

A LITTLE CHARITY GUARANTEES ALMOST ENVY-FREENESS*

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Abstract.

Fair division of indivisible goods is a very well-studied problem. The goal of this problem is to distribute m goods to n agents in a “fair” manner, where every agent has a valuation for each subset of goods. We assume monotone valuations. Envy-freeness is the most extensively studied notion of fairness. However, envy-free allocations do not always exist when goods are indivisible. The notion of fairness we consider here is “envy-freeness up to any good” (EFX) where no agent envies another agent after the removal of any single good from the other agent’s bundle. It is not known if such an allocation always exists.

We show there is always a partition of the set of goods into $n + 1$ subsets (X_1, \dots, X_n, P) where for $i \in [n]$, X_i is the bundle allocated to agent i and the set P is unallocated (or donated to charity) such that we have:

- envy-freeness up to any good,
- no agent values P higher than her own bundle, and
- fewer than n goods go to charity, i.e., $|P| < n$ (typically $m \gg n$).

Our proof is constructive and leads to a pseudo-polynomial time algorithm to find such an allocation. When agents have additive valuations and $|P|$ is large (i.e., when $|P|$ is close to n), our allocation also has a good maximin share (MMS) guarantee. Moreover, a minor variant of our algorithm also shows the existence of an allocation that is $4/7$ *groupwise* maximin share (GMMS): this is a notion of fairness stronger than MMS. This improves upon the current best bound of $1/2$ known for an approximate GMMS allocation.¹

Key words. Discrete Fair Division, EFX, GMMS, Nash Welfare

AMS subject classifications. 68W99, 91B32

1. Introduction. Fair division of goods among competing agents is a fundamental problem in Economics and Computer Science. There is a set M of m goods and the goal is to *allocate* goods among n agents in a *fair* way. An allocation is a partition of M into disjoint subsets X_1, \dots, X_n where X_i is the set of goods given to agent i . When can an allocation be considered “fair”? One of the most well-studied notions of fairness is *Envy-freeness*. Every agent has a *value* associated with each subset of M and agent i *envies* agent j if i values X_j more than X_i . An allocation is *envy-free* if no agent envies another. An envy-free allocation can be regarded as a fair and desirable partition of M among the n agents since no agent envies another; as mentioned in [30], such a mechanism of partitioning land dates back to the Bible.

Unlike land which is divisible, goods in our setting are *indivisible* and an envy-free allocation of the given set of goods need not exist. Consider the following simple example with two agents and a single good that both agents desire: one of the agents has to receive this good and the other agent envies her. Since envy-free allocations need not exist, several relaxations have been considered.

*A preliminary version of these results appeared in SODA 2020 [19].

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¹Very recently and independently, Amanatidis et al. [3] also showed the existence of a $4/7$ -GMMS allocation.

Relaxations. Budish [13] introduced the notion of *EF1*: this is an allocation of goods that is “envy-free up to *one* good”. In an EF1 allocation, agent i may envy agent j , however this envy would vanish as soon as *some* good is removed from X_j . Note that no good is really removed from X_j : this is simply a way of assessing how much i values X_j more than X_i . That is, if i values X_j more than X_i , then there exists some $g \in X_j$ such that i values X_i at least as much as $X_j \setminus \{g\}$. Going back to the example of two agents and a single good, the allocation where one agent receives this good is EF1. It is known that EF1 allocations always exist; as shown by Lipton et al. [29], such an allocation can be efficiently computed.²

Caragiannis et al. [15] introduced a notion of envy-freeness called *EFX* that is stronger than EF1. An EFX allocation is one that is “envy-free up to *any* good”. In an EFX allocation, agent i may envy agent j , however this envy would vanish as soon as *any* good is removed from X_j . Thus every EFX allocation is also EF1 but not every EF1 allocation is EFX.

	a	b	c
Agent 1	1	1	3
Agent 2	1	1	3

Consider the simple example given above: there are three goods a, b, c and two agents with *additive valuations*³ (formally defined in Section 1.1) as described below. Both agents value c three times as much as a or b . Observe that there is no envy-free allocation here. The allocation where agent 1 gets $\{a\}$ and agent 2 gets $\{b, c\}$ is EF1 but not EFX. On the other hand, the allocation where agent 1 gets $\{a, b\}$ and agent 2 gets $\{c\}$ is EFX. Indeed, the latter allocation seems fairer than the former allocation. As said in [14]: “*Arguably, EFX is the best fairness analog of envy-freeness of indivisible items*”. While it is known that EF1 allocations always exist, the question of whether EFX allocations always exist is still an open problem (despite “significant effort”, according to [15]). In fact, Procaccia [31] calls this question the current biggest open problem in fair division:

“This fundamental and deceptively accessible question is open. In my view, it is the successor of envy-free cake cutting as fair division’s biggest problem.”

Plaut and Roughgarden [30] showed that EFX allocations always exist (i) when there are only two agents or (ii) when all n agents have the same valuation function. Moreover, it was shown in [30] that exponentially many value queries may be needed to determine EFX allocations even in the restricted setting where there are only two agents with identical *submodular* valuation functions⁴. It was not known until very recently, if an EFX allocation always exists even when there are only three agents with additive valuations. A positive answer to this question was very recently given by Chaudhury et al. [17]. It was remarked in [30]: “*We suspect that at least for general valuations, there exist instances where no EFX allocation exists*”. Note that a general valuation is one that is *monotone* (defined in Section 1.1.1).

A relaxation of EFX. Very recently, Caragiannis et al. [14] introduced a more relaxed notion of EFX called *EFX-with-charity*. This is a partial allocation that is EFX, i.e., the entire set of goods need not be distributed among the agents. So some goods may be left *unallocated* and it is assumed that these unallocated goods are

²The algorithm in [29] was published in 2004 with a different property and EF1 was proposed in 2011.

³The value of a set $S \subseteq M$ is the sum of values of goods in S .

⁴These are valuation functions with decreasing marginal values.

donated to charity. There is a very simple allocation that is EFX-with-charity where *no* good is assigned to any agent—thus all goods are donated to charity. Obviously, this is not an interesting allocation and one seeks allocations with better guarantees on the allocated and unallocated goods. One such allocation was shown in [14].

Let $X^* = \langle X_1^*, \dots, X_n^* \rangle$ be an optimal *Nash social welfare* allocation on the entire set of goods, i.e., an allocation that maximizes $\prod_{i=1}^n v_i(X_i^*)$, where v_i is agent i 's valuation function. It was shown in [14] that when agents have additive valuations, there always exists an EFX-with-charity allocation $X = \langle X_1, \dots, X_n \rangle$ where every agent receives at least half the value of her allocation in X^* . Interestingly, as shown in [14], $X_i \subseteq X_i^*$ for all i . Unfortunately, there are no upper bounds on the *number* of unallocated goods or on the *value* any agent has for the set of goods donated to charity.

We believe these are important questions to ask. The ideal allocation is one that is EFX and allocates *all* goods; so we would like a guarantee that a large number of goods have been allocated to agents. Moreover, since EFX allocations are relaxations of envy-free allocations, it is in the same spirit that we seek an EFX (partial) allocation where nobody envies the set of unallocated goods. The allocation in [14] gives no guarantee either on the number of unallocated goods or on whether any agent values the set of unallocated goods more than her own bundle. Here we consider the notion of *EFX-with-bounded-charity*. That is, we seek EFX-with-charity allocations with bounds on the set given to charity, i.e., a bound on the *size* and a bound on the *value* of the set of goods donated to charity.

Efficient Allocations. There has also been a significant amount of work done in determining fair allocations that are *efficient*. Efficiency is usually a measure of the overall welfare that an allocation achieves while still being fair. A very common measure of efficiency is *Pareto-optimality*: an allocation X is said to be Pareto-optimal if there is no allocation Y such that $v_i(Y_i) \geq v_i(X_i)$ for all agents i and $v_i(Y_j) > v_i(X_j)$ for some agent j . Intuitively, we cannot make any agent happier without reducing the valuation of some other agent. With this in mind, a natural question to ask is the following: can we get fair allocations (EF1 or EFX) that are also Pareto-optimal?

Caragiannis et al. [15] answered the above question positively for EF1, by showing that an optimal Nash social welfare allocation is EF1 (fair) and Pareto-optimal (efficient). Thereafter, Nash social welfare in itself has been used as a measure of efficiency [14].⁵ Furthermore, Plaut and Roughgarden [30] showed that there exists an instance where no EFX allocation is Pareto-optimal. Thus, the question we wish to answer is: can we get good relaxations of EFX with high Nash social welfare? It turns out that by modifying the first step of our algorithm, we can determine an EFX allocation with bounded charity (fair) that has high Nash social welfare (efficient) (see Subsection 1.1.3 for a more precise and detailed statement).

1.1. Our Results. Let $N = [n]$ be the set of agents. Every agent $i \in [n]$ has a valuation function $v_i : 2^M \rightarrow \mathbb{R}_{\geq 0}$, where M is the set of m goods.

1.1.1. Monotone valuations. We show our main existence result for *monotone* valuation functions, i.e., the only assumption we make on any valuation function v_i is that it is monotone, so $S \subseteq T$ implies $v_i(S) \leq v_i(T)$.

In contrast, the EFX-with-charity allocation in [14] works only for *additive valuations*, i.e., for any $S \subseteq M$ and $i \in [n]$, we have $v_i(S) = \sum_{g \in S} v_i(\{g\})$.

⁵Higher the Nash social welfare of an allocation, the more efficient the allocation is.

We show there always exists an allocation⁶ $X = \langle X_1, \dots, X_n \rangle$ that satisfies the following properties:

1. X is EFX, i.e., for any two agents i, j : $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for any $g \in X_j$;
2. $v_i(X_i) \geq v_i(P)$ for all agents i , where $P = M \setminus \cup_{i=1}^n X_i$ is the set of unallocated goods;
3. $|P| < n$ (recall that n is the number of agents and typically $n \ll m$).

