On the Resolution of Foliation Singularities via Jet Schemes

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Doctor in Philosophy

by
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Abstract

We aim to contribute to the problem of finding desingularisations of codimension-1 holomorphic foliations in arbitrary ambient dimension. To this end, we prove an alternate characterisation of simple singularities, which leads to an equivalent condition for the existence of a desingularisation in the non-dicritical case. We then give some results and conjectures to show when this condition is satisfied.

This thesis is dedicated to my father, Andy Carter, as the culmination of four years' work.

Of making many books there is no end, and much study wearies the body.

— Ecclesiastes 12:12

He who is certain he knows the ending of things when he is only beginning them is either extremely wise or extremely foolish; no matter which is true, he is certainly an unhappy man, for he has put a knife in the heart of wonder.

— Tad Williams,
The Dragonbone Chair

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Introduction

Foliations of complex manifolds arise from holomorphic differential forms on the manifold, where an open submanifold can be written as the disjoint union of the solutions of the corresponding system of differential equations. By duality, a foliation also corresponds to a sheaf of holomorphic vector fields; the open submanifold is the locus where this sheaf is locally free. Its complement is called the singular locus of the foliation.

Just as with algebraic varieties, we seek a classification of foliations up to birational equivalence. (See [3]). For varieties, such a classification first depends on proving the existence of a desingularistion. Desingularisation of varieties, as well as embedded desingularisation of closed subschemes of smooth varieties, was first proved by Hironaka in [17]. The method involves a sequence of blow-ups in smooth centres until we reach simple normal crossings, that is, the scheme is locally isomorphic to a union of co-ordinate hyperplanes. Various simplifications of the proof have since appeared, including by Bierstone and Milman in [2], and by Kollár in [21], although to date desingularisation has only been proved in characteristic zero.

Similarly, in seeking to classify singular holomorphic foliations on a complex manifold X, one first seeks the existence of a resolution – a sequence of blow-ups of X such that the pull-back foliation has the best possible singularities, so-called *simple* singularities, as in Figure 1 below. (In most cases, it is impossible to resolve the foliation to make it smooth. For example, blowing up the singularity in Figure 1 gives two singularities of the same form.)

Existence of resolutions was first proved by Seidenberg in [28], in the case when $\dim X = 2$. In the case $\dim X = 3$, existence of resolutions for codimension-1 foliations has been proved by Cano, firstly for the non-dicritical case (that is, for foliations where the exceptional divisors of any sequence of blow-ups in smooth centres remain tangent) in [5], and then in the general case in [7]. Existence of resolutions for foliations by curves on a 3-fold was proved by Panazzolo and McQuillan in [24]. We note that the paper of Panazzolo and McQuillan provides an example of a foliation by curves which cannot be resolved without passing from manifolds to stacks, so the problem of resolution is in general more difficult for foliations than for schemes.

The existence of resolutions of codimension-1 foliations in arbitrary ambient dimension is still conjectural; there are however a number of partial results pertaining to it: By [6, Theorem 16] a resolution to simple singularities exists if all the singularities of the foliation are so-called *pre-simple* singularities (which we define in Section 10). From the discussion following Statement 9 in the same paper we see that such is the case for any foliation with a holomorphic first integral, a class which by [22] includes any foliation with singular locus of codimension at least three.

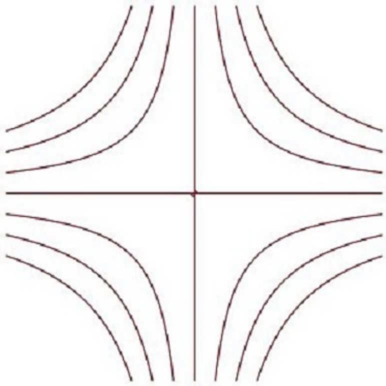

Figure 1: A foliation with simple singularities

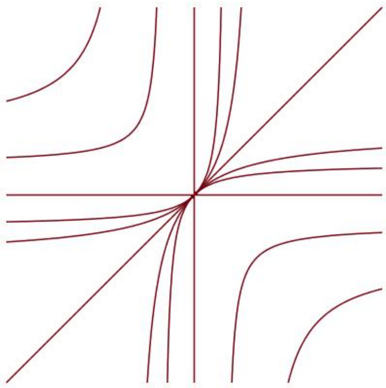

Figure 2: A dicritical foliation

An important tool used in the thesis is the jet space. The space of m-jets of a manifold or analytic space X is the set of equivalence classes of germs of holomorphic functions $\mathbb{C} \to X$, with two germs being equivalent if their corresponding power series have the same truncation to degree m. In the algebraic setting, the jet space can be constructed as a scheme, whose closed points are the morphisms $\mathrm{Spec}(\mathbb{C}[t]/(t^{m+1})) \to X$. Jet spaces have been applied to the study of singularities of schemes, for example by Mustaţă in [26]; this motivates us to introduce the jet spaces of foliations. These are defined in Section 9.1 as those jets of the ambient manifold which pull back the 1-forms defining the foliation to zero. (The notion of the jet space of a foliation is new; however in the smooth case it is equal to the jet space of a directed manifold constructed in [10]).

Using the jet spaces, we can define stronger notions of tangency to a foliation: A complex subspace of the ambient manifold is *strongly tangent* to the foliation if its jet spaces of every order are contained in the jet spaces of the foliation. As opposed to classical notions of tangency, which deal only with the 1-jets (that is, the tangent vectors), or which define tangency as being contained in a finite union of leaves of the foliation, this definition allows us to view as tangent non-reduced spaces, and so better consider the behaviour of the foliation at the singular locus.

The jet spaces of a foliation also allow us to produce a geometrically more natural definition of simple singularities. These are defined for codimension-1 foliations in arbitrary ambient dimension in [6], however the definition is somewhat technical—we therefore present an alternate characterisation, given in Theorem 10.3 for ambient dimension 2, and Corollary 10.28 in the higher dimensional case. These results state that in a neighbourhood of a simple singularity, the foliations has the same geometric structure (as determined by the jet spaces) as a normal crossings divisor.

Armed with these tools, we return to the problem of the existence of resolutions. We restrict to the case of codimension-1 non-dicritical foliations, in arbitrary ambient dimension. The non-dicriticality condition ensures the existence of a separatrix, that is, a leaf of the foliation whose analytic closure passes through the singular locus; assuming mild finiteness conditions, for example, quasi-compactness of the ambient manifold and algebraicity of the defining 1-form, we have that there are finitely many separatrices—we thus define the *total separatrix* of the foliation to be the maximal (formal) subscheme supported on their union which is strongly tangent. Having observed that the endpoints of desingularisations of varieties and resolutions of foliations have isomorphic jets, we are led to prove the following (Theorem 11.15): A non-dicritical foliation admits a resolution to simple singularities if and only if the total separatrix exists and can be resolved to a normal crossings divisor. (As a formal scheme is not guaranteed to have a resolution globally, existence is not enough to prove resolvability. We do however show that if the total separatrix exists the foliation can be resolved on the level of germs.)

This result serves as a generalisation of results such as [9, Theorem 1], which states that non-dicritical foliations on a 3-fold with certain extra properties have the same resolution as their set of separatrices, although it was proved quite independently. The stronger nature of the results in this thesis comes from our defining of strong tangency, which allows us to replace the union of separatrices with something potentially non-reduced.

At the end of the thesis we give a conjecture (Conjecture 11.21) on the structure of the jet spaces of foliations; assuming this we show that the total separatrix always exists (Proposition 11.22). We also give a way of constructing it, which comes with an alternate characterisation of dicriticality, given in terms of the jets (Proposition 12.6).

The thesis is structured as follows: In Section 1 to Section 4, we set out background material and notation regarding commutative algebra, schemes and analytic spaces, and coherent sheaves. In these sections we give all the definitions and results used in the rest of the thesis.

In Section 5 we introduce formal schemes. We take as our definition that of Yasuda in [32], and show how other categories of formal schemes in the literature, especially those desccribed by Grothendieck and by McQuillan, relate to this one. We also state a theorem of Temkin ([31]) regarding desingularisation of formal schemes.

Section 6 introduces jet spaces, with descriptions of both the algebraic and analytic constructions. In Section 7 we introduce linear spaces on manifolds (a generalisation of vector bundles that correspond to coherent sheaves), and then in Section 8 we introduce foliations, giving attention to both the vector field and differential form characterisations.

The final four sections form the main body of the thesis. In Section 9 we introduce the jet spaces of a foliation. We also define the notion of a subscheme being strongly tangent to the foliation (Definition 9.10). In Section 9.4 we define the separatrices. From here onwards, we consider only the codimension-1 case.

In Section 10 we prove the alternate characterisation of simple singularities. In Section 11 we define the total separatrix, and use it to prove results on resolutions. Section 12 gives some results pertaining to dicritical foliations.

Convention. We denote by $\mathbb{N}$ the set of natural numbers $\{1, 2, 3, \ldots\}$. We denote by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$.
All rings are assumed to be commutative with unit.
All schemes are defined over the field of complex numbers $\mathbb{C}$.
Any manifold X is a complex manifold, with holomorphic tangent sheaf $\mathcal{T}_X$, holomorphic tangent bundle TX, and sheaf of holomorphic 1-forms Ω^1_X.
We denote by $\hat{\mathfrak{m}}_n$ the ideal $(x_1, \ldots, x_n) \subset \mathbb{C}[[x_1, \ldots, x_n]]$.

Contents

1 Commutative Algebra

1.1 Basic Ring Theory

For a ring R, we denote by $\mathrm{Spec}(R)$ the set of prime ideals, which is endowed with the topology which has closed sets of the form

$$\{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \supset I\},$$

for some ideal $I \subset R$.

Proposition 1.1. *Let $\phi : R \to S$ be a ring homomorphism.*

(1) If J is an ideal of S, then $\phi^{-1}(J)$ is an ideal of R.

(2) If ϕ is surjective, and I is an ideal of R, then $\phi(I)$ is an ideal of S.

Proof: Trivial. □

Definition 1.2. Let R be a ring. The *nilradical* of R, denoted $\mathfrak{N}_R$, is the ideal of nilpotent elements of R, or equivalently the radical of the zero ideal.

We denote by R_{red} the quotient ring $R/\mathfrak{N}_R$. The ring R is said to be *reduced* if $R_{\mathrm{red}} \cong R$.

Proposition 1.3. *Let $\phi : R \to S$ be a ring homomorphism. Then there is an induced morphism $\bar{\phi} : R_{\mathrm{red}} \to S_{\mathrm{red}}$ given by $\bar{\phi}([x]) = [\phi(x)]$, where $[x] = x + \mathfrak{N}_R$.*

Proof: A ring homomorphism maps nilpotent elements of R to nilpotent elements of S, and so $\bar{\phi}$ is well-defined. The rest of the proof follows naturally. □

Proposition 1.4. *Let $\phi : R \to S$ be a surjective ring homomorphism, and suppose the induced morphism $\bar{\phi} : R_{\mathrm{red}} \to S_{\mathrm{red}}$ is injective. Then $\mathrm{Ker}\,\phi \subset \mathfrak{N}_R$.*

Proof: Let $x \in \mathrm{Ker}\,\phi$. Then $\bar{\phi}([x]) = [\phi(x)] = 0$, and so $[x] = 0$, since $\bar{\phi}$ is injective. Hence $x \in \mathfrak{N}_R$. □

1.2 Complete Rings

Definition 1.5. A *topological ring* is a ring R endowed with a topology such that both addition and multiplication are continuous.

Definition 1.6. Let R be a topological ring. R is *linearly topologised* if there is a fundamental system of open neighbourhoods of 0 consisting of ideals.

Such a system is called a *basis of open ideals.*

Remark 1.7. If a ring R has a descending filtration of ideals $(I_\lambda)_{\lambda\in\Lambda}$, where Λ is a poset, then this defines a unique topology on R for which R is linearly topologised, with the I_λ forming a basis of open ideals.

Definition 1.8. A topological ring R is called *separated* if it is separated as a topological space; in particular, if R is a Hausdorff space.

Proposition 1.9. *Let R be a linearly topologised ring, with basis of open ideals (I_λ). Then R is separated if and only if $\bigcap I_\lambda = 0$.*

Proof: Suppose $\bigcap I_\lambda = 0$. By continuity of addition, to prove R is separated it suffices to show that there are open neighbourhoods separating any x from 0. Let $x \neq 0$. By assumption, there exists some I_λ which does not contain x. Then $I_\lambda, x + I_\lambda$ are disjoint open neighbourhoods of 0 and x respectively, so we are done.

Conversely, suppose there exists some $0 \neq x \in \bigcap I_\lambda$. Any open neighbourhood U of 0 contains some I_λ, and so also contains x. So R is not separated. □

Definition 1.10. Let R be a linearly topologised ring, with basis of open ideals (I_λ). The *completion* of R with respect to this basis is the limit

$$\hat{R} = \varprojlim R/I_\lambda.$$

We endow $\hat{R}$ with the linear topology generated by the kernels of the canonical maps $\hat{R} \to R/I_\lambda$.

Definition 1.11. A linearly topologised ring R is said to be *complete* if the canonical map $R \to \hat{R}$ is an isomorphism.

We observe that $\hat{\hat{R}} \cong \hat{R}$, where the second completion is with respect to the canonical topology from the inverse limit, the ideals in the filtration being $\hat{I_\lambda} = \mathrm{Ker}(\hat{R} \to R/I_\lambda)$, and so the completion of a ring is a complete ring. We further observe that the kernel of the canonical map $R \to \hat{R}$ is $\bigcap I_\lambda$, and so any complete ring is separated.

Definition 1.12. Let R be a linearly topologised ring, with basis of open ideals (I_λ), and let M be a topological R-module (that is, a module with continuous addition and scalar multiplication). The *completion* of M with respect to this basis is the limit

$$\hat{M} = \varprojlim M/I_\lambda M.$$

Again there is a canonical map $M \to \hat{M}$. The module M is said to be complete with respect to (I_λ) if this map is an isomorphism.

Remark 1.13. Let R be a ring, and $I \subset R$ a proper ideal. Then $(I^n)_{n\in\mathbb{N}}$ is a descending filtration of ideals, and so defines a topology. Completing R, or any topological R-module, with respect to this basis of ideals is referred to as completion with respect to I. Similarly, if R is isomorphic to this completion, we say it is complete with respect to I.

Definition 1.14. A *complete local ring* is local ring R which is complete with respect to its unique maximal ideal $\mathfrak{m}$.

Proposition 1.15. *Let R be a Noetherian complete local ring, and $I \subset R$ a proper ideal. Then R/I is a complete local ring.*

Proof: That R/I is local follows from the correspondence theorem for ideals of quotient rings. As completion with respect to the maximal ideal $\mathfrak{m}_R \subset R$ is exact (by [30, Tag 00MA], as all modules are finitely generated), we have an exact sequence of R-modules

$$0 \to \hat{I} \to \hat{R} \to \widehat{R/I} \to 0.$$

As R is complete, $\hat{R} = R$. Also, $\hat{I} = I$. So R/I is also complete as an R-module. Hence

$$R/I \cong \varprojlim (R/I)/\mathfrak{m}_R^n(R/I).$$

As $\mathfrak{m}_R^n(R/I) = \mathfrak{m}_{R/I}^n$, we have that R/I is a complete local ring. □

1.3 Prorings

Definition 1.16. A *proring* is a directed projective system of rings $(R_\lambda)_{\lambda\in\Lambda}$, where Λ is a poset, with morphisms $f_{\lambda\mu} : R_\mu \to R_\lambda$.

Definition 1.17. Let $R = (R_\lambda)_{\lambda\in\Lambda}, S = (S_\mu)_{\mu\in M}$ be two prorings. A *morphism* of prorings $R \to S$ is an element of

$$\mathrm{Hom}(R, S) = \varprojlim_{\mu} \varinjlim_{\lambda} \mathrm{Hom}(R_\lambda, S_\mu),$$

where the latter sets are morphisms of rings.

Such a morphism of prorings $f : R \to S$ is defined by a collection of representative ring homomorphisms $f^\lambda_\mu : R_\lambda \to S_\mu$. Then the diagrams

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ R_\lambda & \xrightarrow{f^\lambda_\mu} & S_\mu \end{array}$$

commute.

Example 1.18. By indexing a projective system over a singleton set, we see that every ring is a proring.

Proposition 1.19. *The category of prorings has all projective limits.*

Proof: This is true of the category of rings, so the result follows by [32, Proposition 1.2]. □

Remark 1.20. By [16, Theorem 1], there is an uncountable projective system of rings, each of which is countably infinite, such that the projective limit is the zero ring. Hence the limit of a projective system of rings in the category of rings may not be isomorphic to its limit in the category of prorings.

Definition 1.21. A proring is said to be *epi* if all the morphisms $f_{\lambda\mu}$ are epimorphisms.

For a topological space X we can define a sheaf of prorings. Such a sheaf can be presented as a projective system of sheaves of rings. Therefore by [32, Proposition 1.2] the category of sheaves of prorings has projective limits.

1.4 Excellent Rings

Definition 1.22. Let R be a ring. The *Krull dimension* of R, denoted $\dim R$, is the supremum of the lengths of all chains of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subset R.$$

Definition 1.23. Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$, which has minimal set of generators $a_1, \ldots, a_n$. If $\dim R = n$, it is said to be a *regular local ring.*

A Noetherian ring R is said to be a *regular ring* if the localisation at every prime ideal is a regular local ring.

Definition 1.24. A ring homomorphism $\phi : R \to S$ is *regular* if:

(1) The functor $M \mapsto M \otimes_R S$ is exact, where M is an R-module. (That is, ϕ is flat).

(2) For any prime ideal $\mathfrak{p} \subset R$, and any finite field extension K of $k_{\mathfrak{p}}$, where $k_{\mathfrak{p}}$ is the residue field of the localisation $R_{\mathfrak{p}}$, the ring $(S \otimes_R k_{\mathfrak{p}}) \otimes_{k_{\mathfrak{p}}} K$ is regular.

Definition 1.25. A Noetherian ring R is a *G-ring* if for every prime ideal $\mathfrak{p} \subset R$, the map from the localisation $R_{\mathfrak{p}}$ to its completion is regular.

Definition 1.26. A ring R is a *J-2 ring* if for every finitely generated R-algebra S, the singular points of $\mathrm{Spec}(S)$ (that is, the prime ideals $\mathfrak{p} \subset S$ for which $S_{\mathfrak{p}}$ is not a regular local ring) form a closed subset.

Definition 1.27. Let $\mathfrak{p}, \mathfrak{p}'$ be two prime ideals of a ring R. Then if any two chains of prime ideals

$$\mathfrak{p} = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}'$$

which are saturated, (that is, the chain cannot be extended while keeping all inclusions strict), have the same length, then R is said to be *catenary.*

R is *universally catenary* if all finitely generated R-algebras are catenary rings.

Definition 1.28. A ring R is *quasi-excellent* if it is a G-ring and a J-2 ring.

A ring R is *excellent* if it is quasi-excellent and universally catenary.

Example 1.29. All fields are excellent rings. Further, all Noetherian complete local rings are excellent. (See [30, Tag 07QW]).

In particular, the power series rings $\mathbb{C}[[x_1, \dots, x_n]]$ and their quotients are excellent.

Example 1.30. A finitely generated algebra over an excellent ring is excellent. (See [30, Tag 07QW]). In particular, rings of the form $\mathbb{C}[x_1, \dots, x_n]/I$ are excellent.

1.5 Modules over Rings

Definition 1.31. Let R be a ring, and let (M_i, ϕ_{ji}) be directed inverse system of R-modules indexed by a poset $\mathcal{I}$. The system (M_i, ϕ_{ji}) is *Mittag-Leffler* if for each $i \in \mathcal{I}$, there exists $j \geq i$ such that for $k \geq j$, $\phi_{ki}(M_k) = \phi_{ji}(M_j)$.

Definition 1.32. Let M be a module over an integral domain R. An element $x \in M$ is a *torsion element* if there exists $r \in R \setminus \{0\}$ such that $rx = 0$.

M is *torsion-free* if 0 is the only torsion element.

Remark 1.33. The notion of being torsion-free can be generalised for modules over arbitrary rings, replacing the condition $r = 0$ with being a zero-divisor.

Definition 1.34. Let M be a module over a ring R. The *dual module* is $M^* = \operatorname{Hom}(M, R)$.

Let M be a module over a ring R. We define the canonical morphism $j_M : M \to M^{**}$ by $j_M(x)(f) = f(x)$. This morphism has kernel $\operatorname{Ker} j_M = \bigcap_{f \in M^*} \operatorname{Ker} f$.

Definition 1.35. Let M be a module over a ring R. M is *torsionless* if the canonical morphism is injective.

M is *reflexive* if the canonical morphism is an isomorphism.

Proposition 1.36. *If R is an integral domain, any torsionless R-module M is torsion-free.*

Proof: Let $x \in M$ and $r \in R \setminus \{0\}$ such that $rx = 0$. Then for all $f \in M^*$, $f(rx) = rf(x) = 0$. As R is an integral domain, it follows that $f(x) = 0$, for all $f \in M^*$. Hence $x \in \operatorname{Ker} j_M = \bigcap_{f \in M^*} \operatorname{Ker} f$. As M is torsionless, it follows that $x = 0$, and we are done. □

The converse is not true in general: Viewing $\mathbb{Q}$ as a $\mathbb{Z}$-module, we see that $\mathbb{Q}$ is clearly torsion-free, but is not torsionless, as $\operatorname{Hom}(\mathbb{Q}, \mathbb{Z}) = 0$.

We do however have the following:

Proposition 1.37. *Let R be an integral domain. A finitely generated torsion-free R-module M is torsionless.*

Proof: See [15, Proposition 4.5.7]. □

Definition 1.38. Let R be a ring, and $f : M \to N$ a homomorphism of R-modules. The *dual* is the homomorphism $f^* : N^* \to M^*$ defined by $f^*(\phi)(x) = \phi(f(x))$.

Lemma 1.39. *Let $f : M \to N$ be a surjective homomorphism of R-modules. Then the dual f^* is injective.*

Proof: If $\phi \in N^*$, and $f^*(\phi) = 0$, then by definition of the dual we have $\phi(f(x)) = 0$, for all $x \in M$. By surjectivity of f, we have $\phi(y) = 0$, for all $y \in N$, i.e. $\phi = 0$. So f^* is injective. □

Lemma 1.40. *Let $f : M \to N$ be a surjective homomorphism of R-modules, with kernel K. By the previous lemma, we can view N^* as a submodule of M^* via the map f^*. Let $\phi \in M^*$. Then $\phi \in N^*$ if and only if $\phi(x) = 0, \forall x \in K$.*

Proof: If $\phi \in M^*$ lies in N^*, we can factorise it as $\phi = \phi' \circ f$, for some $\phi' \in N^*$. Then $\phi(x) = 0$, for all $x \in K$.

Conversely, if $\phi(x) = 0$ for all $x \in K$, then, for $x, y \in M$, we have $\phi(x) = \phi(y)$ whenever $x - y \in K$. So f factors through ϕ, hence $\phi \in N^*$. □

Lemma 1.41. *In the setup of the previous lemma, suppose also that N is torsionless, and let $x \in M$. Then $x \in K$ if and only if $\phi(x) = 0, \forall \phi \in f^*(N^*)$.*

Proof: The forward implication holds by the above lemma. Conversely, let $x \in M$ be such that $\phi(x) = 0$, for all $\phi \in f^*(N^*)$. So $f(x) \in \bigcap_{\phi' \in N^*} \operatorname{Ker} \phi' = \operatorname{Ker} j_N$; as N is torsionless, this implies that $f(x) = 0$, so $x \in K$. □

2 Schemes and Resolutions

Let X be a scheme over $\mathbb{C}$. We assume that X is of finite type and separated (and so is locally Noetherian). The structure sheaf of X is the sheaf of regular functions $\mathscr{O}_X$, and so $(X, \mathscr{O}_X)$ is a locally ringed space.

Recall that X admits an open cover by affine schemes of the form

$$U = \operatorname{Spec}(\mathbb{C}[x_1, \ldots, x_n]/I).$$

Each open subscheme U embeds in $\operatorname{Spec}(\mathbb{C}[x_1, \ldots, x_n]) = \mathbb{A}^n_{\mathbb{C}}$, and so we have local algebraic co-ordinates on X.

Notation 2.1. Let $I = (f_1, \ldots, f_r) \subset \mathbb{C}[x_1, \ldots, x_n]$ be an ideal. The affine scheme $X = \operatorname{Spec}(\mathbb{C}[x_1, \ldots, x_n]/I)$ is also denoted by $\mathbb{V}(I)$ or $\mathbb{V}(f_1, \ldots, f_r)$.

Remark 2.2. The closed points of X correspond to set

$$\{x \in \mathbb{C}^n \mid f_1(x) = \cdots = f_r(x) = 0\},$$

justifying the notation.

Definition 2.3. Let X be a separated scheme of finite type over $\mathbb{C}$. Then X is a *variety* if it is reduced and irreducible.

Definition 2.4. Let X a separated scheme of finite type over $\mathbb{C}$. We denote by X_{sing} the singular locus of X, that is, the set of points $x \in X$ such that $\mathscr{O}_{X,x}$ is not a regular local ring.

Definition 2.5. Let X be a smooth variety. A divisor $E = \sum E_i$ is an *NC (normal crossings) divisor* if, choosing local analytic co-ordinates $x_1, \ldots, x_n$ in a neighbourhood of any point, E is given by an equation of the form

$$x_1 \cdots x_k = 0.$$

It is *simple normal crossings (SNC)* if the co-ordinates can be chosen in a Zariski open neighbourhood, in which case every irreducible component is smooth.

A $\mathbb{Q}$-divisor $\sum \alpha_i E_i$ has *(simple) normal crossing support* if $\sum E_i$ is an (S)NC divisor.

Definition 2.6. Let E be an NC divisor of X, and let $x \in E$. Then $e(E, x)$ is the number of components of E through x in some formal neighbourhood of x.

Example 2.7. Let $X = \mathbb{A}^2$ and $E = \mathbb{V}(y^2 - x^2 - x^3)$. E is NC, but not SNC. Although there is one irreducible component, we have $e(E, 0) = 2$, as we work in a formal neighbourhood of the origin.

Definition 2.8. Let $\pi : X' \to X$ be a birational morphism of schemes. The *exceptional locus* $\mathrm{exc}(\pi)$ of π is the locus of X' over which π is not an isomorphism. It is the sum of the exceptional divisors.

Definition 2.9. Let $\pi : X' \to X$ be a birational morphism of schemes, and let Y be a closed subscheme of X. The pre-image $\pi^{-1}(Y)$ is called the *total transform* of Y. We define the *strict transform* $\pi^*(Y)$ of Y to be the smallest subscheme satisfying $\pi^{-1}(Y) = \pi^*(Y) \cup \mathrm{exc}(\pi)$.

Definition 2.10. Let X be a smooth variety, and Y a closed subscheme of X. A *log resolution*, or *resolution of singularities*, of Y is a projective birational mapping $\pi : X' \to X$, with X' smooth, such that $\pi^{-1}(Y) = \pi^* Y \cup \mathrm{exc}(\pi)$ has simple normal crossing support.

Theorem 2.11. *[17, Main Theorem II] For any smooth variety X and closed subscheme $Y \subset X$, there exists a log resolution of Y.*

3 Complex Spaces

Definition 3.1. A locally ringed space $(X, \mathscr{O}_X)$ is called a $\mathbb{C}$-*space* if all the rings in the sheaf are $\mathbb{C}$-algebras, and $\mathscr{O}_{X,x}/\mathfrak{m}_{X,x} \cong \mathbb{C}$, as $\mathbb{C}$-algebras, for all $x \in X$.

Let $U \subset \mathbb{C}^n$ be open, and $f_1, \ldots, f_k : U \to \mathbb{C}$ holomorphic functions. Let

$$X = \mathbb{V}(f_1, \ldots, f_k) = \{x \in U \mid f_1(x) = \cdots = f_k(x) = 0\}.$$

Define a sheaf of rings on X by

$$\mathscr{O}_X = \mathscr{O}_U/(f_1, \ldots, f_k)|_X.$$

The ringed space $(X, \mathscr{O}_X)$ is a $\mathbb{C}$-space, called a *local model space.*

Definition 3.2. A *complex analytic space*, or *complex space*, is a $\mathbb{C}$-space $(X, \mathscr{O}_X)$ locally isomorphic to a local model space, where the underlying topological space X is assumed to be Hausdorff.

Let $(X, \mathscr{O}_X), (Y, \mathscr{O}_Y)$ be ringed spaces, and $\phi : X \to Y$ a continuous map of the underlying topological spaces. Then for $V \subset Y$ open, we define a presheaf $\phi_* \mathscr{O}_X$ on Y (indeed a sheaf) by

$$\phi_* \mathscr{O}_X(V) = \mathscr{O}(\phi^{-1}(V)).$$

Definition 3.3. Let $(X, \mathscr{O}_X), (Y, \mathscr{O}_Y)$ be complex spaces. A *holomorphic* map $\phi : X \to Y$ is a morphism of locally ringed spaces, i.e. a pair $(\phi, \tilde{\phi})$, where $\phi : X \to Y$ is a continuous map of the underlying topological spaces, and $\tilde{\phi} : \mathscr{O}_Y \to \phi_* \mathscr{O}_X$ is a morphism of sheaves of $\mathbb{C}$-algebras.

Definition 3.4. Let S be a complex space. A *complex space over* S is a pair (X, ϕ), where X is a complex space and $\phi : X \to S$ is holomorphic.

If (X, ϕ) and (Y, ϕ') are complex spaces over S, a morphism $\psi : X \to Y$ in the category of complex spaces over S is a holomorphic map satisfying $\phi' \circ \psi = \phi$.

Example 3.5. Any complex manifold is a complex space, as each point has a neighbourhood isomorphic to the trivial local model, defined by the zero ideal.

Example 3.6. Any scheme of finite type over $\mathbb{C}$ has an associated complex space with the same set of closed points, formed by considering the generating polynomials as holomorphic functions. The topology of the complex space is the Hausdorff topology generated by the Euclidean topology on the charts; it is a refinement of the Zariski topology on the scheme. The structure sheaf is also extended to include all holomorphic functions.

Remark 3.7. We can define normal crossings space (and all the associated definitions) for complex spaces in the same way as for schemes. Resolution of singularities also holds in this category (see [17, Main Theorem II]).

