

HYPERBOLIC GROUPS AND NON-COMPACT REAL ALGEBRAIC CURVES

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ABSTRACT. In this paper we study the spaces of non-compact real algebraic curves, i.e. pairs (P, τ) , where P is a compact Riemann surface with a finite number of holes and punctures and $\tau : P \rightarrow P$ is an anti-holomorphic involution. We describe the uniformisation of non-compact real algebraic curves by Fuchsian groups. We construct the spaces of non-compact real algebraic curves and describe their connected components. We prove that any connected component is homeomorphic to a quotient of a finite-dimensional real vector space by a discrete group and determine the dimensions of these vector spaces.

In memory of Ernest Borisovich Vinberg

1. INTRODUCTION

It is well known that the category of complex algebraic curves is isomorphic to the category of compact Riemann surfaces. For a complex algebraic curve generated by the polynomial $F(x, y)$ with complex coefficients, the corresponding compact Riemann surface P is obtained as the regularisation of the set of complex solutions of the equation $F(x, y) = 0$. If the Riemann surface P has a finite number of holes and punctures, we will say that the corresponding complex curve has holes and punctures.

Similarly the category of real algebraic curves is isomorphic to the category of pairs (P, τ) , where P is a compact Riemann surface and $\tau : P \rightarrow P$ is an anti-holomorphic involution on P . For a real algebraic curve generated by the polynomial $F(x, y)$ with real coefficients, the corresponding pair (P, τ) consists of the compact Riemann surface P as above and the involution τ which is generated by the complex conjugation $(x, y) \mapsto (\bar{x}, \bar{y})$. The set of fixed points of the involution τ is called the *set of real points* of the real curve (P, τ) and is denoted by P^τ . If the Riemann surface P has a finite number of holes and punctures, we will say that (P, τ) is a real curve with holes and punctures. Those holes and punctures that are invariant under the involution τ will be called real holes and real punctures. The remaining holes and punctures occur in pairs which are mapped by the involution τ .

The set of real points of a compact real curve decomposes into pairwise disjoint simple closed smooth curves that are called *ovals*. The *topological type* of the compact real curve (P, τ) is determined by the triple (g, k, ε) , where g is the genus

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of P , k is the number of ovals and $\varepsilon \in \{0, 1\}$ with $\varepsilon = 1$ if $P \setminus P^\tau$ is orientable and $\varepsilon = 0$ otherwise.

We can compactify a real curve (P, τ) by closing every hole with a disk and every puncture with a point and extending τ to an anti-holomorphic involution $\hat{\tau}$ on the resulting surface \hat{P} . The ovals of $(\hat{P}, \hat{\tau})$ are called *compactified ovals*. The *type* of a compactified oval is given by the cyclic sequence of glued in disks and points. The *topological type* of the real curve (P, τ) is determined by the type (g, k, ε) of the corresponding compactified real curve $(\hat{P}, \hat{\tau})$, the numbers $2n_I$ and $2m_I$ of non-real holes and punctures and the types of compactified ovals. The numbers n_R and m_R of real holes and punctures respectively are determined by the type of the real curve. In this paper we will assume that $2g + 2n_I + n_R + 2m_I + m_R > 2$ as the other cases are easier, but require different techniques.

The aim of this paper is to prove the following description of the moduli space of real curves with holes and punctures.

Theorem 1.1. *The moduli space \mathcal{M}_t of all real algebraic curves of type*

$$t = (g, k, \varepsilon | 2n_I, 2m_I, \text{types of compactified ovals}, n_R, m_R)$$

is not empty if and only if $1 \leq k \leq g + 1$ and $k \equiv g + 1 \pmod{2}$ in the case $\varepsilon = 1$ and $0 \leq k \leq g$ in the case $\varepsilon = 0$. Under these conditions the moduli space \mathcal{M}_t has a natural topological structure, is connected and is the quotient $\mathcal{T}_t / \text{Mod}_t$, where \mathcal{T}_t is homeomorphic to a real vector space of dimension $3g - 3 + 3n_I + 2m_I + 2n_R + m_R$ and Mod_t is a discrete group.

