A PDEM-COM framework for uncertainty quantification of backward issues involving both aleatory and epistemic uncertainties

Z Q Wan[1,2], J B Chen[1] and M Beer[2]

[1] State Key Laboratory of Disaster Reduction in Civil Engineering, College of Civil Engineering, Tongji University, Shanghai, China
[2] Institute for Risk and Reliability, Leibniz Universität Hannover, Hannover, Germany

E-mail: chenjb@tongji.edu.cn

Abstract. Uncertainties that exist in nature or due to lack of knowledge have been widely recognized by researchers and engineering practitioners throughout engineering design and analysis for decades. Though great efforts have been devoted to the issues of uncertainty quantification (UQ) in various aspects, the methodologies on the quantification of aleatory uncertainty and epistemic uncertainty are usually logically inconsistent. For instance, the aleatory uncertainty is usually quantified in the framework of probability theory, whereas the epistemic uncertainty is quantified mostly by non-probabilistic methods. In the present paper, a probabilistically consistent framework for the quantification of both aleatory and epistemic uncertainty by synthesizing the probability density evolution method (PDEM) and the change of probability measure (COM) is outlined. The framework is then applied to the backward issues of uncertainty quantification. In particular, the uncertainty model updating issue is discussed in this paper. A numerical example is presented, and the results indicate the flexibility and efficiency of the proposed PDEM-COM framework.

1. Introduction

Computational models have become an indispensable part of modern analysis and design. In general, a computational model is mainly composed of two sub-models, i.e., the **physical model (PM)** and the **uncertainty model (UM)**. The former establishes the physical mechanisms between the input parameters and the output parameters, while the latter characterizes the uncertainties involved in the input parameters. Such a comprehensive computational model involving both two sub-models is usually called as a **stochastic physical model** [1], which is the basic object of uncertainty quantification (UQ). As for the embedded uncertainties, it has been widely recognized that there are two types of uncertainties [2], i.e., the aleatory uncertainty and the epistemic uncertainty. The aleatory type is due to the natural randomness that is irreducible, while the epistemic type is originated from the lack of knowledge, which is, more or less, able to be reduced by the improvement of knowledge or the enrichment of information [3]. Hence, the task of uncertainty model updating, i.e., the process of reducing epistemic uncertainties, plays a significant role in UQ. Among all the model updating methods investigated in the literature, the Bayesian updating framework might be mostly studied and applied by researchers and engineers [4-5].

In the Bayesian updating framework, the transitional Markov Chain Monte Carlo (tMCMC) [6] shows an advantage in convergence over the direct Monte Carlo Simulation (MCS) approach. Besides, the Approximate Bayesian Computation (ABC) is also taken into account to efficiently calculate the likelihood function in the Bayesian updating [5]. Nevertheless, there still exist two obstacles: (1) how to choose a proper distance metric in ABC, and (2) how to reduce repeated physical model calculations in tMCMC?

As for the issue (1), Bi et al [7] compared several different stochastic metrics and indicated that the Bhattacharyya distance performs well in the stochastic model updating [5]. Nevertheless, the Bhattacharyya distance does not satisfy the triangle inequality, which is a major shortage as a metric. To this end, a very similar distance called the Hellinger distance is introduced in this paper to modify the stochastic distance used in ABC.

For the issue (2), physical model evaluations are repeatedly required in each stage chain of tMCMC. This is generally very time-consuming. Notice that the target of physical model evaluations is to obtain a posterior distribution, which can be obtained efficiently by the probability density evolution method (PDEM) [8]. Additionally, in each stage of tMCMC, extra physical model evaluations due to the updating of the uncertainty model can be saved once the technique of change of probability measure (COM) is applied. Consequently, the novel framework called PDEM-COM that recently proposed by Chen & Wan [3] can be embedded in the procedure of tMCMC. By doing so, only a small number of physical model evaluations (usually 300~500 times) is required only once.

In this paper, the PDEM-COM framework is first introduced as a basis. After that, the Bayesian updating framework is briefly summarized, including the ABC and the tMCMC. The incorporation of PDEM-COM into the tMCMC is then elaborated. A numerical example is studied to illustrate the efficiency and accuracy of the proposed PDEM-COM framework on handling with Bayesian updating problems.

2. PDEM-COM framework

Generally speaking, the probability density evolution method (PDEM) provides an approach for uncertainty propagation (UP) when all uncertainties are considered to be aleatory, while the method of the change of probability measure (COM) offers a technique of re-using the evaluated results from physical model solutions when dealing with epistemic uncertainties, e.g., the uncertainty model is imprecise [3]. The basic theories of PDEM and COM are introduced as follows.