4 Coherent Sheaves

Let $(X, \mathscr{O}_X)$ be a ringed space. We may assume that it is a locally ringed space, that is, all the stalks $\mathscr{O}_{X,x}$ are local rings. Let $\mathcal{F}$ be a sheaf of $\mathscr{O}_X$-modules.

Definition 4.1. The sheaf $\mathcal{F}$ is *locally free* if for every point $x \in X$ there is an open neighbourhood $U \ni x$ and a set $\mathcal{I}$ such that

$$\mathcal{F}|_U \cong \bigoplus_{i \in \mathcal{I}} \mathscr{O}_X|_U.$$

Remark 4.2. If $\mathcal{I}$ is finite, its cardinality is constant on connected components of X, and is called the rank of the sheaf.

Definition 4.3. The sheaf $\mathcal{F}$ is *finitely generated* if for every point $x \in X$ there is an open neighbourhood $U \ni x$ such that there is a surjective morphism $\mathscr{O}_X^n|_U \to \mathcal{F}|_U$.

$\mathcal{F}$ is *coherent* if it is finitely generated, and for every open $U \subset X$ and every $p \in \mathbb{N}$, every homomorphism $\mathscr{O}_X^p|_U \to \mathcal{F}|_U$ has a finitely generated kernel.

We now assume that X is either a scheme of finite type over $\mathbb{C}$ or a complex space. In this case, the structure sheaf $\mathscr{O}_X$ is a coherent sheaf of modules over itself. (For the analytic case, this comes from the Oka coherence theorem. For the case of locally Noetherian schemes, see [30, Tag 01XZ]). In this setting, locally free sheaves of finite rank are coherent.

Proposition 4.4. *Let $(X, \mathscr{O}_X)$ be a scheme over $\mathbb{C}$ or complex space which is reduced, and let $\mathcal{F}$ be a coherent sheaf of $\mathscr{O}_X$-modules. Then $\mathcal{F}$ is locally free outside an analytic subset of X.*

Proof: Let $x \in X$. We define the rank of $\mathcal{F}$ at x by

$$\operatorname{rank} \mathcal{F}_x = \dim_{k(x)}(\mathcal{F}_x \otimes_{\mathscr{O}_{X,x}} k(x)),$$

where $k(x)$ is the residue field of $\mathscr{O}_{X,x}$. This is upper semicontinuous (see [14, Example 12.7.2] for the scheme case; the same argument, using Nakayama's lemma, holds in general), and so there exists an open set $U \subset X$ where it is constant. Then $\mathcal{F}|_U$ is a coherent sheaf of constant rank, so it is locally free. □

Remark 4.5. The analytic subset is called the *singular locus* of the sheaf $\mathcal{F}$, denoted $\operatorname{Sing}\mathcal{F}$.

Definition 4.6. Let $(X, \mathscr{O}_X)$ be a ringed space, and $\mathcal{F}$ a sheaf of $\mathscr{O}_X$-modules. $\mathcal{F}$ is *torsion-free* if for all $x \in X$, $\mathcal{F}_x$ is a torsion-free $\mathscr{O}_{X,x}$-module.

If $\mathcal{E}$ is a subsheaf of $\mathcal{F}$, it is called *saturated* if the quotient sheaf $\mathcal{F}/\mathcal{E}$ is torsion-free.

Definition 4.7. Let $(X, \mathscr{O}_X)$ be a ringed space, and $\mathcal{F}$ a sheaf of $\mathscr{O}_X$-modules. We define the *torsion subsheaf* $\operatorname{Tor}(\mathcal{F}) \subset \mathcal{F}$ to be the sheaf generated by the torsion elements.

Definition 4.8. Let $(X, \mathscr{O}_X)$ be a ringed space, and $\mathcal{F}$ a sheaf of $\mathscr{O}_X$-modules. The *dual* of $\mathcal{F}$ is the sheaf $\mathcal{F}^* = \mathscr{H}om_{\mathscr{O}_X}(\mathcal{F}, \mathscr{O}_X)$.

Proposition 4.9. *[25, Corollary 92] Let $(X, \mathscr{O}_X)$ be a ringed space, and $\mathcal{F}$ a sheaf of $\mathscr{O}_X$-modules. For any $x \in X$, $(\mathcal{F}^*)_x \cong (\mathcal{F}_x)^*$.*

As with modules over a ring, there is a canonical morphism $\mathcal{F} \to \mathcal{F}^{**}$.

Definition 4.10. A sheaf of $\mathscr{O}_X$-modules $\mathcal{F}$ is *reflexive* if the canonical morphism $\mathcal{F} \to \mathcal{F}^{**}$ is an isomorphism.

Example 4.11. All locally free sheaves of finite rank are reflexive. (See [25, Proposition 74]).

Definition 4.12. Let $(X, \mathscr{O}_X)$ be a complex space, and $\mathcal{F}$ a sheaf of $\mathscr{O}_X$-modules.

$\mathcal{F}$ is *normal* if for any open $U \subset X$ and any analytic subset $A \subset U$ of codimension at least 2, the restriction map $\mathcal{F}(U) \to \mathcal{F}(U \setminus A)$ is an isomorphism.

Lemma 4.13. *[27, Lemma II.1.1.16] Let $\mathcal{F}$ be a reflexive sheaf, and $\mathscr{E} \subset \mathcal{F}$ a saturated subsheaf. Then $\mathscr{E}$ is normal.*

Proposition 4.14. *Let $(X, \mathscr{O}_X)$ be a ringed space. Then the functorial mapping $\mathcal{F} \mapsto \mathcal{F}_x$, where $\mathcal{F}$ a sheaf of $\mathscr{O}_X$-modules, is exact.*

Proof: See [30, Tag 01AG]. □

Corollary 4.15. *Let $(X, \mathscr{O}_X)$ be a ringed space, and $\mathcal{F}$ a sheaf of $\mathscr{O}_X$-modules. Let $\mathscr{E} \subset \mathcal{F}$ be a subsheaf, and let $x \in X$. Then $(\mathcal{F}/\mathscr{E})_x \cong \mathcal{F}_x/\mathscr{E}_x$.*

Proof: The sequence

$$0 \to \mathscr{E} \to \mathcal{F} \to \mathcal{F}/\mathscr{E} \to 0$$

is exact, and hence by Proposition 4.14, the sequence

$$0 \to \mathscr{E}_x \to \mathcal{F}_x \to (\mathcal{F}/\mathscr{E})_x \to 0$$

is exact. The result follows. □

Corollary 4.16. *Let $\mathcal{F}$ be a coherent sheaf of $\mathscr{O}_X$-modules. Then for each $x \in X$, $\mathcal{F}_x$ is a finitely generated $\mathscr{O}_{X,x}$-module.*

Proof: Fix $x \in X$. As $\mathcal{F}$ is coherent, it is finitely generated, and so there is an open neighbourhood $U \ni x$ and an exact sequence

$$\mathscr{O}_x^n|_U \to \mathcal{F}|_U \to 0.$$

By Proposition 4.14, there is an exact sequence of stalks

$$\mathscr{O}_{X,x}^n \to \mathcal{F}_x \to 0.$$

It follows that $\mathcal{F}_x$ is finitely generated. □

5 Formal Schemes

We now introduce formal schemes. In the literature, several different classes of objects go by this name. The first, which we call classical formal schemes, were introduced by Grothendieck in [13]. This is the most commonly used case in the literature, but is quite restrictive. We therefore take as our definition of formal scheme that introduced by Yasuda in [32]; this is sufficiently general, and indeed includes most other cases of formal schemes as subcategories.

We start by introducing these formal schemes, following Yasuda's paper. We then introduce formal schemes as defined via the functor of points, classical formal schemes, and McQuillan formal schemes (a generalisation of classical formal schemes introduced by McQuillan in [23]), and look at the relations between these categories.

5.1 Formal Schemes via Prorings

Definition 5.1. Let X be a topological space. We say that X is *quasi-separated* if the intersection of any two quasi-compact open subsets is quasi-compact.

We say that X is *qsqc* if it is quasi-separated and quasi-compact.

A *qsqc basis* is a basis of open subsets, all of which are qsqc.

Any scheme has a qsqc basis. An affine scheme, or any Noetherian topological space, is qsqc.

Definition 5.2. An *admissible system of schemes* is a directed inductive system $(X_\lambda)_{\lambda\in\Lambda}$ of schemes such that every morphism $X_\lambda \to X_\mu$ is a bijective closed immersion.

Definition 5.3. A proring $R = (R_\lambda)_{\lambda\in\Lambda}$ is *admissible* if every morphism $R_\lambda \to R_\mu$ is surjective, and induces an isomorphism $(R_\lambda)_{\text{red}} \to (R_\mu)_{\text{red}}$.

Equivalently, R is admissible if $(\operatorname{Spec}(R_\lambda))_{\lambda\in\Lambda}$ is an admissible system of schemes.

An admissible proring R has an associated reduced ring $R_{\text{red}} = (R_\lambda)_{\text{red}}$, for any $\lambda \in \Lambda$.

Definition 5.4. A *locally admissibly proringed space* is a pair $(X, \mathscr{O}_X)$, where X is a topological space with a qsqc basis $\mathfrak{B}$, $\mathscr{O}_X$ is a sheaf of prorings such that $\mathscr{O}_X(U)$ is admissible for all $U \in \mathfrak{B}$, and for each $x \in X$ the stalk $\mathscr{O}_{X_{\text{red}},x}$, where $\mathscr{O}_{X_{\text{red}}}$ is the sheaf of rings defined on $\mathfrak{B}$ by $U \mapsto (\mathscr{O}_X(U))_{\text{red}}$, is a local ring.

Definition 5.5. Let $R = (R_\lambda)_\lambda$ be an admissible proring. We define the *formal spectrum* of R to be the locally admissibly proringed space $\mathrm{Spf}(R)$, which as a topological space is equal to $\mathrm{Spec}(R_{\text{red}}) = \mathrm{Spec}(R_\lambda)$, for all λ, and whose structure sheaf is

$$\mathscr{O}_{\mathrm{Spf}(R)} = \varprojlim \mathscr{O}_{\mathrm{Spec}(R_\lambda)},$$

where the limit is taken in the category of sheaves of prorings.

Definition 5.6. A *formal scheme* is a locally admissibly proringed space locally isomorphic to the formal spectrum of an admissible proring.

A formal scheme which is isomorphic to a formal spectrum is called *affine*.

Example 5.7. Any ring R can be viewed as a proring, in which case it will be admissible, with $\mathrm{Spf}(R) = \mathrm{Spec}(R)$. Thus any scheme can be viewed as a formal scheme.

Remark 5.8. To distinguish from strict formal schemes (i.e. formal schemes which are not schemes), schemes are sometimes called *ordinary schemes*.

Proposition 5.9. *[32, Corollary 2.14] The category of affine formal schemes is equivalent to the dual category of the category of admissible prorings.*

An affine formal scheme X corresponds to an admissible system of affine schemes $(X_\lambda)_{\lambda \in \Lambda}$. As such, we can view X as the limit $X = \varinjlim X_\lambda$. A general formal scheme which is qsqc can also be viewed as the limit of an admissible system of schemes (see [32, Proposition 3.32]). We therefore sometimes write formal schemes in this limit notation.

Definition 5.10. Let $X = \varinjlim X_\lambda$ be an affine formal scheme. X is *countably indexed* if the indexing set Λ can be taken to be countable (say $\Lambda = \mathbb{N}$).

If X is a general formal scheme, we say it is *locally countably indexed* if it admits a cover by countably indexed affine formal schemes.

Remark 5.11. In [32], the term *gentle* is used in place of locally countably indexed.

Definition 5.12. An admissibile proring $R = (R_\lambda)_\lambda$ is called *pro-Noetherian* if every R_λ is Noetherian.

A formal scheme X is called *locally ind-Noetherian* if every $x \in X$ admits an affine neighbourhood $\mathrm{Spf}(R) \subset X$, where R is pro-Noetherian.

Definition 5.13. Let $X = \varinjlim X_\lambda$ and $Y = \varinjlim Y_\mu$ be formal schemes. We say that $X \subset Y$ if for all $\lambda \in \Lambda$ there exists $\mu \in M$ such that $X_\lambda \subset Y_\mu$.

Remark 5.14. Note that this is a more general notion than that of a formal subscheme in [32], as we do not take into account the topologies on the formal schemes.

Proposition 5.15. *Let $(X_\alpha)_{\alpha \in A}$ be an inductive system of formal schemes, all of which have the same underlying topological space. Then the direct limit $\varinjlim X_\alpha$ exists as a formal scheme.*

Proof: We focus on the affine case. Then by duality we have a projective system $((R_\lambda)_{\lambda \in \Lambda_\alpha})_{\alpha \in A}$ of admissible prorings, with surjective morphisms between them. The projective limit R of this system exists as a proring; it remains to show that this is admissible. Now every ring R_λ that appears in the system has the same reduction. Every admissible proring is epi, so by [32, Lemma 1.17] all the morphisms $R_\lambda \to R_\mu$ representing the morphisms in the projective system are epimorphisms, and so induce isomorphims on the reduced rings. So R is admissible, as required.

The general case can be proved by gluing affine formal schemes. □

5.2 Formal Schemes via the Functor of Points

We denote by $\mathbf{F}$ the category of contravariant functors from schemes to sets. We have as full subcategories $\mathbf{F}_{Zar}$, the category of Zariski sheaves, and $\mathbf{F}_{ét}$, the category of étale sheaves.

For every formal scheme X, there is an associated contravariant functor $F_X \in \mathbf{F}$, defined by $F_X(Y) = \mathrm{Hom}(Y, X)$, where the morphisms are of formal schemes.

Theorem 5.16. *[32, Theorem 4.3] The functorial mapping $X \mapsto F_X$ on formal schemes is fully faithful.*

If we write $X = \varinjlim X_\lambda$, then F_X is isomorphic to the inductive limit of the F_{X_λ} in $\mathbf{F}_{Zar}$ or $\mathbf{F}_{ét}$. Thus our notion of formal scheme corresponds to that used in [1, Section 7.11].

5.3 Formal Schemes via Complete Rings

Definition 5.17. A *locally topologically ringed space* is a pair $(X, \mathscr{O}_X)$, where X is a topological space, and $\mathscr{O}_X$ is a sheaf of topological rings whose stalks $\mathscr{O}_{X,x}$ are local rings.

Definition 5.18. Let R be topological ring which is linearly topologised, complete and separated.

R is *admissible* if there exists an open ideal $I \subset R$ such that every neighbourhood of 0 contains I^n for some $n \in \mathbb{N}$. Such an ideal is called an *ideal of definition.*

R is *weakly admissible* if there exists an open ideal $I \subset R$ such that $\lim_{n\to\infty} f^n = 0$, for all $f \in I$. Such an ideal is called a *weak ideal of definition.*

Remark 5.19. The collection of all (weak) ideals of definition forms a fundamental system of neighbourhoods of 0.

Remark 5.20. Any ring with the discrete topology is admissible, with the zero ideal being an ideal of definition.

Any ideal of definition is a weak ideal of definition. So any admissible ring is weakly admissible.

Lemma 5.21. *Let R be a (weakly) admissible ring, and let $(I_\lambda)_{\lambda\in\Lambda}$ be the collection of all (weak) ideals of definition. Then for every $\lambda \in \Lambda$, the image of the morphism* $\operatorname{Spec}(R/I_\lambda) \to \operatorname{Spec}(R)$ *is the set of open prime ideals of R.*

Proof: As R is complete, we have $R \cong \varprojlim R/I_\lambda$, and the topology is the limit topology of discrete topologies. Thus any prime ideal in the image of the map $\operatorname{Spec}(R/I_\lambda) \to \operatorname{Spec}(R)$ is the pre-image of a prime ideal of R/I_λ under the quotient map, and so is open.

Conversely, let $\mathfrak{p} \in \operatorname{Spec}(R)$ be an open prime ideal. As an open neighbourhood of 0, it contains some (weak) ideal of definition I_μ. Now for any $f \in I_\lambda$, we have $\lim_{n\to\infty} f^n = 0$, and so $f^n \in I_\mu$ for sufficiently large n. Hence

$$I_\lambda \subset \sqrt{I_\mu} \subset \sqrt{\mathfrak{p}} = \mathfrak{p}.$$

The result follows. □

Definition 5.22. Let R be a (weakly) admissible ring, and let $(I_\lambda)_{\lambda\in\Lambda}$ be the collection of all (weak) ideals of definition. We define the *formal spectrum* of R to be the locally topologically ringed space $\mathrm{Spf}(R)$, which as a topological space comprises the open prime ideals of R with the subspace topology from $\mathrm{Spec}(R)$, or equivalently is equal to $\mathrm{Spec}(R/I_\lambda)$, and has the structure sheaf

$$\mathscr{O}_{\mathrm{Spf}(R)} = \varprojlim \mathscr{O}_{\mathrm{Spec}(R/I_\lambda)}.$$

Definition 5.23. A *classical formal scheme* is a locally topologically ringed space X locally isomorphic to the formal spectrum of an admissible ring.

A *McQuillan formal scheme* is a locally topologically ringed space X locally isomorphic to the formal spectrum of a weakly admissible ring.

A classical (respectively, McQuillan) formal scheme which is isomorphic to a formal spectrum is called *affine*.

Remark 5.24. Any scheme is a classical formal scheme.

Any classical formal scheme is a McQuillan formal scheme.

For any proring $R = (R_\lambda)_\lambda$, we define $\hat{R}$ to be the limit $\hat{R} = \varprojlim R_\lambda$ in the category of rings, and endow it with the linear topology for which $(\mathrm{Ker}(\hat{R} \to R_\lambda))$ forms a basis of ideals. Then $\hat{R}$ is a complete ring.

Conversely, if S is a complete ring with (I_λ) the collection of open ideals, then $\mathscr{P}(S) = (S/I_\lambda)_\lambda$ is an epi proring.

The mappings $R \mapsto \hat{R}$ and $S \mapsto \mathscr{P}(S)$ are both functorial, and we have $\widehat{\mathscr{P}(S)} \cong S$ for any complete ring S.

Definition 5.25. A proring R is called *mild* if $R \cong \mathscr{P}(S)$ for some complete ring S.

Proposition 5.26. *Any classical or McQuillan formal scheme X corresponds to a unique formal scheme.*

Proof: Without loss of generality, we may assume that X is an affine McQuillan formal scheme. Then $X = \mathrm{Spf}(S)$, for a weakly admissible ring S. Let (I_λ) be the collection of all ideals of definition. Let $R = \mathscr{P}(S) = (S/I_\lambda)_\lambda$. Then R is a uniquely determined proring. By construction, each of the $\mathrm{Spec}(S/I_\lambda)$ has the same underlying topological space, so it follows that R is an admissible proring, and so defines an affine formal scheme. □

Proposition 5.27. *Let X be a locally countably indexed formal scheme. Then X corresponds to a unique McQuillan formal scheme Y. If X is further assumed to be locally ind-Noetherian, then Y is a classical formal scheme.*

Proof: We may assume that X is affine. Then X corresponds to an admissible proring $R = (R_n)_{n \in \mathbb{N}}$. As the indexing set is countable, all the natural maps $\hat{R} \to R_n$ are surjective. As R is by definition epi, by [32, Proposition 5.3], R is mild. Hence $\hat{R}$ is a uniquely determined complete ring.

Set $I_n = \text{Ker}(\hat{R} \to R_n)$. Then the (I_n) form a basis of open ideals. Let $x \in I_n$. This corresponds to a sequence $(x_m)_{m \in \mathbb{N}}$, with $x_m \in R_m$, $f_{mp}(x_p) = x_m$, for $p \geq m$, and $x_k = 0$ for $k \leq n$. So for $m \geq n$, we have $x_m \in \text{Ker}\, f_{mn}$. As f_{mn} induces an isomorphism on the reduced rings, we can apply Proposition 1.4 to see that x_m is nilpotent, for all $m \geq n$. Hence $x^N \in I_K$, for some $K \in \mathbb{N}$ and for sufficiently large N. It follows that I_n is a weak ideal of definition for $\hat{R}$, and so $\hat{R}$ is weakly admissible. It therefore corresponds to an affine McQuillan formal scheme Y.

If X is locally ind-Noetherian, then in the affine case we have that R is pro-Noetherian. As R is mild we have $R = \mathscr{P}(\hat{R})$; applying [32, Proposition 6.6] we see that $\hat{R}$ is admissible. Hence Y is a classical formal scheme. □

Example 5.28. Let X be a smooth variety, and $Z \subset X$ be a closed reduced subscheme given by the ideal sheaf $\mathcal{I}$. For $n \in \mathbb{N}$, let Z^n be the scheme given by the ideal sheaf $\mathcal{I}^n$. The formal scheme $\hat{X}_Z = \varinjlim Z^n$ is called the *formal completion* of X along Z.

The formal completion is a classical formal scheme. In the affine case, with $X = \text{Spec}(R)$ and $Z = \text{Spec}(R/I)$, then $\hat{X}_Z = \text{Spf}(\varprojlim R/I^n)$.

Definition 5.29. Let X be a classical (respectively, McQuillan) formal scheme with affine cover $\bigcup \text{Spf}(R_\alpha)$. A *closed formal subscheme* $Z \subset X$ is the classical (respectively, McQuillan) formal scheme with affine cover $\bigcup \text{Spf}(R_\alpha / I_\alpha)$, where the I_α are closed ideals forming a coherent sheaf.

Proposition 5.30. *Let P be a point in a smooth variety X, and let $Z = \varinjlim Z_n$ be a formal scheme with $Z_n \subset P^{n+1}$, for all $n \in \mathbb{N}$. Then Z is a closed formal subscheme of $\hat{X}_P$.*

Proof: We choose co-ordinates locally on X so that $P = 0$. We let $R = \mathbb{C}[x_1, \ldots, x_n]$, and denote by $\mathfrak{m}$ the ideal $(x_1, \ldots, x_n) \subset R$. Then $P^n =$

$\operatorname{Spec}(R/\mathfrak{m}^n)$, and so $\hat{X}_P = \operatorname{Spf}(\mathbb{C}[[x_1, \ldots, x_n]])$. We write $Z_n = \operatorname{Spec}(R/I_n)$, and $\hat{A} = \varprojlim R/I_n$, so that $Z = \operatorname{Spf}(\hat{A})$.

Moving into the category of rings, we have the following diagram:

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Ker} f_3 & \hookrightarrow & R/\mathfrak{m}^4 & \xrightarrow{f_3} & R/I_3 \\ \downarrow & & \downarrow{\scriptstyle \phi_{32}} & & \downarrow \\ \operatorname{Ker} f_2 & \hookrightarrow & R/\mathfrak{m}^3 & \xrightarrow{f_2} & R/I_2 \\ \downarrow & & \downarrow{\scriptstyle \phi_{21}} & & \downarrow \\ \operatorname{Ker} f_1 & \hookrightarrow & R/\mathfrak{m}^2 & \xrightarrow{f_1} & R/I_1 \end{array}$$

The surjections correspond to the inclusions of schemes. If $i \geq j$, we define $\phi_{ij} = \phi_{i,i-1} \circ \cdots \circ \phi_{j+1,j}$.

If $x \in \operatorname{Ker} f_i$, then by commutativity $f_j(\phi_{ij}(x)) = 0$, for $j < i$; hence $\phi_{ij}(\operatorname{Ker} f_i)$ is an ideal of $R/\mathfrak{m}^{j+1}$ contained in $\operatorname{Ker} f_j$. If $k \geq i > j$, then $\phi_{kj}(\operatorname{Ker} f_k) = \phi_{ij}(\phi_{ki}(\operatorname{Ker} f_k)) \subset \phi_{ij}(\operatorname{Ker} f_i)$. So $(\phi_{kj}(\operatorname{Ker} f_k))_{k=j+1}^{\infty}$ is a decreasing sequence of ideals in $R/\mathfrak{m}^{j+1}$, and therefore stabilises by the Artinian property. So the sequence $(\cdots \to \operatorname{Ker} f_i \to \cdots)$ satisfies the Mittag-Leffler property.

Taking the projective limits of the above diagram, we have, for each $m \in \mathbb{N}$, the diagram

$$\begin{array}{ccccc} \varprojlim \operatorname{Ker} f_i & \hookrightarrow & \mathbb{C}[[x_1, \ldots, x_n]] & \xrightarrow{f} & \hat{A} \\ \downarrow & & \downarrow{\scriptstyle \rho_m} & & \downarrow{\scriptstyle \psi_m} \\ \operatorname{Ker} f_m & \hookrightarrow & R/\mathfrak{m}^{m+1} & \xrightarrow{f_m} & R/I_m \end{array}$$

That the map f is indeed surjective comes from [30, Tag 0598]. We therefore have that $\operatorname{Ker} f = \varprojlim \operatorname{Ker} f_i$, and so $\operatorname{Ker} f = \bigcap_{j=1}^{\infty} \rho_j^{-1}(\operatorname{Ker} f_j)$, which is a closed ideal of $\mathbb{C}[[x_1, \ldots, x_n]]$. The result follows. □

Remark 5.31. We expect that an analogue of this result can be proved with P replaced by a reduced subscheme of arbitrary dimension. However, the given proof does not generalise to this case.

Example 5.32. We use the notation of Proposition 5.30. Let $g_1, \ldots, g_r \in$

$\mathbb{C}[[x_1, \ldots, x_n]]$. Let $I_n = \mathfrak{m}^{n+1} + (g_1^n, \ldots, g_r^n)$, where the superscripts denote truncation to order n. Then $\operatorname{Ker} f_n = (g_1^n, \ldots, g_r^n)/\mathfrak{m}^{n+1}$.

Let $G \in \operatorname{Ker} f$. Then $G^n \in \operatorname{Ker} f_n$, so $G^n = h_{1,n}^n g_1^n + \cdots + h_{r,n}^n g_r^n = (h_{1,n}g_1 + \cdots + h_{r,n}g_r)^n$, for some $h_{i,n} \in \mathbb{C}[[x_1, \ldots, x_n]]$. As $(G^n)^m = G^m, m < n$, we can assume that $h_{i,n}^m = h_{i,m}^m$, and hence that the h_i are independent of n. So $\operatorname{Ker} f \subset (g_1, \ldots, g_r)$. The other inclusion is trivial, so we have $Z = \varinjlim \operatorname{Spec}(R/I_n) = \operatorname{Spf}(\mathbb{C}[[x_1, \ldots, x_n]]/(g_1, \ldots, g_r))$.

Remark 5.33. Thus for formal power series $g_1, \ldots, g_r \in \mathbb{C}[[x_1, \ldots, x_n]]$ we can define the formal scheme

$$\mathbb{V}(g_1, \ldots, g_r) = \operatorname{Spf}(\mathbb{C}[[x_1, \ldots, x_n]]/(g_1, \ldots, g_r)),$$

which is a closed formal subscheme, and has the expected kernel when we move to complete rings. Furthermore, any closed formal subscheme of $\operatorname{Spf}(\mathbb{C}[[x_1, \ldots, x_n]])$ is isomorphic to a formal scheme of this form.

Definition 5.34. A classical (respectively, McQuillan) formal scheme is called *(quasi-) excellent* if it admits an open affine cover by formal spectra of (quasi-) excellent rings.

Remark 5.35. We can similarly define (quasi-) excellent schemes. However, all schemes of finite type over a field are excellent (see Example 1.30), so this property is superfluous in our setting.

As with schemes and complex spaces, we can define normal crossings divisors of formal schemes. We have the following:

Theorem 5.36. *[31, Theorem 1.1.9, Theorem 1.1.13]*

Let X be a quasi-compact, quasi-excellent classical formal scheme, and $Z \subset X$ a closed formal subscheme. Then there is a sequence of blow-ups $f : X' \to X$ with X' smooth, such that $f^{-1}(Z)$ has simple normal crossings support.

Remark 5.37. In this setting, we say that Z has a desingularisation.

6 Jet Spaces

We present two ways of looking at the jets of a space: firstly the algebraic setting, where, following [11], we define the jet scheme and the arc scheme, and secondly the analytic setting, where we view jets as equivalence classes of holomorphic maps. We then show their equivalence in terms of closed points. Following this, we define the jet space of a formal scheme, and then give a third way of construting the jet space, which has some advantages in calculations.

6.1 Algebraic Jets

Definition 6.1. Let X be a scheme of finite type over $\mathbb{C}$, and $m \in \mathbb{N}_0$. The *scheme of m-jets* of X is the scheme $J_m(X)$ which, if it exists, satisfies for every $\mathbb{C}$-algebra A the functorial relation

$$\mathrm{Hom}(\mathrm{Spec}(A), J_m(X)) \cong \mathrm{Hom}(\mathrm{Spec}(A[t]/(t^{m+1})), X).$$

If $m > p$, and $J_m(X)$ and $J_p(X)$ both exist, the truncation morphism $A[t]/(t^{m+1}) \to A[t]/(t^{p+1})$ induces a map

$$\mathrm{Hom}(\mathrm{Spec}(A[t]/(t^{m+1})), X) \to \mathrm{Hom}(\mathrm{Spec}(A[t]/(t^{p+1})), X),$$

and thence a (canonical) projection $\pi_{m,p} : J_m(X) \to J_p(X)$. Where the maps are defined, we have $\pi_{m,p} \circ \pi_{q,m} = \pi_{q,p}$.

$J_0(X) = X$, and so we have projections $\pi_m = \pi_{m,0} : J_m(X) \to X$.

$J_1(X)$ is canonically isomorphic to the tangent bundle TX.