This result was previously only described for compact real algebraic curves (Theorem 2 in [N75] and §2 in [N78]).

2. SEQUENTIAL SETS OF AUTOMORPHISMS

We will recall some standard results in complex analysis (see for example Chapter 5 in [N18]). Consider the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$. Holomorphic and anti-holomorphic automorphisms of \mathbb{H} are isometries with respect to the hyperbolic metric $ds = \frac{|dz|}{\text{Im}(z)}$ on \mathbb{H} . Geodesics in this geometry are half-circles with centre on the real axis and rays orthogonal to the real axis. The group $\text{Aut}(\mathbb{H})$ of bi-holomorphic automorphisms of \mathbb{H} is isomorphic to $\text{PSL}(2, \mathbb{R})$ and consists of Möbius transformations

$$C(z) = \frac{az + b}{cz + d}, \quad \text{where } a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0.$$

Bi-holomorphic automorphisms can be classified with respect to the fixed point behavior of their action on \mathbb{H} . An element is called *hyperbolic* if it has two fixed points, which lie on the boundary $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ of \mathbb{H} . A hyperbolic element with fixed points α, β in \mathbb{R} is of the form

$$[\tau_{\alpha, \beta}(\lambda)](z) = \frac{(\lambda\alpha - \beta)z - (\lambda - 1)\alpha\beta}{(\lambda - 1)z + (\alpha - \lambda\beta)},$$

where $\lambda > 0$. A hyperbolic element with one fixed point at ∞ is of the form

$$[\tau_{\infty, \beta}(\lambda)](z) = \lambda z - (\lambda - 1)\beta$$

or

$$[\tau_{\alpha,\infty}(\lambda)](z) = \frac{1}{\lambda}z - \left(\frac{1}{\lambda} - 1\right)\alpha,$$

where α resp. β is the real fixed points and $\lambda > 0$. The parameter $\lambda > 0$ is called the *shift parameter*. The *axis* $\ell(g)$ of the element $g = \tau_{\alpha,\beta}(\lambda)$ is the geodesic between the fixed points α and β , oriented from β to α if $\lambda > 1$ and from α to β if $\lambda < 1$. The element $g = \tau_{\alpha,\beta}(\lambda)$ preserves the geodesic $\ell(g)$ and moves the points on this geodesic in the direction of the orientation. We call a hyperbolic element $\tau_{\alpha,\beta}(\lambda)$ with $\lambda > 1$ *positive* if $\alpha < \beta$.

An element is called *parabolic* if it has one fixed point, which is on the boundary $\partial\mathbb{H}$. A parabolic element with a fixed point α is of the form

$$[\pi_\alpha(\lambda)](z) = \frac{(1 - \lambda\alpha)z + \lambda\alpha^2}{-\lambda z + (1 + \lambda\alpha)}.$$

A parabolic element with fixed point ∞ is of the form

$$[\pi_\infty(\lambda)](z) = z + \lambda.$$

We call a parabolic element $\pi_\alpha(\lambda)$ *positive* if $\lambda > 0$.

An element that is neither hyperbolic nor parabolic is called *elliptic*. It has one fixed point that is in \mathbb{H} . Given a base-point $x \in \mathbb{H}$ and a real number φ , let $\rho_x(\varphi)$ denote the rotation through angle φ about the point x . Any elliptic element is of the form $\rho_x(\varphi)$, where x is the fixed point.

We will call hyperbolic and parabolic automorphisms of \mathbb{H} *shifts*. Riemann surfaces are bi-holomorphic to quotients \mathbb{H}/Γ , where Γ is a Fuchsian group that consists of shifts. Sequential sets are special generating sets of such Fuchsian groups which were introduced in [N72], p. 151. They can be defined as follows:

Definition 2.1. For two elements C_1 and C_2 in $\text{Aut}(\mathbb{H})$ with finite fixed points in \mathbb{R} we say that $C_1 < C_2$ if all fixed points of C_1 are to the left of any fixed point of C_2 .

