# Insights into Instability of Friction-induced Vibration

# of Multi-Degree-of-Freedom Models

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## Abstract

This study addresses a fundamental problem of linear friction-induced vibration (FiV), the intriguing role of damping on the instability of FiV in the form of mode-coupling. Two well-known classic 2-degree-of-freedom (DoF) slider-belt models are revisited, and a 4-DoF slider-belt model considering different damping configurations in the slider and the belt is investigated. Since proportional damping is very rare in reality, non-proportional damping is given much attention in this paper.

Firstly, the analytical solutions of bifurcation boundary of the critical coefficients of friction of the classic 2-DoF models are derived, which show the subtle role of damping clearly and theoretically explain and clear misunderstandings and confusions on the phenomenon about the drop of the stability boundary from that of the undamped system caused by adding non-proportional damping. Secondly, a further increase of non-proportional damping is found to improve the stability and even bring higher degree of stability than increasing proportional damping (which is a new finding). Thirdly, the influences of the different damping distributions among the system components on the stability of the 4-DoF slider-belt model are explored. The evolution of the bifurcation boundary indicates the importance of the ratio of the horizontal to the vertical damping in the slider and in the belt. For some non-proportional damping values, the bifurcation boundary fluctuates, which could exhibit local optimal values of the ratio of damping in different directions of a system component and can be exploited in design against FiV. These findings are valid for general linear second-order dynamic systems with an asymmetric stiffness matrix.

Key words：Friction-induced vibration; Stability analysis; Mode coupling; Damping.

## Introduction

Friction-induced vibration (FiV) occurs in various scientific fields, engineering applications as well as in daily life. These include music from string instruments, vibration of robot joints, and various well-known automobile brake noise. In many circumstances, FiV results in annoying noise, wear and fatigue. In the last decades, much attention has been given to address FiV and identify the effects of physical parameters on its instability [1, 2]. Recently, FiV is exploited in energy harvesting [3] and tactile perception of robots [4]. A variety of investigations have been conducted to shed light on the essential mechanisms for the instability of FiV [5]. The examined mechanisms fall into four main categories: the negative slope of the friction-velocity relation [6], stick-slip [7], mode coupling [8], and sprag-slip [9].

Two essential ways of inducing the instability of a finite-DoF frictional system are found to be: (1) the characteristics of the law of friction; (2) the values of the mass and stiffness of the system and their geometric arrangement. A negative friction-velocity slope results in negative damping to cause instability [10]. Chen and Zhou [11] found that unstable vibration could even happen during a positive slope stage of a friction law. Stick-slip occurs when the friction at the interfaces follows particular friction laws with the different coefficients of static and kinetic friction. They could be the Coulomb’s law of friction [12], the friction law with a decreasing friction-velocity relationship [13], and even the friction law in which the coefficient of kinetic friction is greater than the static one [14]. Popp and Stelter [15] found that stick-slip could lead to rich dynamic behaviour in a simple linear slider-belt model, including periodic, quasi-periodic and even chaotic vibration. Slip-stick is always observed at a low relative moving speed, and the speed could be constant [16], decelerating [17] or accelerating [18].

Among the study on the ‘geometry-induced’ vibration, sprag-slip motion was firstly demonstrated by Spurr [19] though an elastic inclined beam on a moving belt in a theoretical study. Another research line, parallel to sprag-slip, is about the existence and uniqueness of the solutions of FiV in frictional systems [20, 21]. Mode-coupling instability is an exemplary ‘geometry-induced’ vibration mechanism, which has been generally acknowledged as the primary mechanism for squeal in brakes [22, 23] in recent years. Hoffmann et al. [24] proposed a minimal slider-belt model which clearly showed the bifurcation of its complex eigenvalues. Hultén [25] proposed a new kind of mode-coupling instability which is induced by the two frictional interfaces of in a slider-belt models in the two orthogonal directions rather than the inclined spring in ref. [24]. Chomette and Sinou [26] implemented active control on an extended 4-DoF model from Hultén’s model, and found that the periodic as well as the quasi-periodic vibration due to mode-coupling instability could be sufficiently reduced through linear active control. Chen et al. [27] also suggested that the coupling of the in-plane and out-of-plane modes of the brake was the main reason for car disc brake squeal above 3 kHz. Sahoo and Chatterjee [28] pointed out that horizontal high-frequency excitation could be used to stabilise mode-coupling instability of a slider-belt model and the vertical high-frequency excitation was useful in the suppression of the FiV amplitude.