Lemma 6.2. *If $U \subset X$ is an open set, and $J_m(X)$ exists, then $J_m(U)$ exists, and equals $\pi_m^{-1}(U)$.*

Proof: For A a $\mathbb{C}$-algebra, let $\iota_A : \mathrm{Spec}(A) \to \mathrm{Spec}(A[t]/(t^{m+1}))$ be induced by the truncation map. If a morphism $f : \mathrm{Spec}(A[t]/(t^{m+1})) \to X$ factors through U, then $f \circ \iota_A$ does; conversely, we can cover a general open set with affine open subsets, and so assume $U = \mathrm{Spec}(R)$. If $f \circ \iota_A$ factors through U, we have a ring homomorphism $R \to A$ and so a homomorphism $R \to A[t]/(t^{m+1})$; hence f factors through U. The result follows from the definition of the jet scheme. □

Proposition 6.3. *For every scheme X of finite type over $\mathbb{C}$, and every $m \in \mathbb{N}_0$, $J_m(X)$ exists, is a scheme of finite type over $\mathbb{C}$, and is unique up to a canonical isomorphism.*

Proof: Suppose that X is affine: We can then present it in the form $X = \mathbb{V}(I) \subset \mathbb{A}^n$, where $I = (f_1, \ldots, f_r)$. Now for a $\mathbb{C}$-algebra A, a morphism $\mathrm{Spec}(A[t]/(t^{m+1})) \to X$ corresponds to a ring homomorphism $\phi : \mathbb{C}[x_1, \ldots, x_n]/I \to A[t]/(t^{m+1})$. This map is determined by setting $u_i = \phi(x_i) = \sum_{j=0}^{m} a_{ij} t^j$ such that $f_l(u_1, \ldots, u_n) = 0$ for each l. Equating co-efficients, this gives a system of polynomial constraints $g_{l,p}$ on the a_{ij}, depending on the f_l; we can then define

$$J_m(X) = \mathbb{V}(g_{l,p} \mid 1 \leq l \leq r, 0 \leq p \leq m) \subset \mathbb{A}^{(m+1)n}.$$

Now suppose X is arbitrary (of finite type), with open affine cover $X = U_1 \cup \cdots \cup U_r$. For each i, the jet scheme $J_m(U_i)$ exists, and we have the projection $\pi_m^i : J_m(U_i) \to U_i$. Now by Lemma 6.2, for each i and j, $(\pi_m^i)^{-1}(U_i \cap U_j)$ and $(\pi_m^j)^{-1}(U_i \cap U_j)$ both give the jet scheme $J_m(U_i \cap U_j)$, and so are canonically isomorphic. We can then construct $J_m(X)$ by gluing the jet schemes of the affine charts along these isomorphims; the projections also glue to give the projection $\pi_m : J_m(X) \to X$.

Showing that $J_m(X)$ has the required properties is then straightforward; uniqueness comes from the universal property. □

Definition 6.4. Let X be a scheme, $m \in \mathbb{N}_0$ and $x \in X$. The scheme of m-jets of X above x is the fibre of $J_m(X)$ above x, denoted $J_m(X, x)$. Suppose x lies in an affine open subset $U = \mathrm{Spec}(A)$, where $A = \mathbb{C}[x_1, \ldots, x_n]/I$. It then corresponds to a maximal ideal $\mathfrak{m}_x \subset A$, where by the Nullstellensatz, $\mathfrak{m}_x = (x_1 - a_1, \ldots, x_n - a_n)$. We can identify $J_m(X, x)$ with $J_m(\mathrm{Spec}(A), x)$, this latter defined by morphisms $A \to \mathbb{C}[t]/(t^{m+1}), x_i \mapsto \sum a_{ij} t^j$, setting $a_{i0} = a_i$.

Definition 6.5. Let X be a scheme of finite type over $\mathbb{C}$, and $m \in \mathbb{N}_0$. The *m-jets* of X are the closed points of the scheme of m-jets

$$J_m(X)(\mathbb{C}) = \mathrm{Hom}(\mathrm{Spec}(\mathbb{C}[t]/(t^{m+1})), X).$$

Remark 6.6. By abuse of notation, we sometimes write $J_m(X)$ for $J_m(X)(\mathbb{C})$.

Let $X \subset \mathbb{A}^n$ be an affine scheme. Then $J_m(X) \subset J_m(\mathbb{A}^n) = \mathbb{A}^{(m+1)n}$. We have a system of algebraic co-ordinates $(a_{ij} \mid 1 \leq i \leq n, 0 \leq j \leq m)$ on the jet space, as in the proof of Proposition 6.3.

If $n = 2$, we write a_j for a_{1j} and b_j for a_{2j}.

Example 6.7. Let $C = \mathbb{V}(xy) \subset \mathbb{A}^2$. The jet spaces $J_m(C, 0)$ are the union of the co-ordinate hyperplanes

$$a_1 = a_2 = \cdots = a_{m-1} = 0, a_1 = \cdots = a_{m-2} = b_1 = 0, \\ \ldots, a_1 = b_1 = \cdots = b_{m-2} = 0, b_1 = \cdots = b_{m-1} = 0.$$

Indeed, as in the proof of Proposition 6.3, we set $x = \sum a_i t^i, y = \sum b_i t^i$. Equating co-efficients so that $xy \in (t^{m+1})$, we have $J_2(C, 0) = \mathbb{V}(a_1 b_1)$. Also, we have $J_1(C, 0) = \mathbb{A}^2$.

If the result holds for m, the extra equation to define the $(m+1)$-jets, that is, the co-efficient of t^{m+1} in the new co-ordinates, is

$$(m+1)(a_1 b_m + a_2 b_{m-1} + \cdots + a_{m-1} b_2 + a_m b_1).$$

Each string of equalities kills every summand except $b_{j+1} a_{m-j}, 0 \leq j \leq m-1$, and so is appended by either $b_{j+1} = 0$ or $a_{m-j} = 0$. Replacing m with $m+1$, we see that this gives the equations of the next jet space, so the result follows by induction.

Remark 6.8. The mapping $X \mapsto J_m(X)$ is functorial: For any morphism $f : X \to Y$, and any $m \in \mathbb{N}$, there is an induced morphism $f_m : J_m(X) \to J_m(Y)$ given by $\tau \mapsto f \circ \tau$.

If $p < m$, the diagram

$$\begin{array}{ccc} J_m(X) & \xrightarrow{f_m} & J_m(Y) \\ \downarrow{\scriptstyle \pi^X_{m,p}} & & \downarrow{\scriptstyle \pi^Y_{m,p}} \\ J_p(X) & \xrightarrow{f_p} & J_p(Y) \end{array}$$

commutes.

Taking X to be a closed subscheme of Y, and f the inclusion map, we see that $J_m(X) \subset J_m(Y)$.

Proposition 6.9. *Let $(Y_i)_{i\in\mathcal{I}}$ be closed subschemes of a scheme X of finite type over $\mathbb{C}$. Then $J_m(\bigcap_{i\in\mathcal{I}} Y_i) = \bigcap_{i\in\mathcal{I}} J_m(Y_i)$ for each m.*

Proof: First note that, for each $i \in \mathcal{I}$, $\bigcap_{i\in\mathcal{I}} Y_i$ is a closed subscheme of Y_i. Applying J_m to the inclusion map we get an inclusion $J_m(\bigcap_{i\in\mathcal{I}} Y_i) \to J_m(Y_i)$. So we have $J_m(\bigcap_{i\in\mathcal{I}} Y_i) \subset \bigcap_{i\in\mathcal{I}} J_m(Y_i)$.

Conversely, consider a point $\tau \in \bigcap_{i\in\mathcal{I}} J_m(Y_i)$. For each $i \in \mathcal{I}$, $\tau \in J_m(Y_i)$, and so it corresponds to a morphism $\mathrm{Spec}(\mathbb{C}[t]/(t^{m+1})) \to Y_i$. We then see that τ corresponds to a single morphism $\mathrm{Spec}(\mathbb{C}[t]/(t^{m+1})) \to \bigcap_{i\in\mathcal{I}} Y_i$, and therefore $x \in J_m(\bigcap_{i\in\mathcal{I}} Y_i)$. □

Definition 6.10. Let X be a scheme of finite type over $\mathbb{C}$. The *scheme of arcs* of X is the projective limit $J_\infty(X) = \varprojlim J_m(X)$ of the jet spaces.

As the truncation morphisms $\pi_{m,p} : J_m(X) \to J_p(X)$ are all affine, $J_\infty(X)$ exists as a scheme over $\mathbb{C}$, though not in general of finite type. It comes with canonical projection morphisms $\rho_m : J_\infty(X) \to J_m(X)$ for each $m \in \mathbb{N}_0$.

Lemma 6.11. *If X is a smooth scheme, the projections $\rho_m : J_\infty(X) \to J_m(X)$ are all surjective.*

Proof: If $X = \mathbb{A}^n$, then $J_m(X) = \mathbb{A}^{(m+1)n}$, and the jet truncations $\pi_{m,p}$ are co-ordinate projections, hence are all surjective. A general smooth scheme is locally isomorphic to $\mathbb{A}^n$, so again the jet truncation maps are all surjective. $J_\infty(X)$ is then the limit of a projective system with surjective morphisms, so the canonical maps ρ_m are surjective. □

Definition 6.12. Lat X be a scheme of finite type over $\mathbb{C}$. The *arcs* of X are the closed point of the scheme of arcs

$$J_\infty(X)(\mathbb{C}) = \mathrm{Hom}(\mathrm{Spec}(\mathbb{C}[[t]]), X).$$

Remark 6.13. For any morphism $f : X \to Y$ of schemes of finite type, there is an induced morphism $f_\infty : J_\infty(X) \to J_\infty(Y)$.

If $m \in \mathbb{N}_0$, the diagram

$$\begin{array}{ccc} J_\infty(X) & \xrightarrow{f_\infty} & J_\infty(Y) \\ \downarrow{\scriptstyle \rho_m^X} & & \downarrow{\scriptstyle \rho_m^Y} \\ J_m(X) & \xrightarrow{f_m} & J_m(Y) \end{array}$$

commutes.

Definition 6.14. A subset $D \subset J_\infty(X)$ is called *thin* if $D \subset J_\infty(Y)$, where $Y \subset X$ is a closed subset not containing any irreducible component of X.

Definition 6.15. A subset Y of a topological space X is *constructible* if it is the finite union of subsets which are locally closed, that is, they are the intersection of an open subset and a closed subset.

Definition 6.16. A subset $C \subset J_\infty(X)$ is called a *cylinder* if it is of the form $C = \rho_m^{-1}(S)$, for some $m \in \mathbb{N}_0$ and some constructible subset $S \subset J_m(X)$.

Lemma 6.17. *[11, Lemma 5.1] If $C \subset J_\infty(X)$ is a cylinder, then C is thin if and only if it is contained in $J_\infty(X_{\text{sing}})$.*

Proposition 6.18. *Let $f : X \to Y$ be a proper birational morphism of smooth varieties. Then for each $m \in \mathbb{N}$, the induced morphism $f_m : J_m(X) \to J_m(Y)$ is surjective.*

Proof: Let $Z \subset Y$ be the smallest closed subset such that $f|_{X \setminus f^{-1}(Z)} : X \setminus f^{-1}(Z) \to Y \setminus Z$ is an isomorphism. Let $\tau \in J_m(Y)$. As Y is smooth, the fibre $\rho_m^{-1}(\tau) \subset J_\infty(Y)$ is non-empty; it is a cylinder, and so by Lemma 6.17 is not thin. In particular, $\rho_m^{-1}(\tau) \not\subset J_\infty(Z)$. We can then choose an arc $\gamma \in J_\infty(Y) \setminus J_\infty(Z)$ such that $\rho_m(\gamma) = \tau$.

By [11, Proposition 3.2], f_∞ induces a bijection

$$J_\infty(X) \setminus J_\infty(f^{-1}(Z)) \to J_\infty(Y) \setminus J_\infty(Z),$$

so we have an arc $\gamma' \in J_\infty(X)$ such that $f_\infty(\gamma') = \gamma$. By commutativity, we have

$$\tau = \rho_m^Y(f_\infty(\gamma')) = f_m(\rho_m^X(\gamma')),$$

and the result follows. □

Remark 6.19. This is not true in general. Consider the map $f : \mathbb{A}^1 \to \mathbb{A}^1, x \mapsto x^3$. The induced morphism $f_1 : J_1(\mathbb{A}^1) \to J_1(\mathbb{A}^1)$ is given by $(a_0, a_1) \mapsto (a_0^3, 3a_0^2 a_1)$, which is not surjective, as on the fibre above 0 it is constantly zero.

Proposition 6.20 (Change of Variables)**.** *Let $W \subset X$ be a closed subscheme, and $\pi : X' \to X$ a morphism of schemes. For each $m \in \mathbb{N}$, $J_m(\pi^{-1}(W)) = \pi_m^{-1}(J_m(W))$.*

Proof: Let $\tau \in J_m(\pi^{-1}(W))$. This corresponds to a morphism $\mathrm{Spec}(\mathbb{C}[t]/(t^{m+1})) \to X'$ with image in $\pi^{-1}(W)$. Hence $\pi \circ \tau \in J_m(W)$, so $\tau \in \pi_m^{-1}(J_m(W))$.

Conversely, let $\tau \in \pi_m^{-1}(J_m(W))$. Then $\pi \circ \tau \in J_m(W)$, so corresponds to a morphism $\mathrm{Spec}(\mathbb{C}[t]/(t^{m_1})) \to X$ with image in W. Hence τ corresponds to a morphism with image in $\pi^{-1}(W)$, and so $\tau \in J_m(\pi^{-1}(W))$. □

Keeping the notation of the proposition, we have the following corollaries:

Corollary 6.21. $\pi_m(J_m(W)) \subset J_m(\pi(W))$*, with equality if* π_m *is surjective.*

Corollary 6.22. $\pi_m^{-1}(J_m(W)|_C) = J_m(\pi^{-1}(W))|_{\pi^{-1}(C)}$*, where* $C \subset W$ *is a closed subscheme.*

6.2 Analytic Jets

Let X be a complex space, and $x \in X$. The space of analytic m-jets of X above x, $J_m^{an}(X, x)$, can then be defined as the set of germs of holomorphic maps $f : (\mathbb{C}, 0) \to (X, x)$ modulo the equivalence relation $f \sim g$ if $f^{(i)}(0) = g^{(i)}(0)$ for all $0 \leq i \leq m$. We define $J_m^{an}(X) = \bigcup_{x \in X} J_m^{an}(X, x)$.

Proposition 6.23 (Jet GAGA)**.** *Suppose* X *is a scheme of finite type over* $\mathbb{C}$*. Then* $J_m(X)(\mathbb{C}) \cong J_m^{an}(X)$*.*

Proof: Let $f \in J_m^{an}(X)$. As f is a germ, we can assume its image lies in an open affine set of X, which has local co-ordinates $x_1, \ldots, x_n$. The equivalence relation on the germs means we can define the jet by assigning to each co-ordinate x_i an element of $\mathbb{C}[t]/(t^{m+1})$, and these truncated polynomials must satisfy the defining equations of the affine chart. Thus the vector of co-efficients of these polynomials lies in $J_m(X)(\mathbb{C})$.

Conversely an element of $J_m(X)(\mathbb{C})$ corresponds to a ring homomorphism $\phi : \mathbb{C}[x_1, \ldots, x_n]/I \to \mathbb{C}[t]/(t^{m+1})$, with each x_i mapped to a truncated polynomial of order m. It can thus be presented as a representative of the equivalence class of germs of holomorphic maps that define the analytic jets. □

6.3 Jets of Formal Schemes

Proposition 6.24. *Let* $X = \varinjlim X_\lambda$ *be a formal scheme. Then for each* $m \in \mathbb{N}$ *and* $x \in X$*,* $J_m(X, x) = \bigcup_{\lambda \in \Lambda} J_m(X_\lambda, x)$*.*

Proof: As each X_λ is a subscheme of X, we clearly have the union $\bigcup_\lambda J_m(X_\lambda, x) \subset J_m(X, x)$.

Conversely, let $\tau : \mathrm{Spec}(\mathbb{C}[t]/(t^{m+1})) \to X$ be an m-jet. We may assume that τ factors through an affine subset $U \subset X$, where $U = \mathrm{Spf}((R_\lambda))$. By duality, τ then corresponds to an element of $\varinjlim_\lambda \mathrm{Hom}(R_\lambda, \mathbb{C}[t]/(t^{m+1}))$, which in turn corresponds to a a collection of morphisms $\mathrm{Spec}(\mathbb{C}[t]/(t^{m+1})) \to \mathrm{Spec}(R_\lambda)$. As $J_m(\mathrm{Spec}(R_\lambda)) = J_m(X_\lambda)|_U$, the result follows.

Now suppose $X = \varinjlim Y_\mu$ is another presentation, and let $\tau \in J_m(X_\lambda, x)$. Then τ maps into one of the X_μ: that is, $\tau \in J_m(X_\mu)$. It follows that the jets of X are independent of the limit presentation. □

Corollary 6.25. *Let $X = \varinjlim X_\lambda$ be a direct limit of formal schemes. Then for each $m \in \mathbb{N}$ and $x \in X$, $J_m(X, x) = \bigcup_{\lambda \in \Lambda} J_m(X_\lambda, x)$.*

The algebraic and analytic constructions of the jet space can also be applied to complex spaces and formal schemes defined by formal power series, so we have full generalisation. (As such, we will now only consider the algebraic setting.) In particular, the change of variables formula still holds in these cases.

Proposition 6.26. *Let X be a complex space (not necessarily algebraic), and let $x \in X$. Then for each $m \in \mathbb{N}$, the fibre of the jet space $J_m(X, x)$ is an affine scheme.*

Proof: By applying a linear shift of co-ordinates, we may assume $x = 0$; this induces an affine isomorphism on the jet fibres by Proposition 6.20. Suppose that, in these shifted co-ordinates, X is locally given as $X = \mathbb{V}(g_1, \ldots, g_k), g_i \in \mathbb{C}[[x_1, \ldots, x_n]]$. Then

$$J_m(\mathbb{V}(g_1, \ldots, g_k), 0) = J_m(\mathbb{V}(g_1^m, \ldots, g_k^m), 0),$$

where g_i^m denotes the truncation to degree m. These spaces are affine schemes. □

Example 6.27. Let X be a smooth variety, $Z \subset X$ a closed subscheme, and $z \in Z$. Consider the formal completion $\hat{X}_Z$ along Z. Then $J_m(\hat{X}_Z, z) = J_m(X, z)$.

Indeed, $\hat{X}_Z$ is the formal direct limit of the schemes Z^i, so by Proposition 6.24, $J_m(\hat{X}_Z, z) = \bigcup_{i \in \mathbb{N}} J_m(Z^i, z)$. As Z^i has degree $\geq i$, its jets of lower orders are the full affine space, equal to $J_m(X, z)$.

Lemma 6.28. *Let X be a smooth variety, and let Y_1, Y_2 be two formal subschemes of X with the same underlying topological space. Suppose that $J_m(Y_1) \subset J_m(Y_2)$, for all $m \in \mathbb{N}$. Then $Y_1 \subset Y_2$.*

Proof: Let π_m^X be the truncation map $J_m(X) \to X$. Then $\pi_m^X|_{Y_i} = \pi_m^{Y_i} : J_m(Y_i) \to Y_i$. We have $\pi_m^X(J_m(Y_1)) \subset \pi_m^X(J_m(Y_2))$, and so, as Y_1 and Y_2 have the same underlying topological space, we have $\pi_m^{Y_1}(J_m(Y_1)) \subset \pi_m^{Y_2}(J_m(Y_2))$. This holds for all $m \in \mathbb{N}$, so $Y_1 \subset Y_2$. □

Theorem 6.29. *Let $X = \varinjlim X_\lambda$ be a formal scheme such that for each $\lambda \in \Lambda$, $(X_\lambda)_{\text{red}} = \{P\} \subset Y$, where Y is a smooth variety. Then X is countably indexed, is a quasi-excellent classical formal scheme, and is a closed formal subscheme of $\hat{Y}_{\{P\}}$. In particular X has a desingularisation.*

Proof: We choose co-ordinates such that $P = 0$. Let

$$X_\lambda = \operatorname{Spec}(\mathbb{C}[x_1, \ldots, x_n]/(g_{\lambda,1}, \ldots, g_{\lambda,k_\lambda})).$$

By Proposition 6.26, $J_m(X_\lambda) = J_m(\mathbb{V}(g^m_{\lambda,1}, \ldots, g^m_{\lambda,k_\lambda}))$, where the superscript denotes truncation to order m.

The ideals I_λ corresponding to the X_λ form a descending chain; quotienting each by $\mathfrak{m}^{m+1}$, where $\mathfrak{m}$ denotes the ideal $(x_1, \ldots, x_n) \subset \mathbb{C}[x_1, \ldots, x_n]$, gives a descending chain that stabilises, and which yields the same jets. So, for all $m \in \mathbb{N}$, there exists $\lambda_m \in \Lambda$ such that $J_m(X_\mu) = J_m(X_{\lambda_m})$, for all $\mu \geq \lambda_m$. Therefore for all $m \in \mathbb{N}$, there exists $k_m \in \Lambda$ such that $J_k(X) = J_k(X_{k_m})$, for all $k \leq m$.

We define $S_m(X)$ to be the smallest scheme supported at P such that $J_k(X) \subset J_k(S_m(X))$ for all $k \leq m$. We define $\mathscr{S}(X) = \varinjlim S_m(X)$. From these definitions, we have $S_m(X) \subset X_{k_m}$ for all m, and hence $\mathscr{S}(X) \subset X$. Therefore $J_m(\mathscr{S}(X)) \subset J_m(X)$, for all m. The other inclusion holds by definition, so we have equality. It follows from Lemma 6.28 that $\mathscr{S}(X) = X$, and so X is countably indexed.

That X is a classical formal scheme follows from Proposition 5.27. For each $m \in \mathbb{N}$, $S_m(X) \subset P^{m+1}$, and so X satisfies the conditions of Proposition 5.30. So X is a closed formal subscheme of $\hat{Y}_{\{P\}}$; by the proof of Proposition 5.30, X is the formal spectrum of a quotient of a power series ring, and so is excellent. □

Corollary 6.30. *The result of Proposition* 6.26 *also holds if X is a formal scheme.*

Proof: Take an affine open neighbourhood $U \ni x$. We can write U as $U = \varinjlim Z_\lambda$, where the Z_λ are affine schemes all with the same support, and so we can embed U as a formal subscheme of some smooth affine variety Y. By Theorem 6.29, $U \cap \hat{Y}_{\{x\}}$ is a closed formal subscheme of $\hat{Y}_{\{x\}}$, and so is isomorphic to $\mathbb{V}(g_1, \dots, g_r)$, for some $g_i \in \mathbb{C}[[x_1, \dots, x_n]]$.

By the same argument as in Proposition 6.26,

$$J_m(\mathbb{V}(g_1, \dots, g_k), 0) = J_m(\mathbb{V}(g_1^m, \dots, g_k^m), 0),$$

where g_i^m denotes the truncation to degree m. We then have, using an appropriate change of co-ordinates,

$$J_m(X, x) = J_m(U, x) = J_m(U \cap \hat{Y}_{\{x\}}, x) \cong J_m(\mathbb{V}(g_1, \dots, g_k), 0),$$

which is an affine scheme. □

Proposition 6.31. *Let $f : X \to Y$ be a (formal) isomorphism of complex spaces. Then the induced isomorphisms $J_m(X, x) \to J_m(Y, f(x))$ are algebraic.*

Proof: We choose appropriate co-ordinates such that $f(0) = 0$. Then the induced map $f_m|_0 : J_m(X, 0) \to J_m(Y, 0)$ is the same as that induced by the truncation f^m, and so is algebraic.

As all the jet fibres are algebraic, we can apply this construction at every point, and hence get the result. □

6.4 Jets by Formal Derivatives

We can construct the jet space a third way, compatible with the algebraic and analytic constructions:

Let $W \subset \mathrm{Spec}(\mathbb{C}[x_1, \dots, x_n])$ be an affine algebraic set defined by polynomials $f_1, \dots, f_r$. We have $J_m(W) \subset \mathbb{A}^{(m+1)n}$; denote the co-ordinates of this affine space by $x_i^{(j)}, 1 \leq i \leq n, 0 \leq j \leq m$, where $x_i^{(0)} = x_i$. We can then define a derivation D on the space, sending $x_i^{(j)}$ to $x_i^{(j+1)}$, and setting $x_i^{(m+1)} = 0$.

$J_m(W)$ is then defined by $f_k, f_k', \ldots, f_k^{(m)}, 1 \leq k \leq r$, where the formal derivative is with respect to the derivation D. We sometimes write $D^j f_k$ for $f_k^{(j)}$. We obtain the definition of the jet scheme given in Section 6.1 by setting $x_i^{(j)} \mapsto a_{ij}$.

Suppose we have a morphism of schemes $\pi : X' \to X$, and a subscheme $W = \mathbb{V}(f_1, \ldots, f_r) \subset X$. For each $m \in \mathbb{N}$, there is an induced map $\pi_m : J_m(X') \to J_m(X)$. How π^m pulls back $D^j x_i$ is induced from how it pulls back x_i.

Using this contruction of the jet space, $\pi_m^{-1}(J_m(W)) = \pi_m^{-1}(\mathbb{V}(D^j f_i))$, and $J_m(\pi^{-1}(W)) = \mathbb{V}(D^j(\pi^* f_i)$. By the chain rule, these two sets are equal, and we reprove the change of variables formula.

7 Linear Spaces on Manifolds

Let X be a complex manifold. There is a one-to-one correspondence between rank r vector bundles on X and locally free sheaves of rank r on X: If $\pi : E \to X$ is a vector bundle, then its sheaf of sections $\mathscr{E}$, given by

$$\mathscr{E}(U) = \{\sigma : U \to E \mid \pi \circ \sigma = \mathrm{id}_U, \sigma \text{ holomorphic}\}.$$

is locally free; conversely any locally free sheaf is the sheaf of sections of some vector bundle.

We now seek to generalise this equivalence.

Throughout the sequel, let S be a complex space.

If $\alpha_{\mathbb{C}}, \mu_{\mathbb{C}}$ are the addition and multiplication maps on $\mathbb{C}$, $R = S \times \mathbb{C}$ becomes a ring under the operations $\mathrm{id}_S \times \alpha_{\mathbb{C}}, \mathrm{id}_S \times \mu_{\mathbb{C}}$ on $R \times_S R$.

Definition 7.1. A *linear space* L is a unitary $S \times \mathbb{C}$-module, that is, a complex space over S with operations $+ : L \times_S L \to L, \cdot : (S \times \mathbb{C}) \times_S L = \mathbb{C} \times L \to L$ satisfying the module axioms.

Remark 7.2. If S is a complex manifold, then any vector bundle over S is a linear space.

Definition 7.3. Let $(L, +, \cdot), (L', +', \cdot')$ be linear spaces over S. A *homomorphism* of linear spaces is a holomorphic map $\xi : L \to L'$ such that the diagrams

$$\begin{array}{ccc} L \times_S L & \xrightarrow{\xi \times_S \xi} & L' \times_S L' \\ \downarrow{\scriptstyle +_L} & & \downarrow{\scriptstyle +_{L'}} \\ L & \xrightarrow{\quad \xi \quad} & L' \end{array} \qquad \begin{array}{ccc} \mathbb{C} \times L & \xrightarrow{\mathrm{id} \times \xi} & \mathbb{C} \times L' \\ \downarrow{\scriptstyle \cdot_L} & & \downarrow{\scriptstyle \cdot_{L'}} \\ L & \xrightarrow{\quad \xi \quad} & L' \end{array}$$

commute.

Proposition 7.4. *Let L be a linear space over S, with given map $p : L \to S$; let (T, ξ) be a complex space over S. Then $L \times_S T$ is a linear space over T.*

Proof: $L \times_S T$ exists and is a complex space ([12, Prop 0.28]); projecting onto the second co-ordinate gives a complex space over T. To endow the space with module structure, we need operations

$$(L \times_S T) \times_T (L \times_S T) = L \times_S L \times_S T \to L \times_S T;$$

$$(T \times \mathbb{C}) \times_T (L \times_S T) = L \times \mathbb{C} \times T \to L \times_S T;$$

it is clear then that the linear operations on L carry over to $L \times_S T$, making it into a linear space. □

Corollary 7.5. *The fibres L_s of L are vector spaces over $\mathbb{C}$; homomorphisms induce maps $\xi_s : L_s \to L'_s$ which are linear maps.*

Proof: Let T be the singleton subset $\{s\}$ of S, with ξ the inclusion. Then T is a complex space over S. The fibre L_s is then the space $L \times_S T$, a linear space over T. By definition we can only add up elements of a linear space if they are in the same fibre; as L_s has only fibre, any two elements can be added (and any can be multiplied by complex scalars), so L_s is a vector space. As linear space homomorphisms commute with addition and scalar multiplication, the pullbacks to the fibres will be linear. □

Definition 7.6. Let L, L' be two linear spaces over S, with given maps $p : L \to S$ and $p' : L' \to S$ respectively. Let $U \subset S$ be open. We define $L_U = p^{-1}(U)$ and $L'_U = p'^{-1}(U)$. We can then define $\mathscr{H}om_S(L, L')(U) = \mathrm{Hom}_U(L_U, L'_U)$, the set of homomorphisms; this gives a sheaf of $\mathscr{O}_S$-modules on S.

Theorem 7.7 (Duality Theorem)**.** *Let L be a linear space over a complex space S. The sheaf of linear forms on L, $\mathcal{L}_S(L) = \mathscr{H}om_S(L, S \times \mathbb{C})$, is a coherent sheaf of $\mathscr{O}_S$-modules. The functor $\mathcal{L}$ from the category of linear spaces over S to the category of coherent sheaves of $\mathscr{O}_S$-modules is an antiequivalence.*

Proof: The outline of the proof is given in [12, Theorem 1.6]. See the references there for the full details. □

As a result of the antiequivalence, kernels and cokernels exist in the category of linear spaces over S, and are unique up to isomorphism.

Proposition 7.8. *Let $\xi : L \to L'$ be a morphism of linear spaces over S. Then* $\mathrm{Ker}\,\xi = \{x \in L \mid \xi(x) = 0\}$.

