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On the risk consistency and monotonicity of ruin theory

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Abstract Setting a proper minimum capital requirement is one of the most fundamental problems in the insurance industry. Ruin theory proposes a solution to this problem by identifying the minimum capital that a company needs to hold in order to stay solvent with a high probability. In this note we discuss the ruin theory risk consistency. More precisely we show that the *ruin-consistent Value-at-Risk (VaR)* is not continuous in probability, in $L^p, 0 \leq p < \infty$, and in weak convergence. Furthermore, it is not a monotone measure of risk.

1 Introduction

Ruin theory is based on the pivotal concept of ruin probability, that has been well developed over the last century. In recent years, as various risk measures have been introduced, people propose risk measures as an alternative to ruin probability when determining the minimum capital requirement for a company to stay solvent. However, one wonders how one can link the two theories (risk theory vs risk measures) or unify them. The idea of introducing risk measures in a risk theory context has been explored by Cossette and Marceau [5] and Cossette et al. [6] in a discrete setup and by Trufin et al. [10] and Wüthrich [11] in a continuous environment. Particularly Trufin et al. [10] introduces *ruin-consistent VaR*, thus placing a ruin theory object within the family of risk measures. This risk measure is introduced for individual risks, by associating the smallest amount of capital that can keep the risk of ruin below a certain level. This risk measure possesses very interesting properties that are discussed in Trufin et al. [10]. In this note we explore the monotonicity and continuity of this measure and find that *ruin-consistent VaR* is not monotone and only continuous in the uniform topology.

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2 The setup

Throughout this paper, we fix a probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is a σ-field of measurable sets and P is a probability measure on $\mathcal{F}$. The set of all random variables is denoted by $L^0 = L^0(\Omega)$. Let $p \in (0, \infty]$. For $p \neq \infty$, $L^p = L^p(\Omega)$ denotes the space of real-valued random variables X on Ω such that $E(|X|^p) < \infty$, where E represents the mathematical expectation under P. We consider the following topologies:

1) On L^0, $X_n \to X$ iff $\forall \epsilon > 0, P(|X_n - X| > \epsilon) \to 0$. This is convergence in probability and one can show that this topology is equivalent to a topology induced by the metric $d(X,Y) = E(\min\{|X-Y|, 1\})$.
2) On L^0, $X_n \to X$, iff $P(X_n \to X) = 1$. This is actually point-wise convergence.
3) On L^p, for $0 < p < \infty$, $X_n \to X$ iff $E(|X_n - X|^p) \to 0$.
4) On L^∞, $X_n \to X$ iff $\|X_n - X\|_\infty \to 0$, where

$$\|X\|_\infty = \sup\{m \geq 0 | P(|X| < m) > 0\}.$$

5) Also one can consider convergence in distribution, which does not induce any topology, as $X_n \to X$ iff $F_{X_n} \to F_X$ point-wise.

Note that weak convergence is not associated with any topology, but for distribution invariant functions weak convergence can be regarded as point-wise convergence.

Recall the Cramér-Lundberg model

$$U_t = u + ct - \sum_{i=1}^{N^\lambda(t)} X_k. \tag{2.1}$$

introduced by Lundberg [9]. It is a compound Poisson process that models the surplus process of an insurance company. Here u represents the initial capital, $N^\lambda(t)$ is a counting process with Poisson distribution and parameter λ, X_k's are individual claims that have the same distribution as X and c is the premium rate. In this context, the infinite time ruin probability is, see e.g. Asmussen and Albrecher [1],

$$\Psi(\lambda, c, u, F_X) = P(\inf_{0 \leq t} U_t < 0 | U_0 = u).$$

By definition, it is clear that, for any $a > 0$,

$$\Psi(\lambda, c, u, F_X) = \Psi(\lambda a, ca, u, F_X) = \Psi(1, c/\lambda, u, F_X). \tag{2.2}$$