Various theoretical models, such as the slider-belt model [29], the wobbling disc model [30, 31] and the pad-disc model [32], were developed for studying FiV problems. Hochlenert et al. [33] conducted a dynamic stability analysis of continuous systems. In [34], the wave pattern and the limit cycle of the stick-slip motion of a simplified rotating disc brake system, which included a disc in frictional contact with a pad under uniform pressure, were analysed. Liu and Ouyang [18] found that the deceleration of a rotating disc made the horizontal stick-slip motion of the slider more complicated, accompanied by a significant change of the dynamic behaviour of the vertical vibration of the disc, and acceleration of the disc tended to encourage separation between the slider and the disc. Wei et al. [35] proposed a double-layer pad and disc model and studied the contribution of the friction-layer pad mass, constrained stiffness and the load pressure. They found that large friction-layer pad mass and constrained stiffness would reduce the unstable vibration zone.

Damping which usually plays a stabilising role shows more complex characteristics in friction-induced vibration [36-38]. There are many works that explored the dynamics of frictional systems with damping [35, 39], but damping effects are not the focus or are not even the subject of study in most of these investigations. Damping effects of different components of a system have been investigated [40, 41]. Chevallier et al. [42] revealed that adding viscoelastic damping could stabilise some of the unstable modes, but resulted in new mode-coupling between other modes. Kang [43] introduced a damping shim in his FE model of a disc brake and found that the in-plane torsion mode could easily lead to brake squeal, and the propensity of this kind of squeal could not be suppressed by the damping shim. Fritz et al. [44] studied the effects of proportional modal damping on the unstable modes of an FE model of a disc brake and found that stability boundary would vary with equal damping and unequal damping cases. The unequal damping distribution caused both shifting and smoothing effects of coalescence curves of the two unstable modes, and thus the drop of the critical coefficient of friction was observed. Additionally, they showed that damping in one mode could influence other modes, but the damping effects were far from being comprehensively understood [45]. In the numerical and experiential study of the onset of brake squeal, Massi and Giannini [46] conducted an experimental study of the damping effect on their beam-disc test rig. Cantone and Massi [47] built an FE model of the beam-disc test rig. They found that a homogenous distribution of damping in the beam and the disc played a stabilising role; in contrast, a non-uniform configuration of damping could increase the squeal propensity. Recently, damping in brake joints attracted attention, which was found to play an important role in the dynamics of brake systems and could contribute more than 60% of the component damping [48]. Tiedemann et. al [48] pointed out that joint damping would largely dissipate energy and could be a decisive factor for brake squeal problems.

On the other hand, theoretical research on damping effects has had big advancement. Hoffmann and Gaul [49] found an unusual phenomenon of damping effects on mode-coupling of a 2-DoF model. Shin et al. [50] reported that either increasing the damping of the ‘pad’ or the ‘disc’ potentially reduced the stability of the system. Li et al. [51] considered the LuGre model in the study of a self-excited 3-DoF FiV model. The viscous damping coefficient played an essential role in the tangential and torsional dynamics of the system and could lead to chaotic vibration when damping was very large. Sinou and Jézéquel [52] reported the effects of damping and damping ratio of horizontal to vertical damping of the slider on the stable and unstable zones of a nonlinear slider-belt model with mode-coupling instability.

The understanding of damping effects on FiV problems is [inadequate](javascript:;). It is important to explore instability concerning damping through further theoretical analysis. The first motivation of the current work is inspired by the new results in the revisit to the damping effects on mode-coupling in two well-known FiV model. A further motivation of the current work is to provide a deep understanding on the damping distribution in the friction pair of a frictional system to improve the stability of the system. Thus, in the present paper, firstly a classic 2-DoF slider-belt model with an inclined spring [49] is used to derive analytical expressions of the bifurcation boundary and thus gain insights into its stability behaviour. The true role of damping is clearly identified, particularly the influences from horizontal damping versus vertical damping are investigated in terms of the damping index. Then, the damping effects on mode-coupling instability of another classic 2-DoF slider-belt model are revisited. Finally, a 4-DoF slider-belt system considering damping in both the slider and the belt is proposed. A parametric analysis shows the effects of proportional and non-proportional damping configurations. The changes of the bifurcation boundary, and the zones of mode-coupling instability with various system parameters, such as damping and the damping index of the slider and the belt, are shown, which give clear pictures for avoiding instability and improving stability of FiV in low-DoF frictional systems.

## FiV Model I: a 2-DoF model with an inclined spring

The main objective of the present work is to re-examine the damping effects on the well-known mode-coupling instability of FiV. The essential mechanism of mode-coupling instability can be revealed by a minimal 2-DoF model, which is a variant of the classic models [24, 49] proposed and studied by Hoffmann and his colleagues as shown in Figure 1, which is named as Model Ⅰ. The reason for building and analysing a model different from those studied in [24, 49] is that interesting and very different dynamic behaviour may be discovered.