Proof: The set described is the equaliser of ξ and the zero map $L \to L'$, which exists as a complex subspace of L by [12, Prop 0.33]. Linearity of ξ implies that it is a linear subspace, and thus is a kernel in the category of linear spaces. □

Now let $\mathcal{F}$ be a coherent sheaf of $\mathscr{O}_S$-modules. Then we have an exact sequence

$$\mathscr{O}_S^m \xrightarrow{\eta} \mathscr{O}_S^n \to \mathcal{F} \to 0.$$

The transpose of η defines a homomorphism

$$\xi : S \times \mathbb{C}^n \to S \times \mathbb{C}^m.$$

We define $\mathbb{V}(\mathcal{F}) = \operatorname{Ker} \xi$—this is the linear space corresponding to the coherent sheaf $\mathcal{F}$ under the antiequivalence. We have the following canonical isomorphisms:

$$\begin{aligned} \mathcal{F} &\cong \mathcal{L}(\mathbb{V}(\mathcal{F})), \\ L &\cong \mathbb{V}(\mathcal{L}(L)), \\ \mathscr{H}om_{\mathscr{O}_S}(\mathcal{F}, \mathcal{F}') &\cong \mathscr{H}om_S(\mathbb{V}(\mathcal{F}'), \mathbb{V}(\mathcal{F})), \\ \mathscr{H}om_S(L, L') &\cong \mathscr{H}om_{\mathscr{O}_S}(\mathcal{L}(L'), \mathcal{L}(L)). \end{aligned}$$

If $\xi : L \to L'$ is a morphism of linear spaces over S, and $\eta : \mathcal{L}(L') \to \mathcal{L}(L)$ is the associated map of sheaves, then $\operatorname{Ker} \xi = \mathbb{V}(\operatorname{Coker} \eta)$, and $\operatorname{Coker} \xi = \mathbb{V}(\operatorname{Ker} \eta)$.

Example 7.9. Let X be a complex manifold, let E be a vector bundle of finite rank over X, and let $\mathscr{E}$ be the sheaf of sections of E. Then $\mathcal{L}(E) \cong \mathscr{E}^*$.

In particular, if TX is the tangent bundle, then $\mathcal{L}(TX) \cong \Omega^1_X$, the sheaf of 1-forms.

Lemma 7.10. *Let X be a complex manifold. Then there is a one-to-one correspondence between linear subspaces of TX and quotient sheaves of Ω^1_X.*

Proof: Let $L \subset TX$ be a linear subspace. By the duality theorem, there is a sheaf of modules $\mathcal{L}(L)$ corresponding to L, and a morphism of sheaves $\Omega^1_X \to \mathcal{L}(L)$ corresponding to the inclusion $L \to TX$, and given by the suitable restriction of the linear forms. This morphism is surjective; the sheaf $\mathcal{L}(L)$ is thus isomorphic to the quotient sheaf $\Omega^1_X / \mathscr{K}$, where $\mathscr{K}$ is the kernel sheaf of the morphism.

Conversely, any quotient sheaf $\mathscr{Q}$ of Ω^1_X is a coherent analytic sheaf on M, so corresponds to a linear space $\mathbb{V}(\mathscr{Q})$ over X; by the isomorphisms given in Theorem 7.7, the quotient map corresponds to an injective homomorphism of linear spaces $\mathbb{V}(\mathscr{Q}) \to \mathbb{V}(\Omega^1_X) = TX$. Thus $\mathbb{V}(\mathscr{Q})$ may be thought of as a linear subspace of TX.

The one-to-one correspondence comes from the antiequivalence in the duality theorem. □

Definition 7.11. Let S be an irreducible complex space. A linear space L over S is said to be *irreducible* (as a linear space), if, whenever we write $L = L' \cup L''$, for L', L'' linear spaces over S, then either $L = L'$ or $L = L''$. Otherwise L is *reducible* (as a linear space).

Remark 7.12. If L is irreducible as a complex space, then it will be irreducible as a linear space, but the converse is not true in general, as in the following:

Example 7.13. Consider $L = \mathbb{V}(xy)$, which can be viewed as a linear space over $\mathbb{A}^1$. Clearly it is reducible as a complex space. However, as any linear subspace must contain the $y = 0$ component, a subspace containing more than this must be the whole space by linearity. Hence L is irreducible as a linear space.

Definition 7.14. The *support* of a linear space L over an irreducible complex space S is the closure of $\{x \in S \mid L_x \neq 0\}$, denoted $\operatorname{supp} L$. It is an analytic subset.

Remark 7.15. We similary define the support of a sheaf. We have $\operatorname{supp} \mathcal{L}(Q) = \operatorname{supp} Q$, and $\operatorname{supp} \mathbb{V}(\mathscr{E}) = \operatorname{supp} \mathscr{E}$.

Proposition 7.16. *Let L be a linear space over S. Then L is reducible as a linear space if and only if there is a non-zero, non-isomorphic surjective morphism of linear spaces $L \to Q$, such that* $\dim \operatorname{supp} Q < \dim S$.

Proof: Suppose L is reducible, with decomposition $L = \bigcup_{i \in I} L_i$. Now one of the L_i must have lower-dimensional support; if not we can write L as $L = L' \cup L''$, where L' and L'' both have all of S as their support. Then for each $x \in S$, $L'_x = L_x$ or $L''_x = L_x$; the set of $x \in S$ satisfying one of these conditions is an analytic set, so we have the union of two analytic subsets being the whole of S, and therefore one of them equalling S—so $L' = L$ or $L'' = L$, a contradiction.

The canonical map $L \to L_i$ taken by quotienting out the other components satisfies the conditions of the proposition.

Conversely, suppose we have a surjective map $q : L \to Q$, with $\dim \operatorname{supp} Q < \dim S$. Let $\pi : Q \to S$ be the projection onto the base space. Let $U = Q \backslash \pi^{-1}(\operatorname{supp} Q)$, the zero section outside the support. $\overline{U}$ is then the whole zero section of Q. Now as q is continuous, $\overline{q^{-1}(U)} \subset q^{-1}(\overline{U}) = \operatorname{Ker} q$; as q is assumed to be non-zero, the linear closure L' of $\overline{q^{-1}(U)}$ is then a proper linear subspace of L.

We define a linear space L'' with the same support as Q, equal to $q^{-1}(\operatorname{supp} Q)$ above the support. L'' is a proper linear subspace of L, and we have $L = L' \cup L''$. So L is reducible. □

Proposition 7.17. *Let $(X, \mathscr{O}_X)$ be a complex manifold, and let $\mathcal{F}$ be a coherent sheaf of $\mathscr{O}_X$-modules. Then $\mathcal{F}$ has torsion if and only if there is a subsheaf $\mathcal{E} \subset \mathcal{F}$ such that* $\dim \operatorname{supp} \mathcal{E} < \dim X$.

Proof: See discussion following [18, Definition 1.1.4]. □

Proposition 7.18. *Let $(X, \mathscr{O}_X)$ be a complex manifold. Then there is a one-to-one correspondence between torsion-free coherent sheaves on X and irreducible linear spaces over X with full support.*

Proof: Let $\mathcal{F}$ be a torsion-free sheaf on X, and let $L = \mathbb{V}(\mathcal{F})$ be the associated linear space under duality. Then $\operatorname{supp}(L) = \operatorname{supp}(\mathcal{F}) = X$. Let Q be a linear space such that there exists a non-zero non-isomorphic surjection $L \to Q$. Then $\mathcal{L}(Q)$ is a proper subsheaf of $\mathcal{F}$. As $\mathcal{F}$ is torsion-free, we have $\dim \operatorname{supp} \mathcal{L}(Q) = \dim X$; it follows that L is irreducible.

Conversely, let L be an irreducible linear space on X with full support, and let $\mathcal{F} = \mathcal{L}(L)$ be the associated sheaf. Let $\mathscr{E} \subset \mathcal{F}$ be a non-zero proper subsheaf. Then there is a non-zero non-isomorphic surjection $L \to \mathbb{V}(\mathscr{E})$. As L is irreducible, $\dim \operatorname{supp} \mathbb{V}(\mathscr{E}) = \dim X$; as $\mathcal{F}$ has full support it follows that it is torsion-free. □

8 Foliations

Throughout this section we let X be a complex manifold.

8.1 Foliations by Vector Fields

Definition 8.1. A subsheaf $\mathcal{F} \subset \mathcal{T}_X$ is called a *foliation* if it is saturated and is integrable, that is, $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$, where the Lie bracket is defined on the sections of the sheaf.

Remark 8.2. The condition of being saturated is not preserved under many operations, for example pullback. Hence we sometimes deal with unsaturated foliations. If $\mathcal{F}$ is an unsaturated foliation, we define its saturation $\mathrm{sat}(\mathcal{F})$ to be the smallest saturated sheaf containing it. It is the kernel of the morphism

$$\mathcal{T}_X \to \mathcal{T}_X/\mathcal{F} \to (\mathcal{T}_X/\mathcal{F})/\operatorname{Tor}(\mathcal{T}_X/\mathcal{F}).$$

Definition 8.3. Let $\mathcal{F}$ be a foliation on X. The *singular locus* of $\mathcal{F}$, denoted $\mathrm{Sing}(\mathcal{F})$, is the singular locus of $\mathcal{F}$ as a coherent sheaf, that is, the locus of points of X around which $\mathcal{F}$ is not locally free. It is a complex subspace of X of codimension at least 2.

Remark 8.4. Let $\mathcal{F}$ be a foliation on X, and let $A = \mathrm{Sing}\,\mathcal{F}$. The tangent sheaf $\mathcal{T}_X$ is reflexive, and so the saturation property gives that $\mathcal{F}$ is normal (see Lemma 4.13). As $\mathcal{F}$ is saturated, we also have that A has codimension at least 2, and so for all open $U \subset X$, we have $\mathcal{F}(U) \cong \mathcal{F}(U \setminus A)$. So the global behaviour of $\mathcal{F}$ is defined by its behaviour on the smooth locus.

Theorem 8.5 (Frobenius)**.** *Let $\mathcal{F}$ be a smooth dimension-r foliation on X (that is, $\mathcal{F}$ is locally free of rank r). Thus $\mathcal{F}$ corresponds to a rank r vector subbundle $E \subset TX$. Then X can be written as the disjoint union of connected submanifolds (L_α), where $T_x L_\alpha = E_x$, for all $x \in X$, and locally at each point $x \in X$, there is a system of local holomorphic co-ordinates $x_1, \dots, x_n$ on an open neighbourhood $U \ni x$, such that the components of $U \cap L_\alpha$ can be written as*

$$x_{r+1} = \mu_{r+1}, \dots, x_n = \mu_n,$$

with the μ_i constant.

Definition 8.6. The L_α in Frobenius' theorem are called the *leaves* of the foliation.

Definition 8.7. Let $\mathcal{X}$ be a vector field on a complex manifold X. An *integral curve* of $\mathcal{X}$ is a holomorphic map $\alpha : U \to X$, where $U \subset \mathbb{C}$ is an open domain, such that $\alpha'(t) = \mathcal{X}(\alpha(t))$, for all $t \in U$.

The subsheaf of $\mathcal{T}_X$ defined by a vector field $\mathcal{X}$ is clearly integrable, and so defines a foliation $\mathcal{F}$ if it is also saturated. On the smooth locus of $\mathcal{F}$, the leaves of the foliation are the integral curves of $\mathcal{X}$.

Definition 8.8. Let $\mathcal{X}$ be a vector field on a complex manifold X. A *flow* of $\mathcal{X}$ is a holomorphic function $\phi : U \to X$, where $U \subset X \times \mathbb{C}$ is an open subset such that for each $x \in X$, $U \cap \{x\} \times \mathbb{C}$ is an open domain containing 0, such that for all $x \in X$, the function $\phi(x, \_)$ is an integral curve of $\mathcal{X}$.

8.2 Foliations by 1-forms

Theorem 8.9. *Let X be a complex manifold. Then there is a one-to-one correspondence between torsion free quotient sheaves of the sheaf of 1-forms $\Omega = \Omega^1_X$ and saturated subsheaves of the tangent sheaf $\mathcal{T} = \mathcal{T}_X$.*

Proof: We prove the statement, equivalent by definition of saturatedness, that there is a one-to-one correspondence between saturated subsheaves of $\mathcal{T}$ and saturated subsheaves of Ω. To this end, we define a map

$$F : \{\text{saturated subsheaves of } \Omega\} \to \{\text{saturated subsheaves of } \mathcal{T}\},$$

$F(\mathscr{K}) = (\Omega/\mathscr{K})^*$.

Let $\mathscr{K}$ be a saturated subsheaf of Ω, and $\mathscr{Q}$ the quotient sheaf. Then we have an exact sequence

$$0 \to \mathscr{K} \to \Omega \xrightarrow{q} \mathscr{Q} \to 0.$$

Fix $x \in X$; we now consider the stalks at x and get a sequence of $\mathscr{O}_{X,x}$-modules,

$$0 \to \mathscr{K}_x \to \Omega_x \xrightarrow{q_x} \mathscr{Q}_x \to 0.$$

By Proposition 4.14 this sequence is exact. We also have that $\mathscr{Q}_x$ is torsion-free.

By Lemma 1.39, the dual q_x^* of the quotient map q_x is an injection from $\mathscr{Q}_x^*$ into Ω_x^*, and so we now have another exact sequence:

$$0 \to \mathscr{Q}_x^* \xrightarrow{q_x^*} \Omega_x^* \to \Omega_x^*/\mathscr{Q}_x^* \to 0.$$

We also have, by Lemma 1.40, that if $\tau \in \Omega_x^*$, then $\tau \in \mathscr{Q}_x^*$ if and only if $\tau(\omega) = 0$, for all $\omega \in \mathscr{K}_x$.

Now let $\bar{\tau} \in \Omega_x^*/\mathscr{Q}_x^*$ and $f \in \mathscr{O}_{X,x} \setminus \{0\}$ be such that $f\bar{\tau} = 0$. Then $0 = f\bar{\tau} = \overline{f\tau} \Rightarrow f\tau \in \mathscr{Q}_x^*$. Therefore

$$(f\tau)(\omega) = f\tau(\omega) = 0, \forall \omega \in \mathscr{K}_x \Rightarrow \tau(\omega) = 0, \forall \omega \in \mathscr{K}_x.$$

So $\tau \in \mathscr{Q}_x^*$, hence $\bar{\tau} = 0$. So $\Omega_x^*/\mathscr{Q}_x^* = (\Omega^*/\mathscr{Q}^*)_x$ is torsion-free. As x was arbitrary, it follows that the sheaf $\Omega^*/\mathscr{Q}^*$ is torsion-free, so $\mathscr{Q}^*$ is a saturated subsheaf of $\Omega^* = \mathcal{T}$. Hence the map F is well-defined.

Now suppose $\mathscr{K}, \mathscr{K}'$ are two saturated subsheaves of Ω, with quotient sheaves $\mathscr{Q}, \mathscr{Q}'$, and such that $F(\mathscr{K}) = F(\mathscr{K}')$ (that is, $\mathscr{Q}^* = \mathscr{Q}'^*$). As $\mathscr{K}, \mathscr{K}'$ are saturated subsheaves of Ω, they are normal, as Ω is reflexive (Lemma 4.13). Outside of an analytic set $A \subset X$, $\mathscr{Q}$ and $\mathscr{Q}'$ are both locally free, so reflexive. By the torsion-free property, A has codimension at least 2. We then have, restricting to $X \setminus A$,

$$\mathscr{Q}^* = \mathscr{Q}'^* \Rightarrow \mathscr{Q}^{**} = \mathscr{Q}'^{**} \Rightarrow \mathscr{Q} = \mathscr{Q}' \Rightarrow \mathscr{K} = \mathscr{K}'.$$

Now for $U \subset X$ any open set, this, along with the normality of $\mathscr{K}$ and $\mathscr{K}'$, gives us

$$\mathscr{K}(U) \cong \mathscr{K}(U \setminus A) = \mathscr{K}'(U \setminus A) \cong \mathscr{K}'(U).$$

So $\mathscr{K} \cong \mathscr{K}'$ on all X, hence the map F is injective.

We now define a map

$$G : \{\text{saturated subsheaves of } \mathcal{T}\} \to \{\text{saturated subsheaves of } \Omega\},$$

$G(\mathcal{F}) = (\mathcal{T}/\mathcal{F})^*$.

By the same argument, G is also well-defined and injective.

Let $\mathscr{K}$ be a saturated subsheaf of Ω, and $\mathscr{Q}$ the quotient sheaf. Fix a point $x \in X$. $\mathscr{Q}$ is a torsion-free coherent sheaf, and so $\mathscr{Q}_x$ is a finitely generated torsion-free module over an integral domain by Corollary 4.16, and is therefore torsionless by Proposition 1.37. We can then apply Lemma 1.41 with $M = \Omega_x$ and $N = \mathscr{Q}_x$ to see that for $\omega \in \Omega_x$, $\omega \in \mathscr{K}_x$ if and only if $\tau(\omega) = 0$, for all $\tau \in \mathscr{Q}_x^*$.

Applying Lemma 1.40 with $M = \mathcal{T}_x$ and $N = (\mathcal{T}/\mathscr{Q}^*)_x$, we have that $\omega \in (\mathcal{T}/\mathscr{Q}^*)_x^*$ if and only if $\tau(\omega) = 0$, for all $\tau \in \mathscr{Q}_x^*$.

As x was arbitrary, we have that $\mathscr{K} = (\mathcal{T}/\mathscr{Q}^*)^* = G(F(\mathscr{K}))$, and so $G \circ F$ is the identity map. It follows from set-theoretic arguments that the other composition is also the identity, so we have the one-to-one correspondence required. □

We can now redefine a foliation in terms of differential forms: If $\mathcal{F} \subset \mathcal{T}_X$ is a foliation, we can take the kernel of the quotient sheaf of Ω_X^1 given by the theorem, and define that also as a foliation. It remains to give an integrability condition in this case.

Definition 8.10. Let $\mathcal{G} \subset \Omega_X^1$ be a saturated subsheaf. On some open subset $U \subset X$, this is generated by 1-forms $\omega_1, \dots, \omega_r$, which generate an ideal in $\Omega(U)$, the ring of all differential forms. $\mathcal{G}$ is said to be integrable if this ideal is closed under the exterior derivative.

Lemma 8.11. *Let $\mathcal{G} \subset \Omega_X^1$ be a saturated subsheaf. Then $\mathcal{G}$ is integrable if and only if the subsheaf $\mathcal{F} \subset \mathcal{T}_X$ it corresponds to by Theorem* 8.9 *is integrable.*

Proof: Take an open $U \subset X$. As $\mathcal{G}$ is normal, we can restrict to the case where $\mathcal{G}|_U$ is locally free. $\mathcal{G}(U)$ is the annihilator of $\mathcal{F}(U)$ by Lemmas 1.39, 1.40, and 1.41, so the result holds by [29, Proposition 7.14]. □

Thus any foliation $\mathcal{F}$ is defined uniquely by a saturated subsheaf of the sheaf of 1-forms Ω, and so the foliation is locally defined as the integrable distribution of vector fields annihilated by a collection of 1-forms. A foliation can be given equivalently in either the vector field or 1-form presentations, as convenient.

Furthermore, by Lemma 7.10 and Proposition 7.18 $\mathcal{F}$ corresponds uniquely to an irreducible linear subspace $L_{\mathcal{F}} \subset TX$, which has full support. If $\mathcal{F}$ is given by 1-forms

$$\omega_l = \sum b_{il}(x_1, \dots, x_n) dx_i,$$

then $L_{\mathcal{F}}$ is given by the equations

$$\sum b_{il}(f_1(0), \ldots, f_n(0)) f_i'(0),$$

where f_i are the components of the germs of holomorphic maps $f : \mathbb{C} \to X$ which define the tangent vectors.

Outside the singular locus $\operatorname{Sing} \mathcal{F}$ of the foliation, the associated linear space is a smooth distribution, so by Frobenius' theorem we can write $X \setminus \operatorname{Sing} \mathcal{F}$ as a disjoint union of locally parallel submanifolds; these are called the leaves of the foliation. The fibres of the linear space are the tangent spaces of the leaves.

8.3 Codimension-1 Foliations

We briefly restrict to the case of codimension-1 foliations, where we have the following results.

Lemma 8.12. *Let $\mathcal{G} \subset \Omega^1_X$ be a saturated subsheaf given locally by ω. Then $\mathcal{G}$ is integrable if and only if $\omega \wedge d\omega = 0$.*

Proof: By normality of $\mathcal{G}$ we can choose an open set U such that ω does not vanish on U; we extend to a basis $\omega, dx_2, \ldots, dx_n$ of $\Omega^1(U)$.

If $\mathcal{G}$ is integrable, then we have $d\omega = \omega \wedge \alpha$ for some 1-form α. Hence $d\omega \wedge \omega = 0$ by skew-symmetry of the wedge product.

Conversely, suppose $d\omega \wedge \omega = 0$. We have as a basis for $\Omega^2(U)$

$$\{\omega \wedge dx_i\} \cup \{dx_j \wedge dx_k\}.$$

We write

$$d\omega = \sum b_i \omega \wedge dx_i + \sum c_{jk} dx_j \wedge dx_k.$$

As $d\omega \wedge \omega = 0$, we have

$$\sum c_{jk} dx_j \wedge dx_k \wedge \omega = 0,$$

and thus $c_{jk} = 0$. We therefore have

$$d\omega = \omega \wedge (\sum b_i dx_i).$$

□

Remark 8.13. In the literature which is concerned exclusively with codimension-1 foliations, this characterisation is often given as the definition of integrability.

Lemma 8.14. *Let $\mathcal{G} \subset \Omega^1_X$ be an integrable subsheaf given locally by $\omega = b_1 dx_1 + \cdots + b_n dx_n$, where the b_i are holomorphic. Then $\mathcal{G}$ is saturated if and only if* $\gcd(b_1, \ldots, b_n) = 1$.

Proof: Suppose $\gcd(b_1, \ldots, b_n) = f \neq 1$. Then we can write $\omega = f\omega'$, where ω' is not a section of $\mathcal{G}$. Hence $\mathcal{G}$ is unsaturated by definition.

Now suppose $\mathcal{G}$ is unsaturated. Then there exists a 1-form η, which is not a multiple of ω, and holomorphic functions f, g such that $f\omega = g\eta$. Then g divides fb_i, for each $1 \leq i \leq n$.

Suppose $\gcd(b_1, \ldots, b_n) = 1$. Then g divides $\gcd(fb_1, \ldots, fb_n) = f$, that is, there exists a holomorphic function h such that $f = gh$. Then $\eta = h\omega$, a contradiction. □

Corollary 8.15. *Let $\mathcal{G} \subset \Omega^1_X$ be a sheaf given locally by $\omega = b_1 dx_1 + \cdots + b_n dx_n$, where the b_i are holomorphic. Then $\mathcal{G}$ is integrable if and only if its saturation is.*

Proof: Let $f = gcd(b_1, \ldots, b_n)$. Then $\omega = f\omega'$, where ω' generates a saturated subsheaf. Now

$$\omega \wedge d\omega = (f\omega') \wedge d(f\omega') = (f\omega') \wedge df \wedge \omega' + (f\omega') \wedge (fd\omega') = f^2\omega' \wedge d\omega';$$

as f is holomorphic and non-zero, this implies that $\omega' \wedge d\omega' = 0$ if and only if $\omega \wedge d\omega = 0$. The result follows. □

Lemma 8.16. *Let $\mathcal{G} \subset \Omega^1_X$ be a foliation given locally by $\omega = b_1 dx_1 + \cdots + b_n dx_n$, where the b_i are holomorphic. Then* $\operatorname{Sing} \mathcal{G} = \mathbb{V}(b_1, \ldots, b_n)$.

Proof: Let $x \in X \setminus \mathbb{V}(b_1, \ldots, b_n)$. Then ω does not vanish at x, so by continuity does not vanish in an open neighbourhood U of x. In this neighbourhood ω defines a line bundle, and so U is contained in the smooth locus of the foliation.

Conversely, suppose $x \in \mathbb{V}(b_1, \ldots, b_n)$. Let $L_{\mathcal{F}}$ be the linear space associated to the foliation. Then the fibre $(L_{\mathcal{F}})_x = \mathbb{A}^n$, which is of higher dimension than the general fibre. So $x \in \operatorname{Sing} \mathcal{F}$. □

Corollary 8.17. *Let $\mathcal{G} \subset \Omega^1_X$ be an integrable subsheaf given locally by $\omega = b_1 dx_1 + \cdots + b_n dx_n$, where the b_i are holomorphic. Then* $\operatorname{Sing}\mathcal{G}$ *has codimension 2 in X if and only if $\mathcal{G}$ is saturated.*

Proof: If $\mathcal{G}$ is saturated, then the singular locus has codimension 2 by standard results. If not, then $f = \gcd(b_1, \ldots, b_n) \neq 1$, and $\mathbb{V}(f)$ is a component of the singular locus of codimension 1. □

8.4 Pullback Foliations

Proposition 8.18. *Let $f : X \to Y$ be a holomorphic map of complex manifolds. Then a foliation $\mathcal{F}$ on Y pulls back to a (possibly unsaturated) foliation on X.*

Proof: Suppose on some open set $V \subset Y$ the foliation is given by 1-forms $\omega_1, \ldots, \omega_r$. These pull back to 1-forms $f^*\omega_1, \ldots, f^*\omega_r$ on $f^{-1}(V)$. It remains to show that these generate an ideal closed under the exterior derivative.

Indeed,

$$df^*\omega_j = f^*(d\omega_j) = f^*(\sum \omega_i \wedge \eta_i) = \sum f^*(\omega_i \wedge \eta_i) = \sum (f^*\omega_i) \wedge (f^*\eta_i),$$

proving the result. □

We call this foliation $f^{-1}(\mathcal{F})$.

Definition 8.19. Let $\mathcal{F}$ be a foliation on X, and $V \subset X$ a reduced, irreducible complex subspace. The *restriction* $\mathcal{F}|_V$ of $\mathcal{F}$ to V is the pullback of $\mathcal{F}$ along the inclusion map $\iota : V \to X$.

Definition 8.20. Let X and Y be manifolds, and $\mathcal{F}$ a foliation on X. Let $\mathcal{G}$ be the pullback of $\mathcal{F}$ along the projection $X \times Y \to X$. Then $\mathcal{G}$ is called the *cylinder* over $\mathcal{F}$.

Example 8.21. Let $X = \mathbb{A}^n$, and $\mathcal{F}$ be given by the form $\omega = f_1 dx_1 + \cdots + f_k dx_k, k < n$, where the f_i are functions of $x_1, \ldots, x_k$ only. Then $\mathcal{F}$ is the cylinder over the foliation of $\mathbb{A}^k$ given by ω.

8.5 Normal Forms

Let $\mathcal{F}$ be a foliation on a smooth surface X with an isolated singularity at a point P. We can choose a system of holomorphic co-ordinates on X such that P is the origin of the co-ordinate chart. In a neighbourhood of the origin, the foliation is descibed by a single vector field $\mathcal{X}$, which is unique up to a factor. We can write $\mathcal{X}$ in the form

$$\mathcal{X} = (ax + by + \cdots)\frac{\partial}{\partial x} + (cx + dy + \cdots)\frac{\partial}{\partial y},$$

where $a, b, c, d \in \mathbb{C}$. The linear part of $\mathcal{X}$ is the vector field $(ax + by)\frac{\partial}{\partial x} + (cx + dy)\frac{\partial}{\partial y}$.

Let λ_1, λ_2 be the eigenvalues of the linear part of $\mathcal{X}$, that is, the eigenvalues of the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Depending on these eigenvalues, we can take a system of formal co-ordinates at the origin so that the 1-form ω generating $\mathcal{F}$ can be written in a standard form, called a *formal normal form.* As we now consider a formal neighbourhood, we may assume that $X = \mathbb{A}^2$.

We first define the following:

Definition 8.22. A ordered tuple of complex numbers $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ is called *resonant* if there is a tuple of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $|\alpha| = \sum \alpha_i \geq 2$, such that for some $j \in \{1, \ldots, n\}$, $\lambda_j = \sum_{i=1}^n \alpha_i \lambda_i$.

We call λ a *resonant tuple*; $|\alpha|$ is called the *order* of the resonance.

Lemma 8.23. *Let $p, q \in \mathbb{C}^*$, such that $\frac{p}{q}, \frac{q}{p} \in \mathbb{Q}^+ \setminus \mathbb{N}$. Then the pair (p, q) is non-resonant.*

Proof: Suppose there exists $\alpha, \beta \in \mathbb{N}_0$ such that $\alpha p + \beta q = p$. Then $(\alpha - 1)p + \beta q = 0$. If $\beta = 0, \alpha = 1$, and $(1, 0)$ is not a resonant pair.

If $\beta \neq 0$, then $\frac{\alpha - 1}{\beta} = -\frac{q}{p} < 0$, and hence $\alpha = 0$. Therefore $\beta = \frac{p}{q} \notin \mathbb{Z}$, a contradiction. The other resonance relation fails to hold by a similar argument. $\square$

Definition 8.24. A *formal vector field* is a derivation on $\mathbb{C}[[x_1, \ldots, x_n]]$. It can equivalently be thought of as a tuple in $\mathbb{C}[[x_1, \ldots, x_n]]^n$.

Remark 8.25. All holomorphic vector fields are formal vector fields.

Definition 8.26. Two formal vector fields F, F' are said to be *formally equivalent* if there exists an algebra isomorphism $H : \mathbb{C}[[x_1, \ldots, x_n]] \to \mathbb{C}[[x_1, \ldots, x_n]]$ such that $H_*(x)F(x) = F'(H(x))$, for all $x \in \mathbb{C}^n$, where H_* is the Jacobian matrix $(\frac{\partial h_i}{\partial x_j})$.

Proposition 8.27 (Poincaré Linearisation Theorem)**.** *Let $\mathcal{X}$ be a vector field defining a foliation on the plane. (We can assume without loss of generality that there is an isoated singularity at the origin). If the eigenvalues of the linear part form a non-resonant pair, then $\mathcal{X}$ is formally equivalent to its linear part. In particular, the formal normal form of the corresponding* 1*-form ω has linear co-efficients.*

Proof: See [19, Sections 4B-4C]. □

Proposition 8.28. *[19, Theorem 4.23] Let $\mathcal{X}$ be a vector field defining a foliation on the plane, whose linear part is non-zero but has two zero eigenvalues. Then the formal normal form of ω is $\omega = ydy-(p(x)+yq(x))dx, \operatorname{ord} p \geq 2, \operatorname{ord} q \geq 1$.*

Proposition 8.29. *Let $\mathcal{X}$ be a vector field defining a foliation on the plane with a resonance among its eigenvalues, at least one of which is non-zero. Let λ be the ratio of the eigenvalues. Then we have the following formal normal forms:*

$$\begin{aligned}
&\lambda = 0: && \omega = (x(1+\nu y^l))dy - y^{l+1}dx, \nu \in \mathbb{C}, l \in \mathbb{N}.\\
&\lambda = -\frac{p}{q} \in \mathbb{Q}^- : && \omega = -\lambda y(1+g_2(x^p y^q))dx + x(1+g_1(x^p y^q))dy, g_i \in \hat{\mathfrak{m}}_1.\\
&\lambda = r \in \mathbb{N}, r \geq 2: && \omega = ydx - (rx + ay^r)dy, a \in \mathbb{C}.
\end{aligned}$$

Proof: See [19, Proposition 4.29] and [3, Section 1.1]. □

9 Jet Spaces of Foliations

Throughout, we let X be a complex manifold.

