{\overline{F}_I(u)/\delta} \geq \frac{(1-\epsilon)\delta}{\delta^*}\overline{F}_I(u).$$

Letting first $\epsilon^* \downarrow 0$ so that $\delta^* \uparrow \delta$ and next $\epsilon \downarrow 0$ gives the desired lower bound.

Proof of upper bound $\limsup_{u\to\infty} \dfrac{\psi(u)}{\overline{F}_I(u)/\delta} \leq 1$.
Let $0 < \eta < 1/\alpha$ and define

$$A_1 = \{\tau(u) < u^\eta\}, \quad A_2 = \{S_n > u \text{ for some } n \geq u^\eta\},$$
$$A_3 = \{S_n^\Pi > (1-\epsilon)n\mu \text{ for all } n \geq u^\eta\},$$

where for simplicity of notation we have suppressed the dependence of $\tau(u)$, A_1, A_2, A_3 on u, η, ϵ. The proof is then based on the decomposition

$$\psi(u) = \mathbb{P}(A_1) + \mathbb{P}(A_1^c A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_1^c A_2 A_3) + \mathbb{P}(A_1^c A_2 A_3^c) \tag{4.3}$$

and the similar decomposition for the risk process $S_n^X - (1+\delta)(1-\epsilon)n\mu$, notationally denoted by tildes, which in view of $\mathbb{P}(\widetilde{A}_3) = 1$ and (3.1) takes the form

$$\frac{1}{\delta^*}\overline{F}_I(u) \sim \mathbb{P}(\widetilde{\tau}(u) < \infty) = \mathbb{P}(\widetilde{A}_1) + \mathbb{P}(\widetilde{A}_1^c \widetilde{A}_2), \tag{4.4}$$

where $\delta^* = (1+\delta)(1-\epsilon) - 1$. We shall show that

$$\mathbb{P}(A_1) = \mathrm{o}(F_I(u)), \quad \mathbb{P}(\widetilde{A}_1) = \mathrm{o}(F_I(u)), \tag{4.5}$$
$$\mathbb{P}(A_3^c) = \mathrm{o}(F_I(u)). \tag{4.6}$$

From this we get

$$\frac{1}{\delta^*}\overline{F}_I(u) = \mathbb{P}(\widetilde{A}_2) + \mathrm{o}(F_I(u)), \quad \psi(u) = \mathbb{P}(A_2 A_3) + \mathrm{o}(F_I(u)).$$

But

$$\begin{aligned}\mathbb{P}(A_2 A_3) &\leq \mathbb{P}\big(S_n^X - (1-\epsilon)n > u \text{ for some } n \geq u^\eta;\ A_3\big) \\ &\leq \mathbb{P}\big(S_n^X - (1-\epsilon)n > u \text{ for some } n \geq u^\eta;\big) = \mathbb{P}(\widetilde{A}_2),\end{aligned}$$

so

$$\psi(u) \le \mathbb{P}(\widetilde{A}_2) + \mathrm{o}(F_I(u)) = \frac{1}{\delta^*}\overline{F}_I(u) + \mathrm{o}(F_I(u)).$$

Dividing by $\overline{F}_I(u)/\delta$ and taking $\limsup$ gives the upper bound with limit δ/δ^* instead of 1. Now just let $\epsilon \downarrow 0$ so that $\delta/\delta* \downarrow 1$.

For (4.5), (4.6), the proof of (4.5) is the easier one. Indeed, if $\tau(u) = k < u^\eta$, then $S_k^X > u$ and hence also $S_{u^\eta}^X > u$. Hence it suffices to show that $\mathbb{P}(S_{u^\eta}^X > u) = \mathrm{o}(F_I(u))$. To this end, let $0 < p < \alpha$ be so close to α that $p(1-\eta) > \alpha - 1$. It is standard by L_p-theory that $\mathbb{E}(S_n^X)^p \le n^p\mu^p$. Hence by Markov's inequality

$$\mathbb{P}(A_1) \le \mathbb{P}(S_{u^\eta}^X > u) \le \frac{\mathbb{E}(S_{u^\eta}^X)^p}{u^p} \le \frac{\mu^p}{u^{p-\eta p}} = \mathrm{o}(u^{-(\alpha-1)}) = \mathrm{o}(F_I(u)).$$

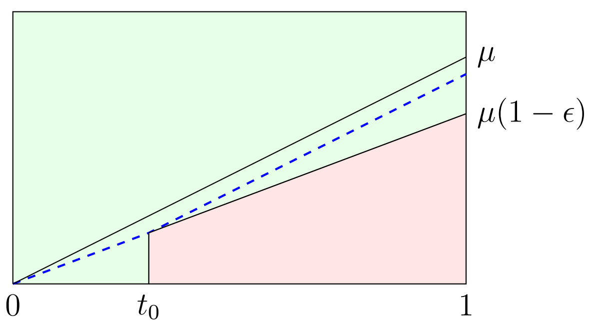

Figure 1: The set B (red)

The key step in the proof of (4.6) is an estimate based on large deviations theory, for which some familiarity with this area and in particular Mogulski's theorem is needed; we refer to Dembo and Zeitouni (2010) for the general mathematical theory and to Asmussen and Steffensen (2020, Section XIII.4) for an elementary introduction stressing applications of the type to follow. Define $B \subset [0,1] \times \mathbb{R}$ as

$$B = \big\{(t,y) : \, t_0 \le t \le 1, \, 0 \le y \le \mu(1-\epsilon_1)t\big\},$$

cf. Fig. 1 and let $\xi_n(t)$ be the $C[0,1]$ function obtained by linear interpolation between the points S_k^X/n, $k = 0, 1, \ldots, n$. Let B_n be the event that $(t, \xi_n(t)) \in B$ for some t. The most likely way for B_n to occur is by ξ_n to follow the blue-dotted path, and the probability for this to occur is in the logarithmic sense e^{-nJ_0} where $J_0 = t_0\kappa^*(\mu(1-\epsilon_1))$ with $\kappa(\theta) = \log \mathbb{E}\mathrm{e}^{\theta X}$ the cumulant function of X (defined for $\theta < 0$) and $\kappa^*(z)$ its convex conjugate (strictly positive for $z < \mu$). Thus, for large n, we have the estimate

$$\mathbb{P}(B_n) = \mathbb{P}(S_k^X \le \mu(1-\epsilon)k \text{ for some } nt_0 \le k \le n) \le \mathrm{e}^{-nJ}, \tag{4.7}$$

for some arbitrarily chosen $J \in (0, J_0)$.

Given $\epsilon > 0$ such that δ^* defined by $1+\delta^* = (1+\delta)(1-\epsilon)$ is strictly positive, we choose $\epsilon_1 > 0$ and $t_0 \in (0,1)$ such that $(1-t_0)(1-\epsilon_1) > 1 - \epsilon/2$. For n sufficiently large, we then have on B_n^c that

$$S_n^\Pi \ge \Pi_{nt_0} + \cdots + \Pi_n \ge \frac{nt_0}{nt_0 + d}(1-t_0)\mu(1-\epsilon_1)n > (1-\epsilon)n$$

and hence that $S_n \leq S_n^X - (1+\delta^*)n\mu$. Therefore for u sufficiently large,

$$\begin{aligned}\mathbb{P}(A_3^c) &= \mathbb{P}(S_n^\Pi \leq (1-\epsilon)n\mu \text{ for some } n \geq u^\eta) \\ &\leq \sum_{n=u^\eta}^{\infty} \mathbb{P}(B_n) = \mathcal{O}(\mathrm{e}^{-u^\eta J}) = o(\overline{F}_I(u)).\end{aligned}$$

5 Proofs for the BAYES-EVP model in the gamma-gamma case

Next we wish to study how the solvency of the insurance company is affected if the company sticks to the static expected value premium principle, $H_n = (1+\delta)H_0$ for all n, when a Bayesian structure is present in the claims. In this case, we proceed with the conditional setup as in (3.2) and (5.13) with

$$S_n = \sum_{k=1}^{n}\bigl(X_k - (1+\delta)H_0\bigr) = -n(1+\delta)H_0 + \sum_{k=1}^{n} X_k.$$

As consequence of the constant premium $(1+\delta)H_0$ in the BAYES-EVP model, two things can happen. 1) The net profit condition is not satisfied: $(1+\delta)H_0 < \mathbb{E}_\zeta[X]$, which for the scale-family happens when $\zeta < c = \frac{\mathbb{E}_1[X]}{(1+\delta)H_0}$. Ruin is then certain. 2) The net profit condition is satisfied: $(1+\delta)H_0 \geq \mathbb{E}_\zeta[X]$ when $\zeta \geq c$. Consider (3.2), reduced for this case to:

$$\frac{1}{n}\log\bigl(\mathbb{E}_\zeta[\mathrm{e}^{aS_n}]\bigr) = -a(1+\delta)H_0 + \kappa_\zeta(a).$$

Hence, $\phi_\zeta(a) = -a(1+\delta)H_0 + \kappa_\zeta(a)$, where $\phi_\zeta(0) = 0$, $\phi_\zeta'(0) = -(1+\delta)H_0 + \mathbb{E}_\zeta[X] \leq 0$, and $\phi_\zeta''(a) = \kappa_\zeta''(a) > 0$. So by Glynn and Whitt (1994), when the net premium condition is satisfied, the adjustment coefficient R_ζ exists as the solution of

$$\phi_\zeta(R_\zeta) = -R_\zeta(1+\delta)H_0 + \kappa_\zeta(R_\zeta) = 0. \tag{5.1}$$

The proposition next tells us how the adjustment coefficient acts asymptotically when ζ approaches the threshold c from upwards and when ζ goes to infinity.