2.1. Probability density evolution method (PDEM)

Without loss of generality, consider a stochastic physical model stated by

$$\boldsymbol{Y} = \mathcal{G}(\boldsymbol{X}, t), \ \boldsymbol{Y}(t = t_0) = \boldsymbol{Y}_0 \tag{1}$$

where $\boldsymbol{X} = (X_1, X_2, \cdots, X_n)^\top$ is a random vector of dimension n, and is represented by an uncertainty model $\mathcal{P}(\boldsymbol{x}; \boldsymbol{\theta})$ where $\boldsymbol{\theta}$ is the vector of distribution parameters. For instance, if $\boldsymbol{X}$ is normally distributed, then $\mathcal{P}(\boldsymbol{x}; \boldsymbol{\theta}) = \mathcal{N}(\boldsymbol{x}; \boldsymbol{\theta})$ and $\boldsymbol{\theta} = (\boldsymbol{\mu}, \boldsymbol{\Sigma})^\top$ where $\boldsymbol{\mu}$ is the mean vector, $\boldsymbol{\Sigma}$ is the covariance matrix and $\mathcal{N}(\cdot)$ is the multivariate Gaussian function. $\mathcal{G}(\cdot)$ is a deterministic function, that usually determined by the embedded physical laws. The quantity of interest (QoI), i.e., $\boldsymbol{Y} = (Y_1, Y_2, \cdots, Y_m)^\top$ is also certainly random. Generally, the stochastic physical model presented in Eq. (1) is probability preserved [9], therefore the joint probability density function (PDF) of $\boldsymbol{X}$ and $\boldsymbol{Y}$, i.e., $p_{YX}(\boldsymbol{y}, \boldsymbol{x}, t)$ satisfies

$$\frac{\partial p_{YX}(\boldsymbol{y}, \boldsymbol{x}, t)}{\partial t} = -\sum_{\ell=1}^{m} \dot{Y}_\ell(\boldsymbol{x}, t) \frac{\partial p_{YX}(\boldsymbol{y}, \boldsymbol{x}, t)}{\partial y_\ell} \tag{2}$$

with the initial condition

$$p_{YX}(\boldsymbol{y}, \boldsymbol{x}, t_0) = \delta(\boldsymbol{Y} - \boldsymbol{Y}_0) p_X(\boldsymbol{x}) \tag{3}$$

where $\delta(\cdot)$ is the Dirac delta function. Eq. (2) with (3) is the so-called generalized density evolution equation (GDEE) [8-9].

The numerical algorithm of PDEM for solving the GDEE includes four major steps:

Step 1.1. Partition of the probability-assigned space [10].

Denote the q-th partition of the probability space of $\boldsymbol{X}$ by Ω_q for $q = 1, 2, \cdots, N$ where $\Omega_X = \cup_{q=1}^{N} \Omega_q$ and $\Omega_q \cap \Omega_p = \varnothing$ for $\forall p \neq q$. For each sub-domain, one representative point is selected, i.e., $\boldsymbol{X} = \boldsymbol{x}_q \in \Omega_q$, and therefore a point set $\mathcal{D} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N\}$ is generated. Each point is further equipped with the corresponding assigned probability defined by

$$P_q = \int_{\Omega_q} \mathcal{P}(\boldsymbol{x}; \boldsymbol{\theta}) \mathrm{d}\boldsymbol{x} \text{ for } q = 1, 2, \cdots, N. \tag{4}$$

In this paper, the GF-discrepancy minimized strategy [11] is adopted to generate such a point set $\mathcal{D}$ and its assigned probability set $\{P_q\}_{q=1}^{N}$. The number of representative point is usually small, e.g., $N = 300 \sim 500$.

Step 1.2. Physical model evaluation in terms of all representative points.

For each $\boldsymbol{X} = \boldsymbol{x}_q$, Eq. (1) is solved and the velocity $\dot{\boldsymbol{Y}}^{(q)} = \left(\dot{Y}_1(\boldsymbol{x}_q, t), \cdots, \dot{Y}_m(\boldsymbol{x}_q, t)\right)^{\top}$ is stored.

Step 1.3. Solution of partially discretized GDEEs.

By integrating Eq. (2) in each sub-domain Ω_q, the partially discretized GDEE

$$\frac{\partial p_{YX}^{(q)}(\boldsymbol{y}, t)}{\partial t} = -\sum_{\ell=1}^{m} \dot{Y}_\ell(\boldsymbol{x}_q, t) \frac{\partial p_{YX}^{(q)}(\boldsymbol{y}, t)}{\partial y_\ell} \tag{5}$$

with the correspondingly partially discretized initial condition

$$p_{YX}^{(q)}(\boldsymbol{y}, t = t_0) = \delta(\boldsymbol{y} - \boldsymbol{y}_0) P_q \tag{6}$$

is solved after substituting $\dot{\boldsymbol{Y}}^{(q)}$ in them for $q = 1, 2, \cdots, N$.