Figure 1. 2-DoF model with an inclined spring

Model Ⅰ consists of a mass-spring-damper part as a slider and a rigid moving belt. The slider is held by a spring *k*1 and a damper *c*1 in the horizontal direction, a contact spring *k*2 and a ground damper *c*2 in the vertical direction and an inclined spring *k*3. The slider and the belt are assumed to be in one-point frictional contact. It is also assumed that no stick occurs, and Coulomb’s law of friction applies with a constant coefficient of kinetic friction *μ*. The friction force *F*T thus is proportional to the normal contact force, which is -*k*2*y*, so *F*T is expressed as -*μk*2*y.* It is further assumed that there is no loss of contact between the belt and the slider, as commonly done in similar classic FiV models. If stick-slip oscillation and/or separation/reattachment is allowed to happen, the FiV model would be a nonlinear non-smooth system and a time-domain analysis is necessary, which is beyond the scope of this paper. Interested readers may read [18, 53, 54].

The equation of motion of the 2-DoF model is expressed as:



in which , ,

where   (), .

### The stability criterion

In this section, the stability of the system is investigated by solving the eigenvalue problem to show the essential correlation of the system parameters to the system stability.

#### Damped system

As the equation of motion of the resulting system (Eq. ) is a second-order homogenous differential equation, the general solution can be expressed as:



in which  and  are the real and imaginary part of the eigenvalue.

By substituting Eq. (2) into Eq. (1), a fourth-degree characteristic polynomial can be obtained:



First of all, the definition of stabiltiy must be reviewed. In linear control theory, (1) if the real part () of any eigenvalue is positive, the system is unstable; (2) if the real part of all the two pairs of eigenvalues is negative, the system is stable; (3) if a real part is zero and all the other real parts are negative, the system is said to be critically (or neutrally, or margianlly) stable. However, this state of critical stability in the sense of control theory often corresponds to a range of system parameter values, particularly the coefficient of friction. It must be state here that the critical coefficient of friction, denoted as **c, is a particualr value of the coefficient of friction at which the system is critically stable but beyond which the system in unstable. Therefore, the stability at the critical coefficient of friction is NOT equivallent to critical stability in the sense of linear control theory, but a narrow subset of the latter. It is a bifurcation point of the eigenvalues of the undamped system. Because of the special charateristic of **c, one has to be very cautious in the stability analysis of FiV, particularly when using Routh-Hurwitz criterion.

By applying Routh-Hurtiwiz criterion to the characteristic equation, the analytical formula of **c can be derived as:



By introducing the ratio of the horizontal damping over the vertical damping, the relation between andis. The analytical formula ofcan be rewritten as:



where(referred to as the damping index) and, which will be used to represent damping characteristics of the frictional systems under study in this paper.

Now, it is clear thatis a function of the basic parameters of the undamped system, damping *D* and damping index.

On the other hand, the zeroth-order term in the characteristic equation must be also checked for stability. Fortunately for Model I, it turns out that for **> 0, the zeroth-order term is always positive. Thus, there is only one bifurcation point obtained from Routh-Hurwitz criterion.

#### Undamped system

Similarly, for the corresponding undamped system, the eigenvalues of the system can be derived:

 

When, *s* are purely imaginary. When, *s* are in two pairs of complex conjugates, and the real part of one pair of *s* is positive. Thus, the condition for bifurcation is:



and  of the undamped system is obtained:



When ** of the system is smaller than **c, the system is stable; on the other hand, when **>**c, the system would be unstable. Furthermore, the value of **c can be recognised as an indicator to estimate the stability level of a system. Systems with larger **c are regarded as more stable than systems with smaller **c.

### Damping effects on stability

As the analytical formula of **c has been obtained, the effects of system parameters on the stability can be analysed. In this work, the effects of damping are focused. Two general damping cases are discussed.

(1) For proportional damping case with(), the expression of **c (Eq.) is rewritten as:



It can be noticed that **c consists of two parts: the first part is independent of damping and the second part contains damping ratio term  squared, which means that with the increase of damping, **c will increase. According to Eq., when, the analytical solution of , which is identical to  of the undamped systems given in Eq. . This is important as it predicts that whether the original system is undamped or damped, adding proportional damping to the system would monotonically enhance stability. The reason to emphasise this phenomenon would be known after comparing with the results of the non-proportional damping case in the following.

(2) For the non-proportional damping case, and the relation between the damping ratios would be(), the expression ofis the same as Eq. . In this case, when,

the critical coefficient of friction,

which is different fromof the corresponding undamped system given in Eq. . More importantly it is smaller than. Thus, the effect of non-proportional damping on the stability of a system is complex, which depends on not only the damping value but also other system parameters. This finding is distinct from that of a similar, well-known 2-DoF FiV model [49].