Proposition 5.1. *R_ζ in (5.1) satisfies*

$$R_\zeta \to cd(\zeta - c), \quad \textit{as} \quad \zeta \downarrow c, \tag{5.2}$$

where $d = 2(1+\delta)H_0/\kappa_1''(0)$. Futher, if either $\kappa_1(a) \uparrow \infty$ as $a \uparrow a_1^*$, or $\kappa_1(a) < \infty$ for all $a < \infty$, then $R_\zeta/\zeta \to \infty$ as $\zeta \uparrow \infty$.

Proof. In terms of $S(\zeta) = R_\zeta/\zeta$, we can write (5.1) as

$$\kappa_1(S(\zeta)) = \zeta S(\zeta)(1+\delta)H_0. \tag{5.3}$$

We first prove the lower bound asymptotic. Criticality of c means that there the drift is 0, hence

$$0 = \phi_c'(0) = \frac{1}{c}\kappa_1'(0/c) - (1+\delta)H_0,$$

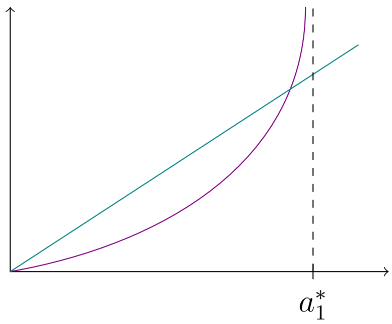

Figure 2: Illustration of $\kappa_1(s)$ and $s \mapsto \zeta rs$.

which leads to

$$\kappa_1'(0) = c(1+\delta)H_0. \tag{5.4}$$

By the implicit function theorem, R_ζ and hence $S(\zeta)$ is C^∞ on (c,∞). Differentiating both sides of (5.3), we get

$$S'(\zeta)\kappa_1'(S(\zeta)) = (S(\zeta) + \zeta S'(\zeta))(1+\delta)H_0.$$

Letting $\zeta = c$ and using $0 = R_c = S(c)$ together with (5.4) this gives $S'(c)c(1+\delta)H_0 = 0 + cS'(c)(1+\delta)H_0$, a tautology. Thus we need to differentiate once more and get

$$S''(\zeta)\kappa_1'(S(\zeta)) + S'(\zeta)^2\kappa_1''(S(\zeta)) = (S'(\zeta) + S'(\zeta) + \zeta S''(\zeta))(1+\delta)H_0.$$

Evaluate in $\zeta = c$ to get

$$S''(c)c(1+\delta)H_0 + S'(c)^2\kappa_1''(0) = (2S'(c) + cS''(c))(1+\delta)H_0.$$

Rearranging yields

$$S'(c) = d \quad \text{where} \quad d = \frac{2(1+\delta)H_0}{\kappa_1''(0)}. \tag{5.5}$$

Hence, $S(\zeta)$ behaves linearly when ζ approaches c, that is (5.2) holds.

For the upper asymptotics, we take another look at (5.3). For a fixed ζ, $S(\zeta)$ is the point > 0 at which the convex function $\kappa_1(s)$ intersects the line $s \mapsto \zeta rs$ where $r = (1+\delta)H_0$. The claim then immediately follows by graphical inspection (see Figure 2), which also gives $S(\zeta) \to a_1^*$ under the assumptions, i.e. $R(\zeta) \sim a_1^*\zeta$. □

Prior to the proof of Theorem 3.3, we need the following lemma.

Lemma 5.2. *There is a $C_- > 0$ and $A > 0$ such that $\psi_\zeta(u) \geq C_- \mathrm{e}^{-R_\zeta u}$ for all $u \geq 0$ and all $c \leq \zeta \leq c + A$. Further, $\psi_\zeta(u) \leq \mathrm{e}^{-R_\zeta u}$ for all $u \geq 0$ and all $c \leq \zeta < \infty$.*

Proof. The upper bound is just the standard Lundberg inequality. The lower bound follows by a small modification of the arguments in Asmussen and Albrecher (2010, p. 93) provided we can show

$$C_- = \inf_{c \leq \zeta \leq c+A} C_-(\zeta) > 0 \quad \text{where} \quad C_-(\zeta) = \inf_{x \geq 0} \frac{\overline{G}_\zeta(x)}{\int_x^\infty \mathrm{e}^{R_\zeta(y-x)}\, G_\zeta(\mathrm{d}y)}. \tag{5.6}$$

where G_ζ is the distribution of $Y_{(\zeta)} = X_{(\zeta)} - (1+\delta)H_0 = X_{(1)}/\zeta - (1+\delta)H_0$. To this end, choose A with $R_\zeta/\zeta \le \varepsilon$ for all $c \le \zeta \le c+A$ (possible by (5.2)), and let $x_\zeta = \zeta(x + (1+\delta)H_0)$. Then

$$\begin{aligned}\int_x^\infty \mathrm{e}^{R_\zeta(y-x)}\, G_\zeta(\mathrm{d}y) \;&=\; \mathbb{P}(Y_{(\zeta)} > x)\mathbb{E}\big[\exp\big\{R_\zeta(Y_{(\zeta)} - x)\big\}\,\big|\, Y_{(\zeta)} > x\big] \\ &=\; \overline{G}_\zeta(x)\mathbb{E}\big[\exp\big\{R_\zeta(X_{(1)} - x_\zeta)/\zeta\big\}\,\big|\, X_{(1)} > x_\zeta\big] \\ &\le\; \overline{G}_\zeta(x)\mathbb{E}\big[\exp\big\{\epsilon(X_{(1)} - x_\zeta)\big\}\,\big|\, X_{(1)} > x_\zeta\big] \;\le\; B\overline{G}_\zeta(x).\end{aligned}$$

This gives the result with $C_- = 1/B$, due to the assumption that appears in Theorem 3.3. □

Proof of Theorem 3.3. Let again c be the threshold for the net profit condition to be satisfied. The unconditional ruin probability, described by the law of total probability, can then be split into a sum,

$$\psi(u) = \int_0^c \psi_\zeta(u)\mathbb{P}(Z \in \mathrm{d}\zeta) + \int_c^\infty \psi_\zeta(u)\mathbb{P}(Z \in \mathrm{d}\zeta). \tag{5.7}$$

Let the first part with area of integration $(0, c)$ be denoted $\psi_{\zeta<c}(u)$ and the second part with area of integration $[c, \infty)$ be $\psi_{\zeta\ge c}(u)$.

Obviously, $\psi_{\zeta<c}(u)$ is constant as

$$\psi_{\zeta<c} = \psi_{\zeta<c}(u) = \int_0^c \mathbb{P}(Z \in \mathrm{d}\zeta) = \mathbb{P}(Z < c).$$

More interesting is the behaviour of $\psi_{\zeta\ge c}(u)$. Recall that in this region, the adjustment coefficient (5.10) exists. Therefore, as stated in Lemma 5.2, the conditional ruin probability $\psi_\zeta(u)$ is upward bounded by $\mathrm{e}^{-R_\zeta u}$ for $u \ge 0$ and $\xi \ge c$. The integral representing $\psi_{\zeta\ge c}$ can likewise be upward bounded as follows

$$\int_c^\infty \psi_\zeta(u)\mathbb{P}(Z \in \mathrm{d}\zeta) \le \int_c^\infty \mathrm{e}^{-R_\zeta u}\mathbb{P}(Z \in \mathrm{d}\zeta) = \int_c^\infty \mathrm{e}^{-R_\zeta u} f_Z(\zeta)\mathrm{d}\zeta.$$