Step 1.4. Synthesizing all the results.
The PDF of QoI is then obtained by

$$p_Y(\boldsymbol{y}, t) = \sum_{q=1}^{N} p_{YX}^{(q)}(\boldsymbol{y}, t) \tag{7}$$

Remark 1. In PDEM, the uncertainty model is assumed to be precise, which means both the exact form of $\mathcal{P}(\cdot)$ and the distribution parameters $\boldsymbol{\theta}$ are determined.

Remark 2. PDEM is also capable of handling static and stochastic physical models, i.e., there is no "real" time t in Eq. (1). On this condition, a virtual process defined by virtual time t is constructed, see [12] for details.

2.2. Change of probability measure (COM)

Suppose there are two different uncertainty models, denoted by $\mathcal{P}^{(1)}(\boldsymbol{x}; \boldsymbol{\theta}^{(1)})$ and $\mathcal{P}^{(2)}(\boldsymbol{x}; \boldsymbol{\theta}^{(2)})$. The corresponding PDFs of QoI $\boldsymbol{Y}$ are written by $p_Y^{(1)}(\boldsymbol{y}, t)$ and $p_Y^{(2)}(\boldsymbol{y}, t)$, respectively, on the condition that Eq. (1) holds for both cases. To obtain the PDF of QoI in these two cases, one could do the PDEM analysis for twice, i.e., the first round is to evaluate $p_Y^{(1)}(\boldsymbol{y}, t)$ as the input PDF is $\mathcal{P}^{(1)}(\boldsymbol{x}; \boldsymbol{\theta}^{(1)})$ and the second round is to compute $p_Y^{(2)}(\boldsymbol{y}, t)$ as the input PDF is $\mathcal{P}^{(2)}(\boldsymbol{x}; \boldsymbol{\theta}^{(2)})$.

Nonetheless, this strategy requires repeated evaluations of the physical model and is apparently uneconomic.

Noticing that two uncertainty models can explicitly differ, e.g., the distribution types of $\mathcal{P}^{(1)}(\cdot)$ and $\mathcal{P}^{(2)}(\cdot)$ are distinct, or the distribution parameters $\boldsymbol{\theta}^{(1)}$ and $\boldsymbol{\theta}^{(2)}$ are disparate, or both are different. Nevertheless, if the probability measure defined by $\mathcal{P}^{(2)}(\cdot)$ that denoted by ν is absolutely continuous with respect to the probability measure defined by $\mathcal{P}^{(1)}(\cdot)$ that denoted by μ, and they are both σ-finite measures on the measurable space $\left(\Omega_X, \mathcal{F}\right)$, then there exists a measurable function [13]

$$\mathcal{T} = \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \tag{8}$$

which is the so-called Radon-Nikodym derivative of ν with respect to μ, and such that the equation

$$\nu(A) = \int_A \mathcal{T}\mathrm{d}\mu, \ A \in \mathcal{F} \tag{9}$$

holds. Eq. (9) ensures a direct change of measure from $\mathcal{P}^{(1)}(\cdot)$ to $\mathcal{P}^{(2)}(\cdot)$, which is easy and known, and also builds a straightforward change from $p_Y^{(1)}(\boldsymbol{y},t)$ to $p_Y^{(2)}(\boldsymbol{y},t)$ without inducing any other extra evaluations.

The numerical algorithm of COM synthesized with PDEM is shortly summarized as follows [3]:

Step 2.1. Accomplishment of one round PDEM analysis.

With a given uncertainty model, e.g., $\mathcal{P}^{(1)}(\cdot)$, one round of PDEM analysis introduced in Section 2.1 is needed to be completed. The point set denoted by $\mathcal{D}^{(1)}$ as well as the computed velocity $\mathcal{V} = \left\{\dot{\boldsymbol{Y}}^{(1)}, \dot{\boldsymbol{Y}}^{(2)}, \cdots, \dot{\boldsymbol{Y}}^{(N)}\right\}$ should be stored.

Step 2.2. Calculation of Radon-Nikodym derivative.

