COSMOLOGICAL KILLING HORIZONS AND THEIR THERMODYNAMICS

PH.D. THESIS

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ABSTRACT

We derive new classes of cosmological solutions of $\mathcal{N} = 2$ supergravity containing planar Killing horizons and develop a novel treatment of the Euclidean action formalism suitable for deriving a thermodynamic partition function for solutions with a time-dependent exterior region.

We consider solutions to Einstein-Maxwell theory and $\mathcal{N} = 2$ supergravity coupled to three vector multiplets, known as the *STU model*. To obtain non-extremal solutions of the STU model, we solve the time-reduced field equations. Lifting back to four dimensions, the resulting static spacetime is incomplete, bounded by a curvature singularity on one side and a Killing horizon on the other. Analytic continuation reveals the existence of dynamic patches in the past and future, with the Kasner geometry recovered in the asymptotic limit. The global structure of the solutions to both Einstein-Maxwell theory and the STU model are shown to be the same. Restricting the integration constants in our solution to the STU model, we show that the scalar fields of the theory can be made constant, yielding the previously derived solutions of Einstein-Maxwell theory.

We find explicit lifts to five, six, ten and eleven dimensions which show that in the extremal limit, the underlying brane configuration is the same as for STU black holes. The extremal limit of the six-dimensional lift is shown to be BPS for special choices of the integration constants. We argue that there is a universal correspondence between spherically symmetric black hole solutions and planar cosmological solutions which can be illustrated using the Reissner-Nordström solution of Einstein-Maxwell theory.

We present a modified implementation of the Euclidean action formalism suitable for studying the thermodynamics of a class of cosmological solutions containing Killing horizons. To obtain a real metric of definite signature, we perform a *triple Wick-rotation* by analytically continuing all spacelike directions. The resulting Euclidean geometry is used to calculate the Euclidean on-shell action which defines a thermodynamic potential. The thermodynamic potential obtained can be used to define an internal energy that obeys the first law of thermodynamics. Our approach is complementary to, but consistent with the isolated horizon formalism.

We conclude with an outline of future work inspired by planar solutions in *Einstein-anti-Maxwell* theory where the sign of the Maxwell coupling is flipped. These solutions are planar black holes rather than cosmological solutions. We show that upon a standard Wick-rotation, the black hole solutions give rise to the same Euclidean action and thermodynamic relations as the planar solutions of Einstein-Maxwell theory. We understand this as an indication of a thermodynamic duality between distinct theories. Considering Einstein-anti-Maxwell theory as a consistent truncation of compactified type II* string theories, we propose that this duality can be generalised to the *anti-STU model* where the sign of the gauge coupling matrix is flipped.

DECLARATION OF ORIGINALITY

I hereby declare that all work described in this thesis is the result of my own research unless reference to others is given. None of this material has previously been submitted to this or any other university. All work was carried out in the Theoretical Physics Division of the Department of Mathematical Sciences, University of Liverpool, UK, during the period of October 2016 until December 2020.

For my family

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PUBLICATIONS

This thesis contains material which has appeared previously in the following publications by the author and collaborators

[1] J. Gutowski, T. Mohaupt, G. Pope, *From static to cosmological solutions of $\mathcal{N} = 2$ supergravity*, JHEP **08** (2019) 172 [1905.09167].

[2] J. Gutowski, T. Mohaupt, G. Pope, *Cosmological Solutions, a New Wick-Rotation, and the First Law of Thermodynamics*, JHEP **03** (2021) 293 [2008.06929].

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1

INTRODUCTION

Black holes are among the most wonderful and exotic objects to have come from theoretical physics. They appear naturally when we ask ourselves what happens when a body becomes so dense that the escape velocity is faster than the speed of light. In astrophysics, black holes appear as the final stage in the life cycle of certain stars, and in general relativity, they appear as some of the simplest solutions to Einstein's equations. In contemporary research, black holes have a central role in guiding us towards a consistent description of quantum gravity.

One of the most surprising mathematical results of general relativity is the appearance of curvature singularities from gravitational processes. In particular, Penrose's singularity theorem asserts that there is a curvature singularity within a trapping region of a spacetime [1]; we can roughly understand this as the assertion that a singularity necessarily lives at the centre of a black hole. In 2020, Penrose received the Nobel prize in physics 'for the discovery that black hole formation is a robust prediction of the general theory of relativity', a remarkable moment reflecting the growing acceptance from the wider academic community in the importance of mathematical research when understanding gravitation.

Penrose's award is preceded by two experimental results which have bookended the work undertaken during our research. In 2016, LIGO and VIRGO announced their joint results measuring gravitational waves emitted from the merger of a binary black hole system [2]. In 2019, the Event Horizon Telescope released a radio telescopic image of the shadow of a supermassive black hole [3]. Extraordinarily, this publication not only spread through the scientific community, but the black hole's shadow was printed onto the front page of newspapers across the world. More than ever, black holes are being understood and accepted as physical objects outside of our imagination while still motivating some of our most abstract and theoretical work.

Classically, we understand a black hole as a region of spacetime from which nothing can escape. As such, the development of black hole thermodynamics is certainly one of the most surprising and interesting twists in the history of black hole physics. The story begins with Bekenstein's conjecture that a black hole has entropy proportional to its surface area [4], motivated by Hawking's area law that states that for any classical physical process, the area of a black hole weakly increases [5]. Hawking's law sits within a group of four geometric laws known as the 'laws of black hole mechanics' which are mathematically rigorous statements about black holes and Killing horizons.[1] Hawking showed that for stationary black hole solutions, the event horizon of

[1] A Killing horizon is a null hypersurface with a normal vector which is a solution to Killing's equation.

a black hole was a Killing horizon [6].

Not long after Bekenstein's conjecture, Hawking set the constant of proportionality by considering a quantum field theory in a curved spacetime background. Hawking was able to show that an observer external to the surface of a black hole would detect the emission of thermal radiation [7] with a temperature[2] and corresponding entropy given by

$$T_H = \frac{\hbar\kappa}{2\pi c k_B}, \qquad S_{BH} = \frac{k_B c^3}{\hbar G}\frac{A}{4}.$$

These relations are particularly beautiful in how they bring together the fundamental constants of special relativity: the speed of light (c), gravitation: Newton's constant (G), quantum mechanics: the reduced Planck's constant ($\hbar$) and statistical mechanics: Boltzmann's constant (k_B). Incredibly, by considering a black hole semi-classically, Hawking was able to show that the 'region of no return' would emit energy and that eventually, the black hole would totally evaporate. The thermodynamic description of black holes led to the re-interpretation of the laws of black hole mechanics as the laws of thermodynamics. The parameters of a black hole solution, such as the mass, surface gravity and area now had thermodynamic interpretations as the internal energy, temperature and entropy respectively, and with this, a series of new thermodynamic and quantum mechanical questions could be asked of black holes.

In classical thermodynamic systems, when a system is considered macroscopically, the entropy measures the amount of energy in the system unable to do work. There is a second interpretation for thermodynamic systems from statistical mechanics, in which the entropy counts the number of microscopic configurations that would produce the same macroscopic system. The puzzle is then: if a black hole is a thermodynamic system, what are the microscopic degrees of freedom being counted by the Bekenstein-Hawking entropy? Another open problem is the *information paradox* [8]. The emission of Hawking radiation causes a black hole to lose energy and eventually totally evaporate. The particles emitted are purely thermal and so carry no information about the system. As such, we can imagine a pure state entering into a black hole region only to eventually be emitted, with all information about the original state having been lost. This loss of information violates unitarity and seems to be a fundamental incompatibility between black holes and quantum mechanics.

The elephant in the room of contemporary theoretical physics is a consistent theory of quantum gravity. The beginning of the twentieth century brought about two incredibly fundamental theories: quantum mechanics and special relativity. The inconsistencies between Newtonian gravity and special relativity were immediately apparent, with the Newtonian potential predicting instantaneous changes. Led by the equivalence principle, Einstein put forward the theory of general relativity [9, 10]. Understanding gravitation as the curvature of a spacetime manifold gave new insight into experimental problems, such as providing the corrections to Mercury's orbit [11], but also predicted more exotic physics such as gravitational waves and black holes. However, despite the successes of general relativity, it is an effective field theory lacking the structure to explain gravitation in the smallest of scales.

[2] The Hawking temperature is proportional to the surface gravity κ which for now can be understood as a constant associated to a Killing horizon.

During the development of general relativity, there was a successful effort to incorporate special relativity into quantum mechanics, leading to quantum field theory. By the 1970s, the electromagnetic force, together with the strong and weak forces, had quantum field theory descriptions, and the unification of these theories produced the standard model of particle physics. The missing piece was gravity, which would not yield to a quantum field theory description due to unresolvable divergent behaviour when analysed perturbatively.

In our efforts to write down a theory for quantum gravity, we should expect black hole thermodynamics to play a central role in what questions we hope to be able to answer, and for the past fifty years, black holes have indeed been guiding our research. Most prominently, noticing that the entropy of a black hole was encoded by its area, rather than its volume, was the starting point for the holographic principle [12, 13], suggesting that a quantum description for gravity could be described as a boundary theory with one less dimension. This insight led to the development of the AdS/CFT conjecture [14, 15] which has produced an incredible amount of work for the past twenty-five years, relating gravitational systems in $(D+1)$ dimensions, to quantum field theories in D dimensions with a strong/weak coupling duality.

Superstring theory [16, 17] is the most prominent and hopeful candidate for a unified quantum theory. String theory describes particles as oscillation modes of a relativistic string and was initially put forward by Veneziano as a model for the scattering amplitudes of strongly interacting particles [18]. When a massless spin-two particle was found in the spectrum of the closed string, string theory began a new life as a possible theory of quantum gravity. In the early eighties, it was found that by including supersymmetry, the so-called superstring could be anomaly free [19, 20], leading to the 'first string revolution' and the realisation of a consistent, perturbative quantum theory including the graviton. However, there was still the problem of a non-perturbative description and black holes were out of string theory's scope.

In the mid-nineties, the 'second string revolution' began after the discovery of string dualities known as S- and T-duality, allowing us to understand the five once-distinct superstring theories as intimately related. It was shown that the strong coupling limit of one string theory was related to the weak coupling limit of another string theory, or eleven-dimensional supergravity, and with this insight came a way to study strongly coupled systems with a perturbative treatment of the dual theory. Furthermore, these dualities suggested that all superstring theories could be collected together into a single eleven-dimensional theory known as M-theory[3] with each theory appearing as an asymptotic expansion in different limits [22].

Of particular interest when studying non-perturbative effects in string theory are a set of string soliton solutions known as p-branes. These $(p+1)$-dimensional objects can be understood as higher-dimensional analogues of charged particles and are fundamental objects of string theory. When a p-brane additionally preserves some fraction of supersymmetry, they are referred to as BPS solitons and are stable. When considered as solutions to the field equations, the BPS p-branes have a simple geometric description.

[3] What the 'M' stands for in M-theory is a common footnote in texts on the subject, with suggestions such as matrix, magic, mystery, membrane and even mother suggested. Ultimately, no single answer is accepted (Witten offers three different answers up to the user's discretion). However, the silliest explanation I found is attributed to Sheldon Glashow, stating he wondered whether "'M' wasn't an upside-down 'W' for Witten" [21].

In this thesis, we are particularly interested in charged black hole solutions of $\mathcal{N} = 2$ supergravity and their thermodynamics. Here, $\mathcal{N}$ counts the number of conserved supercurrents and in four dimensions, there are $4\mathcal{N}$ real supercharges. Although supergravity exists as the low-energy effective field theory of string theory, the study of supergravity predates string theory and instead was first studied by realising the super Poincaré symmetries of supersymmetry as local (or gauged) symmetries. From the point of view of studying black hole solutions and their thermodynamics, we can think of supergravity as the extension of general relativity with additional massless fields, the appearance of which allow for more complex systems of solutions and with that, the potential for more exotic black hole geometries.

Since the bosonic sector of pure $\mathcal{N} = 2$ supergravity is Einstein-Maxwell theory, the Reissner-Nordström solution of general relativity is additionally a solution to $\mathcal{N} = 2$ supergravity. The black hole is parameterised by its mass M and charge Q, and provided that $M \geq Q$, the curvature singularity is hidden behind a Killing horizon. In the special case when $M = Q$, the Hawking temperature vanishes and the solution is said to be extremal. Viewed as a solution of $\mathcal{N} = 2$ supergravity, the extremal Reissner-Nordström solution is a BPS soliton.

Charged black holes have a dual, perturbative description in terms of p-branes of type IIA and type IIB string theory [23]. The embedding of BPS black hole solutions into string theory allows us to return to the question of a microscopic understanding of a black hole's entropy. As the p-branes are supersymmetric configurations, it is possible to perturbatively compute various properties which hold for all values of the coupling. The entropy of the p-brane system can be computed in this way, and to leading order, it can be shown that this matches with the Bekenstein-Hawking area law [24, 25].

The thermodynamic picture offered from string theory and its BPS configurations have the restriction that they necessarily describe black hole solutions with zero temperature.[4] Studying the full realisation of black hole thermodynamics requires non-zero temperature and so there are additional research questions from the supergravity perspective, searching for non-extremal solutions and comparing to known thermodynamic relationships.

Of the four laws of black hole mechanics, the third law is the least understood from both the black hole and thermodynamic perspective. The law comes in two forms which are known as the strict and weak versions. In the strict version, it is said that the zero-temperature limit corresponds to a system with vanishing entropy, whereas the weak version only demands minimal entropy in the extremal limit. From a black hole perspective, the strict third law would impose that the black hole vanishes in the extremal limit, which generally is not the case; take for example the extremal charged black holes with dual string theory realisations.

It is then an interesting question to search for black hole solutions in which the strict third law holds. Recently, a class of four-dimensional, non-extremal, planar symmetric black brane solutions of $\mathcal{N} = 2$ supergravity coupled to n_V vector multiplets were presented [28]. It was found in the extremal limit, the area density of the Killing horizon vanished, reproducing the so-called 'Nernst branes' of [29, 30] and therefore were examples of a black hole solutions obeying

[4] Although near-extremal solutions can be studied with leading order approximations describing the greybody factors of Hawking radiation [26, 27].

the strict third law of thermodynamics.

Finding non-extremal solutions of supergravity theories is generally difficult. A common method to find black hole solutions in supergravity is by employing the Killing spinor equations; a set of first-order equations, generally easier to solve than Einstein's equations. The conditions placed by the Killing spinor equations then restrict the geometry of the solution which can lead towards the derivation of exact, analytic solutions. The price to pay is that the Killing spinor equations ensure the solutions are BPS states and so necessarily have zero Hawking temperature.

The derivation of the non-extremal Nernst branes followed the work of [31, 32], which developed the c-map in a new formulation using special real coordinates. The four-dimensional solutions are dimensionally reduced over a timelike circle, and in the real formulation of special geometry, the resulting equations of motion are symplectically covariant. By making non-trivial restrictions on the field configurations, it is possible to find exact Euclidean instanton solutions that can be lifted back into four dimensions and interpreted. Application of this procedure has led to a series of new non-extremal black hole solutions of $\mathcal{N} = 2$ supergravity [31, 33, 28, 34].

In this thesis, we use the real formulation of special Kähler geometry to derive new classes of solutions of $\mathcal{N} = 2$ supergravity, generalising the work on the Nernst branes by allowing there to be multiple charges. We find that these generalisations no longer obey the strict third law of thermodynamics, but instead yield a vastly different causal structure, containing external spacetime regions which are time-dependent. The Killing horizons of these solutions are understood to be cosmological horizons, and we are given the opportunity to study the laws of black hole mechanics in non-stationary spacetimes. By restricting our solutions, we recover solutions for Einstein-Maxwell theory, which can be understood as the Reissner-Nordström solution, but with planar rather than spherical symmetry.

In the extremal limit, these cosmological solutions can be embedded into higher-dimensional supergravity, understood as p-brane configurations in ten or eleven dimensions. The cosmological solutions can be described thermodynamically, with an internal energy which is conserved and obeys the first law of thermodynamics. However, as the solution is non-stationary, the internal energy cannot be understood as a mass parameter as in conventional black hole thermodynamics.

Considering $\mathcal{N} = 2$ supergravity theories related by flipping the sign of gauge coupling, planar symmetric solutions are derived. It is shown that these distinct solutions have the same partition functions, realising thermodynamically dual Killing horizons. Understanding that these solutions are derived from theories related by T-duality, we can look towards string theory for a deeper understanding of this relationship. Again, we find evidence of black hole thermodynamics highlighting relationships between distinct physical theories. The verification of the first law for our cosmological Killing horizons suggests a deeper and more fundamental thermodynamic interpretation for Killing horizons departing from the conventional solutions first considered fifty years ago.

1.1 OUTLINE

The main content of this thesis is structured into two parts. In Part I, we concentrate on introducing the relevant background material needed to understand the research within this thesis. We split the background into three chapters. In Chapter 2, we introduce general relativity, assuming the reader is familiar with special relativity. The majority of the discussion is focused on the relevant differential geometry for the topic, and the conclusion of this chapter ties these areas together with the postulates of general relativity. We additionally include a discussion of the Lagrangian formulation of gravity. In Chapter 3, we discuss black holes in detail. We describe the geometry of Killing, trapping and event horizons. Using the Schwarzschild solution, we introduce the global structure of a black hole. We then use the Reissner-Nordström solution to give an in-depth discussion of deriving the black hole geometry from the field equations. We give a discussion of what it means to compute mass within general relativity and a general discussion of the laws of black hole mechanics and their relationship to thermodynamics. We conclude the chapter with the Euclidean action formalism, which is used to study the thermodynamic properties of the solutions within this thesis. In Chapter 4, we cover the necessary topics of supergravity. We begin by introducing supersymmetry through the extension of the Poincaré algebra. We build upon this, introducing the $\mathcal{N} = 2$ supergravity Lagrangians, the field content, and a few remarks on what we mean when we talk about supersymmetric black hole solutions. The electric-magnetic duality is discussed and its generalisation that appears for $\mathcal{N} = 2$ vector multiplet theories. Kaluza-Klein dimensional reduction is introduced and we use the double reduction of the STU model from six to four dimensions as an example. The c-map is then discussed, serving as a second example of dimensional reduction as well as introducing a key piece of the solution generating technique we use to find non-extremal solutions. Finally, we overview supergravity in higher dimensions, motivating it from the point of view of string/M-theory and discuss p-branes. We view these p-branes as solutions to the field equations and in particular we focus on the BPS solutions of p-branes and their intersections, relating this back to black hole solutions in lower dimensions.

In Part II, we present the results of this thesis: planar symmetric solutions of the field equations for Einstein-Maxwell theory and the STU model of $\mathcal{N} = 2$ supergravity, and their corresponding thermodynamics. In Chapter 5, we begin by making a planar symmetric, static ansatz for our geometry and find a solution of Einstein-Maxwell theory supported by an electric charge. Studying the static solution closely, we find that it contains a curvature singularity and a Killing horizon. We thus interpret the static patch as the *interior* of our solution in analogy with the interior of the Schwarzschild solution, or alternatively, the region behind the Cauchy horizon of the Reissner-Nordström solution. Analytic continuation through the Killing horizon leads to a second region which we interpret as the *exterior*. Here, the coordinates $\{t, r\}$ switch from timelike/spacelike to spacelike/timelike, and as such, the exterior region is dynamic (non-stationary). The explicit time-dependence of the exterior geometry leads to us naming the solution as a *cosmological* solution. The remainder of the chapter is split between studying properties of the static region of the spacetime — such as the motion of causal geodesics or the conserved charges of the solution — and the global spacetime structure of a generalised class of cosmological solutions.

Understanding the global structure leads to the classification of the horizons of the various solutions, which is vital for the discussion of the thermodynamics in later chapters. We conclude the chapter with a discussion of the extremal limit for the spacetime geometry. We notice that in the extremal limit, the location of the Killing horizon is 'pushed off' to infinity; the static region becomes spacetime filling and the dynamic region is of zero size. We also find the area density of the solution diverges, indicating infinite entropy for the extremal solution.[5] The remaining spacetime contains a timelike singularity without a Killing horizon and so is understood as a naked singularity solution.

In Chapter 6 we turn to study non-extremal, planar symmetric solutions of the STU model of $\mathcal{N} = 2$ supergravity. To solve the field equations, we begin with our four-dimensional theory and make an ansatz to impose staticity and planar symmetry. We then use the c-map, dimensionally reducing the solution over the timelike coordinate to obtain a three-dimensional Euclidean theory. Expressing this using the real formulation of special geometry, we write our field equations in a symplectically covariant manner. After making a restriction of the field content, we find an exact solution to the equations of motion and the Euclidean instanton solution is then uplifted back into four dimensions. Here, we impose regularity conditions on the solution to ensuring the existence of a Killing horizon with finite area density, and physical scalars with no divergent behaviour at the horizon. The resulting four-dimensional solutions are then studied through a series of coordinate changes and we find that qualitatively, the global structure of the solution is identical to that of the planar symmetric solutions of Einstein-Maxwell theory. We find that asymptotically the spacetime geometry is that of the Kasner type-D vacuum solution and that the extremal limit 'undresses' the solution, removing the Killing horizon and leaving behind a static solution with a naked singularity. We conclude the chapter by showing that through making a specific choice in our integration constants, the physical scalars of the theory can be made constant, and the solution simplifies to reproduce the solution of the Einstein-Maxwell theory.

In Chapter 7, we study the extremal limit of these cosmological solutions from a different perspective. We begin by uplifting the four-dimensional, non-extremal solutions into higher dimensions, finding solutions to consistent truncations of five, six, ten and eleven-dimensional supergravity. From the perspective of ten and eleven dimensions, we find that after taking the four-dimensional extremal limit, we can describe the cosmological solutions as smeared brane configurations with only small departures from the canonical examples given in Chapter 4. In six dimensions, we find that by taking the extremal limit together with a charge balancing condition, we recover supersymmetric solutions despite having made no assumptions about supersymmetry while solving the equations of motion in the previous chapter.

In Chapter 8, we present our research on verifying the first law of thermodynamics for our planar symmetric, cosmological solutions. The first law is a differential relationship between the internal energy of the solution and the other thermodynamic quantities. The crux of our discussion is how to obtain a properly normalised mass-like parameter to vary, as the static region of the spacetime is finite, containing a singularity, and the external region of the spacetime is

[5] This divergence occurs simultaneously with the horizon 'vanishing' into the asymptotic distance, and so the physical interpretation of this divergence is not obvious.

neither asymptotically flat nor stationary. We choose to employ the Euclidean action formalism, which is well suited to non-asymptotically flat solutions. The standard methodology of the Euclidean action formalism employs a Wick-rotation of the timelike coordinate, allowing us to study the Euclidean section of the geometry, and from a quantum-mechanical argument, we can derive a thermodynamic potential from the saddle-point approximation of the Euclidean action. From the thermodynamic potential, we can obtain an expression for the thermodynamic internal energy, which becomes the mass-like parameter we need. The issue we find for our classes of solutions is that a Wick-rotation within the static region does *not* produce a smooth Euclidean geometry (due to the singularity) and a Wick-rotation of the timelike coordinate in the exterior region produces a complex line element. To work around this, we obtain a smooth, real Euclidean geometry from the exterior region of the solution through Wick-rotating all three spacelike coordinates. We refer to this technique as the *triple Wick-rotation*. As a consistency check for this procedure, we use the de Sitter solution which contains a Killing horizon when written in static coordinates, but no singularity. We then use the Euclidean action formalism in both the static and dynamic regions of solution and verify the results are the same for both methods. We then turn to the planar symmetric solutions of this thesis to verify the first law of thermodynamics.

One last complication we encounter is how to properly normalise the Euclidean action. When computing the Euclidean action, usually a term is included which corresponds to the subtraction of the background contribution. For solutions which are asymptotically flat, the solution can be considered as isolated and the background subtraction comes from a boundary term computed by embedding the solution into Minkowski space. More generally, there are commonly divergent contributions to the action when evaluating in the asymptotic limit. In these cases, the divergences can removed by including a counter term built from geometric data of the boundary manifold. In fact, for asymptotically flat solutions we find a divergence too and the Minkowski background is the appropriate counter term. Removing the divergences uniquely determines the background subtraction and hence the normalisation of the action. When evaluating the Euclidean action for our planar symmetric solutions, we find there is no natural background or divergent contribution, and hence no natural subtraction term. To remedy this, we introduce a 'boundary condition' which ensures that the electric charge computed from the thermodynamic partition function matches that of the conserved charge computed from Gauss' law. This condition sets an overall numerical normalisation for our partition function, and from this, we find a consistent formulation of the first law of thermodynamics and Smarr's law for both the solutions of Einstein-Maxwell theory and the STU model. At the end of the chapter, we then cover an alternative procedure known as the isolated horizon formalism. This method assumes the form of the first law and derives all thermodynamic quantities from horizon data. We find that this method is consistent with our work from the triple Wick-rotation.

Finally, in Chapter 9, we offer some thoughts on how this research can be continued, focusing on a particular project in which partial progress has been made. The thesis ends with a summary of what we have presented and a few closing thoughts. Part III is dedicated to the appendices of the thesis, containing our conventions and some additional discussions and calculational details relegated from the main text.

Part I

BACKGROUND MATERIAL

2

GENERAL RELATIVITY

Einstein's theory of general relativity allows us to study gravitational physics through understanding how matter curves the spacetime we live within. To understand gravity, we must then understand what it means for a manifold to be curved and the mathematics we use to describe curvature. In this chapter, we introduce the reader to differential geometry, and the mathematical tools we will need while studying black holes. With the ground work of differential geometry, we motivate the postulates of general relativity from looking at the inconsistencies between Newtonian gravity and special relativity. A brief discussion of what it means to solve a gravitational system is given, which is followed by the Lagrangian formulation of general relativity.

The work contained in this section is based on Wald's 'General Relativity' [35], and some old lecture notes written by Reall [36, 37] from my time at the University of Cambridge.

2.1 MANIFOLDS

2.1.1 *Differential manifolds*

If we are to understand gravity through the curvature of spacetime, we must work to understand the spacetime manifold, and how we can perform calculations on it. One of the cornerstones of physics is the application of calculus, which is well defined on the manifold $\mathbb{R}^n$. The aim of differential geometry is to generalise calculus on spaces which we can only treat as $\mathbb{R}^n$ locally.

Beginning with a generic manifold M, if we consider some small, local patch $\mathcal{O} \subset M$, we can reasonably approximate $\mathcal{O}$ as a patch of flat space $\mathcal{U} \subset \mathbb{R}^n$, *i.e.* if we look at a small enough piece of something curved, it appears flat. If we are able to build our curved manifold from a set of overlapping, flat patches, we can perform calculus over the whole of the manifold; such a manifold is known as a *differential manifold*. The construction of a differential manifold is based on the requirement on correctly gluing together each local piece in a smooth way.

Definition 2.1. An n-dimensional differential manifold is a set M with a collection of subsets $\mathcal{O}_\alpha$ such that

(i) Every point in M is in at least one $\mathcal{O}_\alpha$ *i.e.* the collection $\{\mathcal{O}_\alpha\}$ covers M.

(ii) For each $\mathcal{O}_\alpha$, there is an isomorphic map $\phi_\alpha : \mathcal{O}_\alpha \to \mathcal{U}_\alpha$, where $\mathcal{U}_\alpha$ is an open subset of $\mathbb{R}^n$.

(iii) If any two subsets $\mathcal{O}_\alpha, \mathcal{O}_\beta$ overlap, there exists a smooth, bijective map $\phi_\beta \circ \phi_\alpha^{-1}$ which maps from $\mathcal{U}_\alpha \to \mathcal{U}_\beta$.

The maps ϕ_α are coordinate systems for the manifold and are sometimes referred to as charts. For the remainder of the thesis, a manifold M is assumed to be a differential manifold, and for this section we assume our manifold M has dimension n.

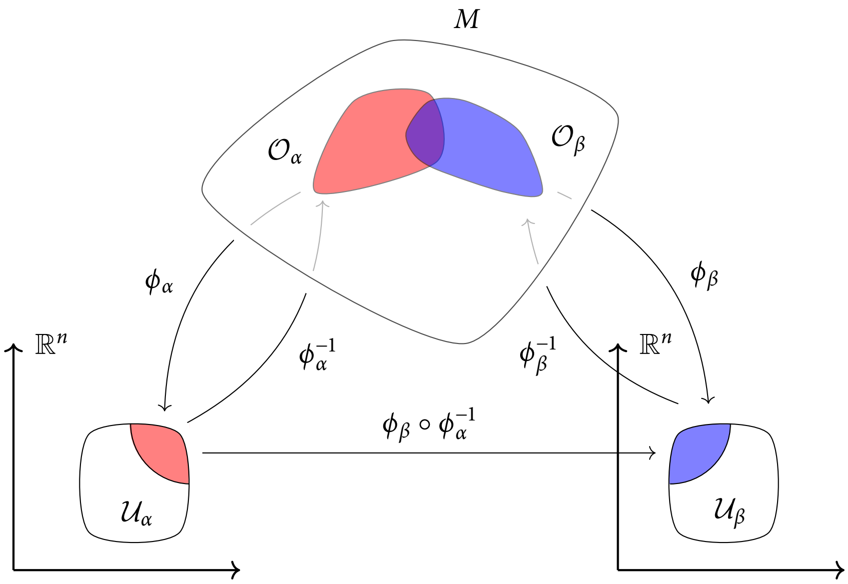


Figure 2.1: An illustration of the mapping $\phi_\beta \circ \phi_\alpha^{-1}$ of two coordinate systems overlapping. A differential manifold requires the mapping between the red and blue sections $\phi_\beta \circ \phi_\alpha^{-1} : \mathcal{U}_\alpha \to \mathcal{U}_\beta$ to be smooth.

2.1.2 *Curves, vectors & tensors*

Both Minkowski and Euclidean manifolds have the structure of a vector space. This means that if we consider some vector, such as the position or velocity of a particle on a flat manifold, we can construct the vector from the manifold itself. For a general manifold, this is not the case.

For a differential manifold, we recover a vector space by taking a point $p \in M$ and looking at all vectors tangent to this point. This defines a vector space $T_p(M)$ of dimension n, known as the *tangent space*. In flat space, the rate of change of a function along a curve at a point p is given as a directional derivative $X_p \cdot (\nabla f)_p$ for a vector X_p. We can carry this notion over to differential manifolds by studying the variation of a smooth curve, with some tangent vector $X_p \in T_p(M)$.

Definition 2.2. A smooth curve in a differential manifold M is a smooth function mapping $\lambda : I \to M$, for an open interval $I \in \mathbb{R}$. We say $\phi_\alpha \circ \lambda$ is a smooth map from $I \to \mathbb{R}^n$ for every chart ϕ_α.

Definition 2.3. Given a smooth curve $\lambda : I \to M$, the *tangent vector* to λ at a point p is a linear map from $M \to \mathbb{R}$ defined by:

$$X_p(f) = \frac{d}{dt}(f \circ \lambda) .$$

Introducing a local coordinate patch $\mathcal{U} \subset M$ with coordinates x^μ, for a curve $\lambda(t)$ parameterised by t, the components of the tangent vector are given by

$$X^\mu = \frac{dx^\mu}{dt}.$$

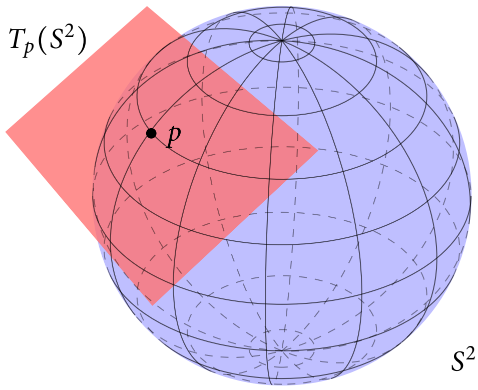


Figure 2.2: Illustration of the manifold S^2 and the tangent space $T_p(S^2)$ at the point p in the manifold.

As well as vectors, which usually are first introduced when studying mechanics, we also work with quantities which map vectors to numbers. These objects are known as *covectors*.

Definition 2.4. Let V be a finite dimensional, real vector space. The *dual vector space* V^* is the collection of linear maps from $V \to \mathbb{R}$.

When V is finite dimensional, the double dual V^{**} of a vector space is isomorphic to V. The dual space to the tangent space $T_p(M)$ is the *cotangent space* $T_p^*(M)$. An element in this space is a covector.

In physics, we not only deal with vectors and covectors but also tensors which are multilinear maps that produce a real number from a set of vectors and covectors. We first contact tensorial objects in mechanics with the energy-momentum tensor, or in electromagnetism with the field strength. In general relativity, the curvature of a spacetime is described using tensors.

Definition 2.5. A tensor of type (r, s) at the point p is a multilinear map:

$$T : \underbrace{T_p^*(M) \times \ldots \times T_p^*(M)}_{r-\text{times}} \times \underbrace{T_p(M) \times \ldots \times T_p(M)}_{s-\text{times}} \to \mathbb{R}.$$

The above discussion considered vectors, covectors and tensors defined at a point p in a manifold M. When considering physical systems, we are concerned with how these objects vary over the manifold. This leads to the concept of a vector field, where we assign a vector X_p for every $p \in M$.

Definition 2.6. A *vector field* is a map X which maps a point p in M to a vector X_p in such a way that X_p varies smoothly from point to point. Consider a smooth function f, for every point $p \in M$, there exists a function $X(f)$ which maps from the manifold $M \to \mathbb{R}$. We say the vector field is smooth if for a smooth function f, the function $X(f)$ is also smooth.

Following this definition, we can think of a covector field as assigning a covector to every point in our manifold and a rank (r, s) tensor field as assigning for each $p \in M$, a (r, s) tensor T_p at p.

Definition 2.7. Given a smooth vector field X, an *integral curve* γ of X, is a curve on M passing through a point p whose tangent everywhere is X.

In a coordinate chart, the existence of an integral curve γ parameterised by t can be written as

$$\frac{dx^\mu}{dt} = X^\mu(x(t)), \qquad x^\mu(0) = x^\mu_p.$$

There exists a unique solution to this differential equation, and hence there is an integral curve for all vector fields X through a point p.

2.1.3 *Metric tensor*

One tensor we will be particularly interested in is the *metric tensor*. Essentially, the metric tensor measures distance on a manifold. We construct the metric tensor as a bilinear scalar function on two vectors such that $g : T_p(M) \times T_p(M) \to \mathbb{R}$ which has the following properties:

Definition 2.8. A *metric tensor* g is a rank (0, 2) tensor which is

(i) Symmetric: $g(X,Y) = g(Y,X), \quad \forall X,Y \in T_p(M)$

(ii) Non-degenerate: $g(X,Y) = 0 \Leftrightarrow Y = 0, \quad \forall X \in T_p(M)$

In a coordinate basis, the metric tensor is given by

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu.$$

Commonly the metric tensor is abbreviated to

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu,$$

which makes more clear the interpretation of the metric as an *infinitesimal distance squared*.

The symmetry of the metric tensor ensures that it is possible to introduce a basis that diagonalises the metric. As the metric is non-degenerate, all diagonal elements will be non-zero and so the metric is guaranteed an inverse g^{-1} with components $g^{\mu\nu}$. An *orthonormal basis* can be found such that all diagonal elements of the metric tensor are ±1. There are many orthonormal bases, but the number of elements which are +1 or −1 is fixed and the collection of these signs fixes the *signature* of the metric.

Definition 2.9. A *pseudo-Riemannian manifold* is a pair (M, g), for a differential manifold M and metric tensor g.

In differential geometry, we are usually concerned with a metric g which is positive-definite. A pseudo-Riemannian manifold with a positive-definite metric g is a *Riemannian manifold*, *i.e.* the metric g has signature $\{+ + \ldots +\}$. In general relativity we consider a *spacetime* with the signature $\{- + \ldots +\}$, such a manifold is known as a *Lorentzian manifold*.[1] For a local patch of

[1] A n-dimensional Lorentzian manifold has a signature $\{\mp, \pm, \ldots, \pm\}$. In relativistic physics — and this thesis — the 'mostly plus' convention is used. In high-energy physics, the 'mostly minus' convention is used where the signature of the metric is $\{+ - \ldots -\}$.

a Riemannian (Lorentzian) manifold, the metric tensor is the flat space Euclidean (Minkowski) metric:

$$\delta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The metric tensor not only acts as a scalar product on two vectors, but also a linear mapping from vectors to covectors. For a pseudo-Riemannian manifold (M, g), given a vector X^μ, we can define a covector $\omega_\mu = g_{\mu\nu}X^\nu$. Similarly, we can define a vector from a covector ω_ν by $X^\mu = g^{\mu\nu}\omega_\nu$.

Definition 2.10. For a Lorentzian manifold (M, g) we classify vectors in the following way. Given a non-zero vector $X \in T_p(M)$, we say a vector is *timelike* if $g(X, X) < 0$, *null* if $g(X, X) = 0$ and *spacelike* if $g(X, X) > 0$.

This terminology follows to curves. We say that a curve is *timelike* when the tangent vector to the curve is everywhere timelike. The same goes for spacelike or null curves.

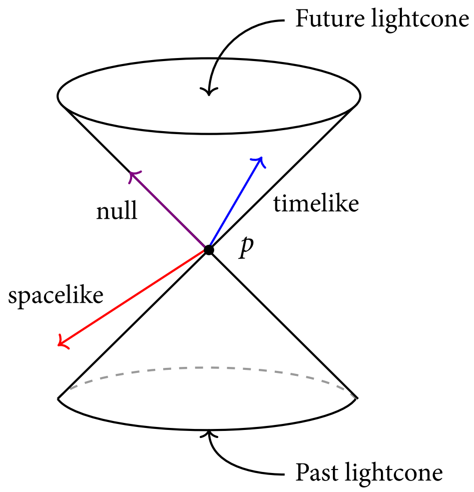


Figure 2.3: The lightcone illustrating the causal structure of a point p in a Lorentzian manifold. Null vectors run along the surface of the lightcone, shown in purple. All timelike vectors (blue) lie within the lightcone and spacelike vectors (red) are exterior to the cone.