Definition 2.2. A triple of shifts (C_1, C_2, C_3) in $\text{Aut}(\mathbb{H})$ is a *sequential set* if their product is

$$C_1 \cdot C_2 \cdot C_3 = 1,$$

and for some element $A \in \text{Aut}(\mathbb{H})$ the elements $\{\tilde{C}_i = AC_iA^{-1}\}_{i=1,2,3}$ are positive, have finite fixed points and satisfy $\tilde{C}_1 < \tilde{C}_2 < \tilde{C}_3$.

Figure 1 illustrates the position of the axes of the elements \tilde{C}_i when all elements are hyperbolic.

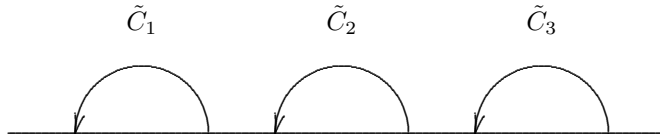


Figure 1: Sequential set

Definition 2.3. A tuple of shifts (C_1, \dots, C_r) in $\text{Aut}(\mathbb{H})$ is a *sequential set* if for any $j \in \{2, \dots, r-1\}$ the triple $(C_1 \cdots C_{j-1}, C_j, C_{j+1} \cdots C_r)$ is a sequential set.

Definition 2.4. A *sequential set of type (g, n, m)* is a $(n + m + 2g)$ -tuple of shifts

$$(C_1, \dots, C_{n+m}, A_1, B_1, \dots, A_g, B_g)$$

in $\text{Aut}(\mathbb{H})$ such that the elements $A_1, \dots, A_g, B_1, \dots, B_g$ and C_1, \dots, C_n are hyperbolic, the elements C_{n+1}, \dots, C_{n+m} are parabolic, and the tuple

$$(C_1, \dots, C_{n+m}, A_1, B_1 A_1^{-1} B_1^{-1}, \dots, A_g, B_g A_g^{-1} B_g^{-1})$$

is a sequential set.

Figure 2 illustrates the position of the axes and fixed points of the elements of a sequential set of type (g, n, m) .

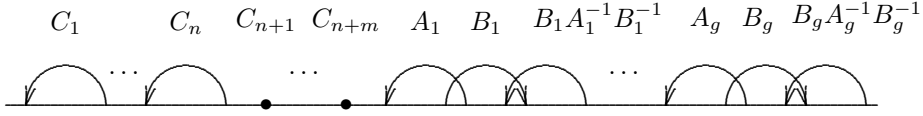


Figure 2: Sequential set of type (g, n, m)

According to Theorem A in [N78] and Theorem 1.1 in [N04], a sequential set \mathcal{V} of type (g, n, m) generates a Fuchsian group $\Gamma(\mathcal{V})$ that consists of shifts such that the surface $P = \mathbb{H}/\Gamma$ is of genus g with n holes and m punctures. We will say that P is a surface of type (g, n, m) .

3. REAL SURFACES WITHOUT REAL HOLES OR REAL PUNCTURES

Let the moduli space $\mathcal{M}_{g,n,m}$ be the space of classes of bi-holomorphic equivalence of Riemann surfaces of type (g, n, m) . According to Theorem 2.1 in [N04], any Riemann surface of type (g, n, m) is bi-holomorphic to a quotient \mathbb{H}/Γ , where Γ is a Fuchsian group generated by a sequential set of type (g, n, m) , hence we can describe the moduli space $\mathcal{M}_{g,n,m}$ via the space of sequential sets of type (g, n, m) . Let $\tilde{\mathcal{T}}_{g,n,m}$ be the set of all sequential sets of type (g, n, m) . The group $\text{Aut}(\mathbb{H}) \cong \text{PSL}(2, \mathbb{R})$ acts on the set $\tilde{\mathcal{T}}_{g,n,m}$ by conjugation. Let $\mathcal{T}_{g,n,m} = \tilde{\mathcal{T}}_{g,n,m} / \text{PSL}(2, \mathbb{R})$ be the quotient of this action. A sequential set of type (g, n, m) consists of $2g + n$ hyperbolic and m parabolic automorphisms which can be described by