Furthermore, the contribution of the damping indexand its combined effect with *D* on the stability of the system are studied. Figure 2 illustrates the results of three kinds of systems: the undamped system, the system with damping in only one direction and the system with damping in both directions. The base parameters are given by , , . Firstly, as shown by the line with star markers and the line with triangle markers, one can see: (1) when the system is damped only in the horizontal or the vertical direction, the system’s stability actually would not be influenced by the damping; (2) no matter which direction damping is added to the system, the system’s stability stays the same. This can also be explained theoretically: whether *D*1 or *D*2 is set to be zero in the analytical formula of **c (Eq. ), one will get identical **c which is a constant value ; (3) adding damping in only one direction produces smaller **c than that of the undamped system and that of the system with damping in both directions. Thus, adding damping in only one direction to the system should be avoided in consideration of improving the stability of the frictional system with mode-coupling.

In addition, the results of the proportionally and non-proportionally damped system are discussed. As shown in Figure 2, non-proportional damping is not as useful as proportional damping with the first increase of *D* when *D* is not large, as **c of the non-proportionally damped systems is smaller than that of the proportionally damped system (). However, when *D* is large enough, some non-proportional damping cases could gain better stability, likeor; while for some other cases withor, the proportionally damped system is more stable all along. Additionally, a smaller damping indexproduces a sharper increase of **c with *D*. Figure 3 shows the results of the effect ofon **c. When<, the effects ofon stability depend on damping value: (1) in the large- *D* cases (*D* =0.06 or *D* =0.08)，**c decreases all the way down with the increase of; (2) For case *D*=0.04, **c firstly decreases, then increases to a local peak at=, and then declines; (3) For case *D*=0.02, **c firstly increases to a peak value at=and then declines. However, when >,uniformly plays a destabilising role regardless of the damping value.

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| Figure 2. The critical coefficient of friction **c changes with damping *D* (Model Ⅰ) | Figure 3. The critical coefficient of friction **c changes with the damping index(Model Ⅰ) |

As the effect of damping *D* and damping indexon the stability is quite intricate, more comprehensive results about the variation of **c with *D* andare illustrated in Figure 4.  is shown to be a curved surface with respect to *D* and. Now it is clear that **c of the proportionally damped system, shown by the black line with triangle markers, appears like a ridge which connects with **c of the undamped system, shown in by blue line with hollow circle markers. The effect of *D* on stability does not change with, but the effect ofon stability depends on the value of *D*. Additionally, the red solid circles denote the maximum value of **c at each combination of *D* and, which shows that the optimal value of is slightly smaller thanof the proportional-damping case at smaller *D*. However, this would not change the fact that whenof a system is larger than, the system will be less stable than the proportionally damped system no matter how much damping is added to the system.

Furthermore, Figures 5 and 6 show the results of systems with another two sets of the basic parameter values: one is with *ω*1=8 and *ω*2=5, and the other is with *ω*1=4 and *ω*2=8. Apparently, the values of *ω*1 and *ω*2 do not affect the behaviour of the combined effect of *D* andon the stability. Systems with different *ω*1 and *ω*2 have distinct ridge positions of the stability boundary, as their corresponding damping index () are different. Thus, it could be deduced that the above findings of damping effects on mode-coupling are not restricted to any specific values of system parameters.



Figure 4. The combined effect of *D* andon **c for bifurcation

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| Figure 5. The combined effect of *D* andon **c for bifurcation (*ω*1=8 and *ω*2=5) | Figure 6. The combined effect of *D* andon **c for bifurcation (*ω*1=4 and *ω*2=8) |

Consequently, damping plays a non-intuitive and complex role in the stability of the system with mode-coupling. Firstly, adding damping in one direction makes the system less stable than the undamped system and adding damping in two directions enhances system stability. The analytical solution of **c reveals that the bifurcation point of the undamped system appears as an isolated point. Due to the drop of **c at *D*=0, systems with non-proportional damping could be less stable than the undamped system at the initial increase of damping. Proportional and non-proportional damping both are found to be a positive way to increase the stability of the damped system, and through a proper design, some non-proportional damping could bring higher stability than proportional damping. The stability of the non-proportionally damped system has a complex relationship with the damping and the damping index. Systems with non-proportional damping are less stable than those with proportional damping in most cases. However, when the modal damping index is less than the damping index of the proportionally damped system, one may design the most satisfying stability of the system by increasing *D*.