Using the metric, we can now calculate the lengths of timelike and spacelike curves within a manifold. Note that null curves have zero 'length'. For a Riemannian manifold (M, g), we can calculate the length of a curve $\lambda : (a, b) \to M$ with tangent X

$$s = \int_a^b dt \sqrt{g(X, X)}. \tag{2.1.1}$$

This may remind the reader of the calculation of a length of a curve in $\mathbb{R}^n$

$$s = \int_a^b dt \sqrt{\frac{d\mathbf{x}}{dt} \cdot \frac{d\mathbf{x}}{dt}},$$

which is simply the case for which $g_{\mu\nu} = \delta_{\mu\nu}$ is the metric for Euclidean space.

For a Lorentzian manifold, the length of a spacelike curve is given by (2.1.1). When the curve λ is timelike, the length of the curve is called the *proper time* and is calculated from

$$\tau = \int_a^b du \sqrt{-g(X,X)}. \tag{2.1.2}$$

When a curve λ is parameterised by the proper time, the tangent to the curve is called the *four-velocity* of the curve. The curve is usually denoted by $u^\mu = dx^\mu/d\tau$. Looking infinitesimally at Equation (2.1.2), we have that

$$d\tau^2 = -g_{\mu\nu}u^\mu u^\nu \quad \Rightarrow \quad g_{\mu\nu}u^\mu u^\nu = -1,$$

and so u^μ is a unit timelike vector.

Given the notion of length along a curve on a manifold, a natural question is what is the length extremising curve between two points $p, q \in M$. In this discussion, we will restrict ourselves to timelike curves, and consider the Euler-Lagrange problem for maximising the proper length of a curve $\lambda(u)$ between two points $p, q \in M$. We write the problem as

$$\tau[\lambda] = \int_0^1 du\, L(x(u), \dot{x}(u)), \quad \lambda(0) = p,\ \lambda(1) = q, \quad \dot{x} = \frac{dx}{du}.$$

We can find the extremal curve by ensuring that the functional

$$L = \sqrt{-g_{\mu\nu}(x(u))\dot{x}^\mu \dot{x}^\nu},$$

satisfies Euler-Lagrange's equation

$$\frac{d}{du}\frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} = 0.$$

We can write this as

$$\frac{d}{du}\left(\frac{1}{L}g_{\mu\nu}\dot{x}^\nu\right) - \frac{1}{2L}g_{\nu\rho,\mu}\dot{x}^\nu \dot{x}^\rho = 0.$$

Making a change of parameterisation of the curve from u, to the proper time τ, we can rewrite the above equation into the form

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) - \frac{1}{2}g_{\nu\rho,\mu}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0,$$

where we have used that

$$L^2 = -g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \left(\frac{d\tau}{du}\right)^2 \quad \Rightarrow \quad \frac{d}{du} = L\frac{d}{d\tau}.$$

Cleaning up the equation and multiplying by the inverse metric, we arrive at the differential equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0. \tag{2.1.3}$$

Where $\Gamma^\mu_{\nu\rho}$ are known as *Christoffel symbols*, given by the expression

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}(g_{\nu\sigma,\rho} + g_{\rho\sigma,\nu} - g_{\nu\rho,\sigma}). \tag{2.1.4}$$

We note that the Christoffel symbols are not tensor components, but that their variation is. We will use this later in Section 2.3.3. Equation (2.1.3) is the *geodesic equation*. Geodesics themselves will be defined below, after we introduce the notions of a connection on a manifold and parallel transport along a curve.

2.1.4 *Symmetries*

Sometimes we will be interested in mapping from one generic manifold to another, *e.g.* when projecting from a manifold onto a lower-dimensional hypersurface. As with the coordinate basis, we can ensure that this mapping is smooth in the following way.

Definition 2.11. Let M, N be two differential manifolds of dimension m and n. A function $\varphi : M \to N$ is smooth if and only if $\phi_\beta \cdot \varphi \cdot \phi_\alpha^{-1}$ for all coordinate basis ϕ_α on M and ϕ_β on N.

The mapping φ allows us to map objects from one manifold onto another. With a map φ we can naturally write a smooth function on N, onto M as follows.

Definition 2.12. Given a smooth mapping $\varphi : M \to N$, the *pull-back* of a function $f : N \to \mathbb{R}$ is defined as $\phi^\star(f) = f \circ \varphi : M \to \mathbb{R}$.

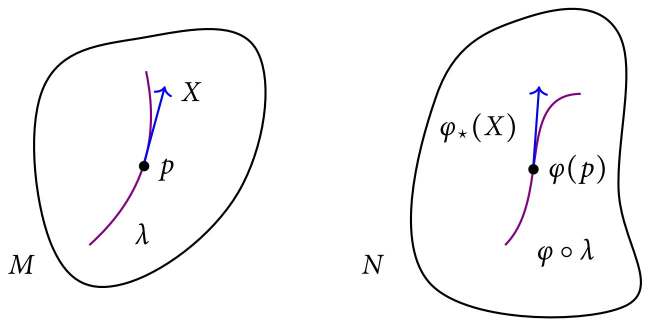


Figure 2.4: Illustration of the mapping between two differential manifolds M and N. The smooth map φ can push-forward a curve λ in M to N, where the tangent vector $X \in T_p(M)$ is mapped to the vector $\varphi_\star(X) \in T_{\varphi(p)}$.

We can also define a *push-forward*, which naturally takes a curve in M and maps it onto N in the following way.

Definition 2.13. Given a smooth map $\varphi : M \to N$, and a point $p \in M$ we can define the *push-forward* of a vector $X^\mu \in T_p(M)$ as the vector $\varphi_\star(X) \in T_{\varphi(p)}(N)$. Given a curve λ on M passing through p, with tangent $X^\mu \in T_p(M)$, the vector $\varphi_\star(X)$ is the tangent to the curve $\varphi \circ \lambda$ in N, at the point $\varphi(p)$.

The push-forward $\varphi_\star$ can been seen as the total derivative of the map: $\varphi_\star = d\varphi$. Pointwise on the manifold we understand the push-forward as a linear map $\varphi_\star : T_p(M) \to T_{\varphi(p)}(N)$. Similarly, we understand the pull-back as a pointwise linear mapping from $T^*_{\varphi(p)}(N) \to T^*_p(M)$. We can extend the action of the pull-back onto a rank $(0, s)$ tensor S as

$$(\varphi^\star(S))(X_1, X_2, \ldots, X_s) = S(\varphi_\star(X_1), \varphi_\star(X_2), \ldots, \varphi_\star(X_s)),$$

for $X_1, X_2, \ldots X_s \in T_p(M)$. Similarly, we extend the action of the push-forward onto a rank $(r, 0)$ tensor as

$$(\varphi_\star(R))(\omega_1, \omega_2, \ldots \omega_r) = R(\varphi^\star(\omega_1), \varphi^\star(\omega_2), \ldots, \varphi^\star(\omega_r)),$$

for $\omega_1, \omega_2, \ldots \omega_r \in T^*_p(N)$. However, we can't extended these actions to mixed rank tensors without a further requirement.

Definition 2.14. We say that the map $\varphi : M \to N$ is a *diffeomorphism* if and only if it is bijective, smooth and has a smooth inverse.

If there exists a diffeomorphism φ then the two manifolds M and N have the same manifold structure with $\dim(M) = \dim(N)$. When the mapping is a diffeomorphism, we can use the inverse φ^{-1} to extend the pull-back onto a tensor T of mixed rank (r, s)

$$(\varphi^\star T)(\omega_1, \ldots \omega_r, X_1, \ldots, X_s) = \\ T((\varphi^{-1})^\star(\omega_1), \ldots, (\varphi^{-1})^\star(\omega_r), \varphi_\star(X_1), \ldots, \varphi_\star(X_s)).$$

For a diffeomorphism φ, we have that $\varphi_\star = (\varphi^\star)^{-1}$, and so we need only either the push-forward or pull-back.

Given a diffeomorphism $\varphi : M \to N$ and a spacetime (M, g, T), the system found after mapping $(N, \varphi_\star(g), \varphi_\star(T))$ is physically indistinguishable from the original system. A tensor T is physically inequivalent to $\tilde{T}$ if and only if there is not a diffeomorphism $\tilde{T} = \varphi^\star(T)$. We then see diffeomorphism invariance as a redundancy in our description of physics in general relativity and so diffeomorphisms are gauge symmetries in general relativity.

An alternative way to view diffeomorphisms is a mapping from the manifold onto itself, where the mapping φ acts as a change of the coordinate basis ϕ_α. When $\varphi : M \to M$ is a diffeomorphism and T is a tensor field on M, the push-back (pull-forward) allows us to compare T at different points in the manifold. We say that if $\varphi^\star(T) = T$ then φ is a symmetry transformation of the tensor field.

On a manifold M, given a vector field X, one can construct a diffeomorphism by looking at the map ϕ_t which sends a point $p \in M$ along an integral curve a parameter distance t. It can be shown that ϕ_t is a diffeomorphism [38]. Using $\phi_t^\star$, one can compare a tensor along the integral curve of X.

Definition 2.15. The *Lie derivative* of a rank (r, s) tensor at $p \in M$ with respect to a vector $X \in T_p(M)$ is defined by

$$(\mathcal{L}_X T)_p = \lim_{p \to 0} \frac{(\phi_{-t}^\star(T_p)) - T_p}{t}. \tag{2.1.5}$$

When the diffeomorphism ϕ_t is a symmetry of the tensor T, then we have

$$\mathcal{L}_X T = 0.$$

We will expand on this later, in Section 2.2.4. The Lie derivative allows us one way to compare tensors at different points in a manifold. In the following section we introduce another way, which leads us naturally to a formal definition of curvature.

2.2 CURVATURE

We first think about curvature as an experience of some two-dimensional surface embedded within three-dimensional flat space. This type of curvature is mathematically captured by the *extrinsic curvature* and is defined formally in Section 2.2.6. As an example, we can imagine a rolled up newspaper. The surface of the paper appears 'curved' to our eye when viewed from our three-dimensional perspective. However, allowing the newspaper to unroll, what remains is a flat surface, or approximately $\mathbb{R}^2$. What we are experiencing is that an object which is flat can be curved into a space of higher dimension.

There is another type of curvature though, which belongs to a manifold without any notion of embedding. We call this the intrinsic curvature of a manifold. We can compare our newspaper example to the peeling of an orange. In three-dimensional space, the surface of the fruit appears round and in this sense has extrinsic curvature as our rolled newspaper did. However, as we peel the orange we might try to flatten the surface onto the table. What we find that it is impossible to flatten the peel into a good approximation of $\mathbb{R}^2$. The surface of a sphere in itself is curved.

In general relativity, we are interested in the intrinsic curvature of spacetime. We do not think of our spacetime as being embedded within another manifold, but rather we see gravity manifest itself as the curvature of spacetime itself.

2.2.1 *Connections*

Definition 2.16. A *covariant derivative* ∇ on a manifold M is a mapping from two smooth vector fields X, Y to a smooth vector field $\nabla_X Y$

$$\nabla : (X, Y) \longrightarrow \nabla_X Y,$$

which obeys the following properties:

$$\begin{aligned} \nabla_X(Y+Z) &= \nabla_X Y + \nabla_X Z, \\ \nabla_{(fX+gY)} Z &= f\nabla_X Z + g\nabla_Y Z, \\ \nabla_X(fY) &= X(f)Y + f\nabla_X Y, \end{aligned}$$

for all vector fields $X, Y, Z \in T(M)$ and smooth functions f, g on M.

We note that the action of ∇ on a smooth function f has been defined by: $\nabla_X f = X(f)$. When written in a components, the covariant derivative of a vector field is given by

$$\nabla_X Y = X^\mu \nabla_\mu (Y^\rho \partial_\rho) = X^\mu \left(\partial_\mu Y^\rho + \Gamma^\rho_{\mu\nu} Y^\nu\right) \partial_\rho,$$

where $\Gamma^\rho_{\mu\nu}$ are the *connection components*, we will see later that for a special connection — the Levi-Civita connection — the connection components in a coordinate basis are given by the Christoffel symbols (2.1.4). Using the Leibniz property of the covariant derivative, we can write down the action of ∇ action on a covector ω

$$\nabla_X(\omega(Y)) = X(\omega(Y)) = (\nabla_X \omega)Y + \omega(\nabla_X Y),$$

which shows we should define the action of the connection on a covector as

$$\begin{aligned} &\nabla_X \omega : T(M) \longrightarrow \mathbb{R}, \\ &(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y). \end{aligned}$$

Written in components

$$(\nabla_X \omega)Y = (X(\omega_\mu) - \Gamma^\rho_{\mu\nu}\omega_\rho X^\nu)Y^\mu.$$

We can extended this to an arbitrary tensor in the following way

$$\nabla_X(T_1 \otimes T_2) = \nabla_X(T_1) \otimes T_2 + T_1 \otimes \nabla_X(T_2).$$

Definition 2.17. A connection is *torsion-free* if the torsion tensor $T(X, Y) = 0$

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

In components, this is equivalent to the condition $\Gamma^{\rho}_{[\mu\nu]} = 0$.

If given only a manifold, we find that the choice of connection ∇ is not unique, however, with the additional structure of a metric, we can define a unique connection for a pseudo-Riemannian manifold.

Definition 2.18. Given a pseudo-Riemannian manifold (M, g), there exists a unique torsion-free connection such that $\nabla g = 0$, called the *Levi-Civita* connection.

We say a connection ∇ is *metric compatible* if $\nabla g = 0$. Deriving the components for the Levi-Civita connection, we find that they are the Christoffel symbols (2.1.4).

2.2.2 *Parallel transport*

Given a manifold M, there is no immediate way of comparing tensors at different points in the manifold. We saw that the Lie derivative allowed us to compare points using diffeomorphisms to compare the value of tensors along integral curves of a vector field. An alternative way is to use a connection ∇ to study how tensors change as we propagate them along some curve γ in M. We will see that this leads us to the notion of a geodesic; a special curve which describes the path of freely falling objects in general relativity.

Definition 2.19. Given an integral curve γ of a vector field X, a tensor T is said to be *parallelly propagated* along γ if $\nabla_X T = 0$.

Taking a vector X_p, we can compare it to X_q by parallelly propagating the vector along some curve γ which joins the points p and q. However, the choice of curve joining two points on a curved manifold is not unique and we find that parallel propagation of a tensor is curve dependent. In other words, picking two distinct curves γ_1 and γ_2 the vector X_p will take on two values $X_{q|1}$ and $X_{q|2}$ after parallel propagation. The path dependence of parallel transport is a measure of a manifold's *intrinsic curvature* and is captured formally by the *Riemann curvature tensor*.

Definition 2.20. The *Riemann curvature tensor* of a connection ∇ is defined by

$$R(X, Y)Z = \left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\right), \tag{2.2.1}$$

for vector fields X, Y, Z.

Before continuing the discussion of parallel transport, we cover some basic properties of the Riemann tensor. We call a connection *flat* when the Riemann tensor vanishes. In a coordinate basis, the components of the Riemann curvature tensor are given by

$$R^{\mu}{}_{\nu\rho\sigma} = s_2\left(\partial_\rho \Gamma^{\mu}{}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\rho\tau}\Gamma^{\tau}{}_{\nu\sigma} - \Gamma^{\mu}{}_{\sigma\tau}\Gamma^{\tau}{}_{\nu\rho}\right), \tag{2.2.2}$$

where we have explicitly included the sign $s_2 = \pm 1$ momentarily. In most conventional differential geometry texts, $s_2 = +1$. However, when working with Supergravity, there are other sign conventions and in the work completed in [39, 40], we found it most appropriate to make the choice $s_2 = -1$. To maintain consistency throughout this thesis, we will use that $s_2 = -1$ for the remainder of the discussion. As $s_2 = -1$ is unconventional from the differential geometry perspective, we will make appropriate comments throughout this chapter to highlight key differences but refer to Appendix A for a thorough discussion of various signs.

The symmetry of the Riemann tensor means that

$$R^{\mu}{}_{\nu(\rho\sigma)} = 0.$$

For a torsion free connection, the Riemann curvature tensor obeys some other useful relationships. The Christoffel symbols obey $\Gamma^{\mu}_{[\mu\nu]} = 0$ and the Riemann tensor has the additional property

$$R^{\mu}{}_{[\nu\rho\sigma]} = 0.$$

Contraction of the first or last two indices of the Riemann tensor is zero, but contracting over the first and third gives the *Ricci tensor*, an important tensor in General relativity

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = -\left(\partial_{\lambda}\Gamma^{\lambda}{}_{\mu\nu} - \partial_{\nu}\Gamma^{\lambda}{}_{\mu\lambda} + \Gamma^{\lambda}{}_{\lambda\tau}\Gamma^{\tau}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\tau}\Gamma^{\tau}{}_{\mu\lambda}\right). \tag{2.2.3}$$

The *Ricci identity*

$$\nabla_{\rho}\nabla_{\sigma}X^{\mu} - \nabla_{\sigma}\nabla_{\rho}X^{\mu} = R^{\mu}{}_{\nu\rho\sigma}X^{\nu}, \tag{2.2.4}$$

allows the interpretation of the Riemann tensor as a measure of the failure for successive operations of the connection to commute.

The *differential Bianchi identity* is given by

$$\nabla_{\rho}R^{\mu}{}_{\nu\sigma\lambda} + \nabla_{\lambda}R^{\mu}{}_{\nu\rho\sigma} + \nabla_{\sigma}R^{\mu}{}_{\nu\lambda\rho} = 0.$$

Contracting first over the indices (μ, ρ)

$$\nabla_{\mu}R^{\mu}{}_{\nu\sigma\lambda} + \nabla_{\lambda}R_{\nu\sigma} - \nabla_{\sigma}R_{\nu\lambda} = 0,$$

then (ν, σ) gives us

$$\nabla^{\mu}G_{\mu\nu} = 0, \qquad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu},$$

where $G_{\mu\nu}$ is the *Einstein tensor*, and will appear later as one side of Einstein's equations. When discussing general relativity, we will see that the differential Bianchi identity shows consistency within general relativity for the conservation of the energy-momentum tensor.

So how can we relate parallel transport and the Riemann tensor? Picking some two-dimensional surface S in the manifold M, we can construct a small loop that passes through a point $p \in S$ in the following way. We choose coordinates (t, s) on S such that the point $p = (0, 0)$. We take vector fields $X = \partial/\partial s$, $Y = \partial/\partial t$ which are linearly independent: $[X, Y] = 0$. We define three more points: (q, r, u) by moving small distances along curves with tangents X, Y such that $q = (\delta s, 0)$, $r = (0, \delta t)$ and $y = (\delta s, \delta t)$. Connecting these four points creates a parallelogram $pqur$.

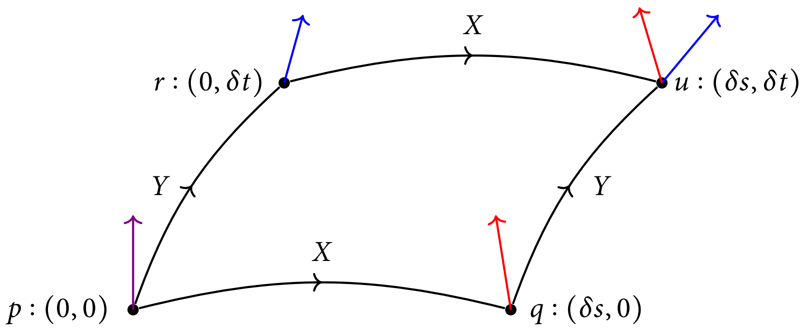


Figure 2.5: A vector V_p^μ (purple) is parallelly propagated along the two curves pqu and pru. The resulting vectors $V_{u|1}^\mu$ and $V_{u|2}^\mu$ (red and blue respectively) are not parallel at the point u. This is due to the intrinsic curvature of the manifold and is captured by the Riemann curvature tensor.

We are now interested in how some vector $V_p \in T_p(M)$ changes as it is parallelly propagated around the loop. We will compare the resulting vectors $V_{u|1}$ and $V_{u|2}$ obtained by parallel propagation along the curves pqu and pru respectively. Along pq we have that $\nabla_X V = 0$ and so

$$\begin{aligned}\frac{dV^\mu}{ds} &= -\Gamma^\mu_{\nu\rho}V^\nu X^\rho,\\ \frac{d^2V^\mu}{ds^2} &= -\partial_\sigma(\Gamma^\mu_{\nu\rho}V^\nu X^\rho)X^\sigma.\end{aligned}$$

Using normal coordinates at the point p, we have $\Gamma^\mu_{\nu\rho}(p) = 0$ and we can expand the vector field as

$$\begin{aligned}V_q^\mu &= V_p^\mu + \left(\frac{dV^\mu}{ds}\right)_p \delta s + \frac{1}{2}\left(\frac{d^2V^\mu}{ds^2}\right)_p \delta s^2 + \mathcal{O}(\delta s^3),\\ &= V_p^\mu - \frac{1}{2}\left(\partial_\sigma\Gamma^\mu_{\nu\rho}V^\nu X^\rho X^\sigma\right)_p \delta s^2 + \mathcal{O}(\delta s^3).\end{aligned} \tag{2.2.5}$$

We can now look at the parallel transport of V_q^μ along the curve qu. The result can be expanded

$$\begin{aligned}V_u^\mu &= V_q^\mu + \left(\frac{dV^\mu}{dt}\right)_q \delta t + \frac{1}{2}\left(\frac{d^2V^\mu}{dt^2}\right)_q \delta t^2 + \mathcal{O}(\delta t^3),\\ &= V_q^\mu - (\Gamma^\mu_{\nu\rho}V^\nu Y^\rho)_q\delta t - \frac{1}{2}\left(\partial_\sigma(\Gamma^\mu_{\nu\rho}V^\nu Y^\rho)Y^\sigma\right)_q \delta t^2 + \mathcal{O}(\delta t^3),\\ &= V_p^\mu - \frac{1}{2}\left(\partial_\sigma\Gamma^\mu_{\nu\rho}V^\nu\left[X^\rho X^\sigma\delta s^2 + Y^\rho Y^\sigma\delta t^2 + 2Y^\rho X^\sigma\delta s\delta t\right]\right)_p + \mathcal{O}(\delta^3).\end{aligned}$$

In the third line we have substituted in the results from (2.2.5) and assumed that $\mathcal{O}(\delta s) = \mathcal{O}(\delta t) = \mathcal{O}(\delta)$. The result of parallel propagation for the vector V_p^μ along the curve pru follows the same steps, and only $X \leftrightarrow Y$ must be swapped in the above equation. We can then look at the difference between these two vectors at u:

$$\begin{aligned}\Delta V_u^\mu \equiv V_{u|1}^\mu - V_{u|2}^\mu &= \left[\partial_\sigma\Gamma^\mu_{\nu\rho}V^\nu(Y^\rho X^\sigma - X^\rho Y^\sigma)\delta s\delta t\right]_p + \mathcal{O}(\delta^3),\\ &= [(\partial_\rho\Gamma^\mu_{\nu\sigma} - \partial_\sigma\Gamma^\mu_{\nu\rho})X^\rho Y^\sigma V^\nu]_p + \mathcal{O}(\delta^3),\\ &= [R^\mu{}_{\nu\rho\sigma}V^\nu X^\rho Y^\sigma]_p + \mathcal{O}(\delta^3),\\ &= [R^\mu{}_{\nu\rho\sigma}V^\nu X^\rho Y^\sigma]_u + \mathcal{O}(\delta^3).\end{aligned}$$

Here we have used Equation (2.2.2) and that $\Gamma^{\mu}_{\nu\rho}(p) = 0$, and in the final line we use that quantities at p and q differ in order δ and so both the LHS and RHS are tensors at the point u. As such, this relationship is basis independent and the Riemann tensor appears as

$$R^{\mu}{}_{\nu\rho\sigma}V^{\nu}X^{c}Y^{d} = \lim_{\delta\to 0}\frac{\Delta V^{\mu}_{u}}{\delta s\delta t}.$$

2.2.3 *Geodesics*

From the notion of parallel transport, we arrive at the definition of a geodesic. We can think about a geodesic as being a path between two points in a manifold that curves as little as possible.

Definition 2.21. For a manifold M with connection ∇, a *geodesic* is an integral curve of a vector field X such that $\nabla_X X = \alpha X$, for an arbitrary function α. When the function $\alpha = 0$ we say that the geodesic is *affinely parameterised*:

$$\nabla_X X = 0. \tag{2.2.6}$$

In other words, an affinely parameterised geodesic is the integral curve of the vector field X which is parallelly propagated along itself.

Looking at the components, we can write down the geodesic equation in a coordinate basis

$$\begin{aligned} X^{\nu}\nabla_{\nu}X^{\mu} &= X^{\nu}(\partial_{\nu}X^{\mu} + \Gamma^{\mu}_{\nu\rho}X^{\rho}), \\ &= \frac{d^2x^{\mu}}{dt^2} + \Gamma^{\mu}_{\nu\rho}\frac{dx^{\nu}}{dt}\frac{dx^{\rho}}{dt}. \end{aligned} \tag{2.2.7}$$

We can view Equation (2.2.7) as a coupled system of n second order differential equations and so from ODE theory, we know that there exists a unique solution given initial values for x^{μ} and dx^{μ}/dt. Therefore, for a given point $p \in M$ and a tangent vector $X^{\mu} \in T_p(M)$ there is always a unique geodesic through the point p with tangent X^{μ}.

In Section 2.1.3, we arrived at the geodesic equation by considering the extremal curve joining two points. However, the existence of a *unique* curve holds only locally. Consider for example the surface of a sphere (see Figure 2.2). The great arcs of a sphere are geodesics and if we take our points (p, q) to be the North and South poles, we find that there is an infinite number of geodesics joining them. This is an example of conjugate points on a Riemannian manifold, for which there exists a one-parameter family of geodesics joining them. We also mention that the length extremising curve is always a geodesic, but not all geodesics are length extremising. Consider again two points on a greats arc of the sphere and the two segments of the great arc which connect them; both of these segments are geodesics. When the points are antipodal, we return to our example of the conjugate points, for all other cases, only one of these geodesics is length minimising.

2.2.4 *Isometries*

Earlier, we used diffeomorphisms to define a symmetry of a tensor and saw that this was echoed in the Lie derivative. When the tensor we consider is the metric tensor, we give the symmetry a special name.

Definition 2.22. For a pseudo-Riemannian manifold (M, g) a diffeomorphism φ is called an *isometry* when its action on the metric tensor is invariant: $\varphi_\star(g) = g$.

Given an isometry φ and the corresponding vector field X, its Lie derivative (2.1.5) vanishes: $\mathcal{L}_X g = 0$. We now want to write this in components. The natural action of a Lie derivative on a smooth function f is $\mathcal{L}_X f = X(f)$. In a coordinate basis, we can write the action of a Lie derivative on a vector, covector and $(0,2)$ tensor as:

$$\begin{aligned}(\mathcal{L}_X Y)^\mu &= [X, Y]^\mu, \\ (\mathcal{L}_X \omega)_\mu &= X^\nu \nabla_\nu \omega_\mu + \omega_\nu \nabla_\mu X^\nu, \\ (\mathcal{L}_X T)_{\mu\nu} &= X^\rho \nabla_\rho T_{\mu\nu} + T_{\mu\rho} \nabla_\nu X^\rho + T_{\rho\nu} \nabla_\mu X^\rho.\end{aligned}$$

When we consider an isometry for a pseudo-Riemannian manifold equipped with the Levi-Civita we can simplify the above equation.

Definition 2.23. Given a one-parameter isometry ϕ_t with corresponding vector field ξ on a pseudo-Riemannian manifold (M, g) it follows that $\mathcal{L}_\xi g = 0$. When the connection ∇ is the Levi-Civita connection, isometries are characterised by *Killing's equation*

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0. \tag{2.2.8}$$

The vector fields ξ satisfying the above equations are known as *Killing vector fields*. We can interpret a Killing vector field as infinitesimal generators of isometries.

Lemma. *For a pseudo-Riemannian manifold equipped with the Levi-Civita connection, a Killing vector obeys the identity*

$$\nabla_\mu \nabla_\nu \xi_\sigma = R^\rho{}_{\mu\nu\sigma} \xi_\rho. \tag{2.2.9}$$

Proof. As the Levi-Civita connection is torsion free, the Killing vector satisfies the Ricci identity (2.2.4)

$$\nabla_\mu \nabla_\nu \xi_\sigma - \nabla_\nu \nabla_\mu \xi_\sigma = -R^\rho{}_{\sigma\mu\nu} \xi_\rho.$$

From the antisymmetry of the Riemann tensor $R^\mu{}_{[\nu\rho\sigma]} = 0$, we can write down that

$$\nabla_{[\mu} \nabla_\nu \xi_{\sigma]} \simeq R^\mu{}_{[\sigma\nu\rho]} \xi^\nu = 0.$$

From Killings equation (2.2.8) we have $\nabla_{[\mu} \xi_{\nu]} = 0$, and so antisymmetry in the above equation is equivalent to the cyclic permutation

$$\nabla_\mu \nabla_\nu \xi_\sigma + \nabla_\sigma \nabla_\mu \xi_\nu + \nabla_\nu \nabla_\sigma \xi_\mu = 0,$$

which can be rearranged using Killing's equation to give

$$\begin{aligned}\nabla_\mu \nabla_\nu \xi_\sigma &= \nabla_\sigma \nabla_\nu \xi_\mu - \nabla_\nu \nabla_\sigma \xi_\mu, \\ &= -R^\rho{}_{\mu\sigma\nu} \xi_\rho, \\ &= R^\rho{}_{\mu\nu\sigma} \xi_\rho,\end{aligned}$$

where in the last line, we have used the skew symmetry of the Riemann tensor. □

Lemma. *If ξ^μ is a Killing vector field and V^μ is tangent to an affinely parameterised geodesic, then the quantity $\xi \cdot V$ is constant along the geodesic.*

Proof. Taking the derivative of $\xi \cdot V$ along a geodesic parameterised by τ we find

$$\begin{aligned}\frac{d}{d\tau}(\xi_\mu V^\mu) = \nabla_V(\xi_\mu V^\mu) &= V^\nu \nabla_\nu(\xi_\mu V^\mu), \\ &= V^\nu V^\mu \nabla_\nu \xi_\mu + \xi_\mu V^\nu \nabla_\nu V^\mu = 0.\end{aligned}$$

Where in the last line, we lose the first term from the antisymmetry in Killing's equation (2.2.8) and the second term vanishes due to the geodesic equation (2.2.6). □

This is particularly interesting when we consider the energy-momentum tensor. Given a Killing vector field ξ^μ and the energy-momentum tensor $T_{\mu\nu}$, we can construct the conserved current $J^\mu = -T_{\mu\nu}\xi^\nu$ such that $\nabla_\mu J^\mu = 0$.

2.2.5 *Congruences*

Definition 2.24. Given a manifold M and an open set $\mathcal{U} \in M$, a congruence in $\mathcal{U}$ is a family of curves such that exactly one curve passes through each point $p \in \mathcal{U}$. We call this a *geodesic congruence* if the curves are geodesics.

Consider a geodesic congruence where all geodesics are either timelike, spacelike or null. When all geodesics of a congruence are of the same type, we can normalise the affine parameter such that the tangent vector U^μ of the geodesic satisfies $U^2 \in \{-1, 1, 0\}$ where the three cases are for timelike, spacelike and null geodesics respectively.

In Section 2.2.2, we used parallel transport to look at the difference of two vectors as a measure of the curvature of the manifold. We can apply a similar discussion when looking at geodesic congruences, looking at neighbouring geodesic curves and measuring the deviation of their tangent vectors remaining parallel along the curve.

Definition 2.25. A one-parameter family of geodesics is a map: $\gamma : [0,1] \times [0,1] \to M$ such that

(i) For fixed parameter s, $\gamma(s, \lambda)$ is a affinely parameterised geodesic .

(ii) The map $(s, \lambda) \mapsto \gamma(s, \lambda)$ is a smooth bijection.

Together, these two points imply that the family of geodesics span a two-dimensional surface Σ, parameterised by the coordinates (s, λ).

Let us consider a one-parameter family of geodesics. We refer to the tangent vectors of the geodesics as U^μ and study the vector S^μ which is tangent to the curves for constant λ. Consider the two-dimensional surface Σ parameterised by the coordinates (s, λ). We can extend these coordinates into a neighbourhood around Σ with coordinates $(s, \lambda, \dots)$ such that

$$S = \frac{\partial}{\partial s}, \quad U = \frac{\partial}{\partial \lambda}, \quad [S, U] = 0, \qquad U^\nu \nabla_\nu S^\mu = S^\nu \nabla_\nu U^\mu.$$

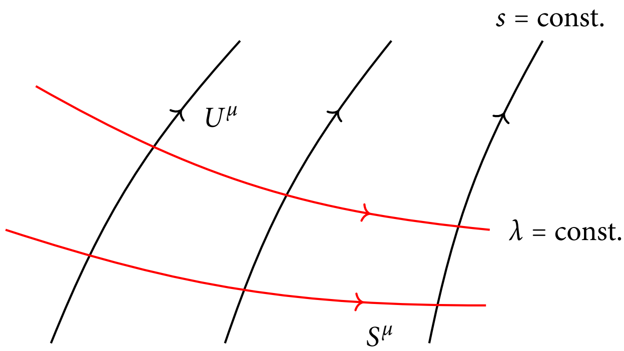


Figure 2.6: A sketch of the two-dimensional surface Σ spanned by the vectors U^μ (black) and S^μ (red). On the surface Σ, we can use the coordinates (s, λ).

This relationship allows us to write down

$$\nabla_U \nabla_U S = R(U, S)U, \tag{2.2.10}$$

where we have used that $\nabla_U U = 0$. Equation (2.2.10) is known as the *geodesic deviation equation* and measures the failure for infinitesimally close geodesics within a one-parameter family to have tangent vectors which remain parallel.

We are interested in the geodesic deviation of geodesic congruences. We know that

$$U^\nu \nabla_\nu S^\mu = S^\nu \nabla_\nu U^\mu = B^\mu{}_\nu S^\nu,$$

where the tensor $B^\mu{}_\nu$ measures the failure of S^ν to be parallelly transported along the geodesics. For a geodesic congruence where all geodesics are of the same type, we have the additional property that U^2 is constant in $\mathcal{U}$, and so we have

$$\frac{1}{2}\nabla_\mu U^2 = U_\nu B^\nu{}_\mu = 0,$$

and so we can look at

$$U \cdot \nabla(U \cdot S) = \cancel{S^\nu U^\mu \nabla_\mu U_\nu} + U_\nu U^\mu \nabla_\mu S^\nu = U_\nu B^\nu{}_\mu S^\mu = 0,$$

where we have cancelled a term using by that U^μ is affinely parameterised. We see that $(U \cdot S)$ is constant along the geodesic.

When considering congruences, there is a gauge freedom in how we pick our affine parameter. Even after setting the value of U^2, we find we can shift λ by a constant value. Scaling λ maintains that U^μ is affinely parametrised, but shifts the displacement vector

$$\tilde{\lambda} = \lambda - a(s), \qquad \tilde{S}^\mu = S^\mu + \frac{da}{ds}U^\mu.$$

For timelike and spacelike congruences, we see that the shift in the affine parameter causes a shift in

$$(U \cdot \tilde{S}) = (U \cdot S) + \frac{da}{ds}U^2.$$

We can then fix the gauge freedom by picking λ such that $(U \cdot S) = 0$. For null congruences where $U^2 = 0$, a shift in λ leaves $(U \cdot S)$ invariant and so we need to introduce extra structure to fix the gauge freedom. We delay the further discussion of null congruences until Section 3.1.5, where we will look at the properties of them in more detail in the context of trapping regions.

2.2.6 *Extrinsic curvature*

Extrinsic curvature is the curvature that we are used to *seeing*. We measure it for lower-dimensional manifolds embedded within our spacetime, *e.g.* the surfaces of the objects we have in front of us can be thought of as embedded within $\mathbb{R}^3$. We see extrinsic curvature in curves we draw onto pages, or the cylinder we see by rolling up a paper. Before formally defining extrinsic curvature, we must formalise what it means for a surface to be embedded within another.

Definition 2.26. Let M and S be manifolds of dimension m, s, with $s < m$. A smooth map $\varphi : M \to S$ is an *embedding* when φ is an injection and for any $p \in S$, there is a neighbourhood $\mathcal{U}$ such that $\varphi^{-1} : \varphi[\mathcal{U}] \to S$ is smooth.

Definition 2.27. Let M and S be manifolds of dimension m, s, $s < m$. If $\varphi : M \to S$ is an embedding, we say that $\phi[S]$ is an *embedded submanifold* of M. If $s = m - 1$ then we say that S is a *hypersurface* denoted by Σ.

For a Lorentzian manifold, a hypersurface Σ is said to be *timelike*, *spacelike* or *null* corresponding to whether the normal to the surface is everywhere timelike, spacelike or null.

When working with the action of a gravitational theory, we will be interested in calculating boundary terms for a manifold M. The boundary of a manifold M is a hypersurface $\partial M = \Sigma$. In the following discussion, we allow Σ to be either timelike or spacelike, with a corresponding unit normal: $n_\mu n^\mu = \epsilon$, where $\epsilon = \pm 1$ for timelike/spacelike Σ respectively. The case for null hypersurfaces will be considered later when discussing the geometry of black holes in Section 3.1.3.

We now have the necessary pieces to define extrinsic curvature. Informally, we can understand the extrinsic curvature as follows. Lets us take a normal vector n^μ at the point $p \in \Sigma$ and parallelly transport it along a curve $\lambda \in \Sigma$ to the point q. The extrinsic curvature measures the failure for this vector to be normal to Σ at q.