9.1 Basic Definitions

Definition 9.1. Let $\mathcal{F}$ be a foliation on X, given by 1-forms $\omega_1, \ldots, \omega_r$. The *jet space* of $\mathcal{F}$ is defined as

$$J_m(\mathcal{F}) = \{\tau \in J_m(X) \mid \tau^*(\omega_l) = 0, 1 \leq l \leq r\}.$$

If on some open subset of X we have local co-ordinates $x_1, \ldots, x_n$, then, following the construction of the jet space of an affine scheme (see Proposition 6.3), we define a morphism $\tau : \mathrm{Spec}(\mathbb{C}[t]/(t^{m+1})) \to X$ by setting $x_i = \sum_{j=0}^m a_{ij}t^j$, and by pullback we have the differential dx_i mapped to $\sum_{j=1}^m j a_{ij} t^{j-1} dt$. The jet space $J_m(\mathcal{F})$ is again defined as the vanishing locus of the polynomial constraints imposed on the a_{ij} to ensure that $\tau^*(\omega) \in (t^m)dt$ for each of the 1-forms ω defining the foliation, noting that as $t^{m+1} = 0$, we have $t^m dt = 0$. We thus see that $J_m(\mathcal{F})$ is a subscheme of $J_m(X)$.

As $J_m(\mathcal{F}) \subset J_m(X)$, we can look at its fibres $J_m(\mathcal{F}, x) = J_m(X, x) \cap J_m(\mathcal{F})$, again constructed in the same way as with jet spaces of schemes.

As in the case of schemes, we can define the jet spaces of foliations by formal derivatives: if $\mathcal{F}$ is a foliation given by 1-forms $\omega_l = \sum b_{il} dx_i, 1 \leq l \leq r$, then, identifying dx_i with x_i', these correspond to forms $\sum b_{il} D x_i$ on the tangent bundle, where D is the derivation in Section 6.4; we also call these ω_l. Then $J_m(\mathcal{F})$ is defined by the vanishing of $\omega_l, D\omega_l, \ldots, D^{m-1}\omega_l, 1 \leq l \leq r$.

Proposition 9.2 (Change of Variables for Foliations)**.** *Let $f : X \to Y$ be a map of complex manifolds, and $\mathcal{F}$ a foliation on Y. Then $J_m(f^{-1}(\mathcal{F})) = f_m^{-1}(J_m(\mathcal{F}))$.*

Proof: Let $\tau : \mathrm{Spec}(\mathbb{C}[t]/(t^{m+1})) \to X \in J_m(f^{-1}(\mathcal{F}))$. Then $\tau^*(f^*\omega_l) = 0$ for each of the ω_l generating $\mathcal{F}$. Then $(f \circ \tau)^*(\omega_l) = 0$, and so $f_m(\tau) \in J_m(\mathcal{F})$. Hence $\tau \in f_m^{-1}(J_m(\mathcal{F}))$.

Conversely, let $\tau \in f_m^{-1}(J_m(\mathcal{F}))$. Then $f \circ \tau \in J_m(\mathcal{F})$, and so $\tau^*(f^*\omega_l) = (f \circ \tau)^*(\omega_l) = 0$, and hence $\tau \in J_m(f^{-1}(\mathcal{F}))$. □

In the case that f is smooth, the above result can be reproved using the chain rule, using the same arguments as in Section 6.4.

Corollary 9.3. *The fibres of the jet spaces of foliations, and the morphisms between them induced by (formal) isomorphisms of the ambient space, are algebraic. In particular, formally equivalent vector fields yield isomorphic jets.*

Proof: If $\mathcal{F}$ is a foliation on X, then $J_m(\mathcal{F}, x) \subset J_m(X, x)$. The result then follows using the same arguments as in Propositions 6.26 and 6.31. □

Lemma 9.4. *Let $\mathcal{F}$ be a foliation given by 1-forms $\omega_1, \ldots, \omega_r$, each of which has an algebraic first integral: that is, for each l, $\omega_l = dg_l$ for some polynomial g_l. Then for any point $x \in \mathbb{V}(g_1, \ldots, g_r)$, $J_m(\mathcal{F}, x) = J_m(\mathbb{V}(g_1, \ldots, g_r), x)$.*

Proof: $J_m(\mathbb{V}(\{g_l\}_l)) = \mathbb{V}(\{g_l, Dg_l, \ldots, D^m g_l\}_l)$, and

$$J_m(\mathcal{F}) = \mathbb{V}(\{Dg_l, \ldots, D^m g_l\}_l),$$

since $\omega_l = dg_l$ and therefore corresponds to Dg_l. Cutting out the equation for g_l in the jet scheme of the foliation gives the result. □

Proposition 9.5. *Let $\mathcal{F}$ be a foliation on X, and V be a reduced, irreducible complex subspace. Set $\mathcal{G} = \mathcal{F}|_V$. Then $J_m(\mathcal{G}) = J_m(\mathcal{F}) \cap J_m(V)$, for all m.*

Proof: By definition of $\mathcal{G}$, and the change of variables formula for foliations, we have

$$J_m(\mathcal{G}) = \iota_m^{-1}(J_m(\mathcal{F})) \subset J_m(\mathcal{F}),$$

where $\iota : V \to X$ is the inclusion; hence $J_m(\mathcal{G}) \subset J_m(\mathcal{F}) \cap J_m(V)$.

Conversely, let $\tau : \mathrm{Spec}(\mathbb{C}[t]/(t^{m+1})) \to V$ be a jet in $J_m(\mathcal{F}) \cap J_m(V)$. Then $\iota \circ \tau$ is a defined composition of maps. Suppose $\mathcal{F}$ is defined by the 1-forms $\omega_1, \ldots, \omega_r$. Then $\mathcal{G}$ is defined by $\iota^*\omega_1, \ldots, \iota^*\omega_r$. Then, for each $1 \le l \le r$,

$$\tau^*(\iota^*\omega_l) = (\iota \circ \tau)^*(\omega_l) = \tau^*(\omega_l) = 0.$$

So $\tau \in J_m(\mathcal{G})$. □

9.2 Motivating Examples

We now let $X = \mathbb{A}^2$, and consider foliations $\mathcal{F}$ given by a 1-form ω and having a single singular point at the origin. To calculate the m-jets above the origin (the singular point), we set

$$x = a_1t + a_2t^2 + \cdots + a_mt^m; y = b_1t + b_2t^2 + \cdots + b_mt^m,$$

and so

$$dx = (a_1 + 2a_2t + \cdots + ma_mt^{m-1})dt; dy = (b_1 + 2b_2t + \cdots + mb_mt^{m-1})dt.$$

By equating co-efficients, we find the a_i, b_i such that the image of ω under this morphism lies in $(t^m)dt$.

Example 9.6. The foliation $\mathcal{F}_1$, given by $\omega_1 = ydx+xdy$, has a first integral xy. Let $C = \mathbb{V}(xy)$. Then by Lemma 9.4, for each m, $J_m(\mathcal{F}_1, 0) = J_m(C, 0)$. These jets were calculated in Example 6.7.

The foliation $\mathcal{F}_2$ given by $\omega_2 = ydx - x^2dy$ has the same jets above the origin, so the jets are not sufficient to determine the foliation.

Example 9.7. Let $\mathcal{F}$ be the foliation given by $\omega = ydx - (x + y)dy$. Computing the jet spaces, we have that $J_m(\mathcal{F}, 0)$ is given as the union of sets

$$b_1 = b_2 = \cdots = b_{m-1} = 0,$$
$$b_1 = b_2 = \cdots = b_{m-2} = a_1 = 0, \ldots,$$
$$b_1 = \cdots = b_{\frac{m-1}{2}} = a_1 = \cdots = a_{\frac{m-1}{2}} = 0,$$

if m is odd, and by

$$b_1 = b_2 = \cdots = b_{m-1} = 0,$$
$$b_1 = b_2 = \cdots = b_{m-2} = a_1 = 0, \ldots,$$
$$b_1 = \cdots = b_{\frac{m}{2}} = a_1 = \cdots = a_{\frac{m-2}{2}} = 0,$$

if m is even.

Indeed, from the equations $J_2(\mathcal{F}, 0) = \mathbb{V}(b_1), J_3(\mathcal{F}, 0) = \mathbb{V}(b_1, a_2b_1 - a_1b_2 - 3b_1b_2)$, so the assertion holds here. If m is odd, the extra equation to define the $(m+1)$-jets is

$$(m-1)(a_m b_1 - a_1 b_m) + \cdots + 2(a_{\frac{m+3}{2}} b_{\frac{m-1}{2}} - \\ a_{\frac{m-1}{2}} b_{\frac{m+3}{2}}) - (m+1)(b_1 b_m + \cdots + b_{\frac{m+1}{2}} b_{\frac{m+3}{2}}) - \frac{m+1}{2} b^2_{\frac{m+1}{2}}.$$

The string of equalites $b_1 = \cdots = b_j = a_1 = \cdots = a_k = 0, k = m - 1 - j$ for a component of the lower order jets (henceforth denoted by $\{j,k\}$), for $j \leq \frac{m-3}{2}$, causes all but the term $a_{k+1}b_{j+1}$ to vanish—so the string of equalities can be extended to either $\{j+1,k\}$ or $\{j,k+1\}$. The string $\{\frac{m-1}{2}, \frac{m-1}{2}\}$ causes all but the term $b^2_{\frac{m+1}{2}}$ to vanish, so it can be extended to $\{\frac{m+1}{2}, \frac{m-1}{2}\}$. Replacing m by $m+1$ gives the equations for an even order jet space.

If m is even, the extra equation is

$$(m-1)(a_m b_1 - a_1 b_m) + \cdots + a_{\frac{m+2}{2}} b_{\frac{m}{2}} - \\ a_{\frac{m}{2}} b_{\frac{m+2}{2}} - (m+1)(b_1 b_m + \cdots + b_{\frac{m}{2}} b_{\frac{m+2}{2}}).$$

The string of equalities $\{j,k\}$ causes all but the term $a_{k+1}b_{j+1}$ to vanish, so it can be extended to either $\{j+1,k\}$ or $\{j,k+1\}$. Replacing m by $m+1$ gives the equations for an odd order jet space. The whole result follows by induction.

Let $\tilde{C} = \mathbb{V}(y^2)$. By simple calculation, we see that $J_m(\tilde{C},0) = \mathbb{V}(b_1,\ldots,b_{\frac{m-1}{2}})$, if m is odd, and $J_m(\tilde{C},0) = \mathbb{V}(b_1,\ldots,b_{\frac{m}{2}})$, if m is even. Comparing with the jets calculated in Example 6.7, we have

$$J_m(\mathcal{F},0) = J_m(\mathbb{V}(y^2),0) \cap J_m(\mathbb{V}(xy),0) = J_m(\mathbb{V}(y^2,xy),0)$$

—the jets of the x-axis with a double point at the origin.

Lemma 9.8. *Consider the affine space $\mathbb{A}^{2n}$, with co-ordinates $a_1,\ldots,a_n$, $b_1,\ldots,b_n$. Denote by $[i,j]$ the class of polynomials $\mu(a_i b_j - a_j b_i)$, for some $\mu \in \mathbb{C}^*$. Then if a_i and b_i are not both zero, $[i,j] = 0, [i,k] = 0 \Rightarrow [j,k] = 0$, $1 \leq i,j,k \leq n$.*

Proof: We have the relations $a_i b_j - a_j b_i = 0, a_i b_k - a_k b_i = 0$. If $a_i = 0$, then we have $a_j = a_k = 0$, and the result holds. Similarly the result holds if $b_i = 0$. We then assume $a_i, b_i \neq 0$. Then $a_j = 0$ if and only if $b_j = 0$, and

$a_k = 0$ if and only if $b_k = 0$; in either case the result holds.

Now suppose none of the terms equals zero. We have $\frac{b_j}{b_i} = \frac{a_j}{a_i}, \frac{b_k}{b_i} = \frac{a_k}{a_i}$. Rearranging we get $\frac{b_k}{b_j} = \frac{a_k}{a_j}$, which implies that $a_j b_k - a_k b_j = 0$. $\square$

Example 9.9. Let $\mathcal{F}$ be the foliation given by $\omega = ydx - xdy$. This has jet spaces $J_m(\mathcal{F}, 0)$ given as the union of the sets

$$[1,2] = [1,3] = \cdots = [1, m-1] = 0, a_1 = b_1 = [2,3] = \cdots = [2, m-2] = 0,$$
$$\ldots, a_1 = \cdots = a_{\frac{m-3}{2}} = b_1 = \cdots = b_{\frac{m-3}{2}} = \left[\frac{m-1}{2}, \frac{m+1}{2}\right] = 0,$$

if m is odd, and by

$$[1,2] = [1,3] = \cdots = [1, m-1] = 0,$$
$$a_1 = b_1 = [2,3] = \cdots = [2, m-2] = 0, \ldots,$$
$$a_1 = \cdots = a_{\frac{m-4}{2}} = b_1 = \cdots = b_{\frac{m-4}{2}} = \left[\frac{m-2}{2}, \frac{m}{2}\right] = \left[\frac{m-2}{2}, \frac{m+2}{2}\right] = 0,$$

if m is even and $m \geq 4$. $J_2(\mathcal{F}, 0) = \mathbb{A}^4$.

Indeed, we see from the equations that $J_3(\mathcal{F}, 0) = \mathbb{V}([1,2]), J_4(\mathcal{F}, 0) = \mathbb{V}([1,2],[1,3])$. If m is odd, the extra equation to define the $(m+1)$-jets is

$$[1, m] + [2, m-1] + \cdots + \left[\frac{m-1}{2}, \frac{m+3}{2}\right];$$

by the previous proposition (assuming a_i, b_i are non-zero unless stated otherwise), the string of equalities for each component is appended by $[j, m+1-j] = 0, 1 \leq j \leq \frac{m-1}{2}$. Replacing m by $m+1$ gives the equation for an even order jet space.

If m even, the extra equation is

$$[1, m] + [2, m-1] + \cdots + \left[\frac{m}{2}, \frac{m+2}{2}\right];$$

again we append the equation $[j, m+1-j] = 0$ to the strings of equalities, $1 \leq j \leq \frac{m-2}{2}$, and we get the extra equation $a_1 = \cdots = a_{\frac{m-2}{2}} = b_1 = \cdots = b_{\frac{m-2}{2}} = [\frac{m}{2}, \frac{m+2}{2}] = 0$. Replacing m by $m+1$ gives the equation for an odd order jet space, and so we have the whole result by induction.

We have $J_m(\mathcal{F}, 0) = \bigcup_{\lambda \in \mathbb{C}} J_m(\mathbb{V}(x(y - \lambda x), 0)$. So there are foliations where a whole family of subschemes are needed to describe the jets at the singularity.

9.3 Tangent Schemes

Definition 9.10. Let $\mathcal{F}$ be a foliation on a complex manifold X, and $C \subset X$ a complex subspace.

C is *weakly tangent* to $\mathcal{F}$ if $J_1(C) \subset J_1(\mathcal{F})$.

C is *strongly tangent* to $\mathcal{F}$ if $J_m(C) \subset J_m(\mathcal{F})$, for all $m \in \mathbb{N}$.

C is *fully tangent* to $\mathcal{F}$ if $J_m(C) = J_m(\mathcal{F})|_C$, for all $m \in \mathbb{N}$.

A complex subspace $C \subset X$ is a *solution* for the foliation $\mathcal{F}$ if locally its defining equations $g_1, \ldots, g_k$ solve all the 1-forms $\omega_1, \ldots, \omega_r$ that determine $\mathcal{F}$; that is, $\omega_l|_C \in (dg_1, \ldots, dg_k)|_C \subset \Omega^1_X|_C$ for each l.

If $\mathcal{F}$ is a codimension-1 foliation given by ω, and $C = \mathbb{V}(f)$ is a hypersurface, C is an *integral hypersurface* for $\mathcal{F}$ if $\omega \wedge df = f\eta$, for some holomorphic 2-form η.

Remark 9.11. The notion of C being weakly tangent is the simplest conception of tangency: It is simply that the tangent vectors of C are tangent to the foliation. This, along with notions of a solution or an integral hypersurface, is classical; in the literature one of these three will be given a the definition for tangency to a foliation. (We note that the definition of a solution given above is more general than that found elsewhere, where a solution is generally required to be a disjoint union of finitely many leaves.)

However, these notions fail to encapsulate the behaviour of the foliation at the the singular locus. To rectify this, we introduce the notion of being strongly tangent, or indeed fully tangent. This allows us to consider the tangency of subspaces with non-reduced structure. We discuss the relations between these notions in the following paragraphs, showing that, when C is reduced, they co-incide over the smooth locus of the foliation.

Remark 9.12. We can also define all these notions of tangency if C is instead a formal subscheme of X. (A formal scheme $C = \varinjlim Y_\lambda$ can be viewed as a subscheme of a complex manifold X if we identify each Y_λ with its associated complex space, and take the formal direct limit of complex subspaces of X). For a formal scheme to be a solution or integral hypersurface, we assume it is given by some collection of formal power series.

Example 9.13. Let $X = \mathbb{A}^2$, $\omega = ydx + xdy$. The leaves of the resultant foliation are of the form $xy = a$; these are clearly solutions, and integral hypersurfaces, for the foliation. All points of the plane are also strongly tangent.

Tangent schemes need not be reduced: let C be the origin counted with multiplicity 2—this is defined by the equations x^2, xy, y^2. As $d(xy) = ydx + xdy$, we see that C is indeed a solution of the foliation.

However the origin counted with multiplicity 3 is not: This is defined by x^3, x^2y, xy^2, y^3, and so the space of 2-jets is all of $\mathbb{A}^4$, which is not contained in $J_2(\mathcal{F}, 0)$, as calculated in Example 9.6.

It follows from the definitions that all notions of tangency (as described in Definition 9.10) are locally defined. Furthermore, if $C \subset X$ is weakly tangent (respectively, strongly tangent, respectively, a solution), then any subspace of C is also weakly tangent (respectively, strongly tangent, respectively, a solution). As the definitions of a solution or integral hypersurface are given with reference to the defining equations, we see that these properties are preserved by taking closures.

Proposition 9.14. *All notions of tangency are invariant under co-ordinate changes.*

Proof: Let $\mathcal{F}$ be a foliation on X, let $\pi : X' \to X$ be a surjective map, with X' smooth, and suppose $C \subset X$ is strongly tangent to $\mathcal{F}$. So $J_m(C) \subset J_m(\mathcal{F})$, for all $m \in \mathbb{N}$. Then by Propositions 6.20 and 9.2,

$$J_m(\pi^{-1}(C)) = \pi_m^{-1}(J_m(C)) \subset \pi_m^{-1}(J_m(\mathcal{F})) = J_m(\pi^{-1}(\mathcal{F})),$$

for all $m \in \mathbb{N}$. So $\pi^{-1}(C)$ is strongly tangent to $\pi^{-1}(\mathcal{F})$. A similar argument holds if C is weakly tangent or fully tangent.

Now suppose $C = \mathbb{V}(g_1, \ldots, g_k)$ is a solution for $\mathcal{F}$. For each ω_l defining $\mathcal{F}$, we have

$$\pi^*\omega_l|_{\pi^{-1}(C)} = \pi^*(\omega_l|_C) = \pi^*((f_i dg_1 + \cdots + f_k dg_k)|_C),$$

where the f_i are holomorphic. This in turn equals

$$(\pi^* f_1 \pi^* dg_1 + \cdots + \pi^* f_k \pi^* dg_k)|_{\pi^{-1}(C)} \in (d(g_1 \circ \pi), \ldots, d(g_k \circ \pi))|_{\pi^{-1}(C)}.$$

So $\pi^{-1}(C)$ is a solution for $\pi^{-1}(\mathcal{F})$.

Suppose $\mathcal{F}$ is codimension-1, and given by ω, and let $C = \mathbb{V}(f)$ be an integral hypersurface. Then $\omega \wedge df = f\eta$, for some holomorphic 2-form η. Then

$$\pi^*\omega \wedge d(f \circ \pi) = \pi^*\omega \wedge \pi^* df = \pi^*(\omega \wedge df) = \pi^*(f\eta) = (f \circ \pi)\pi^*\eta.$$

So $\pi^{-1}(C)$ is an integral hypersurface for $\pi^{-1}(\mathcal{F})$. □

Corollary 9.15. *Suppose $\pi : X' \to X$ is a proper birational morphism (for example, a sequence of blow-ups in smooth centres), and suppose, for some $C \subset X$, that $\pi^{-1}(C)$ is strongly tangent (respectively, weakly tangent, respectively, fully tangent) to $\pi^{-1}(\mathcal{F})$. Then C is strongly tangent (respectively, weakly tangent, respectively, fully tangent) to $\mathcal{F}$.*

Similarly, if $C \subset X'$ is strongly tangent (respectively, weakly tangent, respectively, fully tangent) to $\pi^{-1}(\mathcal{F})$, then $\pi(C)$ is strongly tangent (respectively, weakly tangent, respectively, fully tangent) to $\mathcal{F}$.

Proof: Suppose first that $\pi^{-1}(C)$ is strongly tangent to $\pi^{-1}(\mathcal{F})$. Then for all $m \in \mathbb{N}$, $J_m(\pi^{-1}(C)) \subset J_m(\pi^{-1}(\mathcal{F}))$. By Propositions 6.20 and 9.2 we then have $\pi_m^{-1}(J_m(C)) \subset \pi_m^{-1}(J_m(\mathcal{F}))$. By Proposition 6.18, π_m is surjective, and so $J_m(C) \subset J_m(\mathcal{F})$, for all $m \in \mathbb{N}$; that is, C is strongly tangent to $\mathcal{F}$.

In the second case, if C is strongly tangent to $\pi^{-1}(\mathcal{F})$, then for all $m \in \mathbb{N}$, $J_m(C) \subset J_m(\pi^{-1}(\mathcal{F})) = \pi_m^{-1}(J_m(\mathcal{F}))$, and hence $\pi_m(J_m(C)) \subset \pi_m(\pi_m^{-1}(J_m(\mathcal{F})))$. By Proposition 6.18, π_m is surjective, and so by Corollary 6.21 we have $J_m(\pi(C)) \subset J_m(\mathcal{F})$, for all $m \in \mathbb{N}$. So $\pi(C)$ is strongly tangent to $\mathcal{F}$.

The cases for weakly tangent and fully tangent are proved in the same way. □

Proposition 9.16. *Let $\mathcal{F}$ be a codimension-1 foliation on X. A union of integral hypersurfaces is also an integral hypersurface. The irreducible components of an integral hypersurface are integral hypersurfaces.*

Proof: Suppose $\mathcal{F}$ is given by ω, and $\mathbb{V}(f)$ and $\mathbb{V}(g)$ are integral hypersurfaces. Then $\omega \wedge df = f\eta_1, \omega \wedge dg = g\eta_2$. So

$$\omega \wedge d(fg) = \omega \wedge (gdf + fdg) = g\omega \wedge df + f\omega \wedge dg = gf\eta_1 + fg\eta_2 = fg(\eta_1 + \eta_2).$$

Therefore $\mathbb{V}(fg) = \mathbb{V}(f) \cup \mathbb{V}(g)$ is an integral hypersurface.

Conversely, if S is an integral hypersurface with multiple components, denote one of them by $\mathbb{V}(f)$, so that $S = \mathbb{V}(fg)$; we can assume f and g are coprime. Then we have $\omega \wedge (gdf + fdg) = fg\eta$, and so $g\omega \wedge df = f(g\eta - \omega \wedge dg)$. Now $g \mid (f\omega \wedge dg)$, and by coprimality of f and g, $g \mid \omega \wedge dg$. Hence $\omega \wedge df = f(\eta - \frac{1}{g}\omega \wedge dg)$, where the latter is holomorphic. So $\mathbb{V}(f)$ is an integral hypersurface. $\square$

Proposition 9.17. *The union of weakly tangent schemes to a foliation $\mathcal{F}$ is weakly tangent to $\mathcal{F}$.*

Proof: Let C_1, C_2 be two weakly tangent schemes. If they are disjoint, then clearly their union is weakly tangent. If not, we notice that for $x \in C_1 \cap C_2$, $J_1(C_1 \cup C_2, x) = J_1(C_1, x) + J_1(C_2, x)$, where the addition is addition of spaces of tangent vectors. As $J_1(C_i, x) \subset J_1(\mathcal{F}, x), i = 1, 2$, it follows that $J_1(C_1 \cup C_2, x) \subset J_1(\mathcal{F}, x)$. Hence $C_1 \cup C_2$ is weakly tangent. $\square$

Remark 9.18. Note that this does not hold for strongly tangent schemes: Let $X = \mathbb{A}^2$, $\mathcal{F}$ be given by $\omega = ydx - xdy$, $C_1 = \mathbb{V}(xy)$, and $C_2 = \mathbb{V}(y - x)$. C_1 and C_2 are both strongly tangent, but $C_1 \cup C_2$ is not: It has $x = t + t^2 + t^3, y = t + 2t^2 + t^3$ as a 3-jet over the origin, which does not lie in $J_3(\mathcal{F}, 0)$, by the calculations in Example 9.9.

We now look at equivalences between the notions of tangency. First off, clearly anything fully tangent is strongly tangent, and anything strongly tangent is weakly tangent.

Lemma 9.19. *Let $\mathcal{F}$ be a foliation on X and $C \subset X$ a reduced solution for $\mathcal{F}$. Then C is strongly tangent to $\mathcal{F}$.*

Proof: Let $g_r(x_1, \ldots, x_n), r = 1, \ldots, k$ be the defining equations of C in a local co-ordinate system, and $(a_{ij}) \in J_m(C)$, that is, (a_{ij}) is a tuple of complex numbers defining the morphism $\tau : \operatorname{Spec} \mathbb{C}[t]/(t^{m+1}) \to C \subset X$. Then $g_r \circ \tau = g_r(\sum_{j=0}^m a_{1j}t^j, \ldots, \sum_{j=0}^m a_{nj}t^j) \in (t^{m+1})$, and $\tau^* dg_r = d(g_r \circ \tau) = 0 \in \Omega_{\operatorname{Spec} \mathbb{C}[t]/(t^{m+1})}$. As C is a solution for $\mathcal{F}$, all the 1-forms ω_l determining $\mathcal{F}$ satisfy $\omega_l|_C \in (dg_1, \ldots, dg_k)|_C$, hence $\omega_l|_C$ pulls back to 0 too. As $\operatorname{Im}(\tau) \subset C$, and the inclusion $C \to X$ is a monomorphism, it follows that $\tau^* \omega_l = 0$, and hence $(a_{ij}) \in J_m(\mathcal{F})$. $\square$

Proposition 9.20. *Let $\mathcal{F}$ be a foliation on X, let $U \subset X$ be the smooth locus of the foliation, and let $C \subset X$ be a reduced subspace. Then:*

(1) $C \cap U$ is weakly tangent to $\mathcal{F}$ if and only if it is strongly tangent if and only if it is a solution if and only if it is contained in a disjoint union of finitely many leaves;

(2) If C is of leaf dimension and C is weakly tangent to $\mathcal{F}$, then $C \cap U$ is fully tangent (and a solution);

(3) If $\mathcal{F}$ is codimension 1, and C is a hypersurface, then $C \cap U$ is an integral hypersurface if and only if it is a solution.

Proof: We start by looking at the case where $\mathcal{F}$ is smooth (so $U = X$), and is given by $dx_1, \ldots, dx_k$. We note that the leaves of the foliation are reduced, irreducible solutions, and so are strongly tangent.

(1) A disjoint union of finitely many leaves is a strongly tangent solution, so if C is contained in a disjoint union of finitely many leaves, it too is a strongly tangent solution. By definition, anything strongly tangent is weakly tangent.

Suppose C is weakly tangent and irreducible. As the foliation satisfies the conditions of Lemma 9.4, we have $J_1(C, x) \subset J_1(\mathcal{F}, x) = J_1(L_x, x)$, for each $x \in C \cap U$, and where L_x is the unique leaf through x. So every tangent vector of C is a tangent vector of one of the leaves. However, as C is reduced and irreducible, these vectors must all be tangent to the same leaf, and so C is contained in one of the leaves. Therefore any weakly tangent subscheme is contained in a disjoint union of finitely many leaves.

Now suppose C is weakly tangent to $\mathcal{F}$ (but not necessarily irreducible). Let $g_1, \ldots, g_s$ be the generators of C. By definition, $J_1(C) \subset J_1(\mathcal{F})$, and so

$$(dx_1, \ldots, dx_k) \subset (g_1, \ldots, g_s, dg_1, \ldots, dg_s).$$

Restricting to C yields

$$(dx_1, \ldots, dx_k)|_C \subset (dg_1, \ldots, dg_s)|_C.$$

So C is a solution for $\mathcal{F}$.

Suppose now that C is a solution for $\mathcal{F}$. It is reduced, and so is strongly tangent by Lemma 9.19, and hence weakly tangent.

(2) If C is of leaf dimension, each component of $C \cap U$ must be equal

to one of the leaves, and so be a solution. Therefore, C itself is a solution. Further, the leaves are fully tangent, as the foliation satisfies the conditions of Lemma 9.4.

(3) Suppose we are in the codimension-1 case (so $\mathcal{F}$ is generated by dx_1), and let $\mathbb{V}(f)$ be a reduced integral hypersurface for $\mathcal{F}$. Then we have $dx_1 \wedge df = f\eta$ for some 2-form η, hence f divides $\frac{\partial f}{\partial x_i}$ for $i > 1$. Therefore, as f is represented as a power series, $\frac{\partial f}{\partial x_i} = 0, i > 1$, and so f is dependent only on x_1. Hence $\mathbb{V}(f)$ is a disjoint union of leaves, so is a solution.

By part (2), a reduced hypersurface which is a solution is a union of leaves, and hence an integral hypersurface, as each leaf is an integral hypersurface.