$$3(2g + n) + 2m = 6g + 3n + 2m$$

real parameters. The relation

$$C_1 \cdots C_{n+m} \cdot [A_1, B_1] \cdots [A_g, B_g] = 1$$

implies some restrictions on these parameters, hence a point in $\tilde{\mathcal{T}}_{g,n,m}$ can be described by $6g + 3n + 2m - 3$ real parameters. Taking into account the action of the 3-dimensional group $\text{PSL}(2, \mathbb{R})$, we can conclude that the space $\mathcal{T}_{g,n,m}$ is homeomorphic to $\mathbb{R}^{6g+3n+2m-6}$, see Theorem 4.1 in [N04].

Sequential sets in the same orbit of $\mathrm{PSL}(2, \mathbb{R})$ correspond to bi-holomorphically equivalent Riemann surfaces, hence the moduli space $\mathcal{M}_{g,n,m}$ of Riemann surfaces of type (g, n, m) up to bi-holomorphic equivalence is of the form

$$\mathcal{M}_{g,n,m} = \mathcal{T}_{g,n,m} / \mathrm{Mod}_{g,n,m},$$

where $\mathrm{Mod}_{g,n,m}$ is a discrete group of automorphisms of a surface of type (g, n, m) that acts on $\mathcal{T}_{g,n,m}$. Therefore

$$\mathcal{M}_{g,n,m} \simeq \mathbb{R}^{6g-6+3n+2m} / \mathrm{Mod}_{g,n,m}.$$

This fact is due to O. Teichmüller, see [T], §49 on p. 360 and §143 on p. 458 of the English translation.

Now let us consider real algebraic curves. Recall that the topological type of a real curve without real holes and real punctures is determined by the type (g, k, ε) of the corresponding compactified real curve and the numbers $2n_I$ and $2m_I$ of non-real holes and punctures. A slight modification of the proofs in [N04] (Lemma 2.2) leads to the following description of the corresponding Fuchsian groups. A real curve of type $t = (g, k, \varepsilon | 2n_I, 2m_I)$ can be constructed using a sequential set

$$\mathcal{V} = (C_0, C_1, \dots, C_r, A_1, \dots, A_h, B_1, \dots, B_h)$$

of type $(h, g - 2h + 1 + n_I, m_I)$ and hyperbolic automorphisms

$$\mathcal{D} = (D_1, \dots, D_{g-2h}),$$

where the number h and the hyperbolic automorphisms \mathcal{D} depend on the type t . The hyperbolic automorphisms \mathcal{D} are constructed using reflections in geodesics in the hyperbolic space \mathbb{H} . Let R_j denote the reflection in the axis of the hyperbolic automorphism C_j . For $\varepsilon = 1$, let $h = (g - k + 1)/2$ and $D_j = R_0 R_j$ for $j = 1, \dots, g - 2h = k - 1$. For a hyperbolic automorphism C , let \sqrt{C} be the hyperbolic automorphism such that $(\sqrt{C})^2 = C$. For $\varepsilon = 0$ and $k > 0$, let $h = 0$, $D_j = R_0 R_j$ for $j = 1, \dots, k - 1$ and $D_j = R_0 R_j \sqrt{C_j}$ for $j = k, \dots, g$. For $\varepsilon = 0$ and $k = 0$, let $h = 0$ and $D_j = \sqrt{C_0} R_0 R_j \sqrt{C_j}$ for $j = 1, \dots, g$. Figure 3 shows the axes of the elements with the notation $\tilde{A}_i = R_0 A_i R_0$, $\tilde{B}_i = R_0 B_i R_0$, $C_j = R_0 C_j R_0$.