The ruin probability can be expressed via the Pollaczek-Khintchine formula as

$$\Psi(\lambda, c, u, F_X) = \left(1 - \frac{\lambda E(X)}{c}\right) \sum_{n=0}^{\infty} \left(\frac{\lambda E(X)}{c}\right)^n (1 - F_l^{*n}(u)),$$

where $F_l^{*n}(x)$ is the transform of the tail distribution of F_X,

$$F_l(x) = \frac{1}{E(X)} \int_0^x (1 - F_X(u))\, du,$$

as in Asmussen and Albrecher [1]. The initial capital to control the ruin probability can be determined as the smallest capital for which the company is solvent with probability α:

$$\inf\left\{u\in\mathbb{R}\,\middle|\,P\left(\inf_{t\geq 0}\left(u+ct-\sum_{k=1}^{N^{\lambda}(t)}X_k\right)<0\right)\leq 1-\alpha\right\}. \tag{2.3}$$

Therefore, one can define a new risk measure from (2.3) for $X\sim X_k$ as follows

$$X\mapsto u=\mathrm{VaR}_{\alpha}\left(\sup_{t\geq 0}\left(\sum_{k=1}^{N^{\lambda}(t)}X_k-ct\right)\right). \tag{2.4}$$

In Trufin et al. [10] a *ruin consistent VaR*, denoted by ρ^{α} is introduced as follows

$$\rho^{\alpha}(X)=\rho^{\alpha}_{\eta,\lambda}(X)=\mathrm{VaR}_{\alpha}\left(\sup_{t\geq 0}\left(\sum_{k=1}^{N^{\lambda}(t)}X_k-\eta\lambda E(X)t\right)\right), \tag{2.5}$$

where $\{X_k\}_{k=1,2,...}$ is a sequence of i.i.d random variables with the same distribution as X and $\eta>1$ is a given constant. Indeed, $\rho^{\alpha}(X)$ solves the following equations

$$\alpha=\Psi\left(\lambda,\lambda\eta E(X),\rho^{\alpha}(X),F_X\right)=\Psi\left(1,\eta E(X),\rho^{\alpha}(X),F_X\right). \tag{2.6}$$

3 Main Results

To present the main results of this paper we need to know the ruin probability for a loss of size 1_A for a measurable set A. We start by looking at the ruin probability of a Cramér-Lundberg process, in which each claim is of size 1,

$$U_t=u+ct-N^{\lambda}(t).$$

According to Example 4 and relation (1.3) in Hubalek and Kyprianou [7], the ruin probability of the process U_t is given by $\Psi(\lambda,c,u,F_1)=1-\lambda W(u)$, where $W(u)=\frac{\lambda}{c}\sum_{n=1}^{\lfloor u\rfloor}e^{-\frac{\lambda}{c}(n-u)}\frac{1}{n!}\left(\frac{\lambda}{c}\right)^n(n-u)^n$, with $\lfloor u\rfloor$ the integer part of u. In the Appendix we prove the following proposition that links the ruin probability of claim size 1 and 1_A.

Proposition 1 *Let X_i, $i=1,2,...$ be a sequence of i.i.d. random variables and let $\{A_k\}_{k=1,2,...}$ be a sequence of independent sets such that for some $x>0$, $P(A_k)=x$. If $\{A_k\}_{k=1,2,...}$, $\{X_k\}_{k=1,2,...}$ and $\left\{N_t^{\lambda}\right\}_{t\geq 0}$ are independent, then $Y_t=\sum_{i=1}^{N_t^{\lambda x}}X_i$ and $Y_t^x=\sum_{i=1}^{N_t^{\lambda}}1_{A_i}X_i$ have the same distribution.*

The following corollary is an immediate application of Proposition 1 to the process $t\mapsto u+c-\sum_{i=1}^{N_t^{\lambda}}1_{A_i}X_i$, for an independent sequence of measurable sets $A_i, i=1,2,...$ that is also independent from $\{X_k\}_{k=1,2,...}$ and $\left\{N_t^{\lambda}\right\}_{t\geq 0}$.

Corollary 1 *For any measurable set A that is independent from X, we have*

$$\Psi\left(\lambda, c, u, F_{1_A X}\right)=\Psi\left(\lambda E(1_A), c, u, F_X\right). \tag{3.1}$$

Proposition 2 *The risk measure ρ^α is continuous in topology 4) and is not continuous in topologies 1) ,2), 3) and weak convergence in 5). Furthermore, ρ^α is not monotone.*

Proof Let us start by proving the continuity statements. First, continuity in L^∞ is clear.

