GEOMETRIC CRITERIA FOR REALIZABILITY OF TENSEGRITIES IN HIGHER DIMENSIONS

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Abstract. In this paper we study a classical Maxwell question on the existence of self-stresses for frameworks, which are called tensegrities. We give a complete answer on geometric conditions of at most $(d+1)$-valent tensegrities in d-dimensional space both in terms of discrete multiplicative 1-forms and in terms of "join" and "intersection" operations in projective geometry.

1. Introduction

In this paper we are dealing with a classical question on self-stresses of frameworks in arbitrary dimensions (that were later referred as tensegrities). Our main goal is to find geometric tensegrity existence characterizations on all (generic) k-valent graphs in $\mathbb{R}^d$ ($k \leq d+1$). We do this in two different geometric settings. The first one is based on discrete multiplicative 1-forms which belong to discrete differential geometry. For example, discrete multiplicative 1-forms have been used to characterize discrete Koenigs nets [2]. The second one is via geometric relations written in terms of join/intersection operations in projective geometry.

Tensegrities (Definition 1) are frameworks in equilibrium consisting of rods, cables, and struts linked to each other at vertices p_i. Forces in the framework (represented by vectors $w_{i,j}(p_i - p_j)$) acting along cables are pulling $w_{i,j} < 0$ whereas forces acting along rods are pushing $w_{i,j} > 0$; for the struts both force action is possible. The equilibrium condition means that all forces around each vertex sum up to zero:

$$\sum_{\{j|j\neq i\}} w_{i,j}(p_i - p_j) = 0, \quad \text{for all } i.$$

Although the research on tensegrities was initiated already in 1864 by J.C. Maxwell [18], the term "tensegrity" itself appears much later. Tensegrity is a concatenation of the words "tension" and "integrity". This term was proposed by R. Buckminster Fuller who was inspired

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by the elegance of self-stressed constructions. Tensegrities form an essential part of modern architecture and in arts, they serve as a light structural support (like in a recent sculpture *TensegriTree* in the University of Kent). Tensegrities are traditionally used in the study of cells [10, 1], viruses [4, 22], deployable mechanisms [24], etc.

In the second half of the 20th century the subject of tensegrities became popular in mathematics again: questions of rigidity and flexibility of structures were studied amongst others by R. Connelly, B. Roth, and W. Whiteley in [5, 6, 20, 31], etc. For a general modern overview of the subject we refer to the book [23].

Tensegrities were generalized to spherical and projective geometries (by F.V. Saliola and W. Whiteley [21]); to normed spaces (by D. Kitson and S.C. Power in [15] and by D. Kitson and B. Schulze in [16]); and to surfaces in $\mathbb{R}^3$ (by B. Jackson and A. Nixon in [11]); etc.

Realizability of tensegrities. If the amount of edges is not large enough, a generic realization of a graph in $\mathbb{R}^d$ will not have a non-zero tensegrity. The non-zero tensegrities exist only for specific frameworks (that are actually semi-algebraic sets in the configuration spaces of tensegrities [8]). For instance, a framework for the $K_{3,3}$ graph admits a non-zero tensegrity if and only if all its six points are on a conic.

An algebraic description of realizability conditions for tensegrities was proposed by N.L. White and W. Whiteley in [29, 30]. It was given in terms of bracket rings for the determinants of extended rigidity matrices (see also [27]). This original algebraic approach introduces large polynomial conditions using all vertices of the whole graph (plus several additional vertices taken arbitrarily, they are called *tie-downs*) which can become rather complex and tricky to observe and to analyze. Factorization of these polynomials is a hard problem that remains open since the seminal papers of N.L. White and W. Whiteley [29, 30]. This problem is of high importance in the area as the factors correspond to self-stressed frameworks sharing certain geometric properties (like certain vertices in a line or certain planes intersect, etc). The hope is that the factorization of these polynomials can be deduced from their rigidity nature (that was confirmed by several examples).

Our geometric approach is designed to complement this algebraic approach. It is more localised, here we write single conditions for cycles in the graph which delivers explicit conditions on the arrangements of affine spaces associated to frameworks. Often the last provides factors of the bracket expression. Here we would like to note that the amount of "cycle" conditions to be considered simultaneously is suggested by the combinatorial theory of tensegrities studied by R. Connelly, B. Roth,

W. Whiteley etc. (e.g., for Laman graphs in the two-dimensional plane all conditions should be one-dimensional). A good reference for the combinatorial theory is the book [23].

In their work M. de Guzmán and D. Orden [7] made first steps in the study of geometry of stresses by introducing atom decomposition techniques. In all the studied examples (see, e.g., [8, 30]) there is a simple geometric description for tensegrities in terms of the "meet-join" operations of Cayley algebra. This suggests such a description for all possible graphs. In this paper we develop techniques to write such conditions for the case of k-valent graphs ($k \leq d+1$) in an arbitrary dimension d.

A preliminary investigation of geometric conditions was made in [8]: the authors had introduced two surgeries that result in classification of all the geometric conditions for codimension one strata for graphs with 8 or less vertices. Topological properties of the configuration spaces of all tensegrities for graphs with 4 and 5 vertices were studied in [14]. A complete description of geometric conditions in the two-dimensional case was announced in [13]. Finally, a nice collection of problems on geometry and topology of stratification of tensegrities can be found in [12].

In the present paper we consider less than $d+1$ valent graphs in $\mathbb{R}^d$. We write geometric conditions both in terms of integrability of multiplicative 1-forms (Theorem 11) and in terms of join/intersection operations in projective geometry (Theorem 36).

Organization of the paper. We start in Section 2 with the definition of tensegrities and notions that we use throughout the paper. In Section 3 we discuss discrete multiplicative 1-forms and how exact 1-forms characterize frameworks admitting non-zero self-stresses. In Section 4 we work within join/intersection operations in projective geometry to provide a recursive geometric characterization of tensegrities. Section 5 is devoted to point out a relation between tensegrities and harmonic maps. Finally in Section 6 we study examples of tensegrities in $\mathbb{R}^3$.

2. Notions and definitions

In this section we give the necessary definitions of the setting around tensegrities. Additionally, we provide the notion of general position of the framework so that we can formulate our geometric conditions on frameworks admitting a tensegrity.

2.1. **Definition of tensegrities.** Let us first set the scene by recalling some basic notions before we come to the general definition of tensegrities.

Definition 1. Let G be an arbitrary graph without loops and multiple edges on n vertices.

— Let $V(G) = \{v_1, \ldots, v_n\}$ and $E(G)$ denote the sets of vertices and edges for G, respectively. Denote by $(v_i; v_j)$ the edge joining v_i and v_j.

— Let $B(G)$ be the subset of all 1-valent vertices in $V(G)$, which we refer to as the *boundary of* G.

— Let $Z(G)$ be the subset of all vertices with valence greater than 1 in $V(G)$.

- A *framework* $G(P)$ is a map of the vertices $v_1, \ldots, v_n$ of G onto a finite point configuration $P = (p_1, \ldots, p_n)$ in $\mathbb{R}^d$, such that $G(P)(v_i) = p_i$ for $i = 1, \ldots, n$. We say that there is an *edge* between p_i and p_j if $(v_i; v_j)$ is an edge of G and denote it by $(p_i; p_j)$. Note that the points $p_1, \ldots, p_n$ are not necessarily distinct.
- A *stress* w on a framework is an assignment of real scalars $w_{i,j}$ (called *tensions*) to its edges $(v_i; v_j)$ with the property $w_{i,j} = w_{j,i}$. We also set $w_{i,j} = 0$ if there is no edge between the corresponding vertices.
- A stress w is called a *self-stress* if the following equilibrium condition is fulfilled at every vertex of valence greater than 1, i.e., for all $v_i \in Z(G)$:
$$\sum_{\{j \mid j \neq i\}} w_{i,j}(p_i - p_j) = 0.$$
By $p_i - p_j$ we denote the vector from the point p_j to the point p_i. Note that we do not consider equilibrium for the boundary points $B(G)$. These are the points where the framework is attached to the exterior construction. Therefore, the corresponding forces are compensated by the forces of the exterior construction.
- A pair $(G(P), w)$ is called a *tensegrity* if w is a self-stress for the framework $G(P)$.
- A tensegrity $(G(P), w)$ (or stress w) is said to be *non-zero* if there exists an edge $(v_i; v_j)$ of the framework that has non-vanishing tension $w_{i,j} \neq 0$.
- A tensegrity $(G(P), w)$ (or stress w) is said to be *everywhere non-zero* if each existing edge $(v_i; v_j)$ of the framework has non-vanishing tension $w_{i,j} \neq 0$.

Remark 2. If the set of boundary points $B(G)$ is empty, we have the classical case of tensegrities without boundary.

2.2. **Frameworks in various general positions.** To formulate our geometric conditions on frameworks admitting a non-zero self-stress via discrete multiplicative 1-forms, we need the vertices to lie in general position (Sec. 3). A slightly stronger version of generality is needed to formulate our conditions within the setting of join/intersection operations in projective geometry (Sec. 4).

As a general notion throughout the paper, by $\mathrm{span}(s_1, \ldots, s_k)$ we denote the *affine* or *projective span* of affine/projective spaces $s_1, \ldots, s_k$ and not the linear span as a vector space. By 2*-plane* we denote a two-dimensional affine/projective subspace and by k-plane a k-dimensional affine/projective subspace.