If we take particular notice of the structure of the integral on the right side, namely

$$I_1(u) = \int_c^\infty \mathrm{e}^{\omega_1(\zeta)u}\omega_2(\zeta)\mathrm{d}\zeta,$$

where $\omega_1(\zeta) = -R_\zeta$ and $\omega_2(\zeta) = f_Z(\zeta)$, then we recognise $I_1(u)$ as a Laplace integral, using the terminology of Orszag and Bender (1978, Sections 6.2-6.3). Its asymptotics can easily be evaluated by Laplace's method (the needed differentiability properties for ω_2 hold by assumption and were noted above for ω_1). We only intend to sketch the idea here. For a rigorous treatment, we point to the reference above. Integration by parts yields

$$\int_c^\infty \mathrm{e}^{\omega_1(\zeta)u}\omega_2(\zeta)\mathrm{d}\zeta = \Big[\frac{1}{u}\cdot\frac{\omega_2(\zeta)}{\omega_1'(\zeta)}\cdot \mathrm{e}^{\omega_1(\zeta)u}\Big]_c^\infty - \frac{1}{u}\int_c^\infty \frac{\partial}{\partial\zeta}\Big(\frac{\omega_2(\zeta)}{\omega_1'(\zeta)}\Big)\cdot \mathrm{e}^{\omega_1(\zeta)u}\mathrm{d}\zeta.$$

As $\omega_1(\zeta), \omega_1'(\zeta), \omega_2(\zeta)$ are presumed to be continuous functions on $[c, \infty)$, such that $\omega_1'(\zeta) \neq 0$ for all $\zeta \geq c$ and $\omega_2(c) \neq 0$, and it can be shown that

$$\frac{1}{u}\int_c^\infty \frac{\partial}{\partial\zeta}\Big(\frac{\omega_2(\zeta)}{\omega_1'(\zeta)}\Big) \cdot \mathrm{e}^{\omega_1(\zeta)u}\mathrm{d}\zeta = \mathrm{o}\bigg(\Big[\frac{1}{u} \cdot \frac{\omega_2(\zeta)}{\omega_1'(\zeta)} \cdot \mathrm{e}^{\omega_1(\zeta)u}\Big]_c^\infty\bigg).$$

Hence

$$I_1(u) \sim \Big[\frac{1}{u} \cdot \frac{\omega_2(\zeta)}{\omega_1'(\zeta)} \cdot \mathrm{e}^{\omega_1(\zeta)u}\Big]_c^\infty, \qquad \text{as } u \to \infty.$$

Note that as consequence of Proposition 5.1 we have

$$\frac{\omega_2(\zeta)}{\omega_1'(\zeta)} \to 0 \quad \text{and} \quad \mathrm{e}^{\omega_1(\zeta)u} \to 0, \qquad \text{as } \zeta \to \infty. \tag{5.8}$$

So we arrive at

$$I_1(u) \sim \lim_{\zeta \downarrow c} \frac{-1}{u} \cdot \frac{\omega_2(\zeta)}{\omega_1'(\zeta)}\mathrm{e}^{\omega_1(\zeta)u} = \frac{1}{u} \cdot \frac{\omega_2(c)}{cd},$$

where d is specified in Proposition 5.1. This leads to the following logarithmic limsup inequality:

$$\limsup_{u\to\infty} \frac{\log(\psi_{\zeta \geq c}(u))}{-\log(u)} \leq 1.$$

In similar fashion, from Lemma 5.2 we know that for a given $A > 0$ the conditional ruin probability is downward bounded by $\psi_{\zeta \geq c}(u) \geq C_{-}\mathrm{e}^{-R_\zeta u}$ for $u \geq 0$ and $c \leq \xi \leq c + A$, where $C_{-} > 0$ by assumption. This translates into the inequality for $\psi_{\zeta \geq c}(u)$:

$$\psi_{\zeta \geq c}(u) \geq \int_c^{c+A} \psi_\zeta(u)\mathbb{P}(Z \in \mathrm{d}\zeta) \geq C_{-}\int_c^{c+A} \mathrm{e}^{-R_\zeta u} f_Z(\zeta)\mathrm{d}\zeta.$$

We are again faced with the structure of a Laplace integral,

$$I_2(u) = \int_c^{c+A} \mathrm{e}^{\omega_1(\zeta)u}\omega_2(\zeta)\mathrm{d}\zeta,$$

with the same $\omega_1(\cdot)$ and $\omega_2(\cdot)$ as in (**??**), but with different area of integration. Similar arguments lead to

$$I_2(u) \sim \Big[\frac{1}{u} \cdot \frac{\omega_2(\zeta)}{\omega_1'(\zeta)} \cdot \mathrm{e}^{\omega_1(\zeta)u}\Big]_c^{c+A}, \qquad \text{as } u \to \infty.$$

As

$$\omega_2(c+A) \in [0, \infty), \quad \omega_1(c+A) < 0 \quad \text{and} \quad \omega_1'(c+A) < 0 \text{ exists}$$

we have that once again that

$$I_2(u) \sim \frac{1}{u} \cdot \frac{\omega_2(c)}{cd}.$$

This correspondingly yields the logarithmic liminf inequality:

$$\liminf_{u\to\infty} \frac{\log(\psi_{\zeta \geq c}(u))}{-\log(u)} \geq 1,$$

which concludes the proof. □

The intuition behind the Laplace's method is that as $\omega_1(\zeta) = -R_\zeta$ is monotonically decreasing, it takes on its greatest value in $\zeta = c$. So it is only the neighbourhood of c that contributes to the integral asymptotically.

Now we return to gamma-gamma case with cumulant generating function

$$\kappa_\zeta(a) = \alpha \log\Big(\frac{\zeta}{\zeta - a}\Big) = -\alpha \log(1 - a/\zeta) = \kappa_1(a/\zeta), \qquad \text{for } a < \zeta. \tag{5.9}$$

When F_ζ is gamma, the net premium condition is satisfied if $\zeta \geq c$ where $c = \frac{\alpha_0 - 1}{(1+\delta)\beta_0}$, cf. (A.1).

Lemma 5.3. *In the gamma-gamma case,* (5.1) *has solution*

$$R_\zeta = \Big(\frac{W_0(-c_\zeta e^{-c_\zeta})}{c_\zeta} + 1\Big)\zeta, \qquad \textit{if } \zeta \geq c. \tag{5.10}$$

Proof. For the conditional gamma distribution, the cumulant generating function is given in (5.9), so we can write equation (5.1) as

$$-R_\zeta(1+\delta)H_0 + \alpha \log\Big(\frac{1}{1 - R_\zeta/\zeta}\Big) = 0.$$

Rearranging yields $-c_\zeta \mathrm{e}^{-c_\zeta} = x\mathrm{e}^{x}$, where

$$c_\zeta = \frac{\zeta}{\alpha}(1+\delta)H_0 \geq 1 \quad \text{and} \quad x = -c_\zeta(1 - R_\zeta/\zeta) \in [0,1].$$

The Lambert-W function is defined as the functional inverse of $f(w) = w\mathrm{e}^{w}$ and thus gives us

$$x = W_0(-c_\zeta e^{-c_\zeta}).$$

As explained in Corless et al. (1996), there are two possible branches of the Lambert-W function, an upper (indexed by 0) and a lower (indexed by -1). As $x \in [0,1]$, the relation xe^{x} is described by the upper branch, conversely, as $c_\zeta \geq 1$, the relation $-c_\zeta e^{-c_\zeta}$ is described by the lower, which is why W_0 is not the inverse in the latter case. Substituting back yields the adjustment coefficient (5.10). □

In terms of the terminology and notation of Theorem 3.3, we note that obviously

$$\omega_1(\zeta) = -\Big(\frac{W_0(-c_\zeta e^{-c_\zeta})}{c_\zeta} + 1\Big)\zeta \quad \text{and} \quad \omega_2(\zeta) = \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)}\zeta^{\alpha_0 - 1}\mathrm{e}^{-\beta_0\zeta}, \tag{5.11}$$

satisfies the assumption of Theorem 3.3. Further we have

$$\omega_1'(\zeta) = -\frac{W_0(-c_\zeta e^{-c_\zeta})(1 - c_\zeta)}{c_\zeta(W_0(-c_\zeta e^{-c_\zeta}) + 1)} - 1.$$

As $a_1^* = 1$ and $\kappa_1(a) \to \infty$ as $a \uparrow a_1^*$, we learn by Proposition 5.1 that $\omega_1'(\zeta) \sim \zeta$ when $\zeta \to \infty$.

Now we only need to show that for a gamma$(\alpha, 1)$ distribution, the hazard rate is eventually bounded away from zero. To this end, note that

$$\overline{F}_1(x) = \int_x^\infty \frac{1}{\Gamma(\alpha)} y^{\alpha-1} \mathrm{e}^{-y}\, \mathrm{d}y \sim \frac{1}{\Gamma(\alpha)} x^{\alpha-1}\mathrm{e}^{-x}, \qquad \text{when } x \to \infty.$$

This is asymptotically reflected in the hazard rate as follows,

$$\frac{f_1(x)}{\overline{F}_1(x)} \sim 1, \qquad \text{when } x \to \infty, \tag{5.12}$$

which concludes proving Corollary 3.4.