We know that for any positive measurable set $A \in \mathcal{F}$, based on the previous corollary and (2.2) we have

$$\begin{aligned}\Psi\left(1, \eta E(1_A), u, F_{1_A}\right) &= \Psi\left(\lambda, \eta\lambda E(1_A), u, F_{1_A}\right)=\Psi\left(\lambda E(1_A), \eta\lambda E(1_A), u, F_1\right)\\ &= \Psi\left(1, \eta, u, F_1\right)=\Psi\left(1, \eta E(1), u, F_1\right).\end{aligned}$$

The implication of this equation and (2.6) is that

$$\rho^\alpha\left(1_A\right)=\rho^\alpha\left(1\right), \tag{3.2}$$

for any measurable set A. So if $P(A) \to 0$, then 1_A will converge to zero in all topologies 1), 2), 3) and weak convergence 5) while $\rho^\alpha\left(1_A\right)$ does not. This completes the proof of the continuity statement.

We prove the monotonicity statement by way of contradiction. So, assume that ρ^α is non-decreasing. For a nonzero random variable $X \geq 0$ in L^∞, let us consider a number $0 < c < \|X\|_\infty$. Then, we have $P(\{X > c\}) > 0$. If ρ^α is non-decreasing we have $c\rho^\alpha(1) = c\rho^\alpha(1_{\{X>c\}}) = \rho^\alpha(c1_{\{X>c\}}) \leq \rho^\alpha(X)$. Since c can be any number smaller than $\|X\|_\infty$, this shows that $\|X\|_\infty\rho^\alpha(1) \leq \rho^\alpha(X)$. On the other hand, since $X \leq \|X\|_\infty$, then $\rho^\alpha(X) \leq \rho^\alpha(\|X\|_\infty \times 1) \leq \|X\|_\infty\rho^\alpha(1)$. Therefore, the two inequalities imply that $\rho^\alpha(X) = \|X\|_\infty\rho^\alpha(1)$. Now, for a general non-negative random variable X, for any large number $C > 0$, we have

$$\rho^\alpha(X) \geq \rho^\alpha(\min\{X, C\}) = \|\min\{X, C\}\|_\infty\rho^\alpha(1). \tag{3.3}$$

Now let X be an exponentially distributed random variable, e.g., $F_X(x) = 1 - e^{-x}, x > 0$. It is well understood that $\rho^\alpha(X) < \infty$, but as $X \in L^1 \setminus L^\infty$, by sending $C \to \infty$ in (3.3), we get $\rho^\alpha\left(X\right) = \infty$, which is a contradiction. This completes the proof.

Remark 1 Let us comment on the implications of this proposition. For the continuity statement, one can see that using an adjusted premium rate (i.e., $\eta\lambda E(X)$), the ruin probability, as a tool to measure the risk, is unable to distinguish between two different events, one with 99 percent of chance of generating loss (i.e., $P(A) = 0.99$) and another one with only 1 percent chance of generating loss (i.e., $P(A) = 0.01$). The same is true for the monotonicity, which means that the ruin probability as a measure for risk is not sensitive enough to changes in claim risk.

Remark 2 Let us now take a closer look at the limitations observed in Proposition 2. The risk measure ρ^α can be written as a composition of two mappings: $\rho^\alpha = \mathrm{VaR}_\alpha \circ f$, where

$$f(X) \sim \sup_{t\geq 0} \sum_{k=1}^{N^\lambda(t)} X_k - \eta\lambda E(X)t, \tag{3.4}$$

for an independent sequence $X_k \sim X$, and $f(X)$ is co-monotone with a fixed uniform(0,1) random variable U. Note that we had to use U to make sure f is well defined; however, if we only think of the distribution of $f(X)$, we do not need to consider U.