When the uncertainty model is changed from $\mathcal{P}^{(1)}(\cdot)$ to $\mathcal{P}^{(2)}(\cdot)$, the Radon-Nikodym derivative is computed by re-computing the assigned probability, where Eq. (4) is rewritten by

$$P_q^{(2)} = \int_{\Omega_q^{(1)}} \mathcal{P}^{(2)}\left(\boldsymbol{x};\boldsymbol{\theta}^{(2)}\right)\mathrm{d}\boldsymbol{x}, \ q = 1, 2, \cdots, N \tag{10}$$

where the integral domain $\Omega_q^{(1)}$ denotes the q-th sub-domain defined by the point set $\mathcal{D}^{(1)}$.

Step 2.3. Re-calculation of Step 1.3 and Step 1.4.

It should be emphasized that, since there is no new point set is required, therefore the **Step 1.2** in the PDEM procedure can be skipped over, which means the most time-consuming part, i.e., the step on model evaluation is not needed anymore. Then, the partially discretized GDEEs in **Step 1.3** can be resolved with the new initial conditions

$$p_{YX}^{(q)}\left(\boldsymbol{y}, t = t_0\right) = \delta\left(\boldsymbol{y} - \boldsymbol{y}_0\right)P_q^{(2)} \tag{11}$$

for $q = 1, 2, \cdots, N$ and the stored velocity $\mathcal{V}$ in **Step 2.1**. After that, **Step 1.4** is redone to obtain a new PDF of QoI in terms of the new uncertainty model $\mathcal{P}^{(2)}(\cdot)$.

3. Bayesian updating framework

The Bayesian updating framework is based on the Bayes' theorem

$$\mathcal{M}(\boldsymbol{\theta}\,|\,D) = \frac{\mathcal{L}(D\,|\,\boldsymbol{\theta})}{\mathcal{L}(D)}\mathcal{M}(\boldsymbol{\theta}) \tag{12}$$

where $D = \{y_q\}_{q=1}^N$ denotes N observed outputs y in Eq. (1), $\mathcal{M}(\theta)$ is the prior distribution of θ and $\mathcal{M}(\theta|D)$ is the posterior distribution calculated by the likelihood $\mathcal{L}(D|\theta)$ and the evidence $\mathcal{L}(D)$. The likelihood $\mathcal{L}(D|\theta)$ is undoubtedly a central component in the Bayesian updating framework and is defined as

$$\mathcal{L}(D|\theta) = \prod_{q=1}^N p_Y(y_q|\theta) \tag{13}$$

where $p_Y(y|\theta)$ stands for the conditional PDF of given θ. Actually, if $\mathcal{M}(\cdot)$ is completely in a probabilistic framework [14], Eq. (12) also indicates that with respect to the prior distribution $\mathcal{M}(\theta)$, the posterior distribution $\mathcal{M}(\theta/D)$ has the Radon-Nikodym derivative that calculated by $\mathcal{L}(D|\theta)/\mathcal{L}(D)$.

Remark 3. In this situation, the uncertainty model of input X switches into $\mathcal{P}(x; \mathcal{M}(\theta))$, i.e., a compatible UM that combined with both aleatory and epistemic uncertainties, is taken into account [3], and the mission of the Bayesian updating framework is to reduce the epistemic uncertainties involved in $\mathcal{M}(\theta)$ with observations $D = \{y_q\}_{q=1}^N$.

3.1. Approximate Bayesian Computation (ABC)
However, in Eq. (13) the evaluated PDF is needed for every single observation y_q for $q = 1, 2, \cdots, N$. This is usually hard. Instead, the Approximate Bayesian Computation (ABC) provides an approximate likelihood formula [5]

$$\mathcal{L}(D|\theta) \propto \exp\left\{-\frac{d^2(y|\theta)}{\sigma^2}\right\} \tag{14}$$

where $d(y|\theta)$ is a stochastic metric, which is only dependent on the PDF of Y and the condition of θ; the parameter σ denotes a width factor that determines the scale of the approximate likelihood. As a result, Eq. (14) is more flexible and robust than Eq. (13), but the definition of the stochastic metric $d(\cdot)$ matters [7]. Bi et al [5, 7] found that the Bhattacharyya distance defined as

$$d_\mathrm{B} = -\log\left[\int_{\Omega_Y} \sqrt{p_Y^\mathrm{obs}(y) p_Y^\mathrm{sim}(y)}\,\mathrm{d}y\right] \tag{15}$$

behaves well in ABC, where $p_Y^\mathrm{obs}(y)$ and $p_Y^\mathrm{sim}(y)$ stand for the PDFs estimated from the observations and simulations, respectively.

It is easy to know that: (1) the Bhattacharyya distance is ranged in $[0, +\infty)$, and (2) it does not satisfy the triangle inequality. As for the property (1), when the distribution by simulation is far away from the one by observation, the likelihood would be very small since d_B is large and it can even approach infinity if there is no intersection between $p_Y^\mathrm{obs}(y)$ and $p_Y^\mathrm{sim}(y)$, which results in a very slow speed of convergence in tMCMC. Besides, due to the property (2), tMCMC may fail in search of the global optimum.