Note that our result implies the following simple observation – among the n agents, if there is just *one* agent who has no preferences (say, for agent i , we have $v_i(S) = 0$ for all $S \subseteq M$) then a complete EFX allocation always exists for monotone valuations: Find an EFX allocation satisfying all the three properties mentioned above, on the set of agents $N \setminus \{i\}$. The allocation among the agents in $N \setminus \{i\}$ is EFX (by condition 1) and we now allocate P (the set of unallocated goods) to agent i . Observe that agent i envies nobody as i has the same value for every subset of goods and nobody envies i as nobody envies P (by condition 2).

Our proof is constructive. We start with no goods being allocated to the agents and find the claimed allocation by at most $nmV/\Delta + m$ applications of three simple update rules, where n is the number of agents, m is the number of goods, $V = \max_i v_i(M)$ is the maximum valuation of any agent, and $\Delta = \min_i \min\{|v_i(T) - v_i(S)| : S, T \subseteq M \text{ and } v_i(S) \neq v_i(T)\}$ is the minimum difference between distinct valuations.

The update rules use a *minimal-envied-subset-oracle*: given $S \subseteq M$ such that there is an agent who envies S , find a subset $Z \subseteq S$ such that there is an agent who envies Z and no agent envies a proper subset of Z . This oracle can be realized by a simple algorithm that uses at most nm value queries. Thus we show that for monotone valuations, an EFX allocation that satisfies properties 1-3 can be computed with $\text{poly}(n, m, V, 1/\Delta)$ value queries, i.e., in pseudo-polynomial time. With a slight modification of our pseudo-polynomial time algorithm, we can determine a $(1 - \varepsilon)$ -EFX allocation with bounded charity⁷ in $\text{poly}(n, m, \log(V/\Delta), 1/\varepsilon)$ time.

1.1.2. Additive valuations. The most well-understood class of valuation functions is the set of *additive* valuations. We consider the case when all agents have additive valuations and show that our allocation or very minor variants of our allocation can guarantee several other notions of fairness.

Number of Unallocated Goods and MMS Guarantee. Another interesting and well-studied notion of fairness is *maximin share*. Suppose agent i has to partition M into n *bundles* (or sets) knowing that she would receive the worst bundle with respect to her valuation. Then i will choose a partition of M that maximizes the valuation of the worst bundle (wrt her valuation). The value of this worst bundle is the maximin share of agent i . An important question here is: does there always exist an allocation of M where every agent gets a bundle worth at least her maximin share?

Formally, let N and M be the sets of n agents and m goods, respectively. Using \mathcal{X} to denote the set of all complete allocations, we define the maximin share of an

⁶Henceforth, allocations are *partial* and we will use “complete allocation” to refer to one where all goods are allocated.

⁷ X is a $(1 - \varepsilon)$ -EFX allocation with bounded charity, when we have:

1. For any pair of agents i, j : $v_i(X_i) \geq (1 - \varepsilon) \cdot v_i(X_j \setminus \{g\})$ for any $g \in X_j$;
2. $v_i(X_i) \geq (1 - \varepsilon) \cdot v_i(P)$ for all agents i , and
3. $|P| < n$.

agent (say, i) as follows:

$$MMS_i(n, M) = \max_{\langle X_1, \dots, X_n \rangle \in \mathcal{X}} \min_{j \in [n]} v_i(X_j).$$

The goal is to determine an allocation $\langle X_1, X_2, \dots, X_n \rangle$ of M such that for every i , we have $v_i(X_i) \geq MMS_i(n, M)$. This question was first posed by Budish [13]. Kurokawa et al. [28] showed that such an allocation need not exist, even in the restricted setting of only three agents! Thereafter, *approximate-MMS* allocations were studied [28, 23, 26, 25] and there are polynomial time algorithms to find allocations where for all i , agent i gets a bundle of value at least $\alpha \cdot MMS_i(n, M)$; the best guarantee known for α was $3/4 - \epsilon$ by Ghodsi et al. [26] (for any $\epsilon > 0$) and this was very recently improved to $3/4$ by Garg and Taki [25].

Amanatidis et al. [1] showed that any complete EFX allocation is also a $\frac{4}{7}$ -MMS allocation. We show that our allocation promises better MMS guarantees when the number of unallocated goods is large. Let $X = \langle X_1, \dots, X_n \rangle$ be our allocation as described by properties 1-3 above and P be the set of unallocated goods. For any agent $i \in [n]$, we have:

$$v_i(X_i) \geq \frac{1}{2 - |P|/n} MMS_i(n, M).$$

Hence, larger the number of unallocated goods, better are the guarantees that we get on MMS. The extreme values are $|P| = 0$ and $|P| = n - 1$. When $|P| = 0$, we have a complete EFX allocation and when $|P| = n - 1$, we have an EFX allocation that is an *almost-MMS* allocation: $v_i(X_i) \geq (1 - 1/n) \cdot MMS_i(n, M)$ for all i .

Improved Guarantees for Groupwise MMS. Barman et al. [7] recently introduced a notion of fairness called *groupwise maximin share* (GMMS) which is stronger than MMS. An allocation is said to be GMMS if the MMS condition is satisfied for every subgroup of agents and the union of the sets of goods allocated to them. Formally, a complete allocation $X = \langle X_1, X_2, \dots, X_n \rangle$ is α -GMMS if for any $N' \subseteq N$ and all $i \in N'$, we have $v_i(X_i) \geq \alpha \cdot MMS_i(n', \bigcup_{j \in N'} X_j)$ where $n' = |N'|$. Every GMMS allocation, i.e. $\alpha = 1$, is also a complete EFX allocation [7].

It is known [7] that GMMS strictly generalizes MMS. In particular, it was shown that GMMS allocations rule out some very unsatisfactory allocations that have MMS guarantees. For example, consider an instance with n agents with additive valuations and a set M of $n - 1$ goods and every agent has a valuation of one for each good. Since the number of goods is less than the number of agents, we have $MMS_i(n, M) = 0$ for every agent i . So any allocation is an MMS allocation. It is not hard to see that the only allocations that have a GMMS guarantee are those where $n - 1$ agents get one good each and one agent is left without any goods. See Subsection 2.1 in [7] for more discussion. Naturally, it is a harder problem to approximate GMMS than MMS. While $\frac{3}{4}$ -MMS allocations always exist, the largest α for which α -GMMS allocations are known to exist is $\frac{1}{2}$ [7]. We extend the result of Amanatidis et al. [1] for MMS to show the following:

- A $\frac{4}{7}$ -GMMS allocation always exists and it can be computed in pseudo-polynomial time.

In particular, we show that modifying the last step of our algorithm results in a complete allocation that is $\frac{4}{7}$ -GMMS. As mentioned earlier, very recently and independently, Amanatidis et al. [3] showed the same approximation.

1.1.3. Efficiency of Our Allocation. We remark that our algorithm is very “robust” in the sense that it computes an EFX allocation with bounded charity starting from any partial EFX allocation (trivially, we can start with the *empty* allocation, i.e., no good is assigned to any agent) and every agent in the final allocation has a valuation at least as high as her initial valuation (since we maintain the invariant that the valuation of any agent never decreases in the entire course of the algorithm). This robustness helps us to determine an EFX allocation $X = \langle X_1, \dots, X_n \rangle$ with bounded charity that also has a high Nash social welfare, i.e., a high value of $(\prod_{i=1}^n v_i(X_i))^{1/n}$.

Additive valuations. We show that modifying the starting step of our algorithm ensures that our EFX allocation X with bounded charity also satisfies $v_i(X_i) \geq \frac{1}{2} \cdot v_i(X_i^*)$ as promised in [14], where $X^* = \langle X_1^*, \dots, X_n^* \rangle$ is an optimal Nash social welfare allocation. Here we use the allocation computed in [14] as a black box in our starting step, thus our result can be regarded as an extension of the result in [14].

Subadditive valuations. A valuation v is subadditive if $v(S) + v(T) \geq v(S \cup T)$ for all $S, T \subseteq M$. It follows from recent work by Chaudhury et al. [18] that a careful modification of the first step of our algorithm results in an EFX allocation with bounded charity that also achieves an $O(n)$ approximation of Nash social welfare⁸ when agents have subadditive valuations. Barman et al. [6] showed that it requires an exponential number of value queries to provide a sublinear approximation to Nash social welfare under subadditive valuations. Thus, by choosing the initial partial EFX allocation carefully, for any fixed $\varepsilon > 0$, our algorithm yields a $(1 - \varepsilon)$ -EFX allocation with bounded charity, using a polynomial number of value queries and it achieves the best approximation of Nash social welfare that is possible with a polynomial number of value queries.

1.2. Our Techniques. We now give an overview of the main ideas used to find our EFX allocation. We first recall the algorithm of Lipton et al. [29] for finding an EF1 allocation. They use the notion of an *envy-graph*: here each vertex corresponds to an agent and there is an edge (i, j) if and only if i envies j . The invariant maintained is that the envy-graph is a DAG: a cycle corresponds to a cycle of envy and by swapping bundles along a cycle, every agent in the cycle becomes better-off. More precisely, if $i_0 \rightarrow i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_{\ell-1} \rightarrow i_0$ is a cycle in the envy graph, then reassigning $X_{i_{j+1}}$ to agent i_j for $0 \leq j < \ell$ (indices are to be read modulo ℓ) will increase the valuation of every agent in the cycle. Thus cycles can be eliminated (see Figure 1). Also if there was an edge from s to some i_k where s is not a part of the cycle, this edge just gets directed now from s to i_{k+1} after we exchange bundles along the cycle.

The algorithm in [29] runs in rounds and always maintains an allocation that is EF1. At the beginning of every round, an unenvied agent s (this is a source vertex in this DAG) is identified and an unallocated good g is allocated to s . The new allocation is EF1, as nobody will envy the bundle of s after removing the good g .