The first fundamental form, $\gamma_{\mu\nu}$, and can be constructed given the spacetime metric $g_{\mu\nu}$ and the unit normal n_μ to a hypersurface Σ. We can understand $\gamma^\mu{}_\nu = g^\mu{}_\nu - \epsilon n^\mu n_\nu$ as either projection operator for tensors in M onto Σ, or $\gamma_{\mu\nu}$ as the induced metric on Σ. The second fundamental form is the extrinsic curvature.

Let us consider the parallel transport of a normal vector N_μ along a curve in Σ with a tangent X^μ: $\nabla_X N_\mu = 0$. We can measure the failure of N_μ remaining normal to Σ through considering a second tangent vector: $N \cdot Y \stackrel{?}{=} 0$. We can study how $(N \cdot Y)$ varies along the curve:

$$X(N \cdot Y) = X^\mu \nabla_\mu (Y^\nu N_\nu) = N_\nu X^\mu \nabla_\mu Y^\nu.$$

We see that $N \cdot Y = 0$ remains true along the curve iff $N_\nu X^\mu \nabla_\mu Y^\nu = 0$. This leads to the definition (2.2.11) of the extrinsic curvature. By extending our definition of n_μ in a neighbourhood around a point p in Σ into M in an arbitrary way such that n_μ remains the unit normal, we can write down the extrinsic curvature tensor

$$K(X, Y) = -s_4 n_\mu \left(\nabla_{X_\parallel} Y_\parallel \right)^\mu , \tag{2.2.11}$$

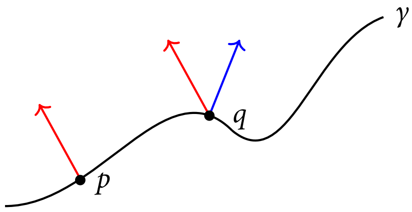


Figure 2.7: The extrinsic curvature measures the failure of a normal to remain normal after parallel transport along a curve. The unit normal at the point $p \in \gamma$ (red) is parallelly propagated along the curve to the point q. The extrinsic curvature of γ causes the deviation of this vector and the unit normal at the point $q \in \gamma$ (blue).

for vector fields X, Y defined in M, and $X_{\|}^{\mu} = \gamma_{\nu}^{\mu} X^{\nu}$ is tangent to Σ. The sign $s_4 = \pm 1$ is not set through the construction of this tensor. In this thesis, we use that $s_4 = 1$, but note that many other authors use $s_4 = -1$ (see for example Equation (5) in [41]).

Using that $n_\rho Y_{\|}^{\rho} = 0$, we can evaluate $K(X, Y)$:

$$K(X, Y) = -n_\rho X_{\|}^{\sigma} \nabla_\sigma Y_{\|}^{\rho} = X_{\|}^{\sigma} Y_{\|}^{\rho} \nabla_\rho n_\sigma = (\gamma_{\mu}^{\sigma} \gamma_{\nu}^{\rho} \nabla_\rho n_\sigma) X^\mu Y^\nu = K_{\mu\nu} X^\mu Y^\nu \,. \tag{2.2.12}$$

This leads to the expressions

$$K_{\mu\nu} = \gamma_{\mu}^{\rho} \gamma_{\nu}^{\sigma} \nabla_\rho n_\sigma = \nabla_\mu n_\nu - \epsilon n^\rho n_\mu \nabla_\rho n_\nu = \gamma_{\mu}^{\rho} \nabla_\rho n_\nu, \tag{2.2.13}$$

for the extrinsic curvature, where we used $n^\rho \nabla_\mu n_\rho = \frac{1}{2} \nabla_\mu (n^\rho n_\rho) = 0$. The boundary can locally be described as the level set of a function f. Then $N_\mu = \partial_\mu f$ is a normal vector field, and the corresponding unit normal vector field n_μ is hypersurface orthogonal and satisfies the Frobenus integrability condition

$$n_{[\mu} \nabla_\nu n_{\rho]} = 0 \,. \tag{2.2.14}$$

Contracting this relation with n^ρ it is straightforward to obtain a relation which allows to show that $K_{\mu\nu}$ is symmetric: $K_{\mu\nu} = K_{\nu\mu}$.

In the literature, it is common to see the extrinsic curvature defined through the relation [41]

$$K_{\mu\nu} = \frac{1}{2} \mathcal{L}_n \gamma_{\mu\nu}. \tag{2.2.15}$$

We can verify that this is in the same form as our above definition:

$$\begin{aligned}
K_{\mu\nu} &= \frac{1}{2} \mathcal{L}_n \gamma_{\mu\nu}, \\
&= \frac{1}{2} (n^c \nabla_\rho \gamma_{\mu\nu} + \gamma_{\mu\rho} \nabla_\nu n^\rho + \gamma_{\rho\sigma} \nabla_\mu n^\rho), \\
&= \frac{1}{2} (n^c \nabla_\rho (g_{\mu\nu} - \epsilon n_\mu n_\nu) + (g_{\mu\rho} - \epsilon n_\mu n_\rho) \nabla_\nu n^\rho + (g_{\rho\sigma} - \epsilon n_\rho n_\nu) \nabla_\mu n^\rho), \\
&= \frac{1}{2} (\nabla_\mu n_\nu + \nabla_\nu n_\mu - \epsilon n^\rho \nabla_\rho (n_\mu n_\nu)), \\
&= \nabla_\mu n_\nu - \epsilon n_\mu n^\rho \nabla_\rho n_\nu = K_{\mu\nu}.
\end{aligned}$$

In our calculations, we need the trace of the extrinsic curvature, K

$$K = g^{\mu\nu} K_{\mu\nu} = \gamma^{ij} K_{ij}, \tag{2.2.16}$$

where the indices i, j run over the coordinates of Σ only. The trace can be calculated from the expression

$$\begin{aligned} K = g^{\mu\nu} K_{\mu\nu} &= \frac{1}{2} g^{\mu\nu} (\nabla_\mu n_\nu + \nabla_\nu n_\mu - \epsilon n^\rho \nabla_\rho (n_\mu n_\nu)), \\ &= \frac{1}{2} (\nabla_\mu n^\mu + \nabla^\mu n_\mu - \cancel{\epsilon n^\rho \nabla_\rho (g^{\mu\nu} n_\mu) n_\nu}), \\ &= \nabla_\mu n^\mu \, . \end{aligned} \tag{2.2.17}$$

2.3 GENERAL RELATIVITY

2.3.1 *Motivation*

Einstein's theory of special relativity revolutionised physics. The understanding that our physical laws should be the same in all inertial frames led to developing our concept three-dimensional space, into a four-dimensional spacetime. An event in special relativity is specified by its position within space and time, which we can think of as a point in Minkowski spacetime. Invariance of inertial frames is captured by Lorentz invariance of the tensors on the manifold.

After relativity, it was understood that physical laws should be Lorentz invariant, and so our classical theories would need to be reviewed from this new viewpoint. The constant speed of light bought in new causality constraints, and from this we understood that the instantaneous changes associated with particle interaction would have to be addressed. It was the theory of electromagnetism that led to special relativity, and could be considered as a ready-made theory in regard to being relativistic. However, this was not the case for other physical theories.

Newton's theory of gravitation was one of these theories. Although very accurate in a low energy limit when the masses were not too large, or the velocities small compared to the speed of light, its formulation is inconsistent with special relativity. The most immediate issue comes from Newton's equation

$$\nabla^2 \Phi = 4\pi G \rho,$$

for the gravitational potential Φ and mass density ρ. As a Lorentz transformation will mix together space and time coordinates, there will be inertial frames involving time derivatives, *i.e.* Newton's laws are not invariant between inertial frames. Another way of seeing this same issue can be found from solutions of the potential

$$\Phi(t, \mathbf{x}) = -G \int d^3\mathbf{x} \, \frac{\rho(t, \mathbf{y})}{|\mathbf{x} - \mathbf{y}|}.$$

We see this describes instantaneous response for the potential at x due to an event at y, which is incompatible with special relativity.

Despite these issues, Newton's work also gives us a clue on how we might work with relativistic gravity. In Newtonian physics, there are two notions of mass. From Newton's second law, we have the *inertial mass* appearing in the relationship $\mathbf{F} = m_I \mathbf{a}$, and the gravitational mass which determines how a body interacts with the gravitational field: $\mathbf{F} = m_G \mathbf{g}$. The equation for the gravitational field defines both the mass m_G and the acceleration due to gravity $\mathbf{g}$, leading to a scaling ambiguity between them. We fix this by imposing that $m_I = m_G$, which is known as the *weak equivalence principle*. Equality of these two couplings leads to the differential equation

$$\ddot{\mathbf{x}} = \mathbf{g}(t, \mathbf{x}).$$

which has a unique solution given a test body's initial position and velocity. Newton's theory doesn't give a *why* to this equivalence, but the Eötvös experiment shows that $m_I/m_G - 1 \simeq \mathcal{O}(10^{-12})$ [42].

It is this equivalence that leads us to the idea of describing gravity as curvature. Let us imagine two bodies which are dramatically different in their composition. The weak equivalence principle states that if these two bodies begin with the same initial position and velocity, they will travel along the same path within spacetime. The independence of a body's composition on the journey they travel through during free fall suggests that the gravitational field is determined by spacetime alone. It is this simple insight that inspired the conjecture that the gravitational force should be seen as a geometric theory of the spacetime.

Newton's equivalence principle is stated for test bodies moving within inertial frames, but how does this extend to areas of physics with additional laws, such as the movement of charged particles? Einstein generalised the equivalence principle in the following way

Definition 2.28. The *Einstein Equivalence Principle* states that

(i) The weak equivalence principle is valid.

(ii) In a local inertial frame, all non-gravitational experiments will be indistinguishable from the same experiment carried out in a Minkowski inertial frame.

We are led to realise that any physical experiment carried out under uniform acceleration would be equivalent to that of one done within a uniform gravitational field.

So to understand gravity after special relativity, Einstein proposed the theory of general relativity; capturing this equivalence with using geometry. The intrinsic, observer independent nature of spacetime should be described by a spacetime metric, but unlike special relativity, we should not require this to be flat Minkowski space. Einstein prosed that our experience of gravity is captured by the deviation from flatness of the metric of spacetime. Moreover, the curvature of the spacetime metric is caused by the matter within it, and so we relate the curvature of space to a conserved tensor known as the energy-momentum tensor. In essence '*Spacetime tells matter how to move, matter tells spacetime how to curve*' [43]. We can formalise this paragraph with the postulates of general relativity.

Postulates of General Relativity

(i) Spacetime is a four-dimensional Lorentzian manifold.

(ii) Free particles fall along geodesics of the Levi-Civita connection. Massive particles follow timelike geodesics, massless particles follow null geodesics.

(iii) Matter content of a physical system is captured by a $(0, 2)$ rank symmetric tensor $T_{\mu\nu}$ called the *energy-momentum* tensor, which is conserved: $\nabla^{\mu} T_{\mu\nu} = 0$.

(iv) The curvature of spacetime is related to the energy-momentum tensor by Einstein's equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -8\pi G T_{\mu\nu}. \tag{2.3.1}$$

The minus sign on the right-hand side comes from our convention $s_3 = -1$ which is explained in more detail within Appendix A.

We might ask how the form of (2.3.1) came to be, or whether it could be generalised. When looking for a symmetric tensor of rank $(0,2)$ to be related to the stress-energy tensor, the immediate guess would be the Ricci tensor: $T_{\mu\nu} = cR_{\mu\nu}$, for some constant c. However, the conservation of the energy-momentum tensor then implies that $\nabla^\mu R_{\mu\nu} = 0$. This, together with the contracted Bianchi identity implies $\nabla^\mu R = 0$, and hence $\nabla^\mu T = 0$. This would require the energy-momentum tensor to be constant throughout the universe, which obviously isn't the case. Knowing $G_{\mu\nu}$ and the contracted Bianchi identity, we see the Einstein tensor as the natural candidate, but is this the most general solution? The answer comes from Lovelock [44] in the following theorem:

Theorem 2.29 (Lovelock)**.** *Let $A_{\mu\nu}$ be a symmetric tensor with the properties*

- *Metric dependence: $A_{\mu\nu}$ is a function of only $g_{\mu\nu}$, $\partial_\rho g_{\mu\nu}$ and $\partial_\rho\partial_\sigma g_{\mu\nu}$*
- *Conservation: $\nabla^\mu A_{\mu\nu} = 0$*

Then there exist constants α, β such that

$$A_{\mu\nu} = \alpha G_{\mu\nu} + \beta g_{\mu\nu}.$$

Lovelock's theorem shows us that Einstein's tensor can be generalised and we can write Einstein's equations as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}. \tag{2.3.2}$$

The constant Λ is called the *cosmological constant* and was predicted (and then retracted!!) by Einstein while he was developing his theory of gravity. With $\Lambda \neq 0$, Einstein's equations no longer reduce to Newtonian physics, but if Λ is sufficiently small then the deviation is negligible.

For historical reasons we have kept Newton's constant (G) explicit during the above discussion, while maintaining that the speed of light $c = 1$. We now return to units $G = c = 1$, unless otherwise stated.

2.3.2 *Solving Einstein's equations*

Einstein's equations (2.3.1) describe how spacetime is curved by the matter within it. The solutions to Einstein's equations are given by the metric of the spacetime. The Riemann tensor contains first and second order derivatives of the metric tensor, and so finding solutions of Einstein's equations is equivalent to solving non-linear second order differential equations. The non-linearity of the differential equations separates gravitational solutions from other physical theories *e.g.* the electromagnetic field equations, or Schrödinger's equation.

Due to this non-linearity, exact analytic solutions to Einstein's equations are hard to find, and in general can only be found after making a series of assumptions about the metric tensor. The assumptions for the metric tensor are referred to as the metric ansatz which are formed by writing down the most general metric satisfying a set of assumed isometries. As we saw in Section 2.2.4, symmetries of the metric tensor manifest as the existence of Killing vector fields of the manifold.

The first non-trivial solution to the vacuum Einstein equations was found by Schwarzschild in 1916 [45]. To a good approximation, stars and planets are spheres and so a reasonable assumption is that the spacetime geometry containing these massive bodies has the same symmetries as a sphere.

Definition 2.30. A spacetime is *spherically symmetric* if its isometry group contains an SO(3) subgroup with two-sphere orbits. In other words, a manifold is spherically symmetric if it posseses the symmetries of the two-sphere, which has a metric

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

By assuming spherical symmetry of the metric, Schwarzschild was able to analytically solve the vacuum equations and describe the geometry of spacetime exterior to a spherically symmetric distribution of uncharged matter.

Theorem 2.31 (Birkhoff). *The unique spherically symmetric solution of the vacuum Einstein equations is the* Schwarzschild solution. *In the coordinates* (t, r, θ, ϕ) *the Schwarzschild metric is given by*

$$ds^2 = -\left(1 - \frac{2M}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.3.3}$$

where M *is a real, positive constant corresponding to the mass of the solution.*

Proof. Hawking and Ellis [6]. □

The Schwarzschild solution is found by assuming spherical symmetry, but there is additional isometry of the spacetime. As the metric (2.3.3) is time-independent, there is an additional Killing vector: $k^\mu = (\partial_t, 0, 0, 0)$.

Definition 2.32. A spacetime is *stationary* is it admits a Killing vector field[2] k^μ that is everywhere timelike: $k^\mu k^\nu g_{\mu\nu} < 0$.

Definition 2.33. A spacetime is *static* if it admits a hypersurface-orthogonal Killing vector field. If a spacetime is static, it is also stationary.

We see that the Schwarzschild solution is a static solution and the full isometry group of the metric is $\mathbb{R} \times \mathrm{SO}(3)$. From Birkhoff's theorem, we realise that the gravitational field exterior to any spherical symmetric distribution of matter is therefore time-independent.

Before moving on, we make a few more comments about the line element (2.3.3). We see that the constant M parameterises the solution, and that as we take the limit for $M \to 0$, the resulting metric is that of Minkowski. The Schwarzschild solution is also *asymptotically flat*, which informally can be understood as a spacetime which approaches Minkowski spacetime in the limit

[2] A note for Killing vector notation, we will use ξ^μ when talking of a generic Killing vector field, and k^μ when the Killing vector field is also a timelike vector field.

$r \to \infty$. A formal definition for the asymptotic region of a spacetime will be discussed in Section 3.2.5.

In Section 3.3, we derive the *Reissner–Nordström* solution, a generalisation which allows for the matter distribution to be electromagnetically charged. As a result, we postpone further discussion of the Schwarzschild solution and its various properties until later in the thesis.

2.3.2.1 *Maximally Symmetric Spacetimes*

To illustrate how assuming the symmetries of the spacetime produces solutions to Einstein's equations, we show how imposing *maximal symmetry* on a spacetime almost uniquely determines the Riemann tensor.

Definition 2.34. A *maximally symmetric* spacetime is a manifold equipped with the maximum number of linearly independent Killing vector fields.

Proposition. *For an n-dimensional manifold, the maximum number of linearly independent Killing vector fields is given by*

$$\frac{n(n+1)}{2}.$$

Proof. Given a pseudo-Riemannian manifold (M, g) equipped with the Levi-Civita connection, we are interested in counting the number of linearly independent vector fields which satisfy Killing's equation (2.2.8). We say that a set of Killing vector fields are *linearly independent* if they are linearly independent as vector fields, *i.e.* for a set of constants α_i, the condition

$$\sum_i \alpha_i \xi^i_\mu = 0,$$

implies that $\alpha_i = 0$.

From the relationship (2.2.9), we see that a Killing vector field is determined uniquely by the data of $\xi_\mu(p)$ and $\nabla_\mu \xi_\nu(p)$ for $p \in M$. We can understand this statement from how (2.2.9) relates the second derivative of a Killing vector to itself. As a result, given the zeroth and first derivatives of the Killing vector at a point $p \in M$, we can expand and obtain the full solution for Killing vector via the Taylor series.

For a n-dimensional manifold, there can be at most n linearly independent vector fields $\xi_\mu(p)$ at a point $p \in M$. Likewise, there can be at most $n(n-1)/2$ values for $\nabla_\mu \xi_\nu(p)$ due to the anti-symmetry of Killing's equation. Combining these results, we obtain that there are

$$n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2},$$

linearly independent Killing vectors. □

Definition 2.35. A pseudo-Riemannian manifold (M, g) is *homogeneous* if there exists a group of isometries which acts transitively, *i.e.* any two points $p, q \in M$ can be mapped to each other by a metric preserving symmetry.

Definition 2.36. We say that Riemannian (Lorentzian) manifold is *isotropic at a point p* if the isotropy group at $p \in M$ acts transitively on the unit sphere (pseudo-sphere) in $T_p(M)$. If a manifold is isotropic for every point in the manifold, it is also homogeneous.

We take a moment to expand on the second remark in definition 2.36. Take a homogenous Riemannian space. Isotropy at a point $p \in M$ can be understood as rotation about the point, *i.e.* the Killing vectors generating the isometry are within $\mathrm{SO}(n)$. Rotating a sphere about a point which is not the sphere's origin acts as a translation of the sphere. If every point in the manifold is invariant under rotation, then it must also be invariant under the translation between two points.

Proposition. *A manifold M is homogeneous and isotropic if and only if it is a maximally symmetric manifold*

Proof. S. Kobayashi [46], S. Gallot [47] □

If a maximally symmetric manifold is isotropic at a point $p \in M$, then tensors in the tangent space at that point will be invariant under Lorentz transformations. As the only invariants of the Lorentz group are the Minkowski metric (or products thereof) we can use the symmetries of the Riemann tensor to write the most general form at the point p in the manifold

$$R_{\mu\nu\rho\sigma}(p) = f(p)\left(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}\right).$$

Allowing the coordinate system to be arbitrary, we replace $\eta_{\mu\nu}$ with a generic metric, but we also know that for all $p \in M$, the spacetime is isotropic, and so we can write down a general expression for the Riemann tensor as

$$R_{\mu\nu\rho\sigma}(x) = f(x)\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right),$$

for some function $f(x)$. Contracting the Riemann tensor, we find that this function is related to the Ricci scalar

$$R_{\mu\nu} = (n-1)f(x)g_{\mu\nu}, \qquad R = n(n-1)f(x).$$

We can then write the Riemann tensor for a maximally symmetric spacetime in terms of curvature tensors

$$R_{\mu\nu\rho\sigma} = \frac{R}{n(n-1)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right).$$

Written this way, we see that the Ricci tensor is

$$R_{\mu\nu} = \frac{1}{n}Rg_{\mu\nu}. \tag{2.3.4}$$

The contracted Bianchi identity gives that

$$\nabla^{\mu}G_{\mu\nu} = \left(\frac{1}{n} - \frac{1}{2}\right)g_{\nu\sigma}\nabla^{\mu}R = 0,$$

where we have used that the Levi-Civita connection is metric compatible. From this, we see that for $n > 2$, the Ricci scalar R must be a constant, and we can write

$$R_{\mu\nu\rho\sigma} = \frac{k}{n(n-1)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right),$$

for a constant k. We do not elaborate on the cases for $n \leq 2$. If a Riemann tensor satisfies the above relationship, it is said to be a *manifold of constant curvature*. As R is constant, from (2.3.4) we see that the Ricci tensor is proportional to the metric tensor and so a maximally symmetric space is also an *Einstein space* [48].

We thus find that imposing maximally symmetry on our spacetime leads to solutions of Einstein's equations having constant curvature. We can write down Einstein's tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = R\left(\frac{1}{n} - \frac{1}{2}\right)g_{\mu\nu} + \Lambda g_{\mu\nu},$$

and see that maximally symmetric spacetimes are solutions of the vacuum Einstein equations with a cosmological constant (2.3.2), where we find

$$\Lambda = R\left(\frac{1}{2} - \frac{1}{n}\right).$$

For a fixed signature, there are only three possible metrics with constant curvature which depend on the sign of R. For a Riemannian manifold, a space with $R < 0$ is the sphere, for $R = 0$, the manifold is flat Euclidean space and for $R > 0$, the space is the hyperboloid. For Lorentzian signature, the case for $R = 0$ is flat Lorentzian space, for $R < 0$, the spacetime is called *de Sitter* and the *anti-de Sitter* for $R > 0$. Let us note explicitly that a space of positive curvature has $R < 0$ and a space with negative curvature has $R > 0$. This mismatch of signs comes from our conventions outlined in Appendix A where we explain that a space with positive curvature has $\mathrm{sign}(R) = s_1 s_3 = -1$ in our conventions.

The metrics for these maximally symmetric solutions are induced via an embedding of $\mathbb{R}^{n+1}$. As an example, one can find the four-dimensional de Sitter metric, beginning with a $(1+4)$-dimensional Lorentzian spacetime

$$ds^2 = -dt^2 + dx_1^2 + \ldots + dx_4^2,$$

from the embedding

$$-t^2 + x_1^2 + \ldots x_4^2 = r^2.$$

This can be solved through making the choice

$$\begin{aligned} t &= r \sinh\tau, \\ x_1 &= r\cosh\tau\sin\chi\sin\theta\sin\phi, \quad & x_3 &= r\cosh\tau\sin\chi\cos\theta, \\ x_2 &= r\cosh\tau\sin\chi\sin\theta\cos\phi, \quad & x_4 &= r\cosh\tau\cos\chi. \end{aligned}$$

The resulting line element is:

$$ds^2 = r^2\left[-d\tau^2 + \cosh^2\chi\left(d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right)\right].$$

One can also obtain the metric for these spaces by solving Einstein's equations. As a worked example, let us study three-dimensional Riemannian space. We are interested in solutions to the differential equation

$$R_{\mu\nu} = 2kg_{\mu\nu}.$$

The metric describes a homogenous and isotropic manifold, so to start we make an ansatz that the manifold is spherically symmetric with coordinates (r, θ, ϕ)

$$ds^2 = e^{2F(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

From (2.1.4) we can find all non-zero components of the Christoffel symbols

$$\begin{aligned}
&\Gamma^r{}_{rr} = \partial_r F(r), \quad \Gamma^r{}_{\theta\theta} = -re^{-2F}, \quad \Gamma^r{}_{\phi\phi} = -r\sin^2\theta e^{-2F}, \\
&\Gamma^\theta{}_{r\theta} = \frac{1}{r}, \qquad \Gamma^\phi{}_{r\phi} = \frac{1}{r}, \qquad \Gamma^\theta{}_{\phi\phi} = -\cos\theta\sin\theta, \quad \Gamma^\phi{}_{\theta\phi} = \cot\theta.
\end{aligned} \tag{2.3.5}$$

Collecting these and inserting them into (2.2.3), we obtain only three non-zero components of the Ricci tensor

$$R_{rr} = -\frac{2\partial_r F(r)}{r}, \quad R_{\theta\theta} = e^{-2F(r)}\left(1 - e^{2F(r)} - r\partial_r F(r)\right), \quad R_{\phi\phi} = R_{\theta\theta}\sin^2\theta,$$

which yields two independent first order differential equations

$$\begin{aligned}
2ke^{2F(r)} &= -\frac{2\partial_r F(r)}{r}, \\
2kr^2 &= e^{-2F(r)}\left(1 - e^{2F(r)} - r\partial_r F(r)\right).
\end{aligned}$$

From the first equation we have

$$\partial_r F(r) = -rke^{2F(r)},$$

which allows us to solve the second equation algebraically, to obtain

$$e^{2F(r)} = \frac{1}{1+kr^2}.$$

Notice that we have found a solution for the metric without solving a differential equation! The metric for a maximally symmetric, three-dimensional Riemannian manifold is given by

$$ds^2 = \frac{dr^2}{1+kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

The size of the curvature k can be fixed with a coordinate redefinition, such that there are only three solutions for $k \in \{0, \pm 1\}$. When $k = 0$, we obtain flat space, as we would expect for a solution with zero curvature. For $k = -1$, we obtain the metric on the sphere, and the hyperboloid for $k = 1$.

2.3.2.2 *Planar Symmetry*

Spherical symmetry is the natural ansatz when looking for solutions to Einstein's equations, but it's not the only assumption we can start with. The main results of this thesis come from considering static solutions which are planar symmetric.

Definition 2.37. A spacetime is *planar symmetric* if its isometry group contains an $\mathrm{E}(2)$ subgroup with $\mathbb{R}^2$ orbits. In other words, a manifold is planar symmetric if it possesses the symmetries of two-dimensional Euclidean space.

Unlike spherically symmetric solutions, for a four-dimensional spacetime, imposing planar symmetry will not ensure that the spacetime is asymptotically flat. We interpret spacetimes which are not non-asymptotically flat as solutions which have an ever present contribution to the energy-momentum tensor. The de Sitter and anti-de Sitter solutions are the simplest examples of this, where the cosmological constant causes constant curvature throughout the entire spacetime.

In Chapter 5, the Einstein-Maxwell solution is considered, and by making an ansatz for planar, rather than spherical symmetry, we find dramatic changes in the resulting metric when compared to the Reissner-Nordström solution derived in Section 3.3. The changes between these solutions carries through to the causal structure, the number of horizons and their classification.

If one considers solutions of Einstein's equations in dimensions greater than four, one can obtain asymptotically flat solutions with planar symmetry. In Section 4.5.3, we will see how these solutions relate to brane solutions of supergravity. Thinking within the context of brane solutions allows the interpretation of planar symmetric solutions in lower dimensions as the dimensional reduction of brane configurations of ten and eleven-dimensional supergravity. Finding solutions with planar symmetry in four dimensions is can then be understood as finding 'large brane' solutions of supergravity [49].

Imposing spherical symmetry gave the Schwarzschild solution to the vacuum Einstein equations, and Birkhoff's theorem showed that this was the unique solution. If the spacetime is assumed to be stationary, the unique planar symmetric solution of the vacuum Einstein's equations is the Taub solution with the corresponding line element [50]

$$ds^2 = \frac{1}{\sqrt{1+kz}}\left(-dt^2 + dz^2\right) + \sqrt{1+kz}(dx^2 + dy^2).$$

In the limit of $k \to 0$, we see Taub's line element reduces to the Minkowski solution.

Another planar symmetric solution comes from the Kasner solutions, which are anisotropic cosmological solutions that depend only on some timelike coordinate t. The Kasner solution is described by [51]

$$ds^2 = -dt^2 + \sum_{i=1}^{n-1} t^{p_i} dx_i^2,$$

where the exponents p_i are called the *Kasner exponents*. This metric is an exact solution of Einstein's equations when the Kasner exponents obey

$$\sum_{i=1}^{n-1} p_i = 1, \qquad \sum_{i=1}^{n-1} p_i^2 = 1.$$

Selecting the constants $p_1 = p_2 = \frac{2}{3}$, and $p_3 = -\frac{1}{3}$ the above line element can be written as

$$ds^2 = -dt^2 + t^{-\frac{1}{3}} dz^2 + t^{\frac{2}{3}}(dx^2 + dy^2),$$

and we see that this solution is planar symmetric over the coordinates (x, y).

2.3.3 *Einstein-Hilbert action*

Einstein's equations describe the dynamic content of general relativity. Given an energy-momentum tensor, one can solve the differential equations $G_{\mu\nu} = -8\pi T_{\mu\nu}$, and obtain a metric for the spacetime. Despite this, it is still useful to develop a Lagrangian for general relativity. Later, we will use the Lagrangian formulation to construct a partition function for various black hole solutions, which we then use to study the thermodynamics of black holes. More generally, a Lagrangian description for general relativity starts to reach towards the tools and descriptions one would expect for a theory of quantum gravity. If one wanted to calculate the path integral for a gravitational theory, one would need an action and thus a Lagrangian or Hamiltonian of the classical theory. In this section, we keep the various signs s_i, which are a collection of conventions one picks when studying general relativity. These are described in more detail in Appendix A. Keeping these signs in our calculations will allow for the comparison of various conventions between the computations of the thesis to external resources.

We would expect our Lagrangian to be of the form:

$$S_{EH} = \frac{1}{16\pi}\int_M d^4x\sqrt{-g}L,$$

for some scalar L. An obvious choice would be to use the Ricci scalar R, which produces the *Einstein-Hilbert* action

$$S_{EH}[g] = \frac{s_1 s_3}{16\pi}\int_M d^4x\sqrt{-g}R[g]. \tag{2.3.6}$$

This is the action for a vacuum spacetime with no cosmological constant. Note that the dynamical field for the Einstein-Hilbert action is the metric tensor, and so we not only vary the Ricci scalar R, but also the volume form itself.

We now show one can derive Einstein's equations by varying the action

$$\begin{aligned} \delta S_{EH} &= \frac{s_1 s_3}{16\pi}\int_M d^4x\ \delta(\sqrt{-g}R), \\ &= \frac{s_1 s_3}{16\pi}\int_M d^4x(\delta\sqrt{-g}R + \sqrt{-g}\delta R) \end{aligned} \tag{2.3.7}$$

To calculate the variation, we look at each term independently. We begin by looking at the variation of the metric determinant. First we write the metric inverse in terms of the determinant and the metric cofactors

$$g^{\mu\nu} = g^{-1}(A^{\mu\nu})^T = g^{-1}A^{\nu\mu} \Rightarrow g = g_{\mu\nu}A^{\mu\nu}.$$

We can then write the variation of the determinant

$$\delta g = \frac{\partial g}{\partial g_{\mu\nu}}\delta g_{\mu\nu} = A^{\mu\nu}\delta g_{\mu\nu} = g g^{\mu\nu}\delta g_{\mu\nu}.$$

Application of the chain rule yields

$$\begin{aligned} \delta\sqrt{-g} &= -\frac{1}{2}\frac{\delta g}{\sqrt{-g}}, \\ &= -\frac{1}{2}\frac{g g^{\mu\nu}}{\sqrt{-g}}\delta g_{\mu\nu}, \\ &= \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}, \qquad\qquad g^{\mu\nu}\delta g_{\mu\nu} = -g_{\mu\nu}\delta g^{\mu\nu}, \\ &= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}. \end{aligned}$$

We now look at the second term, and calculate the variation of the Ricci scalar. We can break this into a variation of the Ricci tensor and the metric

$$\delta R = \delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}.$$

The first term requires no more work. To calculate the Ricci tenor's variation, we have to take a few steps back. The Riemann tensor, written in terms of Christoffel symbols is

$$R^{\mu}{}_{\nu\rho\sigma} = s_2 \left(\partial_\rho \Gamma^{\mu}{}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\tau}{}_{\nu\sigma} \Gamma^{\mu}{}_{\tau\rho} - \Gamma^{\tau}{}_{\nu\rho} \Gamma^{\mu}{}_{\tau\sigma} \right).$$

The variation of the Riemann tensor is then given by

$$\delta R^{\mu}{}_{\nu\rho\sigma} = s_2 \left(\partial_\rho \delta\Gamma^{\mu}{}_{\nu\sigma} - \partial_\sigma \delta\Gamma^{\mu}{}_{\nu\rho} + \delta\Gamma^{\tau}{}_{\nu\sigma} \Gamma^{\mu}{}_{\tau\rho} + \Gamma^{\tau}{}_{\nu\sigma} \delta\Gamma^{\mu}{}_{\tau\rho} - \delta\Gamma^{\tau}{}_{\nu\rho} \Gamma^{\mu}{}_{\tau\sigma} - \Gamma^{\tau}{}_{\nu\rho} \delta\Gamma^{\mu}{}_{\tau\sigma} \right).$$

As mentioned, the Christoffel symbol is not a tensor, but its variation is, and taking the covariant derivative we find that

$$\nabla_\sigma (\delta\Gamma^{\mu}{}_{\nu\rho}) = \partial_d \delta\Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\sigma\tau} \delta\Gamma^{\tau}{}_{\nu\rho} - \Gamma^{\tau}{}_{\nu\sigma} \delta\Gamma^{\mu}{}_{\rho\tau} - \Gamma^{\tau}{}_{\rho\sigma} \delta\Gamma^{\mu}{}_{\nu\tau},$$

which can be used to simplify the above expression, given

$$\delta R^{\mu}{}_{\nu\rho\sigma} = s_2 \left(\nabla_\rho (\delta\Gamma^{\mu}{}_{\nu\sigma}) - \nabla_\sigma (\delta\Gamma^{\mu}{}_{\nu\rho}) \right).$$

The variation of the Ricci tensor is then found by contracting the indices:

$$s_2 s_3 \delta R_{\mu\nu} = \delta R^{\rho}{}_{\mu\rho\nu} = s_3 \left(\nabla_\rho (\delta\Gamma^{\rho}{}_{\mu\nu}) - \nabla_\nu (\delta\Gamma^{\rho}{}_{\rho\mu}) \right).$$

The last step is to write out the variation of the Christoffel symbol in terms of variation of the metric. Varying the Christoffel symbol from (2.1.4) results in

$$\delta\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} \delta g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) + \frac{1}{2} g^{\rho\sigma} (\partial_\mu \delta g_{\nu\sigma} + \partial_\nu \delta g_{\mu\sigma} - \partial_\sigma \delta g_{\mu\nu}).$$

The first term can be simplified by writing

$$\delta g^{\rho\sigma} = -g^{\rho\sigma} g^{\tau\lambda} \delta g_{\tau\lambda}.$$

To find

$$\delta\Gamma^{\rho}{}_{\mu\nu} = -\Gamma^{\rho}{}_{\mu\nu} g^{\tau\lambda} \delta g_{\tau\lambda} + \frac{1}{2} g^{\rho\sigma} (\partial_\mu \delta g_{\nu\sigma} + \partial_\nu \delta g_{\mu\sigma} - \partial_\sigma \delta g_{\mu\nu}).$$

Then with come careful rearranging of indices we obtain

$$\delta\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\nabla_\mu \delta g_{\nu\sigma} + \nabla_\nu \delta g_{\mu\sigma} - \nabla_\sigma \delta g_{\mu\nu}).$$

Finally, contracting the variation with the metric tensor we find

$$\begin{aligned}
g^{\mu\nu} \delta R_{\mu\nu} &= s_3 \left(g^{\mu\nu} \nabla_\rho (\delta\Gamma^{\rho}{}_{\mu\nu}) - g^{\mu\nu} \nabla_\nu (\delta\Gamma^{\rho}{}_{\rho\mu}) \right), \\
&= s_3 \left(\nabla_\rho (g^{\mu\nu} \delta\Gamma^{\rho}{}_{\mu\nu}) - \nabla_\nu (g^{\mu\nu} \delta\Gamma^{\rho}{}_{\rho\mu}) \right), \\
&= s_3 \nabla_\rho (g^{\mu\nu} \delta\Gamma^{\rho}{}_{\mu\nu} - g^{\mu\rho} \delta\Gamma^{\nu}{}_{ba}), \\
&= \frac{s_3}{2} \nabla_\rho (g^{\mu\nu} g^{\rho\sigma} (\nabla_\mu \delta g_{\nu\sigma} + \cancel{\nabla_\nu \delta g_{\mu\sigma}} - \nabla_\sigma \delta g_{\mu\nu}) \\
&\quad - g^{\mu\rho} g^{\nu\sigma} (\nabla_\nu \delta g_{\mu\sigma} + \cancel{\nabla_\mu \delta g_{\nu\sigma}} - \nabla_\sigma \delta g_{\mu\nu})), \\
&= s_3 \nabla^\mu (\nabla^\nu \delta g_{\mu\nu} - g^{\rho\sigma} \nabla_\mu \delta g_{\rho\sigma}).
\end{aligned}$$

In summary, the two terms varied from the Einstein-Hilbert action are given by

$$\begin{aligned} \delta\sqrt{-g} &= \frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g_{\mu\nu}, \\ \delta R &= -R^{\mu\nu}\delta g_{\mu\nu} + s_3\nabla^\mu(\nabla^\nu\delta g_{\mu\nu} - g^{\rho\sigma}\nabla_\mu\delta g_{\rho\sigma}), \end{aligned} \tag{2.3.8}$$

which when substituted into (2.3.7), we find that the variation of the Einstein-Hilbert action is given by

$$\begin{aligned} \delta S_{EH} &= \frac{s_1 s_3}{16\pi}\int_M d^4x\,(\delta\sqrt{-g}R + \sqrt{-g}\delta R), \\ &= \frac{s_1 s_3}{16\pi}\int_M d^4x\sqrt{-g}\left(\frac{1}{2}g_{\mu\nu}R - R^{\mu\nu}\right)\delta g_{\mu\nu}, \\ &+ \frac{s_1}{16\pi}\int_M d^4x\sqrt{-g}\nabla^\mu(\nabla^\nu\delta g_{\mu\nu} - g^{\rho\sigma}\nabla_\mu\delta g_{\rho\sigma}). \end{aligned}$$