Now for the general case, the foliation restricted to its smooth locus is locally isomorphic to a smooth foliation in the given co-ordinates; the result follows from the local nature of the tangency definitions, and Proposition 9.14. □

Corollary 9.21. *Reduced integral hypersurfaces of codimension-*1 *foliations are weakly tangent.*

Proof: From the proposition, a reduced integral hypersurface is a solution over the smooth locus, and so is weakly tangent over the smooth locus. It is also weakly tangent over the singular locus, as for $x \in \operatorname{Sing}\mathcal{F}$, $J_1(\mathcal{F}, x) = J_1(X, x)$, so we are done. □

Non-reduced integral hypersurfaces also exist: If $\mathbb{V}(f)$ is an integral hypersurface, so is $\mathbb{V}(f^r), r \in \mathbb{N}$. (Set $f = g$ in the proof of the first part of Proposition 9.16). We can also thicken single components.

Conversely, if $\mathbb{V}(f^r)$ is an integral hypersurface, so is $\mathbb{V}(f)$:

$$\omega \wedge d(f^r) = f^r\eta \Rightarrow rf^{r-1}\omega \wedge df = f^r\eta \Rightarrow \omega \wedge df = f\frac{1}{r}\eta.$$

Corollary 9.22. *If $C \subset X$ is reduced, irreducible subspace, which is weakly tangent to $\mathcal{F}$ and of the same dimension as the leaves, then C is a solution.*

Proof: By the proposition, $C \cap U$ is a solution, where U is the smooth locus; as C is reduced and irreducible, it is the closure of $C \cap U$, and so is itself a solution. □

Proposition 9.23. *Let $\mathcal{F}$ be a foliation given by 1-forms $\omega_1, \ldots, \omega_r$, and $\mathcal{F}'$ the unsaturated foliation given by $f\omega_1, \ldots, f\omega_r$, for some holomorphic function f. If $C = \mathbb{V}(g_1, \ldots, g_k)$ is tangent to $\mathcal{F}$, for any of the notions of tangency given in Definition* 9.10*, then $C' = \mathbb{V}(fg_1, \ldots, fg_k)$ is tangent to $\mathcal{F}'$.*

Proof: If C is a solution for $\mathcal{F}$, then $\omega_l|_C \in (dg_1, \ldots, dg_k)|_C$, for all l. Hence

$$(f\omega_l)|_C \in (fdg_1, \ldots, fdg_r)|_C = (d(fg_1), \ldots, d(fg_k))|_C,$$

and $(f\omega_l)|_{\mathbb{V}(f)} = 0 \in (d(fg_1), \ldots, d(fg_k))|_{\mathbb{V}(f)}$; so C' is a solution for $\mathcal{F}'$.

In codimension 1, if $C = \mathbb{V}(g)$ is an integral hypersurface for $\mathcal{F}$, then $\omega \wedge dg = g\eta$, and hence

$$f\omega \wedge d(fg) = f\omega \wedge (gdf + fdg) = fg\omega \wedge df + f^2 g\eta = fg(\omega \wedge df + f\eta).$$

So C' is an integral hypersurface for $\mathcal{F}'$.

Now suppose C is strongly tangent to $\mathcal{F}$. We can work pointwise: We choose a point $x \in C$ —without loss of generality, we may assume $f(x) = 0$; otherwise, the local nature of the jet space construction makes the result trivial.

We work with the derivative description of the jets. By our hypothesis, we have for all m

$$\mathbb{V}(\omega_l, D\omega_l, \ldots, D^{m-1}\omega_l)|_x \supset \mathbb{V}(g_j, Dg_j, \ldots, D^m g_j)|_x = \mathbb{V}(Dg_j, \ldots, D^m g_j)|_x,$$

$1 \leq l \leq r, 1 \leq j \leq k$.

Now, by the product rule, and in analogy with the jets of $\mathbb{V}(xy)$, $J_m(\mathcal{F}', x)$ is the union of the sets cut out by the equations

$$Df = \cdots = D^{m-1}f = 0, Df = \cdots = D^{m-2}f = \omega_l = 0, \ldots,$$
$$Df = \omega_l = \cdots = D^{m-3}\omega_l = 0, \omega_l = D\omega_l = \cdots = D^{m-2}\omega_l = 0,$$

each restricted to the point x, where $1 \leq l \leq r$.

Similarly, $J_m(C', x)$ is the union of the sets cut out by the equations

$$Df = \cdots = D^{m-1}f = 0, Df = \cdots = D^{m-2}f = Dg_j = 0, \ldots,$$
$$Df = Dg_j = \cdots = D^{m-2}g_j = 0, Dg_j = \cdots = D^{m-1}g_j = 0,$$

each restricted to the point x, where $1 \le j \le k$. From the hypothesis, we see that $J_m(C', x) \subset J_m(\mathcal{F}', x)$, so we are done.

The cases for weak tangency and full tangency use the same argument.

□

Corollary 9.24. *Let $X = \varinjlim X_\lambda$ be a direct limit of formal schemes. If each X_λ is strongly tangent to a foliation $\mathcal{F}$, then X is also strongly tangent to $\mathcal{F}$.*

Proof: $J_m(X, x) = \bigcup_{\lambda \in \Lambda} J_m(X_\lambda, x) \subset J_m(\mathcal{F}, x)$, by strong tangency of the X_λ. Hence X is strongly tangent. □

9.4 Separatrices and Dicriticality

Definition 9.25. Let $\mathcal{F}$ be a singular foliation on a manifold X. A *separatrix* is a reduced, irreducible complex subspace of X, of dimension equal to that of the leaves of $\mathcal{F}$, which intersects the singular locus and is strongly tangent to the foliation.

Remark 9.26. A separatrix is in fact the closure of a leaf of the foliation that extends holomorphically through the singular locus, and so is a solution for the foliation. In the codimension 1 case, it is an integral hypersurface.

Remark 9.27. In the literature, separatrices are only defined for codimension-1 foliations, as either the holomorphic closure of a leaf through the singular locus, or as a reduced, irreducible integral hypersurface intersecting the singular locus. By Lemma 9.19 and Corollary 9.22, we can relax the requirement that a separatrix be strongly tangent to requiring only weak tangency, and so in the codimension-1 case these definitions are all equivalent.

Remark 9.28. We also allow for *formal separatrices*, which are formal subschemes of X strongly tangent to the foliation (and which are reduced, irreducible, of leaf dimension, and which intersect the singular locus).

Example 9.29. Consider the foliations on $X = \mathbb{A}^2$ given by the following 1-forms:

(1) $\omega = ydx + xdy$. This has separatrices $\{x = 0\}$ and $\{y = 0\}$.

(2) $\omega = ydx - xdy$. Here every line through the origin is a separatrix.

(3) $\omega = ydx-(x+y)dy$. The only separatrix is $\{y = 0\}$. The other leaves of the foliation cannot be extended holomorphically through the origin, so do not satisfy the definition.

(4) $\omega = (y - x)dx - x^2dy$. This has one convergent separatrix, $\{x = 0\}$. It also has a formal separatrix, given by the formal power series $y = \sum_{k=0}^{\infty} k!x^{k+1}$.

We now give some results on the existence of separatrices. We henceforth assume that X is quasi-compact, and the foliation is codimension-1, and generated by an algebraic 1-form. (So in particular the singular locus has finitely many irreducible components.)

Theorem 9.30. *[4] Suppose* $\dim X = 2$. *Then any foliation on* X *has a separatrix.*

The same is not true in higher dimensions: Let $X = \mathbb{A}^3$, and, for $m \geq 2$, let $\mathcal{F}_m$ be the foliation given by

$$\omega_m = (x^m y - z^{m+1})dx + (y^m z - x^{m+1})dy + (z^m x - y^{m+1})dz.$$

Then none of the foliations $\mathcal{F}_m$ have any separatrices at the origin. (See [20]).

To proceed, we introduce the following notion:

Definition 9.31. A codimension-1 foliation $\mathcal{F}$ on X is said to be *dicritical* if there exists a sequence of blow-ups in smooth centres, where the centre of each blow-up is contained in the singular locus, after which a component of the exceptional divisor is transversal to the transformed foliation (that is, is not an exceptional divisor).

Otherwise $\mathcal{F}$ is called *non-dicritical.*

Example 9.32. Let $\mathcal{F}$ be the foliation on $\mathbb{A}^2$ given by $\omega = ydx - xdy$. Then $\mathcal{F}$ is dicritical. Indeed, blowing up at the origin we get the form $-x^2dv$, which describes a foliation whose saturation is transversal to the exceptional divisor.

Proposition 9.33. *[8, Theorem 4] Let $\mathcal{F}$ be a germ of a codimension-*1 *foliation on $\mathbb{A}^n$. Then $\mathcal{F}$ is dicritical at* 0 *if and only if there exist a germ of an irreducible surface Z, which is not tangent to $\mathcal{F}$, and an infinite collection of germs of distinct curves $((\Gamma_i, x_i))_{i\in\mathbb{N}}$ such that:*

(a) Each Γ_i is tangent to $\mathcal{F}$, and contained in Z;

(b) Each x_i is contained in $\operatorname{Sing}\mathcal{F}$, *with* $\lim x_i = 0$*;*

(c) For each $i \in \mathbb{N}$, $(\Gamma_i, x_i) \neq (\operatorname{Sing}\mathcal{F}, x_i)$.

Remark 9.34. By taking $n = 2$, we see that a foliation on a surface is dicritical if and only if it has infinitely many separatrices. In this case, the germs of a countable subset of the separatrices can be taken as the Γ_i in the proposition.

Example 9.35. The foliations $\mathcal{F}_m$ on $\mathbb{A}^3$ defined above are dicritical. Indeed, blowing up at the origin we get the form

$$x^{m+2}((u^m v - 1)du + (v^m - u^{m+1})dv).$$

The exceptional divisor is seen to be transversal to the saturated foliation.

Theorem 9.36. *[8, Theorem 5] Any non-dicritical foliation has a separatrix.*

In [8] the theorem is given in terms of germs. However, if the germ of the foliation has a germ of a separatrix, then expanding from a formal to an open neighbourhood of the origin, we see that the hypersurface will still be a separatrix to the full foliation.

Definition 9.37. Let $\mathcal{F}$ be a codimension-1 foliation on X. A hypersurface $V \subset X$ is said to be *truly transversal* to $\mathcal{F}$ if it is smooth, reduced and irreducible; if it is not tangent to $\mathcal{F}$; and if the restriction foliation $\mathcal{F}|_V$ is saturated.

The condition that the restriction foliation is saturated means that V does not contain any codimension-2 component of the singular locus, and its intersection with any leaf has codimension at least 2. By taking hypersurfaces of sufficiently high degree, a truly transversal hypersurface can be found through every point $x \in X$.

Proposition 9.38. *A non-dicritical foliation on a quasi-compact manifold has only finitely many separatrices.*

Proof: Suppose $\mathcal{F}$ has infinitely many separatrices $(L_\alpha)_{\alpha \in A}$. We first assume the L_α pass through a point $x \in \operatorname{Sing} \mathcal{F}$.

Let $V \subset X$ be a truly transversal hypersurface passing through x. As each L_α is the closure of a leaf of $\mathcal{F}$, $L_\alpha \cap V$ is a leaf of $\mathcal{F}|_V$; moreover $(L_\alpha \cap V) \cap (\operatorname{Sing} \mathcal{F} \cap V) \neq \emptyset$, and so each $L_\alpha \cap V$ is a separatrix of $\mathcal{F}|_V$. (If any of the L_α is a formal separatrix, then $L_\alpha \cap V$ is a formal separatrix of $\mathcal{F}|_V$. So $\mathcal{F}|_V$ is a foliation with infinitely many separatrices on a manifold of lower dimension.

We apply this procedure recursively, and get a surface $W \subset X$, not tangent to $\mathcal{F}$, such that $\mathcal{F}|_W$ has infinitely many separatrices. Therefore $\mathcal{F}|_W$ is dicritical, and hence $\mathcal{F}$ is dicritical by [8, Proposition 5].

If the L_α cannot be taken to all pass through the same point, then they pass through a sequence of points in the singular locus converging to a limit. Take a non-tangent, irreducible hypersurface V containing these points. (It need not be truly transversal). The intersections $L_\alpha \cap V$ are separatrices of $\mathcal{F}|_V$. Applying the procedure recursively to get a surface, we see that $\mathcal{F}$ satisfies Case (V) of [8, Section 2.1]. By [8, Theorem 4], $\mathcal{F}$ is dicritical. $\square$

10 Singularities of Codimension 1 Foliations

We now consider the case where X is a quasi-compact complex manifold, and $\mathcal{F}$ is a codimension-1 foliation on X given by an algebraic 1-form. We study the behaviour of the singular loci. In particular, we seek the best possible singularities, in order to provide an endpoint for a resolution process. (Achieving a smooth foliation by repeated blowing up is often impossible. For example, if $\mathcal{F}$ is the foliation of $\mathbb{A}^2$ given by $\omega = ydx + xdy$, blowing up at the origin yields a foliation with two singularities of the same type.)

In this section we introduce reduced and simple singularities, which turn out to be the optimal singularities required. We use the jet spaces to characterise their behaviour. We then state the known results about existence of resolutions.

10.1 Preliminaries

Example 10.1. Let $C = \mathbb{V}(x_1 \cdots x_n) \subset \mathbb{A}^n, n \geq 2$, and suppose $m \geq n$. Then, setting $x_i = \sum a_{ij} t^j$, we have

$$J_m(C, 0) = \bigcup_{j_1+\cdots+j_n=m-n+1} \bigcap_{1\leq i\leq n, 1\leq j\leq j_i} \{a_{ij} = 0\}.$$

Proof: We prove by induction on n. The base case $n = 2$ is proved in Example 6.7.

For $n \geq 3$ and $m \geq n$, the co-efficient of t^m, which vanishes for jets of order m and higher, is

$$y_{n-1}a_{n,m-n+1} + y_n a_{n,m-n} + y_{n+1}a_{n,m-n-1} + \cdots + y_{m-1}a_{n,1},$$

where y_j is the co-efficient of t^j obtained from $y = x_1 \cdots x_{n-1}$. (Note that $y_1 = \cdots = y_{n-2} = 0$.)

Then $J_m(C, 0)$ is given by the union of the sets cut out by the equations

$$a_{n,1} = a_{n,2} = \cdots = a_{n,m-n+1} = 0, a_{n,1} = \cdots = a_{n,m-n} = y_{n-1} = 0, \ldots,$$
$$a_{n,1} = y_{n-1} = y_n = \cdots = y_{m-2} = 0, y_{n-1} = y_n = \cdots = y_{m-1} = 0,$$

by the same argument as in Example 6.7.

By the induction hypothesis, the equations $y_{n-1} = \cdots = y_p = 0$ give

$$\bigcup \{a_{11} = \cdots = a_{1j_1} = \cdots = a_{n-1,1} = \cdots = a_{n-1,j_{n-1}} = 0\},$$

with the union taken over indices with $j_1 + \cdots j_{n-1} = p-(n-1)+1 = p-n+2$.

Now we have $j_n = m-1-p$, and so $j_1 + \cdots + j_n = m-n+1$, as required. □

Now let $x = (x_1, \ldots, x_n)$ be a non-zero point of C; let I be the set of indices of its non-zero entries. Let $C' = \mathbb{V}(\prod_{i \notin I} x_i)$. Then $J_m(C, x) = J_m(C', 0)$.

10.2 The 2-dimensional Case

Definition 10.2. Let $\mathcal{F}$ be a foliation on a smooth surface with an isolated singularity at a point P, which can be taken to be the origin of some co-ordinate chart. Let $\mathcal{X}$ be a vector field describing the foliation in a neighbourhood of the origin; let λ_1, λ_2 be the eigenvalues of the linear part of $\mathcal{X}$. If one of these eigenvalues, λ_2 say, is non-zero, and the ratio $\lambda = \frac{\lambda_1}{\lambda_2}$ is not a positive rational number, then the singularity is said to be *reduced*.

If in addition, $\lambda \notin (-\infty, 0]$, the foliation is said to be in the *Poincaré domain*.

If $\lambda \in (-\infty, 0)$, the foliation is said to be in the *Siegel domain*.

If $\lambda = 0$, the singularity is called a *saddle-node*.

Theorem 10.3. *Let $\mathcal{F}$ be a foliation on a smooth surface X given locally by a 1-form ω. Then a singular point P is reduced if and only if $J_m(\mathcal{F}, P)$ is isomorphic to $J_m(\mathbb{V}(xy), 0)$, for all $m \in \mathbb{N}$. If the singular point is reduced, then choosing formal co-ordinates at P, the separatrices of $\mathcal{F}$ are given by $x = 0$ and $y = 0$, and their union is fully tangent.*

Proof: As all the singularities are isolated, we consider each in turn, and may assume that $P = 0$ in some co-ordinate system. We restrict to a neighbourhood of 0 on which the foliation is given by a vector field $\mathcal{X}$, whose linear part has eigenvalues λ_1, λ_2. Then, taking a formal co-ordinate system at 0, we write ω in one of the formal normal forms from Section 8.5. Throughout the proof, though we are using formal co-ordinate changes, we can assume the induced isomorphisms on the jet fibres converge by Corollary 9.3.

(A) The reduced cases. Here we may assume that $\lambda_2 \neq 0$; let $\lambda = \frac{\lambda_1}{\lambda_2}$.

Case 1: $\lambda \in \mathbb{C} \setminus ((-\infty, 0] \cup \mathbb{Q}^+)$ (Poincaré domain): $\omega = ydx - \lambda xdy$.

We set $x = \sum a_i t^i, y = \sum b_i t^i, dx = (\sum i a_i t^{i-1})dt, dy = (\sum i b_i t^{i-1})dt$, as in the examples in Section 9.2. Expanding out $ydx - \lambda xdy = 0$, we see that the co-efficient of t^k (which vanishes for jets of order $k+1$ and higher) is $\sum_{j=1}^{k}(k - j + 1 - j\lambda)b_j a_{k+1-j}$. Since $\lambda \notin \mathbb{Q}^+$, none of the co-efficients in these sums vanishes; we then see that the result holds in this case, from Example 10.1.

Case 2.1: $\lambda \in \mathbb{R}^- \setminus \mathbb{Q}^-$ (Siegel domain): $\omega = ydx - \lambda xdy$. Same proof as above.

Case 2.2: $\lambda = -\frac{p}{q} \in \mathbb{Q}^-$, $p, q \in \mathbb{N}$ coprime (Siegel domain):
$\omega = -\lambda y(1 + g_2(x^p y^q))dx + x(1 + g_1(x^p y^q))dy, g_i \in \hat{\mathfrak{m}}_1$.

We start by noting that for foliations given by 1-forms of the form $x^m y^n dy$, where $m \geq 1, m+n \geq 2$, the co-efficient of t^k is zero for $k < m+n$. By symmetry, a similar result holds for the 1-forms $x^m y^n dx$. Now in this case, we have $\omega = xdy - \lambda ydx + \sum \omega_l$, where the ω_l are of the above forms. Let k be the smallest integer where the ω_l contribute a non-zero co-efficient for t^k. Then for $m \leq k$, the m-jets of the foliation above the origin are equal to $J_m(\mathbb{V}(xy), 0)$.

Let $K \geq k$, and suppose the result holds for jets of lower order. Then we have the relations $b_1 = \cdots = b_j = a_1 = \cdots = a_{K-1-j} = 0$ for $0 \leq j \leq K-1$. Every summand in the co-efficient of t^K contributed by the ω_l is of the form $ra_{i_1} \cdots a_{i_m} b_{j_1} \cdots b_{j_n} a_r$ or $ra_{i_1} \cdots a_{i_m} b_{j_1} \cdots b_{j_n} b_r$, where the sum of the indices equals K. The equalities from the lower order jets necessarily cause these contributions to vanish: If such a summand is not annihilated by one of the strings of equalities, it must have b terms only of index $\geq j+1$ and a terms only of index $\geq K-j$, contradicting that the sum of the indices is K. Therefore they do not affect the equations of the $(K+1)$-jets, and so we have $J_{K+1}(\mathcal{F}, 0) = J_{K+1}(\mathbb{V}(xy), 0)$; the result follows by induction.

Case 3: $\lambda = 0$ (saddle-node): $\omega = (x(1 + \nu y^l))dy - y^{l+1}dx, \nu \in \mathbb{C}, l \in \mathbb{N}$.

Noting that the equation $xdy = 0$ gives the same jets above the origin as $xy = 0$ (setting $\lambda = 0$ in Case 1), this case reduces to the one above, so we are done.

For the result on the separatrices, see [3, Section 1.1]. The union of the separatrices is $C = \mathbb{V}(xy)$, and so $J_m(C, 0) = J_m(\mathcal{F}, 0)$. Other points of C are in the smooth locus of the foliation, and so C is fully tangent.

(B) The non-reduced cases.

Case 4.1: The linear part of $\mathcal{X}$ is zero: $\omega = p(x,y)dx + q(x,y)dy$, $\operatorname{ord} p \geq 2, \operatorname{ord} q \geq 2$.

We see that $J_2(\mathcal{F}, 0) = \mathbb{A}^4$, and so is not isomorphic to $J_2(\mathbb{V}(xy), 0)$.

Case 4.2: $\lambda_1 = \lambda_2 = 0$, but the linear part of $\mathcal{X}$ is non-zero (cuspidal case): $\omega = ydy - (p(x) + yq(x))dx$, $\operatorname{ord} p \geq 2, \operatorname{ord} q \geq 1$.

Then $J_2(\mathcal{F}, 0) = \mathbb{V}(b_1) \subset \mathbb{A}^4$, which has one fewer component than $J_2(\mathbb{V}(xy, 0))$, so there is no isomorphism.

Henceforth we assume $\lambda_2 \neq 0$.

Case 5: $\lambda_1 = \lambda_2$: $\omega = ydx - xdy$ or $\omega = ydx - (x+y)dy$, as in this case, $\mathcal{X}$ is linearisable, that is, formally equivalent to a linear vector field.

We have calculated the jets at the origin in these cases (Examples 9.9 and 9.7), and we see there is no isomorphism to $J_m(\mathbb{V}(xy), 0)$.

Case 6: $\frac{\lambda_1}{\lambda_2}, \frac{\lambda_2}{\lambda_1} \notin \mathbb{Z}$: $\omega = ydx - \lambda xdy$, $\lambda = \frac{\lambda_1}{\lambda_2}$, since by Lemma 8.23, the pair (λ_1, λ_2) is non-resonant, so $\mathcal{X}$ is linearisable.

$J_m(\mathcal{F}, 0)$ is cut out by the equations $\sum_{j=1}^{K}(K - j + 1 - j\lambda)b_j a_{K+1-j}, 1 \leq K \leq m-1$. For $m < \lambda_1 + \lambda_2$, none of the co-efficients in these sums vanishes, so we have $J_m(\mathcal{F}, 0) = J_m(\mathbb{V}(xy), 0)$. For the jets of order $m = \lambda_1 + \lambda_2$, the extra defining equation has a co-efficient of 0 for the $b_{\lambda_2} a_{\lambda_1}$ term, hence, as for the order below, there is a component given by

$$b_1 = \cdots = b_{\lambda_2 - 1} = a_1 = \cdots = a_{\lambda_1 - 1} = 0,$$

which is of higher dimension than $J_m(\mathbb{V}(xy), 0)$, so there is no isomorphism.

Case 7: $\frac{\lambda_1}{\lambda_2} = r \in \mathbb{Z}, r \geq 2$: $\omega = ydx - (rx + ay^r)dy, a \in \mathbb{C}$.

If $a = 0$, we are in the same case as above in terms of jets.

Let $a \neq 0$. We blow up the singularity. On the chart $x = yt$, we get $\omega = y(ydt - ((r-1)t + ay^{r-1})dy)$, which is of the same form. So we take r iterated blow-ups. On one chart we have $x = y^r t$. Then we have $\omega = y^r(ydt - ady)$, which gives a smooth (unsaturated) foliation with fully tangent curve through the origin $\mathbb{V}(y^{r+1})$, by Proposition 9.23 applied to the leaf $\mathbb{V}(y)$.

On the other charts, we have $x = y^j u, 0 \leq j \leq r-1$, which gives $\omega = y^j(ydu - ((r-j)u + ay^{r-j})dy)$, followed by $y = ut$. This gives

$$\omega = u^{j+1}t^j(((1 - r + j)t - au^{r-j-1}t^{r-j+1})du - ((r-j)u + au^{r-j}t^{r-j})dt).$$

The linear part has ratio of eigenvalues $\frac{1-r+j}{r-j} \leq 0$, so the saturation is reduced. Therefore the unsaturated foliation has fully tangent curve through the origin $\mathbb{V}(u^{j+2}t^{j+1})$, by Proposition 9.23 applied to $\mathbb{V}(ut)$, which is fully tangent by the argument for reduced foliations above.

Blowing back down, we see that the original foliation has fully tangent curve $C = \mathbb{V}(y^{r+1}, xy)$, by Corollary 9.15. Now $J_{r+1}(\mathbb{V}(y^{r+1}), 0) = \mathbb{V}(b_1)$, and so $J_{r+1}(\mathcal{F}, 0) = J_{r+1}(C, 0) = J_{r+1}(\mathbb{V}(y^{r+1}, 0) \cap J_{r+1}(\mathbb{V}(xy), 0)$ has one fewer irreducible component than $J_{r+1}(\mathbb{V}(xy), 0)$, by Example 6.7, so there is no isomorphism.

By reversing the roles of λ_1 and λ_2, the case where $\frac{\lambda_1}{\lambda_2}$ is the reciprocal of a natural number reduces to the case above, and we are done. □

10.3 The Higher Dimensional Case

When looking at singularities of higher dimensional foliations, we first introduce the following concepts.

Definition 10.4. Let $f : X \to Y$ be a holomorphic map of complex manifolds, and define the sheaf $\Omega^1_{X|Y}$ to be the cokernel of the induced map $df : f^*\Omega^1_Y \to \Omega^1_X$. The dual of this sheaf is called the *relative tangent sheaf* of f.

Definition 10.5. Let $\mathcal{F}$ be a foliation on a complex manifold X given locally by a 1-form ω, (so in particular ω generates a saturated sheaf, and $\omega \wedge d\omega = 0$), and let $x \in X$. The *dimensional type* $\tau(\mathcal{F}, x)$ is the codimension in T_xX of $\mathcal{D}_{\mathcal{F}}(x) = \{\mathcal{X}(x) \mid \omega(\mathcal{X}) = 0\}$, where $\mathcal{X}$ is a germ of a vector field at x.

Remark 10.6. The space $\mathcal{D}_{\mathcal{F}}(x)$ can also be defined for foliations $\mathcal{F}$ of higher codimension.

Proposition 10.7. *Let $\mathcal{F}$ be a codimension-1 foliation on X, and let $x \in X$. The dimension type $\tau(\mathcal{F}, x)$ is the minimal number of formal co-ordinates needed to write a generator ω of the foliation in a neighbourhood of x.*

Remark 10.8. This result is used without proof in many papers, including [6]. We give a proof below, making use of the following lemma.

Lemma 10.9. *Let $\mathcal{F}$ be a codimension-1 foliation on X, and let $x \in X$. Then in a neighbourhood of x, $\mathcal{F}$ is the cylinder over a foliation on a $\tau(\mathcal{F}, x)$-dimensional submanifold of X.*

Proof of Lemma 10.9*:* Let $r = \dim \mathcal{D}_{\mathcal{F}}(x)$. We use induction on r. If $r = 0$, then we write $\mathcal{F}$ as the trivial cylinder over itself, and there is nothing to prove.

Suppose $r \geq 1$. Then there is a holomorphic vector field $\mathcal{X}$ in a neighbourhood of x which is tangent to $\mathcal{F}$ and does not vanish at x. We can choose holomorphic co-ordinates $(x_1, \ldots, x_n)$ at x such that $x = (0, \ldots, 0) \in \mathbb{C}^n$ and $\mathcal{X} = \frac{\partial}{\partial x_1}$. Let $\pi : \mathbb{C}^n \to \mathbb{C}$ be the projection onto the first co-ordinate.

Let $\mathcal{F}' \subset \mathcal{F}$ be the subsheaf of vector fields also tangent to the fibres of π. The leaves of $\mathcal{F}$ containing the integral curves of $\mathcal{X}$ intersect the fibres of π transversally, so $\mathcal{F}'$ is saturated. As both $\mathcal{F}$ and the relative tangent sheaf of π are integrable, so is $\mathcal{F}'$. The flow generated by $\mathcal{X}$ maps the smooth part of $\mathcal{F}$ to itself; by normality, this extends to all of $\mathcal{F}$. The fibres of π are also mapped to each other by $\mathcal{X}$, and so $\mathcal{F}'$ is invariant under the flow of $\mathcal{X}$. Thus $\mathcal{F}$ is a cylinder over $\mathcal{F}|_{\pi^{-1}(0)}$.

Now as $\mathcal{F}'$ is a subsheaf of $\mathcal{F}$, we have $\mathcal{D}_{\mathcal{F}|_{\pi^{-1}(0)}}(0) = \mathcal{D}_{\mathcal{F}'}(0) \subset \mathcal{D}_{\mathcal{F}}(0)$. $\mathcal{D}_{\mathcal{F}'}(0)$ does not contain $\mathcal{X}$; by the cylindrical structure of $\mathcal{F}$, all the vector fields in $\mathcal{D}_{\mathcal{F}}$ can be written as the sum of a multiple of $\mathcal{X}$ and a vector field tangent to the restriction. Hence $\dim \mathcal{D}_{\mathcal{F}'}(0) = r - 1$.

By induction, $\mathcal{F}|_{\pi^{-1}(0)}$ is a cylinder over a foliation on a submanifold of $\pi^{-1}(0)$ of dimension $(n-1) - (r-1) = n - r$. As $\mathcal{F}$ itself is a cylinder over this foliation, we have the result. □

Proof of Proposition 10.7*:* Let t be the minimum number of formal co-ordinates needed to write ω. By the lemma, we have $t \leq n - r = \tau(\mathcal{F}, x)$. Indeed, the flows $\phi_1, \ldots, \phi_r$ of the vector fields $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_r}$ tangent to $\mathcal{F}$ map ω to itself, because of the cylindrical structure, so $\omega(\phi_{i,t}(z) = \omega(z)$. Also $d(\phi_{i,t}(z_j)) = dz_j$.