Therefore, a very similar metric called the Hellinger distance [18]

$$d_\mathrm{H} = \sqrt{1 - \int_{\Omega_Y} \sqrt{p_Y^\mathrm{obs}(y) p_Y^\mathrm{sim}(y)}\,\mathrm{d}y} \tag{16}$$

is adopted in this paper to help overcome all obstacles as mentioned above. The Hellinger distance is ranged in $[0,1]$ and the triangle inequality is satisfied, as well.

3.2. Transitional Markov chain Monte Carlo (tMCMC)

There are two major challenges in the Bayesian updating framework. One is to calculate the evidence

$$\mathcal{L}(D) = \int_{\Omega_\Theta} \mathcal{L}(D|\theta)\mathcal{M}(\theta)\mathrm{d}\theta \tag{17}$$

which is usually hard to be integrated when the distribution $\mathcal{M}(\theta)$ is sharp or peaked. An alternative approach is to do it iteratively, i.e., the original Bayesian updating formula in Eq. (12) is rewritten as [6]

$$\mathcal{M}^{(j)}(\theta|D) \propto \mathcal{L}(D|\theta)^{h_j} \mathcal{M}(\theta) \tag{18}$$

for $0 = h_0 \leq h_1 \leq \cdots \leq h_M = 1$ and $j = 0,1,\cdots,M$, where $\mathcal{M}^{(j)}(\theta|D)$ is called the j-th intermediate distribution. Obviously, Eq. (18) is a typical iterative scheme that starts with the prior distribution $\mathcal{M}(\theta)$ when $h_j = 0$ and ends with the posterior distribution $\mathcal{M}(\theta|D)$ when $h_j = 1$, and M is the total number of stages or chains in tMCMC. An adaptive method to choose the stage number h_j is adopted in this paper, which ensures a very fast convergence of tMCMC [6].

3.3. An acceleration of tMCMC via PDEM-COM

For each stage in tMCMC, θ is resampled and $p_Y^{\mathrm{sim}}(y)$ is re-evaluated to generate the new likelihood function. The methods of MCS based or PDEM based all require new model evaluations, which is prohibitive if the computational model is large and complex. Is it possible to reduce model evaluations? The answer is founded in the recent work done by Chen & Wan [3]. In their work as introduced in Section 2, the PDEM-COM, which provides an accelerating scheme to deal with issues involved with both aleatory and epistemic uncertainties, is embedded herein to accelerate the procedure of tMCMC.

The basic idea of acceleration of tMCMC via PDEM-COM is summarized as follows:

Step 3.1. Setting a reference.

As an initiation, a reference $\theta^{\mathrm{ref}} \in \mathcal{M}(\theta)$ is set. For instance, $\mathcal{M}(\theta)$ is usually considered to be a hypercube, i.e., $\mathcal{M}(\theta) = \left[\underline{\theta_1},\overline{\theta_1}\right] \times \left[\underline{\theta_2},\overline{\theta_2}\right] \times \cdots \times \left[\underline{\theta_s},\overline{\theta_s}\right]$ where $\underline{\theta_\square}$ and $\overline{\theta_\square}$ are the lower and upper bounds, respectively, and $\times$ is the Cartesian product. Then, θ^{ref} can be picked as a midpoint or any other proper point, i.e.,

$$\theta^{\mathrm{ref}} = \left(\frac{\underline{\theta_1}+\overline{\theta_1}}{2}, \frac{\underline{\theta_2}+\overline{\theta_2}}{2}, \cdots, \frac{\underline{\theta_s}+\overline{\theta_s}}{2}\right)^\top \tag{19}$$

Then, the uncertainty model is chosen as $\mathcal{P}(x;\theta^{\mathrm{ref}})$.

Step 3.2. Initial analysis via PDEM.

With $\mathcal{P}(x;\theta^{\mathrm{ref}})$, an initial analysis by PDEM is done, and the PDF of Y is evaluated and denoted by $p_Y^{\mathrm{ini}}(y)$. The PDF of Y from the observations, i.e., $p_Y^{\mathrm{obs}}(y)$ is also estimated where the technique of probability density estimation via PDEM is adopted [15].

Step 3.3. Acceleration of tMCMC via COM.

On each stage (chain) of tMCMC, the uncertainty model, i.e., $\mathcal{P}(x;\mathcal{M}^{(j)}(\theta|D))$ is updated. Instead of re-sampling x from it, x is kept unchanged (as the same in **Step 3.1**) but the assigned

probability is updated via COM. In consequence, there is no need for new model evaluations but only a new calculation of the assigned probability, which is practically much faster [3, 16].