The Reallocation Operation. We now highlight a key difference between an EF1 allocation and an EFX allocation. From the algorithm of Lipton et al. [29], it is clear that given an EF1 allocation on a set M_0 of goods, one can determine an EF1 allocation on $M_0 \cup M_1$, for any $M_1 \subseteq M \setminus M_0$, by simply adding goods in M_1 one-by-one to the existing bundles and changing the owners (if necessary) in a clever way.

⁸In fact, it achieves an $O(n)$ approximation of a much larger class of efficiency functions, namely, the generalized p -mean welfare function. Independently, Barman et al. [6] also showed how to achieve an $O(n)$ approximation for the generalized p -mean welfare (hence also Nash social welfare).

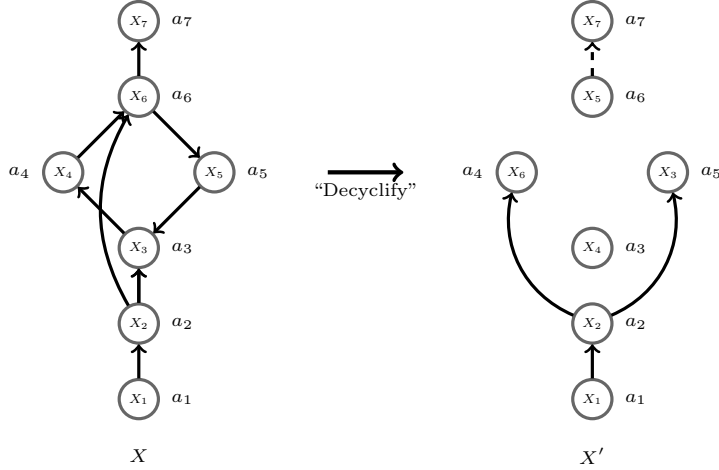


FIG. 1. *Eliminating cycles in the envy-graph. There is a cycle in allocation X : $a_3 \rightarrow a_4 \rightarrow a_6 \rightarrow a_5 \rightarrow a_3$. The bundles are exchanged along the cycle in allocation X' and the agents along the cycle strictly improve their valuations. The envy edges outgoing from the agents in the cycle may or may not be there in X' (the edge from a_6 to a_7), while the envy edges which were incoming to the agents in the cycle, re-orient themselves (instead of the edge from a_2 to a_3 (and from a_2 to a_6) in X , there is the edge from a_2 to a_5 (from a_2 to a_4) in X').*

Intuitively, we never need to cut or merge the bundles formed in any EF1 allocation. We can just append the unallocated goods appropriately to the current bundles.

The above strategy is very far from true for EFX. Consider the example given below where there are three agents with additive valuations and four goods a, b, c , and d ; let $\varepsilon \in (0, 1)$.

	a	b	c	d
Agent 1	ε	1	1	2
Agent 2	1	ε	1	2
Agent 3	1	1	ε	2

An EFX allocation for the first three goods has to give exactly one of a, b, c to each of the three agents. However an EFX allocation for all the four goods has to allocate the singleton set $\{d\}$ to some agent (say, agent 1) and say, $\{a\}$ to agent 2 and $\{b, c\}$ to agent 3. Thus the allocation needs to be *cut and merged*. When there are many agents, each with her own valuation, figuring out the cut-and-merge operations is the difficult step. Here we implement our global reallocation operation as follows.

Improving Social Welfare. Suppose we have an EFX allocation $X = \langle X_1, \dots, X_n \rangle$ on some subset $M_0 \subset M$. We would now like to add a good $g \in M \setminus M_0$. However we will not be able to guarantee an EFX allocation on $M_0 \cup \{g\}$. What we will ensure is that either case (i) or case (ii) occurs:

- (i) We have an EFX allocation $X' = \langle X'_1, \dots, X'_n \rangle$ on a subset of $M_0 \cup \{g\}$ such that $v_i(X'_i) \geq v_i(X_i)$ for all i and for at least one agent j we have $v_j(X'_j) > v_j(X_j)$. Thus $\sum_{i \in [n]} v_i(X'_i) > \sum_{i \in [n]} v_i(X_i)$; in other words, the *social welfare* strictly improves.
- (ii) We have an EFX allocation on $M_0 \cup \{g\}$ and the social welfare does not decrease.

Hence in each step of our algorithm, we either increase social welfare or we increase the number of allocated goods without decreasing social welfare—thus we always make progress. This is similar to the approach used by Plaut and Roughgarden [30] to guarantee the existence of $\frac{1}{2}$ -EFX when agents have subadditive valuations. We now outline how we ensure that one of cases (i) and (ii) has to happen.

For simplicity of exposition, let us assume the envy-graph corresponding to our starting EFX allocation X has a single source s . Add g to s 's bundle: if nobody envies s up to any single good, i.e., if $v_i(X_i) \geq v_i(X_s \cup \{g\} \setminus \{g'\})$ for all $i \in [n]$ and all $g' \in X_s \cup \{g\}$, then we are in an easy case as we have an EFX allocation on $M_0 \cup \{g\}$. In this case, we “decycle” the envy-graph (if cycles are created) and continue. Observe that swapping bundles along a cycle in the envy-graph increases social welfare.

Most Envious Agent. Assume now that there are one or more agents who envy s up to some good after g is allocated to s . To resolve this, we introduce the concept of *most envious agent*. If there is an agent who envies s up to some good then there is some $S \subset X_s \cup \{g\}$ that is *envied* (we use \subset to denote a proper subset). That is, there is an agent who values S more than her bundle. Let Z be any inclusion-wise minimal subset of $X_s \cup \{g\}$ that is envied. So there exists an agent who envies Z and no proper subset of Z is envied by any agent. Any agent who envies Z is a most envious agent of $X_s \cup \{g\}$. Whenever there is an agent who envies s up to some good, observe that there has to exist a most envious agent of $X_s \cup \{g\}$.

Let Z be an inclusion-wise minimal subset of $X_s \cup \{g\}$ that is envied and let t be an agent such that $v_t(X_t) < v_t(Z)$. Recall the assumption that s is the only source, so there is a path $s \rightarrow i_1 \rightarrow \dots \rightarrow i_{k-1} \rightarrow t$ in the envy-graph. We do a leftwise shift of bundles along this path: so s gets i_1 's bundle, and for $1 \leq r \leq k-1$, i_r gets i_{r+1} 's bundle (where $i_k = t$), and finally t gets Z . The goods in $X_s \cup \{g\} \setminus Z$ are thrown back into the pool of unallocated goods (see Figure 2).

Observe that every agent in this path is strictly better-off now than in the allocation X and no other agent is worse-off. Moreover, since no proper subset of Z is envied, there is no agent envying any other agent up to any single good. Thus we have a desired EFX allocation X' . When there are multiple sources, we can adapt this technique *provided* there are enough unallocated goods; in particular, the number of unallocated goods must be at least the number of sources in the envy-graph. We describe this in detail in Section 2.

We would like to contrast the above approach with other EFX algorithms [30, 14]. The $\frac{1}{2}$ -EFX algorithm by Plaut and Roughgarden [30] either merges g (the new good) with an existing bundle or allocates the singleton set $\{g\}$ to an agent. The EFX-with-charity algorithm by Caragiannis et al. [14] takes an allocation of maximum Nash social welfare as input and then permanently removes some goods from the instance. We regard the notion of “most envious agent” that shows a natural way of breaking up a bundle to preserve envy-freeness up to any good as one of the innovative contributions of our work.

Our Other Results. Regarding our result with approximate MMS guarantee, if the number of unallocated goods in our EFX allocation is large, then the number of sources also has to be large: these are unenvied agents. Moreover, no agent envies the set of unallocated goods. Suppose for now that $|P| = n - 1$. This means every agent is a source. So no agent envies the bundle of any other agent and also the set of unallocated goods. We assumed valuations to be additive here, so this means

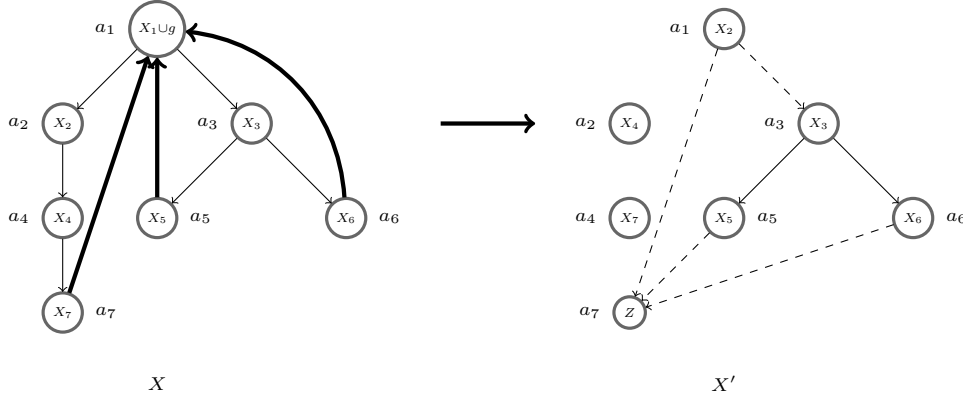


FIG. 2. Illustration of the update rule. The figure on the left indicates the envy-graph corresponding to the allocation X . Thin edges indicate weak envy edges: a thin edge from a_i to a_j signifies that a_i envies a_j , however a_i does not envy any proper subset of a_j 's bundle. The thick edges indicate strong envy edges: a thick edge from a_i to a_j indicates that a_i also envies some proper subset of a_j 's bundle. Initially the envy graph has only thin edges with a_1 as the only source. Agents a_7 , a_5 and a_6 strongly envy a_1 when we give a_1 the good g and a_7 is a most envious agent. The figure on the right indicates the envy graph of the allocation X' that we obtain after applying the update rule. We shift the bundles along the path $a_1 \rightarrow a_2 \rightarrow a_4 \rightarrow a_7$ and give a_7 a subset Z of $X_5 \cup \{g\}$ that she envies and no proper subset of Z is envied by any agent. All the agents along the path (a_1, a_2, a_4, a_7) strictly improve their valuations. All the thick edges disappear as no agent envies a proper subset of Z (there may or may not be thin edges directed towards a_7 , depending on whether a_1 , a_5 or a_6 envy Z).