Hence the co-efficient functions of $\omega = \sum_{i=1}^{n} f_i dz_i$ only depend on $z_{r+1}, \ldots, z_n$. Finally, $\omega(\frac{\partial}{\partial z_i}) = 0, i = 1, \ldots, r$, and so $f_j = 0, j = 1, \ldots, r$.

If $z_1, \ldots, z_n$ is a system of co-ordinates at x, and ω can be written in terms of just the first t co-ordinates, then $\frac{\partial}{\partial z_{t+1}}, \ldots, \frac{\partial}{\partial z_n}$ annihilate ω at x. So $r \geq n - t$. □

Proposition 10.10. *Let $\mathcal{F}$ be a codimension-*1 *foliation on X, defined by ω, and let $x \in X$. Then $\tau(\mathcal{F}, x) = 1$ if and only if $x \in X \setminus \mathrm{Sing}(\mathcal{F})$.*

Proof: If $\tau(\mathcal{F}, x) = 1$, then by Proposition 10.7 there are co-ordinates in a neighbourhood of x such that ω is given only in terms of the first co-ordinate

x_1. There is only one such form yielding a saturated sheaf, namely dx_1, and so the foliation around x is smooth.

Conversely, if x is a smooth point of the foliation, then by Frobenius' theorem $\mathcal{F}$ is given by the form dx_1 in some neighbourhood of x, and so $\tau(\mathcal{F}, x) = 1$. $\square$

Example 10.11. Let $X = \mathbb{A}^3$, and let $\mathcal{F}$ be given by $\omega = yzdx + xzdy + xydz$. The singular locus is the union of the three co-ordinate axes. The form ω is annihilated only by the vector fields in the span of $x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}$.

All these vector fields vanish at the origin, so we have $\tau(\mathcal{F}, 0) = 3$. We now consider the singular point $p = (0, 0, 1)$. We see that $\mathcal{D}_{\mathcal{F}}(p)$ is spanned by $\frac{\partial}{\partial z}$, and so $\tau(\mathcal{F}, p) = 2$. By symmetry, all other points of the singular locus have dimensional type 2. The points of the smooth locus have dimensional type 1.

Let $\mathcal{F}$ be a codimension 1 foliation on a complex manifold X, let $x \in X$ be a point with dimensional type $t = \tau(\mathcal{F}, x)$, and let E be a normal crossings divisor of X through x with each component tangent to $\mathcal{F}$.

Proposition 10.12. *In this setting, E has at most t components through x.*

Proof: In a neighbourhood of x, $\mathcal{F}$ is a cylinder over a foliation on a t-dimensional subspace of X. As each component of E is tangent to $\mathcal{F}$, E is a cylinder over an SNC divisor of this subspace, and so has at most t components. $\square$

Taking appropriate holomorphic co-ordinates at x, as in [6, Section 4], there is a subset $A \subset \{1, \ldots, t\}$ such that we can write $E = \mathbb{V}(\prod_{i \in A} x_i)$, and the foliation is generated by

$$\omega = \left(\prod_{i \in A} x_i\right)\left(\sum_{i \in A} b_i \frac{dx_i}{x_i} + \sum_{i \in \{1,\ldots,t\} \setminus A} b_i dx_i\right),$$

where $b_i = b_i(x_1, \ldots, x_t)$ are germs of holomorphic functions without common factor. Indeed, as the dimensional type is t at x, we can write ω in the first t co-ordinate as $\omega = \sum_{i \in A} f_i dx_i + \sum_{i \notin A} f_i dx_i$. As the components of E are tangent, we have $x_i \mid f_j$, for all $i \in A$ and all $j \neq i$. Setting $b_i = \frac{f_i}{\prod_{j \in A, j \neq i} x_j}$ yields the result.

Proposition 10.13. *Let $\mathcal{F}$ be a foliation on X, and let E be an SNC divisor with each component tangent to $\mathcal{F}$. Then E itself is a solution for $\mathcal{F}$.*

Proof: We choose holomorphic co-ordinates so E and $\mathcal{F}$ are given in the form above. If $E = \mathbb{V}(\prod_{i\in A} x_i)$, then for each component $\mathbb{V}(x_i)$ of E, we have $\omega|_{x_i=0} = b_i d(\prod_{i\in A} x_i)|_{x_i=0}$, and the result follows. □

It follows from Lemma 9.19 that E is strongly tangent.

Lemma 10.14. *Let $\mathcal{F}$ be a foliation on X given by ω and let $C \subset X$ be a smooth irreducible hypersurface. Suppose for some point $x \in C$, $J_m(C, x) \subset J_m(\mathcal{F}, x)$, for all m. Then C is a solution for $\mathcal{F}$.*

Proof: We choose a co-ordinate system around x so that $x = 0$ and $C = \mathbb{V}(x_1)$. We write $\omega = b_1 dx_1 + \cdots + b_t dx_t$, for holomorphic functions b_i. If C is not a solution, then there exists $i > 1$ such that x_1 does not divide b_i.

Using the derivation description of the jet spaces (see Section 6.4 and Section 9.1), we have that $J_m(\mathcal{F}, 0) = \mathbb{V}(\omega, D\omega, \ldots, D^{m-1}\omega, x_1, \ldots, x_n)$. As ω is not a multiple of x_1, for high enough m, $D^m\omega$ has summands containing no x_k terms for $k > 1$, and no $D^j x_1$ terms, contradicting that

$$\mathbb{V}(x_1, Dx_1, \ldots, D^m x_1, x_2, \ldots, x_n) \subset J_m(\mathcal{F}, 0), \forall m.$$

□

Corollary 10.15. *The same result holds if C is an SNC divisor with smooth components.*

Proof: Each component of C satisfies the conditions of the above lemma, so is a solution for $\mathcal{F}$. As C is SNC, it too is a solution by Proposition 10.13. □

Definition 10.16. [6, Section 4] Let $\mathcal{F}$ be a codimension-1 foliation on a complex manifold X, let $x \in X$ be a point with dimensional type $t = \tau(\mathcal{F}, x)$, and let E be a normal crossings divisor of X through x with each component tangent to $\mathcal{F}$. We write the generator of the foliation in the form

$$\omega = \left(\prod_{i\in A} x_i\right)\left(\sum_{i\in A} b_i \frac{dx_i}{x_i} + \sum_{i\in\{1,\ldots,t\}\setminus A} b_i dx_i\right),$$

with the b_i holomorphic.

The *adapted order* is $\nu(\mathcal{F}, E; x) = \min\{\operatorname{ord}_x b_i\}$.

The *adapted multiplicity* $\mu(\mathcal{F}, E; x)$ is the order at x of the ideal generated by

$$\{b_i\}_{i\in A} \cup \{x_j b_i\}_{i\notin A, j=1,\dots,n},$$

that is, the minimum of the orders at x of all the functions in the ideal.

Remark 10.17. As composition with an invertible holomorphic function does not change the order, the adapted order and adapted multiplicity are independent of the normal form chosen.

Definition 10.18. [6, Definition 4] In the above situation, $x \in \operatorname{Sing}\mathcal{F}$ is a *pre-simple* singularity of $\mathcal{F}$ adapted to E if and only if one of the following occurs:

$\nu(\mathcal{F}, E; x) = 0$;

$\nu(\mathcal{F}, E; x) = \mu(\mathcal{F}, E; x) = 1$, and for some $i \in A$, the linear part of b_i does not depend only on $\{x_i \mid i \in A\}$.

Proposition 10.19. *[6, Proposition 12] Let $x \in \operatorname{Sing}\mathcal{F}$ be a pre-simple singularity (that is, pre-simple adapted to some SNC divisor E), with $\tau(\mathcal{F}, x) = t$. Then in a formal co-ordinate system $x_1, \dots, x_n$ at x, $\mathcal{F}$ is locally generated by a 1-form in one of the following normal forms:*

(A): $\omega = x_1 \cdots x_t(\sum_{i=1}^t \lambda_i \frac{dx_i}{x_i})$, $\lambda_i \in \mathbb{C}^*$;

(B): $\omega = x_1 \cdots x_t(\sum_{i=1}^k p_i \frac{dx_i}{x_i} + \Psi(x_1^{p_1} \cdots x_k^{p_k}) \sum_{i=2}^t \lambda_i \frac{dx_i}{x_i})$, where $1 \leq k \leq t$, $1 \leq p_1, \dots, p_k \in \mathbb{N}$ (we can assume them to have no common factor), $\lambda_i \in \mathbb{C}$, with $\lambda_{k+1}, \dots, \lambda_t \in \mathbb{C}^*$, and $\Psi \in \hat{\mathfrak{m}}_1$, which we can assume to be non-vanishing except at 0;

(C): $\omega = x_2 \cdots x_t(dx_1 - x_1 \sum_{i=2}^k p_i \frac{dx_i}{x_i} + x_2^{p_2} \cdots x_k^{p_k} \sum_{i=2}^t \lambda_i \frac{dx_i}{x_i})$, where $k \geq 2$, $p_2, \dots, p_k \in \mathbb{N}$, and $\lambda_i \in \mathbb{C}$, with $\lambda_{k+1}, \dots, \lambda_t \in \mathbb{C}^*$.

These three cases are mutually exclusive; case (A) can be seen as case (B) with $k = 0$.

In cases (A) and (B), we can take $E = \mathbb{V}(x_1 \cdots x_t)$; in case (C), we can take $E = \mathbb{V}(x_2 \cdots x_t)$.

A 1-form ω in one of these normal forms is said to be of the form (A), (respectively, (B) or (C)). A singular point x of a foliation $\mathcal{F}$ is said to be of the form (A), (respectively, (B) or (C)), if in a neighbourhood of x, $\mathcal{F}$ is generated by such a 1-form.

Definition 10.20. A pre-simple singularity is *simple* if we are in the case (A) or (B) above, and the tuple $(\lambda_{k+1}, \ldots, \lambda_t)$ is non-resonant in the sense that for all maps $\phi : \{k+1, \ldots, t\} \to \mathbb{N}_0$, not constantly zero, $\sum_{j=k+1}^{t} \phi(j)\lambda_j \neq 0$.

Remark 10.21. If $\dim X = 2$, then simple singularities correspond to reduced singularities.

Notation 10.22. Let $\mathcal{I}^n$ denote the set $\{(i,j) \in \mathbb{Z}^2 \mid 1 \leq i < j \leq n\}$.
For $k < n$, let $\mathcal{I}_k^n$ denote $\mathcal{I}^n \setminus \{(1, k+1), \ldots, (1, n)\}$.

Let $\mathcal{F}$ be a foliation with a pre-simple singularity at the point x, with $\tau(\mathcal{F}, x) = t$, and let U be an open (or formal) neighbourhood of the point on which the foliation is given by one of the normal forms in Proposition 10.19 (written in the first t co-ordinates). Then, in cases (A) and (B), $\operatorname{Sing} \mathcal{F} \cap U = \bigcup_{(i,j) \in \mathcal{I}^t} \{x_i = x_j = 0\}$, and in case (C), $\operatorname{Sing} \mathcal{F} \cap U = \bigcup_{(i,j) \in \mathcal{I}_k^t} \{x_i = x_j = 0\}$.

If $y \in \operatorname{Sing} \mathcal{F}$, we denote the number of the first t co-ordinates which equal zero by $z(y)$.

Lemma 10.23. *Let $\mathcal{F}$ be a foliation with a pre-simple singularity of type (A) or (B) at the point x, with $\tau(\mathcal{F}, x) = t$, and let U be an open (or formal) neighbourhood of the point on which the foliation is given by one of the normal forms in Proposition* 10.19*. Let $y \in \operatorname{Sing} \mathcal{F}$. Then $\tau(\mathcal{F}, y) = z(y)$.*

Proof: Let ω be the 1-form generating $\mathcal{F}$ near x. If $y_i \neq 0$, then $\frac{\partial}{\partial x_i}$ annihilates the 1-form at y. Thus $\tau(\mathcal{F}, y) \leq z(y)$. However, y lies in an SNC divisor E with $e(E, y) = z(y)$, hence the result holds by Proposition 10.12. □

Definition 10.24. Let $\mathcal{F}$ be a foliation given locally by the 1-form $\omega = b_1(x_1, \ldots, x_t)dx_1 + \cdots + b_t(x_1, \ldots, x_t)dx_t$. For a fixed point $y = (y_1, \ldots, y_t)$, we define the 1-form

$$\begin{aligned} \omega_y^i = b_1(x_1, \ldots, y_i, \ldots, x_t)dx_1 + \cdots + b_{i-1}(x_1, \ldots, y_i, \ldots, x_t)dx_{i-1} + \\ b_{i+1}(x_1, \ldots, y_i, \ldots, x_t)dx_{i+1} + \cdots + b_t(x_1, \ldots, y_i, \ldots, x_t)dx_t; \end{aligned}$$

denote by $\mathcal{F}_y^i$ the foliation generated by ω_y^i. Extra indices are added recursively: $\omega_y^{(i,j)} = (\omega_y^i)_y^j$, etcetera, and the same for $\mathcal{F}_y^{(i,j)}$

Lemma 10.25. *Let $\mathcal{F}$ be a foliation given by ω, with a pre-simple singularity at the point x, with $\tau(\mathcal{F}, x) = t$, and let U be an open (or formal) neighbourhood of the point on which the foliation is given by one of the normal forms in Proposition* 10.19. *Let $y \in \operatorname{Sing}\mathcal{F}$, and let $I \subset \{1, \ldots, t\}$ be the list of indices of non-zero co-ordinates of y. Then $\mathcal{F}$ is generated in a neighbourhood of y by ω_y^I.*

Proof: If $i \in I$, then $\frac{\partial}{\partial x_i}$ annihilates ω at y. Then by the proof of Lemma 10.9, in a neighbourhood of y, $\mathcal{F}$ is a cylinder over $\mathcal{F}|_{\{x_i = y_i | i \in I\}}$. This foliation is given by the form ω_y^I. □

Theorem 10.26. *Let $\mathcal{F}$ be a codimension-*1 *foliation on X, and let $x \in \operatorname{Sing}\mathcal{F}$ be pre-simple singularity with $\tau(\mathcal{F}, x) = t$. Then x is a simple singularity if and only if there is an open (or formal) neighbourhood $U \ni x$ such that in a local co-ordinate system on U, $x = 0$, $\operatorname{Sing}\mathcal{F} \cap U \subset \operatorname{Sing}\mathbb{V}(x_1 \cdots x_t)$, and $J_m(\mathcal{F}, y)$ is isomorphic to $J_m(\mathbb{V}(x_1 \cdots x_t), y)$ for all $y \in \operatorname{Sing}\mathcal{F} \cap U$ and all $m \in \mathbb{N}_0$.*

Proof: First, by Proposition 10.19 we can choose a neighbourhood U of x on which, by a formal co-ordinate change, we can write the generator ω in one of the above normal forms, and say $x = 0$ and that the singular locus is in the right form. Again, Corollary 9.3 says that the induced isomorphism on the jet fibres converges.

In calculating the jets, we set $x_i = \sum a_{ij} t^j$, and so $dx_i = (\sum j a_{ij} t^{j-1}) dt$, and we equate co-efficients to get the co-efficient of t^j.

Now let us suppose x is a simple singularity. We have the following cases:

(A): The co-efficient that vanishes for jets of order m or higher at the origin can be calculated as

$$\sum_{\{(i_1,\ldots,i_t) | i_1 + \cdots + i_t = m\}} \left(\sum_{j=1}^{t} \lambda_j i_j \right) a_{1,i_1} a_{2,i_2} \cdots a_{t,i_t}.$$

By non-resonance of the λ_i, none of the co-efficients in the sum vanishes; hence we have $J_m(\mathcal{F}, 0) = J_m(\mathbb{V}(x_1 \cdots x_t), 0)$.

Now let y be some other singular point, lying in U, with $\tau(\mathcal{F}, y) = s$, and with I the list of indices of its non-zero co-ordinates. Then $J_m(\mathcal{F}, y) = J_m(\mathcal{F}_y^I, 0)$. As $\omega_y^I = \prod_{1 \le i \le t, i \notin I} x_i (\sum_{1 \le i \le t, i \notin I} \lambda_i \frac{dx_i}{x_i})$ is also of the form (A), with the λ_i non-resonant, this in turn is isomorphic to $J_m(\mathbb{V}(x_1 \cdots x_s), 0) \cong J_m(\mathbb{V}(x_1 \cdots x_t), y)$.

(B): We write $\omega = \omega' + \Psi\omega''$. For $m < p_1 + \cdots + p_k + t - 1$, (where the notation is the same as in Proposition 10.19), the jets of order m are given solely by the ω' term. Write $\omega' = x_{k+1} \cdots x_t \eta$, where $\eta = p_1 x_2 \cdots x_k dx_1 + \cdots + p_k x_1 \cdots x_{k-1} dx_k$. This is of the form (A), and the tuple $(p_1, \ldots, p_k)$ is non-resonant. Letting $\mathcal{G}$ denote the foliation generated by η, we have $J_m(\mathcal{G}, 0) = J_m(\mathbb{V}(x_1 \cdots x_k), 0)$, and hence $J_m(\mathcal{F}, 0) = J_m(\mathbb{V}(x_1 \cdots x_t), 0)$ by Proposition 9.23.

So for low order, $J_m(\mathcal{F}, 0) = \{a_{11} = \cdots = a_{1j_1} = a_{21} = \cdots = a_{2j_2} = \cdots = a_{t1} = \cdots = a_{tj_t} = 0 \mid j_1 + \cdots + j_t = m - t + 1\}$. Suppose this holds for all orders $m \leq K$. Now every summand in the co-efficient of t^K contributed by the $\Psi\omega''$ term is the product of $p_1 + \cdots p_k + t - 1$ terms of the form a_{ij}, where the sum of the second indices equals K. The equalities from the lower order jets cause these terms to vanish; else there is a summand with a_i terms only of order $> j_i$, contradicting that the sum of the indices is K.

Hence by induction, $J_m(\mathcal{F}, 0) = J_m(\mathbb{V}(x_1 \cdots x_t), 0)$ for all m.

Again let y be some other singular point, lying in U, with $\tau(\mathcal{F}, y) = s$, and with I the list of indices of its non-zero co-ordinates. Then $J_m(\mathcal{F}, y) = J_m(\mathcal{F}_y^I, 0)$. If $I \supset \{1, \ldots, k\}$, then

$$\omega_y^I = \prod_{k+1 \leq i \leq t, i \notin I} x_i \left(\sum_{k+1 \leq i \leq t, i \notin I} \lambda_i \frac{dx_i}{x_i} \right)$$

is of the form (A), with the λ_i non-resonant. Otherwise

$$\omega_y^I = \prod_{1 \leq i \leq t, i \notin I} x_i \left(\sum_{1 \leq i \leq k, i \notin I} p_i \frac{dx_i}{x_i} + \Psi \left(\prod_{1 \leq i \leq k, i \notin I} x_i^{p_i} \prod_{1 \leq i \leq t, i \in I} y_i^{p_i} \right) \sum_{2 \leq i \leq t, i \notin I} \lambda_i \frac{dx_i}{x_i} \right)$$

is of the form (B). In either case, we have $J_m(\mathcal{F}, y) \cong J_m(\mathbb{V}(x_1 \cdots x_s), 0) \cong J_m(\mathbb{V}(x_1 \cdots x_t), y)$.

Conversely, suppose that x is a pre-simple singularity that is not simple. We again have the following cases:

(A): Non-simplicity means that the tuple $(\lambda_1, \ldots, \lambda_t)$ is resonant: Let $(r_1, \ldots, r_t)$ be a tuple of non-negative integers (not all zero), such that $\sum_{i=1}^t r_i \lambda_i = 0$. (We can choose the r_i so that their sum is minimal among all such tuples). Let $K = r_1 + \cdots + r_t$.

Again, the co-efficient that vanishes for jets of order m or higher at the origin is

$$\sum_{\{(i_1,\ldots,i_t)|i_1+\cdots+i_t=m\}} \left(\sum_{j=1}^t \lambda_j i_j \right) a_{1,i_1} a_{2,i_2} \cdots a_{t,i_t}.$$

For $m < K$, none of the co-efficients in the sum vanishes; hence we have $J_m(\mathcal{F}, 0) = J_m(\mathbb{V}(x_1 \cdots x_t), 0)$. For the jets of order K, the extra defining equation has a co-efficient of 0 for the $a_{1,r_1} a_{2,r_2} \cdots a_{t,r_t}$ term, hence, as for the order below, there is a component given by

$$a_{11} = \cdots = a_{1,r_1-1} = a_{21} = \cdots = a_{2,r_2-1} = \cdots = a_{t1} = \cdots = a_{t,r_t-1} = 0,$$

which is of higher dimension than $J_K(\mathbb{V}(x_1 \cdots x_t), 0)$, so there is no isomorphism.

(B): We have $k < t - 1$, otherwise the non-resonance condition on the λ_i is trivial, and the singularity is automatically simple. Consider the point $y = (y_1, \ldots, y_k, 0, \ldots, 0)$, where $y_i \neq 0$ for all $i = 1, \ldots, k$. This is a singular point with dimensional type $t - k$, and $I = (1, \ldots, k)$. The 1-form ω_y^I is of the form (A) and is resonant. So we have

$$J_m(\mathcal{F}, y) = J_m(\mathcal{F}_y^I, 0) \ncong J_m(\mathbb{V}(x_1 \cdots x_{t-k}), 0) \cong J_m(\mathbb{V}(x_1 \cdots x_t), y)$$

for some $m \in \mathbb{N}$.

(C): The point $y = (0, 0, y_3, \ldots, y_t)$, where $y_i \neq 0$ for all $i = 3, \ldots, t$, is a singular point of dimensional type 2, about which the foliation is given by ω_y^I for $I = (3, \ldots, t)$. Now $\omega_y^I = x_2 dx_1 - (p_2 x_1 - \lambda_2 \prod_{i=3}^k y_i^{p_i} x_2^{p_2}) dx_2$, which by Theorem 10.3, Case 7, does not yield jet spaces isomorphic to those of an SNC divisor. The result follows. $\square$

Theorem 10.27. *Let $\mathcal{F}$ be a codimension-1 foliation on X, and let $x \in \operatorname{Sing} \mathcal{F}$ be a singularity with $\tau(\mathcal{F}, x) = t$ which is not pre-simple. Then there exists a natural number m such that $J_m(\mathcal{F}, x)$ is not isomorphic to $J_m(\mathbb{V}(x_1 \cdots x_t), 0)$.*

Proof: That x is not a pre-simple singularity means that for any SNC divisor $E \subset X$, either E is not tangent to $\mathcal{F}$, or x is not pre-simple adapted to E.

Suppose there is a divisor E with t components through x and tangent to $\mathcal{F}$. Non-pre-simplicity implies that $\nu(\mathcal{F}, E; x) \geq 1$, which means that the unadapted order $\nu(\mathcal{F}, \emptyset; x) \geq t$. Hence the co-efficients of the pullback of ω under the map $x_i = \sum a_{ij} s^j$ are zero for all $k \leq t$, and so $J_t(\mathcal{F}, x) = \mathbb{A}^{tn}$, and there is no isomorphism to $J_t(\mathbb{V}(x_1 \cdots x_t), 0)$.

Now suppose our divisor E has $t-1$ components: We can write it as $E = \mathbb{V}(x_1 \cdots x_{t-1})$. There are three cases for non-pre-simplicity:

(i) $\nu(\mathcal{F}, E; x) \geq 2$;

(ii) $\mu(\mathcal{F}, E; x) \geq 2$.

In both these cases we have $\nu(\mathcal{F}, \emptyset; x) \geq t$, and so there is no isomorphism as in the first case.

(iii) $\nu(\mathcal{F}, E; x) = \mu(\mathcal{F}, E; x) = 1$, and for all $i = 1, \ldots, t-1$, the linear part of b_i is independent of x_t. Some of these linear parts are non-zero; denote them by $l_1, \ldots, l_s$ (relabelling co-ordinates if necessary). Then the equation defining the t-jets is

$$a_{1,1} \cdots a_{t-1,1} \left(l_1 \left(\sum a_{i,j} t^j \right) + \cdots + l_s \left(\sum a_{i,j} t^j \right) \right) = 0.$$

This is either zero, or else gives t components in $t-1$ variables—in either case, there is no isomorphism to $J_t(\mathbb{V}(x_1 \cdots x_t), 0)$, which is SNC (see Example 10.1).

Now suppose that the only SNC divisors tangent to $\mathcal{F}$ through x have at most $t-2$ components. If we have $J_m(\mathcal{F}, x) \cong J_m(\mathbb{V}(x_1 \cdots x_t), 0)$, then by Corollary 10.15, the pre-image of $\mathbb{V}(x_1 \cdots x_t)$ under the isomorphism is a solution for $\mathcal{F}$, a contradiction. □

Combining Theorems 10.26 and 10.27, we have:

Corollary 10.28. *Let $\mathcal{F}$ be a codimension-1 foliation on X, and let $x \in \operatorname{Sing} \mathcal{F}$ be singularity with $\tau(\mathcal{F}, x) = t$. Then x is a simple singularity if and only if there is an open (or formal) neighbourhood $U \ni x$ such that in a local co-ordinate system on U, $x = 0$, $\operatorname{Sing} \mathcal{F} \cap U \subset \operatorname{Sing} \mathbb{V}(x_1 \cdots x_t)$, and $J_m(\mathcal{F}, y)$ is isomorphic to $J_m(\mathbb{V}(x_1 \cdots x_t), y)$ for all $y \in \operatorname{Sing} \mathcal{F} \cap U$ and all $m \in \mathbb{N}_0$.*

Moreover, if x is simple, $\operatorname{Sing}\mathcal{F} \cap U = \operatorname{Sing}\mathbb{V}(x_1 \cdots x_t)$*, and the union of the separatrices of $\mathcal{F}$ at x is given by* $\mathbb{V}(x_1 \cdots x_t)$*, which is fully tangent in U.*

Remark 10.29. If we were to define simple singularities by the jet condition of Corollary 10.28, we could not then recover the normal forms from Proposition 10.19. Indeed, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are two foliations on X with the same singular locus, and such that each singular point of either foliation is a simple singularity, then we have $J_m(\mathcal{F}_1, x) \cong J_m(\mathcal{F}_2, x)$ for each $x \in X$ and each $m \in \mathbb{N}$, and yet the global jet spaces are not isomorphic if the foliations are given by different normal forms.

Definition 10.30. Let $\mathcal{F}$ be a singular codimension-1 foliation on X. A *resolution of singularities* for $\mathcal{F}$ is a sequence of blow-ups $f : X' \to X$, where X' is a manifold and $\operatorname{sat}(f^{-1}(\mathcal{F}))$ has only simple singularities.

Theorem 10.31. *Let $\mathcal{F}$ be a singular codimension-*1 *foliation on X. Then $\mathcal{F}$ is known to have a resolution of singularities in the following cases:*
$\dim X = 2$ *[28];*
$\dim X = 3$ *[7];*

Remark 10.32. Resolution of singularities, along with the appropriate notion of simple singularities, for codimension-2 foliations on a threefold (in the category of stacks) have been defined and proved to exist in [24].

11 Jets and Separatrices of Non-dicritical Codimension 1 Foliations

Throughout this section, we assume that X is a quasi-compact complex manifold, and that $\mathcal{F}$ is a codimension-1 foliation on X given by an algebraic 1-form. Unless stated otherwise, we assume that $\mathcal{F}$ is non-dicritical. In this case, $\mathcal{F}$ has finitely many separatrices by Proposition 9.38

In this section, we begin by introducing the notion of the total separatrix of a foliation $\mathcal{F}$. We then prove a series of results culminating in Theorem 11.15: that the existence of a desingularisation of the total separatrix is equivalent to the existence of a resolution of the foliation itself. We then give some results and conjectures pertaining to the existence and uniqueness of the total separatrix.

Definition 11.1. Let $\mathcal{F}$ be a foliation on X, and let Z be the union of the separatrices of $\mathcal{F}$. We define

$$\mathscr{C}(\mathcal{F}) = \{Y \subset X \mid \operatorname{supp}(Y) = Z, Y \text{ is strongly tangent to } \mathcal{F}\},$$

where the elements of $\mathscr{C}(\mathcal{F})$ are formal subschemes of X.

Remark 11.2. $\mathscr{C}(\mathcal{F}) \neq \emptyset$ if and only if Z itself is strongly tangent to $\mathcal{F}$.

Lemma 11.3. *If $\mathscr{C}(\mathcal{F}) \neq \emptyset$, then it has a maximal element.*

Proof: By Corollary 9.24, all ascending chains of strongly tangent formal subschemes through the singular locus have a strongly tangent upper bound, namely, the direct limit; the result follows by Zorn's lemma. □

Definition 11.4. Such a formal scheme, if it exists, is called the *total separatrix* of $\mathcal{F}$. It is the maximal strongly tangent formal scheme passing through the singular locus.

A non-dicritical foliation with a total separatrix is called *totally separable.*

Lemma 11.5. *If there exists a scheme C supported on the union Z of the separatrices of $\mathcal{F}$ which is fully tangent to $\mathcal{F}$, then C is the total separatrix.*

Proof: As C is fully tangent and supported on Z, we have $J_m(C) = J_m(\mathcal{F})|_Z$, for all $m \in \mathbb{N}$. For any scheme S supported on Z and strongly tangent to $\mathcal{F}$, we have for all $m \in \mathbb{N}$ that $J_m(S) \subset J_m(\mathcal{F})|_Z$, and therefore

$J_m(S) \subset J_m(C)$, for all $m \in \mathbb{N}$. As S and C have the same support, we have that $S \subset C$ by Lemma 6.28, and so C is maximal among all such schemes. Therefore C serves as a total separatrix for $\mathcal{F}$. $\square$

Example 11.6. If a foliation has a first integral, that is, it is generated by a form $\omega = df$ for some function f, then it has total separatrix $\mathbb{V}(f)$. (See Lemma 9.4). In particular, the total separatrix need not be irreducible.

Example 11.7. The foliation of the plane generated by $\omega = ydx-(x+y)dy$ has total separatrix $\mathbb{V}(xy, y^2)$. (See Example 9.7). So the total separatrix need not be reduced.

Example 11.8. The foliation of the plane given by $\omega = (y-x)dx - x^2dy$ has total separatrix $\mathbb{V}(xy - \sum_{m=0}^{\infty} m!x^{m+2})$. So the total separatrix need not be an ordinary scheme.