Now let us see which part can cause the limitations observed in Proposition 2. An interesting implication of (3.2) is that for any measurable set A with $P(A) > 0$, the random variable $f(1_A)$ has the same distribution as $f(1) \sim \sup_{t\geq 0}(N^{\lambda}(t) - \eta\lambda t)$. This shows that if we replace VaR by any law-invariant mapping in (2.5), the relation (3.2) still holds. In addition, if the law-invariant mapping is positive homogenous and non-decreasing, the relation (3.3) also holds. For instance, as suggested in Cheridito et at. [4] we can replace VaR by CVaR, which gives a sub-additive risk measure. This means the limitations in Proposition 2 are not associated with VaR, but with the mapping f in (3.4). There are different practical implications from this observation. First, the adjusted premium $\eta\lambda E(X)$, even though it seems to be a fair choice, is not sufficient to deal with the aggregate risk. Second, for measuring the risk of a random process, we need to use more information than the ruin distribution $f(X)$ alone. For instance, we can also account for the individual loss. Third, a risk measure on the random processes performs better if it depends on the whole path; this is an approach that is studied in different articles, see Cheridito et at. [3] and Assa [2].

Remark 3 Property 3.1(ii) in Trufin et al. [10], that cannot hold true according to Proposition 2, states that ρ^α is stop-loss increasing i.e., $\rho^\alpha(X) \leq \rho^\alpha(Y)$ if $g(X) \leq g(Y)$ for all increasing and convex functions g. The proof in Trufin et al. [10] is one line and states that: "*If $X \preceq_{icx} Y$, then $\psi_X(u) \leq \psi_Y(u)$ for all u. This classical result of risk theory can be found, e.g., in standard textbooks as Kaas et al. (2008, Section 7.4.2)*". Kaas et al. [8] also require $E(X) = E(Y)$, so the condition $X \preceq_{icx} Y$ in Trufin et al. [10] should be $X \preceq_{cx} Y$.

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4 Appendix, proof of Proposition 1

Let $Y_t = \sum_{k=1}^{N_t^{x\lambda}} X_k$ and $Y_t^x = \sum_{k=1}^{N_t^{\lambda}} 1_{A_k} X_k$. To prove the proposition it is enough to show that $P\left(Y_t^x - Y_s^x \le y\right) = P\left(Y_t - Y_s \le y\right)$ for all $s < t$:

$$
\begin{aligned}
P\left(Y_t^x - Y_s^x \le y\right) &= P\left(\sum_{k=N_s^\lambda+1}^{N_t^\lambda} 1_{A_k} X_k \le y\right) = P\left(\sum_{k=1}^{N_{t-s}^\lambda} 1_{A_k} X_k \le y\right) \\
&= \sum_{m=0}^{\infty} P\left(\sum_{k=1}^{m} 1_{A_k} X_k \le y\right) P\left(N_{t-s}^\lambda = m\right) \\
&= \sum_{m=0}^{\infty} \sum_{l=0}^{m} P\left(\sum_{k=1}^{l} X_k \le y\right) \binom{m}{l} x^l \left(1-x\right)^{m-l} \frac{e^{-\lambda(t-s)} \left(\lambda\left(t-s\right)\right)^m}{m!} \\
&= e^{-\lambda(t-s)} \sum_{l=0}^{\infty} \left(\frac{x^l\left(1-x\right)^{-l}}{l!}\right) P\left(\sum_{k=1}^{l} X_k \le y\right) \\
&\qquad \times \left(\lambda\left(1-x\right)\left(t-s\right)\right)^l \sum_{m=l}^{\infty} \frac{1}{(m-l)!} \left(\lambda\left(1-x\right)\left(t-s\right)\right)^{m-l} \\
&= e^{-\lambda x(t-s)} \sum_{l=0}^{\infty} \left(\frac{\left(\lambda\left(t-s\right)x\right)^l}{l!}\right) P\left(\sum_{k=1}^{l} X_k \le y\right) \\
&\qquad \times \sum_{m=0}^{\infty} \frac{e^{-\lambda(1-x)(t-s)} \left(\lambda\left(1-x\right)\left(t-s\right)\right)^m}{m!} \\
&= e^{-\lambda x(t-s)} \sum_{l=0}^{\infty} \left(\frac{\left(\lambda\left(t-s\right)x\right)^l}{l!}\right) P\left(\sum_{k=1}^{l} X_k \le y\right) = P\left(\sum_{k=1}^{N_{t-s}^{\lambda x}} X_k \le y\right)
\end{aligned}
$$