$v_i(X_i) \geq v_i(M)/(n + 1)$ for each agent i . Thus we have:

$$\begin{aligned} v_i(X_i) &\geq \frac{v_i(M)}{n + 1} = (1 + 1/n)^{-1} \cdot \frac{v_i(M)}{n} \\ &\geq (1 - 1/n) \cdot MMS_i(n, M), \end{aligned}$$

where the constraint that $v_i(M)/n \geq MMS_i(n, M)$ holds for additive valuations. We show our result for approximate-MMS allocation and our improved bound for approximate-GMMS allocation in Section 3.

1.3. Related Work. Fair division of divisible resources is a classical and well-studied subject starting from the 1940's [32]. Fair division of indivisible goods among competing agents is a young and exciting topic with recent work on EF1 and EFX allocations [15, 9, 30, 11, 14], approximate maximin share allocations [13, 12, 2, 8, 28, 26, 23], and approximation algorithms for maximizing Nash social welfare and generalizations [21, 20, 16, 4, 22, 5].

As mentioned earlier, Caragiannis et al. [15] introduced the notion of EFX and it is now known that EFX allocations always exist for three agents with additive valuations [17]. Whether EFX allocations always exist with general valuations or with a larger number of agents is an enigmatic open problem. It was shown in [15] that there always exists an EF1 allocation that is also Pareto-optimal and Barman et al. in [9] showed a pseudo-polynomial time algorithm to compute such an allocation.

Applications. Fair division of goods or resources occurs in many real-world scenarios and this is demonstrated by the popularity of the website Spliddit (<http://www.spliddit.org>) that implements mechanisms for fair division where users can log in, define what needs to be divided, and enter their valuations. This website guarantees

an EF1 allocation that is also Pareto-optimal and since its launch in 2014, it has been used tens of thousands of times [15]. We refer to [27, 30] for details on the diverse applications for which Spliddit has been used: these range from rent division and taxi fare division to credit assignment for an academic paper or group project. Another such website is Fair Outcomes, Inc. (<http://www.fairoutcomes.com>). An interesting application is also *Course Allocate* used at Wharton School that guarantees certain fairness properties to allocate courses among students [30].

1.4. Improvements with respect to the conference version. In the conference version of the paper [19], we gave a pseudo-polynomial time algorithm to determine an EFX allocation with bounded charity when agents have “gross-substitute” valuations. In the current version we show a pseudo-polynomial time algorithm to determine an EFX allocation with bounded charity for all monotone valuations. This is realized with a relaxed definition of “most envious agent” (see Definition 2.3) that helps us to implement the *minimal-envied-subset* oracle efficiently (see Subsection 2.1).

2. Existence of an EFX-Allocation with Bounded Charity. We prove our main result on EFX-with-bounded-charity allocations in this section. We will define three *update rules*. Each update rule takes a pair (X, P) consisting of an allocation X and a set P of unallocated goods (we will call P the *pool*) and returns a modified pair (X', P') .

Each application of an update rule will ensure that either (i) the *social welfare* $\phi(X) = \sum_{i \in [n]} v_i(X_i)$ of the current allocation increases or (ii) the size of the pool decreases and the social welfare does not decrease, so $|P'| < |P|$ in this case. Hence the update process will terminate. The overall structure of the algorithm is given in Algorithm 2.1.

Algorithm 2.1 Algorithm for Computing an EFX-Allocation

Postcondition: X is EFX, $|P| < n$, and $v_i(P) \leq v_i(X_i)$ for all $i \in [n]$.

- 1: $X_i \leftarrow \emptyset$ for $i \in [n]$; $P \leftarrow M$;
 - 2: **while** one of the update rules shown in Algorithm 2.2 is applicable **do**
Invariant: X is EFX and the envy-graph G_X is acyclic
 - 3: Let U_ℓ be an applicable update rule;
 - 4: $(X, P) \leftarrow U_\ell(X, P)$;
 - 5: Decycle the envy-graph;
 - 6: **end while**
-

In order to define our update rules, we need the concepts of *envy-graph* and *most envious agent* for a bundle of goods. These were discussed in Section 1.2 and we formally define them below.

DEFINITION 2.1. *The envy-graph G_X for an allocation $X = \langle X_1, X_2, \dots, X_n \rangle$ has the set of agents as vertices and there is a directed edge from agent i to agent j if and only if $v_i(X_i) < v_i(X_j)$.*

The notion of envy-graph was introduced in [29] and it is well-known that cycles can be removed from the envy-graph without destroying desirable properties (see Lemma 2.2). Thus we can maintain G_X as a DAG.

For each agent s , we define the *reachability component* $C(s)$ as the set of all agents reachable from s in the envy-graph. The *sources* of the envy-graph are the vertices with indegree zero. For ease of notation, we will use $B \setminus g$ and $B \cup g$ to denote $B \setminus \{g\}$ and $B \cup \{g\}$, respectively.

LEMMA 2.2. *Let $i_0 \rightarrow i_1 \rightarrow \dots \rightarrow i_{k-1} \rightarrow i_0$ be a cycle in the envy-graph. Consider the allocation X' where $X'_{i_\ell} = X_{i_{\ell+1}}$ (indices are modulo k) for $\ell \in \{0, \dots, k-1\}$ and $X'_j = X_j$ for $j \notin \{i_0, \dots, i_{k-1}\}$. If X is EFX, then X' is also EFX. Moreover, $\phi(X') > \phi(X)$.*

Proof. Consider any agent i . We have $v_i(X'_i) \geq v_i(X_i)$ with strict inequality if i lies on the cycle. So $\sum_{i \in [n]} v_i(X'_i) > \sum_{i \in [n]} v_i(X_i)$. Thus $\phi(X') > \phi(X)$.

Since X' is just a permutation of X , for any agent j there exists some agent j' such that $X'_j = X_{j'}$. Therefore, since X is EFX, for any good $g \in X_{j'}$ (or equivalently X'_j) we have $v_i(X'_j \setminus g) = v_i(X_{j'} \setminus g) \leq v_i(X_i) \leq v_i(X'_i)$. Thus X' is also EFX. \square

Let $S \subseteq M$. Suppose there exists an agent who considers S more valuable than her own bundle. Then we will call S an *envied set*. The following definition formalizes the notion of “most envious agent”.

DEFINITION 2.3. *Let Z be an inclusion-wise minimal envied subset of S , i.e., (1) there exists an agent i such that $v_i(Z) > v_i(X_i)$ and (2) for all $j \in [n]$ we have $v_j(X_j) \geq v_j(Z')$ for all $Z' \subset Z$. The agent i is called a most envious agent of S .*

We are now ready to present our three update rules U_0 , U_1 , and U_2 , in Algorithm 2.2.

Rule U_0 : Rule U_0 is the easiest of the update rules. It is applicable whenever adding a good from the pool to some source of G_X does not destroy the EFX-property (see Algorithm 2.2).

LEMMA 2.4 (Rule U_0).

- (a) *Rule U_0 returns an EFX allocation. An application of the rule does not decrease social welfare and decreases the size of the pool.*
- (b) *If rule U_0 is not applicable, then for any source i of G_X and good $g \in P$, there is an agent $j \neq i$ such that $v_j(X_i \cup g) > v_j(X_j)$. In particular, an inclusion-wise minimal envied subset of $X_i \cup g$ has size at most $|X_i|$.*

Proof. The first part of (a) follows directly from the precondition of the rule. The second part holds since the valuations are monotone and because $|P'| = |P| - 1$.

The first sentence in part (b) is obvious. We come to the second sentence. Since adding g to X_i destroys the EFX-property, there must be some $g' \in X_i \cup g$ and some $j \in [n]$ such that $v_j(X_i \cup g \setminus g') > v_j(X_j)$ for some $j \in [n]$. Thus an inclusion-wise minimal envied subset of $X_i \cup g$ has size at most $|X_i|$. \square

Rule U_1 : Rule U_1 is applicable whenever there is an agent who values the pool higher than her current bundle (see Algorithm 2.2).

LEMMA 2.5 (Rule U_1). *Rule U_1 increases the social welfare and returns an EFX allocation.*

Proof. Since there is an agent who values the pool higher than her own bundle, there is an inclusion-wise minimal envied subset Z of P and an agent i such that $v_i(X_i) < v_i(Z)$. Let X' be the allocation defined in Algorithm 2.2, line 7. Then $v_i(X'_i) > v_i(X_i)$ and $v_j(X'_j) = v_j(X_j)$ for $j \neq i$. Thus $\phi(X') > \phi(X)$. It remains to show that the allocation X' is EFX, i.e., for every pair of agents j and k and any good $g \in X'_k$, we have $v_j(X'_k \setminus g) \leq v_j(X'_j)$.