4. Numerical example

In this section, a composite Gaussian function of two dimensions, which is commonly studied as a toy function for metamodeling and reliability analysis [17], is tested as a benchmark. The function is explicitly written by

$$Y = \max\left(g_1(X), g_2(X)\right) \tag{20}$$

where

$$\begin{cases} g_1(X) = X_1^2 + X_2 - 8, \\ g_2(X) = X_1/5 + X_2 - 6 \end{cases} \tag{21}$$

and $X = (X_1, X_2)^\top$ is a random vector following the independently normal distribution. To investigate the proposed method, a target (precise) uncertainty model is set as $X_1 \sim \mathcal{N}(6,1)$ and $X_2 \sim \mathcal{N}(5,1)$, and 300 observations are randomly obtained from it, i.e., $D_Y^{\mathrm{obs}} = \left\{y_q\right\}_{q=1}^{300}$. Then, an imprecise uncertainty model is assumed as $X_1 \sim \mathcal{N}([4,7],[0.5,1.5])$ and $X_2 \sim \mathcal{N}([4,7],[0.5,1.5])$, which represents a mixture of both aleatory and epistemic uncertainties. Consequently, the uncertainty model is characterized by

$$\mathcal{M}\left(\boldsymbol{\theta} = (\mu_1, \mu_2, \sigma_1, \sigma_2)^\top\right) \tag{22}$$

where $\mu_1, \mu_2 \in [4,7]$ are the mean values of X_1, X_2 and $\sigma_1, \sigma_2 \in [0.5,1.5]$ are the standard deviations of X_1, X_2, respectively.

4.1. Preliminary works via PDEM

There are some preliminary works needed to be done via PDEM before executing the Bayesian updating framework. Firstly, the PDF of observations D_Y^{obs} is estimated via PDEM [15], i.e., $p_Y^{\mathrm{obs}}(y)$. Then, 300 representative points are selected via the GF-minimized strategy [11], denoted by $D_X^{\mathrm{ini}} = \left\{x_q\right\}_{q=1}^{300}$, and Eq. (20) is computed, then the PDF of Y, i.e., $p_Y^{\mathrm{ini}}(y)$ is also calculated by solving the GDEE in Eq. (2) with the assigned probability $\left\{P_q^{\mathrm{ini}}\right\}_{q=1}^{300}$.

4.2. A comparison between Bhattacharyya distance and Hellinger distance in Bayesian updating

Before verifying the proposed method, a comparison between the Bhattacharyya distance and Hellinger distance is studied. The initial samples of $\boldsymbol{\theta}$ is uniformly generated from $\mathcal{M}(\boldsymbol{\theta})$ as illustrated in Figure 1 (the prior distribution), and the sample size is set to 1000. The posterior distributions via the Bhattacharyya distance and the Hellinger distance are plotted in Figures 2 and 3.
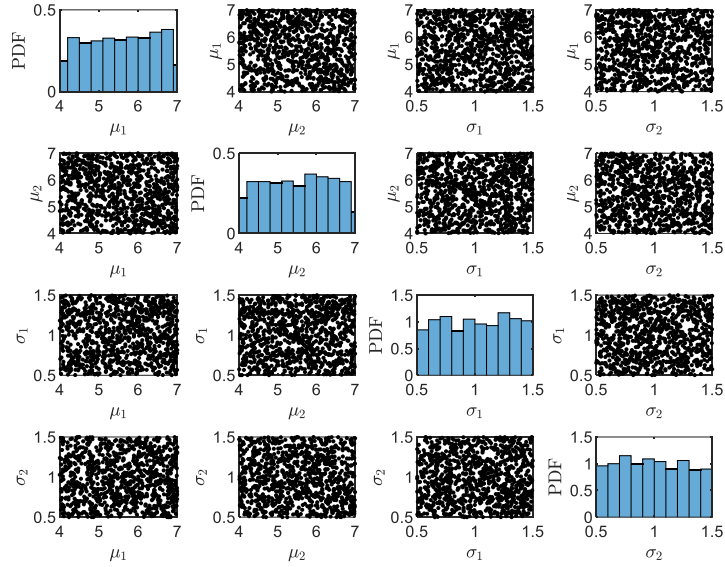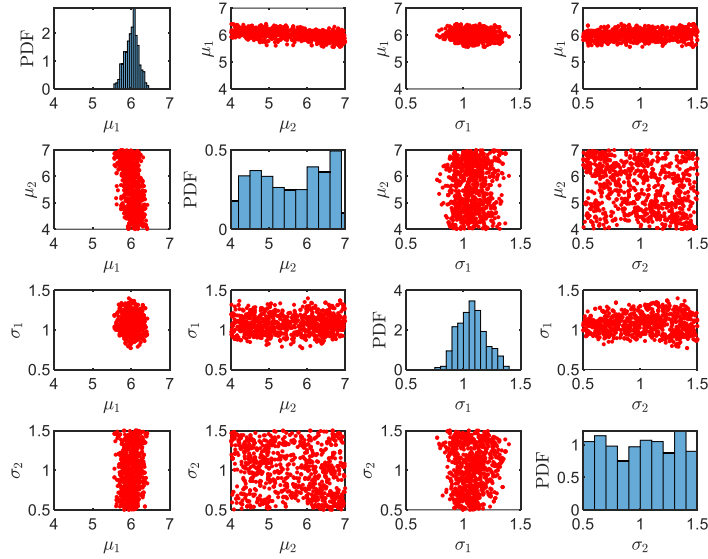
Figure 1. Prior distribution of θ.