Example 1 (Half-normal)**.** Consider the scale family set-up with F_ζ being the half-normal distribution (the absolute value of a standard normal random variable). Then F_ζ has density

$$f_\zeta(x) = \zeta\sqrt{\frac{2}{\pi}}\mathrm{e}^{-\zeta^2 x^2/2},$$

and we have $\kappa_1(a) = \log 2 + a^2/2 + \log\Phi(a)$ with $a_1^* = \infty$. Note that the hazard rate has asymptotics

$$\frac{f_1(x)}{\overline{F}_1} \sim x.$$

As the hazard rate asymptotically behaves like the identity, this ensures the condition on $X_{(1)}$ in Theorem 3.3. For any prior differentiable on $[c, \infty)$ with $f_Z(c) \neq 0$, the asymptotic ruin probability is given by Theorem 3.3.

Example 2 (Inverse Gaussian)**.** Consider the inverse Gaussian$(\lambda_\zeta, \mu_\zeta)$ distribution with density

$$f_\zeta(x) = \sqrt{\frac{\lambda_\zeta}{2\pi x^3}} \exp\Big\{-\frac{\lambda_\zeta}{2\mu_\zeta^2}x - \frac{\lambda_\zeta}{2x} + \frac{\lambda_\zeta}{\mu_\zeta}\Big\}.$$

Note that when $\lambda_\zeta = \lambda/\zeta$ and $\mu_\zeta = \mu/\lambda$ for $\mu, \lambda > 0$, the distribution belongs to the scale family, with c.g.f.

$$\kappa_1(a) = \frac{\lambda}{\mu}\Big(1 - \sqrt{1 - \frac{2\mu^2}{\lambda}a}\Big). \tag{5.13}$$

This is non-steep at $a_1^* = \lambda/2\mu^2$, more precisely with limit $\kappa_1(a_1^*) = \lambda/\mu$. The upper asymptotics of Proposition 5.1 does therefore not hold in this case. Instead, we are able to solve for R_ζ explicitly. For the inverse Gaussian distribution (5.1) becomes

$$-R_\zeta(1+\delta)H_0 + \frac{\lambda}{\mu}\Big(1 - \sqrt{1 - \frac{2\mu^2}{\lambda}\frac{R_\zeta}{\zeta}}\Big) = 0.$$

Rearranging and solving for roots, easily leads to

$$R_\zeta = 2\frac{\lambda}{\mu}\frac{\big((1+\delta)H_0\zeta - \mu)\big)}{(1+\delta)^2 H_0^2\zeta}.$$

Note that $R_\zeta \to 2\frac{\lambda}{\mu(1+\delta)H_0}$, so a potential prior needs to go towards zero quicker than R_ζ turns flat. Put differently, convergence of the first limit in (5.8) is a criterium for a potential prior in order for Theorem 3.3 to hold for the inverse Gaussian distribution.

We further need the existence of $\varepsilon, B > 0$ such that

$$\begin{aligned}
&\mathbb{E}[\mathrm{e}^{\varepsilon(X_{(1)}-x)} \mid X_{(1)} > x] \\
&\quad = \frac{\mathrm{e}^{-\varepsilon x}\mathrm{e}^{\lambda/\mu-\lambda/\theta}\Big(\Phi\Big(-\sqrt{\frac{\lambda}{x}}\Big(\frac{x}{\theta}-1\Big)\Big)+\mathrm{e}^{2\lambda/\theta}\Phi\Big(-\sqrt{\frac{\lambda}{x}}\Big(\frac{x}{\theta}+1\Big)\Big)\Big)}{\Phi\Big(-\sqrt{\frac{\lambda}{x}}\Big(\frac{x}{\mu}-1\Big)\Big)+\mathrm{e}^{2\lambda/\mu}\Phi\Big(-\sqrt{\frac{\lambda}{x}}\Big(\frac{x}{\mu}+1\Big)\Big)} \\
&\quad \le B
\end{aligned}$$

for all $x > 0$, where $\theta = \frac{\sqrt{\lambda}\mu}{\sqrt{\lambda - 2\varepsilon\mu^2}} \ge \mu$.

Note: JT: Not sure that I believe this to be true. Søren, can you "gennemskue" if any ε and B satisfy this criteria? Otherwise, we have no upwards bound on the ruin prob

6 Properties of the conditional ruin probability in the BAYES-CRED model.

In the BAYES-CRED model, we have already in Theorem 3.5 stated that the limit in (3.2) holds with (3.4). In the following we want to study the properties of this $\phi_\zeta(a)$.

Proposition 6.1. (i) *ϕ_ζ is convex with $\phi'(0) = -\delta\mathbb{E}_\zeta X < 0$.*

(ii) *If $\phi_\zeta(a^*_\zeta) > 0$ the solution $R_\zeta \in (0, a^*_\zeta]$ of (5.13) exists and $\psi_\zeta(u) \sim_{\log} \mathrm{e}^{-R_\zeta u}$ as $u \to \infty$.*

(iii) *If $\phi_\zeta(a^*_\zeta) < 0$ then $\psi_\zeta(u) \le_{\log} \mathrm{e}^{-a^*_\zeta u}$ as $u \to \infty$. Further, if $\mathbb{P}_\zeta(X_1 > x) \sim_{\log} \exp(-\zeta x)$ where $\zeta \le a^*_\zeta$, then $\psi(u) \sim_{\log} \exp(-a^*_\zeta u)$.*

Proof. In (i), convexity is clear either because ϕ is a limit of convex functions or by direct inspection: $\kappa_\zeta\big(\exp\{a[1-(1+\delta)z]\}\big)$ is convex for any z, hence so is any mixture. That $\phi'(0) = -\delta\mathbb{E}X < 0$ follows from simple calculus. Part (ii) then follows since if $\phi_\zeta(a^*_\zeta) \ge 0$, the existence and uniqueness of $R_\zeta \in (0, a^*_\zeta)$ is clear.

For (iii), let $S^b_n = S_n + nb$ where $0 < -\phi_\zeta(a^*_\zeta)/a^*_\zeta < b < -\phi_\zeta(a^*_\zeta)/a^*_\zeta + \epsilon$ as $\phi_\zeta(a^*_\zeta) < 0$. Then

$$\frac{1}{n}\log\big(\mathbb{E}\mathrm{e}^{aS^b_n}\big) \to \phi^b_\zeta(a) = \phi_\zeta(a) + ab \text{ when } n \to \infty.$$

By convexity of ϕ_ζ argued in (i), $\phi_\zeta(a^*_\zeta) > \phi'_\zeta(0)a^*_\zeta$. The function ϕ^b_ζ must also be convex with

$$\phi^{b\prime}_\zeta(0) = \phi'_\zeta(0) + b \;<\; \phi'_\zeta(0) - \phi_\zeta(a^*_\zeta)/a^*_\zeta + \varepsilon < 0,$$

for ε small enough. Further, by construction of b, $\lim_{a\uparrow a^*_\zeta}\phi^b_\zeta(a) \ge 0$. Hence a solution $R^b_\zeta \in (0, a^*_\zeta]$ of $\phi^b_\zeta(R^b_\zeta) = 0$ exists. All other conditions of Glynn and Whitt (1994) are clear, and so we can conclude that

$$\psi_\zeta(u) \;\le\; \psi^b_\zeta(u) \;\sim_{\log}\; \mathrm{e}^{-R^b_\zeta u}.$$

The upper bound now follows by letting $\epsilon \downarrow 0$ since then $R^b_\zeta \to a^*_\zeta$.

If we further assume that $\mathbb{P}(X_1 > x) \sim_{\log} \exp(-\zeta x)$, then

$$\begin{aligned}\psi(u) \geq \mathbb{P}(X_1 > (1+\delta)H_0 + u) &\sim_{\log} \exp(-\zeta((1+\delta)H_0 + u)) \\ &\sim_{\log} \exp(-\zeta u) \geq \exp(-a^*_\zeta u).\end{aligned}$$

□

Remark 6.2. The assumption $\mathbb{P}_\zeta(X_1 > x) \sim_{\log} \exp(-\zeta x)$ is equivalent to $\mathbb{P}_\zeta(X_1 > x) = \exp\{-\zeta x(1+o(1))\}$ which holds in particularly when the distribution function takes on the shape $\mathbb{P}_\zeta(X_1 > x) = \exp(-\zeta x)x^\alpha L(x)$ where $\alpha \in \mathbb{R}$ and $L(x)$ is slowly varying.