The first term is recognised to be left-hand side Einstein's equations. The second term is a total derivative and for certain boundary conditions can be assumed to be zero. If we do not impose boundary conditions, we can write down the second term as an integral over the boundary ∂M of M where we define the volume form for the hypersurface Σ for an *outward* pointed normal vector as:

$$d\Sigma_\mu = \epsilon n_\mu\sqrt{|\gamma|}d^3x.$$

Using Stoke's theorem:

$$\begin{aligned} \int_M d^4x\sqrt{-g}\nabla_\mu J^\mu &= \int_M d^4x\,\partial_\mu(\sqrt{-g}J^\mu), \\ &= \int_{\partial M} J^\mu d\Sigma_\mu, \\ &= \int_{\partial M} d^3x\,\epsilon n^\mu\sqrt{|\gamma|}J_\mu. \end{aligned}$$

Which when applied to our solution gives:

$$\delta S_{EH} = \frac{s_1 s_3}{16\pi}\int_M d^4x\sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu} + \frac{s_1\epsilon}{16\pi}\int_{\partial M} d^3x\,1\sqrt{|\gamma|}n^\mu(\nabla^\nu\delta g_{\mu\nu} - g^{\rho\sigma}\nabla_\mu\delta g_{\rho\sigma}). \tag{2.3.9}$$

This boundary term is canceled through the addition of the Gibbons-Hawking-York term:

$$S_{GHY} = \frac{s_1\epsilon}{8\pi}\int_{\partial M} d^3x\,\sqrt{|\gamma|}K, \tag{2.3.10}$$

where γ is the determinant of the induced metric of the hypersurface Σ and K is the corresponding trace of the extrinsic curvature, defined in (2.2.16). To motivate the inclusion of this term we now consider:

$$\delta S_{GHY} = \frac{s_1\epsilon}{8\pi}\int_{\partial M} d^3x\,\delta(\sqrt{|\gamma|}K) = \frac{s_1\epsilon}{8\pi}\int_{\partial M} d^3x\left(\delta(\sqrt{|\gamma|})K + \sqrt{|\gamma|}\delta K\right).$$

As before, we will solve this variation term by term, beginning with the variation of the unit normal. This is achieved by writing a generic normal covector $N_\mu = \partial_\mu f$ for some function f which is constant on Σ and then appropriately normalising to find the unit covector:

$$n_\mu = \frac{\epsilon\partial_\mu f}{\sqrt{g^{\nu\rho}\partial_\nu f\partial_\rho f}}.$$

Varying this with respect to the spacetime metric we find:

$$
\begin{aligned}
\delta n_\mu &= \delta\left(\frac{\epsilon\partial_\mu f}{\sqrt{g^{\nu\rho}\partial_\nu f\partial_\rho f}}\right), \\
&= -\frac{1}{2} n_\mu \delta g^{\nu\rho} g_{\nu\rho}, \\
&= \frac{1}{2}\epsilon n_\mu n^\nu n^\rho \delta g_{\mu\nu}, \qquad\qquad \epsilon n_\mu n^\rho = g^\rho_\mu - \gamma^\rho_\mu, \\
&= \frac{1}{2} n^\nu \delta g_{\mu\nu} + c_\nu, \qquad\qquad c_\nu = -\frac{1}{2}\gamma^\rho_\mu \delta g_{\nu\rho} n^\nu,
\end{aligned}
$$

where c_ν is orthogonal to n_ν. Before continuing we write down a few useful identities which we will use:

$$
\begin{aligned}
n^\mu n^\nu n^\rho &= 0, \\
\delta n_\mu n^\mu n^\nu &= \delta n^\nu.
\end{aligned}
$$

Next we can calculate the variation of the induced metric with respect to the spacetime metric:

$$
\begin{aligned}
\delta\gamma^{\mu\nu} &= \delta(g^{\mu\nu} - \epsilon n^\mu n^\nu), \\
&= \delta g^{\mu\nu} - \epsilon\delta n^\mu n^\nu - \epsilon n^\mu \delta n^\nu, \\
&= \delta g^{\mu\nu} - \cancel{\epsilon\delta n_\rho n^\rho n^\mu n^\nu} - \cancel{\epsilon n^\mu \delta n_\rho n^\rho n^\nu}, \\
&= \delta g^{\mu\nu},
\end{aligned}
$$

and by the same reasoning:

$$
\delta\gamma^\mu_\nu = \delta(\delta^\mu_\nu - \epsilon n^\mu n_\nu) = 0.
$$

Using identical methods as for the spacetime metric we find:

$$
\delta\sqrt{|\gamma|} = -\frac{1}{2}\sqrt{|\gamma|}\gamma_{\mu\nu}\delta\gamma^{\mu\nu}.
$$

The last thing to calculate is the variation of K:

$$
\begin{aligned}
\delta K = \delta(\gamma^{\mu\nu}K_{\mu\nu}) &= \delta(\gamma^{\mu\nu}\gamma^\rho_\mu \nabla_\rho n_\nu), \\
&= \delta\gamma^{\mu\nu}\gamma^\rho_\mu \nabla_c n_\nu + \cancel{\gamma^{\mu\nu}\delta\gamma^\rho_\mu \nabla_c n_\nu} + \gamma^{\mu\nu}\gamma^\rho_\mu \delta(\nabla_c n_\nu), \\
&= \delta\gamma^{\mu\nu}K_{\mu\nu} + \gamma^{\mu\nu}\gamma^\rho_\mu \delta(\partial_\rho n_\nu - \Gamma^\sigma{}_{\nu\rho} n_d), \\
&= \delta\gamma^{\mu\nu}K_{\mu\nu} - \gamma^{\mu\nu}\gamma^\rho_\mu \delta\Gamma^\sigma{}_{\nu\rho} n_d + \gamma^{\mu\nu}\gamma^\rho_\mu \nabla_\rho(\delta n_\nu), \\
&= \delta K_{(1)} + \delta K_{(2)} + \delta K_{(3)}.
\end{aligned}
$$

To make this easier on the eyes, we now go through this expression term by term. The first term is fine as it is. The second term can be expanded:

$$
\begin{aligned}
\delta K_{(2)} &= -\gamma^{\mu\nu}\gamma^\rho_\mu \delta\Gamma^\sigma{}_{\nu\rho} n_\sigma, \\
&= -\gamma^{\mu\nu}\gamma^\rho_\mu (\frac{1}{2} g^{\sigma\tau}(\nabla_\nu \delta g_{\rho\tau} + \nabla_\rho \delta g_{\nu\tau} - \nabla_\tau \delta g_{\nu\rho})) n_\sigma, \\
&= -\frac{1}{2}\gamma^{\mu\nu}\gamma^\rho_\mu n^\tau (\nabla_\nu \delta g_{\rho\tau} + \nabla_\rho \delta g_{\nu\tau} - \nabla_\tau \delta g_{\nu\rho}),
\end{aligned}
$$

and the third term can be expanded to

$$\begin{aligned}
\delta K_{(3)} &= \gamma^{\mu\nu}\gamma^{\rho}_{\mu}\nabla_{\rho}(\delta n_{\nu}),\\
&= \gamma^{\mu\nu}\gamma^{\rho}_{\mu}\nabla_{\rho}(\frac{1}{2}n^{d}\delta g_{\nu\sigma} + c_{\nu}),\\
&= \frac{1}{2}\gamma^{\mu\nu}\gamma^{\rho}_{\mu}n^{d}\nabla_{\rho}\delta g_{\nu\sigma} + \frac{1}{2}\delta g_{\nu\sigma}\gamma^{\mu\nu}\gamma^{\rho}_{\mu}\nabla_{\rho}n^{d} + \gamma^{\mu\nu}\gamma^{\rho}_{\mu}\nabla_{\rho}c_{\nu},\\
&= \frac{1}{2}\gamma^{\mu\nu}\gamma^{\rho}_{\mu}n^{d}\nabla_{\rho}\delta g_{\nu\sigma} + \frac{1}{2}\delta g_{\nu\sigma}K^{\nu\sigma} + D^{\nu}c_{\nu}.
\end{aligned}$$

Where D_{μ} is the covariant derivative projected onto Σ. Adding these all together we find that:

$$\begin{aligned}
\delta K &= \delta K_{(1)} + \delta K_{(2)} + \delta K_{(3)},\\
&= -K^{\mu\nu}\delta\gamma_{\mu\nu} - \frac{1}{2}\gamma^{\mu\nu}\gamma^{\rho}_{\mu}n^{\tau}(\nabla_{\nu}\delta g_{\rho\tau} + \cancel{\nabla_{\rho}\delta g_{\nu\tau}} - \nabla_{\tau}\delta g_{\nu\rho}),\\
&+ \cancel{\frac{1}{2}\gamma^{\mu\nu}\gamma^{\rho}_{\mu}n^{d}\nabla_{\rho}\delta g_{\nu\sigma}} + \frac{1}{2}\delta g_{\nu\sigma}K^{\nu\sigma} + D^{\nu}c_{\nu},\\
&= -\frac{1}{2}K^{\mu\nu}\delta\gamma_{\mu\nu} - \frac{1}{2}\gamma^{\mu\nu}\gamma^{\rho}_{\mu}n^{d}(\nabla_{\nu}\delta g_{\rho\sigma} - \nabla_{\sigma}\delta g_{\nu\rho}) + D_{\mu}c^{\mu},\\
&= -\frac{1}{2}K^{\mu\nu}\delta\gamma_{\mu\nu} - \frac{1}{2}\gamma^{\mu\nu}n^{\rho}(\nabla_{\mu}\delta g_{\nu\rho} - \nabla_{\rho}\delta g_{\mu\nu}) + D_{\mu}c^{\mu}.
\end{aligned}$$

Where we have relabelled some dummy indices and cancelled out one term. We are now in a position to accumulate all of these pieces to find that:

$$\begin{aligned}
\delta S_{GHY} &= \frac{s_1\epsilon}{8\pi}\int_{\partial M} d^3x\ \delta(\sqrt{|\gamma|}K),\\
&= \frac{s_1\epsilon}{8\pi}\int_{\partial M} d^3x\left[-\frac{1}{2}\sqrt{|\gamma|}\gamma_{\mu\nu}\delta\gamma^{\mu\nu}K - \frac{1}{2}\sqrt{|\gamma|}\gamma^{\mu\nu}n^{\rho}(\nabla_{\mu}\delta g_{\nu\rho} - \nabla_{\rho}\delta g_{\mu\nu}) + \sqrt{|\gamma|}D_{\mu}c^{\mu}\right],\\
&= \frac{s_1\epsilon}{16\pi}\int_{\partial M} d^3x\sqrt{|\gamma|}\left[\left(K\gamma_{\mu\nu} - K^{\mu\nu}\right)\delta\gamma^{\mu\nu} - \frac{1}{2}\gamma^{\mu\nu}n^{\rho}(\nabla_{\mu}\delta g_{\nu\rho} - \nabla_{\rho}\delta g_{\mu\nu}) + D_{\mu}c^{\mu}\right].
\end{aligned}$$

Rewriting for comparison the variation of the Einstein Hilbert action:

$$\delta S_{EH} = \frac{s_1 s_3}{16\pi}\int_M d^4x\sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu} + \frac{s_1\epsilon}{16\pi}\int_{\partial M} d^3x\sqrt{|\gamma|}n^{\mu}(\nabla^{\nu}\delta g_{\mu\nu} - g^{\rho\sigma}\nabla_{\mu}\delta g_{\rho\sigma}),$$

we see that the inclusion of S_{GHY} will exactly cancel the boundary term which we found in the Einstein-Hilbert variation. This leads us to define a new action:

$$S = \frac{s_1 s_3}{16\pi}\int_M d^4x\sqrt{g}R + \frac{s_1\epsilon}{8\pi}\int_{\partial M} d^3x\sqrt{|\gamma|}K, \tag{2.3.11}$$

which when varied with respect to the spacetime metric gives a variational principle:

$$\delta S = \frac{s_1 s_3}{16\pi}\int_M d^4x\sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu} + \frac{s_1\epsilon}{16\pi}\int_{\partial M} d^3x\sqrt{|\gamma|}\left[\left(K\gamma_{\mu\nu} - K^{\mu\nu}\right)\delta\gamma^{\mu\nu} + D_{\mu}c^{\mu}\right].$$

We now make a few comments on this action. We know that by Einsteins equations $G_{\mu\nu} = 0$ in a vacuum, and so that term will disappear. It is usually assumed that the metric has zero variation on the boundary $\delta M = \Sigma$. This is to say that $\delta\gamma_{\mu\nu} = 0$. This will remove the second term, and as the last term is a total derivative on the boundary spacetime it is safely ignored. When this boundary condition isn't assumed, we have a further requirement of the extrinsic curvature of the hypersurface:

$$\left(K\gamma_{\mu\nu} - K^{\mu\nu}\right) = 0.$$

This is known as the *Israel Junction Condition*.

The above discussion assumes that we are working with a vacuum: $T_{\mu\nu} = 0$, but to fully recover Einstein's equations we must include a matter term into our Lagrangian. We allow this to be a generic contribution

$$S_m = \int_M d^4x\sqrt{-g}\mathcal{L}_m.$$

The energy-momentum tensor is defined by

$$T^{\mu\nu} := s_1 \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}},$$

and so we find that the variation of the matter term can be written as

$$\delta S_m = \frac{s_1}{2} \int_M d^4x\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu}.$$

Defining the total action as the Einstein-Hilbert term and the matter term together

$$S = S_{EH} + S_m = \int_M d^4x\sqrt{-g}\left(\frac{s_1 s_3 R}{16\pi} + \mathcal{L}_m\right), \tag{2.3.12}$$

its variation is found to be

$$\begin{aligned} \delta S &= \frac{s_1 s_3}{16\pi} \int_M d^4x\sqrt{-g}\left(\frac{1}{2}g_{\mu\nu}R - R^{\mu\nu}\right)\delta g_{\mu\nu} + \frac{s_1}{2}\int_M d^4x\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu}, \\ &= \frac{s_1}{16\pi}\int_M d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}R - R^{\mu\nu} + s_3 8\pi T^{\mu\nu}\right)\delta g_{\mu\nu}, \end{aligned}$$

where for clarity we suppress the boundary conditions. We see that a vanishing variation implies

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = s_3 8\pi T^{\mu\nu},$$

or in other words, Einstein's equations are recovered from varying the action (2.3.12).

3

BLACK HOLES

Black holes are an incredible consequence of general relativity naturally arising from the study of dense distributions of matter. We find that when the mass density of a gravitational object reaches a critical limit,[1] a region of spacetime appears in which it is impossible for causal information to escape. These regions are what we call as black holes, and their boundaries are known as event horizons.

One of the most extraordinary advances in modern theoretical physics was the development of black hole thermodynamics. The story begins geometrically, in the development of the laws of black hole mechanics [52, 5], a set of four laws which had a remarkable similarity to the laws of thermodynamics. The relations were similar enough to lead Bekenstein to conjecture a proportionality between a black hole's area and its entropy [4]. Through studying gravitational solutions semi-classically, Hawking was able set the constant of proportionality. Considering a curved spacetime quantum field, Hawking found that a black hole emitted thermal radiation as a black body [7], making concrete our understanding of the geometric rules derived from black holes as thermodynamic relations. The core of this thesis is work put towards better understanding this relationship between general relativity and thermodynamics.

In this chapter, we introduce the reader to many of the tools we will need while studying the planar symmetric solutions of $\mathcal{N} = 2$ supergravity. We begin in Section 3.1, looking at the mathematical structure of a horizon. We follow this in Section 3.2 by considering the causal structure of a black hole and then develop a global description of the Schwarzschild black hole. In Section 3.3, we look at the Einstein-Maxwell theory, and derive a black hole solution which is charged under the Maxwell field, known as the Reissner-Nordström solution. This is followed by a discussion of what we mean by the conserved energy of a gravitational solution in Section 3.4. We follow this with an overview of the laws of black hole mechanics and their relationship to the laws of thermodynamics in Section 3.5. In Section 3.6, we show using the Euclidean action formalism that one can derive thermodynamic quantities from the saddle-point approximation of the gravitational partition function, and use Reissner-Nordström black hole as a worked example.

[1]To understand the requirement of high density: in Section 2.3.2, we wrote down the Schwarzschild solution, describing a spherical symmetric distribution of matter. For the Schwarzschild spacetime, the event horizon is located for $r = 2M$, but for almost all matter distributions, this point in spacetime is within the distribution of matter where the Schwarzschild solution is no longer valid. Hence, we can only expect black holes to appear once the matter distribution is suitably dense such that all the matter is contained in a ball with radius $r < 2M$.

3.1 HORIZONS

A black hole is a region of spacetime for which it is impossible for a timelike or null curve to escape. In essence, the gravitational field is so strong, that one would need to travel faster than the speed of light to escape the region. We call the boundary of this region an *event horizon*, which we understand as the point of no return. This 'definition' of a black hole is sufficient to understand what we mean when we talk about a black hole, but due to the work of Hawking, Penrose and others, this definition can be made more formal. The language and tools needed to do this go beyond the scope of our discussion, and so we reference [6, 35] as two key textbooks for this result. As a black hole region is impossible to escape, we realise that to properly define the event horizon, we must understand the black hole geometry globally. It is not enough to say you cannot currently escape, you have to know that whatever happens, you never will.

In this section we introduce two horizons closely related to event horizons, the Killing horizon and the trapping horizon.[2] Killing horizons are surprisingly simple to locate given a manifold's line element, and when the spacetime is stationary, Hawking showed that the event horizon is a Killing horizon [6]. The trapping horizon, unlike the event horizon, can be defined locally, and we will find the notion of a trapping horizon useful when studying black hole thermodynamics. It has been proven that the trapping horizon is always contained within the event horizon [35].

3.1.1 *Null hypersurfaces*

For a manifold M, with a metric g, the vector field $n^\mu = \nabla^\mu f$ is normal[3] to a surface Σ defined by constant f, for any function f such that $df \neq 0$ on Σ.

Definition 3.1. A *null hypersurface* $\mathcal{N}$ is a hypersurface whose normal is everywhere null.

For any null hypersurface $\mathcal{N}$, with a normal n_μ, a vector X^μ tangent to $\mathcal{N}$ satisfies $X \cdot n = 0$. As the normal vector is null: $n \cdot n = 0$, we see that the normal is itself a tangent vector for some null curve in $\mathcal{N}$.

Proposition. *The integral curves of the normal vector field of a null hypersurface are geodesics.*

Proof. Let $\mathcal{N}$ be a null hypersurface for f = constant. We can write the unit normal vector $n = \tilde{f} df$, for some function $\tilde{f}$. For some general normal vector field $N = df$, n^μ and N^μ have the same integral curves.

Taking the covariant derivative of the norm of N evaluated on $\mathcal{N}$, we can write down

$$\nabla_\mu (N^\nu N_\nu)|_{\mathcal{N}} = 2N^\nu \nabla_\mu N_\nu = 2N^\nu \nabla_\nu N_\mu, \tag{3.1.1}$$

where we have used that $\nabla_\nu N_\mu = \nabla_\nu \nabla_\mu f = \nabla_\mu \nabla_\nu f = \nabla_\mu N_\nu$. As we also know that $N^\nu N_\nu = 0$, N^2 is constant on $\mathcal{N}$. From this, we know that the gradient of N_μ is normal to $\mathcal{N}$

$$\nabla_\mu (N^\nu N_\nu)|_{\mathcal{N}} = 2hN^\nu, \tag{3.1.2}$$

[2] The trapping horizon is the boundary of the trapping region, which we will define in better detail in Section 3.1.4. Penrose proved the singularity theorem [1], which states that within a trapping region at least one geodesic is future-inextendible, signifying the presence of a singularity. An extraordinary result showing the physical significance of spacetime singularities.

[3] Although technically the normal vector field need not be a gradient in general, see for example [53].

for some function h. Combining (3.1.1) with (3.1.2) we can write

$$N^\nu \nabla_\nu N_\mu = 2hN^\nu,$$

which we read as the geodesic equation for a non-affinely parameterised geodesic (2.2.6). We thus see integral curves of N^μ, and therefore n^μ, are geodesics. Appropriately picking the normalisation of the curve parameter, one can find an affinely parameterised geodesic. □

Definition 3.2. The *generators* of a null hypersurface $\mathcal{N}$ are the null geodesics $x^\mu(\lambda)$, with affine parameter λ, such that the tangent vectors to $x^\mu(\lambda)$ are normal $\mathcal{N}$.

Definition 3.3. A null hypersurface $\mathcal{N}$ is a *Killing horizon* when there exists a Killing vector ξ^μ normal to $\mathcal{N}$.

Hawking proved the rigidity theorem [6], which states that for a stationary black hole, the event horizon is also a Killing horizon, for a review, see [54]. We will comment again on event horizons in Section 3.2, but note here that often we work with the existence of a Killing horizon rather than an event horizon, due to the simplicity in locating a Killing horizon by considering the line element of a gravitational solution.

3.1.2 *Surface gravity*

As $\xi^2 = 0$ on a Killing horizon, then the gradient of ξ^2 will also be zero along the null hypersurface. From this, we can write down the relationship [55]

$$\nabla_\mu(\xi^\nu \xi_\nu) = -2\kappa \xi_\mu. \tag{3.1.3}$$

The function κ is known as the *surface gravity* of the Killing horizon.

Rearranging the above definition for the surface gravity by using Killing's equation, we obtain

$$\begin{aligned} \nabla_\mu(\xi^\nu \xi_\nu) &= 2\xi^\nu \nabla_\mu \xi_\nu, \qquad \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \quad \text{(Killing's Equation)} \\ &= -2\xi^\nu \nabla_\nu \xi_\mu \\ \xi^\nu \nabla_\nu \xi_\mu &= \kappa \xi^\mu. \end{aligned}$$

This allows us to geometrically interpret the surface gravity κ as the measure of the failure of λ to be an affine parameter, where $\xi^\mu = \partial_\lambda$. We see that the Killing vector ξ^μ obeys the non-affine geodesic equation (2.2.6).

We can understand the surface gravity from another perspective. Let us imagine a spaceship at rest in some static, asymptotically flat spacetime, some distance above the Killing horizon. To maintain its position within space, we can intuitively understand the spaceship as accelerating so as not to fall towards the black hole. To use a more formal language, a particle at rest in a static spacetime follows the orbit of a timelike Killing vector field k^μ. As these orbits are not geodesics, we understand this particle as accelerating.

Now instead of imagining the ship's acceleration being generated by its engines, let us imagine the ship is held in place by some rigid rod extending off to an anchor point at infinity. An observer at infinity measures the force holding the ship in place as a tension T in the rod. In the limit of the spaceship being located on the Killing horizon, the tension measured at the anchor point tends towards the surface gravity: $T \to \kappa$. We can then say that the surface gravity κ is the force needed to hold a body in place on the Killing horizon for an observer at infinite distance. The local force experienced by the spaceship is a different matter. The local tension, *i.e.* the force exerted by the ship onto the rod, diverges as the ship approaches the Killing horizon. Note for this interpretation, we have required the spacetime to be static. For rotating solutions, such as the Kerr solution [56], this interpretation is no longer valid.

3.1.3 *Expansion of null hypersurfaces*

In Section 2.2.5, we discussed geodesic congruences and showed the gauge freedom for the affine parameter was fixed for timelike and spacelike congruences by setting $U \cdot S = 0$. For null congruences, we need to fix the gauge freedom by constraining the action of S on two vector fields. In the following discussion, we look at a congruence containing the generators of a null hypersurface $\mathcal{N}$. In this case, the displacement vector is tangent to $\mathcal{N}$ such that $U \cdot S = 0$ always holds.

To see how to set the gauge freedom, let us choose a spacelike hypersurface Σ which intersects each geodesic of the congruence only once. We define our second vector field N^μ such that on the hypersurface we have

$$N^2 = 0, \qquad N \cdot U = -1.$$

We can understand N^μ as a vector field in M by extending it off Σ through parallel transport along U^μ: $\nabla_U N^\mu = 0$. If U^μ is tangent to outgoing radial null geodesics, we understand N^μ as tangent to ingoing radial null geodesics, and vice-versa.

Given our two vector fields N and U, we can now fix the gauge freedom in λ by picking the displacement vector such that

$$U \cdot S = 0, \qquad N \cdot S = 0.$$

Under this condition, the displacement vector S^μ spans a two-dimensional subspace tangent to both U^μ and N^μ. We can define a projection onto this tangent space using

$$P^\mu{}_\nu = \delta^\mu{}_\nu + N^\mu U_\nu + U^\mu N_\nu,$$

and we will refer to the projected quantities using a hat

$$\hat{B}^\mu{}_\nu = P^\mu{}_\rho B^\rho{}_\sigma P^\sigma{}_\nu .$$

We note that for null congruences which do not contain generators of a null hypersurface *i.e.* when the displacement vector is not tangent to the hypersurface, we would have to project the displacement vector,

$$\hat{S}^\mu = P^\mu{}_\nu S^\nu, \qquad S^\mu = \alpha U^\mu + \beta N^\mu + \hat{S}^\mu,$$

where we have additionally written out the form of the displacement vector for a generic null congruence.

We can interpret $\hat{B}^\mu{}_\nu$ as a matrix in the two-dimensional space tangent to U^μ and N^μ. We can decompose this matrix into its algebraically irreducible pieces

$$\hat{B}^\mu{}_\nu = \frac{1}{2}\theta P^\mu{}_\nu + \hat{\sigma}^\mu{}_\nu + \hat{\omega}^\mu{}_\nu,$$

where we have split the matrix into its trace, traceless symmetric and anti-symmetric parts:

$$\begin{aligned} \theta &= \hat{B}^{\mu}{}_{\mu}, && \textit{Expansion}, \\ \hat{\sigma}_{\mu\nu} &= \hat{B}_{(\mu\nu)} - \frac{1}{2} P_{\mu\nu} \hat{B}^{\rho}{}_{\rho} && \textit{Shear}, \\ \hat{\omega}_{\mu\nu} &= \hat{B}_{[\mu\nu]} && \textit{Twist}. \end{aligned}$$

We interpret these values in the following way. Let us take two vector fields $V_{\pm}^{\mu}$ which are orthogonal to U^{μ} and N^{μ}. The two vectors $V_{\pm}^{\mu}$ define an area element in the space tangent to both U^{μ} and N^{μ} given by the expression

$$A = \varepsilon^{\mu\nu\rho\sigma} U_{\mu} N_{\nu} V_{+|\rho} V_{-|\sigma}.$$

The *shear* measures the change of shape of this area while maintaining its magnitude. We can think of this as describing the geodesics moving apart in one direction and towards each other in another. The *expansion*, measures the change in the magnitude of the area. When $\theta > 0$, we say understand that the geodesics are moving away from each other (expanding) and for $\theta < 0$ they come together (contraction). To see how the expansion measures the change in the area,, we can vary the area A with respect to our affine parameter to find:

$$\begin{aligned} \frac{dA}{d\lambda} &= U \cdot \nabla A = \varepsilon^{\mu\nu\rho\sigma} U_{\mu} N_{\nu} \left[U \cdot \nabla \left(V_{+|\rho} V_{-|\sigma} \right) \right], \\ &= \varepsilon^{\mu\nu\rho\sigma} U_{\mu} N_{\nu} \left[\hat{B}^{\lambda}{}_{\rho} V_{\lambda|+} V_{\sigma|-} + \hat{B}^{\lambda}{}_{\sigma} V_{\lambda|+} V_{\rho|-} \right], \\ &= \varepsilon^{\mu\nu\rho\sigma} U_{\mu} N_{\nu} \hat{B}^{\lambda}{}_{\rho} \left(V_{\lambda|+} V_{\sigma|-} - V_{\sigma|+} V_{\lambda|-} \right), \\ &= \varepsilon^{\mu\nu\rho\sigma} U_{\mu} N_{\nu} \hat{B}^{\lambda}{}_{\lambda} V_{\rho|+} V_{\sigma|-} = A\theta. \end{aligned}$$

From this, we can then understand the expansion as measuring the rate of increase of the area with respect to the affine parameter λ.

The *twist* $\hat{\omega}_{\mu\nu}$, measures how the geodesics twist around each other. For congruences containing the generators null hypersurface $\mathcal{N}$, the twist is everywhere zero on $\mathcal{N}$. If $\hat{\omega}_{\mu\nu} = 0$, then U^{μ} is hypersurface orthogonal [35].

An alternative and useful expression for the expansion is given by

$$\theta = \nabla_{\mu} U^{\mu} = g^{\mu\nu} \nabla_{\mu} U_{\nu}, \tag{3.1.4}$$

which can be derived from

$$\theta = P^{\mu}{}_{\nu} B^{\nu}{}_{\rho} P^{\rho}{}_{\mu},$$

expanding out all the terms and using that

$$U^{\mu} \nabla_{\mu} U^{\nu} = U^{\mu} \nabla_{\nu} U^{\mu} = 0.$$

This representation of the expansion shows us that θ is independent of the vector field N^{μ} and should be understood as an intrinsic property of the congruence.

Later when discussing the laws of black hole mechanics, we will consider a congruence containing the generators of a Killing horizon. With this additional property for the vector fields, we find that on the Killing horizon, all of these quantities vanish.

Proposition. *The expansion, shear and rotation for a geodesic congruence containing the generators of a Killing horizon are all zero evaluated on the Killing horizon:* $\theta = \hat{\sigma} = \hat{\omega} = 0$.

Proof. As the generators of the Killing horizon are hypersurface orthogonal, the rotation $\hat{\omega} = 0$, as discussed before. What remains is to show that the symmetric part of $\hat{B}_{(\mu\nu)} = 0$. We denote the Killing vector field ξ^μ, which is normal to the Killing horizon $\mathcal{N}$. On $\mathcal{N}$, we can relate the Killing vector field to tangents of the generators of the hypersurface U^μ with some generic function h on the hypersurface: $\xi^\mu = hU^\mu$.

To calculate the expansion and shear, we use that our Killing horizon is specified by the zero of some function $f = 0$. This allows us to write down the generator

$$U^\mu = h^{-1}\xi^\mu + fV^\mu,$$

for some vector field V^μ. Taking the covariant derivative, we find

$$B_{\mu\nu} = \nabla_\mu U_\nu = \partial_\mu(h^{-1})\xi_\nu + h^{-1}\nabla_\mu\xi_\nu + \partial_\mu(f)V_\nu + f\nabla_\mu V_\nu,$$

Taking only the symmetric part, and evaluating on the hypersurface, we find

$$B_{(\mu\nu)}\big|_{\mathcal{N}} = \xi_{(\nu}\partial_{\mu)}h^{-1} + V_{(\nu}\partial_{\mu)}f,$$

where we have used Killing's Equation (2.2.8) and that $f = 0$. However, when we project this onto the orthogonal space $T_\perp$ we see that all terms vanish

$$\hat{B}_{(\mu\nu)} = P^\rho{}_\mu B_{(\rho\sigma)} P_\nu{}^\sigma = 0,$$

as both the Killing vector ξ^μ and the derivative $\partial_\mu f$ are parallel to the vector field U^μ. □

3.1.4 *Trapping and apparent horizons*

For an observer in a spacetime, how can they decide if they're within a black hole region? The event horizon is defined as a region of spacetime in which it is impossible to send a signal to infinity. The requirement of properly dealing with *impossible* and *infinity* necessitates a global description for the spacetime. For certain solutions, we may only have a local patch of the solution we wish to probe, and in these scenarios, we cannot describe the black hole region.

A *trapping horizon* is a way to talk about the local structure of a spacetime. It is the boundary of a trapping region, which can be thought of as a local analogue of a black hole region. Let us pick a spacelike hypersurface Σ within our spacetime, and on that, identify a closed hypersurface S. For every point on S, we have two future-directed null vector fields $\ell^\mu_\pm$ which correspond to two families of null geodesics. We can understand each family of geodesics generating two null hypersurfaces $\mathcal{N}_+$ and $\mathcal{N}_-$. In the cases we are concerned with, null geodesics with tangents $\ell_\pm$ will correspond to the ingoing and outgoing light rays within the spacetime. We can identify a trapped surface by calculating the expansion of these vector fields.

Definition 3.4. A closed, spacelike hypersurface S is a *trapped surface* if both families of null geodesics orthogonal to the hypersurface have an everywhere negative expansion. When the expansions are everywhere non-positive, we say that this surface is *marginally* trapped.

Looking at the spacelike hypersurface Σ, we can find the *trapped regions* $\mathcal{T}$ by looking at the union of all trapped surfaces $S \subset \Sigma$. The boundary of this region is called the *apparent horizon* $\mathcal{A} = \partial\mathcal{T}$. Calculating the expansions on the apparent horizon, we find that it is a marginally trapped surface.

Definition 3.5. Using the foliation of a spacetime M with codimension-one spacelike hypersurfaces Σ_t, one can find the set of all trapped regions $\mathcal{T}_t$. The set of their boundaries $\mathcal{A}_t = \partial\mathcal{T}$ are the codimension-two apparent horizons. We define the *trapping horizon* $\mathcal{T}_{\mathcal{H}}$ as the union of all apparent horizons.

It can be shown that if weak cosmic censorship conjecture is correct, then the trapping region is contained within the black hole region [6]. This implies that the apparent horizon, and hence the trapping horizon, lies either on or inside of the event horizon. Thus, we understand that if a local observer measures the existence of a trapped surface, and hence the apparent horizon, they can conclude that they lie within the black hole region.

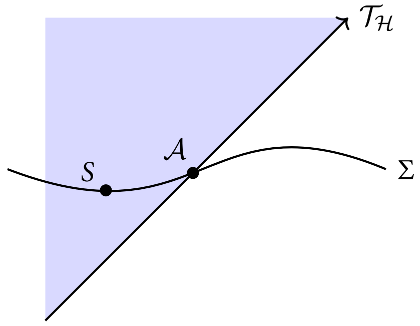


Figure 3.1: Illustration of a trapping horizon in a spacetime M. Σ is a codimension-one spacelike hypersurface. A trapped surface S is a closed codimension-two spacelike hypersurface on Σ. The union of all trapped surfaces along Σ is the trapped region $\mathcal{T}$, it's boundary is the apparent horizon $\mathcal{A}$. Evolving to the past and future from Σ yields a set of trapped regions, shaded blue, the boundary of this region is the trapping horizon $\mathcal{T}_{\mathcal{H}}$.

3.1.5 *Classification of trapping horizons*

By considering the signs of the expansion of future-directed ingoing and outgoing null vector fields $\ell_\pm$, we can write down four distinct trapping horizons based on the work of [57, 58]. Denoting the expansions of $\ell_\pm$ as $\theta_\pm$, we distinguish the horizons in the following way. We say a trapping horizon is a *future horizon* when $\theta_+ = 0$ and $\theta_- < 0$ on $\mathcal{T}_{\mathcal{H}}$, and a *past horizon* when $\theta_- = 0$ and $\theta_+ > 0$ on $\mathcal{T}_{\mathcal{H}}$. We can further distinguish trapping horizons by considering how the sign of $\theta_\pm$ changes as we cross the horizon in the direction of $\ell_\mp$. We can calculate this sign with the Lie derivative: $\mathcal{L}_{\ell_\mp}\theta_\pm$, evaluated on the horizon. When $\mathcal{L}_{\ell_\mp}\theta_\pm < 0$, we say that the horizon is an *outer horizon*. Using the convention that the non-trapping region bounded by this horizon has $\theta_+ > 0$ and $\theta_- < 0$, we say that the horizon is an *inner horizon* when $\mathcal{L}_{\ell_\mp}\theta_\pm > 0$. In summary, we can identify the four types of horizons as:

(i) Future outer horizons: $\theta_+ = 0,\ \theta_- < 0,\ \mathcal{L}_{\ell_-}\theta_+ < 0$. The sign of θ_+ changes from positive to negative with increasing ℓ_-. We understand that 'outside the horizon', the outgoing congruence is expanding, while 'inside the horizon' both congruences contract. Therefore future outer horizons can be taken as local definitions of black holes.

(ii) Past outer horizons: $\theta_- = 0,\ \theta_+ > 0,\ \mathcal{L}_{\ell_+}\theta_- < 0$. The sign of θ_- changes from positive to negative with increasing ℓ_+. We understand that 'outside the horizon' the ingoing congruence is expanding, while 'inside the horizon' both congruences expand. We can understand outer horizons as local definitions of white holes.[4]

(iii) Future inner horizons: $\theta_+ = 0,\ \theta_- < 0,\ \mathcal{L}_{\ell_-}\theta_+ > 0$. The sign of θ_+ changes from negative to positive with increasing ℓ_-. The inside region is non-trapping while in the outside region, both congruences contract. We can understand future inner horizons as the local definition of contracting cosmologies, where all null congruences become converging for large enough distances from the observer.

(iv) Past inner horizons: $\theta_- = 0,\ \theta_+ > 0,\ \mathcal{L}_{\ell_+}\theta_- > 0$. The sign of θ_- changes from positive to negative with increasing ℓ_+. The interior region is non-trapping while in the exterior region both congruences expand. We can therefore understand past inner horizons as a local definition of expanding cosmologies, where all null congruences become expanding for large enough distances from the observer.

We will return to these classifications when we discuss the temperature associated to trapping horizons.

3.1.6 *Vaidya spacetime*

We now take a brief detour to consider an example which illustrates a case when the event horizon and trapping horizon do not coincide, following a discussion from [55].