Definition 3. A framework $G(P)$ is *linearly generic* if for every vertex (whose degree or valence we denote by k) the following two conditions hold:

- the k edges emanating from this vertex span a $(k-1)$-plane;
- every subset of $k-1$ edges emanating from this vertex spans this $(k-1)$-plane.

Remark 4. The valences of a linearly generic framework in $\mathbb{R}^d$ do not exceed $d+1$.

For the geometric characterization of tensegrities in terms of discrete multiplicative 1-forms (Section 3), the property on frameworks of being linearly generic is all we need. As for our characterization as formulated within join/intersection operations in projective geometry (Section 4), we have to include one further notion of general position.

Definition 5. A framework $G(P) \subset \mathbb{R}^d$ is *in 3D-general position* if the following two conditions hold:

- $G(P)$ is linearly generic, and
- every 4-tuple of vertices in every cycle of $G(P)$ spans a 3-plane.

Note that the second condition in the previous definition does not imply that the framework must lie in $\mathbb{R}^3$. Just every 4-tuple of vertices in a cycle span a 3-plane.

3. Characterizing k-valent tensegrities in terms of ratios

In this section we give a geometric characterization for linearly generic at most k-valent graphs in $\mathbb{R}^d$ ($k \leq d+1$) admitting a non-zero self-stress. It turns out to be practical to first provide a characterization for trivalent graphs before then generalizing it to k-valent graphs. Throughout this section all graphs G are connected. Our goal is to

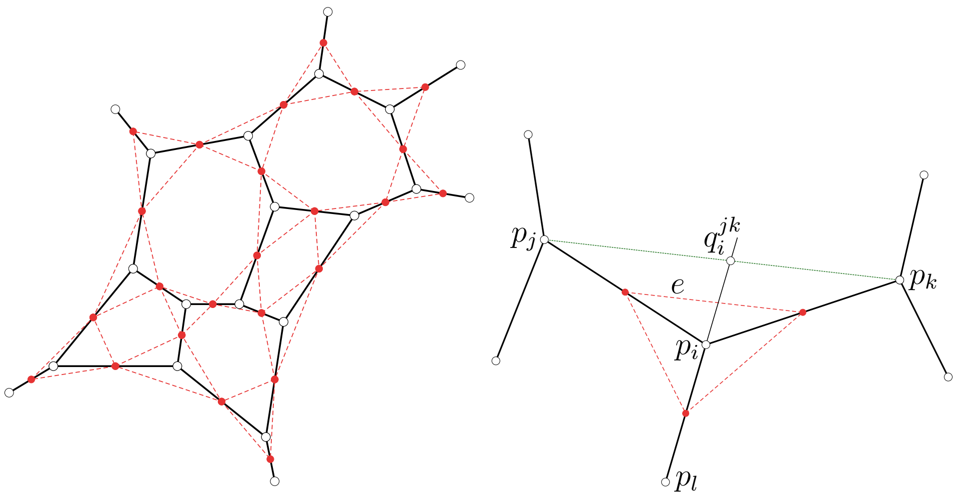


FIGURE 1. *Left*: A graph G (black lines; white vertices) and its line graph $L(G)$ (red dashed edges; red vertices). *Right*: A flat vertex star p_i, p_j, p_k, p_l. The edge e of the line graph $L(G)$ corresponds to the angle (v_j, v_i, v_k). The intersection point q_i^{jk} of the straight lines through $p_j p_k$ and $p_i p_l$ determines the value of the multiplicative 1-form q on that edge by $q(v_j, v_i, v_k) = (p_j - q_i^{jk}) : (q_i^{jk} - p_k)$.

show that the product of certain ratios is 1 if and only if the framework admits a non-zero tensegrity (Theorem 11).

3.1. **Tensegrities over trivalent graphs.** Our geometric characterization of a linearly generic framework to be a non-zero tensegrity is defined on the cycles of the underlying graph. The important notion here is the one of a discrete multiplicative 1-form which is well known in discrete differential geometry. We follow the definition in [2].

Definition 6. A real valued function $q : \vec{E}(G) \to \mathbb{R} \setminus \{0\}$ ($\vec{E}(G)$ denotes the set of oriented edges of the graph G) is called a *multiplicative 1-form*, if $q(-e) = 1/q(e)$ for every $e \in \vec{E}(G)$. It is called *exact* if for every cycle $e_1, \ldots, e_k$ of directed edges the values of the 1-form multiply to 1, i.e.,

$$q(e_1) \cdot \ldots \cdot q(e_k) = 1.$$

The following definition is about a particular subdivision of a graph.

Definition 7. The *line graph* $L(G)$ of a general graph G has the following properties. Its vertices are in a one-to-one correspondence with the edges of G. The edges of $L(G)$ connect two vertices if and only if the two respective edges of G are emanating from the same vertex. See Fig. 1 (left) and Fig. 5 (left).

We now aim at constructing a multiplicative 1-form on the oriented edges of the line graph $L(G)$ of a trivalent graph G corresponding to a linearly generic framework $G(P)$.

Each edge e of $L(G)$ connects the midpoints of edges of the form $(v_j; v_i)$ and $(v_i; v_k)$, as illustrated in Figure 1 (right). We can therefore denote the oriented edges of $L(G)$ by triplets of the form $e = (v_j, v_i, v_k)$ with the property that the negatively oriented edge is $-e = (v_k, v_i, v_j)$.

Let us denote the third edge emanating from v_i by $(v_i; v_l)$. The framework being linearly generic implies that the corresponding vertices p_i, p_j, p_k, p_l lie in a common 2-plane. Furthermore, the framework being linearly generic implies that the straight line connecting $p_i p_l$ intersects the line connecting $p_j p_k$ in a point q_i^{jk}. Consequently, this point gives rise to an affine ratio of the form

$$q(v_j, v_i, v_k) := \frac{p_j - q_i^{jk}}{q_i^{jk} - p_k}, \tag{1}$$

as ratio of parallel vectors. Clearly, $q(v_j, v_i, v_k) = 1/q(v_k, v_i, v_j)$ which implies that q is a multiplicative 1-form on the oriented edges of the line graph $L(G)$.

Theorem 8. Let $G(P)$ be a linearly generic trivalent framework. Then there is a stress w on $G(P)$ such that the framework $(G(P), w)$ is a non-zero tensegrity if and only if the 1-form q given by Equation (1) on the line graph $L(G)$ is exact.

Proof. Let us first assume that $(G(P), w)$ is a non-zero tensegrity. Since at every inner vertex p_i of a trivalent tensegrity the sum of forces adds up to zero we obtain

$$w_{i,j}(p_i - p_j) + w_{i,k}(p_i - p_k) + w_{i,l}(p_i - p_l) = 0. \tag{2}$$

The point q_i^{jk} lies on the straight line through $p_i p_l$ and can therefore be written in the form

$$q_i^{jk} = p_i + \lambda(p_i - p_l)$$

for some $\lambda \in \mathbb{R}$. Inserting Equation (2) yields

$$\begin{aligned} q_i^{jk} &= p_i + \lambda\Big(\frac{w_{i,j}}{w_{i,l}}(p_j - p_i) + \frac{w_{i,k}}{w_{i,l}}(p_k - p_i)\Big) \\ &= \Big(1 - \lambda\frac{w_{i,j}}{w_{i,l}} - \lambda\frac{w_{i,k}}{w_{i,l}}\Big)p_i + \lambda\frac{w_{i,j}}{w_{i,l}}p_j + \lambda\frac{w_{i,k}}{w_{i,l}}p_k. \end{aligned}$$

Since q_i^{jk} must lie on the line through $p_j p_k$ we obtain for $\lambda = \frac{w_{i,l}}{w_{i,j}+w_{i,k}}$ and therefore the affine combination

$$q_i^{jk} = \frac{w_{i,j}}{w_{i,j} + w_{i,k}}p_j + \frac{w_{i,k}}{w_{i,j} + w_{i,k}}p_k.$$

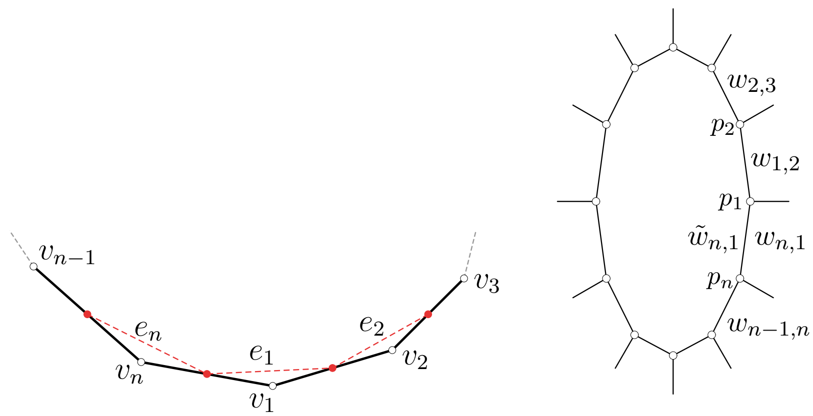


FIGURE 2. *Left*: Notations of edges in a cycle in the line graph $L(G)$. *Right*: A cycle in a trivalent graph. After prescribing a tension $w_{1,2}$ we can compute the tension at edge $(v_n; v_1)$ in two ways: First, by enforcing equilibrium at p_1 (and getting $\tilde{w}_{n,1}$), and second by transporting the tension along the cycle (resulting in $w_{n,1}$).