Since X is EFX, this is obvious if neither j nor k is equal to i . If $j = i$, then $v_i(X'_i) > v_i(X_i) \geq v_i(X_k \setminus g) = v_i(X'_k \setminus g)$ for all $g \in X'_k$ (or equivalently $g \in X_k$). Finally, we consider $k = i$. Since Z is an inclusion-wise minimal envied subset of P , we have $v_j(X'_j) = v_j(X_j) \geq v_j(Z \setminus g) = v_j(X'_i \setminus g)$ for any $g \in Z$, where $v_j(X_j) \geq v_j(Z \setminus g)$

Algorithm 2.2 The Update Rules

-
- 1: **function** U_0 (allocation X , pool P)
Precondition: There is a good $g \in P$ and an agent i such that allocating g to i results in an EFX allocation.
 - 2: Allocate g to i , i.e., $X'_i \leftarrow X_i \cup g$, $P' \leftarrow P \setminus g$, and $X'_j = X_j$ for $j \neq i$.
 - 3: **return** (X', P') .
 - 4: **end function**

 - 5: **function** U_1 (allocation X , pool P)
Precondition: There is an agent k such that $v_k(P) > v_k(X_k)$.
 - 6: Let i be a most envious agent of P ; let Z be an inclusion-wise minimal envied subset of P such that i satisfies $v_i(Z) > v_i(X_i)$.
 - 7: Set $X'_i = Z$ and $X'_j = X_j$ for $j \neq i$.
 - 8: Set $P' = X_i \cup (P \setminus Z)$.
 - 9: **return** (X', P') .
 - 10: **end function**

 - 11: **function** U_2 (allocation X , pool P)
Precondition: There is an $\ell \geq 1$, distinct goods $g_0, g_1, \dots, g_{\ell-1}$ in P , distinct sources $s_0, s_1, \dots, s_{\ell-1}$ of G_X and distinct agents $t_0, t_1, \dots, t_{\ell-1}$ such that t_i is a most envious agent of $X_{s_i} \cup g_i$ and $t_i \in C(s_{i+1})$ for $0 \leq i \leq \ell - 1$ (indices are to be interpreted modulo ℓ).
 - 12: Let Z_i be an inclusion-wise minimal envied subset of $X_{s_i} \cup g_i$ such that $v_{t_i}(Z_i) > v_{t_i}(X_{t_i})$ for $0 \leq i \leq \ell - 1$.
 - 13: Set $P' = (P \setminus \bigcup_{i=0}^{\ell-1} \{g_i\}) \bigcup_{i=0}^{\ell-1} ((X_{s_i} \cup g_i) \setminus Z_i)$.
 - 14: Let $u_0^i \rightarrow \dots \rightarrow u_{m_i}^i$ be the path of length m_i from $s_i = u_0^i$ to $t_{i-1} = u_{m_i}^i$ in $C(s_i)$ for $0 \leq i \leq \ell - 1$.
 - 15: Set $X'_{u_k^i} = X_{u_{k+1}^i}$ for all $k \in \{0, \dots, m_i - 1\}$ and all $i \in \{0, \dots, \ell - 1\}$.
 - 16: Set $X'_{t_i} = Z_i$ for all $i \in \{0, \dots, \ell - 1\}$.
 - 17: Set $X'_j = X_j$ for all other j .
 - 18: **return** (X', P') .
 - 19: **end function**
-

follows from the inclusion-wise minimality of Z as an envied set. \square

Rule U_2 : Rule U_2 is our most complex rule. It is applicable if for some $\ell \geq 1$, there are distinct goods $g_0, g_1, \dots, g_{\ell-1}$ in P , distinct sources $s_0, s_1, \dots, s_{\ell-1}$ of G_X and distinct agents $t_0, t_1, \dots, t_{\ell-1}$ such that for each i : (1) t_i is a most envious agent of $X_{s_i} \cup g_i$ and (2) t_i is reachable from s_{i+1} (indices are to be interpreted modulo ℓ). We first show that rule U_2 is always applicable when rule U_0 is not applicable and the pool contains at least n goods.

LEMMA 2.6. *If $|P| \geq n$ and rule U_0 is not applicable then there is an $\ell \geq 1$, distinct goods $g_0, g_1, \dots, g_{\ell-1}$ in P , distinct sources $s_0, s_1, \dots, s_{\ell-1}$ of G_X , and distinct agents $t_0, t_1, \dots, t_{\ell-1}$ such that t_i is a most envious agent of $X_{s_i} \cup g_i$ and $t_i \in C(s_{i+1})$ for $i \in \{0, \dots, \ell - 1\}$ (indices are modulo ℓ).*

Proof. Since rule U_0 is not applicable, for every source s of G_X and every good $g \in P$, we have $v_j(S') > v_j(X_j)$ for some $S' \subset X_s \cup g$ and $j \in [n]$. Let s_0 be an arbitrary source of G_X and g_0 be an arbitrary good in P . Construct a sequence of

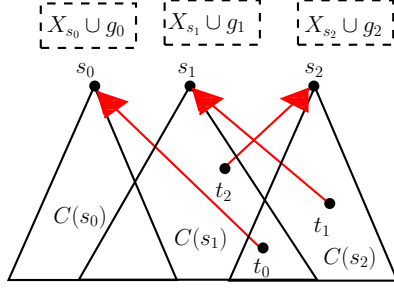


FIG. 3. We have t_i as a most envious agent of $X_{s_i} \cup g_i$. Recall that $C(s_i)$ is the set of all agents reachable from s in the envy-graph. Neither $t_0 \in C(s_0)$ nor $t_1 \in C(s_0) \cup C(s_1)$. We have $t_2 \in C(s_0) \cup C(s_1) \cup C(s_2)$. Note that $j = 1$ is the largest index in $\{0, 1, 2\}$ such that $t_2 \in C(s_j)$. The cycle (see the proof of Lemma 2.6) is defined by s_1, s_2, g_1, g_2, t_1 , and t_2 .

triples (s_{i+1}, g_{i+1}, t_i) , $i \geq 0$ defined as follows. Assume we have defined s_i and g_i . Let t_i be a most envious agent of $X_{s_i} \cup g_i$. If $t_i \notin C(s_0) \cup \dots \cup C(s_i)$, let s_{i+1} be a source such that $t_i \in C(s_{i+1})$ and let g_{i+1} be a good in P distinct from g_0 to g_i . If $t_i \in C(s_0) \cup \dots \cup C(s_i)$ then stop the construction of the sequence and let j be the maximum index in $\{0, \dots, i\}$ such that $t_i \in C(s_j)$. Set $\ell = i - j + 1$ and return $s_j, \dots, s_i, g_j, \dots, g_i$ and t_j, \dots, t_i ,⁹ see Figure 3 for an illustration.

The construction is well-defined since $|P| \geq n$ and hence we cannot run out of goods. The sources and goods are pairwise distinct by construction. The agents t_0 to t_{i-1} are also distinct by construction. Finally, agent t_i is distinct from any agent t_k for $j \leq k < i$ since $t_i \in C(s_j)$ and by definition, $t_k \notin C(s_0) \cup \dots \cup C(s_k)$ and so $t_k \notin C(s_j)$. \square

For each i , let $u_0^i \rightarrow u_1^i \rightarrow \dots \rightarrow u_{m_i}^i$ be the path of length m_i from $s_i = u_0^i$ to $t_{i-1} = u_{m_i}^i$ in $C(s_i)$. Rule U_2 assigns (i) $X'_{u_k^i} = X_{u_{k+1}^i}$ for all $k \in \{0, \dots, m_i - 1\}$ and all $i \in \{0, \dots, \ell - 1\}$ and (ii) $X'_{t_i} = Z_i$ for all $i \in \{0, \dots, \ell - 1\}$, where Z_i is defined in Algorithm 2.2 (see line 12). For all other j , we have $X'_j = X_j$.

LEMMA 2.7 (Rule U_2). *Rule U_2 increases social welfare and returns an EFX allocation.*

Proof. We first observe that the valuations of the agents for their bundles have either increased or remained the same (since either the agents are left with their old bundles or assigned bundles that they envied). In particular, the valuations of all the agents in $\bigcup_{i=0}^{\ell-1} \bigcup_{k=0}^{m_i} \{u_k^i\}$ for their bundles are strictly larger, where the vertices u_k^i are defined above. Thus $\phi(X') > \phi(X)$.

It remains to show that the allocation X' is EFX, i.e., for every pair of agents j and k and any good $g \in X'_k$ we have $v_j(X'_k \setminus g) \leq v_j(X'_j)$. Let $T = \{t_0, t_1, \dots, t_{\ell-1}\}$. For every agent $k \notin T$ we have $X'_k = X_{k'}$ for some k' . Now consider two cases depending on k :

- $k \notin T$: Note that the agents' valuation for their current bundle (in X') is at least as good as their valuation for their old bundle (in X). We have $v_j(X'_j) \geq v_j(X_j) \geq v_j(X_{k'} \setminus g) = v_j(X'_k \setminus g)$ for any $g \in X'_k$ (or equivalently, $g \in X_{k'}$).
- $k \in T$: Let $k = t_i$ for some i . We have $v_j(X'_j) \geq v_j(X_j) \geq v_j(Z_i \setminus g)$ for

⁹We took j to be the maximum index such that $t_i \in C(s_j)$ so that the cycle defined by $s_j, \dots, s_i, g_j, \dots, g_i$ and t_j, \dots, t_i is simple.

any $g \in Z_i$ (by the inclusion-wise minimality of Z_i as an envied set) and $v_j(Z_i \setminus g) = v_j(X'_{i_i} \setminus g) = v_j(X'_k \setminus g)$ for any $g \in X'_k$. \square

We can now summarize. Let $V = \max_i v_i(M)$ be the maximum valuation of any agent and $\Delta = \min_i \min\{|v_i(T) - v_i(S)| : S, T \subseteq M \text{ and } v_i(S) \neq v_i(T)\}$ be the minimum difference between distinct valuations. Each application of rule U_1 or rule U_2 increases the social welfare by at least Δ and hence there can be at most nV/Δ applications of these rules. Each application of rule U_0 decreases the size of the pool by one and so there can be at most $m - 1$ successive applications of rule U_0 at any intermediate stage in the algorithm. Hence we can conclude that the number of iterations of Algorithm 2.1 is at most $mnV/\Delta + m$. Thus we have shown the following theorem.