Corollary 6.3. *If $a^*_\zeta = \infty$ and F_ζ has unbounded support, then $\phi_\zeta(a) \uparrow \infty$ as $a \uparrow a^*_\zeta$. In particular, the adjustment coefficient exists.*

Proof. The unbounded support implies that $\kappa_\zeta(a)/a \uparrow \infty$ as $a \uparrow a^*_\zeta$. Split the domain of integration in (3.4) into two; $(0, z_0)$ og (z_0, ∞) with $z_0 = 1/(1+\delta)$, i.e. $\phi_\zeta(a) = I_1(a) + I_2(a)$ where

$$I_1(a) = \int_0^{z_0} \mathrm{e}^{-t}\kappa_\zeta\Big(a\big[1-(1+\delta)t\big]\Big)\,\mathrm{d}t, \quad I_2(a) \int_{z_0}^{\infty} \mathrm{e}^{-t}\kappa_\zeta\Big(a\big[1-(1+\delta)t\big]\Big)\,\mathrm{d}t.$$

Due to monotone convergence, using that $1-(1+\delta)z > 0$ for $z \in (0, z_0)$, we get $I_1(a)/a \to \infty$ as $a \to \infty$, in particular $I_1(a) \to \infty$. Further by convexity of κ_ζ and $\kappa'_\zeta(0) = \mu(\zeta) > 0$,

$$I_2(a) \;>\; -\kappa'_\zeta(0)a\eta, \qquad \text{where } \eta = \int_{z_0}^{\infty} \mathrm{e}^{-z}\big[(1+\delta)z - 1\big]\,\mathrm{d}z > 0.$$

Putting these estimates together gives $\phi_\zeta(a) \uparrow \infty$. □

The corollary above has the implication, that if $\phi_\zeta(a) \uparrow \infty$ then the adjustment coefficient must exist.

Example 3 (Truncated exponential)**.** Consider the case where the reference measure $\lambda(\mathrm{d}x)$ in (2.3) is uniform on (0,1). Then the normalisation function is

$$\kappa(\zeta) = \log\Big(\int_0^1 \exp(-\zeta x)\mathrm{d}x\Big) = \log\Big(\frac{1-\mathrm{e}^{-\zeta}}{\zeta}\Big), \quad -\infty < \zeta < \infty.$$

For $\zeta > 0$, F_ζ becomes the exponential(ζ) distribution truncated to having support $(0, 1)$. The prior has kernel

$$f_Z(\zeta; s_1, s_2) \propto \Big(\frac{\zeta}{1-e^{-\zeta}}\Big)^{s_1} e^{s_2\zeta},$$

for $\zeta \in \mathbb{R}$ and suitable parameters $(s_1, s_2) \in \{\mathbb{R}^+ \times \mathbb{R}^+ : s_2 < s_1\}$. The conditional cumulant generating function is

$$\kappa_\zeta(a) = \kappa(\zeta + a) - \kappa(\zeta) = \log\Big(\frac{\zeta}{\zeta + a} \cdot \frac{\exp(\zeta + a) - 1}{\exp(\zeta) - 1}\Big). \tag{6.1}$$

The form of $\kappa_\zeta(a)$ implicates that $\phi_\zeta(a)$ in (3.2) cannot be found explicitly. However, $a^*_\zeta = \infty$ and $\kappa'_\zeta(0) < 0$, $\kappa_\zeta(a) \sim a \to \infty$ as $a \to \infty$, so by convexity R_ζ always exists. To get an idea on the shape of $\phi_\zeta(a)$, see Figure 3 (left panel).

Solving numerically for the adjustment coefficient, R_ζ, as a function of ζ leads to Figure 3 (right panel). Here we observe that the adjustment coefficient is smallest for numerically low values of ζ. Intuitively, for $\zeta \to -\infty$ the truncated exponential distribution will tend to the degenerate distribution in 0. As the claim size then is constant, all risk is eliminated and, consequently, the ruin probability is 0. Similarly, when $\zeta \to \infty$, the truncated exponential distribution will tend to the degenerate distribution in 1, also yielding a ruin probability of 0. Hence, R_ζ does not vary monotonically from 0 to ∞ as in the gamma-gamma case, but has a minimum at some ζ^*.

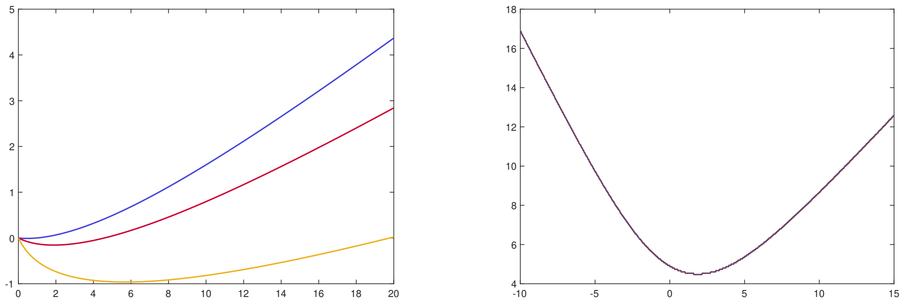

Figure 3: Left panel: The function $\phi_1(a)$ for the exponential truncated distribution and $\delta = 0.1, 0.5, 2$ from top to bottom. Right panel: R_ζ as function of ζ for $\delta = 0.5$.

Example 4 (Rayleigh)**.** Consider the case where the reference measure is Rayleigh, i.e. $\lambda(\mathrm{d}x) = x\mathrm{e}^{-x^2/2}\,\mathrm{d}x$. The exponentially tilted distributed is then

$$f_\zeta(x) = x\exp(-x^2/2 - \zeta x - \kappa(\zeta)),$$

where

$$\kappa(\zeta) = \log\Big(\int_0^\infty x\exp(-x^2/2 - \zeta x)\mathrm{d}x\Big) = \log\big(1 + \sqrt{2\pi}\zeta\exp(\zeta^2/2)(\Phi(\zeta) - 1)\big),$$

using integration by parts and square completion. The prior is then of shape

$$f_Z(\zeta; s_1, s_2) \propto \frac{\exp(-s_1\zeta)}{(1 + \sqrt{2\pi}\zeta\exp(\zeta^2/2)(\Phi(\zeta) - 1))^{s_2}}, \qquad \text{for } \zeta \in \mathbb{R} \text{ and } s_1, s_2 \in \mathbb{R}^+.$$

The conditional cumulant generating function can be expressed in terms of the normalisation function,

$$\begin{aligned}\kappa_\zeta(a) &= \kappa(\zeta + a) - \kappa(\zeta) \\ &= \log\bigg(\frac{1 + \sqrt{2\pi}(\zeta + a)\mathrm{e}^{(\zeta+a)^2/2}(\Phi(\zeta + a) - 1)}{1 + \sqrt{2\pi}\zeta\mathrm{e}^{\zeta^2/2}(\Phi(\zeta) - 1)}\bigg).\end{aligned} \tag{6.2}$$

For conditionally Rayleigh distributed claim sizes, the limit function in (3.2) can be rewritten as

$$\begin{aligned}\phi_\zeta(a) = &-\log(1+\sqrt{2\pi}\zeta e^{\zeta^2/2}(\Phi(\zeta)-1) + \frac{1}{a(1+\delta)}\exp\Bigl(-\frac{a+\zeta}{a(1+\delta)}\Bigr)\times \\ &\int_{-\infty}^{\zeta+a}\exp\Bigl(\frac{y}{a(1+\delta)}\Bigr)\log(1+\sqrt{2\pi}y e^{y^2/2}(\Phi(y)-1)\mathrm{d}y.\end{aligned} \tag{6.3}$$

In the left panel of Figure 4, $\phi_1(a)$ is depicted for different values of δ. As $\kappa_\zeta(a)$ for the Rayleigh distribution has $a_\zeta^* = \infty$ and the Rayleigh distribution has unbounded support, Corollary 6.3 states that an adjustment coefficient will exist. Figure 4, right panel, shows R_ζ solved numerically. As for the truncated exponential case, there is a minimum at some ζ^* with $R_{\zeta^*} > 0$, and similar remarks as there apply.

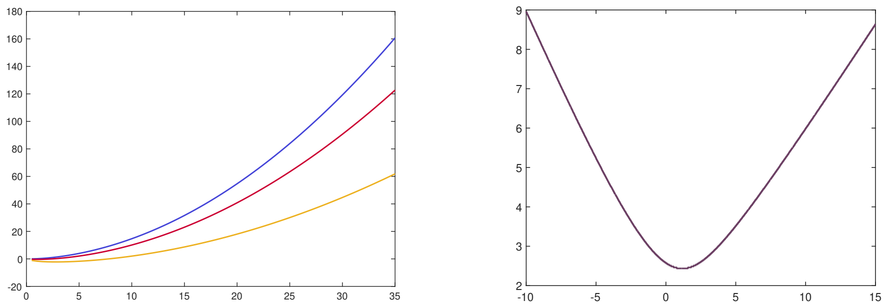

Figure 4: Left panel: The function $\phi_1(a)$ for the Rayleigh distribution and $\delta = 0.1, 0.5, 2$ from top to bottom. Right panel: R_ζ as function of ζ for $\delta = 0.5$.