Consequently, for our q in Equation (1) we obtain

$$q(v_j, v_i, v_k) = \frac{w_{i,k}}{w_{i,j}}. \tag{3}$$

To show the exactness of q we have to show that the product of all values along any cycle multiply to 1. So let $(e_1, \ldots, e_n)$ be a cycle of the line graph $L(G)$ where e_i are oriented edges. There is a corresponding cycle $(v_1, \ldots, v_n)$ in G such that e_i corresponds to the angle (v_{i-1}, v_i, v_{i+1}), where we take the indices modulo n (see Figure 2 left). We compute the product of corresponding values of q:

$$\begin{aligned}\prod_{i=1}^{n} q(e_i) = \prod_{i=1}^{n} q(v_{i-1}, v_i, v_{i+1}) = \prod_{i=1}^{n} \frac{w_{i,i+1}}{w_{i,i-1}} = \prod_{i=1}^{n} \frac{w_{i,i+1}}{w_{i-1,i}} \\ = \frac{w_{1,2}}{w_{n,1}} \cdot \frac{w_{2,3}}{w_{1,2}} \cdot \ldots \cdot \frac{w_{n-1,n}}{w_{n-2,n-1}} \cdot \frac{w_{n,1}}{w_{n-1,n}} = 1,\end{aligned}$$

which shows the first direction of the statement.

As for the other direction, let us first note that prescribing one tension $w_{i,j}$ in a trivalent vertex of a linearly generic framework uniquely determines the other two tensions as well since Equation (2) is then a linear combination of two linearly independent vectors with coefficients $w_{i,k}$ and $w_{i,l}$. Consequently, after choosing one tension $w_{i,j}$ we can transport it to any other vertex along any connected path. This way we could define a stress w on the graph G, if this construction would be well-defined, i.e., if transporting the tension along different paths to the same edge would result in the same tensions. Or equivalently, if we transport the tension around any cycle we would have to get back to the same tension with which we started.

So let us take an arbitrary cycle $(v_1, \ldots, v_n)$. We choose a non-zero tension $w_{1,2}$ on the edge $(v_1; v_2)$ which immediately determines the tension $\tilde{w}_{n,1}$ on the edge $(v_n; v_1)$ due to the equilibrium condition shown in Equation (2). See also Figure 2 (right). The value of the multiplicative 1-form on the oriented edge (v_n, v_1, v_2) of $L(G)$ therefore has the value

$$q(v_n, v_1, v_2) = w_{1,2}/\tilde{w}_{n,1}.$$

On the other hand $w_{1,2}$ determines the tension $w_{2,3}$ as edge emanating from v_1. Repeating this propagation process we define all tensions in the cycle including the last one $w_{n,1}$. We have therefore defined the tension at $(v_n; v_1)$ twice: from the "left" and from the "right" as $\tilde{w}_{n,1}$ and $w_{n,1}$. Now the question is whether those tensions are the same.

Our assumption is that the multiplicative 1-form q is exact which implies

$$1 = \prod_{i=1}^{n} q(v_{i-1}, v_i, v_{i+1}) = \frac{w_{1,2}}{\tilde{w}_{n,1}} \cdot \frac{w_{2,3}}{w_{1,2}} \cdot \ldots \cdot \frac{w_{n-1,n}}{w_{n-2,n-1}} \cdot \frac{w_{n,1}}{w_{n-1,n}} = \frac{w_{n,1}}{\tilde{w}_{n,1}},$$

and therefore $w_{n,1} = \tilde{w}_{n,1}$. Consequently, we can consistently define a stress w (uniquely up to scaling) on $G(P)$ such that the framework $(G(P), w)$ is a non-zero tensegrity. □

The following corollary follows immediately from Theorem 8 and its proof, in particular from Equation (3).

Corollary 9. Let $G(P)$ be a linearly generic trivalent framework and let w be a non-zero stress on $G(P)$. Then the framework $(G(P), w)$ is a non-zero tensegrity if and only if the 1-form

$$\tilde{q}(v_j, v_i, v_k) := \frac{w_{i,k}}{w_{i,j}},$$

defined on the line graph $L(G)$ is exact. □

3.2. **Special cases of trivalent cycles.** In this section we will consider two special cases of cycles and briefly reflect on what Theorem 8 means in these cases.

$\underline{n = 3}$: In that case the cycle is a triangle and the points q_i^{jk} lie on the edges of the triangle opposite to p_i. Consequently, the exactness of the 1-form on that cycle is precisely the setting of the classical Ceva's theorem (see e.g., [19]). Therefore, the three lines $p_1 q_1^{2,3}$, $p_2 q_2^{3,1}$, and $p_3 q_3^{1,2}$ intersect in one point (cf. [13] and see Figure 3 left).

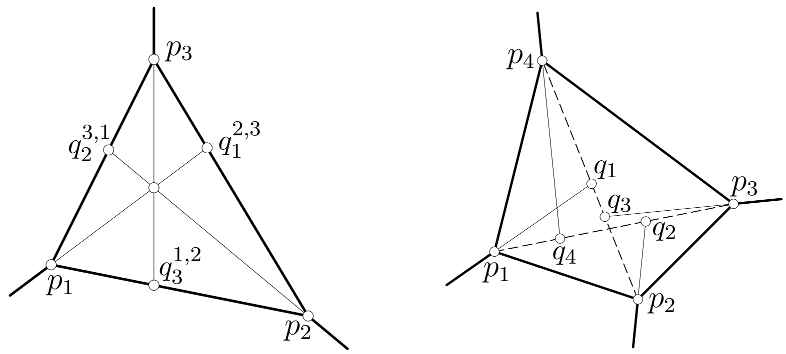


FIGURE 3. *Left*: The cycle is a triangle. The exactness of the 1-form on that triangle is equivalent to the three "outward" pointing edges intersecting in one point, i.e., Ceva's configuration. *Right*: The cycle is a quadrilateral. Then the three "outward" pointing edges intersect the respective diagonals in points q_i. We abbreviate $q_1^{2,4}$ simply by q_1 etc. The exactness of the 1-form is equivalent to $\mathrm{cr}(q_1, p_4, q_3, p_2) = \mathrm{cr}(q_2, p_3, q_4, p_1)$.

$\underline{n=4}$: In the case of a quadrilateral the points q_i^{jk} lie on the diagonals (see Figure 3 right). Exactness of the 1-form on that cycle is equivalent to

$$1 = q(v_4, v_1, v_2) \cdot q(v_1, v_2, v_3) \cdot q(v_2, v_3, v_4) \cdot q(v_3, v_4, v_1),$$

which is further equivalent to

$$q(v_4, v_1, v_2) \cdot q(v_2, v_3, v_4) = \frac{1}{q(v_1, v_2, v_3) \cdot q(v_3, v_4, v_1)},$$

and further to

$$\frac{p_4 - q_1^{4,2}}{q_1^{4,2} - p_2} \cdot \frac{p_2 - q_3^{2,4}}{q_3^{2,4} - p_4} = \frac{q_2^{1,3} - p_3}{p_1 - q_2^{1,3}} \cdot \frac{q_4^{3,1} - p_1}{p_3 - q_4^{3,1}}.$$

The last equation is an equation of cross-ratios, namely

$$\mathrm{cr}(q_1^{4,2}, p_4, q_3^{2,4}, p_2) = \mathrm{cr}(q_2^{1,3}, p_3, q_4^{3,1}, p_1). \tag{4}$$

Example 10. It is well known (see e.g., [8]) that the complete graph $K_{3,3}$, which is trivalent, with vertices in $\mathbb{R}^2$ is a tensegrity if and only if the vertices lie on a conic (see Figure 4). That property can also be shown easily within our setting of exact multiplicative 1-forms as follows. According to Steiner's definition of conics, the property of six points lying on a conic is equivalent to the four lines p_5p_i and p_6p_i for $i = 1, \ldots, 4$ being related by a projectivity (a projective map). Or equivalently that means that the cross-ratios of these pair of four lines are the same. Let us consider the cycle (v_1, v_2, v_3, v_4) with four vertices.

Then Equation (4) holds for this cycle which we will use in the following computation. Further, we have

$$\begin{aligned}&\mathrm{cr}(p_5p_2, p_5p_3, p_5p_4, p_5p_1) = \mathrm{cr}(q_2, p_3, q_4, p_1) \overset{(4)}{=} \mathrm{cr}(q_1, p_4, q_3, p_2)\\ =&\mathrm{cr}(p_6p_1, p_6p_4, p_6p_3, p_6p_2) = \mathrm{cr}(p_6p_2, p_6p_3, p_6p_4, p_6p_1),\end{aligned}$$

where the last equality holds because $\mathrm{cr}(a, b, c, d) = \mathrm{cr}(d, c, b, a)$.

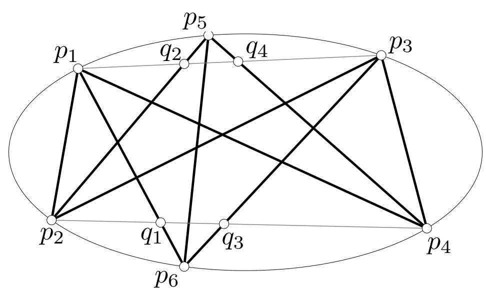


FIGURE 4. Six points $p_1, \ldots, p_6$ in the 2-plane $\mathbb{R}^2$ form a tensegrity if and only if the six points lie on a conic.