Let us consider a spherically symmetric black hole that for some finite time absorbs some null dust, increasing the mass of the black hole. Taking the Schwarzschild and allowing the mass to be a function of the null coordinate v, we obtain the *ingoing Vaidya solution* with a line element in Eddington-Finkelstein coordinates given by [59]

$$ds^2 = -\left(1 - \frac{2M(v)}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2.$$

This is a solution to Einstein's equations describing null dust which itself can be understood as a pressureless fluid. We assume that the black hole absorbs this null dust for a finite interval, where the mass parameter of the solution is given by

$$M(v) = \begin{cases} M_1, & v \leq v_1, \\ M(v), & v_1 < v < v_2, \\ M_2, & v \geq v_2. \end{cases}$$

[4] We discuss white holes in some more detail in Section 3.2.1, but for now we can understand them as the time-reversal of a black hole.

We assume $M_2 > M_1$ and that the growth of $M(v)$ is smooth. In [55], the expansion for the null congruences is computed and it can be shown that the apparent horizon is always located at $r = 2M(v)$.

The event horizon for $v > v_2$ is, as expected, located at $r = 2M_2$, where we can consider this as the Schwarzschild solution of mass $M = M_2$. The surprising result comes at $v < v_2$. The event horizon is the causal boundary from future null infinity and so is located for $r = 2M_2$ throughout the spacetime, as it is the null hypersurface extending from the result of $v > v_2$. We see that the location of the event horizon requires all future knowledge of some non-stationary system, which is in direct contrast to the locations of the apparent horizons which we can compute given any time slicing.

For $v > v_2$, when there is no longer any dynamic changes to the solution, it should be clear that the trapping and event horizons coincide. However, for $v < v_2$, the horizons are distinct, with the trapping horizon contained within the event horizon. We can then understand that some observer within the spacetime could locally measure themselves as exterior to the Trapping horizon, but ultimately still unable to escape the black hole region if they are interior to the event horizon. An illustration of this is given in Figure 3.2.

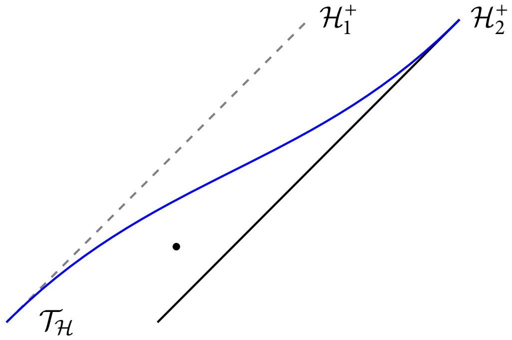


Figure 3.2: An illustration of the horizons for a Vaidya black hole solution being irradiated by a null dust. The dashed event horizon $\mathcal{H}_1^+$ would be the boundary if the mass remained at $M = M_1$. The blue curve depicts the trapping horizon built from the union of apparent horizons each located for $r = 2M(v)$. The event horizon $\mathcal{H}_2^+$ is the boundary of the causal past of future null infinity and is determined by the final mass M_2 of the solution. We see that an observer $\mathcal{O}$ (depicted as a point) outside of the trapping horizon in this non-stationary solution can still be trapped within the black hole.

3.2 CAUSAL STRUCTURE OF BLACK HOLES

In this section, we use the Schwarzschild solution to further discuss blacks holes from the perspective of their causal structure. We start from the line element (2.3.3) and see that the region of spacetime we are able to probe is limited by coordinate singularities. Through making coordinate changes, we will extend the validity of the coordinate range and hence a greater perspective of the Schwarzschild solution. To aid our discussion, we reproduce the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

We start looking at the Schwarzschild solution by determining for what ranges our coordinate system is valid. In particular, we look for divergence of the components of the metric. We see the usual problems for $\theta = 0$ and $\theta = \pi$, which stem from the necessity for more than one coordinate chart to cover S^2. We also notice two divergences associated to $r = 2M$, and $r = 0$, in which the metric components g_{rr} and g_{tt} diverge respectively. As we work with differential manifolds, we must therefore limit the coordinate range of the radial coordinate r to the domain $2M < r < \infty$.[5] What we find is that the divergence for $r = 2M$ comes from inappropriate coordinates (as is for the case of the coordinate chart of the two-sphere), but the singularity at $r = 0$ is a property of the solution itself.

So given a solution with a metric with apparently divergent components, how can we distinguish the physical singularities from coordinate singularities? To identify a singularity, we look for a geodesic which is not extendible within the spacetime. In Penrose's singularity theorem, it is shown that within a trapping region there is always at least one null geodesic which has a finite length [1]. The termination of the geodesic highlights a singularity, and for the Schwarzschild solution, this occurs at $r = 0$. An alternative way to look for singularities is to study divergences of the curvature scalars of the solution. As scalars are diffeomorphism invariant, we know these divergences do not stem from a bad choice of coordinates. The Schwarzschild solution has $R = 0$, and so the Ricci scalar will yield no information about the singularities. Another scalar we study is called the Kretschmann scalar

$$K = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}, \tag{3.2.1}$$

which is quadratic in the Riemann tensor, and so quartic in the metric tensor. We do not detail the calculations, but it can be found [60] that for the Schwarzschild solution, the Kretschmann scalar is

$$K = \frac{48M^2}{r^6}.$$

We see that K is finite for $r = 2M$, but diverges in the limit for $r = 0$, signalling that regardless of the coordinate system, there is singular behaviour at $r = 0$.

We note briefly that we must remember the conditions for which the Schwarzschild solution is valid. When solving Einstein's equations, we assume that the spacetime is spherically symmetric, stationary and a *vacuum* solution, *i.e.* the energy-momentum tensor is zero. This means that for a standard spherically symmetric distribution of matter (such as a star, or even our own planet) the Schwarzschild metric is only valid in the exterior of the matter content. For everything but black holes, the surface of the matter distribution is located for $r_0 >> 2M$. This means that the singular point of the solution is beyond the scope of validity. For black holes, which this thesis is concerned with, the coordinate and physical singularities must be distinguished and understood. The true nature of the physical singularities which appear in solutions to Einstein's equations require models extending beyond the classical limit of general relativity into theories of quantum gravity. From the point of view of this thesis, we gently acknowledge the presence of singularities and their relationship to general relativity, but restrict our discussion to smooth manifolds. To

[5] One could instead limit the solution to the range $0 < r < 2M$, but for this line element, the coordinate r would be timelike and the solution would not be stationary. We discuss this in more detail throughout the thesis.

do this, we choose to limit our coordinates in a way that the singularities such as the one at $r = 0$ are evaded.

Our work now is to understand how we can write down the Schwarzschild solution using an alternative coordinate system that allows us to probe the spacetime for $0 < r \leq 2M$.

3.2.1 *Eddington-Finkelstein coordinates*

To study the Schwarzschild solution on the horizon $r = 2M$ and beyond that towards the singularity to $r \to 0$, we use a coordinate system which is motivated by the null geodesics.

Null radial geodesics obey the relation (2.2.7)

$$0 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \quad \Rightarrow \quad dt^2 = \left(\frac{r}{r - 2M}\right)^2 dr^2 = dr_\star^2,$$

where we have defined the new 'tortoise' coordinate

$$r_\star = r + 2M \log\left|\frac{r - 2M}{r}\right|, \qquad 0 < r_\star < \infty.$$

Radial null geodesics obey $d(t \mp r_\star) = 0$ and so we can define new *ingoing* and *outgoing* null coordinates from

$$v = t + r_\star, \quad u = t - r_\star, \quad -\infty < u \leq v < \infty.$$

where the coordinates (v, r, θ, ϕ) and (u, r, θ, ϕ) are known as *ingoing* and *outgoing Eddington-Finkelstein coordinates* respectively.

3.2.1.1 *Ingoing Eddington-Finkelstein coordinates*

As an example, we can use ingoing coordinates to rewrite the metric as

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

In this new coordinate system, we see that all components are smooth for $r > 0$, and that the coordinate singularity at $r = 2M$ has been removed.

We can now extend the Schwarzschild solution for $r < 2M$. If we want, we can even reverse the coordinate transformation after allowing $r < 2M$ back to the coordinates (t, r, θ, ϕ). This analytic extension of the Schwarzschild solution for $r > 2M$ to $r < 2M$ using a coordinate transformation will be a trick used throughout this thesis. We note that after changing to $r < 2M$ we have

$$0 < r < 2M \;:\; \left(1 - \frac{2M}{r}\right) < 0,$$

and so the Killing vector $k^\mu = \partial/\partial t$ is no longer timelike! After crossing the horizon, the timelike/spacelike properties of the coordinates (t, r) switch. This is easier to see if we write the metric functions such that they are positive valued for the domain of the coordinates, and so the term in the metric with an overall negative sign can be quickly identified as the timelike coordinate

$$ds^2 = \left(\frac{2M}{r} - 1\right) dt^2 - \left(\frac{2M}{r} - 1\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

For the patch given by $r > 2M$, we were able to define a time orientation from the Killing vector. Using ingoing Eddington-Finkelstein coordinates, the Killing vector is given by $\partial/\partial v$ and is spacelike for $r < 2M$. We can recover a time orientation by noticing that $\pm\partial/\partial r$ is globally null. The time orientation is fixed by setting the sign, ensuring that ∂_r shares a lightcone with the Killing vector for $r > 2M$:

$$k \cdot \left(\pm\frac{\partial}{\partial r}\right) = \pm g_{vr} = \pm 1,$$

and so we see that $-\partial/\partial r$ is a good time orientation for all $r > 0$ when working with Eddington-Finkelstein coordinates.

3.2.1.2 *Black Hole region*

This new patch of spacetime, covered by the coordinates (t, r, θ, ϕ) for $0 < r < 2M$, is the black hole region of the spacetime. The boundary of this region, given by the hypersurface $r = 2M$ is the event horizon. Using the ingoing Eddington-Finkelstein coordinates, we can show that all causal curves which have points for $r \leq 2M$ will be unable to escape to $r > 2M$. Formally, we can write this statement in the following way

Proposition. *For all future-directed causal curves (timelike or null) $x(\lambda)$ with $r(\lambda_0) \leq 2M$, we will have $r(\lambda) \leq 2M$ for $\lambda > \lambda_0$.*

Proof. The tangent vector X^μ is a future oriented causal vector. Using our time orientation, we know that

$$\left(-\frac{\partial}{\partial r}\right) \cdot X = -g_{\mu r} X^\mu = -X^v = -\frac{dv}{d\lambda} \leq 0, \quad \Rightarrow \quad \frac{dv}{d\lambda} \geq 0, \tag{3.2.2}$$

and so along any future-directed causal curve, v is non-decreasing. To see whether the curve is trapped within the region $r \leq 2M$, we can look at

$$X^2 = -\left(1 - \frac{2M}{r}\right)\left(\frac{dv}{d\lambda}\right)^2 + 2\frac{dv}{d\lambda}\frac{dr}{d\lambda} + r^2\left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right],$$

to study the behaviour of $dr/d\lambda$. When $r < 2M$, we can rearrange the above expression such that all terms on the right-hand side are non-negative:

$$-2\frac{dv}{d\lambda}\frac{dr}{d\lambda} = -X^2 + \left(\frac{2M}{r} - 1\right)\left(\frac{dv}{d\lambda}\right)^2 + r^2\left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right],$$

and so we can see that

$$\frac{dr}{d\lambda}\frac{dv}{d\lambda} \leq 0. \tag{3.2.3}$$

To understand the sign of $dr/d\lambda$, let us assume that $dr/d\lambda > 0$. For the relation (3.2.3) to hold and be consistent with (3.2.2), we must have $dv/d\lambda = 0$ and hence

$$0 = -X^2 + r^2\left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right],$$

but as both terms here on the right-hand side are non-negative, they must both be zero. This means that if $dr/d\lambda > 0$, the only non-zero term of X^μ is

$$X^\mu = X^r = \frac{dr}{d\lambda} > 0.$$

Now, the only non-zero component is X^μ is X^r, which we see is positive and so X^μ is *past directed*. This is a contradiction from our initial assumption and hence, by proof by contradiction, for a future-directed causal curve we must have

$$\frac{dr}{d\lambda} \leq 0.$$

We see that when $r \leq 2M$, $dr/d\lambda$ is monotonically decreasing and so for a future-directed curve with $r(\lambda_0) \leq 2M$, we will have $r(\lambda) \leq 2M$ for all $\lambda > \lambda_0$. □

3.2.1.3 *Outgoing Eddington-Finkelstein coordinates*

Using ingoing Eddington-Finkelstein coordinates, we analytically continued the Schwarzschild metric to describe the region of spacetime for $r < 2M$ and showed that this region, with the boundary for $r = 2M$, describes a black hole region of the spacetime. Using the outgoing coordinates (u, r, θ, ϕ) instead, we can write down the metric for the spacetime as

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2du dr + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

From this metric, we can again analytically continue the Schwarzschild solution for $r < 2M$. This is a distinct patch of spacetime from the black hole region.

To see the difference, we can look at surfaces of constant $u = t - r_\star$, where the outgoing null geodesics obey $dr/d\tau = 1$. As such, as we increase proper time, the geodesics propagate from $r > 0$, through $r = 2M$ and off to infinity. More than this, if we followed a similar argument to the one given above for the black hole region, we find that it is *impossible* for a signal not to reach $r \geq 2M$ from this region of space; any causal geodesic within the region $r \leq 2M$ will cross the horizon at $r = 2M$ in finite proper time. We can understand this region as the time-reversal of a black hole, and refer to it as a *white hole*.

3.2.2 *Kruskal coordinates*

By performing coordinate changes, we see that we have been able to extend the range of the coordinates on our manifold to cover new regions of spacetime. We can formalise this with the notion of analytic extension.

Definition 3.6. A spacetime (M, g) is *extendable* if it is isometric to a proper subset of a spacetime $(\bar{M}, \bar{g})$. We refer to $(\bar{M}, \bar{g})$ as the analytic extension. We call the spacetime $(\bar{M}, \bar{g})$ the *maximal analytic extension* when $(\bar{M}, \bar{g})$ cannot be extended.

So far, we have seen that with ingoing or outgoing Eddington-Finkelstein coordinates we can extend the Schwarzschild solution to find an additional region for $r < 2M$. We can think of the Schwarzschild solution as (M, g) and the extended spacetime which includes the black (white) hole region as $(\bar{M}, \bar{g})$.

We now show that by performing an additional coordinate transformation, we can write down the *maximal extension* of Schwarzschild solution, including the three regions previously discussed, and a fourth patch with a second asymptotically flat region. The maximal extension of the Schwarzschild solution is known as the Kruskal spacetime.

We begin by making a coordinate change from the Schwarzschild solution using the null coordinates

$$v = t + r_\star, \qquad u = t - r_\star, \qquad -\infty < u, v < \infty,$$

with a resulting metric given by

$$ds^2 = -f(r)dudv + r^2 d\Omega^2, \qquad f(r) = 1 - \frac{2M}{r}, \tag{3.2.4}$$

where we should understand that the radial coordinate $r = r[r_\star(u,v)]$ is a function of our null coordinates. To remove the degeneracy for $r = 2M$, we use Kruskal coordinates

$$U = -e^{-u\kappa}, \qquad V = e^{v\kappa}.$$

The surface gravity (3.1.3) is calculated to be

$$f(r_h) = 0 \Rightarrow r_h = 2M, \qquad \kappa = \frac{1}{2}\partial_r f(r)\Big|_{r=r_h} = \frac{1}{4M}.$$

Null Kruskal coordinates are therefore given by

$$\begin{aligned} U &= +e^{-u/(4M)}, & 0 < U < \infty \quad &\leftrightarrow \quad -\infty < u < \infty, \\ V &= -e^{v/(4M)}, & -\infty < V < 0 \quad &\leftrightarrow \quad -\infty < v < \infty. \end{aligned}$$

Performing the coordinate redefinition, we obtain the metric

$$ds^2 = -\frac{32M^3 e^{-r/(2M)}}{r(U,V)} dUdV + r^2 d\Omega^2, \tag{3.2.5}$$

where the coordinates (t, r) can be expressed as functions of (U, V) by

$$\begin{aligned} UV &= -e^{r_\star/(2M)} = -e^{r/(2M)}\left(\frac{r}{2M} - 1\right), \\ \frac{V}{U} &= -e^{t/(2M)}. \end{aligned}$$

We refer to the region for $U > 0,\ V < 0$ as region I, which covers the same patch of spacetime as the original Schwarzschild coordinates. Through selecting the signs of the coordinates (U, V), we cover new regions of the spacetime, which are separated by Killing horizons located at $U = 0$ and $V = 0$. In Figure 3.3, the four regions obtained through setting the signs of (U, V) are illustrated.

Region II can be obtained by extending (3.2.4) for $r < 2M$ by setting

$$r_\star = r + 2M \log\left|\frac{r}{2M} - 1\right|,$$

which takes on values for $0 > r_\star > -\infty$ for $0 < r < 2M$. Here, the Kruskal coordinates are given by

$$U = +e^{-u/(4M)}, \qquad V = +e^{v/(4M)}.$$

In region II, $U, V > 0$, while u, v can take all real values individually. However, the ranges of both (u, v), and (U, V) are restricted by $r > 0$. Regions I and II together cover the spacetime covered by ingoing Eddington-Finkelstein coordinates.

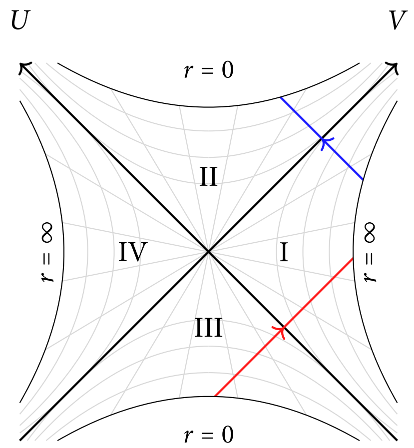


Figure 3.3: Kruskal diagram for the Schwarzschild solution. Surfaces of constant r are hyperbola and surfaces of constant t are straight lines. Also included are the ingoing (blue) and outgoing (red) null geodesics which are future-pointing.

Allowing $(U,V) < 0$ brings us to region III. Regions I and III cover the spacetime found using outgoing Eddington-Finkelstein coordinates.

The new region of spacetime is region IV, where the null Kruskal coordinates take values $U > 0$ and $V < 0$. In this region, one can reintroduce null coordinates u, v by

$$\begin{aligned} U &= +e^{-u/(4M)}, \qquad & 0 < U < \infty \qquad &\leftrightarrow \quad \infty > u > -\infty, \\ V &= -e^{v/(4M)}, \qquad & -\infty < V < 0 \qquad &\leftrightarrow \quad \infty > v > -\infty. \end{aligned}$$

Observe that (u,v) are directed opposite to (U,V) in region IV. If we go back from (u,v), to $(t,r_\star)$ and (t,r), the metric assumes the same local form (2.3.3) as in region I, but globally, there is a difference. Compared to region I, (t,r) point the opposite way: t downwards, r leftwards. Ingoing lightfronts move in positive V = negative v direction. Outgoing lightfronts move in the positive U = negative u direction. The association of (U,V) with in/out-moving lightfronts is reversed compared to region I.

The global spacetime is time-orientable and time-reversal symmetric and so we should not conclude that time is flowing backwards in region IV. If we choose a global time orientation that points in the same direction as t in Region I, then physical time in region IV is measured by $-t$. We conclude that region IV is a copy of region I, with a flipped time orientation. Notice that this second copy of an asymptotically flat region of spacetime is spacelike separated from region I, and so no causal curve will ever travel between them. We note that taking a surface of constant t through the Kruskal spacetime is a hypersurface passing through region I into region IV. This manifold has a topology of $\mathbb{R} \times S^2$ and is known as an *Einstein-Rosen bridge*, or in popular culture as a 'wormhole'.

3.2.3 *Classification of horizons*

This brings us to a good point to work through an example of the classification of the horizons (see Section 3.1.5) of the Schwarzschild solution through studying the expansion of null geodesics. Let us take a two-dimensional surface S in our spacetime such that all tangent vectors are spacelike. At a point $p \in S$, there will be exactly two *future-directed* null vectors ℓ_+, ℓ_- orthogonal to S. These null geodesics form two hypersurfaces.

Using the coordinates (U, V), we can write down the Killing vector field associated with the staticity of the spacetime. In (t, r) coordinates, this is given by[6]

$$k = \frac{\partial}{\partial t}, \qquad k^\flat = -f(r)dt, \qquad k \cdot k = -f(r).$$

Using that

$$\begin{aligned} VdU - UdV &= \frac{e^{(v-u)/(4M)}}{4M}(du + dv), \\ &= \frac{e^{r_\star/(2M)}}{2M}dt, \\ &= \frac{e^{r/(2M)}}{2M}\left(\frac{r}{2M} - 1\right)dt. \end{aligned}$$

We can calculate the covector of the Killing vector k using Kruskal coordinates:

$$\begin{aligned} k^\flat &= -f(r) \cdot 2Me^{-r/(2M)}\left(\frac{r}{2M} - 1\right)^{-1}(VdU - UdV), \\ &= \frac{4M^2}{r}e^{-r/(2M)}(-VdU + UdV), \end{aligned}$$

or the corresponding vector

$$k = \frac{1}{4}\left(-U\frac{\partial}{\partial U} + V\frac{\partial}{\partial V}\right).$$

To check our result, we can compute

$$k \cdot k = -\left(1 - \frac{2M}{r}\right),$$

and see that this matches with the previous calculation.

For the Kruskal spacetime, we can consider S as the sphere at a point in M by considering surfaces of constant (U, V). We thus have two null hypersurfaces which are formed for constant $U = U_0$ and $V = V_0$. We can parameterise these with

$$\ell_+^\flat = -16M^3dU, \qquad \ell_-^\flat = -16M^3dV,$$

where the constants have been chosen to ensure the following vector fields are future-directed, together with a factor to simplify the form of the vectors. Using (3.2.5), we can raise the indices and obtain two *future-directed* null vector fields

$$\ell_+ = re^{r/(2M)}\left(\frac{\partial}{\partial V}\right)^\mu, \qquad \ell_- = re^{r/(2M)}\left(\frac{\partial}{\partial U}\right)^\mu.$$

We can verify $\ell_\pm$ are future-directed, by computing

$$k \cdot \ell_+ = 4M^2U, \qquad k \cdot \ell_- = -4M^2V,$$

which are both negative in region I, where $U < 0$ and $V > 0$.

[6]We use 'musical' notation to distinguish between vectors and the corresponding covectors (one-forms).

We are interested in the expansion of these geodesics. Using the formula

$$\theta_\pm = \nabla_\mu \ell^\mu_\pm = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\ell^\mu_\pm\right),$$

we find that

$$\theta_+ = 2e^{r/(2M)}\frac{\partial r}{\partial V}, \qquad \theta_- = 2e^{r/(2M)}\frac{\partial r}{\partial U}.$$

We can find an exact form for the partial derivatives by looking at the following

$$\begin{aligned} VdU + UdV &= \frac{e^{(v-u)/(4M)}}{4M}\left(du - dv\right), \\ &= -\frac{e^{2r_\star/(2M)}}{2M}dr_\star, \\ &= -\frac{e^{2r/(2M)}}{2M}\left(\frac{r}{2M}-1\right)\left(1-\frac{2M}{r}\right)^{-1}dr, \\ &= -\frac{r}{4M}e^{r/(2M)}dr, \end{aligned}$$

such that

$$\frac{\partial r}{\partial U} = -\frac{4M}{r}e^{-r/(2M)}V, \qquad \frac{\partial r}{\partial V} = -\frac{4M}{r}e^{-r/(2M)}U,$$

and so the expansions are given by

$$\theta_+ = -\frac{8M^2}{r}U, \qquad \theta_- = -\frac{8M^2}{r}V.$$

From the above expressions, we see that the signs of $\theta_\pm$ are determined by the sign of U, V. On the horizon between regions I and II, we have that $U = 0$ and hence $\theta_+ = 0$. Similarly, the horizon between region I and III is given by $V = 0$, and so $\theta_- = 0$. To determine the type of horizon, we look at the Lie derivative, evaluated on the horizon. For $U = 0$, we have that $\theta_+ = 0$, and

$$\begin{aligned} \mathcal{L}_{\ell_-}\theta_+ &= re^{r/(2M)}\frac{\partial}{\partial U}\left(-\frac{8M^2}{r(U,V)}U\right), \\ &= -8M^2e^{r/(2M)} + 8M^2Ue^{r/(2M)}\frac{1}{r}\frac{\partial r}{\partial U}. \end{aligned}$$

Evaluated on $U = 0$ gives us the expression

$$\mathcal{L}_{\ell_-}\theta_+ = -8M^2e^{r/(2M)} < 0.$$

By similar arguments, we find that on the horizon $V = 0$ that

$$\mathcal{L}_{\ell_+}\theta_- = -8M^2e^{r/(2M)} < 0.$$

This shows we have two horizons with the following property. In region I, we have $U < 0$ and $V > 0$. For the horizon given by $U = 0$, we have:

$$\theta_+ = 0, \quad \theta_- < 0, \quad \mathcal{L}_{\ell_-}\theta_+ < 0,$$

which is a *future outer horizon*. For the horizon set by $V = 0$, we have the following data for the expansions

$$\theta_+ > 0, \quad \theta_- = 0, \quad \mathcal{L}_{\ell_+}\theta_- < 0,$$

which is a *future inner horizon*.

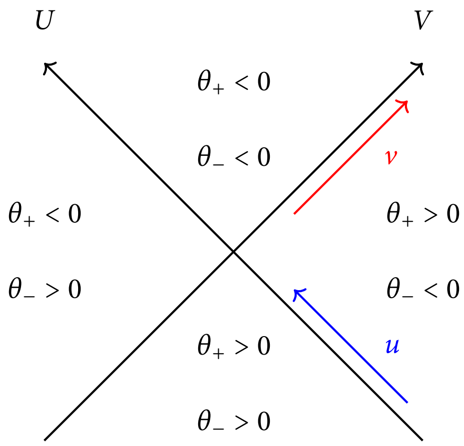


Figure 3.4: Signs of the expansions $\theta_\pm$ in the four quadrants of the Kruskal diagram of the Schwarzschild solution. The red arrow is outwards pointing and flows with increasing v, the blue arrow is inwards pointing and flows with increasing u.

3.2.4 *Penrose-Carter diagram*

The Kruskal coordinates of the Schwarzschild solution give a representation for the global structure of the spacetime. We are able to plot the horizons, singularities and surfaces of constant coordinates, but the asymptotic region built of the 'points at infinity' can't be included into the Kruskal diagram.

The aim of a Penrose-Carter diagram is to compactify the spacetime in such a way that points at infinity are mapped to a finite point while maintaining the lightcone structure of the spacetime. To do this, we make a conformal transformation of the metric $\bar{g} = \Omega^2 g$, where Ω is a smooth function defined in the spacetime M. This new metric $\bar{g}$ allows the extension from $(M, \bar{g})$ to a larger, unphysical spacetime $\bar{M}$. To bring infinity to constant values, we require that $\Omega \to 0$ in the asymptotic limit; we can then understand the physical spacetime M in $\bar{M}$ with a boundary ∂M for $\Omega = 0$. In the following discussion, we perform this transformation on Minkowski spacetime as an example and then proceed to the Schwarzschild solution.

3.2.4.1 *Minkowski spacetime*

Using ingoing and outgoing null coordinates

$$v = t + r, \qquad u = t - r, \qquad -\infty < u \leq v < \infty,$$

we can write down the Minkowski solution as

$$g = -dudv + \frac{1}{4}(v-u)^2\left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

The aim now is to make a coordinate transformation such that the range of the null coordinates is finite. We can do this using the transformation

$$u = \tan x, \qquad v = \tan y, \qquad -\frac{\pi}{2} < x \leq y < \frac{\pi}{2},$$

such that the metric is in the form

$$g = (2\cos x \cos y)^{-2}\left[-4dxdy + \sin^2(y-x)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right].$$

Notice that in this form, all our coordinates have a finite range and that there is a conformal factor in the metric. By picking $\Omega = 2\cos x \cos y$, we can write down

$$\bar{g} = \Omega^2 g = -4dxdy + \sin^2(y-x)\left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

We now perform the coordinate transformation

$$T = x + y \in (-\pi, \pi), \qquad R = y - x \in [0, \pi),$$

such that the metric is of the form

$$\bar{g} = -dT^2 + dR^2 + \sin^2 R\left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

In this form, we understand the metric $\bar{g}$ having the topology $I \times S^3$ for the interval $T \in I = (-\pi, \pi)$. If instead we had $T \in \mathbb{R}$, the above metric would be the Einstein static universe with the topology $\mathbb{R} \times S^3$ [35]. We understand $\bar{g}$ as a section of the Einstein static universe for bounded T.

Plotting the spacetime above onto the (T, R) plane, we obtain the Penrose-Carter diagram for Minkowski space (Figure 3.5) where each point on the diagram represents a two-sphere. The boundary of the diagram corresponds to either $r = 0$, or to the points at infinity.

To understand the 'infinite distance' points, we can study the radial geodesics of the spacetime. Radial null geodesics are the null curves of the flat metric $ds^2 = -dT^2 + dR^2$ and so run as straight lines at an angle of 45°. These null curves originate from the null surface $\mathcal{J}^-$ (given by a surface of constant $T - R$) pass through the origin for $r = 0$ and end at the null surface $\mathcal{J}^+$, given by a surface of constant $T + R$. We refer to $\mathcal{J}^+$ as *future null infinity* and $\mathcal{J}^-$ as *past null infinity*. All timelike geodesics originate from the point i^- and end at i^+ which are named *past timelike infinity* and *future timelike infinity* respectively. We see that these points are located for $(T, R) = (\pm\pi, 0)$ on the Penrose-Carter diagram, and are therefore the points $(u, v) = (\pm\infty, \pm\infty)$ written in terms of the original null coordinates for the metric g. Finally, all spacelike curves originate from the point i^0, which we refer to as *spatial infinity* located for $(T, R) = (0, \pi)$, or $(u, v) = (-\infty, \infty)$.

3.2.4.2 *Kruskal spacetime*

We can carry out the same kind of coordinate transformation on the metric (3.2.5) to 'compactify' the Kruskal diagram to obtain the Penrose-Carter diagram for the Schwarzschild solution. It is possible to do this carefully with a set of coordinate transformations as we have done previously; luckily for us though, the explicit work isn't necessary. As the Schwarzschild solution is asymptotically flat, the structure of the boundary of the Penrose-Carter diagram for the Schwarzschild solution will be of the same form as the Minkowski diagram. As there are two copies of an asymptotic spacetime for the Kruskal spacetime, there will then be two distinct sets of infinite boundaries for the Penrose-Carter diagram.

Null geodesics will still be lines running at 45°, originating at $\mathcal{J}^-$, only now we find that not all null geodesics will terminate at $\mathcal{J}^+$. Future-directed null geodesics which pass through the event

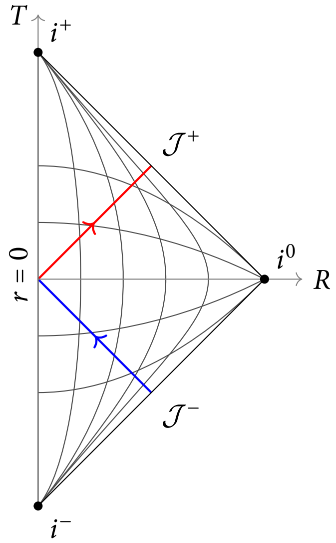


Figure 3.5: Penrose-Carter diagram for Minkowski spacetime. The points $i^{\pm}$ are future (past) timelike infinity, at the points $(T, R) = (\pm\pi, 0)$, i^0 is spacial infinity at the point $(T, R) = (0, \pi)$ and $\mathcal{J}^{\pm}$ are future (past) null infinity and are the null surfaces for surfaces of constant $T \pm R$.

horizon will inevitably fall to the singularity at $r = 0$. Conventionally, the conformal factor Ω used the set the asymptotic boundary is also picked such that the singularity $r = 0$ is represented as a horizontal line. As this point is not within our spacetime, it is represented by a wavey line in the diagram. Timelike geodesics still originate from i^-, but as with the null geodesics, only those which do not pass the event horizon will end at i^+. This same story is repeated under time reversal. All causal geodesics will end at $\mathcal{J}^+$ or i^+, but the contribution to these points will come from $\mathcal{J}^-$ and i^-, or from the white hole horizon.

Putting this all together, we can draw the Penrose-Carter diagram for the Schwarzschild solution, shown in Figure 3.6. We note that this method of constructing the Penrose-Carter diagram piece by piece and comparing with known structures, is how the diagrams are conventionally built. We will apply this same method in Section 3.3, as well as in our novel solutions in Chapter 5 and Chapter 6.

3.2.5 *Asymptotic flatness*

Leading to this point, we have mentioned that the Schwarzschild solution is asymptotically flat, with the understanding that at an infinite distance, the manifold (M, g) looks like the Minkowski spacetime. Here we discuss this in a bit more detail.

A common way to study systems is by assuming that they are isolated. By understanding Minkowski space as the background of a solution, we can think of asymptotic flatness as the statement that from an infinite distance from a source, spacetime appears to be empty.

In this thesis, we consider planar symmetric solutions to the field equations, and for all the solutions we derive, we find that the solutions are not asymptotically flat. Asymptotic flatness is a key assumption for the derivation of many quantities in black hole physics. In a sense, we can

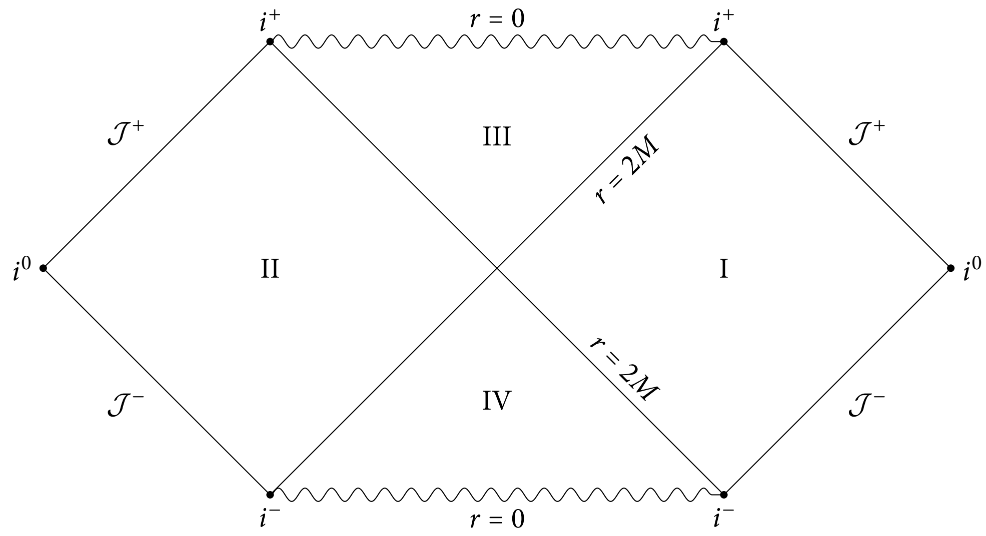


Figure 3.6: Penrose-Carter diagram for the Schwarzschild solution.

understand the existence of an asymptotically flat region to the solution as giving us a natural background to work from.

From the perspective of the metric tensor, we say that the spacetime is asymptotically flat if there exists a system of coordinates $\{t, x, y, z\}$, where $r^2 = x^2 + y^2 + z^2$ such that the metric tensor can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x^\mu),$$

and the tensor $h_{\mu\nu}$ depends on r in the following ways

$$\begin{aligned} \lim_{r\to\infty} h_{\mu\nu}(x^\mu) &\quad \to \quad \mathcal{O}(r^{-1}), \\ \lim_{r\to\infty} h_{\mu\nu,\rho}(x^\mu) &\quad \to \quad \mathcal{O}(r^{-2}), \\ \lim_{r\to\infty} h_{\mu\nu,\rho\sigma}(x^\mu) &\quad \to \quad \mathcal{O}(r^{-3}). \end{aligned}$$

However, this coordinate-dependent method to identify asymptotically flat spacetimes is difficult to work with in general. This method will be enough for our work, but we note that the best way to classify asymptotically flat spacetimes is by showing that the structure of null infinity $\mathcal{J}^\pm$ is the same as for the Minkowski solution. More details on this are given in [6, 35].

3.3 REISSNER-NORDSTRÖM SOLUTION

3.3.1 *Solving the equations of motion*

In this section, we derive the static, spherically symmetric solution to Einstein-Maxwell theory to produce a charged black hole known as the *Reissner-Nordström* solution. The action for Einstein-Maxwell theory is given by

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}(-R - F^2). \tag{3.3.1}$$

Where $F = dA$ is the 2-form field strength. We note here that we adopt the relativistic convention for the gauge coupling which is $g = 4\pi$ and $G = 1$, chosen to unify the couplings from the Ricci and Maxwell terms. We will allow the black hole to be supported by both electric and magnetic charges. Einstein's equations are given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\left(F_{\mu\rho} F_{\nu}{}^{\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right),$$

where we have used that the energy-momentum tensor is

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} = \frac{1}{8\pi} \left(F_{\mu\rho} F_{\nu}{}^{\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right). \tag{3.3.2}$$

Using that $T_{\mu\nu}$ is traceless, we can rewrite Einstein's equations as

$$R_{\mu\nu} = -8\pi T_{\mu\nu}, \tag{3.3.3}$$

and Maxwell equations are

$$\nabla_{\mu} F^{\mu\nu} = 0, \qquad \nabla_{[\mu} F_{\nu\rho]} = 0.$$

We begin our solution by writing down a metric ansatz compatible with a stationary, spherically symmetric spacetime

$$ds^2 = -e^{2F(r,t)} dt^2 + e^{2H(r,t)} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2),$$

where the timelike direction t, is orthogonal to all spacelike directions. Considering the electromagnetic field strength, our assumption of spherical symmetry results in the only non-zero field components being the radial ones. For the electric field we have

$$E_r = F_{tr} = -F_{rt} = \alpha(t,r),$$

and for the magnetic field we use

$$B_{\mu} = g_{\mu\nu} \epsilon^{t\nu\rho\sigma} F_{\rho\sigma},$$

to write down the form of the radial component of the magnetic field

$$B_r = \frac{2 g_{rr}}{r^2 \sin\theta} F_{\theta\phi}.$$

For B_r to have no θ component, the field strength must be of the form[7]

$$F_{\theta\phi} = \beta(t,r) r^2 \sin\theta,$$

which we can use to write down the most general field strength tensor for a spherically symmetric solution

$$F_{\mu\nu} = \begin{pmatrix} 0 & \alpha(t,r) & 0 & 0 \\ -\alpha(t,r) & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta(t,r) r^2 \sin\theta \\ 0 & 0 & -\beta(t,r) r^2 \sin\theta & 0 \end{pmatrix}.$$

[7] Note here that the explicit r^2 term appears for cosmetic reasons which become apparent in the following calculations.