7 Proofs for and examples in the BAYES-CRED model

Lemma 7.1. *As $\lambda \to \infty$, it holds that $\mathbb{E}\bigl[e^{-\lambda(1-\epsilon)Z};\, Z \le B\bigr] \sim_{\log} \mathbb{E}e^{-\lambda Z}$. Further $\mathbb{E}e^{-\lambda(1\pm\epsilon)Z} \sim (1+\mathcal{O}(\epsilon))\mathbb{E}e^{-\lambda Z}$ for $0 < \epsilon < 1/2$.*

Proof. Choose $0 < A < B < 1$ with $\mathbb{P}(A < Z \le B) > 0$. Then the first statement follows from

$$\mathbb{E}e^{-\lambda Z} \;\ge e^{-\lambda A}\mathbb{P}(A < Z \le B)\,, \quad \mathbb{E}\bigl[e^{-\lambda(1-\epsilon)Z};\, Z \le B\bigr] \le e^{-\lambda B}\,. \tag{7.1}$$

For the second, we may assume $B > 4A$ and get

$$\mathbb{E}\bigl[e^{-\lambda(1-\epsilon)Z};\, Z \le B\bigr] \;\le\; e^{\epsilon\lambda B}\mathbb{E}\bigl[e^{-\lambda Z};\, Z \le B\bigr] \;\le\; e^{\epsilon\lambda B}\mathbb{E}e^{-\lambda Z}$$

and, for $\lambda >$???,

$$\mathbb{E}\bigl[e^{-\lambda(1-\epsilon)Z};\, Z > B\bigr] \;\le e^{-\lambda B/2} \;\le\; e^{-2\lambda A} \;=\; o\bigl(\mathbb{E}e^{-\lambda Z}\bigr) \tag{7.2}$$

where the last step in (7.2) follows from (7.1). Combining these estimates give

$$\limsup_{\lambda\to\infty} \frac{\mathbb{E}e^{-\lambda(1-\epsilon)Z}}{\mathbb{E}e^{-\lambda Z}} \;\le 1 + \epsilon B\,.$$

Noting that $\mathbb{E}e^{-\lambda(1-\epsilon)Z} \geq \mathbb{E}e^{-\lambda Z}$ completes the proof for $\mathbb{E}e^{-\lambda(1-\epsilon)Z}$. The one for $\mathbb{E}e^{-\lambda(1+\epsilon)Z}$ follows by replacing λ by $\lambda(1+\epsilon)/(1-\epsilon)$. $\square$

Proof of Theorem 3.6. Let $\psi_\zeta^H(u)$ be the ruin probability when H_0 in the credibility premium (1.2) is replaced by H. The scale parameter property of ζ is reflected in the conditional ruin probability as $\psi_\zeta^0(u) = \psi_1^0(\zeta u)$ and $\psi_\zeta(u) = \psi_1^{H_0/\zeta}(\zeta u)$.

Then obviously $\psi_\zeta^H(u) \leq \psi_\zeta^0(u)$ for all $H \geq 0$ and $u > 0$. Further, for a given $0 < \epsilon < R_1$, Corollary 7.3 and (7.3) implies that there exists an $u_+ < \infty$ such that

$$\log \psi_1^0(u) \leq -(R_1 - \epsilon)u,$$

for $u \geq u_+$. Therefore we can find $C_+ < \infty$ such that

$$\psi_1^0(u) \leq C_+ e^{-(R_1-\epsilon)u},$$

for all $u \geq 0$. We then get

$$\begin{aligned}\psi(u) &= \int_0^\infty \psi_\zeta(u) f_Z(\zeta)\,d\zeta \leq \int_0^\infty \psi_1^0(\zeta u) f_Z(\zeta)\,d\zeta \\ &\leq \int_0^\infty C_+ e^{-(R_1-\epsilon)u\zeta} f_Z(\zeta)\,d\zeta \ \sim_{\log}\ \mathbb{E}e^{-uR_1 Z},\end{aligned}$$

where we used Lemma 7.1 in the last step.

Regarding the lower bound, we first note that $\psi_1^{H_0/\zeta}(u) \geq \psi_1^{H_0}(u) = \psi_1(u)$ for all $\zeta \in (0,1)$ and $u \geq 0$. Hence

$$\psi(u) \geq \int_0^1 \psi_\zeta(u) f_Z(\zeta)\,d\zeta = \int_0^1 \psi_1^{\zeta H_0}(\zeta u) f_Z(\zeta)\,d\zeta \ \geq\ \int_0^1 \psi_1(\zeta u) f_Z(\zeta)\,d\zeta.$$

Similarly as above, we can find $C_- > 0$ such that $\psi_1(u) \geq C_- e^{-(R_1+\epsilon)u}$ for all $u \geq 0$, and so

$$\psi(u) \geq C_- \int_0^1 e^{-(R_1+\epsilon)u\zeta} f_Z(\zeta)\,d\zeta \sim_{\log} \mathbb{E}e^{-(R_1+\epsilon)uZ} \sim_{\log} \mathbb{E}e^{-uR_1 Z},$$

where we used Lemma 7.1 twice. $\square$

We now return to our key example with F_ζ being gamma(α, ζ). The cumulant generating function is then given by (5.9), and therefore $\alpha_\zeta^* = \zeta < \infty$. First we study the existence of an adjustment coefficient. Recall for the following that $\gamma = -\int_0^\infty e^{-y} \log(y)\,dy$ is Euler's constant.

Proposition 7.2. *It holds that*

$$\begin{aligned}\phi_\zeta(a_\zeta^*) &= \lim_{a \uparrow a_\zeta^*} \phi_\zeta(a) = \alpha(\gamma - \log(1+\delta)), \\ \phi'_\zeta(a_\zeta^* -) &= \lim_{a \uparrow a_\zeta^*} \phi'(a) = \frac{\alpha}{1+\delta} \int_0^\infty \frac{1}{y} e^{-y}\,dy - 1 = \infty.\end{aligned}$$

Proof. For the gamma-case, the limit in (3.2) is

$$\phi_\zeta(a) = \alpha \int_0^\infty \mathrm{e}^{-y} \log\Big(\frac{\zeta}{\zeta - a(1-(1+\delta)y)}\Big)\,\mathrm{d}y.$$

Evaluating in $a^*_\zeta = \zeta$ yields

$$\phi_\zeta(\zeta) = -\alpha \int_0^\infty \mathrm{e}^{-y} \log\big((1+\delta)y\big)\,\mathrm{d}y = \alpha\Big(\int_0^\infty \mathrm{e}^{-y}\log(y)\,\mathrm{d}y - \log(1+\delta)\Big)$$

where we recognise the integral expression as γ. The rest of the proof is easy calculus. □

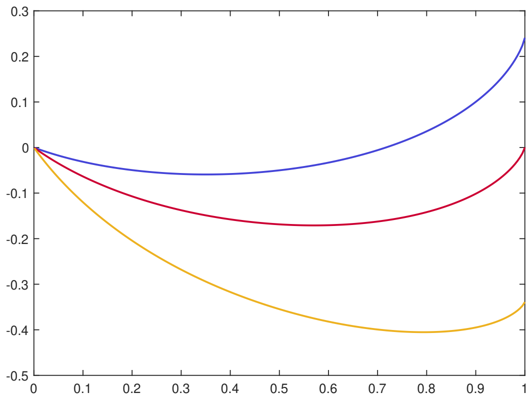

Figure 5: The function $\phi_1(\gamma)$ for $0 \le \gamma \le 1$, $\alpha = 1$ and $\delta = 0.4, 0.781, 1.5$ from top to bottom.

The situation is illustrated in Figure 5 for $\zeta = 1$. That $\phi'_\zeta(1-) = \infty$ is not visible at the given resolution, but is confirmed by a zooming in omitted here, and actually, this feature is unimportant for the following. It follows that $\phi_\zeta(a^*_\zeta) > 0$ if and only if $\log(1+\delta) < \gamma$, i.e. $\delta < \mathrm{e}^\gamma - 1 = 0.781\ldots$.