THEOREM 2.8. *For monotone valuations, there is always an allocation X and a pool P of unallocated goods such that*

- X is EFX,
- $v_i(X_i) \geq v_i(P)$ for all agents i , and
- $|P|$ is less than the number of sources in the envy-graph; in particular, we have $|P| < n$.

Algorithm 2.1 determines such an allocation in at most $mnV/\Delta + m$ iterations.

2.1. Finding an inclusion-wise minimal envied subset. We now describe how to implement a *minimal-envied-subset* oracle efficiently, i.e., given an envied set $S \subseteq M$, we need to efficiently find a subset Z of S that is envied by some agent and no proper subset of Z is envied by any agent. Algorithm 2.3 finds such a set $Z \subseteq S$ for monotone valuations.

Algorithm 2.3 Algorithm for finding an inclusion-wise minimal envied subset

Precondition: There is an agent who values $S \subseteq M$ more than her bundle.

Postcondition: $Z \subseteq S$ such that there is some agent i with $v_i(Z) > v_i(X_i)$ and we have $v_j(X_j) \geq v_j(Z')$ for all $Z' \subset Z$ and $j \in [n]$.

```

1:  $Z = S$ ;
2: for every agent  $i$  do
3:   for every good  $g \in Z$  do
4:     if  $v_i(X_i) < v_i(Z \setminus g)$  then
5:        $Z = Z \setminus g$ ;
6:     end if
7:   end for
8: end for
9: Return  $Z$ .
```

In Algorithm 2.3, if the “if condition” is not invoked for any agent i and good $g \in Z$, then Z is an inclusion-wise minimal envied subset. Suppose agent i envies $Z \setminus g$ for some $g \in Z$. Then we continue our check with $Z \setminus g$ as the new Z . Since the new Z is a subset of the old Z , observe that for each agent $i \in [n]$ and each good $g \in Z$, our algorithm needs to check exactly once if $v_i(X_i) \geq v_i(Z \setminus g)$ or not. Thus our algorithm makes nm value queries. We can conclude the following theorem.

THEOREM 2.9. *For monotone valuations, we can determine an allocation X and a pool of unallocated goods P that satisfies the 3 conditions in Theorem 2.8 using at most $mn \cdot (mnV/\Delta + m)$ value queries.*

2.2. An FPTAS to Determine an “Almost” Desired Allocation. Algorithm 2.1 is pseudo-polynomial, since the increase in individual valuations of the agents when we perform the update rules could be very small. Suppose we just wanted an “almost” EFX property, i.e., given any $\varepsilon > 0$, for every pair of agents i and j , we are happy to ensure that $(1 + \varepsilon) \cdot v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for all $g \in X_j$ and also $(1 + \varepsilon) \cdot v_i(X_i) \geq v_i(P)$ for every i . We now show an algorithm that determines such an allocation with $\text{poly}(n, m, 1/\varepsilon, \log(V/\Delta))$ many value queries.

This algorithm is obtained from our previous algorithm by making appropriate changes to the definitions of (i) inclusion-wise minimal envied subset, (ii) most envious agent, and (iii) update rules U_0 , U_1 , and U_2 . We now describe these changes.

Given a set S such that there is an agent k with $(1 + \varepsilon) \cdot v_k(X_k) < v_k(S)$, we say $Z \subseteq S$ is an inclusion-wise minimal envied subset if there is an agent i such that $(1 + \varepsilon) \cdot v_i(X_i) < v_i(Z)$ and for all agents $j \in [n]$, we have $(1 + \varepsilon) \cdot v_j(X_j) \geq v_j(Z')$ for any $Z' \subset Z$, i.e., for all proper subsets of Z . Agent i is called a most envious agent of the set S .

Note that an inclusion-wise minimal envied subset Z of S and a corresponding most envious agent (as defined above) can be determined using nm value queries by changing the “if” condition in Algorithm 2.3 from checking if $v_i(X_i) < v_i(Z \setminus g)$ to checking if $(1 + \varepsilon) \cdot v_i(X_i) < v_i(Z \setminus g)$. We now describe the appropriate changes to the update rules U_0 , U_1 , and U_2 .

- Rule U_0 : This rule is applicable whenever there is an unallocated good g and a source s in the envy-graph G_X such that adding g to s does not destroy the “almost” EFX property, i.e., for all $j \in [n] \setminus \{s\}$, we have $(1 + \varepsilon) \cdot v_j(X_j) \geq v_j((X_s \cup g) \setminus g')$ for all $g' \in X_s \cup g$. Rule U_0 allocates g to s (as done in the function U_0 in Algorithm 2.2).
- Rule U_1 : This rule is applicable whenever there is an agent i such that $(1 + \varepsilon) \cdot v_i(X_i) < v_i(P)$. Rule U_1 behaves exactly the same way as the function U_1 in Algorithm 2.2 – the only changes are in the modified definitions of a most envious agent and an inclusion-wise minimal envied subset.
- Rule U_2 : The preconditions of rule U_2 are exactly same as in Algorithm 2.2, but with the modified definition of a most envious agent. Rule U_2 behaves exactly the same way as the function U_2 in Algorithm 2.2 – the only changes are in the modified definitions of a most envious agent and an inclusion-wise minimal envied subset.

The algorithm. We now argue that with these changes, we can determine an “almost” EFX allocation with bounded charity using $\text{poly}(m, n, 1/\varepsilon, \log(V/\Delta))$ many value queries. Our algorithm always maintains an allocation where for every pair of agents i and j , we have $(1 + \varepsilon) \cdot v_i(X_i) \geq v_i(X_j \setminus g)$ for all $g \in X_j$. If any of the three update rules U_0, U_1, U_2 is applicable, then we apply that rule and decycle the envy-graph. Observe that application of any of these update rules returns an allocation X such that for every pair of agents (i, j) , we have $(1 + \varepsilon) \cdot v_i(X_i) \geq v_i(X_j \setminus g)$ for all $g \in X_j$. Thus we maintain the invariant of the algorithm.

Bounding the number of iterations. The application of any of these update rules and the subsequent envy-cycle elimination does not decrease the valuation of any agent for her bundle. Moreover, each application U_1 or U_2 will ensure that at least one agent’s valuation for her bundle either increases from 0 to Δ or by a multiplicative factor of $1 + \varepsilon$. So the total number of applications of U_1 and U_2 is $O(n \cdot \log_{1+\varepsilon}(V/\Delta))$. Since each application of U_0 reduces the size of the pool by one, we cannot have more than m consecutive applications of U_0 . Hence the total number of iterations of the

algorithm is $O(mn \cdot \log_{1+\varepsilon}(V/\Delta))$, which is $O((mn/\varepsilon) \cdot \log(V/\Delta))$.

When the preconditions of the update rules are no longer valid, we arrive at our desired allocation. Since each iteration involves $O(mn)$ value queries, we can determine our desired allocation with $O(m^2n^2/\varepsilon \cdot \log(V/\Delta))$ value queries. Thus we can conclude the following theorem.

THEOREM 2.10. *For monotone valuations, given any $\varepsilon > 0$, using $O(m^2n^2/\varepsilon \cdot \log(V/\Delta))$ value queries, we can find an allocation $X = \langle X_1, X_2, \dots, X_n \rangle$ and a pool P of unallocated goods such that*

- for any pair of agents i and j , we have $(1 + \varepsilon) \cdot v_i(X_i) \geq v_i(X_j \setminus g)$ for all $g \in X_j$,
- for any agent i , we have $(1 + \varepsilon) \cdot v_i(X_i) \geq v_i(P)$, and
- $|P| < n$.¹⁰

2.3. New Proof of a Result from [30] for Identical Valuations. For agents with identical (monotone) valuations, it was shown by Plaut and Roughgarden [30] that an allocation that maximizes the minimum valuation, then maximizes the size of this bundle, then maximizes the second minimum valuation, then maximizes the size of this bundle, and so on is EFX. Thus a complete allocation that is EFX always exists when all the agents have identical valuations.

We now show that Algorithm 2.1 gives another proof of the above result. Recall that Algorithm 2.1 consists of applying three update rules U_0, U_1, U_2 – whichever of these is applicable. We will now show that when all the agents have identical valuations and rule U_0 is not applicable, then the precondition of rule U_2 is satisfied as long as there is some unallocated good. Let $X = \langle X_1, \dots, X_n \rangle$ be the current allocation and $P = M \setminus \cup_{i=1}^n X_i$ be the set of unallocated goods in X .

LEMMA 2.11. *Let s be any source vertex in the envy-graph G_X . If $|P| \geq 1$ and rule U_0 is not applicable then s can be chosen as a most envious agent of $X_s \cup g$ for any $g \in P$.*

Proof. Let $g \in P$ and s be any source in the envy-graph G_X . Since rule U_0 is not applicable, there is an agent t such that $v(X_t) < v(S')$ for some $S' \subset S = X_s \cup g$, where v is the common valuation function of all agents. Let $Z \subset S$ be an inclusion-wise minimal envied subset of S such that $v(X_t) < v(Z)$. Since s is a source in G_X , we have $v(X_s) \leq v(X_t)$. So $v(X_s) < v(Z)$; thus s can be chosen as a most envious agent of $X_s \cup g$. \square

Lemma 2.11 implies that while $P \neq \emptyset$, either rule U_0 or rule U_2 is applicable. Whenever we apply rule U_2 , we add any good g in P to the bundle of a source s in G_X and determine an inclusion-wise minimal envied set $Z \subset X_s \cup g$. Note that $v(Z) > v(X_s)$. We then throw the goods in $(X_s \cup g) \setminus Z$ back into the pool P and set $X_s = Z$. This makes agent s strictly better-off and no agent is worse-off, thus we have made progress. So when Algorithm 2.1 terminates, we have an EFX allocation with $P = \emptyset$. Thus we have a complete allocation $X = \langle X_1, \dots, X_n \rangle$ that is EFX.