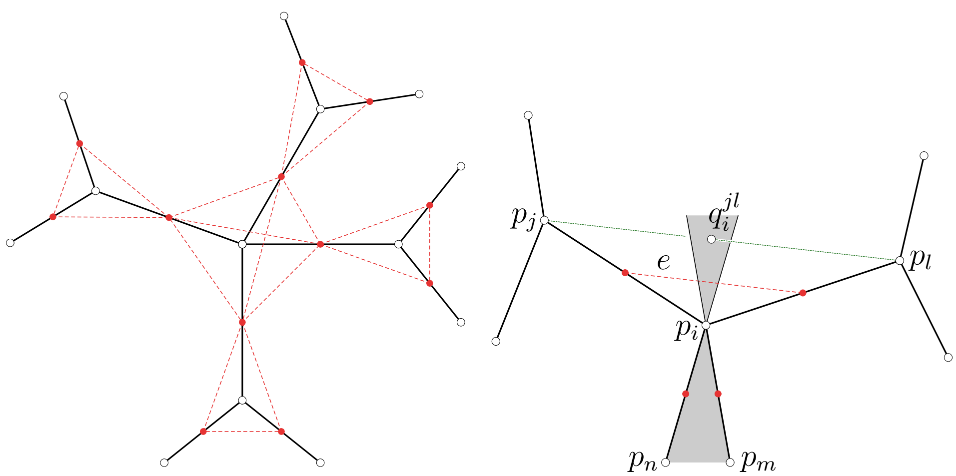


FIGURE 5. *Left*: The line graph $L(G)$ of a *general* graph G. The new edges (red dashed) connect midpoints of old adjacent edges. A vertex star of valence three generates three new edges, a vertex star of valence four generates six new edges. *Right*: We construct the discrete multiplicative 1-form on edges of the line graph by intersecting the line p_jp_l with the affine subspace $\mathrm{span}(\bigcup_{m\neq j,l} p_m)$.

3.3. **Tensegrities over k-valent graphs.** We will now generalize the geometric characterization of trivalent tensegrities (of Section 3.1) to linearly generic k-valent tensegrities in $\mathbb{R}^d$ with $k \leq d+1$.

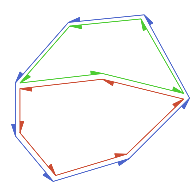
Let us consider a k-valent vertex star with inner vertex v_i and adjacent vertices $v_1, \ldots, v_k$. Again we can denote an oriented edge of the line graph $L(G)$ by (v_j, v_i, v_l) (with $1 \leq j \neq l \leq n$). See also Figure 5 (right). Since G is linearly generic, the subspaces $\operatorname{span}(p_i, p_j, p_l)$ and $\operatorname{span}(\bigcup_{\substack{m\neq j,l \\ v_m\sim v_i}} p_m)$ intersect in a line L, where $v_m \sim v_i$ means v_m is adjacent to v_i. Consequently, this line L intersects the line $p_j p_l$ in a point q_i^{jl}. In the trivalent case, L is simply the line $p_i p_l$. Analogously to the trivalent case we define the discrete multiplicative 1-form as

$$q(v_j, v_i, v_l) := \frac{p_j - q_i^{jl}}{q_i^{jl} - p_l}. \tag{5}$$

Now the proof of Theorem 8 can be repeated basically word by word which implies the following theorem.

Theorem 11. Let $G(P)$ be a linearly generic framework in $\mathbb{R}^d$ with vertices of valence at most $d+1$. Then there is a stress w on $G(P)$ such that the framework $(G(P), w)$ is a non-zero tensegrity if and only if the 1-form q given by Equation (5) on the line graph $L(G)$ is exact. $\square$

Note that if a cycle can be decomposed by two other cycles with common edges, the product of the multiplicative 1-form values of the first cycle equals the product of the decomposing cycles. The values of common edges that appear in the decomposing cycles in reversed orientations are reciprocal to each other and simply cancel out in the product. That leads us to the following remark.



Remark 12. In the previous theorem and in Theorem 8 it is sufficient to check the criterion only for generator loops of the first homology group $H_1(G)$ of the graph (if we consider the graph as topological space), because all other loops can be decomposed by those. These conditions for different generators of $H_1(G)$ may still coincide, as the realisability codimension in the space of all (including zero-force load) tensegrities is defined by the combinatorics of the graphs (e.g., it is one condition for Laman graphs in the plane). Some of the conditions will correspond to different strata.

4. Join/intersection conditions for frameworks in 3D-general position

In this section we construct join/intersection conditions for frameworks whose all 4-tuples of vertices in any cycle span a 3-plane. We start in Section 4.1 with the case of frameworks for so-called framed cycles. We introduce WU-surgeries on framed cycles that preserve the property to admit a non-zero tensegrity and that reduce the amount of vertices of framed cycles. These properties will lead to explicit expressions in terms of join/intersection operations in projective geometry. Further, in Section 4.2 we prove that a sufficiently generic framework admits a non-zero tensegrity if and only if all its associated framed cycle frameworks admit non-zero tensegrities (Theorem 26). Finally, in Section 4.3 we briefly recall the basic notions join/intersection in projective geometry and construct conditions within that framework for the existence of tensegrities for given graphs that are not generically flexible (Theorem 36). All frameworks in this section are in $\mathbb{R}^d$ with $d \geq 3$.

4.1. **Framed cycles and their frameworks.** We start this section with basic definitions, some properties of framed cycles and their generic frameworks. Further, we introduce WU-surgeries that take frameworks in 3D-general position to generic frameworks of framed cycles. We show also that WU-surgeries preserve the property of admitting a non-zero tensegrity.

4.1.1. *General definitions.* We say that a graph is a *cycle* if it is homeomorphic to a circle.

Definition 13. Let $C = (c_1, \dots, c_n)$ and $B = (b_1, \dots, b_n)$ be two n-tuples of points. A *framed cycle* $C_B = (C, B)$ is the cycle $c_1, \dots, c_n$ with attached edges $b_i c_i$ for $i = 1, \dots, n$.

Definition 14. We say that a framework $C_B(P)$ of a framed cycle C_B is *in 3D-general flat position* if

- $C_B(P)$ is linearly generic (see Definition 3);
- there are no four points of $C(P)$ contained in a 2-plane (only for the cycle C).

Remark 15. Notice that linear genericity in particular implies that all edges emanating from the same vertex of a framed cycle are contained in a 2-plane; and that $G(P)(b_i) \neq G(P)(c_i)$ for all admissible i.

4.1.2. *A preliminary observation.* Let us formulate a preliminary statement for the definition of WU-surgeries.

Recall that a cycle $C_B(P)$ is in 3D-general flat position if it is linearly generic (every three edges emanating from the same vertex span a 2-plane) and if there are no four points of $C_B(P)$ contained in a 2-plane (see Definition 14).

Proposition 16. Let a framed cycle framework $C_B(P)$ be in 3D-general flat position. Let also

$$G(P)(b_i) = e_i, \quad G(P)(c_i) = r_i, \quad i = 1, \ldots, n.$$

Then we have the following two statements:

- The line $e_{i-1}e_i$ is not contained in the 2-plane $r_{i-2}r_{i-1}r_{i+1}$;

Denote by $\hat{e}_{i-1}$ the (projective) intersection point of the line $e_{i-1}e_i$ and the 2-plane $r_{i-2}r_{i-1}r_{i+1}$ (see Figure 6). Then additionally we have:

- $\hat{e}_{i-1} \notin r_{i-2}r_{i-1}$;
- $\hat{e}_{i-1} \notin r_{i-1}r_{i+1}$.

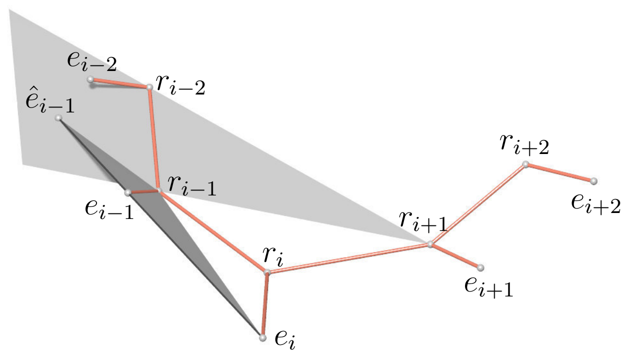


FIGURE 6. Definition of $\hat{e}_{i-1}$.

Remark 17. The theory of tensegrities (or equivalently the theory of infinitesimal rigidity) is projectively invariant. So we do not consider special cases of parallel objects. They are not parallel after an appropriate choice of an affine chart.

Proof of Proposition 16. First of all, the point e_{i-1} is not in the 2-plane

$$r_{i-2}r_{i-1}r_{i+1}$$

(cf. Figure 6), as otherwise

$$\operatorname{span}(r_{i-2}, r_{i-1}, r_i) = \operatorname{span}(r_{i-2}, r_{i-1}, e_{i-1}) = \operatorname{span}(r_{i-2}, r_{i-1}, r_{i+1})$$

(here the first equality holds as vectors $r_{i-2}e_{i-1}$, $r_{i-1}e_{i-1}$ and r_ie_{i-1} are adjacent to the same edge and, therefore, linear genericity of $C_B(P)$ implies that they are contained in a 2-plane; the second equality holds by the assumption $e_{i-1} \in r_{i-2}r_{i-1}r_{i+1}$) which would imply that the points $r_{i-2}, r_{i-1}, r_i, r_{i+1}$ are contained in a 2-plane, and therefore $C(P)$ is not in a 3D-general flat position. Therefore, the line $e_{i-1}e_i$ is not in the 2-plane $r_{i-2}r_{i-1}r_{i+1}$.