Figure 2. Posterior distribution of θ via Bhattacharyya distance.

It can be seen that the Bayesian updating procedure via the Hellinger distance has found the posterior distributions that are much narrower than the results via the Bhattacharyya distance when all computational parameters are set as the same.

4.3. Bayesian updating via ABC-tMCMC combined with PDEM-COM

It is mentioned again that the analysis in Section 4.2 is done in the MCS-based pathway, which is expensive since for each chain of tMCMC the model evaluations are executed [5]. For this reason, the PDEM-COM is adopted, and the updating results are shown in Figure 4.

It is seen apparently that, the updating results by PDEM-COM are in a good convergence compared with the results computed by MCS shown in Figure 3. It should be emphasized again that,

the proposed PDEM-COM in ABC-tMCMC for Bayesian updating framework has at least the following two advantages:

(1) The total time of the updating procedure can be evaluated in advance since there is no requirement of extra model evaluations.

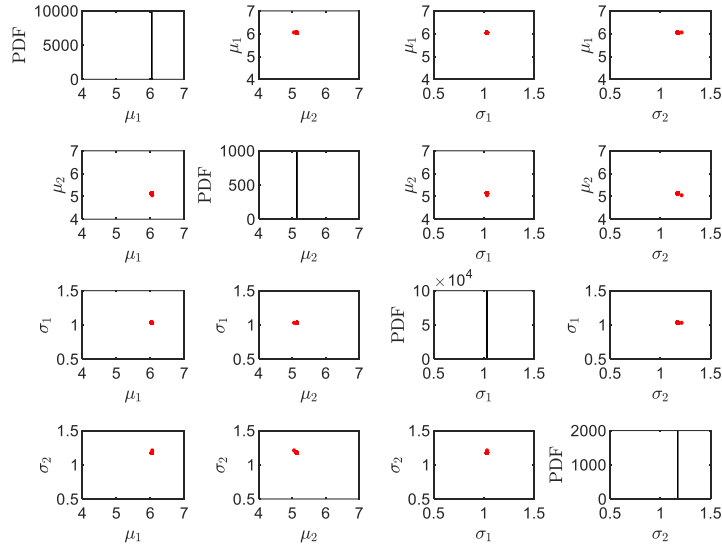(2) The PDF generated by PDEM is of high accuracy, therefore the computed stochastic distance is also exact.

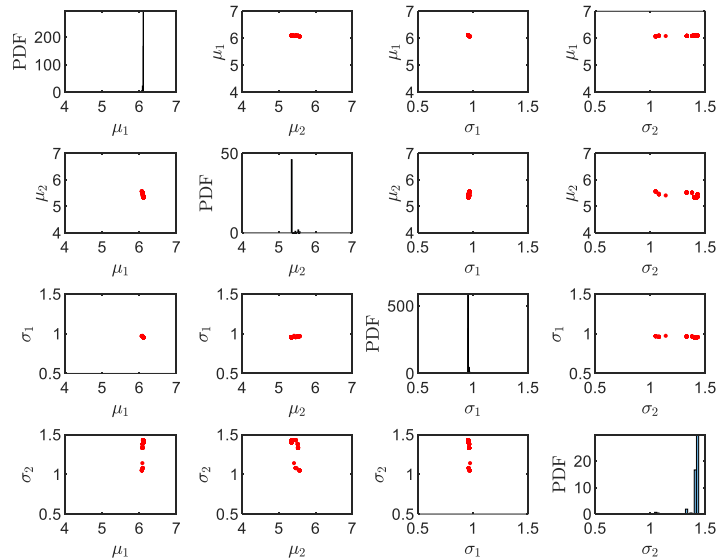**Figure 3.** Posterior distribution of θ via Hellinger distance by MCS.