Corollary 7.3. *If $\delta < \mathrm{e}^\gamma - 1$, then a solution $R_\zeta \in (0,\zeta)$ of (5.13) exists and $\psi_\zeta(u) \sim_{\log} \mathrm{e}^{-R_\zeta u}$ as $u \to \infty$. If $\delta > \mathrm{e}^\gamma - 1$, then $\psi_\zeta(u) \sim_{\log} \mathrm{e}^{-\zeta u}$.*

Proof. Follows from Proposition 6.1, since $\mathbb{P}_\zeta(X_1 > x) \sim_{\log} \exp(-\zeta x)$ as

$$\frac{\log(\mathbb{P}(X_1 > x)}{\log(\exp(-\zeta x))} = \frac{\log(\int_x^\infty \frac{\zeta^\alpha}{\Gamma(\alpha)} y^{\alpha-1}\exp(-\zeta y)\mathrm{d}y)}{-\zeta x}$$
$$= 1 - \frac{\alpha-1}{\zeta x} \to 1, \qquad \text{as } x \to \infty,$$

using the L'Hospital's rule. □

The gamma distribution belongs to the scale family where ζ simply acts as a scale parameter, cf. (5.9). This relation between the cumulant generating functions for a general ζ and $\zeta = 1$ implies $\phi_\zeta(a) = \phi_1(a/\zeta)$ or equivalently $\phi_\zeta(\zeta a) = \phi_1(a)$. So $\phi_1(R_1) = 0$ entails $\phi_\zeta(\zeta R_1) = 0$ and thus

$$R_\zeta = \zeta R_1, \tag{7.3}$$

provided the adjustment coefficient exists. The gamma-gamma case obviously satisfies the conditions of Theorem 3.6, which for a gamma(α_0, β_0) prior gives that

$$\psi(u) \sim_{\log} \mathbb{E}[\mathrm{e}^{-uR_1 Z}] \sim_{\log} \frac{1}{u^{\alpha_0}},$$

which provides the arguments for Corollary 3.7

Example 5 (Half-normal)**.** Now return to the half-normal distribution in Example 1. By Corollary 6.3, R_1 always exists and so Theorem 3.6 applies.

If the prior density is taken as $f_Z(\zeta) = f_{\zeta_0}(\zeta) = \zeta_0\sqrt{2/\pi}\mathrm{e}^{-\zeta^2\zeta_0^2/2}$ for some ζ_0, we have $f_Z(\zeta) \to \zeta_0\sqrt{2/\pi}$ as $\zeta \downarrow 0$. Up to constant, this is the same as for a gamma distribution with $\alpha = 1$, and we can conclude that $\psi(u) \sim_{\log} 1/u$.

Example 6 (Inverse Gaussian)**.** For the inverse Gamma distribution, we recall that $\kappa_1(a)$ given by (5.13) is non-steep at $a_1^* = \lambda/2\mu^2$. This gives $\phi_1(a) = \infty$ for $a > a_1^*$ and $< \infty$ for $a \le a_1^*$. Thus, R_1 will exists if and only if

$$\begin{aligned} 0 \le \psi_1(a_1^*) &= \int_0^\infty \mathrm{e}^{-t}\frac{\lambda}{\mu}\Big(1 - \sqrt{1 - [1 - (1+\delta)t]}\Big)\,\mathrm{d}t \\ &= \frac{\lambda}{\mu}\int_0^\infty \mathrm{e}^{-t}\big(1 - \sqrt{1+\delta}\sqrt{t}\big)\,\mathrm{d}t = \frac{\lambda}{\mu}\big(1 - \sqrt{1+\delta}\Gamma(3/2)\big), \end{aligned}$$

meaning $\delta \le 4/\pi - 1 = 0.273\ldots$. Equality is, however, excluded for Theorem 3.6 to apply since $\kappa_1'(a^*) = \infty$ violates the conditions of Glynn and Whitt (1994).

When the inverse Gaussian (λ, μ) distribution is instead used as prior, inspection of (5.13) shows that $\mathbb{E}\mathrm{e}^{-aZ} \sim_{\log} \exp\big\{-\sqrt{2\lambda\mu^2 a}/\mu\big\}$. Thus if the F_ζ satisfy the assumptions of Theorem 3.6, we can conclude that $\psi(u) \sim_{\log} \mathrm{e}^{-C\sqrt{u}}$ where $C = \sqrt{2\lambda\mu^2 R_1}/\mu$.

Example 7. Consider the lognormal(μ, σ^2) distribution. This family fits into the scale family set-up corresponding to $\zeta = \mathrm{e}^{-\mu}$ but due to all exponential moments being infinite, we can not use it to model claim sizes. When using it as prior, Corollary 2.3 of AJRN gives $\mathbb{E}\mathrm{e}^{-aZ} \sim_{\log} \mathrm{e}^{-(\log a)^2/2\sigma^2}$ and thus $\psi(u) \sim_{\log} e^{-(\log u)^2/2\sigma^2}$.

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A Calculations for the gamma-gamma model

The conjugate prior is likewise gamma(α_0, β_0) leading to posterior gamma(α_n, β_n) given $X_1, \ldots, X_n$ with parameters

$$\alpha_n = \alpha_0 + n\alpha, \qquad \beta_n = \beta_0 + \sum_{i=1}^{n} X_i.$$

In the following, let

$$\Gamma(b) = \int_0^\infty t^{b-1}\exp(-t)\,\mathrm{d}t\,, \quad \gamma(b,z) = \int_0^z t^{b-1}\exp(-t)\,\mathrm{d}t, \quad \Gamma(b,z) = \Gamma(b) - \gamma(b,z)$$

be the Gamma function, and the lower resp. upper incomplete Gamma function. The mean and variance as function of the unknown claim parameter are

$$\mu(\zeta) = \alpha/\zeta, \qquad \sigma^2(\zeta) = \alpha/\zeta^2.$$

The collective premium is then

$$H_0 = \mathbb{E}[X] = \mathbb{E}\Big[\frac{\alpha}{Z}\Big] = \alpha\int_0^\infty \frac{1}{\zeta}\frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)}\zeta^{\alpha_0-1}\mathrm{e}^{-\beta_0\zeta}\,\mathrm{d}\zeta = \frac{\alpha\beta_0}{\alpha_0-1}. \tag{A.1}$$

Similarly, we can find

$$\begin{aligned}
\sigma^2 &= \mathbb{E}\Big[\frac{\alpha}{Z^2}\Big] = \frac{\alpha\beta_0^2}{(\alpha_0-1)(\alpha_0-2)},\\
\tau^2 &= \mathbb{V}\mathrm{ar}\Big[\frac{\alpha}{Z}\Big] = \sigma^2 - \mu^2\\
&= \frac{\alpha^2\beta_0^2}{(\alpha_0-1)(\alpha_0-2)} - \frac{\alpha_0^2\beta_0^2}{(\alpha-1)^2} = \frac{\alpha^2\beta_0^2}{(\alpha_0-1)^2(\alpha_0-2)},
\end{aligned}$$

and so $d = \sigma^2/\tau^2 = (\alpha_0-1)/\alpha$ in (1.2). That the Bayes and credibility premiums coincide is then confirmed since the posterior mean of Z is α_n/β_n and the posterior mean of X therefore

$$\frac{\alpha\beta_n}{\alpha_n-1} = \frac{\alpha(\beta_0+n\overline{X}_n)}{\alpha_0+n\alpha-1} = \frac{n}{n+d}\overline{X}_n + \frac{d}{n+d}H_0, \tag{A.2}$$

where $d = (\alpha_0-1)/\alpha$.

Using again (A.1), we get the tail of the unconditional distribution of X as

$$\begin{aligned}
\mathbb{P}(X > x) &= \int_0^\infty \mathbb{P}(X > x \mid Z = \zeta)f_Z(\zeta)\,\mathrm{d}\zeta\\
&= \int_0^\infty \frac{\Gamma(\alpha,\zeta x)}{\Gamma(\alpha)}\frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)}\zeta^{\alpha_0-1}\mathrm{e}^{-\beta_0\zeta}\,\mathrm{d}\zeta.
\end{aligned}$$

Asymptotically $\Gamma(\alpha,\zeta x) \sim (\zeta x)^{\alpha-1}\mathbb{E}(-\zeta x)$, and therefore

$$\begin{aligned}
\mathbb{P}(X > x) &\sim \frac{\beta_0^{\alpha_0}x^{\alpha-1}}{\Gamma(\alpha_0)\Gamma(\alpha)}\int_0^\infty \zeta^{\alpha_0+\alpha-2}\mathrm{e}^{-(\beta_0+x)\zeta}\,\mathrm{d}\zeta\\
&= k\frac{x^{\alpha-1}}{(x+\beta_0)^{\alpha_0+\alpha-1}} \sim \frac{k}{x^{\alpha_0}}.
\end{aligned}$$

for an appropriate $k > 0$. That is, X has a Pareto-like tail, and this gives immediately that an adjustment coefficient can not exist in the unconditional model since

$$\psi(u) \geq \mathbb{P}(X_1 - H_0 > u) \sim \frac{k}{u^{\alpha_0}}.$$

This estimate may seen quite conservative, but Theorem 3.5 shows that it actually gives the correct order of magnitude $u^{-\alpha_0}$ of $\psi(u)$.