Secondly, if $\hat{e}_{i-1} \in r_{i-2}r_{i-1}$, then the points $e_i, e_{i-1}, r_{i-2}, r_{i-1}$ are in a 2-plane. Now the point r_i is in this 2-plane as it is in the span of $e_{i-1}, r_{i-1}, r_{i-2}$; and additionally r_{i+1} is in this 2-plane as it is in the span of e_i, r_{i-1}, r_i. Therefore, $r_{i-2}, r_{i-1}, r_i, r_{i+1}$ are in this 2-plane, which contradicts to flat 3D-genericity of the cycle.

Finally, if $\hat{e}_{i-1} \in r_{i-1}r_{i+1}$, then the points $e_i, e_{i-1}, r_{i-1}, r_{i+1}$ are in a 2-plane. Now the point r_i is in this 2-plane as it is in the span of e_i, r_{i-1}, r_{i+1}; and additionally r_{i-2} is in this 2-plane as it is in the span of e_{i-1}, r_{i-1}, r_i. Therefore, $r_{i-2}, r_{i-1}, r_i, r_{i+1}$ are in this 2-plane, which contradicts to flat 3D-genericity of the cycle.

This concludes the proof of all statements of the proposition. □

For the definition of WU-surgeries we need an index-symmetric statement. The following corollary is just the index-symmetric version of Proposition 16.

Corollary 18. Let a framed cycle framework $C_B(P) = (C(P), B(P))$ be in a 3D-general flat position. Let also

$$G(P)(b_i) = e_i, \quad G(P)(c_i) = r_i, \quad i = 1, \ldots, n.$$

Then we have the following two statements:

- The line e_ie_{i+1} is not in the 2-plane $r_{i-1}r_{i+1}r_{i+2}$; Denote by $\hat{e}_{i+1}$ the (projective) intersection point of the line e_ie_{i+1} and the 2-plane $r_{i-1}r_{i+1}r_{i+2}$. Then additionally we have:
- $\hat{e}_{i+1} \notin r_{i+1}r_{i+2}$;
- $\hat{e}_{i+1} \notin r_{i-1}r_{i+1}$.

Proof. After swapping the indexes $i \to n - i$ for all i in C_B we arrive at the statement of Proposition 16 for $n - i$. □

4.1.3. WU-*surgeries.* Let us continue with the definition of WU-surgeries.

Definition 19. Consider a framed cycle

$$C_B = \big((c_1, \ldots, c_n), (b_1, \ldots, b_n)\big),$$

and its framework

$$C_B(P) = \big((r_1, \ldots, r_n), (e_1, \ldots, e_n)\big)$$

in 3D-general flat position. Let $i \in 1, \ldots, n$. The WU-*surgery* of the cycle C at node i is the cycle

$$\mathsf{WU}_i(C_B(P)) = \big((r_1, \ldots, r_{i-2}, r_{i-1}, r_{i+1}, r_{i+2}, \ldots, r_n), \\ (e_1, \ldots, e_{i-2}, \hat{e}_{i-1}, \hat{e}_{i+1}, e_{i+2}, \ldots, e_n)\big),$$

where

$$\hat{e}_{i-1} = e_i e_{i-1} \cap r_{i+1} r_{i-1} r_{i-2};$$
$$\hat{e}_{i+1} = e_i e_{i+1} \cap r_{i-1} r_{i+1} r_{i+2},$$

(see Figure 7).

Remark 20. Due to Proposition 16 and Corollary 18, the points $\hat{e}_{i-1}$ and $\hat{e}_{i+1}$ are uniquely defined.

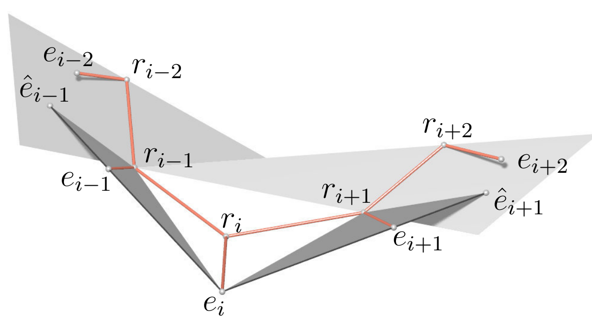


FIGURE 7. The construction of WU_i-surgery. Here we exclude the vertices e_i and r_i and replace e_{i-1} and e_{i+1} respectively by $\hat{e}_{i-1}$ and $\hat{e}_{i+1}$.

Corollary 21. A WU-surgery takes a framework of a framed cycle in 3D-general flat position to a framework of a framed cycle in 3D-general flat position.

Proof. The set of $C(P')$-vertices after the surgery is a subset of $C(P)$ therefore, there are no four points of $C(P')$ in a 2-plane.

By construction we have

$$\hat{e}_{i-1} \in \operatorname{span}(r_{i-2}, r_{i-1}, r_{i+1}) \quad \text{and} \quad \hat{e}_{i+1} \in \operatorname{span}(r_{i-1}, r_{i+1}, r_{i+2}).$$

Further, by Proposition 16 every two vectors from

$$\{r_{i-1} - \hat{e}_{i-1}, r_{i-1} - r_{i-2}, r_{i-1} - r_{i+1}\}$$

are not collinear.

Finally, by Corollary 18 every two vectors of

$$\{r_{i+1} - \hat{e}_{i+1}, r_{i+1} - r_{i+2}, r_{i+1} - r_{i-1}\}$$

are not collinear. Therefore, $\mathsf{WU}_i(C_B(P))$ is in 3D-general flat position. □

4.1.4. *Static properties of* WU*-surgeries.* We continue with the following important property of WU-surgeries.

Proposition 22. Let C_B be a framed cycle of length m and $i \in \{1, \ldots, m\}$. A framework $C_B(P)$ in 3D-general flat position admits a non-zero tensegrity if and only if $\mathsf{WU}_i(C_B(P))$ admits a non-zero tensegrity for every admissible i.

Proof. Let $C_B(P)$ admit a tensegrity $G(C_B(P), w)$. First, we construct a framed cycle tensegrity $(C^3_{B,i}(P), \hat{w})$. Let

$$C^3_{B,i}(P) = \big((r_{i-1}, r_i, r_{i+1}), (e_i, e_i, e_i)\big)$$

(see Figure 8).

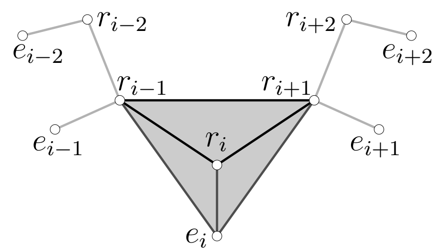


FIGURE 8. The framed cycle $C^3_{B,i}(P)$ is illustrated by the shaded area. Notice that the boundary points for this cycle all coincide with e_i.

The stress $\hat{w}$ is defined from the following condition: at all edges adjacent to r_i the stress $\hat{w}$ coincides with the stress w for $(C_B(P), w)$. It is clear that the tensions $\hat{w}$ on the remaining edges of $C^3_{B,i}(P)$ are defined in the unique way.

Let us now subtract $(C^3_{B,i}(P), \hat{w})$ from $(C_B(P), w)$. We have:

- zero stresses at all vertices adjacent to r_i.
- the sum of vectors of forces $\lambda r_{i-1}e_{i-1}$ and $\mu r_{i-1}e_i$ (for some non-zero λ and μ) should be in the 2-plane spanned by $r_{i-2}, r_{i-1}, r_{i+1}$ and therefore it is in the line $r_{i-1}\hat{e}_{i-1}$.
- the sum of vectors of forces $\lambda r_{i+1}e_{i+1}$ and $\mu r_{i+1}e_i$ (for some non-zero λ and μ) should be in the 2-plane spanned by $r_{i-1}, r_{i+1}, r_{i+2}$ and therefore it is in the line $r_{i+1}\hat{e}_{i+1}$.

Since the points $r_{i-2}, r_{i-1}, r_i, r_{i+1}$ span a 3-plane, the 2-planes $r_{i-1}e_{i-1}e_i$ and $r_{i-2}r_{i-1}r_{i+1}$ intersect by a line.

Symmetrically, the 2-planes $r_{i+1}e_{i+1}e_i$ and $r_{i-1}r_{i+1}r_{i+2}$ intersect by a line.

Therefore, the resulting tensegrity is a non-zero tensegrity on the framework $\mathsf{WU}_i(C_B(P))$.

Now let us assume that there is a non-zero tensegrity on $\mathsf{WU}_i(C_B(P))$. Then we consider a tensegrity $(C^3_{B,i}(P), \tilde{w})$, where $C^3_{B,i}(P)$ is the framed 3-cycle framework as above; the self-stress $\tilde{w}$ is defined by linearity starting from the fact that at edge $r_{i-1}r_{i+1}$ it coincides with the self-stress at $r_{i-1}r_{i+1}$ for $\mathsf{WU}_i(C_B(P))$.