Figure 4. Posterior distribution of θ via Hellinger distance calculated by PDEM-COM.

5. Conclusions

As for the typical backward issues in uncertainty quantification (UQ), i.e., the issues of uncertainty model updating, the Bayesian updating framework can be adopted. In this framework, the transitional Markov Chain Monte Carlo (tMCMC) is equipped to confirm the convergence of Bayesian updating procedure, and the Approximate Bayesian Computation (ABC) is adopted to overcome the difficulty of the high-dimensional integral of the likelihood function. Specifically, a novel tool combining the

probability density evolution method (PDEM) synthesized with the change of probability measure (COM) is introduced to further reduce the repeated model evaluations. The main conclusions are summarized as follows:

(1) The basic mathematical properties of stochastic metrics utilized in ABC are essentially important. The studied case in this paper indicates that the Hellinger distance is advantageous in convergence over the Bhattacharyya distance.

(2) The proposed PDEM-COM reduces the round of model evaluation from many to one. Besides, due to this convenience, one has access to obtain a time estimation of the Bayesian updating process, which is usually not easy by the Monte Carlo Simulation-based methods.

Some issues that remained and needed to be further studied, including, e.g., how to choose a proper reference uncertainty model in the PDEM-COM framework, and how to deal with multi outputs.

References

[1] Li J 1996 *Stochastic Structural Systems: Analysis and Modeling* (Beijing: Science Press) (in Chinese)

[2] Ang A H-S and Tang W 2007 *Probability Concepts in Engineering: Emphasis on Applications to Civil and Environmental Engineering* (Hoboken: John Wiley & Sons)

[3] Chen J B and Wan Z Q 2019 A compatible probabilistic framework for quantification of simultaneous aleatory and epistemic uncertainty of basic parameters of structures by synthesizing the change of measure and change of random variables *Struct. Saf.* **78** 76-87

[4] Stein M, Beer M and Kreinovich V 2013 Bayesian approach for inconsistent information *Inf. Sci.* **245** 96-111

[5] Bi S F, Broggi M and Beer M 2019 The role of the Bhattacharyya distance in stochastic model updating *Mech. Syst. Signal Proc.* **117** 437-52

[6] Ching J Y and Chen Y-C 2007 Transitional Markov Chain Monte Carlo method for Bayesian model updating, model class selection, and model averaging *J. Eng. Mech.* **133** 816-32

[7] Bi S F, Prabhu S, Cogan S and Atamturktur S 2017 Uncertainty quantification metrics with varying statistical information in model calibration and validation *AIAA J.* **55** 3570-83

[8] Li J and Chen J B 2009 *Stochastic Dynamics of Structures* (Singapore: John Wiley & Sons)

[9] Li J and Chen J B 2008 The principle of preservation of probability and the generalized density evolution equation *Struct. Saf.* **30** 65-77

[10] Chen J B, Ghanem R and Li J 2009 Partition of the probability-assigned space in probability density evolution analysis of nonlinear stochastic structures *Probab. Eng. Eng. Mech.* **24** 27-42

[11] Chen J B, Yang J Y and Li J 2016 A GF-discrepancy for point selection in stochastic seismic response analysis of structures with uncertain parameters *Struct. Saf.* **59** 20-31

[12] Chen J B and Li J 2007 The extreme value distribution and dynamic reliability analysis of nonlinear structures with uncertain parameters *Struct. Saf.* **29** 77-93

[13] Grigoriu M 2012 *Stochastic Systems: Uncertainty Quantification and Propagation* (London: Springer-Verlag)

[14] Sullivan T J 2015 *Introduction to Uncertainty Quantification* (Switzerland: Springer International Publishing)

[15] Chen J B and Li J 2006 Probability density evolution method for probability density function estimation of random variables *J. Tongji. Univ. Nat. Sci.* **34** 433-37 (in Chinese)

[16] Wan Z Q, Chen J B, Li J and Ang A H-S 2020 An efficient new PDEM-COM based approach for time-variant reliability assessment of structures with monotonically deteriorating materials *Struct. Saf.* **82** 101878

[17] Bichon B J, McFarland J M and Mahadevan S 2011 Efficient surrogate models for system reliability analysis of systems with multiple failure modes *Reliab. Eng. Syst. Saf.* **96** 1386-95

[18] Liese F and Vajda I 2006 On divergences and informations in statistics and information theory *IEEE T. Inform. Theory* **52** 4394-412