From this and Equation (3.3.2), we compute the non-zero components of the energy-momentum tensor

$$\begin{aligned}
T_{tt} &= \frac{1}{8\pi}\left(\alpha^2(t,r)e^{-2H} + \beta^2(t,r)e^{2F}\right), \\
T_{rr} &= -\frac{1}{8\pi}\left(\alpha^2(t,r)e^{-2F} + \beta^2(t,r)e^{2H}\right), \\
T_{\theta\theta} &= \frac{r^2}{8\pi}(\alpha^2 e^{-2(F+H)} + \beta^2), \\
T_{\phi\phi} &= T_{\theta\theta}\sin^2\theta.
\end{aligned} \tag{3.3.4}$$

We now turn our attention to calculating the components of the Ricci scalar. Using (2.1.4), and the components of the inverse metric

$$g^{\mu\nu} = \begin{pmatrix} -e^{-2F(t,r)} & 0 & 0 & 0 \\ 0 & e^{-2H(t,r)} & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix},$$

the non-zero values of (2.1.4) are

$$\begin{aligned}
&\Gamma^t{}_{tt} = \partial_t F(t,r), && \Gamma^t{}_{tr} = \partial_r F(t,r), && \Gamma^t{}_{rr} = e^{2(H-F)}\partial_t H(t,r), \\
&\Gamma^r{}_{tt} = e^{2(F-H)}\partial_r F(t,r), && \Gamma^r{}_{tr} = \partial_t H(t,r), && \Gamma^r{}_{rr} = \partial_r H(t,r), \\
&\Gamma^r{}_{\theta\theta} = -re^{-2H}, && \Gamma^r{}_{\phi\phi} = -r\sin^2\theta e^{-2H}, && \Gamma^\theta{}_{r\theta} = \frac{1}{r}, \\
&\Gamma^\theta{}_{\phi\phi} = -\cos\theta\sin\theta, && \Gamma^\phi{}_{r\phi} = \frac{1}{r}, && \Gamma^\phi{}_{\theta\phi} = \cot\theta.
\end{aligned}$$

The Riemann tensor expressed in terms of the Christoffel symbols is given in (2.2.2) and when contracted yields the expression for the Ricci tensor

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} = -\left(\partial_\lambda\Gamma^\lambda{}_{\mu\nu} - \partial_\nu\Gamma^\lambda{}_{\mu\lambda} + \Gamma^\lambda{}_{\lambda\tau}\Gamma^\tau{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\tau}\Gamma^\tau{}_{\mu\lambda}\right). \tag{3.3.5}$$

Substituting in the values for the Christoffel symbols, the non-zero components of the Ricci tensor are found to be

$$\begin{aligned}
R_{tt} &= -e^{2(F-H)}\left(\frac{\partial F}{\partial r}\left(\frac{\partial F}{\partial r} - \frac{\partial H}{\partial r} + \frac{2}{r}\right) + \frac{\partial^2 F}{\partial r^2}\right) - \frac{\partial H}{\partial t}\left(\frac{\partial F}{\partial t} - \frac{\partial H}{\partial t}\right) - \frac{\partial^2 H}{\partial t^2}, \\
R_{rr} &= -e^{2(H-F)}\left(\frac{\partial^2 H}{\partial t^2} + \frac{\partial H}{\partial t}\left(\frac{\partial H}{\partial t} - \frac{\partial F}{\partial t}\right)\right) - \frac{\partial F}{\partial r}\left(\frac{\partial H}{\partial r} + \frac{\partial F}{\partial r}\right) + \frac{\partial^2 F}{\partial r^2} - \frac{2}{r}\frac{\partial H}{\partial r}, \\
R_{\theta\theta} &= e^{-2H}\left(r\frac{\partial F}{\partial r} - r\frac{\partial H}{\partial r} + 1\right) + 1, \\
R_{\phi\phi} &= R_{\theta\theta}\sin^2\theta, \\
R_{tr} &= -\frac{2}{r}\frac{\partial H}{\partial t}.
\end{aligned} \tag{3.3.6}$$

With (3.3.3) and that $T_{tr} = 0$, we can deduce that

$$R_{tr} = \frac{2}{r}\frac{\partial H}{\partial t} = 0, \quad \Rightarrow \quad H(r,t) = h(r).$$

Adding components from (3.3.4) we find that

$$T_{rr} + e^{2(h-F)}T_{tt} = 0,$$

which when substituted into (3.3.3), we obtain

$$R_{rr} + e^{2(h-F)} R_{tt} = \frac{2}{r}\left(\frac{\partial F}{\partial r} - \frac{\partial h}{\partial r}\right) = 0.$$

This allows us to pull apart the functional dependence of $F(t,r)$ and find

$$\frac{\partial F}{\partial r} - \frac{\partial h}{\partial r} = 0 \quad \Rightarrow \quad F(r,t) = g(t) - h(r).$$

Through redefinition of the coordinate t, we can absorb $g(t)$ such that $F(t,r) = f(r) = -h(r)$ and the line element for our solution takes the form

$$ds^2 = -e^{2f(r)} dt^2 + e^{-2f(r)} dr^2 + r^2 d\Omega^2.$$

Before continuing, we note here that through the assumption that our solution is spherically symmetric and stationary, we obtain a static line element with no extra assumptions.

The last differential equation to solve is

$$R_{\theta\theta} = -8\pi T_{\theta\theta}, \tag{3.3.7}$$

but before we do this, let's look at the Maxwell equations to determine the exact form of the functions $\alpha(t,r)$ and $\beta(t,r)$.

Maxwell's equations can be simplified using the identity

$$\begin{aligned}\nabla_\mu F^{\mu\nu} &= \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} F^{\mu\nu}\right), \\ &= \frac{1}{r^2 \sin\theta} \partial_\mu \left(r^2 \sin\theta F^{\mu\nu}\right).\end{aligned}$$

In particular, if we look at the r-component we obtain

$$\partial_t F_{tr} = 0 \quad \Rightarrow \quad \alpha(t,r) = \alpha(r). \tag{3.3.8}$$

For the t-component of the equation we obtain:

$$\partial_r (r^2 F^{rt}) = \partial_r (g^{tt} g^{rr} F_{tr}) = 0 \quad \Rightarrow \quad \alpha(r) = \frac{C}{r^2}.$$

We can fix the value of the constant using Gauss' law. Doing this we find

$$\alpha(r) = \frac{Q}{r^2},$$

where Q is the total electric charge of the black hole.

To obtain the form for $\beta(t,r)$, we use the Bianchi identity

$$\nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} + \nabla_\rho F_{\mu\nu} = 0.$$

When the connections are expanded in terms of Christoffel symbols, we find we are left only with partial derivatives

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0.$$

Choosing our indices such that $\mu = t$, $\nu = \phi$ and $\rho = \theta$ we find

$$\partial_t F_{\theta\phi} = 0 \quad \Rightarrow \quad \beta(t,r) = \beta(r).$$

If we instead pick $\mu = r$, $\nu = \theta$ and $\rho = \phi$, we obtain the differential equation

$$\partial_r(r^2\beta) = 0 \quad \Rightarrow \quad \beta(r) = \frac{P}{r^2},$$

where P is the total magnetic charge of the black hole.

Turning our attention back to (3.3.7) we find that:

$$\begin{aligned} R_{\theta\theta} &= \partial_r(re^{2f}) - 1 = -8\pi T_{\theta\theta}, \\ &\Rightarrow \partial_r(re^{2f}) = 1 - 8\pi\frac{r^2}{8\pi r^4}(Q^2+P^2), \\ &\Rightarrow \partial_r(re^{2f}) = 1 - \frac{1}{r^2}(Q^2+P^2). \end{aligned}$$

Integrating with respect to r, we obtain

$$e^{2f(r)} = 1 - \frac{\mu}{r} + \frac{e^2}{r^2}, \tag{3.3.9}$$

where $e = \sqrt{Q^2+P^2}$ is the total charge. To set the final integration constant μ, we can take the limit in which the charges tend to zero and compare our result with Newtonian gravity by taking the weak field approximation

$$ds^2 = -\left(1+\frac{\mu}{r}\right)dt^2 + \left(1+\frac{\mu}{r}\right)^{-1}dr^2 + r^2 d\Omega^2. \tag{3.3.10}$$

The line element for Newtonian gravity is known [35]

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)dr^2 + r^2 d\Omega^2, \tag{3.3.11}$$

where the gravitational potential Φ is given by[8]

$$\Phi = -\frac{M}{r}.$$

By comparing (3.3.11) with (3.3.10) we can identify

$$\mu = -2M,$$

where M is the mass of the black hole. Substituting this into (3.3.9) we obtain the *Reissner-Nordström* solution

$$\begin{aligned} ds^2 &= -\left(1-\frac{2M}{r}+\frac{e^2}{r^2}\right)dt^2 + \left(1-\frac{2M}{r}+\frac{e^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega^2, \\ A &= -\frac{Q}{r}dt - P\cos\theta d\phi, \quad e = \sqrt{Q^2+P^2}, \end{aligned} \tag{3.3.12}$$

where A is the gauge potential from which we obtain the field strength $F = dA$. The solution (3.3.12) is the general solution of the coupled Maxwell-Einstein equations for a spherically symmetric spacetime, and can be proven as the unique solution with a generalisation of Birkhoff's

[8] We reiterate that we are working in natural units where Newton's constant $G = 1$.

theorem [61]. The Reissner-Nordström solution is parametrised by three constants: the mass M, the electric charge Q and magnetic charge P.

In Section 3.4 we will discuss in more detail how the mass parameter of a black hole solution can be derived, but for now we will use that the mass is exactly given by the parameter M. We can compute the conserved electric and magnetic charges of the black hole with the integrals

$$\mathcal{Q} = \lim_{r\to\infty} \frac{1}{4\pi} \int_{S^2} \star F, \qquad \mathcal{P} = \lim_{r\to\infty} \frac{1}{4\pi} \int_{S^2} F. \tag{3.3.13}$$

These integrals can be understood as the curved spacetime generalisation of Gauss' law. These integrals will pop up again and again throughout the thesis, but in Section 4.2 we consider Abelian gauge fields and their conserved charges. As an example, we can use the field strength computed from (3.3.12) and the Hodge-star to compute the conserved electric charge

$$\begin{aligned} \mathcal{Q} &= \lim_{r\to\infty} \frac{1}{4\pi} \int_{S^2} \star F, \\ &= \lim_{r\to\infty} \frac{1}{4\pi} \int g^{tt} g^{rr} F_{tr} r^2 \sin\theta d\theta d\phi, \\ &= Q. \end{aligned} \tag{3.3.14}$$

In a similar way, the magnetic charge is given by

$$\mathcal{P} = \lim_{r\to\infty} \frac{1}{4\pi} \int_{S^2} F = P. \tag{3.3.15}$$

3.3.2 *Causal structure*

Unlike the Schwarzschild solution, we see the function

$$f(r) = \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right) = \frac{(r-r_+)(r-r_-)}{r^2}, \qquad r_\pm = M \pm \sqrt{M^2 - e^2}, \tag{3.3.16}$$

will have two zeros, provided that $M > e$. For the case with $M = e$, there will be a single repeated zero for $r = M$. This solution is known as the *extremal solution* and will be further discussed in Section 3.3.3. For the case of $M < e$, there are no zeros for (3.3.16) in the domain $r > 0$. The Kretschemann scalar for the Reissner-Nordström solution is given by [60]

$$K = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48M^2r^2 - 96Me^2r + 56e^2}{r^8},$$

which shows the presence of a curvature singularity for $r = 0$. We then understand the case for $M < e$ being a solution with a singularity without a horizon. These are known as *naked singularities* and are generally deemed unphysical as they violate the cosmic censorship hypotheses.[9] As such, we will assume that $M \geq e$ for the following discussion. If one were to imagine a ball of matter with $M < e$, we would find the electromagnetic repulsion would prevent gravitational collapse. For quantum particles such as the electron with $Q > M$, we cannot employ these classical arguments.

[9]The cosmic censorship hypotheses comes in two forms. The weak statement is that no singularities can be seen from future null infinity, and so must be hidden behind a horizon; this is what we will refer to by cosmic censorship in this thesis. The strong cosmic censorship hypothesis states that generically, the maximal Cauchy development of some initial dataset is inextendible. The strong cosmic censorship hypothesis – despite the name – does not imply the weak version and Penrose's version [62] of the strong version was disproven in [63].

The solution (3.3.12) derived from the Einstein-Maxwell action is well defined for $r > r_+$. The singularities for $r = r_\pm$ are coordinate singularities and it is possible to analytically continue through the horizons for $r = r_\pm$ with appropriate coordinate transformations to describe spacetime regions which extend from $0 < r < \infty$.

We can perform an Eddington-Finkelstein transformation by identifying our tortoise coordinate

$$dr_\star = \frac{dr}{f(r)} \qquad \Rightarrow \qquad r_\star = r + \frac{1}{2\kappa_+}\log\left|\frac{r-r_+}{r_+}\right| + \frac{1}{2\kappa_-}\log\left|\frac{r-r_-}{r_-}\right|, \tag{3.3.17}$$

with ingoing and outgoing null coordinates

$$v = t + r_\star, \qquad u = t - r_\star,$$

such that when expressed using ingoing coordinates the line element can be written as

$$ds^2 = -f(r)dv^2 + 2dvdr_\star + r^2 d\Omega^2.$$

This line element is smooth for $r > 0$. In our Eddington-Finkelstein coordinates, the timelike Killing vector for $r > r_+$ is given by

$$k = \frac{\partial}{\partial v}, \qquad k^\flat = -f(r)dv + dr, \qquad k^2 = -f(r).$$

The null hypersurfaces located for $r = r_\pm$ are therefore Killing horizons. We denote the Killing horizon for $r = r_+$ as the *outer horizon* and it can be shown that the region for $r < r_+$ is a black hole region with the event horizon located at $r = r_+$ [55]. The Killing horizon for $r = r_-$ is the inner horizon, and is formally known as a *Cauchy horizon*. A Cauchy horizon is a null hypersurface which is also the boundary for the domain of validity of a Cauchy problem. Cauchy horizons are unstable [64], but as we will not be researching the stability of the Killing horizons we discuss in this thesis, we leave further discussion to the given references.

In this coordinate system, we can compute the surface gravity (3.1.3) for the horizons located for $r = r_\pm$ and find that

$$\begin{aligned}\kappa_\pm &= \frac{1}{2}\partial_r f(r)\bigg|_{r=r\pm}, \\ &= \frac{(r-r_+)+(r-r_-)}{2r^2} - \frac{(r-r_+)(r-r_-)}{r^3}\bigg|_{r=r\pm}, \\ &= \frac{r_\pm - r_\mp}{2r_\pm^2}.\end{aligned} \tag{3.3.18}$$

Studying the line element given by the ingoing Eddington-Finkelstein coordinates, we see that the coordinates (t, r) will be timelike/spacelike depending on the sign of the function $f(r)$. Starting in the region $r > r_+$, we can see that in the limit of $r \to \infty$, the line element (3.3.12) approaches the Minkowski solution. Crossing the event horizon at $r = r_+$, $f(r) < 0$, the coordinates (t, r) are spacelike and timelike respectively. The region of spacetime between $r_+ > r > r_-$ is therefore time-dependent. The Cauchy horizon for $r = r_-$ is a point in time, which will eventually be crossed for all causal paths.[10] There is a physical singularity at $r = 0$, which is a point in space which can then be avoided.[11]

In Poisson's textbook [55], the Kruskal coordinate system for the Reissner-Nordström solution is given great care. However, it is possible to construct the Penrose-Carter diagram for the spacetime patch-wise, based off the discussion above. The diagram is given in Figure 3.7, with a conformal factor picked such that the timelike singularities are depicted with a vertical wavey line. Thinking of some future-directed causal curve, one can follow the path of a geodesic which begins from $\mathcal{J}^-$ in region VIII, moves through the event horizon into region III, and inevitably crosses the Cauchy horizon and reaches the region with a singularity in region IV. The regions VII and VIII are isomorphic, as are regions II and IV. Unlike the Schwarzschild solution though, the causal curve is not doomed to remain in the region containing the singularity and can cross the horizon at $r = r_-$ into region I, which is isomorphic to region III. Here, the horizon $r = r_+$ is a point in time and the causal curve must continue through into region V or VI, which are isomorphic not only with each other, but also to VII and VIII. We see that the causal path has traversed from one asymptotically flat patch into another one! We might think that this contradicts the assertion that the null hypersurface ar $r = r_+$ is an event horizon, seeing as the path has 'escaped' but what we must consider is that the regions VIII and VI are distinct in that one cannot communicate between the regions. Once a causal path crosses the horizon in region VIII, it is impossible for it to reach $\mathcal{J}^+$ in region VIII.

3.3.3 *Extremal solution*

For the special case of $M = e$, the solution is said to be *extremal*. The line element for the extremal Reissner-Nordström solution is given by

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega^2, \tag{3.3.19}$$

where for $r > M$, the spacetime is static and the hypersurface for $r = M$ is a Cauchy horizon. The singularity at $r = M$ is a coordinate singularity, and by computing the Kretschmann Scalar

$$K = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{8M^2(7M^2 - 12MR + 6r^2)}{r^8},$$

we can identify $r = 0$ as a physical singularity.

We may analytically continue the line element through making the coordinate transformation

$$r_* = r + 2M \log\left|\frac{r - M}{M}\right| - \frac{M^2}{r - M},$$

and introducing an ingoing Eddington–Finkelstein coordinate $v = t + r_*$. From this, we can write down a line element which extends past $r < M$ towards the singularity at $r = 0$

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dv^2 + 2dvdr + r^2 d\Omega^2.$$

[10] We can think of the inevitably of crossing $r = r_-$ in the same way that the curvature singularity is unavoidable for causal curves that cross $r = 2M$ in the Schwarzschild solution.

[11] This is in contrast to the spacelike singularity of the Schwarzschild solution which is a point of time and is thus reached for all causal geodesics within the black hole region.

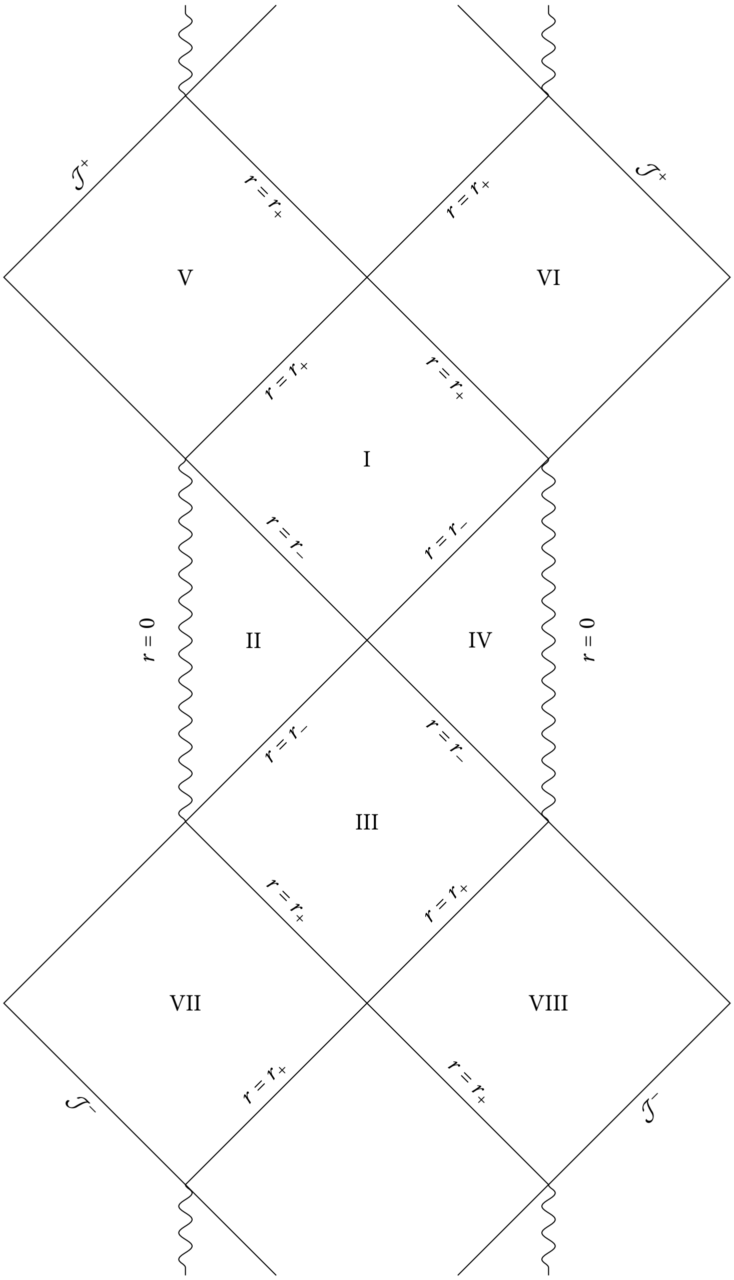


Figure 3.7: Segment of the Penrose-Carter diagram for Reissner-Nordström solution. New patches can be reached in an infinite stack from above and below the diagram.

The extremal solution has an interesting behaviour for surfaces of constant t. Unlike Schwarzschild, where a surface of constant time is an Einstein-Rosen bridge connecting two spacetime regions, a surface of constant t is an infinite throat. This can be seen by calculating the proper distance from a point $x > M$ to $M + \epsilon$:

$$s = \int_{M+\epsilon}^{x} \frac{rdr}{r-M} = \left[M\log(r-M) + r \right]_{M+\epsilon}^{x},$$

which diverges as $\epsilon \to 0$. Looking more closely at the near horizon metric, we can understand this better. For $r = M(1+\epsilon)$, the metric (3.3.19) can be expanded for $\epsilon << 1$ and we find

$$ds^2 = -\epsilon^2 dt^2 + \frac{M^2}{\epsilon^2} d\epsilon^2 + M^2 d\Omega^2.$$

This is the *Robinson-Bertotti* metric, which is isometric to $AdS_2 \times S^2$.

As with the non-extremal solution, the singularity is timelike and so avoidable for causal paths. We can think of the extremal solution as the limit where both horizons for $r = r_\pm$ coincide, and the time-dependent region of the spacetime is lost. In the extremal Reissner-Nordström solution, an observer in region I has two choices. It can either follow a causal path towards $\mathcal{J}^+$, or it may cross the Cauchy horizon at $r = M$. After crossing the horizon, unlike in the Schwarzschild solution, the spacetime region remains static. This is because the coordinate r remains spacelike throughout the geometry as $f(r) \geq 0$ for all r. The singularity is located for $r = 0$, which is simply a point in space and can be avoided. The causal observer can then continue through the horizon at $r = M$ into a new, asymptotically flat spacetime (region I′). See Figure 3.8, for an illustration.

We can see that the area of the black hole is finite: $A = 4\pi r_h^2 = 4\pi M^2$, but if we compute the surface gravity on the Killing horizon using (3.1.3) we find that

$$\begin{aligned}
\kappa &= \frac{1}{2}\partial_r \left(1 - \frac{M}{r}\right)^2 \bigg|_{r=M}, \\
&= \frac{M}{r^2}\left(1 - \frac{M}{r}\right)\bigg|_{r=M}, \\
&= 0.
\end{aligned}$$

We refer to Killing horizons with vanishing surface gravity as an *extremal event horizon* or a *degenerate Killing horizon*. In this way, we can talk of taking the extremal limit as the limit of vanishing surface gravity, we will use this later in Section 6.5. We see that in the limit $e \to M$, the surface gravity calculated in (3.3.18) vanishes as expected.

3.4 CALCULATING MASS IN GENERAL RELATIVITY

Our first experience of gravity in any formal way involves the notion of weight and with this, the concept of mass. It's then certainly a strange fact that when we consider a generic pseudo-Riemannian manifold (M, g) in general relativity, there's not an obvious way to write down a sensible quantity for the mass. The diffeomorphism invariance within general relativity prevents us from assigning a total momentum four-vector, and as a result, we cannot the measure mass parameter *à la* special relativity. If we wish to have a meaningful definition for a mass-like quantity in general relativity, we need some additional structure.

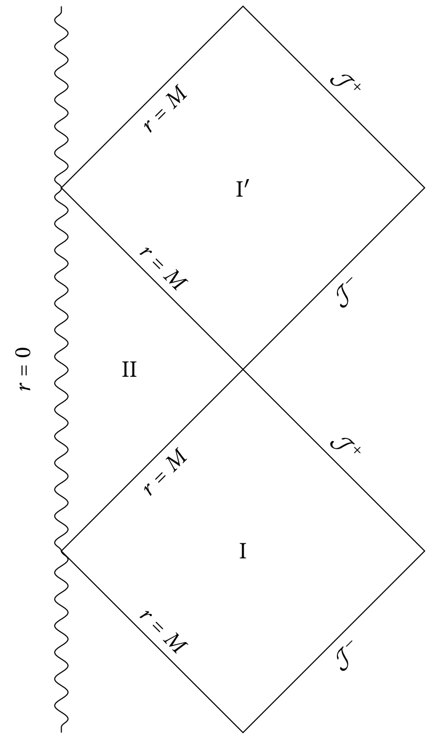


Figure 3.8: Segment of the Penrose-Carter diagram for extremal Reissner-Nordström solution. As with the non-extremal solution, this diagram can be extended above and below, infinitely.

When we consider special relativity, we study some classical field with an associated energy-momentum tensor $T_{\mu\nu}$. Taking a slice of constant time to obtain a spacelike hypersurface Σ, with a unit normal n^μ, the total energy associated with time translations generated by a Killing field k^μ is given by

$$E = \int_\Sigma T_{\mu\nu} n^\mu k^\nu.$$

As the energy-momentum tensor is conserved: $\partial_\mu T^{\mu\nu} = 0$, we understand E to be independent of the choice of time slicing. This independence of the Cauchy surface Σ is vital for the energy to be considered as a globally defined.

When we allow spacetime to be curved, the condition $\nabla_\mu T^{\mu\nu} = 0$ yields a local conservation law, but there is no invariant integral of $T^{\mu\nu}$ over an arbitrary region as $T^{\mu\nu}$ is not a differential form. As a result, we do not *in general* have a global conservation law. An exception is when we have the additional structure of Killing vector field k^μ in the spacetime. We will elaborate on this further when we discuss the Komar mass. When a spacetime is asymptotically flat, the Arnott-Deser-Misner (ADM) construction can be used to define a total mass [65]. In this formalism, the asymptotic limit is used to define a mass parameter in virtue of our understanding of a total momentum four vector in special relativity. In essence, asymptotically flat solutions can be considered isolated and the ADM mass measures a global energy density through calculating quantities in this limit. For spacetimes admitting Killing horizons with an asymptotically flat static region, the ADM formalism is equivalent to the Komar construction [66]. As we will not be concerned with asymptotically flat spacetimes, we do not further comment on the ADM formalism.

After discussing the Komar mass, we then offer an alternative method to obtain a mass-like quantity quasi-locally. The Brown-York mass can be calculated in a spatially bounded region of spacetime providing that it is stationary. This method will be particularly important to us for the planar solutions we study in this thesis, where it found that the static regions of spacetime are of finite size.

When we interpret the law of black hole mechanics as a thermodynamic laws, we will see the mass parameter of black hole solutions plays the role of the internal energy of the system. As such, when we are looking to verify the first law of black hole mechanics it will be vital to compute a meaningful quantity associated to the mass parameter. Despite the usefulness of both the Komar mass and the Brown-York mass from the point of view of classifying the solutions within this thesis, we will see that they both suffer from an inability to be self-consistently normalised. Ultimately it is black hole thermodynamics itself that offers a way past this through the Euclidean action formalism. We expand on this in Section 3.6.

3.4.1 *Komar mass*

We begin our discussion of the Komar mass considering a spacetime (M, g) which is stationary, *i.e.* it admits a timelike Killing vector field k^μ. We can understand the Komar mass as being the Noether charge associated to the symmetry associated with the Killing vector. Using the Killing vector, we can define a conserved current

$$J_\mu = -T_{\mu\nu} k^\nu, \qquad d \star J = 0,$$

where we use the language of forms, as it natural when considering Stoke's theorem. The total energy on a spacelike hypersurface Σ is given by the integral

$$E_\Sigma = -\int_\Sigma \star J.$$

As the current is conserved, we can write the difference of two integrals over Σ and Σ' bounding a region N of spacetime and apply Stoke's theorem

$$E_{\Sigma'} - E_\Sigma = -\int_{\partial N} \star J = -\int_N d \star J = 0.$$

From (2.2.9), we can associate covariant derivatives of the Killing vector field to the Riemann tensor. Contracting (2.2.9), we can relate the Killing vector to the Ricci tensor

$$\nabla^\nu \nabla_\nu k_\mu = -R_{\mu\nu} k^\nu,$$

and with Einstein's equations we have

$$-R_{\mu\nu} k^\nu = 8\pi \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) k^\nu.$$

We can see that the right-hand side of this term is conserved, and so we can write

$$d \star dk = 8\pi \star \tilde{J}, \qquad \tilde{J}_\mu = 2\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) k^\nu,$$

where we have used Killing's equation to write

$$(\star d \star dk)_\mu = -\nabla^\nu \nabla_\mu k_\nu + \nabla^\nu \nabla_\nu k^\mu = 2\nabla^\nu \nabla_\nu k^\mu = -2R_{\mu\nu}k^\nu.$$

We see that $\star\tilde{J}$ is exact and conserved, so we can write down the energy has an integral over the boundary of the hypersurface

$$E_\Sigma = -\int_\Sigma \star\tilde{J} = -\frac{1}{8\pi}\int_\Sigma d \star dk = -\frac{1}{8\pi}\int_{\partial\Sigma} \star dk. \tag{3.4.1}$$

We now have a covariant expression for a conserved charge generated by a timelike isometry, and we can interpret this as a measure of the total energy of spacetime. In its current form, this expression is dependent on the normalisation of the Killing vector field. In [67], it is shown that this conserved charge can be considered as a mass for asymptotically flat spacetimes. In particular, it is shown that when the spacetime is asymptotically Schwarzschild at spatial infinity, the surface integral is evaluated in the asymptotic limit yields the P^0 term from the momentum four-vector in the asymptotic limit. This leads to the definition of the Komar mass

Definition 3.7. For an asymptotically flat, stationary spacetime (M, g), the *Komar mass* is the conserved charge associated with the timelike Killing vector field k^μ

$$M_K = \lim_{r\to\infty} -\frac{1}{8\pi}\int_{S^2} \star dk. \tag{3.4.2}$$

where the coordinate system is picked such that $r \to \infty$ is the asymptotically flat region.

When a spacetime is equipped with a timelike Killing vector field, but is not asymptotically flat, a local-Komar like integral can be calculated without taking the limit to $r \to \infty$. We can still interpret this as a conserved quantity associated with the energy, but the overall normalisation is left unset. In Section 5.2.3 and Section 6.5.3, we use this local form of the Komar integral to calculate a position dependent, mass-like parameter for our planar symmetric solutions.

Example: Reissner-Nordström solution

As an example, let us calculate the Komar mass of the Reissner-Nordström solution. The Killing covector field in the static patch is given by

$$k = -f(r)dt, \qquad dk = \partial_r f(r) dt \wedge dr.$$

Computing the Hodge-star of dk we find

$$(\star dk)_{\mu\nu} = \sqrt{-g}\, \varepsilon_{\mu\nu\rho\sigma} g^{\rho\tau} g^{\sigma\lambda} (dk)_{\tau\lambda},$$

as the only non-zero term of dk is the t, r component, the only non-zero term of the Hodge dual is

$$(\star dk)_{\theta\phi} = -r^2 \sin\theta \cdot \partial_r f(r) = \left[-2M + \frac{2e^2}{r}\right]\sin\theta.$$

Substituting this into (3.4.2), we find that

$$
\begin{aligned}
M_K &= \lim_{r\to\infty} -\frac{1}{8\pi}\left[-2M + \frac{2e^2}{r}\right]\int_{S^2} \sin\theta, \\
&= -\frac{4\pi}{8\pi}\lim_{r\to\infty}\left(-2M + \mathcal{O}(r^{-1})\right), \\
&= M,
\end{aligned}
$$

as expected.

3.4.2 *Brown-York mass*

Much like we find that the local description for trapping horizons is more practical than the definition of event horizons, the Brown-York energy [68] is useful in that we calculate it quasi-locally. The mass is derived via Hamilton-Jacobi analysis of the action functional for a gravitational system, in [68], it is shown that quasi-local energy is also the value of the Hamiltonian that generates unit magnitude proper-time translations. When the spacetime is also asymptotically flat, the Brown-York energy matches the mass derived from the ADM formalism, if the boundary of the calculation is taken to asymptotic infinity. In this section, we do not derive the form of the Brown-York energy, but instead write down the integral from [68] and introduce the various components of the expression, such that we can calculate the Brown-York energy for our planar solutions throughout this thesis.

The Brown-York quasi-local energy is found from[12]

$$
E_{BY} = -\frac{1}{8\pi}\int_B \sqrt{\sigma}(\mathtt{k} - \mathtt{k}_0), \tag{3.4.3}
$$

where we use the notation of $\mathtt{k}$ as the extrinsic curvature of the codimension-two manifold B embedded within M, where σ is the metric of B and $\mathtt{k}_0$ is a background contribution. In the following discussion, we will better introduce all these pieces to the reader.

In this formalism, we consider a physical spacetime (M, g), which is topologically a hypersurface Σ, foliated over a real line interval: $M = \mathbb{R}\times\Sigma$. We denote boundary of Σ as B, a codimension-two manifold in M. Taking the product of B with the timelike worldlines orthogonal to Σ produces the codimension-one hypersurface ${}^3B$, a component of the three-boundary of M. The full boundary of M includes the end points of timelike worldlines.

To calculate (3.4.3), we will need the geometric data of B in terms of the known data of (M, g). We take a future pointing unit vector u^μ, normal to the foliation Σ. A tensor T is said to be spatial when $T\cdot u = 0$. The metric $g_{\mu\nu}$ induces a metric on Σ, which, when regarded as a tensor $h_{\mu\nu}$ on M, is a spatial tensor. The induced covariant derivative $\mathcal{D}_\mu$ for spatial tensors is found through projection $\mathcal{D}_\mu = h^\nu_\mu \nabla_\nu$. The extrinsic curvature of Σ as an embedded submanifold of M is denoted $K_{\mu\nu}$. We use the notation $h_{ij},\ K_{ij}$, where i, j run from one to the dimension of Σ, when regarding the metric and extrinsic curvature of Σ as tensors on Σ.

[12] Alternative treatments of the quasi-local energy include the lapse function N into the expression [69], which has an effect on non-asymptotically flat spacetimes. In this section we follow the work of Brown and York and do not include the lapse function.

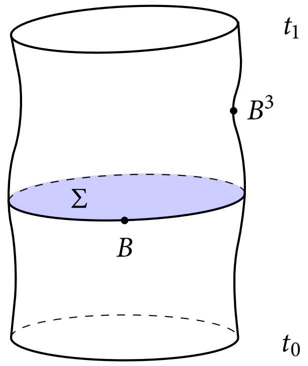


Figure 3.9: The manifold M, which topologically $\mathbb{R} \times \Sigma$. Taking a surface of constant t, we obtain the codimension-one hypersurface Σ. The boundary of Σ is the codimension-two hypersurface B. Taking the product $\mathbb{R} \times B$, we obtain the codimension-one hypersurface B^3.

The ADM decomposition [65] of the metric is given by

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + V^i dt)(dx^j + V^j dt), \tag{3.4.4}$$

for a lapse function N and shift vector V^i.

We proceed in the same way with the three-boundary ${}^3B$ by considering the outward pointing unit vector n^μ, normal to ${}^3B$. The metric induced by $g_{\mu\nu}$ is denoted γ_{mn} when regarded as a tensor on ${}^3B$ and $\gamma_{\mu\nu}$ when regarded as a horizontal tensor on M, *i.e.* as a tensor T on M satisfying $n \cdot T = 0$.

The boundary B, which is the intersection of Σ and ${}^3B$, has a metric $\sigma_{\mu\nu}$ which can be induced from either of the codimension-one manifolds or the spacetime itself. The extrinsic curvature $\mathtt{k}_{\mu\nu}$ of B – the vital part needed to calculate (3.4.3) – is computed using the embedding of B in Σ:

$$\begin{aligned} \mathtt{k}_{\mu\nu} &= \sigma^{\alpha}_{\mu}\mathcal{D}_\alpha n_\nu, \\ &= \gamma^{\alpha}_{\mu} h^{\beta}_{\nu} h^{\rho}_{\alpha} \nabla_\rho n_\beta. \end{aligned} \tag{3.4.5}$$

We will also need the trace $\mathtt{k} = \sigma^{\mu\nu}\mathtt{k}_{\mu\nu}$ in our later calculations.