3. Guarantees on Efficiency and Other Notions of Fairness. In this section we assume that all agents have additive valuations. We show that a minor variant

¹⁰Observe that $(1 + \varepsilon) \cdot v_i(X_i) \geq v_i(X_j \setminus g)$ for all $g \in X_j$, implies that $v_i(X_i) \geq (1 - \varepsilon) \cdot v_i(X_j \setminus g)$ for all $g \in X_j$ and $(1 + \varepsilon) \cdot v_i(X_i) \geq v_i(P)$ implies that $v_i(X_i) \geq (1 - \varepsilon) \cdot v_i(P)$ (as $\frac{1}{1+\varepsilon} \geq 1 - \varepsilon$). Therefore, Theorem 2.10 implies that we can determine a $(1 - \varepsilon)$ -EFX allocation with bounded charity with $O\left(\frac{n^2m^2}{\varepsilon} \cdot \log\left(\frac{V}{\Delta}\right)\right)$ value queries.

of our algorithm finds an allocation that satisfies all the conditions of Theorem 2.8 along with a good guarantee on Nash social welfare *and* maximin share (MMS).

Guarantee of High Nash Social Welfare. We claimed in Section 1 that for additive valuations, it can also be ensured that for each i , we have $v_i(X_i) \geq \frac{1}{2} \cdot v_i(X_i^*)$ where $X^* = \langle X_1^*, \dots, X_n^* \rangle$ is an optimal Nash social welfare allocation and X is the allocation in Theorem 2.8. This is easy to see from Algorithm 2.1:

- rather than initialize $X_i = \emptyset$, we will initialize X_i to the bundle corresponding to the allocation determined by the algorithm in [14].

So we have $v_i(X_i) \geq \frac{1}{2} \cdot v_i(X_i^*)$ to begin with and this is a partial EFX allocation. As the algorithm progresses, our invariant is that $v_i(X_i)$ never decreases for any i . So if $X' = \langle X'_1, \dots, X'_n \rangle$ is the final allocation computed by our algorithm, then we have $v_i(X'_i) \geq \frac{1}{2} \cdot v_i(X_i^*)$ for all $i \in [n]$.

PROPOSITION 3.1. *Given a set N of agents with additive valuations and a set M of goods, there exists an allocation $X = \langle X_1, \dots, X_n \rangle$ and a pool P of unallocated goods that satisfy all the conditions of Theorem 2.8 and $v_i(X_i) \geq \frac{1}{2} v_i(X_i^*)$ for all $i \in N$, where $X^* = \langle X_1^*, \dots, X_n^* \rangle$ is an optimal Nash social welfare allocation.*

3.1. An Approximate MMS Allocation if P is Large. We now show that if $|P|$ (the number of unallocated goods in our allocation) is sufficiently large, then our EFX allocation X has a very good MMS guarantee. Recall that our algorithm continues till $|P|$ is smaller than the number of sources in the envy-graph G_X and recall that sources are *unenvied* agents. In particular, if $|P| = n - 1$, then the number of sources in G_X is n ; so no agent envies another. That is, for each i , we have $v_i(X_i) \geq v_i(X_j)$ for all $j \in [n]$. Moreover, $v_i(X_i) \geq v_i(P)$. So we have

$$\begin{aligned} v_i(X_i) &\geq \frac{v_i(M)}{n+1} \\ &= \left(1 + \frac{1}{n}\right)^{-1} \cdot \frac{v_i(M)}{n} \\ &\geq \left(1 + \frac{1}{n}\right)^{-1} \cdot MMS_i(n, M), \end{aligned}$$

where for every agent i , the inequality $MMS_i(n, M) \leq v_i(M)/n$ holds for additive valuations. We formalize the above intuition in Theorem 3.3. The following proposition from [23] will be useful. It states that if we exclude any set of agents and at most the same number of any goods from N and M , respectively, the maximin share of any remaining agent can only increase.

PROPOSITION 3.2 ([23]). *Let N be a set of n agents with additive valuations and M be a set of m goods. If $N' \subseteq N$ and $M' \subseteq M$ are such that $|N \setminus N'| \geq |M \setminus M'|$ then for any agent $i \in N'$, we have $MMS_i(n', M') \geq MMS_i(n, M)$ where $n' = |N'|$.*

THEOREM 3.3. *Given a set N of n agents with additive valuations and a set M of m goods, there exists an allocation $X = \langle X_1, \dots, X_n \rangle$ and set P of unallocated goods that satisfies:*

- the 3 conditions stated in Theorem 2.8;
- $v_i(X_i) \geq \frac{1}{2} v_i(X_i^*)$ for all $i \in N$, where X^* is an optimal Nash social welfare allocation;
- $v_i(X_i) \geq MMS_i(n, M) / (2 - \frac{k}{n})$ for every $i \in N$, where $k = |P|$.

Proof. Let (X, P) be the allocation guaranteed by Proposition 3.1. Hence the first two conditions given in the theorem statement are satisfied by (X, P) . So what we need to show now is that for any agent i , we have $v_i(X_i) \geq MMS_i(n, M) / (2 - \frac{k}{n})$.

We fix some agent i and let $N' \subseteq N$ be the set of agents consisting of all sources of G_X , agent i , and all other agents j with $|X_j| \geq 2$. Let M' be the set of goods allocated to the agents in N' . Observe that every agent in $N \setminus N'$ is allocated at most one good and so $|N \setminus N'| \geq |M \setminus (M' \cup P)|$. By Proposition 3.2, it holds that $MMS_i(n', M' \cup P) \geq MMS_i(n, M)$ where $n' = |N'|$. Thus, it suffices to show that $v_i(X_i) \geq MMS_i(n', M' \cup P) / (2 - \frac{k}{n})$.

Consider any agent $j \in N'$ with $|X_j| \geq 2$. Because X is EFX, it holds that $v_i(X_i) \geq v_i(X_j \setminus \{g\})$ for all $g \in X_j$. Since the valuations are additive, we have

$$v_i(X_i) \geq \left(1 - \frac{1}{|X_j|}\right) \cdot v_i(X_j) \geq \frac{1}{2} \cdot v_i(X_j).$$

We know the following inequalities hold:

$$(3.1) \quad v_i(X_i) \geq v_i(P),$$

$$(3.2) \quad v_i(X_i) \geq v_i(X_j) \text{ for all } j \text{ that were sources in } G_X,$$

$$(3.3) \quad 2v_i(X_i) \geq v_i(X_j) \text{ for all other } j \in N'.$$

Recall that the number of sources is at least $|P| + 1 = k + 1$. Summing up all inequalities in (3.1)-(3.3) and using the fact that v_i is additive, we have

$$(2(n' - (k + 1)) + k + 2) \cdot v_i(X_i) \geq v_i(M' \cup P).$$

Hence we have

$$\begin{aligned} v_i(X_i) &\geq \frac{v_i(M' \cup P)}{2n' - k} \\ &= \frac{v_i(M' \cup P)}{n'} \cdot \frac{n'}{2n' - k} \\ &\geq MMS_i(n', M' \cup P) \cdot \frac{n'}{2n' - k} \quad (\text{since } v_i \text{ is additive}) \\ &= MMS_i(n', M' \cup P) / (2 - \frac{k}{n'}) \\ &\geq MMS_i(n', M' \cup P) / (2 - \frac{k}{n}) \quad (\text{since } n' \leq n). \end{aligned}$$

Thus $v_i(X_i) \geq MMS_i(n, M) / (2 - \frac{k}{n})$ for every $i \in N$, where $k = |P|$. \square

3.2. An Improved Bound for Approximate-GMMS. As mentioned in Section 1, a new notion of fairness called *groupwise maximin share* (GMMS) was recently introduced by Barman et al. [7]. We formally define a GMMS allocation below.

DEFINITION 3.4. *Given a set N of n agents and a set M of m goods, an allocation $X = \langle X_1, X_2, \dots, X_n \rangle$ is α -GMMS if for every $N' \subseteq N$ and all $i \in N'$, we have:*

$$v_i(X_i) \geq \alpha \cdot MMS_i(n', \bigcup_{j \in N'} X_j) \text{ where } n' = |N'|.$$

Observe that a GMMS allocation is also an MMS allocation. Since MMS allocations do not always exist in a given instance [28], GMMS allocations also need not always exist. Interestingly, $\frac{1}{2}$ -GMMS allocations always exist [7]. We now describe

how to modify our allocation so as to result in a complete allocation that is $\frac{4}{7}$ -GMMS and also $\frac{1}{2}$ -EFX. An allocation $Y = \langle Y_1, \dots, Y_n \rangle$ is a $\frac{1}{2}$ -EFX if for any pair of agents i, j , we have $v_i(Y_i) \geq \frac{1}{2} \cdot v_i(Y_j \setminus \{g\})$ for any $g \in Y_j$. Plaut and Roughgarden [30] showed that when all agents have subadditive valuations, a $\frac{1}{2}$ -EFX allocation always exists. Theorem 2.8 along with the simple modification of allocation X to allocation Y described below gives an alternative proof that a $\frac{1}{2}$ -EFX allocation always exists for subadditive valuations.

Let $X = \langle X_1, \dots, X_n \rangle$ be the allocation and P be the pool of unallocated goods that satisfy the conditions of Proposition 3.1. Without loss of generality, assume that agent 1 is a source in the envy-graph G_X . Define the complete allocation $Y = \langle Y_1, \dots, Y_n \rangle$ as follows:

- * $Y_1 = X_1 \cup P$ and $Y_i = X_i$ for all $i \neq 1$.

Theorem 3.5 shows that Y is our desired allocation. The proof of Theorem 3.5 is similar to [1, Proposition 3.4]. We also remark that one can use the proof of [1, Proposition 3.4] to show that any EFX allocation is $4/7$ -GMMS. However note that Y is not necessarily an EFX allocation. But it has sufficiently nice properties so that we can still show that it is $4/7$ -GMMS. For the sake of convenience, we will refer to goods in the proof of Theorem 3.5 as *items*.