B Proof of Theorem 3.5

We shall need the following lemma:

Lemma B.1. (a) *For $n \in \mathbb{N}$ and $d > 0$, there is a constant γ_d such that*

$$\sum_{k=1}^{n} \frac{1}{k+d} = \log\Bigl(\frac{n+d}{1+d}\Bigr) + \gamma_d + \mathcal{O}(1/n). \tag{B.1}$$

(b) *For $N \leq j \leq n$*

$$\sum_{k=j}^{n} \frac{1}{k+d} = \log(n/j) + \mathcal{O}(1/N), \tag{B.2}$$

where $\mathcal{O}(1/N)$ is uniform in n, j.

Proof. Part (a) is standard for $d = 0$ (or more generally $d \in \mathbb{N}$), where $\gamma_1 = \gamma = -\int_0^\infty \log(x)\mathrm{e}^{-x}\mathrm{d}x = 0.5772\ldots$ is Euler's constant. For a general d, let

$$\gamma_{n,d} = \sum_{j=1}^{n-1} \int_j^{j+1} \Bigl(\frac{1}{j+d} - \frac{1}{x+d}\Bigr)\mathrm{d}x\,, \quad \gamma_d = \sum_{j=1}^{\infty} \int_j^{j+1} \Bigl(\frac{1}{j+d} - \frac{1}{x+d}\Bigr)\mathrm{d}x.$$

Then $\gamma_{n,d} \to \gamma_d$ which is well-defined since the terms are $\mathcal{O}(1/j^2)$. This also gives $\gamma_{d,n} = \gamma_d + \mathcal{O}(1/n)$.

We can then evaluate the sum in (a) as follows,

$$\sum_{k=1}^{n} \frac{1}{k+d} = \int_1^n \Bigl(\frac{1}{x+d}\Bigr)\mathrm{d}x + \gamma_{n,d} + \frac{1}{n+d} = \log\Bigl(\frac{n+d}{1+d}\Bigr) + \gamma_d + \mathcal{O}(1/n).$$

For the sum in (b), assume that $j \geq N$ and consider

$$\begin{aligned}
\sum_{k=j}^{n} \frac{1}{k+d} &= \sum_{k=1}^{n} \frac{1}{k+d} - \sum_{k=1}^{j-1} \frac{1}{k+d} \\
&= \log\Bigl(\frac{n+d}{d+1}\Bigr) + \gamma_d + \mathcal{O}(1/n) - \Bigl(\log\Bigl(\frac{j+d-1}{d+1}\Bigr) + \gamma_d + \mathcal{O}(1/j)\Bigr) \\
&= \log\Bigl(\frac{n+d}{j+d-1}\Bigr) + \mathcal{O}(1/N),
\end{aligned}$$

where

$$
\begin{aligned}
\log\Big(\frac{n+d}{j+d-1}\Big) &= \log\Big(\frac{n(1+d/n)}{j(1+d/j-1/j)}\Big) \\
&= \log\Big(\frac{n}{j}\Big) + \log(1+d/n) - \log(1+j/n) = \log\Big(\frac{n}{j}\Big) + \mathcal{O}(1/N).
\end{aligned}
$$

□

Proof of Theorem 3.5. The claim surplus can be written out explicitly as

$$
\begin{aligned}
S_n &= \sum_{k=1}^{n}(X_k - \Pi_{k-1}) = \sum_{k=1}^{n} X_n - (1+\delta)\sum_{k=0}^{n-1}\Big[\frac{1}{k+d}\sum_{j=1}^{k} X_j + \frac{d}{k+d}\ \mu\Big] \\
&= \sum_{k=1}^{n} X_n - (1+\delta)\sum_{j=1}^{n-1} X_j\Big(\sum_{k=j}^{n-1}\frac{1}{k+d}\Big) - (1+\delta)\mu d\sum_{k=0}^{n-1}\frac{1}{k+d},
\end{aligned}
$$

which for a fixed N by independence of the X_j wih $j < N$ and $j \geq N$ gives that

$$
\frac{1}{n}\log \mathbb{E}_\zeta \mathrm{e}^{aS_n} \ = \ \frac{1}{n}\log \mathbb{E}_\zeta \mathrm{e}^{aX_n} + \frac{1}{n}\log \mathbb{E}_\zeta \mathrm{e}^{aB'_{n,N}} + \frac{1}{n}\log \mathbb{E}_\zeta \mathrm{e}^{aB''_{n,N}} + \frac{1}{n}\log \mathrm{e}^{aA_n}\,, \quad \text{(B.3)}
$$

where

$$
\begin{aligned}
A_n &= -(1+\delta)\mu d\sum_{k=0}^{n-1}\frac{1}{k+d} \\
B'_{n,N} \ &= \ \sum_{j=1}^{N-1} X_j\Big\{1-(1+\delta)\Big(\sum_{k=j}^{n-1}\frac{1}{k+d}\Big)\Big\}\,, \\
B''_{n,N} \ &= \ \sum_{j=N}^{n-1} X_j\Big\{1-(1+\delta)\Big(\sum_{k=j}^{n-1}\frac{1}{k+d}\Big)\Big\}\,.
\end{aligned}
$$

Here the last term in (B.3) is o(1) since $A_n = \mathcal{O}(\log n)$ and the first equals $\kappa_\zeta(a)/n \to 0$. Also

$$
\frac{1}{n}\log \mathbb{E}_\zeta \mathrm{e}^{aB'_{n,N}} \ \leq \ \frac{1}{n}\log \mathbb{E}_\zeta \mathbb{E}\{a(X_1+\cdots+X_{N-1})\} \ = \ \frac{(N-1)\kappa_\zeta(a)}{n} \to 0.
$$

Thus we only are left with studying the third term in (B.3), which by Lemma B.1, (b) we can write as

$$
\begin{aligned}
&\frac{1}{n}\log\mathbb{E}_\zeta\Big[\mathbb{E}\Big\{a\Big(\sum_{j=N}^{n-1} X_j\big[1-(1+\delta)\big(\log((n-1)/j)+\mathcal{O}(1/N)\big)\big]\Big)\Big\}\Big] \\
&\quad = \frac{1}{n}\sum_{j=N}^{n-1}\kappa_\zeta\Big(a\big[1-(1+\delta)\big(\log((n-1)/j)+\mathcal{O}(1/N)\big)\big]\Big) \\
&\quad = \frac{1}{n}\sum_{j=N}^{n-1}\kappa_\zeta\Big(a\big[1-(1+\delta)\log((n-1)/j)\big]\Big) + R_{n,N},
\end{aligned}
$$

where by a suitable version of Taylor's approximation

$$|R_{n,N}| \le \sup_{b \le a+\mathcal{O}(1/N)} \kappa_\zeta'(b) a(1+\delta)\mathcal{O}(1/N).$$

But $X \ge 0$ implies that $\kappa_\zeta(b) \ge \theta b$ for some θ and hence by convexity $\kappa_\zeta'(b)$ is bounded as $b \to -\infty$. This implies that $R_{n,N}$ is another $\mathcal{O}(1/N)$ term. Furthermore

$$\frac{1}{n}\Big|\sum_{j=1}^{N-1} \kappa_\zeta\Big(a\big[1-(1+\delta)\log((n-1)/j)\big]\Big)\Big| \;\le\; \frac{(N-1)\kappa_\zeta(a)}{n} \;\to 0.$$

and a Riemann sum approximation gives

$$\begin{aligned}&\frac{1}{n}\sum_{j=1}^{n-1} \kappa_\zeta\Big(a\big[1-(1+\delta)\log((n-1)/j)\big]\Big)\\ &\qquad \to \int_0^1 \kappa_\zeta\Big(a(1-(1+\delta)\log(1/x)\Big)\mathrm{d}x \;=\; \phi_\zeta(a)\,,\end{aligned}$$

where we substituted $t = \log(1/x)$ in the last step.

Putting the above estimates together, we get

$$\begin{aligned}\limsup_{n\to\infty}\Big|\frac{1}{n}\log \mathbb{E}_\zeta \mathrm{e}^{aS_n} - \phi_\zeta(a)\Big| \;&=\; \limsup_{n\to\infty}\Big|\frac{1}{n}\log \mathbb{E}_\zeta \mathrm{e}^{aB_n''} - \phi_\zeta(a)\Big|\\ =\; \limsup_{n\to\infty}\Big|\frac{1}{n}\Big(\sum_{j=1}^{N-1} &+ \sum_{j=N}^{n-1}\Big)\kappa_\zeta\Big(a\big[1-(1+\delta)\log((n-1)/j)\big]\Big)\\ &+ R_{n,N} - \phi_\zeta(a)\Big| \;=\; \limsup_{n\to\infty}|R_{n,N}| \;=\; \mathcal{O}(1/N)\,.\end{aligned}$$

Letting $N \to \infty$ completes the proof. □

List of Corrections