Similarly, by subtracting $(C^3_{B,i}(P), \tilde{w})$ from $(\mathsf{WU}_i(C_B(P)), w)$ and summing the boundary force vectors at r_{i-1} and r_{i+1} we get a non-zero tensegrity for $C_B(P)$. □

4.2. **On existence and uniqueness of tensegrities for frameworks in 3D-general position.** Recall that in this paper we work only with connected graphs. The uniqueness of tensegrities (up to a scalar) can be formulated as follows.

Proposition 23. All tensegrities on a linearly generic framework are proportional. In addition every non-zero tensegrity is everywhere non-zero.

Proof. The proof is straightforward as tensions at every vertex of a linearly generic framework are defined in the unique way up to a scalar. All stresses at this vertex are either all zero or all non-zero. □

Before to formulate a criterion of existence of a tensegrity we give the following definition.

Definition 24. Consider a linearly generic framework $G(P)$ and a cycle C in G (without self-intersections). Furthermore, consider a framed cycle $C_B = (C, B)$. We say that a framed cycle framework

$$C_B(\tilde{P}) = \big((r_1, \ldots, r_n), (e_1, \ldots, e_n)\big)$$

is *associated* to $G(P)$ if

- $r_i = p_i$ at all corresponding points of C and G;
- for the boundary points we have:

  $$e_i \in \operatorname{span}(p_i, p_{i-1}, p_{i+1}) \cap \operatorname{span}(p_i, p_{i,1}, \ldots, p_{i,k}),$$

  where $v_i v_{i,j}$ correspond to all edges adjacent to v_i except for the two edges $v_i v_{i-1}$ and $v_i v_{i+1}$, and $p_i p_{i,j}$ are their realizations in $G(P)$.
- In addition we require that $e_i \neq r_i$ for $i = 1, \ldots, n$.

Remark 25. Let us note that associated framed cycles are very specific tie-downs introduced in [29] by N.L. White and W. Whiteley. In the original construction of N.L. White and W. Whiteley there is no fixed rule to pick the directions of tie-downs (i.e., framings), whereas in our construction the directions of framings are determined by the framework.

Indeed, consider a linearly generic framework $G(P)$ and a cycle C in G. Note that the vertices of an associated framed cycle C_B are the vertices of G. Now the directions of edges $e_i r_i$ are uniquely determined by the framework. The only freedom e_i can still have is as follows: it can slide along the line

$$\operatorname{span}(p_i, p_{i-1}, p_{i+1}) \cap \operatorname{span}(p_i, p_{i,1}, \ldots, p_{i,k}).$$

Here the position of e_i on that line is not important as it does not change the force-loads on the cycle itself.

The criterion of existence of a tensegrity can be formulated in the following way.

Theorem 26. A linearly generic connected framework admits a non-zero tensegrity if and only if all its associated framed cycle frameworks admit a non-zero tensegrity.

Proof. Assume that a framework admits a non-zero tensegrity. Then the associated framed cycle frameworks admit a non-zero tensegrity directly by Proposition 23.

Let now all associated framed cycle frameworks of $G(P)$ admit a non-zero tensegrity. Let us iteratively construct a non-zero tensegrity for $G(P)$.

We start with any vertex of degree greater than 1 and set the stress on one of its edges to 1. Therefore, the stresses for the other edges are defined in the unique way.

Assume now that we have constructed the stresses for the edges adjacent to all vertices of $V' \subset V$. In addition we assume that every pair of vertices in V' is connected by a path in G within V'.

Let us now consider some edge $v'v$ such that $v' \in V'$ and $v \in V \setminus V'$. If $v \in B$ then there is no equilibrium condition on stresses, we just add v to V'. Let now v' be k-valent ($k > 1$) with edges $vv_1, \ldots, vv_k$ adjacent to v. Consider the following two cases for these edges.
Case 1: $v_i \notin V'$. Then the stress at vv_i is not yet defined. Hence we define it from the equilibrium condition for v.
Case 2: $v_i \in V'$. Then there exists an associated framed cycle framework $C_B(\tilde{P})$ whose non-boundary vertices all correspond to vertices in $V' \cup \{v\}$ and that passes through v and v_i via edge vv_i. First of all, it has a non-zero self-stress by the theorem assumption. Secondly, this self-stress is proportional to the stresses defined on the edges adjacent to V' (since all the vertices but one are in V', and the equilibrium conditions in V' are fulfilled simultaneously for the self-stress on the

cycle $C_B(\tilde{P})$ and the partially constructed stress). So the stress at vv_i defined from v_i before coincides with the stress at vv_i defined by the equilibrium in v.

Now we add v to V' and continue to the next vertex of $V \setminus V'$. Note that after adding v to V' all the vertices of the new V' are connected by edge paths of G via vertices of V'.

At each step of iteration we add a new vertex and define the stresses on the edges adjacent to it (if they were not defined before) such that the equilibrium condition is fulfilled.

Since G is connected, the process terminates and we have a tensegrity on $G(P)$ at the end of the process. □

4.3. **Join/intersection condition for the existence of tensegrities.** Finally, we have all tools to formulate geometric conditions for the existence of non-zero tensegrities for frameworks in 3D-general position in terms of join/intersection operations and conditions within projective geometry. Let us first briefly recall the notions of join and intersection.

4.3.1. *Join/intersection operations and relations.* Let us briefly recall the notions of join and intersection within projective geometry. First of all the elements of projective geometry on $\mathbb{R}P^d$ are all the k-planes of all possible dimensions $k \leq d$.

There are two operations in projective geometry that are called *join* and *intersection* operations and denoted by $\vee$ and $\wedge$, respectively.

We will use the "dimension-operator" dim with respect to *projective* dimension of projective subspaces and refer to the dimension of *linear* subspaces by $\dim_l$. Projective subspaces $\pi \subset \mathbb{R}P^d$ of dimension k are represented by $(k+1)$-dimensional subspaces $U \in \mathbb{R}^{d+1}$, consequently, $\dim(\pi) = \dim_l(U) - 1$.

Definition 27. Given projective subspaces $\pi_1, \ldots, \pi_n \subset \mathbb{R}P^d$ of arbitrary dimensions. The *join* and *intersection* operations for these subspaces are respectively as follows:

$$\pi_1 \vee \ldots \vee \pi_n := \operatorname{span}(U_1 \cup \ldots \cup U_n);$$
$$\pi_1 \wedge \ldots \wedge \pi_n := \bigcap_{i=1}^{n} \pi_i.$$

Remark 28. Note that there is another approach to our problem using bracket algebra and Grassmann-Cayley algebra. It is developed, e.g., in [9, 17, 25, 28, 29]. The expressions in bracket algebra are written as polynomials of brackets (i.e., minors of a matrix) while the translation of our approach to the bracket ring would result in systems of simple brackets equal to zero. We refer an interested reader to the above mentioned papers.

Finally, let us formulate relations on the subspaces of projective geometry.

Definition 29. Given projective subspaces $\pi_1, \ldots, \pi_n$. We say that

$$\pi_1 \wedge \ldots \wedge \pi_n = \text{true},$$

if there exist projective subspaces π_i' with $\dim \pi_i' = \dim \pi_i$ for $i = 1, \ldots, n$ such that

$$\dim(\pi_1 \wedge \ldots \wedge \pi_n) > \dim(\pi_1' \wedge \ldots \wedge \pi_n').$$

Otherwise we say that

$$\pi_1 \wedge \ldots \wedge \pi_n = \text{false}.$$

Here we consider the dimension of an empty set to be -1.

Example 30. Consider three projective lines ℓ_1, ℓ_2, ℓ_3 in a two-dimensional projective space. Then

$$\ell_1 \wedge \ell_2 \wedge \ell_3 = \text{true},$$

if and only if these three lines have projectively at least one point in common.

4.3.2. *Join/intersection conditions for framed cycles.* Let us first start with a join/intersection condition for a framed cycle on three vertices.

Definition 31. Let $C_B(P)$ be a framework of a framed cycle in 3D-general position

$$\big((r_1, r_2, r_3), (e_1, e_2, e_3)\big).$$

Then the *join/intersection condition* for C is

$$r_1e_1 \wedge r_2e_2 \wedge r_3e_3 = \text{true}.$$

Let us now expand the notion of join/intersection condition to framed cycle frameworks of arbitrary length.

Definition 32. Let C_B be a framed cycle of length $n \geq 3$, and let $C_B(P)$ be its framework in 3D-general position. Then the *join/intersection condition* for $C_B(P)$ is as follows

$$r_1e_1^{(n-3)} \wedge r_2e_2 \wedge r_3e_3^{(n-3)} = \text{true},$$

where $e_1^{(n-3)}$ is defined recursively by

$$\begin{aligned} e_1^{(0)} &= e_1; \\ e_1^{(k)} &= e_{n-k+1}e_1^{(k-1)} \wedge \big(r_{n-k} \vee r_1 \vee r_2\big), \end{aligned}$$

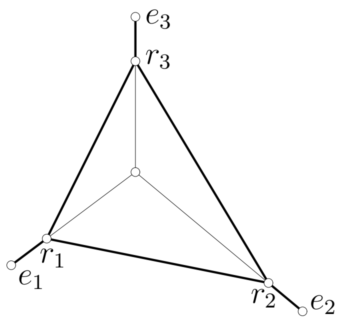


FIGURE 9. A framed cycle consisting of a triangle r_1, r_2, r_3 with external forces $w_i(e_i - r_i)$ is a tensegrity if and only if the three lines $r_i e_i$ meet in a point.

and $e_3^{(n-3)}$ is defined recursively by

$$\begin{aligned} e_3^{(0)} &= e_n; \\ e_3^{(k)} &= e_{n-k} e_3^{(k-1)} \wedge \big(r_{n-k-1} \vee r_{n-k} \vee r_1\big). \end{aligned}$$

Remark 33. For simplicity here and below we write uw instead of $u \vee v$.