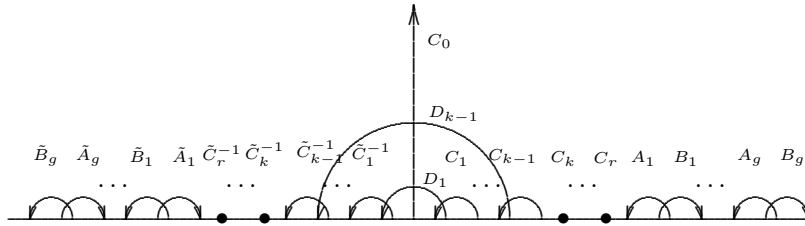


Figure 3: Sequential set for a real curve of type $(g, k, \varepsilon | 2n_I, 2m_I)$

We can show that automorphisms in $\mathcal{V} \cup \mathcal{D}$ generate a Fuchsian group $\Gamma = \Gamma(\mathcal{V} \cup \mathcal{D})$ such that the Riemann surface $P = \mathbb{H}/\Gamma$ is of type $(g, 2n_I, 2m_I)$. Let $\sigma = R_0$ for $k > 0$ and $\sigma = \sqrt{C_0} R_0$ for $k = 0$, then $\sigma \Gamma \sigma = \Gamma$. It follows that σ generates a real curve (P, τ) of type $(g, k, \varepsilon | 2n_I, 2m_I)$. Moreover, it can be shown that every real curve of type t can be constructed in this way.

The hyperbolic metric on the upper half-plane \mathbb{H} induces a hyperbolic metric on the quotient $P = \mathbb{H}/\Gamma$. This metric reflects the geometry of the Fuchsian group Γ . For instance, among all simple closed curves around the hole that corresponds to the generator C_j there is a unique shortest curve c_j . This curve is the image of the axis of C_j under the natural projection $\mathbb{H} \rightarrow P$.

Recall that real curves of type $t = (g, k, \varepsilon | 2n_I, 2m_I)$ are generated by sequential sets of type $(h, g - 2h + n_I, m_I)$, whereby conjugate sequential sets correspond to bi-holomorphically equivalent real curves. The space $\mathcal{T}_{h, g-2h+1+n_I, m_I}$ of conjugacy classes of such sequential sets is homeomorphic to a real vector space of dimension

$$6h + 3(g - 2h + 1 + n_I) + 2m_I - 6 = 3g - 3 + 3n_I + 2m_I.$$

The moduli space $\mathcal{M}_{(g, k, \varepsilon | 2n_I, 2m_I)}$ is obtained as the quotient of $\mathcal{T}_{h, g-2h+1+n_I, m_I}$ by the discrete group Mod_t of homotopy classes of those automorphisms that commute with τ . Thus $\mathcal{M}_{(g, k, \varepsilon | 2n_I, 2m_I)}$ is homeomorphic to

$$\mathbb{R}^{3g-3+3n_I+2m_I} / \text{Mod}_t.$$

4. REAL SURFACES OF GENUS ZERO

We will now prove Theorem 1.1 for real surfaces of genus zero without non-real holes or punctures. In this case the compactified real curve has exactly one oval and the number of holes and punctures on this oval determines the type t of the real surface. Let us consider a set ℓ_1, \dots, ℓ_r , $r > 2$, of pairwise disjoint geodesics in the hyperbolic plane \mathbb{H} such that the end points of ℓ_i are to the left of the endpoints of ℓ_{i+1} . Let R_i be the reflection in ℓ_i . Product of two hyperbolic reflections is parabolic if their axes share exactly one endpoint and hyperbolic if the closures of their axes are disjoint. We can choose the geodesics ℓ_i in such a way that the distribution of parabolic and hyperbolic elements among the products

$$C_r = R_r R_1, \quad C_1 = R_1 R_2, \quad C_2 = R_2 R_3, \quad \dots, \quad C_{r-1} = R_{r-1} R_r$$

corresponds to the distribution of punctures and holes in the topological type t , see Figure 4.