Furthermore, another classic linear slider-belt model (referred to as Model II in this paper) firstly proposed by Hultén [25] and shown in Figure 7 is also investigated. In Model II, a mass is in contact with a right-angled moving belt at two points. The horizontal and the vertical ground damping are introduced into it by the present authors. Compared with Model Ⅰ, the asymmetrical stiffness terms of Model II come from the two [orthogonal](javascript:;) friction forces rather than the included spring. By applying the same analysis procedure of Model Ⅰ to Model II, the analytical stability boundaries of *μ*c of the proportional and non-proportional damped systems are obtained. Again, interesting behaviour is discovered. Firstly, the new findings on proportional damping effects and most non-proportional damping effects of Model Ⅰ remain valid for Model II. This confirms that the new findings of the damping effect on the stability are not a particular phenomenon that is restricted to a specific model, but rather a universal phenomenon in mode-coupling instability. Furthermore, a particularly non-proportional damping case is when the system has damping in only one direction. Such a kind of systems are naturally unstable at any friction level. Thus, adding damping in only one direction in Model II, regardless of its magnitude, is counter-productive. This is another counter-intuitive finding.



Figure 7. 2-DoF model with two sliding interfaces

## FiV Model Ⅲ: a 4-DoF model with damping in both the slider and the belt

### Mechanical model

To gain further insights into the effects of damping on FiV, a 4-DoF slider-belt model is proposed, shown in Figure 8. Unlike Models I and II that involves a non-oscillatory belt, this new model allows belt vibration and thus include the dynamics of the belt. Two modelling aspects are worth mentioning here: (1) damping is introduced in the slider as well as in the belt, and at the same time the model is kept simple which could explicitly show the damping effects; (2) the model is an idealised mathematical model for demonstrating the instability mechanism rather than a geometrically faithful model of a real frictional system. The main idea of the model is that in addition to the features of common low-DoF classic FiV model in which a slider *m*1 is held against a moving belt *m*2 and is also constrained by a horizontal, vertical, and inclined spring (denoted respectively by *k*1, *k*2 and *k*5), vibration of the belt is introduced, which is constrained by a horizontal spring *k*3 and a vertical spring *k*4. Similar FiV models for time-domain nonlinear analysis [55] were studied before.

In this model, damping in both the slider and the belt are considered, as shown in Figure 8. A preload *F* is applied to bring the slider into contact with the belt before FiV starts. The normal contact force is . The same assumptions of no stick and no loss of contact are also made, and the friction is assumed to follow Coulomb’s law of friction with constant **. Thus, the friction force can be described as .



Figure 8. 4-DoF model

The equation of motion of the model therefore is written as:



in which

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is an asymmetry matrix due to the tangential friction force, which could induce instability of the system. By dividing the mass term *mi* (*i*=1, 2) on both sides of each equation in Eq. , more convenient relations using relative damping ratio are given in:



where,,,,,, ,.

### Stability analysis and results

The stability of the static-steady solution is investigated by calculating the eigenvalues *s* of the characteristic equation of the system, given in Eq. .



As Model Ⅲ is more complicated than Model Ⅰ and Model Ⅱ in Section 2, the relations of the bifurcation boundary with system parameters cannot be derived and given in an analytical expression. Thus, a numerical method is used to calculate the eigenvalues of the system. The basic parameters used are: *m*1=5, *m*2=15,,,,,,, and the following results are obtained with, but the essential conclusions for the damping effects with other relations betweenand, and other values of the basic parameters are consistent. Thus, typical results at selected parameter values are shown for the concise of the paper.

#### Damping effects on the evolution of the eigenvalues

Figure 9 shows the change of the eigenvalues of the system with proportional damping for different damping values. Taking parameter ** as the control parameter, two modes of the system would become unstable due to mode-coupling. The typical phenomenon of mode-coupling that two different modes merge together at **c is shown. Besides, it can be seen that adding damping lowers the growth rate of the system, which tends to make the system more stable.

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| Figure 9. The real (left) and imaginary part (right) of the eigenvalues versus the coefficient of friction *μ* for various *D*. (the proportional damping case) | |

Damping effects on the evolution of the eigenvalue of three kinds of non-proportional damping cases with ** are studied. Scenario 1 represents the case that only the belt of the system is damped; Scenario 2 represents the case that only the slider is damped; and Scenario 3 represents the case that both the belt and the slider are damped. Figures 10 and 11 illustrate the results of Scenarios 1, 2, 3 with two different damping and damping index. For the non-proportional damping cases, the imperfect merging phenomena are shown. By comparing the growth rate at the bifurcation point of Scenarios 1 and 2, one can find that Scenario 2 (damping in the belt) appears to be less stable than Scenario 1 (damping in the slider). The growth rate of Scenario 1 is close to but slightly higher than that of the fully damped system (Scenario 3), which is an expected result that a fully damped system should be more stable. On the other hand, when the critical coefficient of friction is examined, **c of the non-proportional damping case could be smaller or larger that **c of the corresponding undamped system, which crudely depends on *D* and damping index of the slider() and damping index of the belt().