Proposition 34. A framed cycle framework $C_B(P)$ in 3D-general flat position has a non-zero tensegrity if and only if $C_B(P)$ fulfills the join/intersection condition.

Proof. The condition is written by iteratively application of WU-surgeries to the last vertex of C, reducing C_B to a triangular framed cycle in general flat position. Namely the resulting flat cycle is

$$\mathsf{WU}_4(\ldots \mathsf{WU}_n(C_B(P))\ldots).$$

The existence of a non-zero tensegrity is equivalent to the existence of a non-zero tensegrity after WU-surgeries by Proposition 22.

So the statement of proposition is reduced to triangular cycles. The statement for a triangular cycle (which has to be planar) is classical (see e.g., [13]). $\square$

Let us write explicitly the join/intersection conditions for cycles on 3 and 4 vertices.

Example 35. If $n = 3$, then we have

$$r_1e_1 \wedge r_2e_2 \wedge r_3e_3 = \text{true}.$$

If $n = 4$, then we have (see Figure 10)

$$\big[r_1 \vee \big(e_4e_1 \wedge (r_3 \vee r_1 \vee r_2)\big)\big] \wedge r_2e_2 \wedge \big[r_3 \vee \big(e_3e_4 \wedge (r_2 \vee r_3 \vee r_1)\big)\big] = \text{true}.$$

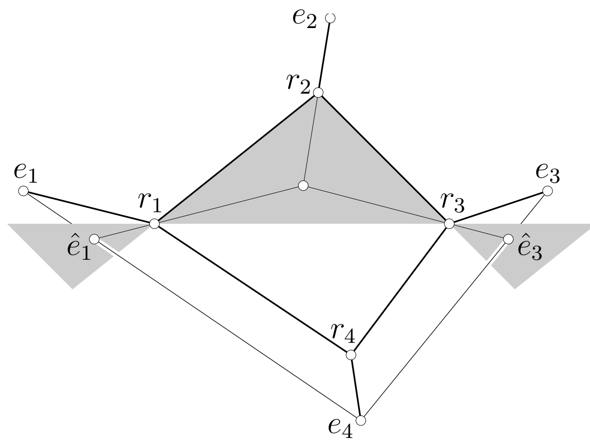


FIGURE 10. Illustration of Example 35 for $n = 4$.

4.3.3. *Join/intersection criteria for tensegrities in 3D-general position.* The following theorem and its proof is the recipe to write the join/intersection criteria for tensegrities in 3D-general position.

Theorem 36. The framework $G(P)$ in 3D-general position admits a non-trivial tensegrity, if and only if all the join/intersection conditions for all its associated framed cycle frameworks are fulfilled.

Proof. The join/intersection conditions for $G(P)$ are written according to Definition 32. Due to Theorem 26 and Proposition 34 they are equivalent to the existence of a non-zero tensegrity on $G(P)$.

It remains to add the following detail to the above construction. In order to generate the boundary $\tilde{e}_i$ of an associated framed cycle framework $C_B(\tilde{P})$, one should take the intersection of the span of two edges in the cycle passing through $\tilde{r}_i = r_i$ (namely $r_{i-1}r_i$ and r_ir_{i+1}) and the span of all other edges adjacent to r_i, say $r_ir_{i,1}, \ldots, r_ir_{i,k}$. Let us denote the resulting line by ℓ. In terms of join/intersection operators ℓ is written as

$$\ell = \big(r_i \vee r_{i,1} \vee \ldots \vee r_{i,k}\big) \wedge \big(r_{i-1} \vee r_i \vee r_{i+1}\big).$$

Finally, we pick up a point $G(P)(b_i)$ on ℓ distinct to r_i. For instance, set

$$\tilde{e}_i = \ell \wedge \big(r_{i,1} \vee \ldots \vee r_{i,k}\big).$$

□

Remark 37. In analogy to Remark 12 it is sufficient to check the criterion only for generator loops of the first homology group $H_1(G)$ of the graph (if we consider the graph as topological space), because all other loops can be decomposed by those. These conditions for different generators of $H_1(G)$ may still coincide. Some of the conditions will correspond to different strata.

5. Tensegrities and discrete harmonic maps

In this section we relate tensegrities to the notion of discrete harmonic functions and demonstrate an alternative way to obtain tensegrities with just positive tensions.

The discrete Laplace operator (the graph Laplacian) acts on maps $f : G \to \mathbb{R}^d$ defined on arbitrary graphs G by with real valued weights $w_{i,j} \in \mathbb{R}$

$$(\Delta f)(v_i) := \sum_{v_j \sim v_i} w_{i,j}(f(v_i) - f(v_j)),$$

where we sum over neighboring vertices v_j of v_i. This *discrete Laplace operator* has been used in several applications of geometry processing as well as in discrete complex analysis and discrete minimal surface theory (see e.g., [2, 3]). The weights $w_{i,j} \in \mathbb{R}$ are chosen depending on the application. Prominent examples are the cotangent-weights or the area of Voronoi cells around the vertex v_i. Furthermore, the choice of the weights implies which properties of the discrete Laplace operator "inherits" from its smooth counterpart [26].

Definition 38. A function $f : G \to \mathbb{R}^d$ is called *discrete harmonic* if

$$(\Delta f)(v_i) = 0$$

for all vertices $v_i \in Z(G)$.

A real valued discrete harmonic function over some rectangular subgrid of the $\mathbb{Z}^2$ lattice is illustrated by Figure 11.

In the setting of tensegrities the function f describes the coordinates of the position of the vertices in space and the weight assignment $w_{i,j}$ represents the stress at each edge $(v_i; v_j)$. Consequently, we will allow positive and negative weights for tensile and compression forces. Therefore, it follows from the definition of tensegrities (Definition 1) that they can be seen as zeroes of the discrete Laplace operator for maps defined on the vertices of a graph. In this sense tensegrities are harmonic maps with respect to the discrete Laplace operator.

Proposition 39. Tensegrities with arbitrary combinatorics and with only positive tensions $w_{i,j} > 0$ can be obtained as the minimum of the discrete Dirichlet energy

$$\sum_{v_i \in G} \sum_{v_j \sim v_i} w_{i,j} \|p_i - p_j\|^2, \tag{6}$$

viewing the coordinates of the vertices p_i as variables.

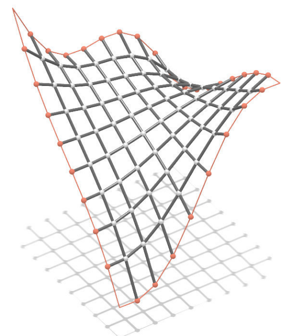


FIGURE 11. Illustration of a discrete harmonic real valued function as solution of a Dirichlet boundary value problem. The discrete harmonic function f is defined over a rectangular patch U of the $\mathbb{Z}^2$ lattice $f : \mathbb{Z}^2 \supset U \to \mathbb{R}$ with weights $w = 1$ at each edge. The values of f can be computed by minimizing an energy (cf. Prop. 39).

This proposition holds since the energy is bounded below and critical points are solutions of the linear system

$$\sum_{v_j \sim v_i} w_{i,j}(p_i - p_j) = 0, \tag{7}$$

for all i. So if the number of vertices is big enough and the boundary vertices are fixed, the minimum is unique. Therefore, a tensegrity with just positive tensions can be interpreted as critical point of an energy.

Example 40. Let us consider a rectangular patch U and let us further fix the values on the boundary of U. We are looking for a harmonic function $f : U \subset \mathbb{Z}^2 \to \mathbb{R}$ with respect to the discrete Laplacian with constant positive weights that solves this Dirichlet problem. According to Proposition 39 we find the solution by minimizing

$$\sum_{v_i \in U} \sum_{v_j \sim v_i} w_{i,j} \|f_i - f_j\|^2,$$

where f_i are considered as variables of this energy function. The values f_i which belong to the minimum are the values of the harmonic function f at v_i. We illustrate the graph $(v_i, f_i) \in \mathbb{R}^2 \times \mathbb{R}$ of a harmonic function f in Figure 11.

Example 41. For the combinatorics of any cell decomposition of a disc we obtain a tensegrity with everywhere unit tensions by fixing the positions of the boundary vertices and minimizing the quadratic energy in Equation (6). The tensegrity is the solution to the linear system (7). An illustration of such a cell decomposition can be found in Figure 12 (left). To check whether this framework is a tensegrity with the machinery provided by Section 3 or Section 4 requires to check the "ratio" condition or the "join/intersection" condition for a set of cycles that generates the first homology group $H_1(G)$.

Example 42. Figure 12 (right) illustrates a twisted strip represented by a net with regular quadrilateral combinatorics which is attached to two interlinked circles. The topology or the combinatorics of the graph does not play any role in the analysis of a framework whether it is a tensegrity. Neither, the "global" topology nor combinatorics of the framework is of importance in the Equations (6) and (7), just the local combinatorics of the vertex stars.