THEOREM 3.5. *Given a set N of n agents with additive valuations and a set M of m items, the allocation Y defined above satisfies:*

- Y is $\frac{4}{7}$ -GMMS and $\frac{1}{2}$ -EFX.
- $v_i(Y_i) \geq \frac{1}{2}v_i(X_i^*)$ for all $i \in N$ where X^* is the optimal Nash social welfare allocation.¹¹

Proof. Observe that the bound on Nash social welfare holds for allocation X and thus for allocation Y (since $v_i(Y_i) \geq v_i(X_i)$ for all $i \in [n]$). The proof that Y is $\frac{1}{2}$ -EFX is straightforward. Recall that $Y_1 = X_1 \cup P$ and $v_i(X_i) \geq v_i(X_1)$ and $v_i(X_i) \geq v_j(P)$ for all $i \in [n]$. Thus for any subadditive valuation v_i , we have $v_i(Y_1) \leq v_i(X_1) + v_i(P) \leq 2v_i(Y_i)$. For $j \neq 1$, we have $v_i(Y_i) \geq v_i(X_i) \geq v_i(X_j \setminus g) = v_i(Y_j \setminus g)$ for all $g \in Y_j$. Hence Y is $\frac{1}{2}$ -EFX.

What we need to show now is the guarantee on GMMS. That is, we need to show that for every $\tilde{N} \subseteq N$ and all $i \in \tilde{N}$, we have $v_i(Y_i) \geq \frac{4}{7}MMS_i(\tilde{n}, \tilde{M})$ where $\tilde{n} = |\tilde{N}|$ and $\tilde{M} = \bigcup_{j \in \tilde{N}} Y_j$.

Fix some $i \in \tilde{N}$. Define N' to be the subset of \tilde{N} that contains i and all agents that have been allocated at least two items in Y , i.e., $j \in N'$ if and only if $j = i$ or $|Y_j| \geq 2$. Let $M' = \bigcup_{j \in N'} Y_j$.

Note that Y allocates all items of $\tilde{M} \setminus M'$ to agents in $\tilde{N} \setminus N'$. Since every agent in $\tilde{N} \setminus N'$ has been allocated at most one item, we have $|\tilde{N} \setminus N'| \geq |\tilde{M} \setminus M'|$. Proposition 3.2 tells us that $MMS_i(n', M') \geq MMS_i(\tilde{n}, \tilde{M})$ where $n' = |N'|$. Thus it suffices to show $v_i(Y_i) \geq 4/7 \cdot MMS_i(n', M')$.

For $j \in N'$, call a bundle Y_j *good* if $j \in \{1, i\}$ or $|Y_j| = 3$; call it *bad* otherwise. Thus Y_j is bad if $j \notin \{1, i\}$ and $|Y_j| = 2$. Call items in good bundles “good” and items in bad bundles “bad”. Let x be the number of bad items in M' . Since the bad items are contained in bundles of size two, the good items in M' come from $n' - x/2$ good bundles of Y . As long as $x > n'$, we will apply a reduction step. Each reduction step

¹¹Related results were independently obtained by Garg and Taki [24] where it was shown that for **additive** valuations, there is an EFX-allocation after donating at most $n - 1$ items to charity. However, no bound on the value of the items donated to charity was shown. Thus a $4/7$ -GMMS-allocation after removing $n - 1$ items from the original set of items was obtained by them.

will reduce the number of bad items in M' by two, the number of agents by one, will not decrease the MMS_i -value, and will leave the quantity $n' - x/2$ and set of good items in M' unchanged.

Let $Z = \langle Z_1, Z_2, \dots, Z_{n'} \rangle$ be an optimal MMS partition for agent i of the set M' of items. If there are more than n' bad items in M' , then there is a set Z_k with at least *two* bad items, say g_1 and g_2 . We distribute the items in $Z_k \setminus \{g_1, g_2\}$ arbitrarily among the other sets in Z . So we have a partition of the set $M' \setminus \{g_1, g_2\}$ of items into $n' - 1$ many bundles. The value for agent i of any remaining bundle did not decrease. We set M' to $M' \setminus \{g_1, g_2\}$ and decrement n' . Note that we reduced the number of bad items by two, the number of agents by one, did not decrease $MMS_i(n', M')$ and the good items in M' still come from the $n' - x/2$ good bundles in Y . We keep repeating this reduction until M' contains at most n' bad items. At this point, we have a set M' of items and an integer n' with the following properties:

1. The number x of bad items in M' is at most n' .
2. $MMS_i(n', M') \geq MMS_i(\tilde{n}, \tilde{M})$, and
3. The set of good items in M' has not changed. The good items come from $n' - x/2$ good bundles in Y .

We will next relate the value of good and bad items to the value of Y_i .

CLAIM 3.6. *We have*

(a) *For any bad item g , $v_i(g) \leq v_i(Y_i)$.*

(b) *If $j \neq 1$ and Y_j is a good bundle then $v_i(Y_j) \leq 3/2 \cdot v_i(Y_i)$.*

The proof of Claim 3.6 is given below. Now we are ready to show the bound on GMMS. We will show that $v_i(Y_i) \geq \frac{4}{7}v_i(M')/n'$. Since $v_i(M')/n' \geq MMS_i(n', M') \geq MMS_i(\tilde{n}, \tilde{M})$, we get the desired bound.

We have x bad items in M' . The good items in M' come from $n' - x/2$ good bundles. Also $x \leq n'$. The total value of the good items for agent i is at most

$$\frac{3}{2} \left(n' - \frac{x}{2} - 2 \right) \cdot v_i(Y_i) + 2v_i(Y_i) + v_i(Y_i) = \frac{3}{2} \left(n' - \frac{x}{2} \right) \cdot v_i(Y_i).$$

This is because either (i) $1 \in N'$ and then there are $n' - x/2 - 2$ good bundles different from Y_1 and Y_i or (ii) $1 \notin N'$ and so there are $n' - x/2 - 1$ good bundles different from Y_i . We know from Claim 3.6 that each good bundle Y_j , where $j \neq 1$, has value at most $3/2 \cdot v_i(Y_i)$ and we showed earlier that Y_1 has value at most $2v_i(Y_i)$.

Also, the total value of the bad items for agent i is at most $x \cdot v_i(Y_i)$, since there are x many bad items and each bad item is worth at most $v_i(Y_i)$ (by Claim 3.6). Therefore,

$$\begin{aligned} v_i(M') &= v_i(\text{bad items in } M') + v_i(\text{good items in } M') \\ &\leq x \cdot v_i(Y_i) + \frac{3}{2} \left(n' - \frac{x}{2} \right) \cdot v_i(Y_i) \\ &= \frac{6n' + x}{4} \cdot v_i(Y_i) \\ &\leq \frac{7n'}{4} \cdot v_i(Y_i). \end{aligned} \quad \square$$

Proof of Claim 3.6. We prove (a) and (b) below.

(a) Let $Y_j = \{g, g'\}$ be the bundle containing g . Since $j \neq 1$ (by the definition of a bad item), $v_i(Y_i) \geq v_i(X_i) \geq v_i(X_j \setminus g') = v_i(Y_j \setminus g') = v_i(g)$.

(b) Let $g \in Y_j$ be such that $v_i(g)$ is minimal. Then $v_i(Y_j \setminus g) \leq v_i(Y_i)$ and $v_i(g) \leq v_i(Y_j)/|Y_j|$. Thus

$$\begin{aligned} v_i(Y_i) &\geq \left(1 - \frac{1}{|Y_j|}\right) \cdot v_i(Y_j) \geq \left(1 - \frac{1}{3}\right) \cdot v_i(Y_j) \\ &= \frac{2}{3} \cdot v_i(Y_j). \end{aligned}$$

This finishes the proof of Claim 3.6. \square

4. Conclusions and Open Problems. We studied the existence of EFX allocations when agents have monotone valuations. We showed that we can ensure such an allocation always exists when we donate a small number of goods that nobody envies to charity. The major open problem here is whether complete EFX allocations always exist. Plaut and Roughgarden [30] remarked that an instance with *no* complete EFX allocation may be easier to find in the general setting of monotone valuations. Our result on “almost-EFX” allocations for monotone valuations allows one to hope that complete EFX allocations always exist, at least for more structured valuations such as additive.

We also showed that we get guarantees in terms of other notions of fairness when agents have additive valuations. To the best of our knowledge, allocations with good guarantees (i.e., constant factor approximation) on Nash social welfare and MMS (as well as GMMS) were not known prior to our work. It would also be interesting to investigate whether these guarantees can be improved or if instances can be constructed where our guarantees are tight. We believe that our work is just the beginning towards determining an allocation that gives good guarantees with respect to several notions of fairness: an allocation that is *universally* fair.

Very recently, Amanatidis et al. [3] have announced an allocation that has good approximation guarantees simultaneously with respect to four notions of fairness when valuation functions are additive: in particular, their allocation is $(\phi - 1)$ -EFX and $2/(\phi + 2)$ -GMMS, where $\phi \approx 1.618$ is the golden ratio. Moreover, a fine-tuned version of their algorithm also achieves $4/7$ -GMMS which matches our result (see Theorem 3.5). They also show that for additive valuations, when m is at most $n + 2$, where m is the number of goods and n is the number of agents, GMMS (and hence, EFX) allocations always exist. Even more recently, Berger et al. [10] showed that when agents have additive valuations, an EFX allocation (X, P) where $|P| \leq n - 2$ always exists; moreover, when $n = 4$, $|P| \leq 1$, i.e., at most one good is unallocated.

Acknowledgments. We are grateful to the reviewers for their very helpful comments and suggestions. We also thank the reviewers of the conference version of our paper in SODA 2020 for their helpful comments.

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