#### Proportional and non-proportional damping effects of completely damped systems

Now the effects of different damping combinations of the slider and the belt are examined. Figures 12 and 13 illustrate the effects of the damping index of the sliderand damping *D*. Two expected results are that: (1) the bifurcation point is shown to increase with damping *D* smoothly from *D*=0 onwards in the proportional damping case as shown in Figure 13 (a) and (b); (2) the discontinuous drop of **c at *D*=0 from the undamped system to the non-proportionally damped system can be seen.

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| Figure 10. The real (left) and imaginary part (right) of the eigenvalues versus the coefficient of friction *μ* for different non-proportional cases. (scenario 1:,; scenario 2:,; scenario 3:,,,) | |
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| Figure 11. The real (right) and imaginary part (left) of the eigenvalues versus the coefficient of friction *μ* for different non-proportional cases. (scenario 1:,; scenario 2:,; scenario 3:,,,) | |

Now the effects of different damping combinations of the slider and the belt are examined. Figures 12 and 13 illustrate the effects of the damping index of the sliderand damping *D*. Two expected results are that: (1) the bifurcation point is shown to increase with damping *D* smoothly from *D*=0 onwards in the proportional damping case as shown in Figure 13 (a) and (b); (2) the discontinuous drop of **c at *D*=0 from the undamped system to the non-proportionally damped system can be seen.



1. vsand *D*

 

(b) vs *D* (c) vs

Figure 12. Evolution of critical coefficient of friction  vs damping *D* and the slider’s damping index ()

Besides, several observations can be made in Figures 12 and 13. The bifurcation boundary in terms of **c is an irregular surface influenced by bothand *D*. The outline of the boundary surface does not change with the value of. However, differentwould produce different trends ofversus *D*. The value ofwhen *c*1=*c*2 is denoted by, and the value ofwhen *c*3=*c*4 is denoted by. With current parameter values, is  andis 2. As shown in Figure 12 (b) (=0.05), **c can firstly decrease and then increase with the increase of *D* or firstly increase and then decrease with the increase of *D*, or monotonously increase with *D*, which depends on the value of. Large tends to produce higher **c when *D* is smaller. However, when *D* is at a large value, the system would have better stability whentakes a small value, clearly shown by the curled-up part of the surface in Figure 12 (a). Whentakes a large value, such as the results shown in Figure 13 (b) (), adding damping would increase **c for both the proportional and non-proportional cases. As to the effect ofon stability, it can be seen in Figure 12 (c) and Figure 13 (c) that except some cases whentakes some very small values, there is a local optimal value of(for maximum **c) which is smaller than(*c*1=*c*2). The value of optimalwould be indeed useful whentakes a large value, as **c around the optimal zone ofis larger than that of the undamped system. Time-domain results of two cases when *β*s=5, *β*b=0.05, *μ*=1.1, given in Appendix Ⅱ, validate the stability analysis.



1. vsand *D*

 

(b) vs *D* (c) vs

Figure 13. Evolution of critical coefficient of friction **c vs damping *D* and the slider’s damping index (belt damping index ).

Figure 14 shows the change of **c in terms of *D* andfor two diffidentcases. The obvious observation is that there is a ridge of. Unlike the influence of, the variation of **c withis firstly growing and then decreasing uniformly with most *D* values. Additionally, the ridge representing the maximum **c occurs at different values ofwhich depends on the value of *D*, but it is not larger than. Furthermore, it is found that when, for example, in the results shown in Figure 14 (b), the ridge of **c actually connects of the undamped case results (blue line with dots) at one place. This means that similar to the proportional damping case, the stability of certain non-proportionally damped systems could also be smoothly improved by adding damping when and at a certain.

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| (a) | (b) |

Figure 14. Evolution of critical coefficient of friction **c vs damping *D* and the slider’s damping index

To better assess the influence of damping index of the slider and the belt (and), Figure 15 illustrates the results of **c with varyingand. Whenandare not close to zero, there are two ridges on the surface. This indicates that there is a series of good combinations ofandfor enhancing stability, which are slightly smaller than the correspondingand. The first possible optimal combination appears whenandare not too small, and this is actually the proportional damping case shown by the red dot in Figure 15 (a) which gives the largest **c. When *D* is large, there would be a second optimal combination. It happens whenis very close to zero andtakes a certain value slightly smaller than, which represents the case that the horizontal damping of the slider is much larger than its vertical damping. The stability of such a system (even if it is non-proportionally damped) could have higher stability than the proportional damping case.