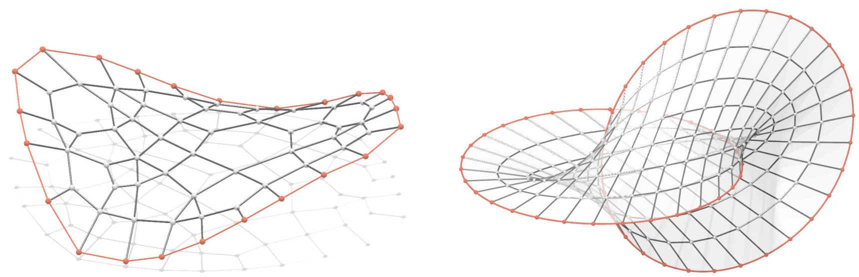


FIGURE 12. *Left*: A tensegrity with the combinatorics of an arbitrary cell decomposition of a disc (cf. Example 41). *Right*: Two circles are the boundaries of a tensegrity with regular quadrilateral combinatorics (cf. Example 42).

6. EXAMPLES

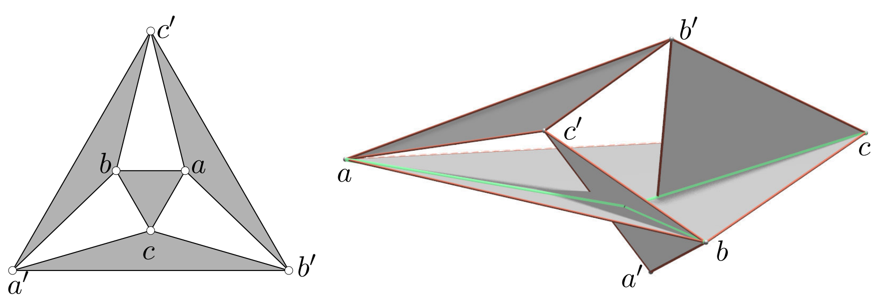


FIGURE 13. *Left*: The combinatorics of an octahedron. *Right*: An octahedron in $\mathbb{R}^3$. Its edges form a tensegrity if and only if any four alternate face planes, i.e., four 2-planes of the configurational type abc, $ab'c'$, $a'bc'$, $a'b'c$, are concurrent in a point. The intersection lines of the first 2-plane with the three latter 2-planes are illustrated by the green lines. See also Proposition 43.

We conclude the paper with a brief description of two non-trivial three-dimensional examples. In fact however, our methods can be applied to all linearly generic graphs (Def. 3), or in 3D-general position (Def. 5), respectively. For graphs where the first homology group $H_1(G)$

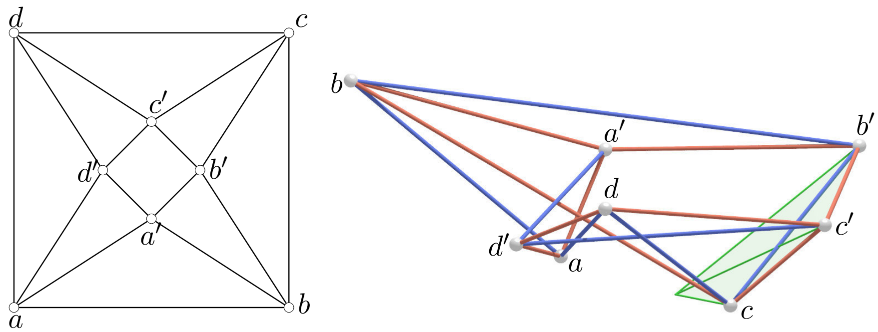


FIGURE 14. *Left*: The combinatorics of a four-sided antiprism. *Right*: A tensegrity in $\mathbb{R}^3$ with the combinatorics of a four-sided antiprism. The rods (= edges with positive weights) are red, the cables (= edges with negative weights) are blue. For the framework to be a tensegrity we have to check cycles with three and four vertices (see Example 45). For example the condition on the cycle $cc'b'$ is that the 2-planes $cc'b'$, bcd, $a'bb'$, $c'dd'$, must be concurrent. The intersection lines of the first 2-plane with the three latter 2-planes are illustrated by the green lines.

can be generated from cycles of length three or four, the conditions to check are written out in Example 35 explicitly.

Proposition 43. An octahedral framework (a, b, c, a', b', c') in $\mathbb{R}^3$ is a tensegrity if and only if four *alternate*, i.e., face planes in a combinatorial configuration like abc, $ab'c'$, $a'bc'$, $a'b'c$, are concurrent in a point (see Figure 13 and cf. [30]).

Let us give two new proofs of this classical statement in terms of multiplicative 1-forms and in terms of join/intersection operations.

Proof 1 (via multiplicative 1*-forms).* Let us consider the cycle with three vertices (a, b, c). The necessary condition for that cycle to be part of a tensegrity is that the product of the three values of the 1-form multiply to 1 which is equivalent to Ceva's theorem (see Section 3.2 for $n = 3$). Consequently, the three lines

$$\begin{aligned}&\operatorname{span}(a, b', c') \cap \operatorname{span}(a, b, c),\\&\operatorname{span}(a', b, c') \cap \operatorname{span}(a, b, c),\\&\operatorname{span}(a', b', c) \cap \operatorname{span}(a, b, c),\end{aligned}$$

must intersect in one point and therefore all four 2-planes intersect in one point. □

Proof 2 (within join/intersection relations). Let

$$\begin{aligned}\ell_1 &= ab'c' \wedge abc;\\ \ell_2 &= a'bc' \wedge abc;\\ \ell_3 &= a'b'c \wedge abc.\end{aligned}$$

Our condition for a triangle abc is (cf. 35 for $n = 3$)

$$\ell_1 \wedge \ell_2 \wedge \ell_3 = \text{true}.$$

This is to say that $b'c'a$, $c'a'b$, $a'b'c$, and abc indeed meet in a point. □

Remark 44. As one can notice, one can apply proofs 1 and 2 of Proposition 43 to any other triangle in the octahedron. In fact all these conditions would be equivalent.

Example 45. Let us consider a graph with the combinatorics of a four-sided antiprism (Figure 14 left). To determine whether a framework with such combinatorics is a tensegrity involves checking cycles with three and four vertices. Cycles with three vertices have been considered also in the previous example (Proposition 43). The configuration for the cycles with four vertices is written down explicitly in Example 35. An illustration of a tensegrity with the combinatorics of a four-sided antiprism is depicted in Figure 14 (right). The rods (= edges with positive weights) are depicted in red, the cables (= edges with negative weights) are depicted in blue.

Example 46. Let us decompose a two-dimensional domain D homeomorphic to a disk into k cells that are either triangles or quadrilateral. Let G be the graph corresponding to the 1-skeleton of this decomposition. Then

$$H_1(G) = \mathbb{Z}_k.$$

Finally we consider frameworks in $\mathbb{R}^3$ representing G (in 3D-general flat position). One can pick all triangular and quadrilateral cycles of G corresponding to all triangles and quadrilaterals in the decomposition of D. All the conditions for these cycles will be of two types described in Example 35. (Here one should substitute suitable vertices of the cycles to the corresponding expressions for r_i and e_i. Recall that the vertices r_i are the corresponding vertices of the graph, and the vertices e_i are defined from Definition 24.)

Example 47. Let us consider the example of the quadrilateral graph $G(m, n)$ on the torus with sides m and n (cf. Figure 15). We have:

$$H_1(G(m, n)) = \mathbb{Z}_{mn+1}.$$

One can pick $mn - 1$ quadrilateral cycles, one longitude cycle, and one latitude cycle. All conditions of the quadrilateral cycles will be of the second type described in Example 35. The conditions for longitude and latitude cycles will be similar to the ones described in Example 35 but longer (they are constructed by the iterations of Definition 32).

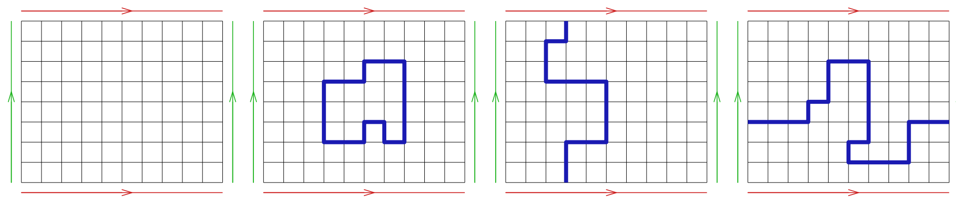


FIGURE 15. Illustration of a torus. *Left*: A torus is homeomorphic to a rectangle with opposite sides identified as illustrated. We are considering a framework with the combinatorics of the vertices of a rectangular sub-patch of the $\mathbb{Z}^2$ lattice, with edges connecting neighbouring vertices, and with opposite vertices of the rectangle glued together like a torus. *Second to third*: Illustrations of the three different types of cycles on that torus.

Remark 48. In the examples of Figure 12: for the left one it is enough to write triangular, quadrilateral, pentagonal, and hexagonal conditions; the right picture is very similar to the torus. Here one can pick mostly quadrilateral cycles, one longitude cycle and one latitude cycle. More generally, this technique is applicable for all graphs with linearly generic frameworks discussed in this paper (see Definition 3).

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