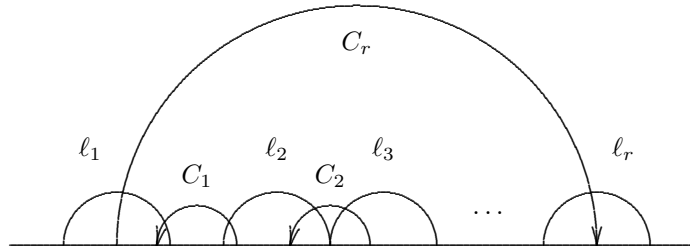


Figure 4: Real curve of genus zero

Consider the shifts

$$F_1 = R_1 R_i = C_1 \cdots C_{i-1} \quad \text{and} \quad F_2 = R_{i+1} R_r = C_{i+1} \cdots C_{r-1}.$$

Then (F_1, C_i, F_2) is a sequential set and hence so is (C_1, \dots, C_r) . Therefore the shifts C_1, \dots, C_r generate a Fuchsian group Γ and $P_r = \mathbb{H}/\Gamma$ is a Riemann surface of genus zero. Moreover, $R_i \Gamma R_i = \Gamma$, hence the reflections R_i induce an anti-holomorphic involution $\tau : P_r \rightarrow P_r$ such that all holes and punctures are real with respect to τ and their distribution corresponds to the distribution prescribed in the topological type t .

We will now prove that this construction gives all real curves (P_r, τ) of genus zero with only real holes and punctures. To this end, given such a real curve, we want to identify the corresponding geodesics ℓ_1, \dots, ℓ_r in \mathbb{H} . Consider a connected component of $P_r \setminus P_r^r$. Let Q_r be its pre-image under the uniformisation map $\mathbb{H} \rightarrow P_r$. The boundary of Q_r consists of geodesic segments in \mathbb{H} . We can extend these geodesic segments to geodesics ℓ_i .

Sets of geodesics are determined by their endpoints. The number of endpoints is $2n + m$, where n and m are the numbers of holes and punctures in the type t respectively. Taking into account the action of the 3-dimensional group $\mathrm{PSL}(2, \mathbb{R})$ of automorphisms of \mathbb{H} on the set of sets of geodesics, we can conclude that the set of orbits of this action is homeomorphic to \mathbb{R}^{2n+m-3} , thus $\mathcal{M}_t \simeq \mathbb{R}^{2n+m-3}/\mathrm{Mod}$, where Mod is a discrete group determined by the numbering of the holes.

We will now construct the space \mathcal{M}^* of real curves of genus zero with 3 holes of which exactly one is real. Let us revisit the previous construction with $r = 3$. We consider a set ℓ_1, ℓ_2, ℓ_3 of pairwise disjoint geodesics in the hyperbolic plane \mathbb{H} such that the end points of ℓ_i are to the left of the endpoints of ℓ_{i+1} . Let R_i be the reflection in ℓ_i . We can choose the geodesics ℓ_i in such a way that all products $C_3 = R_3 R_1, C_1 = R_1 R_2, C_2 = R_2 R_3$ are hyperbolic. Let c_i be the axis of C_i for $i = 1, 2, 3$. In this case the geodesics ℓ_1, ℓ_2, ℓ_3 and c_1, c_2, c_3 bound a right-angled hexagon Q_3 in \mathbb{H} . The type of the hexagon Q_3 is determined up to bi-holomorphic equivalence by the lengths of its sides that are contained in the geodesics c_1, c_2, c_3 , see Theorem 3.5.13 in [R]. These lengths are given by $(\lambda_1/2, \lambda_2/2, \lambda_3/2)$, where $\lambda_i = \lambda(C_i)$ is the shift parameter of the hyperbolic element C_i . They are completely determined by the lengths of the minimal geodesics around the holes of the surface P_3 . If $\lambda_1 = \lambda_2$ then the hexagon Q_3 is symmetric with respect to the geodesic orthogonal to c_3 , see Figure 5. The reflection in this geodesic generates an anti-holomorphic involution $\sigma : P_3 \rightarrow P_3$ that interchanges the holes c_1 and c_2 and maps the hole c_3 to itself. Note that λ_1 and λ_3 completely determine the real curve (P_3, σ) and hence the space \mathcal{M}^* of such real curves is homeomorphic to \mathbb{R}^2 .