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| (a) ; | (b) |

Figure 15. Evolution of critical coefficient of friction **c vs the belt’s damping indexand the slider’s damping index 

Furthermore, the stabilising zones about damping indexandwith respect to the undamped system are shown in Figure 16. By taking *D*=0.1 as an example, the zone inside the blue dashed line on the right and the zone above the blue dashed line on the top mean that the stability of the system with the corresponding combination ofandis higher than the stability of the corresponding undamped system. Furthermore, it can be seen that adding more damping to a system leads to a larger range ofandthat improve the stability of the system. In comparison,has a larger selective range thanfor enhancing the stability of the system.



Figure 16. The optimal zones of the damping index of the sliderand the belt

#### Non-proportional damping effects of partly damped systems

In the end, one kind of special non-proportional cases, when damping in one direction or in one part of the 4-DoF model is absent, are analysed:

(1) Figure 17 (a) shows that if damping is added in only one direction of either the slider or the belt, the stability of the systems that have damping in only one direction is identical and does not change with the value of the added damping. Thus, this phenomenon is present in the two 2-DoF models and the 4-DoF model.

(2) Figure 17 (b) shows the two cases when only one component in the system is damped. One can see that when only the belt is damped, the stability of the system is not affected by the damping index or *D* of the belt. However, when damping is added only to the slider, **c of the system increases with *β*s and reaches the peak value at, and the stability can be improved moderately by adding damping. However, adding damping in only one part of the system, especially only in the belt, is not as good as the proportional damping case and some non-proportional damping cases shown above (e.g., Figure 13), in consideration of **c for stability bifurcation.

(3) Figure 17 (b)-(c) show the results when damping is only added in the horizontal or the vertical direction of the slider or belt individually. It can be clearly seen that when the slider or the belt is damped only in the vertical direction (Figure 17 (b)), vertical damping of the slider *D*2 plays a negative role in stabilisation. On the other hand, when the slider and the belt is damped only in the horizontal direction (Figure 17 (c)), the stability of the system is not affected by the horizontal damping. So neither damping cases should be recommended.

(4) Figure 17 (e)-(f) illustrates four cases when only one direction of the system is not damped. Figure 17 (e) shows the results when horizontal damping of the slider (*D*1) is absent. One can find that *D*2 plays a negative role in stabilisation, and the role of *D*3 and *D*4 is positive at low values, but could become negative at high values. Figure 17 (f) shows the results when *D*2 is absent. In this case, all other damping (*D*1, *D*3, *D*4) firstly improve the stability and then degrade the stability. Additionally, Figure 17 (g) shows the results when horizontal damping of the disc (*D*3) is absent. It can be deduced that small *D*1, *D*2 and *D*4 decrease the stability while large values of them improve the stability and even could make a more stable system than the undamped system. Finally, Figure 17 (h) shows the results when *D*4 is absent. Apparently, *D*1 plays a positive role in stabilising the system, and *D*3 makes no contribution to the stability. Furthermore, *D*2 is found to firstly decrease the stability at low values and then increase the stability at high values.

Therefore, the situation when any damping is zero in a part of a frictional system should be avoided in design as damping in some direction could have a negative effect on system stability.

|  |  |
| --- | --- |
| (a) only one direction is damped; | (b) only the slider or the belt is damped; |
| (c) only the horizontal direction of the slider and the belt is damped; | (d) only the vertical direction of the slider is damped; |
| (e) only the horizontal direction of the slider is not damped; | (f) only the vertical direction of the slider is not damped; |
| (g) only the horizontal direction of the belt is not damped; | (h) only the vertical direction of the belt is not damped; |

Figure 17. Evolution of critical coefficient of frictionwith damping present in a part or in one direction.

## Conclusions

In this paper, the fundamental role of damping, particularly the ratios of damping (the damping index) on the stability of three representative low-degree-of-freedom systems undergoing linear friction-induced vibration (FiV) due to mode-coupling is revealed. An analytical approach is applied to determine the bifurcation boundary of the region of stability. Two classic 2-DoF slider-belt models are revisited at first, and new and interesting results are found. Then a 4-DoF slider-belt model is studied in detail. Damping values and damping distribution over the slider and the belt in different directions are found to play significant roles in the stability as well as the zones of stability of the systems. The role of non-proportional damping looks particularly interesting and seems counter-intuitive.

The results found in this work have confirmed some of the current understanding of damping, and more importantly have revealed new effects of damping. The main conclusions are drawn as follows:

(1) The critical coefficient of friction for stability bifurcation (**c) of the undamped system actually appears as an isolated point on the plane with **c and a damping parameter respectively being the vertical and horizontal axes. It is found that the drop of the stability boundary of non-proportionally damped system in relation to the undamped system is a general phenomenon in friction systems with mode-coupling.

(2) Proportional damping plays a positive role in stabilisation regardless of the damping value.