Lemma 11.9. *Let $S \subset X$ be a scheme. Let $\pi : X' \to X$ be a surjective morphism of schemes, with $S' = \pi^{-1}(S)$. Let $Z \subset X'$ be a scheme with the same support as S', such that S' is a strict subscheme of Z. Then S is a strict subscheme of $\pi(Z)$.*

Proof: The scheme S' is the fibre product $S' = S \times_X X'$. Clearly $S \subset \pi(Z)$; if there is equality, then the diagram

$$\begin{array}{ccc} Z & \hookrightarrow & X' \\ \downarrow & & \downarrow \pi \\ S & \hookrightarrow & X \end{array}$$

commutes, and so by the universal property of fibre products, $Z \cong S'$, a contradiction. The conclusion of the lemma follows. $\square$

Lemma 11.10. *Consider the function $f : D \to \mathbb{C}, z \mapsto z^\lambda$, for some $\lambda \in \mathbb{C}$ with $\mathfrak{Re}\lambda \geq 0$, and where $D \subset \mathbb{C}$ is an open domain on which the function is well-defined, with $0 \in \bar{D}$. (For example, D can be taken as a slitted disc $\Delta \setminus [0,1]$.) If $\mathfrak{Re}\lambda > 0$, then $\lim_{z\to 0} f(z) = 0$. If $\mathfrak{Re}\lambda = 0$, then the limit is undefined.*

Proof: We write $\lambda = a + bi, a, b \in \mathbb{R}, a \geq 0$, and the variable z as $z = re^{i\theta}$. Then $z^\lambda = r^a r^{bi} e^{ai\theta} e^{-\theta b}$. The limit as z tends to zero is the limit as r tends to zero: The θ terms are non-zero and so do not contribute, and the term

$r^{bi} = e^{b \ln ri}$ is non-zero but has no limit as $r \to 0$. Therefore we consider the r^a term. If $a > 0$, then $\lim_{r\to 0} r^a = 0$, and so the limit for z is also zero. If $a = 0$, then $\lim_{r\to 0} r^a = 1$; the limit for z is thus undefined. □

Proposition 11.11. *Pre-simple singularities of types (A) and (B) that are not simple are dicritical.*

Proof: We consider type (A) first. By a suitable choice of formal co-ordinates, the foliation is given locally about the singular point by

$$\omega = \lambda_1 x_2 \cdots x_t dx_1 + \lambda_2 x_1 x_3 \cdots x_t dx_2 + \cdots + \lambda_t x_1 \cdots x_{t-1} dx_t,$$

where there is a resonance relation $\sum_{i=1}^{t} r_i \lambda_i = 0, r_i \in \mathbb{N}_0$.

Claim: Without loss of generality, we may assume that there is at least one $i \in \{1, \ldots, t\}$ with $\mathfrak{Re}\lambda_i > 0$, and at least one with $\mathfrak{Re}\lambda_i < 0$.

Indeed, if all the real parts have the same sign, then as the resonance relation also holds for the real parts, we must have $r_i = 0$ whenever $\mathfrak{Re}\lambda_i \neq 0$. The other λ_i have non-zero imaginary part, and as the imaginary parts also satisfy the resonance relation, there must be at least one with $\mathfrak{Im}\lambda_i > 0$ and at least one with $\mathfrak{Im}\lambda_i < 0$. As we can divide ω through by $\sqrt{-1}$, the claim is proved.

Now ω has solutions of the form $x_1^{\lambda_1} \cdots x_t^{\lambda_t} = \mu, \mu \in \mathbb{C}$. Take a fixed $\mu \neq 0$. By the claim, we can re-arrange the equation so that on both sides, the $x_i^{\lambda_i}$ terms have $\mathfrak{Re}\lambda_i \geq 0$, with at least one strictly positive term. By Lemma 11.10's results on limits, it follows that the origin lies on this leaf. As μ was arbitrary, we have that every leaf of the foliation passes through the origin, and so is a separatrix. Therefore the foliation is dicritical about the singularity.

For type (B), we note that the foliation is of type (A) around a component of the singular locus, so we are done. □

Proposition 11.12. *Let $\mathcal{F}$ be a foliation, and let x be a singular point with $\tau(\mathcal{F}, x) = t$, which is adapted to an SNC divisor E with t components but not pre-simple. Then there are non-reduced schemes supported on E that are strongly tangent to $\mathcal{F}$ in a neighbourhood of x.*

Proof: In a neighbourhood of x, choose holomorphic co-ordinates so that $x = 0$ and the components of the divisor are given by $x_i = 0$. $\mathcal{F}$ is given by the 1-form

$$\omega = x_2 \cdots x_t b_1 dx_1 + x_1 x_3 \cdots x_t b_2 dx_2 + \cdots + x_1 \cdots x_{t-1} b_t dx_t,$$

where $b_i = b_i(x_1, \ldots, x_t)$ are holomorphic functions of order at least 1. (See Proposition 10.12.) We now blow up at the origin: For $j = 1, \ldots, t$, the map is given in the corresponding chart by $x_i = x_j v_i, i \neq j$.

In the jth chart, the pre-image foliation is given by the form

$$x_j^{t-1} \prod_{i=1, i\neq j}^{t} v_i \left(\sum_{i \neq j} x_j b_i \frac{dv_i}{v_i} + (b_1 + \cdots + b_t) dx_j \right),$$

where $b_i = b_i(x_j v_1, \ldots, x_j v_{j-1}, x_j, x_j v_{j+1}, \ldots, x_j v_t)$. As the order of each b_i is at least 1, in the new co-ordinates we can factor out a further copy of x_j. It is clear that $\mathbb{V}(x_j)$ is a solution for the transformed foliation, as is $\mathbb{V}(v_i)$ for all $i \neq j$—they are therefore strongly tangent by Lemma 9.19. These hyperplanes form the components of an SNC divisor, so by Propositions 10.13 and 9.23, we therefore have that $\mathbb{V}(v_1 \cdots v_{j-1} x_j^{t+1} v_{j+1} \cdots v_t)$ is strongly tangent to the transformed foliation. Blowing back down each chart by replacing v_i with $\frac{x_i}{x_j}$, we see by Corollary 9.15 that $\mathbb{V}(x_1^2 x_2 \cdots x_t, \ldots, x_1 \cdots x_{t-1} x_t^2)$ is strongly tangent to $\mathcal{F}$. □

Lemma 11.13. *Suppose a foliation $\mathcal{F}$ has total separatrix E, an NC divisor. Then for each point $x \in \operatorname{Sing} \mathcal{F}$, $e(E, x) = \tau(\mathcal{F}, x)$ (where $e(E, x)$ is the number of local components of E through x).*

Proof: Suppose not. Let $\tau(\mathcal{F}, x) = t$, and choose holomorphic co-ordinates about x so that $\mathcal{F}$ is given by $\omega = b_1 dx_1 + \cdots + b_t dx_t$, where b_i are holomorphic, and that $x_1 = 0, \ldots, x_k = 0$, are the components of E through x. By Proposition 10.12, we have $e(E, x) \leq \tau(\mathcal{F}, x)$, and so here we have $k < t$.

For $1 \leq i \leq k$, we can write $b_i = x_1 \cdots \hat{x}_i \cdots x_k b_i'$, where the hat denotes omission; for $i > k$ we can write $b_i = x_1 \cdots x_k b_i'$. Thus the vector fields

$$x_2 b_1' \frac{\partial}{\partial x_2} - x_1 b_2' \frac{\partial}{\partial x_1}, \ldots, x_k b_1' \frac{\partial}{\partial x_k} - x_1 b_k' \frac{\partial}{\partial x_1},$$
$$b_1' \frac{\partial}{\partial x_{k+1}} - x_1 b_{k+1}' \frac{\partial}{\partial x_1}, \ldots, b_1' \frac{\partial}{\partial x_t} - x_1 b_t' \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_{t+1}}, \ldots, \frac{\partial}{\partial dx_n}$$

annihilate ω at x. As the dimensional type is t, we conclude that $b_1'(x) = 0$.

Using a different collection of vector fields, we also have that $b_i'(x) = 0$, for all $i \in \{2, \dots, k\}$, and hence $\operatorname{ord} b_i \geq 1, 1 \leq i \leq k$.

We now blow up at x, with the charts given by $x_i = x_j v_i, i = 1, \dots, t$.

If $j \leq k$, then in the jth chart, the pre-image foliation is given by the form

$$x_j^{k-1}\left(\sum_{i=1, i\neq j}^{k} v_1 \cdots v_{j-1} x_j v_{j+1} \cdots v_k b_i' \left(\frac{dv_i}{v_i} + \frac{dx_j}{x_j}\right) + \right.$$
$$\left. v_1 \cdots v_{j-1} v_{j+1} \cdots v_k b_j' dx_j + \sum_{i=k+1}^{t} v_1 \cdots v_{j-1} x_j v_{j+1} \cdots v_k b_i'(x_j dv_i + v_i dx_j)\right).$$

As the order of each $b_i, 1 \leq i \leq k$, is at least 1, in the new co-ordinates we can factor out a further copy of x_j. It is clear that $\mathbb{V}(x_j)$ is a solution of the transformed foliation, and hence strongly tangent by Lemma 9.19, as is $\mathbb{V}(v_i)$ for all $i \neq j, i \leq k$. By Propositions 10.13 and 9.23, we therefore have that $\mathbb{V}(v_1 \cdots v_{j-1} x_j^{k+1} v_{j+1} \cdots v_k)$ is strongly tangent to the transformed foliation.

If $j > k$, then in the jth chart, the pre-image foliation is given by the form

$$x_j^k\left(\sum_{i=1}^{k} v_1 \cdots v_k b_i' \left(\frac{dv_i}{v_i} + \frac{dx_j}{x_j}\right) + \right.$$
$$\left. \sum_{i=k+1, i\neq j}^{t} v_1 \cdots v_k b_i'(x_j dv_i + v_i dx_j) + v_1 \cdots v_k b_j' dx_j\right).$$

Again we see that $\mathbb{V}(v_1), \dots, \mathbb{V}(v_k)$ are strongly tangent to the transformed foliation; so is $\mathbb{V}(x_j)$, as we can factor out a copy of x_j from each of the $b_i, 1 \leq i \leq k$. Thus by Propositions 10.13 and 9.23 $\mathbb{V}(v_1 \cdots v_k x_j^{k+1})$ is strongly tangent to the transformed foliation. Blowing back down each chart, we see by Corollary 9.15 that

$$\mathbb{V}(x_1^2 x_2 \cdots x_k, \dots, x_1 \cdots x_{k-1} x_k^2, x_1 \cdots x_k x_{k+1}, \dots, x_1 \cdots x_k x_t)$$

is strongly tangent to $\mathcal{F}$, thus contradicting maximality of E. □

Proposition 11.14. *A foliation $\mathcal{F}$ has all singularities being simple if and only if it is totally separable, and its total separatrix is a normal crossings divisor.*

Proof: Suppose all singularities are simple. Then by Theorem 10.3 (for $\dim X = 2$) and Corollary 10.28 (for $\dim X \geq 3$), at each singular point x, there exists an SNC divisor E_x, whose components are the separatrices, and which is fully tangent on a neighbourhood of x. The E_x can be glued together to give an NC divisor C, which is fully tangent to $\mathcal{F}$. Thus C is the total separatrix.

Conversely, if the total separatrix C exists and is NC, then by Lemma 11.13, the number of local components of C through a singular point x is $\tau(\mathcal{F}, x)$. Proposition 11.12 excludes the case that x is non-pre-simple; Proposition 11.11 excludes the case that x is pre-simple of type (A) or (B) but not simple. Suppose x is pre-simple of type (C). Then there exists a point y in a neighbourhood of x with $\tau(\mathcal{F}, y) = 2$ and $e(C, y) = 1$, (see the proof of Theorem 10.26). This is a contradiction to Lemma 11.13, so we have the result. □

Theorem 11.15. *Let $\mathcal{F}$ be a non-dicritical codimension-1 foliation on X. Then $\mathcal{F}$ admits a resolution to simple (or reduced, if $\dim X = 2$) singularities if and only if $\mathcal{F}$ is totally separable, and its total separatrix C admits a resolution to a normal crossings support divisor. In this case, the resolutions can be assumed to be the same map.*

Proof: Suppose C exists, and admits a log resolution $\pi : X' \to X$. Let $\hat{\mathcal{F}} = \mathrm{sat}(\pi^{-1}(\mathcal{F}))$. Suppose $\hat{\mathcal{F}}$ is given locally by the form ω, so that $\pi^{-1}(\mathcal{F})$ is given by $f\omega$, for some holomorphic function f. If $\pi^{-1}(C)$ is given by the function g, we let $\hat{C}$ be the complex space defined by $\frac{g}{f}$. (This is indeed holomorphic, as f defines the non-reduced structure from the exceptional divisors. We can recover $\pi^{-1}(C)$ from $\hat{C}$ by adding in the exceptional divisors.)

Now $\hat{C}$ is SNC, and is strongly tangent to $\hat{\mathcal{F}}$ (by Proposition 10.13, as each component is a separatrix and hence strongly tangent). Moreover, $\hat{C}$ is the total separatrix. Indeed, if there exists a larger strongly tangent scheme supported on $\hat{C}$, then by Proposition 9.23, adding in the exceptional divisors of π yields a scheme strongly tangent to $\pi^{-1}(\mathcal{F})$ and strictly larger than $\pi^{-1}(C)$; by Corollary 9.15 blowing down yields a scheme strongly tangent to $\mathcal{F}$ and strictly larger than C by Lemma 11.9, a contradiction.

Therefore by Proposition 11.14, $\hat{\mathcal{F}}$ has all singularities simple, so π is a resolution of singularities for $\mathcal{F}$.

Conversely, suppose there is a resolution $\pi : X' \to X$ such that $\hat{\mathcal{F}} = \mathrm{sat}(\pi^{-1}(\mathcal{F}))$ has only simple singularities. By Proposition 11.14, $\hat{\mathcal{F}}$ is totally separable, and its total separatrix E is NC and fully tangent. By Proposition 9.23, adding in the exceptional divisors of π yields a scheme fully tangent to $\pi^{-1}(\mathcal{F})$; blowing down yields a scheme C, which is fully tangent to $\mathcal{F}$ by Corollary 9.15, and hence the total separatrix for $\mathcal{F}$ by Lemma 11.5, and which is resolved by π. □

Corollary 11.16. *If $\mathcal{F}$ admits a resolution, then its total separatrix is fully tangent.*

Definition 11.17. Let $\mathcal{F}$ be a codimension-1 foliation on X given by ω, and suppose in the some co-ordinate chart we have $0 \in \mathrm{Sing}\,\mathcal{F}$. The *germ* of $\mathcal{F}$ at 0, denoted $\mathcal{F}_0$, is given by writing ω in the germs of the co-ordinate functions at 0. It is defined on the formal completion $\hat{X}_0$.

By choosing appropriate co-ordinates, we can define the germ of $\mathcal{F}$ at any singular point. Now suppose $\mathcal{F}$ is totally separable, with total separatrix C, and let $P \in \mathrm{Sing}\,\mathcal{F}$. Then $C \cap \hat{X}_P$ is the total separatrix for the germ $\mathcal{F}_P$. It satisfies the hypotheses of Theorem 6.29, and so admits a desingularisation. By Theorem 11.15, $\mathcal{F}_P$ can be desingularised (as a sheaf on the formal completion).

Proposition 11.18. *Let $\mathcal{F}$ be a non-dicritical codimension-1 foliation on X. If $\mathcal{F}$ is totally separable, then its total separatrix C is fully tangent.*

Proof: Choose a point $P \in \mathrm{Sing}\,\mathcal{F}$. Then $C \cap \hat{X}_P$ is the total separatrix for the germ $\mathcal{F}_P$. $\mathcal{F}_P$ can then by desingularised, so by Corollary 11.16, $C \cap \hat{X}_P$ is fully tangent to $\mathcal{F}_P$. Then for all $m \in \mathbb{N}$

$$J_m(C,P) = J_m(C \cap \hat{X}_P, P) = J_m(\mathcal{F}_P, P) = J_m(\mathcal{F}, P).$$

As this holds for any singular point P, and C is clearly fully tangent over the smooth locus, it follows that C is fully tangent to $\mathcal{F}$. □

Corollary 11.19. *If $\mathcal{F}$ is totally separable, then the total separatrix is unique.*

Proof: Suppose C, C' are two candidates for the total separatrix. They are both fully tangent, and both have the same support. Then for any $x \in$

$C, m \in \mathbb{N}$, $J_m(C,x) = J_m(\mathcal{F},x) = J_m(C',x)$. It follows from Lemma 6.28 that $C = C'$. □

It then remains to show whether the total separatrix indeed exists. By Remark 11.2 and Lemma 11.3, this is equivalent to the union Z of the separatrices of $\mathcal{F}$ being strongly tangent.

We introduce the following notation:

Notation 11.20. Let $\mathcal{F}$ be a non-dicritical codimension-1 foliation on X. We denote by $\mathfrak{V}(\mathcal{F})$ the set of hypersurfaces $V \subset X$ which are truly transversal to $\mathcal{F}$. (See Definition 9.37).

We then have the following conjecture:

Conjecture 11.21. Let $\mathcal{F}$ be a non-dicritical codimension-1 foliation on X, and let Z be the union of its separatrices. Then for each $m \in \mathbb{N}$, $\bigcup_{V \in \mathfrak{V}(\mathcal{F})} J_m(Z) \cap J_m(V)$ is dense in $J_m(Z)$.

Assuming Conjecture 11.21, we have the following:

Proposition 11.22. *Let $\mathcal{F}$ be a non-dicritical codimension-1 foliation on X, and let Z be the union of its separatrices. Then Z is strongly tangent to $\mathcal{F}$. In particular, $\mathcal{F}$ is totally separable.*

Proof: We use induction on the dimension of X. If $\dim X \leq 3$, the result holds by Theorems 10.31 and 11.15. Now assume $\dim X = n$, and that the result holds for foliations on manifolds of lower dimension. Let V be a truly transversal hypersurface. Then $Z \cap V$ is a union of leaves of the foliation $\mathcal{F}|_V$ on V. By hypothesis, the union of all separatrices of $\mathcal{F}|_V$ is strongly tangent, and so any union of separatrices is strongly tangent. As leaves which are not separatrices are disjoint, we have

$$J_m(Z) \cap J_m(V) = J_m(Z \cap V) \subset J_m(\mathcal{F}|_V) = J_m(\mathcal{F}) \cap J_m(V),$$

for each $m \in \mathbb{N}$. Hence $\bigcup_{V \in \mathfrak{V}(\mathcal{F})} J_m(Z) \cap J_m(V) \subset J_m(\mathcal{F})$. By Conjecture 11.21, we can take the closure on both sides, and have $J_m(Z) \subset J_m(\mathcal{F})$. □

Remark 11.23. A priori, we cannot assume that the total separatrix is locally countably indexed, so total separability is not enough to prove desingularisation.

12 Dicritical Foliations

Let X be a quasi-compact complex manifold of dimension n, and let $\mathcal{F}$ be a codimension-1 foliation on X given by an algebraic 1-form. In this section we also allow for dicritical foliations.

Lemma 12.1. *Let Y be a formal scheme supported on a point $P \in X$, and let $r \in \mathbb{N}$. Suppose that $J_k(Y,P) = \mathbb{A}^{kn}$ for all $k \leq r$. Then $P^{r+1} \subset Y$.*

Proof: Suppose Y is an ordinary scheme, and consider the intersection of all schemes satisfying the condition on the jets. This is contained in Y. P^{r+1} also satisfies the condition; by looking at the generators we see that any proper subscheme of P^{r+1} does not. Therefore P^{r+1} is equal to the intersection, and is contained in Y.

Now suppose $Y = \varinjlim Y_\lambda$. By the proof of Theorem 6.29, one of the Y_λ satisfies the condition on the jets; hence $P^{r+1} \subset Y_\lambda \subset Y$. □

Proposition 12.2. *Let Y be a formal scheme supported on a smooth hypersurface $H \subset X$, and let $r \in \mathbb{N}$. Suppose that $J_k(Y,x) = \mathbb{A}^{kn}$ for all $k \leq r$ and all $x \in H$. Then $H^{r+1} \subset Y$.*

Proof: Let $x \in H$. Then for $k \leq r$, $J_k(Y \cap \hat{X}_x, x) = J_k(Y,x) \cap J_k(\hat{X}_x, x) = \mathbb{A}^{kn}$. Then $Y \cap \hat{X}_x$ satisfies the assumptions of Lemma 12.1, and so contains x^{r+1}. This holds for every point $x \in H$, so it follows that Y contains H^{r+1}. □

Lemma 12.3. *Let $\mathcal{F}$ be a foliation, and suppose there is a fully tangent formal scheme Y supported on the singular locus. Then for any sequence of blow-ups $\pi : X' \to X$, $\pi^{-1}(\mathcal{F})$ has a fully tangent formal scheme supported on its singular locus.*

Proof: By Proposition 9.14, $\pi^{-1}(Y)$ is fully tangent to $\pi^{-1}(\mathcal{F})$, and is supported on its singular locus. □

Assuming Conjecture 11.21, we have the following:

Proposition 12.4. *Let $\mathcal{F}$ be a foliation on X. Then $\mathcal{F}$ is non-dicritical if and only if there is a (unique) fully tangent formal scheme Y supported on the singular locus $\Sigma = \operatorname{Sing} \mathcal{F}$.*

Proof: Suppose $\mathcal{F}$ is non-dicritical. Then by Proposition 11.22, $\mathcal{F}$ admits a total separatrix C, which by Proposition 11.18 is fully tangent. Then $C \cap \hat{X}_\Sigma$ is fully tangent and supported on the singular locus.

If Y is another fully tangent formal scheme supported on Σ, then for each $m \in \mathbb{N}$ and $x \in Y$, $J_m(Y, x) = J_m(\mathcal{F}, x) = J_m(C \cap \hat{X}_\Sigma, x)$. As they have the same support, it follows that $Y = C \cap \hat{X}_\Sigma$.

Now suppose $\mathcal{F}$ is dicritical. Then there is a sequence of blow-ups $\pi : X' \to X$ such that one of the exceptional divisors of π is transversal to the leaves of $\mathrm{sat}(\mathcal{G})$, where $\mathcal{G} = \pi^{-1}(\mathcal{F})$. On the smooth locus of $\mathrm{sat}(\mathcal{G})$ we can write a generator as dx_2 in some co-ordinate chart; as the exceptional divisor is transversal to this, we can choose holomorphic co-ordinates on some open set $U \subset X'$, contained in the smooth locus of $\mathrm{sat}(\mathcal{G})$, such that the underlying reduced scheme of the divisor is $\mathbb{V}(x_1)$. Hence $\mathcal{G}$ is given by the 1-form $x_1^r dx_2$, for some $r \in \mathbb{N}$.

Suppose there is a fully tangent formal scheme Y supported on $\mathrm{Sing}\,\mathcal{F}$. Then by Lemma 12.3, there is a formal scheme Y' fully tangent to $\mathcal{G}|_U$ and supported on $H = \mathbb{V}(x_1)$. Now for $k \leq r$, $J_k(Y', x) = \mathbb{A}^{kn}$, for all $x \in H$. So by Proposition 12.2, $H^{r+1} = \mathbb{V}(x_1^{r+1}) \subset Y'$. However, H^{r+1} is not strongly tangent to $\mathcal{G}$: The $2r+1$-jet given by $x_1 = t^2, x_2 = t, x_3 = \cdots = x_n = 0$ is a jet in $J_{2r+1}(\mathbb{V}(x^{r+1}), 0)$ but not in $J_{2r+1}(\mathcal{G}, 0)$. This is a contradiction, so the formal scheme Y does not exist. □

A candidate for the formal scheme Y in Proposition 12.4 can be constructed as follows:

Let $\mathcal{F}$ be a foliation on X with singular locus Σ. We define $S_m(\mathcal{F})$ to be the smallest formal subscheme supported on Σ such that $J_k(\mathcal{F}, x) \subset J_k(S_m(\mathcal{F}), x)$ for all $k \leq m$ and all $x \in \Sigma$. We define the *hull of jets* to be the formal scheme $\mathscr{S}(\mathcal{F}) = \varinjlim S_m(\mathcal{F})$. As $S_m(\mathcal{F}) \subset \Sigma^{m+1}$, we have $\mathscr{S}(\mathcal{F}) \subset \hat{X}_\Sigma$.

Proposition 12.5. *$\mathscr{S}(\mathcal{F})$ is the smallest formal scheme supported on Σ such that $J_k(\mathcal{F}, x) \subset J_k(\mathscr{S}(\mathcal{F}), x)$ for all $k \in \mathbb{N}$ and all $x \in \Sigma$.*

Proof: Let Y be another such formal scheme. For each $m \in \mathbb{N}$, we have $J_k(\mathcal{F}, x) \subset J_k(Y, x)$ for all $k \leq m$ and all $x \in \Sigma$. Hence $S_m(\mathcal{F}) \subset Y$, for all $m \in \mathbb{N}$, and so $\mathscr{S}(\mathcal{F}) \subset Y$. □

We can then reformulate Proposition 12.4 as follows (again assuming Conjecture 11.21):

Proposition 12.6. *Let $\mathcal{F}$ be a foliation on X with singular locus Σ. Then $\mathcal{F}$ is dicritical if and only if $\mathscr{S}(\mathcal{F})$ is not strongly tangent to $\mathcal{F}$.*

Proof: If $\mathcal{F}$ is dicritical, then by Proposition 12.4 there is no formal scheme supported on Σ which is fully tangent. In particular, $\mathscr{S}(\mathcal{F})$ is not fully tangent, and hence not strongly tangent.

Conversely, if $\mathscr{S}(\mathcal{F})$ is not strongly tangent, then by Proposition 12.5, any formal scheme Y supported on Σ with $J_m(\mathcal{F}, x) \subset J_m(Y, x)$ for all $m \in \mathbb{N}$ and all $x \in \Sigma$ is not strongly tangent, and hence none of them is fully tangent. By Proposition 12.4, $\mathcal{F}$ is dicritical. □

If $\mathscr{S}(\mathcal{F})$ is strongly tangent, then $\mathcal{F}$ is non-dicritical. We have $\mathscr{S}(\mathcal{F}) = C \cap \hat{X}_\Sigma$, where C is the total separatrix. The codimension-1 components of $\mathscr{S}(\mathcal{F})$ are the germs of the separatrices, so if we start with $\mathscr{S}(\mathcal{F})$ we can reconstruct the total separatrix by taking the union with those separatrices which are ordinary schemes.

Proposition 12.7. *Let $\mathcal{F}$ be a dicritical foliation on X which has a resolution to simple singularities. Then there is a family of strongly tangent subschemes $(C_\alpha)_{\alpha \in A}$ of X such that $J_m(\mathcal{F}, x) = \bigcup_{\alpha \in A} J_m(C_\alpha, x)$, for all $m \in \mathbb{N}$ and all $x \in \operatorname{Sing} \mathcal{F}$.*

Proof: We can take π to be a partial resolution of $\mathcal{F}$, such that $\mathcal{G} = \operatorname{sat}(\pi^{-1}(\mathcal{F}))$ is non-dicritical. Then $\mathcal{G}$ has a resolution to simple singularities (by completing the resolution of $\mathcal{F}$), and so by Theorem 11.15 is totally separable, with fully tangent total separatrix. Let D be the exceptional divisor of π. As $\mathcal{F}$ is dicritical, at least one of the components of D is transversal to the leaves of $\mathcal{G}$.

For each $x \in D$, let L_x be either the unique leaf of $\mathcal{G}$ through x, or the total separatrix of $\mathcal{G}$, as appropriate. Then for each $m \in \mathbb{N}$, $J_m(\mathcal{G}, x) = J_m(L_x, x)$. So by Proposition 9.23, $J_m(\pi^{-1}(\mathcal{F}), x) = J_m(L_x \cup D, x)$. Using Corollary 9.15 we blow back down to get $J_m(\mathcal{F}, x) = \bigcup J_m(\pi(L_x \cup D), x)$, which yields the result. □

Example 12.8. Let $X = \mathbb{A}^2$, and let $\mathcal{F}$ be given by $\omega = y^2 dx - x^2 dy$. This has a single dicritical singularity at the origin. The leaves of the foliation are $\{x = 0\}$, $\{y = 0\}$, $\{y = x\}$, and $\{xy + kx - ky = 0\}, k \in \mathbb{C}^*$, all of which are separatrices.

We blow up at the origin. In the first chart we set $y = xv$, and get a new foliation given by $x^2((v^2 - v)dx - xdv)$. This has two singularities: $(0, 0)$, which is reduced, and $(0, 1)$, which is again dicritical.

We blow up this second singularity. In one chart we set $v - 1 = xt$, which leaves us with $x^4(t^2dx - dt)$. The saturation of this foliation is smooth, and has leaves $\mathbb{V}(t)$ and $\mathbb{V}(xt + kt + 1), k \in \mathbb{C}$. In the other chart we set $x = (v - 1)s$, which gives $(v - 1)^4 s^2(vds + sdv)$, the saturation of which has a single simple singularity.

So the jets of the unsaturated foliation along the exceptional divisor are of the form $J_m(\mathbb{V}(x^4(xt + kt + 1)), (0, -1/k))$ and $J_m(\mathbb{V}(x^4 t), (0, 0))$ in one chart, and $J_m(\mathbb{V}(v(v - 1)^4 s^3), (0, 1))$ in the other. Blowing down, the jets are of the form $J_m(\mathbb{V}(x^3 v(v - 1)))$ and $J_m(\mathbb{V}(x^3(xv + kv - k)))$.

In the second chart of the initial blow-up, we set $x = yb$ to get the form $y^2((b - b^2)dy + ydb)$. This also has two singularities: $(0, 0)$, which is reduced, and corresponds to the reduced singularity in the first chart, and $(1, 0)$, which is dicritical. Applying the same method as before, we see that the jets are of the form $J_m(\mathbb{V}(b(b - 1)y^3))$ and $J_m(\mathbb{V}(y^3(yb + kb - k)))$.

Blowing down again, we have

$$J_m(\mathcal{F}, 0) = J_m(\mathbb{V}(x^2y - xy^2), 0) \cup \bigcup_{k \in \mathbb{C}^*} J_m(\mathbb{V}(x^2(xy + kx - ky), y^2(xy + kx - ky)), 0).$$

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