The last piece of (3.4.3) to comment on is the background normalisation term $\mathtt{k}_0$. When working with the action, there is an inherent ambiguity on the boundary, which in its most general form is built from data on γ_{mn}. In [68], the subtraction term is set by ensuring the energy density depends only on canonical variables. The quasi-local energy is shown to obey additivity, and so we can understand $\mathtt{k}_0$ as a term removing the contribution from flat Minkowski space, which intuitively we would expect to have a zero energy density. For the derivation, and an explanation of how the action functional relates to the extrinsic curvature of the boundary B, we refer to [68].

It is then shown that calculating the energy when boundary is taken in the asymptotic limit, the quasi-local energy is the same as the ADM mass for asymptotically flat spacetimes. Alternatively, taking the Newtonian limit of the quasi-local energy for a spherical distribution of mass of radius R and mass M, it is shown that the quasi-local energy takes the form

$$E_{BY} \simeq M + \frac{2M}{R}.$$

We interpret these as the contributions from the matter energy density and the Newtonian gravitational potential energy, and so we can see E as a natural description for the total energy of the system within this boundary.

Example: Reissner-Nordström solution

It is instructive to perform the calculation for the Brown-York quasi-local energy for a solution we already have experience with, before applying it to the novel solutions of this thesis. In this section, we calculate the quasi-local energy of the Reissner-Nordström solution, and show that by taking the asymptotic limit we recover the Komar mass calculated in Section 3.4.1.

We will need to calculate the trace of the extrinsic curvature $\mathtt{k}$ for both the Reissner-Nordström metric and $\mathtt{k}_0$ for the Minkowski metric. In the following we will use the metric:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + d\Omega^2, \qquad f(r) = \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right),$$

but keep the function $f(r)$ general as much as we can, which will help for calculations later. We also realise we can get results for the Minkowski solution by allowing $f(r) = 1$. Comparing the Reissner-Nordström metric with the ADM decomposition (3.4.4) we can identify the lapse and shift functions

$$N^2 = f(r), \qquad V^i = 0.$$

The outward point unit normal we consider is $n^\mu = \left(0, \sqrt{f(r)}, 0, 0\right)$, such that $n^2 = 1$.

To calculate the Brown-York energy, we need to calculate the extrinsic curvature of the manifold B embedded into the spacetime M with (3.4.5). Computing this and taking the trace, we find

$$\mathtt{k} = \sigma^{\mu\nu}\mathtt{k}_{\mu\nu} = \frac{2\sqrt{f(r)}}{r}, \qquad \mathtt{k}_0 = \frac{2}{r}.$$

For Reissner-Nordstöm, the boundary B, with metric $\sigma_{\alpha\beta}$ is the two-sphere with radius r set for the boundary location.

We now have all the necessary pieces to calculate the Brown-York energy

$$\begin{aligned} E_{BY} &= -\frac{1}{8\pi}\int_B \sqrt{\sigma}(\mathtt{k} - \mathtt{k}_0), \\ &= -\frac{1}{8\pi}\int_B r^2 \sin\theta \left(\frac{2\sqrt{f(r)}}{r} - \frac{2}{r}\right), \\ &= r\left(1 - \sqrt{f(r)}\right). \end{aligned}$$

We see that in the limit of setting the boundary for $r \to \infty$, the quasi-local energy takes the form

$$M_{BY} := \lim_{r\to\infty} E_{BY} \approx \left(M + \frac{M^2 - e^2}{2r} + \mathcal{O}(r^{-2})\right),$$

and so the constant contribution is the mass parameter of the solution M, which is precisely the ADM mass for asymptotically flat solutions. We see that the energy contributions from the $\mathcal{O}(r^{-1})$ order terms can be understood as the gravitational binding energy, and the electrostatic binding energy. We note here that if we were to include the lapse function to the quasi-local energy following [69], both of these sub-leading contributions would be positive.

3.5 BLACK HOLE THERMODYNAMICS

In this section, we introduce the laws of black hole mechanics, a set of geometric relationships built from the properties of black hole solutions. A truly remarkable feature of black holes is uncovered through these rules; by interpreting the surface gravity and the area of the black hole as the temperature and entropy of some thermodynamic system, the laws of black hole mechanics can be understood as the laws of thermodynamics. This similarity has a deeper physical connection and by considering black holes semi-classically, the thermodynamic interpretation of a black hole can be formalised. The work started with the concept of the entropy of a black hole being proportional to its area, rather than its volume. In 1972, Bekenstein conjectured [4] that the entropy of a black hole would be proportional to the area of the black hole divided by the Planck length squared. Soon after, Hawking was able to set the constant of proportionality and proved Bekenstein's conjecture by considering a quantum field theory in a curved background [7]. Looking at quantum fluctuations near the event horizon, Hawking proved that stationary black holes emitted thermal radiation as a blackbody. We will discuss the temperature of black holes and the seeming contradiction of a black hole emitting heat in more detail in Section 3.5.2.

In this section, we outline the four rules of black hole mechanics and remark on their similarity to the laws of thermodynamics. In Appendix B, we expand on the thermodynamic laws themselves and elaborate on a few extra details that will be useful when we consider the Euclidean action formalism in Section 3.6.

3.5.1 *Laws of black hole mechanics*

Zeroth law of black hole mechanics The zeroth law is the statement that the surface gravity is constant across a future event horizon. As we have most of the pieces already discussed to prove this, we offer a formal proof of this law.

Theorem 3.8 (Zeroth Law of Black Hole Mechanics). *The surface gravity κ is constant on a future event horizon $\mathcal{H}^+$, for a stationary spacetime obeying the dominant energy condition [52].*

Proof. In this proof, we use Hawking's theorem that an event horizon $\mathcal{H}^+$ is a Killing horizon with respect to a Killing vector field ξ^μ [6], and so we will show that the surface gravity κ is constant on a Killing horizon $\mathcal{N}$. This roughly follows the proof given in [52].

Raychaudhuri's equation [55] relates the expansion, shear and twist of a null congruence, with generators U^μ to the Ricci tensor by

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}^{\mu\nu}\hat{\sigma}_{\mu\nu} + \hat{\omega}^{\mu\nu}\hat{\omega}_{\mu\nu} - R_{\mu\nu}U^\mu U^\nu.$$

On a Killing horizon we have seen that, $\theta = \hat{\sigma} = \hat{\omega} = 0$ and Raychaudhuri's equation simplifies to

$$0 = R_{\mu\nu}\xi^\mu \xi^\nu \Big|_{\mathcal{N}} = 8\pi T_{\mu\nu}\xi^\mu \xi^\nu,$$

where we have used Einstein's equation and $\xi^2|_{\mathcal{N}} = 0$ in the last step. From this, we see that on the Killing horizon, we can write down a current $J_\mu = -T_{\mu\nu}\xi^\nu$. The dominant energy condition states that $-T^{\mu}_{\nu}X^{\nu}$ is future-directed for all future-directed vector fields X^{μ}. As the Killing vector field is future-directed, so is J_{μ} and hence the current must be parallel to the Killing vector field.

We can then antisymmetrise and with (2.2.9) and write down

$$0 = \xi_{[\mu}J_{\nu]} = -\frac{1}{8\pi}\xi_{[\mu}R_{\mu]\rho}\xi^{\rho} = \frac{1}{8\pi}\xi_{[\mu}\partial_{\nu]}\kappa,$$

and so $\partial_{\mu}\kappa$ must be proportional to ξ_{μ} and hence $X^{\mu}\partial_{\mu}\kappa = 0$ for a vector field X^{μ} tangent to $\mathcal{N}$. We then understand κ is constant on the Killing horizon. □

Looking at this result, with the not-so-subtle naming of it as the *zeroth law*, we can think about the zeroth-law of thermodynamics, which states that for a body in equilibrium, the temperature is constant.

First Law of Black Hole Mechanics The no hair theorem of Wheeler [43, 70] states that stationary black hole solutions are completely encoded by their mass M, electromagnetic charges $\mathcal{Q}_i$ and their angular momentum J. By varying these parameters, we find a differential relationship

$$\frac{\kappa}{8\pi}dA = dM - \Omega_H J - \mu d\mathcal{Q}, \tag{3.5.1}$$

where the Ω_H is the angular velocity of a rotating black hole solution and μ is the chemical potential, which can be computed from the asymptotic value of the time component of the gauge-fixed vector potential [71]. For all the solutions within this thesis, $\Omega_H = J = 0$, but we include it here for generality; the canonical example of spinning black holes is the Kerr solution [56]. We can understand the first law of black hole mechanics as the statement of energy conservation for a black hole solution. It has been proved for stationary black hole solutions by Sudasky and Wald [72]. In this thesis, we will show that the first law is also satisfied for a class of Killing horizons where the external, asymptotic region of the spacetime is time-dependent. Hartle and Hawking [73] proved an alternative version of the first law by perturbing a black hole solution by adding some small mass δM to a black hole and letting the spacetime settle back to a stationary solution, and an alternative proof is found in [52] by Bardeen, Carter and Hawking.

As an example, we can verify the first law for the Reissner-Nordström solution, where in Section 3.3 we found that $\mathcal{Q} = Q$, $r_{\pm} = M \pm \sqrt{M^2 - Q^2}$ and for the Killing horizon at $r = r_+$, the surface gravity is $\kappa = (r_+ - r_-)/r_+^2$:

$$\begin{aligned}
\frac{\kappa}{8\pi}dA = \frac{\kappa}{2}d(r_+^2) &= \frac{\sqrt{M^2-Q^2}}{r_+}d\left(M + \sqrt{M^2-Q^2}\right) \\
&= \frac{\sqrt{M^2-Q^2}}{r_+}\left[dM + \frac{1}{2\sqrt{M^2-Q^2}}(2MdM - 2QdQ)\right] \\
&= \left(\frac{\sqrt{M^2-Q^2}}{r_+} + \frac{M}{r_+}\right)dM - \frac{QdQ}{r_+} \\
&= dM - \mu d\mathcal{Q},
\end{aligned}$$

where we have used Equation (3.6.4) for the chemical potential μ.

Understanding the mass as encoding the total energy of the black hole, this relationship is tantalisingly close to the first law of thermodynamics

$$dE = TdS + \mu_i d\mathcal{Q}_i - PdV,$$

where the $\mu_i d\mathcal{Q}_i$ terms encode the work terms associated to the system, and PdV is the term associated to the pressure / volume of the closed system. For us to relate these two differential relationships, we must associate TdS with the geometric terms $\frac{1}{8\pi}\kappa dA$.

Second Law of Black Hole Mechanics The second law of black hole mechanics is Hawking's area law [5]. Informally, this law states that if one assumes both the cosmic censorship conjecture and the null energy condition, the area A of an event horizon $\mathcal{H}$ does not decrease with time

$$\delta A \geq 0.$$

We have not formally defined either of these two assumptions in this thesis as they are not vital to the work in the main body. As a sketch, the (weak) cosmic censorship is the statement that all singularities in the spacetime are hidden behind a horizon and the null energy condition states that for a null vector field X^μ we have the inequality $T_{\mu\nu}X^\mu X^\nu \geq 0$.

The first law seemed to suggest that the area and entropy were proportional. We can follow this into the second law and realise Hawking's area theorem is then the statement that for a physical process, the entropy of a black hole would be always increasing. This relationship is perhaps the most interesting. Hawking's area theorem is a rigorous geometric statement from the geometry of spacetime, but the second law of thermodynamics is a classical result which is believed to hold for systems with a large number of particles.

There is a slight modification to this story when we take seriously the notion of thermal radiation from a black hole with temperature proportional to κ. Over time, a black hole will eventually evaporate and while doing so, lose entropy through the thermal radiation. Taking this into account, the second law is generalised such that combination of the black hole's entropy together with the entropy encoded into the radiation is non-decreasing: $\delta(S_{BH} + S_T) \geq 0$[74]. We will not comment further on this.

Third Law of Black Hole Mechanics The third law is less well defined, from both the thermodynamic and black hole perspectives, and in fact comes in two versions: the strong and the weak third law. Unlike the last three, we will state the third law as a thermodynamic law and then refer to the geometric formulation for the third law(s) as they currently stand.

The weak version of the third law states that the temperature of a system cannot be reduced to zero in a finite number of steps, in which the entropy is at the minimum value. A stronger version put forward by Planck [75] states that the entropy vanishes when the system is brought to zero temperature.

From a geometric perspective, Israel put forward a formulation [76] that it is impossible to reduce the surface gravity to zero in a finite number of steps. However, there is no form of proof for the third law, in fact, we already have a counter example with the extremal Reissner-Nordström solution. In this solution, the surface gravity of the solution vanishes, and yet there is still a black hole with a finite size! The third law may have ambiguities, but it's still an active area of research in black hole physics [77, 78, 79, 80]. In [29, 30, 28, 34] a class of black hole solutions dubbed *Nernst branes* were found in $\mathcal{N} = 2$ gauged supergravity with the special property that they obeyed the strong version of the third law. These solutions were the starting-off point for the class of solutions found in [39], the research of which is one of the main results of this thesis. Further discussion of Nernst branes is delayed until Chapter 6. We can summarise these results into Table 3.1 [35].

Law	Black Hole Mechanics	Thermodynamics
Zeroth	κ is constant over the horizon of a stationary black hole	T is constant for a body in equilibrium
First	$dM = \frac{1}{8\pi}\kappa dA + \mu dQ + \Omega_H dJ$	$dE = TdS + \mu_i dC_i - PdV$
Second	$\delta A \geq 0$ for physical processes	$\delta S \geq 0$ for physical processes
Third	The surface area is minimised for extremal black holes	The entropy is minimised at absolute zero

Table 3.1: A summary of the laws of black hole mechanics and a comparison of their form to the laws of thermodynamics.

3.5.2 *Temperature of horizons*

Realising the laws of black hole mechanics as thermodynamic relations leads to a new interpretation for geometric parameters associated to the black hole solution. From the zeroth and first law, we are led to related the surface gravity to temperature, and from the first and the second laws, we relate the area of a black hole as a measure of its entropy. We can formalise this in the following way

$$T = \alpha\kappa, \qquad S = \frac{A}{8\pi\alpha},$$

where we allow some constant α, which we cannot set from the mechanical laws alone. This leads to a strange contradiction. By definition, a black hole is a region of spacetime from which nothing can be emitted, but the interpretation of a black hole as a thermodynamic object leads us to interpret it as some blackbody radiating with a temperature T_H.

It was Hawking in 1974 [7] that was able to reconcile this by studying the black hole semi-classically. By considering matter quantum mechanically, Hawking showed that a black hole radiates as a blackbody with a temperature

$$T_H = \frac{\kappa\hbar}{2\pi k_B c}, \tag{3.5.2}$$

where we have re-introduced all physical constants briefly. There is something beautiful in the collision of so many areas of physics in this one equation. Hawking's derivation sets the value of the constant α, and with this validated Bekenstein's conjecture that the entropy of a black hole should be proportional to its area divided by the Planck length squared [4]

$$S_{BH} = \frac{k_B A c^3}{4G\hbar} = \frac{k_B A}{4\ell_P^2}. \tag{3.5.3}$$

For the remainder of our discussion, we return back to our conventions of suppressing physical constants such that

$$T = \frac{\kappa}{2\pi}, \qquad S = \frac{A}{4}.$$

Hawking's work means that given a Killing horizon, one can calculate the surface gravity of the horizon and hence the temperature. However, given a Killing horizon $\mathcal{N}$, with a Killing vector field ξ, and surface gravity κ, $\mathcal{N}$ is also a Killing horizon of $\alpha\xi$ with surface gravity $\alpha^2\kappa$ for any constant α. This means that the surface gravity is not a unique property of $\mathcal{N}$ and depends on the normalisation of the Killing vector. As the norm of the Killing vector vanishes on the hypersurface, there is no way to normalise ξ on $\mathcal{N}$ in a natural way.

When a spacetime is stationary and asymptotically flat, there is a natural normalisation that allows us to set the time-translation Killing vector k with that of Minkowski space:

$$\lim_{r\to\infty} k^2 = -1.$$

However, when a spacetime does not admit an asymptotically flat spacetime, the surface gravity, and hence the temperature, must be set in another setting.

An alternative calculation for the Hawking temperature comes from studying the Euclidean section of a black hole solution by Wick-rotating the time coordinate $t \to -i\tau$. In the next section, we show that the Wick-rotation goes beyond just a calculation for the temperature but all the way to a thermodynamic partition function.

After Wick-rotation, the coordinate τ must be periodically identified $\tau \simeq \tau + \beta$, where $\beta = T_H^{-1}$ is the inverse temperature. This identification is imposed to ensure the absence of a conical singularity at the origin of the Euclidean manifold.

Taking a black hole solution with a line element of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\vec{X}^2,$$

where $d\vec{X}^2$ is a generic line element for a two-dimensional space. Conventionally this would be the two-sphere S^2 and later it will be the plane $\mathbb{R}^2$ for our planar symmetric solutions. The function $f(r)$ has a zero for $f(r_h) = 0$, with a non-zero derivative. We can write down the approximation for line element near the Killing horizon at $r = r_h$ by Taylor-expanding

$$ds^2 = -\partial_r f(r_h)(r - r_h)dt^2 + \frac{dr^2}{\partial_r f(r_h)(r - r_h)} + r_h^2 d\vec{X}^2.$$

Wick-rotating $t \to -i\tau$, we can write down the line element as

$$ds^2 = \rho^2 d\theta^2 + d\rho^2 + r_h^2 d\vec{X}^2,$$

where we have made the additional coordinate transformation

$$\rho^2 = \frac{4(r - r_h)}{\partial_r f(r_h)}, \qquad \theta = \frac{1}{2}\partial_r f(r_h)\tau.$$

We see that the conical singularity from the Wick-rotation is avoided as long as we make the identification

$$\theta \simeq \theta + 2\pi, \quad \Rightarrow \quad \tau \simeq \tau + \frac{4\pi}{\partial_r f(r_h)},$$

and so we find that the temperature of the solution found from Wick-rotating is given by

$$T_H = \frac{\partial_r f(r_h)}{4\pi}.$$

As a consistency check for this calculation, we can calculate the surface gravity using the Killing vector filed $k^\mu = \partial_t$ and Equation (3.1.3) and we find

$$\kappa = \frac{1}{2}\partial_r f(r_h),$$

where in order to evaluate the result on the horizon at $r = r_h$, it is necessary to perform an Eddington-Finkelstein transformation on the line element first. With this expression for κ, we can see that the temperature calculated from the Wick-rotation matches (3.5.2). We can notice that $\theta = \kappa\tau$, but when studying the line element, the surface gravity enters the line element quadratically and so this procedure doesn't actually determine whether T_H is positive or negative.

The ambiguity in the scaling and sign of the temperature of a black hole solution ultimately requires the full computation of Hawking temperature using curved spacetime quantum field [7], or the tunnelling effect for a quantum particle [81]. For stationary, asymptotically flat spacetimes, Hawking's work holds and computation of the temperature can be done with either of the above methods. However, for the solutions we consider within this thesis, we will need to extend our work suitably to understand the Killing horizons of our solutions.

Deriving the temperature of a Killing horizon by considering the semi-classical emission of black body radiation is a complicated set of calculations, especially as it seems that all we want is to set the sign of the temperature for non-standard black hole solutions. However, there is a known relation for the sign of the temperature for the four variants of trapped horizons defined in Section 3.1.5. In particular, we will be concerned with regions of spacetime which depended on a timelike coordinate t. For these solutions, the absence of a timelike Killing vector field inhibits the canonical computation of the surface gravity. However, by following the work of [82], the surface gravity can be computed using the Kodama vector [83]. With a valid expression for the surface gravity, following the work of [57, 58], we can show that once the category of trapping horizon is identified the sign of the temperature is fixed.

The Kodama-Hayward surface gravity can be computed when the metric has the structure

$$ds^2 = \gamma_{ij}(x)dx^i dx^j + C^2(x)d\vec{X}^2\,, \quad i = 0, 1,$$

where γ_{ij} and C only depend on the coordinates $(x^0, x^1) = (t, r)$. For spherically symmetric spacetimes, $d\vec{X}^2 = d\Omega_2$ is the standard metric on the two-sphere. In our later calculations, we will allow planar symmetry and hence $d\vec{X}^2$ is the standard metric on $\mathbb{R}^2$. The surface gravity in the Kodama-Hayward formalism is

$$\kappa = \frac{1}{2\sqrt{-\gamma}}\partial_i\left(\sqrt{-\gamma}\gamma^{ij}\partial_i C\right) = \frac{1}{2}\Delta_\gamma C\,.$$

For later reference, we compute the Kodama-Hayward surface gravity for line elements of the form

$$ds^2 = -\frac{dt^2}{f(t)} + f(t)dr^2 + t^2 d\vec{X}^2,$$

which include the dynamic patches of the solutions considered within this thesis. Performing the computation, we find that the surface gravity is given by

$$\kappa = -\frac{1}{2}\partial_t f(t_h), \qquad f(t_h) = 0. \tag{3.5.4}$$

Following [57, 58], trapping horizons and their Kodama-Hayward surface gravity subdivide into four cases as the sign of the surface gravity is proportional to

$$\kappa \propto -\mathcal{L}_{\ell_\mp}\theta_\pm.$$

Thus outer horizons have positive surface gravity, while inner horizons have negative surface gravity. By computing the Hawking temperature of a trapping horizon using the Parikh-Wilczek tunnelling method, [81, 57, 58], we find an additional sign between the Hawking temperature and the surface gravity for trapping horizons with $T_H \propto \pm\kappa$, with the $(+)$ sign for future horizons and the $(-)$ sign for past horizons. We can summarise these signs

(i) *Future outer horizons*

$$\theta_+ = 0, \quad \theta_- < 0, \quad \mathcal{L}_{\ell_-}\theta_+ < 0, \qquad \kappa > 0, \quad T_H > 0.$$

(ii) *Past outer horizons*

$$\theta_- = 0, \quad \theta_+ < 0, \quad \mathcal{L}_{\ell_+}\theta_- < 0, \qquad \kappa > 0, \quad T_H < 0.$$

(iii) *Future inner horizons*

$$\theta_+ = 0, \quad \theta_- < 0, \quad \mathcal{L}_{\ell_-}\theta_+ > 0, \qquad \kappa < 0, \quad T_H < 0.$$

(iv) *Past inner horizons*

$$\theta_- = 0, \quad \theta_+ < 0, \quad \mathcal{L}_{\ell_+}\theta_- > 0, \qquad \kappa < 0, \quad T_H > 0.$$

The net effect is that future outer horizons (black holes), and past inner horizons (expanding cosmologies) have positive temperature, while future inner horizons (contracting cosmologies) and past outer horizons (white holes) have negative temperature.

Negative temperature has been argued to indicate the absence of Hawking radiation, since future inner and past outer horizons cannot separate virtual particle pairs created by vacuum fluctuations, thus not enabling the Hawking effect [57, 58]. However, in thermodynamics, the inverse temperature is related to the entropy S and internal energy E by

$$\beta = \frac{\partial S}{\partial E}.$$

Therefore, negative temperature can occur if one drops the usual assumption that the entropy increases monotonically with the energy. A toy model for negative temperature is provided by a system with finite maximum energy [84]. Taking a system with two energy eigenstates $E_1 < E_2$

as the simplest example, this will be in a maximally ordered state ($S = 0$) if all particles are either in the lower or in the higher state, while a maximally disordered state is realised when half of the particles are in either state. Upon heating up such a system, entropy and temperature first increase, with the temperature reaching $+\infty$ when entropy becomes maximal. Upon further heating, the entropy decreases and the temperature jumps at the turning point from $+\infty$ to $-\infty$. After this point, it increases, approaching 0 from below when reaching a situation where all particles are in the higher state. Thus negative temperatures are 'higher' than positive temperatures and correspond to 'population inversion.' We will see later that some of the horizons we are interested in have negative surface gravity and negative temperature, and that this is necessary in order for the first law to take its standard form when using our triple Wick-rotated Euclidean formalism.

3.6 EUCLIDEAN ACTION FORMALISM

One of the main results of this thesis is the verification of the first law of Killing horizon mechanics for a class of planar symmetric, cosmological solutions of Einstein-Maxwell and $\mathcal{N} = 2$ supergravity theories. As mentioned, these solutions are neither asymptotically flat, nor stationary in the exterior regions of the spacetime, complicating our ability to find a meaningful mass parameter from which we could verify the first law. Both the Komar mass and Brown-York mass can be used to find local mass-like parameters within the finite, static region of our solutions, but both of these methods have a freedom in their normalisation, and so the first law – a differential relationship – cannot be studied without imposing an additional condition.

The Euclidean action formalism [85, 86] is a method to study the thermodynamics of black holes through the formal identification of the gravitational partition function and the thermodynamic partition function. This procedure leads to the computation of a thermodynamic potential by its relation to the Euclidean action of the gravitational theory. We can calculate various thermodynamic parameters such as the mass, charge, entropy and temperature from derivatives of the thermodynamic potential. More details are given in Appendix B. In our solutions, we find the natural thermodynamic potential related to the Euclidean action for theories with electromagnetic charges is the grand potential $\Omega(\beta, \mu)$ and via a Legendre transformation, we compute the free energy $F(\beta, \mathcal{Q})$.

For planar symmetric solutions, a normalisation ambiguity remains in the Euclidean action formalism, but we can now set it with the additional condition that the charge calculated from the grand potential matches the charge calculated from Gauss' law. This sets an overall numerical factor for the Euclidean action, and hence the remaining thermodynamic variables. For asymptotically flat spacetimes, this normalisation is unnecessary as we have the Minkowski solution at infinity to normalise as a background contribution. With the normalisation set, we show the thermodynamic entropy matches the Bekenstein area law, and we derive a mass parameter. This mass parameter can be varied and the first law verified.

In this section, we introduce the Euclidean action formalism using the Reissner-Nordström solution as an example. We work through a computation of the Euclidean action, together with the background terms and thus derive the thermodynamic potential. We use this to rederive the charge, mass and entropy of the Reissner-Nordström solution and show that these results match with our previous calculations.

In Chapter 8, we develop the Euclidean action formalism such that it is suitable for the classes of cosmological solutions we study in this thesis. Precisely, we find that the natural procedure of Wick-rotating the time coordinate leads to a complex line element, and hence a complex-valued thermodynamic potential. To circumvent this, we introduce the so-called *triple Wick-rotation*, in which we obtain a negative-definite line element through Wick-rotating all spacelike coordinates. We delay further discussion of this procedure until after introducing the standard formalism.

3.6.1 *Gravitational and thermodynamic partition functions*

The thermodynamic canonical partition function $Z(\beta)$ for a system with a Hamiltonian $\hat{H}$ is defined by

$$Z(\beta) := e^{-\beta F} = \mathrm{Tr} e^{-\beta \hat{H}} \, ,$$

where F is the free energy and β is the inverse temperature. For a system with a conserved charge $\mathcal{Q}$, the thermodynamic potential depends on the conserved charge in addition to its dependence on temperature, $F = F(\beta, \mathcal{Q})$. The grand canonical ensemble is defined by keeping the charge constant and letting the corresponding intensive thermodynamic variable, the chemical potential μ, fluctuate. The corresponding thermodynamic partition function is the grand canonical partition function:

$$\mathcal{Z}(\beta, \mu) := e^{-\beta \Omega} = \mathrm{Tr} e^{-\beta \hat{H}} \, ,$$

where $\Omega(\beta, \mu)$ is the grand potential. For the following discussion, we suppress the usual contribution of a volume / pressure term. In gravitational systems, the cosmological constant can be interpreted as a pressure-like term when considering the enthalpy of the solution [87], but for the purposes of our discussion, we will not be including it. For more details, we refer to Appendix B.

To illustrate the correspondence between partition functions of quantum (field) theories and thermodynamic partition functions, we consider the case of a quantum particle. The time-evolution operator admits a path integral representation involving the classical action

$$\langle x | \, e^{-itH} \, | x' \rangle = \int \mathcal{D}x e^{iS[x]},$$

where we have set $\hbar = 1$. By Wick-rotating the time coordinate $t \to -i\beta$ and taking the trace, which in the path integral corresponds to integrating over paths periodic in time, one obtains

$$\mathrm{Tr} e^{-\beta H} = \int \mathcal{D}x e^{-S_E[x]} = e^{-\beta F},$$

where β is interpreted as inverse temperature, and where F is the free energy.

It is straightforward, at least at a formal level, to extend this prescription to quantum field theories. In a quantum theory including gravity, the path integral is performed over the space of all metrics g, as well as over the matter fields φ,

$$Z = \int \mathcal{D}g \mathcal{D}\varphi e^{-S_E[g, \varphi]} \, .$$

This might on the surface look like we need to have a well-defined quantum model for gravity to continue, but it is enough to consider the *saddle point approximation*. By expanding the path-integral around a solution of the classical equations of motion, we keep only the tree-level contributions and so obtain a semi-classical approximation to the path-integral. This leads to the expression $Z \simeq e^{-S_E}$, where the Euclidean action S_E is evaluated on an on-shell field configuration satisfying suitable boundary conditions [88].

Employing this, we obtain a relation between the Euclidean on-shell action and the free energy:

$$\log(Z) \simeq -S_E \simeq -\beta F \quad \Rightarrow \quad F \simeq \frac{S_E}{\beta}.$$

When gauge fields are present, the standard boundary conditions are such that the total charge $\mathcal{Q}$ is fixed, and the Euclidean action depends on the associated chemical potential μ. This means that the Euclidean action is of the form $S_E = S_E(\beta, \mu)$, and one obtains the following relation between the Euclidean on-shell action and the grand potential $\Omega(\beta, \mu)$:

$$\log(\mathcal{Z}) \simeq -S_E \simeq -\beta\Omega, \quad \Rightarrow \quad \Omega \simeq \frac{S_E}{\beta}.$$

The grand potential and free energy are related by a Legendre transformation: $\Omega = F - \mu\mathcal{Q}$, and we will use this relation to derive the free energy from the Euclidean action of the charged black hole solutions we consider. We note here that we could also calculate the free energy directly from the Euclidean action by including an additional boundary term into the action such that the chemical potential is fixed and the charge fluctuates [88]. We will not use this method in the following discussion.

When considering the Euclidean action, one also needs to consider the sign of S_E such that we can make a statement about the boundedness of the partition function. For general gravitational systems, the Euclidean action has an indefinite sign [89]. However, the authors show that through using a conformal transformation and considering up to the one loop contributions, the gravitational action has a converging path integral. An alternative treatment in [89] shows that in static solutions, estabilished theory from flat quantum field theory can be carried forward to curved spacetimes. Practically, we can consider the operator $\exp(-S_E\beta)$ as bounded when $S_E > 0$, and in Chapter 8, we consider the boundedness while computing the Euclidean action for each of our solutions.

3.6.2 *Wick-rotating the gravitational action*

In this section, we cover the calculation of the Euclidean action for the Einstein-Maxwell theory with a cosmological constant. This covers all the necessary pieces we will need for the Reissner-Nordström example we conclude this section with, as well as giving a good backbone for us to build from when we introduce the triple Wick-rotation in Section 8.1. For our calculations, we follow [41] for the gravitational action, and generalise this by including the cosmological constant and the Maxwell action:

$$\begin{aligned} S &= S_{\text{bulk}} + S_{\text{GHY}}, \\ &= -\frac{1}{16\pi}\int_M \sqrt{|g|}(R-2\Lambda)d^4x - \frac{1}{16\pi}\int_M \sqrt{|g|}F_{\mu\nu}F^{\mu\nu}d^4x \\ &\quad + \frac{\epsilon}{8\pi}\int_{\partial M}\sqrt{|\gamma|}(K-K_0)d^3x\,. \end{aligned} \tag{3.6.1}$$

The middle line is the bulk term, containing the Einstein-Hilbert action with the Ricci scalar R, a cosmological constant Λ and the Maxwell term. The final line is the Gibbons-Hawking-York boundary term S_{GHY} [90, 85], which is needed to cancel boundary terms arising from the variation of the Einstein-Hilbert action if spacetime is not closed (compact without boundary). The spacetime metric g induces a metric γ on the boundary ∂M. K is trace of the extrinsic curvature of ∂M as an embedded submanifold of spacetime M. The constant ϵ takes the values $\epsilon = \pm 1$ for boundaries with unit normals which are either spacelike $(+)$ or timelike $(-)$. To obtain a finite value for the on-shell action, we include a background term K_0. For an asymptotically flat spacetime, K_0 is the extrinsic curvature of the boundary embedded into a flat spacetime, which ensures that the action of Minkowski space, which is a solution for $\Lambda = 0$, is zero rather than divergent. In the de Sitter calculation, considered in Section 8.2, K_0 is replaced by a counter term, which we build by hand to remove divergences. For the planar symmetric solution, there are no divergences, and so K_0 is dropped from the action and replaced by an overall numerical factor, set using a condition on the conserved charge.

To obtain the Euclidean action, we apply the Wick-rotation $t \to -i\tau$ to (3.6.1) to map $\exp(iS) \to \exp(-S_E)$. Following [41] we first consider the gravitational terms. The bulk gravitational term receives a factor of $-i$ from the measure:

$$-\frac{1}{16\pi}\int_M \sqrt{g}(R-2\Lambda)d^4x \to i\frac{1}{16\pi}\int_M \sqrt{g}(R-2\Lambda)d^4x.$$

For the transformation of the GHY-term we need to distinguish two cases.

(i) For surfaces with a timelike unit normal:

$$\epsilon = -1, \qquad K \to iK \qquad \sqrt{\gamma}d^3x \to \sqrt{\gamma}d^3x.$$

(ii) For surfaces with a spacelike unit normal:

$$\epsilon = 1, \qquad K \to K, \qquad \sqrt{\gamma}d^3x \to -i\sqrt{\gamma}d^3x.$$

The resulting Euclidean Gibbons-Hawking-York term is the same for both types of hypersurfaces and transforms as

$$+\frac{\epsilon}{8\pi}\int_{\partial M}\sqrt{|\gamma|}(K-K_0)d^3x \to -i\frac{1}{8\pi}\int_{\partial M}\sqrt{|\gamma|}(K-K_0)d^3x.$$

We now consider the Maxwell field. Before Wick-rotation, we use that as we are considering the saddle-point approximation, the Maxwell action is evaluated on-shell which allows us to rewrite its contribution as a total derivative[13]

$$F^{\mu\nu}F_{\mu\nu} = 2\nabla_\mu(A_\nu F^{\mu\nu}).$$

[13] In terms of differential forms, $F \wedge \star F = dA \wedge \star F = d(A \wedge \star F)$, if $d \star F = 0$.

Applying Stoke's theorem, we can write the bulk contribution as an integral over the boundary

$$-\frac{1}{8\pi}\int_M \sqrt{|g|}\nabla_\mu \left(A_\nu F^{\mu\nu}\right) d^4x = \frac{1}{8\pi}\int_{\partial M} F^{\mu\nu} A_\mu d\Sigma_\nu,$$

where the volume element on the boundary is defined as $d\Sigma_\mu = n_\mu \sqrt{|\gamma|} d^3x$ and n^μ is the outward-pointing unit normal vector. Applying a Wick-rotation, we find the Maxwell action transforms as

$$\frac{1}{8\pi}\int_{\partial M} F^{\mu\nu} A_\mu d\Sigma_\nu \to -i\frac{1}{8\pi}\int_{\partial M} F^{\mu\nu} A_\mu d\Sigma_\nu \,,$$

where note explicitly that each pieces transforms as: $d\Sigma_\mu \to -i d\Sigma_\mu$, $A_\mu \to -iA_\mu$, and $F^{\mu\nu} \to iF^{\mu\nu}$.

Taking all contributions together, the Euclidean action is

$$\begin{aligned} S_E = -iS_{\text{Wick-rotated}} = &\frac{1}{16\pi}\int_M \sqrt{g}(R-2\Lambda)d^4x, \\ &-\frac{1}{8\pi}\int_{\partial M}\sqrt{|\gamma|}(K-K_0)d^3x - \frac{1}{8\pi}\int_{\partial M} F^{\mu\nu}A_\mu d\Sigma_\nu \,. \end{aligned} \tag{3.6.2}$$

Example: Reissner-Nordström Solution

As an example of this procedure, we verify the first law of thermodynamics for the Reissner-Nordström solution using the Euclidean action formalism. We will simplify the calculations by working with a purely electric solution: $P = 0$. In [88], it is shown that electric and magnetic black holes are dual, and that we can work with one or the other.[14] Furthermore, for the solutions we discuss later in the thesis, we will use Hodge dualisation to ensure all charges are electric and so this example will be sufficient.

The Reissner-Nordström solution is found from the Einstein-Maxwell action (3.3.1), repeated here:

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}(-R-F^2).$$

The stationary, spherically symmetric solution was found to have a line element (3.3.12)

$$ds^2 = -\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2 + \left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega^2 \quad A = -\frac{Q}{r}dt.$$

As the gauge potential stands, A_t is undefined on the horizon and so we gauge fix this by ensuring it has zero value on the horizon:[15]

$$A_t = -\frac{Q}{r} + \frac{Q}{r_+}.$$

With this gauge we now Wick-rotate our solution $t \to -i\tau$:

$$\begin{aligned} ds^2 &= \left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)d\tau^2 + \left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dr^2 + r^2 d\Omega^2, \\ A &= -i\left(-\frac{Q}{r}+\frac{Q}{r_+}\right)d\tau. \end{aligned}$$

In Figure 3.10, an illustration of the Wick-rotation is given. The outer horizon r_+ becomes the origin of the Wick-rotated spacetime and the boundary is located for some distant r_0, which we will evaluate in the asymptotic limit $r_0 \to \infty$.

[14] If a black hole has both electric and magnetic charge, a symplectic transformation on the gauge fields can be made such that it is purely electric / magnetic. See Section 4.2 for more details on the electromagnetic duality.

[15] See [34] Appendix F for a detailed explanation.