(3) The role of non-proportional damping is non-monotonic and quite dependent on the model details. For the damped 2-DoF models, increasing non-proportional damping is found to improve the stability and could even lead to higher stability than the corresponding systems with proportional damping; For the 4-DoF model, the effects of non-proportional damping may change with the magnitude of damping, and the ratio of the horizontal to vertical damping of the slider ( *β*s ) and the belt ( *β*b ). In some range of *β*s and *β*b, the damped system could be less stable than its corresponding undamped system, e.g., small *β*b and small or large *β*s.

(4) Adding damping in only one part of the system (slider or and belt) or in any single direction (horizontal or vertical) should be avoided from the viewpoint of improving stability.

(5) There are optimal values of the horizontal to vertical damping ratio of one component (the optimal *βD* for the 2-DoF models and the optimal *β*s and *β*b for the 4-DoF model), which depend on both of the spatial configuration of the model and the model parameters. Thus, although adding proportional damping is found to be a comparatively safe and an effective option, systems could be redesigned to gain optimal stability through adding a certain level of non-proportional damping, which is a completely new finding.

(6) In contrast with the optimal damping ratios (the optimal *βD,* *β*s and *β*b), the damping index of each component when the horizontal and the vertical damping are equal (denoted by ,or) are a more straightforward parameter for stability control. In the case of the 4-DoF model, ifand, increasing *β*s and *β*b would destabilise the system. Thus,,orcan be used as performance indicators for good damping configurations for general FiV models.

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## Appendix Ⅰ

The Routh-Hurwitz coefficients (the elements on the second to the fifth rows in the first column of Routh array) are denoted by *Hi* (*i*=1, 2, 3, 4) and they are needed in the Routh-Hurwitz criterion for stability analysis.

For Model I, they are









*H*1, *H*2 and *H*4 for both models are always positive, while *H*3 being zero provides the condition for determining the critical coefficient of friction in damped cases.

Appendix Ⅱ

As the FiV model is linear, the results fall into two categories: (1) decreasing vibration for stable systems and (2) increasing vibration for unstable systems. Two examples of the time-domain results with different damping *D* are shown in Figures A1-A2. When *D*=0.04 (Figures A1), the system exhibits stable vibration. However, when *D* increases to 0.08, the vibration of the system grows (Figure A2). The values of the critical coefficients of friction when *D*=0.04 and *D*=0.08 are *μ*c=1.11 and *μ*c=1.03 respectively.

|  |  |
| --- | --- |
|  |  |
| (a) horizontal vibration of the slider | (b) vertical vibration of the belt |

Figure A1. Time history results with *D*=0.04 (*β*s=5, *β*b=0.05, *μ*=1.1).

|  |  |
| --- | --- |
|  |  |
| (a) horizontal vibration of the slider | (b) vertical vibration of the belt |

Figure A2. Time history results with *D*=0.08 (*β*s=5, *β*b=0.05, *μ*=1.1)

These time-domain results confirm the results of the stability analysis in Section 3.2.1.

List of Figure Captions

Figure 1. 2-DoF model with an inclined spring

Figure 2. The critical coefficient of friction **c changes with damping *D* (Model Ⅰ)

Figure 3. The critical coefficient of friction **c changes with the damping index(Model Ⅰ)

Figure 4. The combined effect of *D* andon **c for bifurcation

Figure 5. The combined effect of *D* andon **c for bifurcation (and)

Figure 6. The combined effect of *D* andon **c for bifurcation (and)

Figure 7. 2-DoF model with two sliding interfaces

Figure 8. 4-DoF model

Figure 9. The real (left) and imaginary part (right) of the eigenvalues versus the coefficient of friction *μ* for various *D*. (the proportional damping case)

Figure 10. The real (left) and imaginary part (right) of the eigenvalues versus the coefficient of friction *μ* for different non-proportional cases. (green dash line: scenario 1 with,; red solid line: scenario 2 with,; black dot line: scenario 3 with,,, )

Figure 11. The real (right) and imaginary part (left) of the eigenvalues versus the coefficient of friction *μ* for different non-proportional cases. (green dash line: scenario 1 with,; red solid line: scenario 2 with,; black dot line: scenario 3 with,,,)

Figure 12. Evolution of critical coefficient of friction **c vs damping *D* and the slider’s damping index()

Figure 13. Evolution of critical coefficient of friction **c vs damping *D* and the slider’s damping index(belt damping index )

Figure 14. Evolution of critical coefficient of friction **c vs damping *D* and the slider’s damping index 

Figure 15. Evolution of critical coefficient of friction **c vs the belt’s damping indexand the slider’s damping index

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Figure 17. Evolution of critical coefficient of friction with damping present in a part or in one direction