Now let us construct the space \mathcal{M}_s^* of real curves (P, τ) of genus zero with two non-real holes and r real holes and punctures, generating an oval of type s . Consider the minimal closed geodesic c_0 that separates non-real holes from real holes and punctures. The geodesic c_0 divides the real curve (P, τ) into two real curves (P_{r+1}, τ) and (P_3, σ) of the types considered above. On the other hand, starting with the real curves (P_{r+1}, τ) and (P_3, σ) , we can glue them together to form a real curve of type t if the length of the minimal closed geodesic around the real hole on (P_3, σ) is equal to the length of the minimal closed geodesic around the hole C_1 on (P_{r+1}, τ) . Therefore $\mathcal{M}_t \simeq \mathbb{R}^{2n_R+m_R-3}/\mathrm{Mod}$, where n_R and m_R are the numbers of real holes and real punctures in the type s respectively.

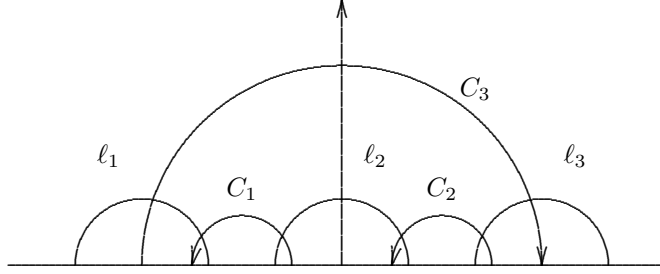


Figure 5: Real curve of genus zero with 3 holes

5. REAL SURFACES OF ANY TYPE

We will now consider the space of real surfaces of any type. Such real surfaces can be obtained by identifying pairs of non-real symmetric holes on the kinds of surfaces considered in sections 3 and 4.

Consider a real curve (P, τ) of type t without real holes or real punctures. Let c be a minimal closed geodesic around a non-real hole on (P, τ) . Let ρ be the length of c . Let c_0 be as described in section 3. Among all geodesic segments connecting the curves c and c_0 there is a unique shortest segment. Let x be the end point of this segment on the curve c .

Consider a real curve (Q, σ) of type s with exactly two non-real holes and without non-real punctures. Let u and $\sigma(u)$ be the minimal simple closed geodesics around the non-real holes on (Q, σ) . We can choose (Q, σ) in such a way that the length of u and $\sigma(u)$ is equal to ρ . Let c_0 be the minimal closed geodesic on (Q, σ) that separates non-real holes from real holes and real punctures. Among all geodesic segments connecting the curves u and c_0 there is a unique shortest segment. Let y be the end point of this segment on the curve u .

The family of isometries φ mapping the geodesic c to the geodesic u is real one-dimensional and can be parametrised by the distance between the points $\varphi(x)$ and y . We can identify the holes $\tau(c)$ and $\sigma(u)$ via the isometry $\sigma \circ \varphi \circ \tau$. Thus we can glue a real curve (Q, σ) of genus zero and type s into a non-real hole of a real surface (P, τ) of type t . Repeatedly gluing in real surfaces of genus zero we can obtain a real curve of any type.

Consider the space $\mathcal{M}_{(t|s_1, \dots, s_r)}$ of real curves of topological type $(t|s_1, \dots, s_r)$, where t is the topological type of a real curve without holes and punctures and s_i are the types of ovals. The results of sections 4 and 5 imply that

$$\mathcal{M}_{(t|s_1, \dots, s_r)} = \mathcal{M}_t \times \mathbb{R}^{2n_R + m_R}.$$

Recall that we have shown in section 3 that

$$\mathcal{M}_t \simeq \mathbb{R}^{3g-3+3n_I+2m_I} / \text{Mod}_t,$$

where $t = (g, k, \varepsilon | 2n_I, 2m_I)$.

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