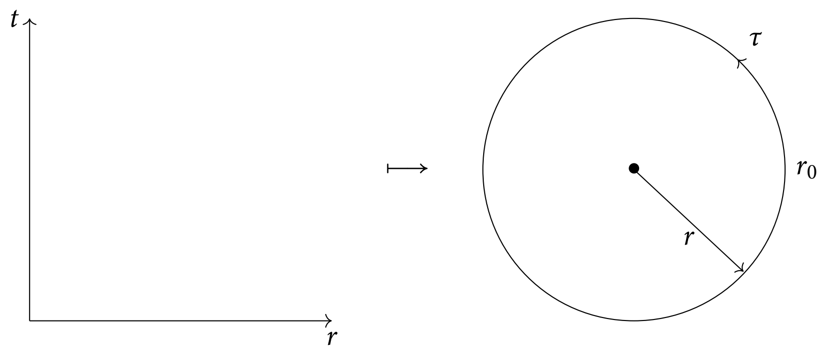


Figure 3.10: 2D representation of the map from the Lorenztian spacetime to Euclidean space. By Wick-rotating the time coordinate, our metric becomes positive-definite and τ is a compact dimension with period $\tau \sim \tau + \beta$. By setting the origin of the Euclidean space at the horizon we can identify β with the inverse of the Hawking temperature. The radial coordinate r now extends from the horizon into the asymptotic region. A boundary is placed at the large distance r_0.

The Euclidean action including all boundary terms is given by (3.6.2), which we repeat here

$$S_E = \frac{1}{16\pi}\int_M \sqrt{g}Rd^4x - \frac{1}{8\pi}\int_{\partial M}\sqrt{|\gamma|}(K-K_0)d^3x - \frac{1}{8\pi}\int_{\partial M}F^{\mu\nu}A_\mu d\Sigma_\nu.$$

As the Reissner-Nordström has zero Ricci scalar, the action will have contributions from only the boundary terms. We begin by taking a surface of constant $r = r_0$, and calculate the Gibbons-Hawking-York term by calculating the trace of the extrinsic curvature

$$\sqrt{\gamma}K = \left(2r_0 - 3M + \frac{Q^2}{r_0}\right),$$

where $\sqrt{\gamma}$ is the volume element of the codimension-one manifold. In the limit of $r_0 \to \infty$, we see that this contribution is divergent. However, the inclusion of the background term K_0 normalises this. To calculate this piece, we use the flat space embedding as the background metric

$$ds^2 = \left(1 - \frac{2M}{r_0} + \frac{Q^2}{r_0^2}\right)d\tau^2 + dr^2 + r^2 d\Omega^2,$$

and performing an identical calculation, we find

$$\sqrt{\gamma}K_0 = \left(2r_0 - 2M + \frac{Q^2 - M^2}{r_0}\right).$$

Combining these terms and taking the boundary to the asymptotic limit, the Gibbons-Hawking York contribution is finite, and given by

$$\begin{aligned}-\frac{1}{8\pi}\int_{\partial M}\sqrt{|\gamma|}(K-K_0)d^3x &= \frac{4\pi\beta}{8\pi}\left(2r_0 - 2M + \frac{Q^2-M^2}{r_0} - 2r_0 + 3M - \frac{Q^2}{r_0}\right),\\ &= \frac{\beta M}{2} + \mathcal{O}(r_0^{-1}).\end{aligned}$$

All that remains is to include the contribution from the Maxwell field. For the boundary located at constant r_0 we find that:

$$d\Sigma_\mu = n_\mu\sqrt{|\gamma|}d^3x = n_r\sqrt{f(r_0)}r_0^2\sin\theta d^3x = r_0^2\sin\theta d^3x, \quad n = \frac{1}{\sqrt{f(r)}}dr.$$

The contraction of the field strength with the gauge potential is calculated

$$
\begin{aligned}
F^{\tau r} A_\tau &= g^{\tau\tau} g^{rr} F_{\tau r} A_\tau, \\
&= \frac{Q^2}{r_+ r^2} - \frac{Q^2}{r^3}.
\end{aligned}
$$

Putting this together, we can write down the Maxwell field contribution in the limit of $r_0 \to \infty$

$$
\begin{aligned}
-\frac{1}{8\pi} \int_{\partial M} F^{\mu\nu} A_\mu d\Sigma_\nu &= -\frac{1}{8\pi} \int_{\partial M} \left(\frac{Q^2}{r_+} - \frac{Q^2}{r_0} \right) \sin\theta, \\
&= -\frac{\beta}{2} \frac{Q^2}{r_+}.
\end{aligned}
$$

From this, we find that the on-shell Euclidean action for the Reissner-Nordström theory is given by

$$
S_E = \frac{\beta}{2}(M - Q\mu), \tag{3.6.3}
$$

where μ is the chemical potential associated with the asymptotic limit of the time component of the gauge potential

$$
\mu := \lim_{r\to\infty} A_t = \frac{Q}{r_+}. \tag{3.6.4}
$$

We can use the Euclidean action to write down the grand potential from the partition function:

$$
\log \mathcal{Z} = -S_E, \quad \Omega = -\frac{1}{\beta} \log \mathcal{Z} \quad \Rightarrow \quad \Omega(\mu, \beta) = \frac{S_E}{\beta} = \frac{1}{2}(M - Q\mu).
$$

From the grand potential, we can calculate the conserved charge from

$$
\mathcal{Q} = -\left(\frac{\partial \Omega}{\partial \mu} \right)_\beta = Q,
$$

and we notice that this quantity matches the derived conserved charge (3.3.14) that we found using Gauss' law. Later, when we do not consider asymptotically flat spacetimes, we will assert that these two quantities are the same, fixing an overall normalisation constant which appears in the Euclidean action.

For the Reissner-Nordström solution, we already have a good physical parameter for all the thermodynamic variables $\{M, S, T_H, Q, \mu\}$, but for the sake of this example, we will take the quantities $\{T_H, \mu\}$ which we can define from horizon data, and derive the remaining variables from the Helmholtz free energy. The free energy can be calculated from the Legendre transformation:

$$
\Omega = F - \mu \mathcal{Q} \quad \Rightarrow \quad F(\mathcal{Q}, \beta) = \frac{1}{2}(M + Q\mu).
$$

Partial derivatives of the free energy can be used to calculate the internal energy, chemical potential and entropy. We can use our results from Section 3.5 as a consistency check for the Euclidean action formalism. For more details on the various parameters we can derive from a given potential, we refer to Appendix B. Later, when considering the planar solutions and their thermodynamics, will use the Euclidean action formalism to derive the values of these quantities.

The entropy is found from

$$\begin{aligned}
S = \beta^2\left(\frac{\partial F}{\partial\beta}\right)_Q &= \frac{\beta^2}{2}\left(\frac{\partial M}{\partial\beta} + Q\frac{\partial\mu}{\partial\beta}\right),\\
&= \frac{\beta^2}{2}\left(\frac{\partial M}{\partial\beta} + \frac{Q}{r_+^2}\frac{\partial r_+}{\partial\beta}\right),\\
&= \beta^2\left(\frac{\left(M\sqrt{M^2-Q^2}+M^2-Q^2\right)^2}{4\pi\left(\sqrt{M^2-Q^2}+M\right)^4}\right),\\
&= \pi(M+\sqrt{M^2-Q^2})^2 = \pi r_+^2,
\end{aligned}$$

which exactly matches with the Bekenstein-Hawking entropy law (3.5.3).

The internal energy E is calculated

$$\begin{aligned}
E = \left(\frac{\partial(\beta F)}{\partial\beta}\right)_Q &= \beta\left(\frac{\partial F}{\partial\beta}\right)_Q + F,\\
&= \beta\left(\frac{\left(M\sqrt{M^2-Q^2}+M^2-Q^2\right)^2}{4\pi\left(\sqrt{M^2-Q^2}+M\right)^4}\right) + \frac{1}{2}(M+\mu Q),\\
&= \left(\frac{M\sqrt{M^2-Q^2}+M^2-Q^2}{2(M+\sqrt{M^2-Q^2})}\right) + \left(\frac{M\sqrt{M^2-Q^2}+M^2+Q^2}{2(M+\sqrt{M^2-Q^2})}\right) = M,
\end{aligned}$$

and as we expect for the Reissner-Nordström solution, this matches with the mass parameter of the solution. Lastly we can find a value for the chemical potential by keeping the temperature fixed (and therefore r_+ fixed)

$$\begin{aligned}
\mu = \left(\frac{\partial F}{\partial Q}\right)_\beta &= \frac{\partial}{\partial Q}\left(\frac{M}{2} + \frac{Q^2}{2r_+}\right),\\
&= \frac{Q}{r_+},
\end{aligned}$$

and note that this matches with the derived value (3.6.4).

Armed with our thermodynamic variables, we can now verify the first law (3.5.1) through varying the internal energy and checking that

$$dE = T_H dS + \mu dQ.$$

To do this, we want to write down the equation of state $E(S,\mathcal{Q})$, and vary the energy. However, due to the functional form, it's a little bit easier to write down $S(E,\mathcal{Q})$ and instead vary this. The entropy $S = \pi r_+^2$ can be varied

$$dS = \partial_E(\pi r_+^2)dE + \partial_Q(\pi r_+^2)dQ, \qquad r_+ = M + \sqrt{M^2-Q^2}.$$

The partial derivatives can be computed

$$\begin{aligned}
\frac{\partial(\pi r_+^2)}{\partial E} &= \frac{2\pi\left(\sqrt{M^2-Q^2}+M\right)^2}{\sqrt{M^2-Q^2}} = \beta,\\
\frac{\partial(\pi r_+^2)}{\partial Q} &= -\frac{2\pi Q\left(\sqrt{M^2-Q^2}+M\right)}{\sqrt{M^2-Q^2}} = -\mu\beta,
\end{aligned}$$

which allows us to write down

$$dS = \partial_E(\pi r_+^2)dE + \partial_Q(\pi r_+^2)dQ = \beta dM - \mu\beta dQ \quad \Rightarrow \quad dE = T_H dS + \mu dQ,$$

and as expected, the first law is satisfied. We can also verify Smarr's law [75] and show that

$$E = 2T_H S + \mu \mathcal{Q}.$$

4

SUPERGRAVITY AND HIGHER DIMENSIONS

In this thesis, our main focus is to understand solutions of supergravity theories containing planar symmetric Killing horizons and their corresponding thermodynamics. In the previous chapters, we have discussed how starting with a Lagrangian containing some matter content and an ansatz imposing a set of symmetries on the manifold, allows us to derive a metric describing the space-time of the solution. From this perspective, the role of supergravity is to suggest a Lagrangian as a starting point from which we build our solutions in Chapter 6.

This chapter aims to introduce supergravity from the perspective of extending the possible physical models we consider as a general relativist, while also giving a backbone to some of the various comments and calculations that relate our four-dimensional solutions to contemporary research in string theory and higher-dimensional supergravity.

We begin with Section 4.1, introducing supergravity from the perspective of the supersymmetry algebra and use this to define the matter content we would expect in a physical theory, focusing on $\mathcal{N} = 2$ supergravity in four dimensions. In Section 4.2, we take a brief detour and discuss the electromagnetic duality and its generalisation, which appears in $\mathcal{N} = 2$ vector multiplets. This will be particularly important when we consider the thermodynamics of our charged solutions derived in Chapter 8. In Section 4.3, we introduce Kaluza-Klein reduction, which is an incredibly useful tool for representing higher-dimensional supergravity in lower dimensions. From a phenomenological perspective, the process of dimensional reduction allows us to understand higher-dimensional theories such as string theory and M-theory in a way that matches our experience of a universe in four dimensions. In Section 4.4, the c-map is introduced, which is a crucial result in finding non-extremal solutions of supergravity in four dimensions. The chapter concludes with Section 4.5, which gives an overview of higher-dimensional supergravity, aiming to introduce the reader to p-branes and their relationship with black hole solutions.

4.1 $\mathcal{N} = 2$ SUPERGRAVITY

In this section, we provide an introduction to supergravity sufficient to motivate the supergravity actions which serve as a starting point from which we derive planar symmetric solutions. We begin in Section 4.1.1 by motivating supersymmetry as the extension of the Poincaré algebra and study the massive and massless representations following [91]. In Section 4.1.2, we then specify to $\mathcal{N} = 2$ supergravity in four dimensions and present the Lagrangian of the bosonic content of

$\mathcal{N} = 2$, two-derivative supergravity coupled to n_V vector multiplets.

4.1.1 *Supersymmetry algebra*

The laws of physics are understood to be invariant under translations (generated by the momentum operator P_μ) and the Lorentz transformations (boosts and rotations, generated by $M_{\mu\nu}$). From these generators, one can build the Lie algebra of the Poincaré group

$$\begin{aligned} &[P_\mu, P_\nu] = 0, \\ &[M_{\mu\nu}, P_\rho] = i\left(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu\right), \\ &[M_{\mu\nu}, M_{\rho\sigma}] = i\left(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}\right). \end{aligned} \tag{4.1.1}$$

Supersymmetry comes naturally from the question of whether we can include additional operators which extend the Lie algebra. In 1967, Coleman and Mandula [92] proved their no-go theorem that the most general *bosonic* symmetries of the S-matrix must commute with the Poincaré algebra if we wish to maintain non-zero scattering amplitudes, *i.e.* these additional generators transform as scalars. Of course, a no-go theorem is only as strong as its assumptions, and in 1971, Golfand and Likhtman were able to extended the algebra by including anti-commuting, *fermionic* generators [93]. The inclusion of fermionic generators generalises the Lie algebra to a graded Lie algebra, which is defined by

$$\mathcal{O}_a \mathcal{O}_b - (-1)^{\chi_a \chi_b} \mathcal{O}_b \mathcal{O}_a = i C^c_{ab} \mathcal{O}_e,$$

for operators $\mathcal{O}_a$, where $\chi_a = 0$ for *bosonic* generators and $\chi_a = 1$ for *fermionic* generators, and C^c_{ab} are the structure constants. In 1975, Haag, Lopuszanski and Sohnius generalised Coleman and Mandula's no-go theorem stating that non-trivial quantum field theories have the super Poincaré alebgra as the most general algebra, and any additional symmetries will commute as internal scalars [94].

The super Poincaré algebra is the extension of (4.1.1) which includes the additional fermionic generators, also known as *supercharges* which we take to be Weyl spinors, transforming in the fundamental representation of $\mathrm{SL}(2, \mathbb{C})$ and its complex conjugate. They are denoted by:

$$Q^A_\alpha, \; \bar{Q}^A_{\dot\alpha}, \quad \alpha, \dot\alpha \in \{1, 2\}, \quad A \in \{1, \ldots, \mathcal{N}\},$$

where the indices $\alpha, \dot\alpha$ are spinorial indices, raised and lowered with $\delta_{\alpha\dot\beta}$, and $\mathcal{N}$ counts 'the amount of supersymmetry'; more precisely, we say that a theory has $4\mathcal{N}$ real supercharges. The generators are related by $\bar{Q}^A_{\dot\alpha} = \epsilon_{\dot\alpha}{}^{\beta} (Q^A_\beta)^\star$.[1] Minimal supersymmetry is for $\mathcal{N} = 1$, and maintaining that the highest spin states have spin-2 requires $\mathcal{N} \leq 8$ [95].

Qualitatively, we can think of these fermionic operators generating a fermion from a boson and vice versa:

$$Q \,|\text{fermion}\rangle = |\text{boson}\rangle, \qquad Q \,|\text{boson}\rangle = |\text{fermion}\rangle.$$

[1] If we consider the Weyl spinors as independent, we have four complex degrees of freedom, and $\mathcal{N} = 2$ supersymmetry realised in our algebra. By asserting the Weyl spinors are each other's complex conjugates, we reduce to two complex, or four real degrees of freedom which can be packaged together into a single Majorana spinor or the two related Weyl spinors introduced in the main text.

These generators extend the Poincaré algebra with the following (anti-)commutation relations

$$\begin{aligned}
&[P_\mu, Q^A_\alpha] = 0, \\
&[M_{\mu\nu}, Q^A_\alpha] = -\frac{i}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q^A{}_\beta, \\
&[M_{\mu\nu}, \bar{Q}^A_{\dot\alpha}] = -\frac{i}{2}(\bar\sigma_{\mu\nu})_{\dot\alpha}{}^{\dot\beta} \bar{Q}^A{}_{\dot\beta}, \\
&\{Q^A_\alpha, \bar{Q}^B_{\dot\beta}\} = 2\delta^{AB}(\sigma_\mu)_{\alpha\dot\beta}P^\mu, \\
&\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta}Z^{AB}.
\end{aligned} \tag{4.1.2}$$

The explicit form of the Pauli matrices is

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $\sigma_{\mu\nu} = \frac{1}{2}\sigma_{[\mu}\sigma_{\nu]}$. The relations (4.1.2) together with (4.1.1) form the super Poincaré algebra. The operators Z^{AB} are known as central charges, and they commute with all elements of the super Poincaré algebra. Note that this extension leaves the Poincaré algebra as a subalgebra, and so supersymmetry only adds additional structure to the physics we already understand.

There is also an internal symmetry called *R-symmetry* which is the group of transformations on the supercharges that leave the superalgebra invariant. For the case of $\mathcal{N} = 1$, the R-symmetry group is $\mathrm{U}(1)$ and we can see this in the automorphism

$$Q_\alpha \to e^{i\lambda}Q_\alpha, \qquad \bar{Q}_{\dot\alpha} \to e^{-i\lambda}\bar{Q}_{\dot\alpha}.$$

As the phases for Q and $\bar{Q}$ are opposite, and the only non-trivial commutation relation is for $\{Q, \bar{Q}\}$, we see immediately the algebra will be left unchanged. For the case of $\mathcal{N} > 1$, we can understand the elements of the R-symmetry group $S \in G_R$ as those for which

$$[Q^A_\alpha, T_i] = S^A{}_B\, Q^B_\alpha, \qquad [\bar{Q}^A_{\dot\alpha}, T_i] = -\bar{Q}^B_{\dot\alpha}\, S^A{}_B.$$

The general linear transformation S is restricted by the condition that it must commute with the supercharges. When the central charges of the theory vanish, the R-symmetry group is simply $\mathrm{U}(\mathcal{N})$. However, when $Z^{AB} \neq 0$, the R-symmetry is a subgroup $G_R \subset \mathrm{U}(\mathcal{N})$ and must be studied on a case-by-case basis.

Considering the Poincaré algebra, we think of the irreducible representations as describing particles. For supersymmetric theories, the particle representations are combined together to form *multiplets* which are populated by *superparticles* which have the same mass, but different spin/helicity. Under action of the supersymmetry algebra, these superparticles transform into each other. The content of the multiplets is dependent on $\mathcal{N}$. As the supercharges commute with the momentum operator and generate fermions from bosons (or vice-versa), we can understand that any bosonic (fermionic) superparticle will be accompanied by at least one fermionic (bosonic) superparticle of the same mass, which we refer to as the *superpartners*. It can be shown that for any supersymmetric theory, the total number of bosons and fermions is the same [91].

While working with quantum field theories, the underlying spacetime is assumed to be flat and the physical system is invariant under global super Poincaré symmetries. In this case, we refer to supersymmetry as *global* or *rigid supersymmetry*. If we instead consider a theory of gravity, the super Poincaré symmetry is realised as a local (gauge) symmetry, and we call this local supersymmetry, or *supergravity*.

Supersymmetry transformations on fields are parameterised by a spinor ϵ. In supergravity, the spinor parameter $\epsilon(x)$ itself becomes a spacetime dependent function. The graviton $g_{\mu\nu}$, which is a spin-two massless particle, has $\mathcal{N}$ massless spin-$\frac{3}{2}$ gravitino fields $\phi^A_{\mu\alpha}$ as superpartners which are the gauge fields for local supertransformations.

Making the restriction to $\mathcal{N} = 2$, which is the starting point for the solutions in Chapter 6, the anti-commutation relations of the supercharges simplifies to the form

$$\begin{aligned} \{Q^A_\alpha, \bar{Q}^B_{\dot\beta}\} &= 2\delta^{AB}(\sigma_\mu)_{\alpha\dot\beta}P^\mu, \\ \{Q^1_\alpha, Q^2_\beta\} &= -\{Q^2_\alpha, Q^1_\beta\} = 2\epsilon_{\alpha\beta}|Z|, \qquad 2|Z| := |Z^{12}|, \end{aligned} \tag{4.1.3}$$

where by making a U(1) phase transformation, it is possible to ensure that Z^{12} is real.

Massive representations

This discussion follows [49]. Let us begin with massive representations for $M^2 > 0$, for which we can boost to the rest frame $P_\mu = (-M, 0, 0, 0)$, such that

$$\sigma^\mu P_\mu = M\sigma_0 = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}.$$

We can apply this simplification to (4.1.3) and find that the algebra is written as

$$\begin{aligned} \{Q^A_\alpha, \bar{Q}^B_{\dot\beta}\} &= 2M\delta_{\alpha\dot\beta}\delta^{AB}, \\ \{Q^1_\alpha, Q^2_\beta\} &= -\{Q^2_\alpha, Q^1_\beta\} = 2\epsilon_{\alpha\beta}|Z|. \end{aligned}$$

Next, we take the supercharges and use them to construct the following fermionic operators

$$a_\alpha = \frac{1}{\sqrt{2}}\left(Q^1_\alpha + \epsilon_\alpha{}^{\dot\beta}\bar{Q}^2_{\dot\beta}\right), \qquad b_\alpha = \frac{1}{\sqrt{2}}\left(Q^1_\alpha - \epsilon_\alpha{}^{\dot\beta}\bar{Q}^2_{\dot\beta}\right).$$

Looking at their anti-commutation relations, we find

$$\{a_\alpha, \bar{a}_{\dot\beta}\} = 2(M + |Z|)\delta_{\alpha\dot\beta}, \qquad \{b_\alpha, \bar{b}_{\dot\beta}\} = 2(M - |Z|)\delta_{\alpha\dot\beta}.$$

We are interested in the irreducible representations of the Poincaré algebra, which are complicated to determine as the algebra is not semi-simple. The upshot of Wigner's classification [96] is that after setting the mass of the particle (which can be understood as fixing the eigenvalue of Casimir operator built from the momentum operator) there is a remaining symmetry of rotations determined by the so-called *little group*. As such, the irreducible representations of massive particles are represented by the spin (s) of the particle (its representation $[s]$ of the little group SU(2)).[2] Setting the spin can be understood as fixing the eigenvalue of the second Casimir operator built from the Pauli–Lubanski operator.

[2] Considering only bosons, the little group of massive particles is SO(3), but in supersymmetry we necessarily also consider fermions and the inclusion of half-integer spin particles requires working with the double cover of SO(3) which is Spin(3) ≅ SU(2).

Setting $M > 0$ and working with the irreducible representations of the little group, we interpret $a_\alpha,\ b_\beta$ as the annihilation operators

$$a_\alpha\,|s\rangle = 0, \qquad b_\beta\,|s\rangle = 0,$$

and $\bar{a}_{\dot\alpha},\ \bar{b}_{\dot\beta}$ as the creation operators. We can then build a suitable basis of irreducible representations by

$$\mathcal{B} = \{\bar{a}_{\dot\alpha_1}\ldots\bar{b}_{\dot\beta_1}\ldots|s\rangle\}.$$

Maintaining that our representations are unitary requires that absence of negative norm states. This enforces the bound

$$M \geq |Z|,$$

known as the BPS bound, named after Bogomol'nyi, Prasad and Sommerfield [97, 98]. States which saturate this bound are known as BPS states, our massive representations fall into two classes.

When $M > |Z|$, the full algebra is unitary; as we have four creation operators, we have a total of $2^4 \cdot \dim[s]$ states. When the bound is saturated, we find null states which must be removed to maintain unitarity. This can be achieved by setting the operators $b_\beta = \bar{b}_{\dot\beta} = 0$ such that the basis of irreducible representations is modified to the form

$$\mathcal{B}' = \{\bar{a}_{\dot\alpha_1}\ldots|s\rangle\}.$$

As we have only two creation operators, we have only $2^2 \cdot \dim[s]$ states in the BPS-multiplets.

Massless representations

We now consider massless states, for which irreducible representations of the Poincaré group are labelled by their helicity λ: the quantum number of their representation in the little group $SO(2)$. We can change the helicity through acting with the supercharges $Q^A_\alpha, \bar{Q}^A_{\dot\alpha}$. As the fermionic operators commute with the momentum operator, we can consider all helicities in a frame in which $P_\mu = (-E, 0, 0, E)$ such that

$$\sigma^\mu P_\mu = E(\sigma_0 + \sigma_3) = \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix}.$$

Referring back to (4.1.3), we see that $Q^A_2 = \bar{Q}^A_{\dot 2} = 0$, and that $Z^{AB} = 0$, such that the central charges annihilate all states. We have two remaining generators which we normalise as

$$a^A = \frac{1}{2\sqrt{E}} Q^A_1, \qquad \bar{a}^A = \frac{1}{2\sqrt{E}} \bar{Q}^A_{\dot 1},$$

which can be understood as raising or lowering the helicity by one half respectively. With two operators, we can expect $2^2 \cdot \dim[\lambda]$ states per representation. The anti-commutation relations are given by

$$\{a^A, \bar{a}^B\} = \delta^{AB}, \qquad \{a^A, a^B\} = \{\bar{a}^A, \bar{a}^B\} = 0.$$

helicity	state		# states
λ_{min}	$\lvert\lambda_{\text{min}}\rangle$		1
$\lambda_{\text{min}}+\frac{1}{2}$	$\bar{a}^1\lvert\lambda_{\text{min}}\rangle$	$\bar{a}^2\lvert\lambda_{\text{min}}\rangle$	2
$\lambda_{\text{min}}+1$	$\bar{a}^1\bar{a}^2\lvert\lambda_{\text{min}}\rangle$		1

Table 4.1: The four states obtained by acting with raising operators on the Clifford vacuum.

To build an irreducible representation, we pick a state of minimal helicity $\lvert E,\lambda_{\text{min}}\rangle$ which is annihilated by all a^A. We refer to $\lvert E,\lambda_{\text{min}}\rangle = \lvert\lambda_{\text{min}}\rangle$ as the Clifford vacuum[3] [91] and by acting on it with our raising operators, we can build our states. We summarise the generation of the four possible states into Table 4.1. We see that for $\mathcal{N}=2$, we have 4 massless superparticles present in each irreducible representation which split into 2 bosons and 2 fermions. Maintaining that $\lambda_{\text{max}} \leq 2$, we can build all possible base states of the massless representations through picking λ_{min}, which is displayed in Table 4.2.

		λ_{max}						
		2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1
	2	1						
	$\frac{3}{2}$	2	1					
	1	1	2	1				
	$\frac{1}{2}$		1	2	1			
helicity	0			1	2	1		
	$-\frac{1}{2}$				1	2	1	
	-1					1	2	1
	$-\frac{3}{2}$						1	2
	-2							1

Table 4.2: All possible base states of the massless representations of the $\mathcal{N}=2$ super Poincaré algebra [91].

Taken on their own, the multiplets formed through choosing some minimum helicity will not be charge-parity (CP) invariant [49]. Thus, the physical multiplets we consider will be the pairs of CP conjugated multiplet which then include the corresponding antiparticles. As a concrete example, let us see how this pairing produces the multiplets of $D=4$, $\mathcal{N}=2$ supergravity of

[3] We could have in fact used this terminology before. The relations between the fermionic creation and annihilation operators are equivalent to those of a Clifford algebra and the standard Clifford relations can be obtained by taking suitable linear combinations of a^A and $\bar{a}^A$. In this sense, the Clifford algebras can be realised using the fermionic ladder operators. This is discussed in detail in Appendix 5.A of [16].

which we will be primarily concerned with.[4]

- Supergravity multiplet

$$(g_{\mu\nu}, \psi^A_{\mu\alpha}, A_\mu)$$

  The supergravity multiplet is built from combining the massless representations with maximum helicity 2 and −1. This consists of the graviton $g_{\mu\nu}$, two Weyl vector-spinors, called gravitini ψ^1_μ, ψ^2_μ and a vector field A_μ known as the graviphoton.

- Vector multiplet

$$(A_\mu, \lambda^A, z)$$

  The vector multiplet is built from combining the massless representations with maximum helicity 1 and 0. It consists of a massless vector field A_μ, two Weyl spinors λ^1, λ^2 and a complex scalar field z.

- Hyper multiplet

$$(\gamma^A, q^u)$$

  A hypermultiplet is built from two representations with maximal helicity-$\frac{1}{2}$ and consists of two Weyl spinors γ^1, γ^2 and four real scalar fields q^1, q^2, q^3, q^4.

4.1.2 $\mathcal{N} = 2$ *supergravity Lagrangians*

We are now ready to study the supergravity Lagrangians which serve as the starting point for the planar solutions we derive in Chapter 6. In this thesis, we consider only the bosonic field configurations, and hence the fermions in our multiplets are effectively set to zero. The removal of the fermionic matter is a form of *consistent truncation*, which is a removal of some sub-set of the matter content of a theory such that solutions of the truncated theory are also solutions of the full theory.

Let us first consider the bosonic matter content of pure supergravity. We see that with the gravitini $\psi^A_\mu = 0$, the supergravity multiplet contains the graviton $g_{\mu\nu}$ and a single gauge field A_μ. This is precisely the same field content as the Einstein-Maxwell theory (see Equation 3.3.1). We can thus interpret the Reissner-Nordström solution derived in Section 3.3 as a solution of $\mathcal{N} = 2$ pure supergravity. There is an important distinction to make here though. The Reissner-Nordström solution may be a solution to supergravity, but we do not say that the Reissner-Nordström solution is a supersymmetric solution. In Section 4.1.3, we revisit this comment and discuss the extremal Reissner-Nordström solution as a supersymmetric solution.

The bosonic content of a vector multiplet is made from the vector field A_μ and a complex scalar field z. Here we consider the Lagrangian of $\mathcal{N} = 2$ supergravity coupled to n_V vector multiplets which will have a bosonic field content of the metric, n_V complex scalar fields and

[4]Technically, the hyper multiplets will be set to zero when we solve the equations of motion in Section 6.2, but we include them here as we will find that after the dimensional reduction of a four-dimensional theory coupled to vector multiplets, the resulting three-dimensional theory can be described as supergravity coupled to hyper multiplets.

(n_V+1) vector fields. Its Lagrangian is given by [99, 100]

$$\begin{aligned}\mathbf{e}_4^{-1}\mathcal{L} &= -\frac{1}{2\kappa_4^2}R - \frac{1}{\kappa_4^2}g_{A\bar{B}}\partial_\mu z^A\partial^\mu\bar{z}^{\bar{B}} + \frac{1}{4}F^I_{\mu\nu}\tilde{G}^{\mu\nu}_I,\\ &= -\frac{1}{2\kappa_4^2}R - \frac{1}{\kappa_4^2}g_{A\bar{B}}\partial_\mu z^A\partial^\mu\bar{z}^{\bar{B}} + \frac{1}{4}\mathcal{I}_{IJ}F^I_{\mu\nu}F^{J|\mu\nu} + \frac{1}{4}\mathcal{R}_{IJ}F^I_{\mu\nu}\tilde{F}^{J|\mu\nu}.\end{aligned} \tag{4.1.4}$$

where we use the indices $A, B \in \{1,\ldots,n_V\}$ and $I, J \in \{0,\ldots,n_v\}$, the n_V complex scalars are represented by z^A, and the (n_V+1) gauge fields appear in the field strengths $F^I_{\mu\nu}$. R is the Ricci scalar and $\mathbf{e}_4 = \sqrt{-g}$ is the vierbein. The gravitational coupling $\kappa_4^2 = 8\pi G$, and although within a relativity context we often make the choice $G=1$, it is more common for $\kappa_4^2 = 1$ from a supergravity perspective. In this section, we will keep κ_4 explicit. We denote the dual field strengths as $G^{\mu\nu}_I$, which can be written in terms of the field strength and it's Hodge-dual

$$G^{\mu\nu}_I = \mathcal{R}_{IJ}F^{J|\mu\nu} - \mathcal{I}_{IJ}\tilde{F}^{J|\mu\nu},$$

where we use a tilde to represent Hodge-dualisation

$$(\star F^I)^{\mu\nu} =: \tilde{F}^{I|\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{I|\rho\sigma}.$$

Lastly, we have coupling matrices for our matter fields. The scalar field coupling is $g_{A\bar{B}}$, and the gauge field coupling is split into the real and imaginary components: $\mathcal{N}_{IJ} = \mathcal{R}_{IJ} + i\mathcal{I}_{IJ}$.

The derivation of the Lagrangian (4.1.4) is often found through the process of gauge fixing a theory of superconformal vector multiplets coupled to superconformal supergravity. To properly contextualise some of the following discussion, we offer a sketch of the relationship between these theories but do not offer a full derivation. We refer to [101, 102] for a comprehensive overview.

As we were able to extend the Poincaré algebra by including fermionic generators, the same story can be played out with the conformal algebra which itself extends the Poincaré algebra by including scale transformations (also known as dilatations) and special conformal transformations. As the superconformal algebra has additional symmetries, to recover Poincaré supergravity we should expect to gauge fix the redundant degrees of freedom.

To understand the relationships between these theories, we can consider the multiplets of the superconformal theory. In this discussion we are interested in Weyl multiplet and the superconformal vector multiplets, however, the full discussion requires the inclusion of at least one hyper multiplet too which contributes as a conformal compensator, which we mention again below. The bosonic content of the superconformal vector multiplets is unchanged, containing a vector field A_μ and a complex scalar X. However, the complex scalar fields have the additional symmetry

$$X \to \lambda X, \qquad \lambda \in \mathbb{C}^\star \simeq \mathbb{R}^{>0} \times \mathrm{U}(1).$$

To obtain the Poincaré theory from the superconformal one, we must break both the dilatations generated by $\mathbb{R}^{>0}$ and the overall phase transformation from the U(1).

The Weyl multiplet is built from the gauge fields that appear when gauging the rigid theory to obtain superconformal supergravity. Included within the Weyl multiplet are the gauge fields ${e_\mu}^a$, which label the translations and ${\omega_\mu}^{ab}$, which label the Lorentz symmetries. To recover the

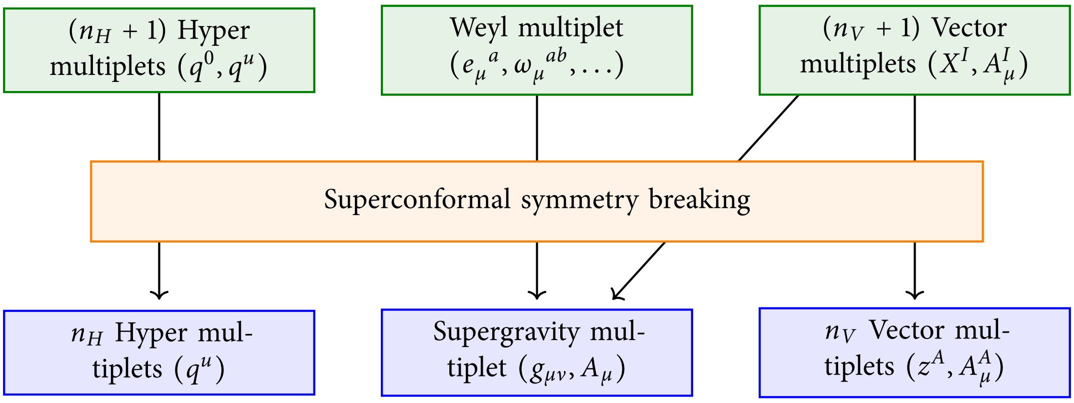


Figure 4.1: Diagram of the packaging of the bosonic content of the multiplets of $\mathcal{N} = 2$ Poincaré supergravity from the superconformal theory. By gauge fixing the superconformal $\mathcal{N} = 2$ theory, we obtain the field content of $\mathcal{N} = 2$ Poincaré supergravity, with the Weyl multiplet and a single vector multiplet being packaged to give the supergravity multiplet.

Poincaré theory, we must understand $e_\mu{}^a$ as the vielbein (or the tetrad) and $\omega_\mu{}^{ab}$ as the spin connection. In other words, we require that the local gauge translations of the superconformal theory are equivalent to the diffeomorphisms of the Poincaré theory. This is achieved by making the so-called *conventional constraint* relating various gauge fields of the superconformal theory to each other. In particular $e_\mu{}^a$ and $\omega_\mu{}^{ab}$ are no longer independent, but related through the expression for the torsion two-form [103]. The conventional constraint requires the matter content of the Weyl multiplet together with two additional multiplets, known as *conformal compensators*. To recover the graviphoton of the supergravity multiplet, we inherit a gauge field by including a superconformal vector multiplet and the second conformal compensator is often taken to be a hyper multiplet.

It can then be shown that Poincaré supergravity is gauge equivalent to the local superconformal supergravity theory after imposing the conventional constraint and including the conformal compensators. The upshot is that in order to consider Poincaré supergravity coupled to n_V vector multiplets and n_H hyper multiplers, we gauge fix superconformal supergravity coupled to $(n_V + 1)$ superconformal vector multiplets and $(n_H + 1)$ superconformal hyper multiplets. An illustration of this is given in Figure 4.1.

Including the additional vector multiplet leads to a miss-match in the number of scalar fields in the superconformal theory: X^I, $I \in 0, \ldots, n_V$ and the super Poincaré theory: z^A, $A \in 1, \ldots, n_V$. In fact, viewed from another perspective, one of the benefits with working with the superconformal theory is that the number of gauge fields and scalar fields is balanced. We will explain the utility of this in more detail from the context of finding planar symmetric solutions in Chapter 6.

Working with the complex fields X^I, we must then remember we have spurious degrees of freedom which are set through two gauge-fixing conditions. The first is known as the *dilatation gauge* or D-gauge, which corresponds to setting a scale for the theory, the second is a U(1) symmetry associated to phase transformations of X^I. From these complex scalars X^I, the physical degrees of freedom can be recovered from the ratios

$$z^A = \frac{X^A}{X^0}, \quad \bar{z}^A = \frac{\bar{X}^A}{\bar{X}^0},$$

which are understood to be the scalars of the Poincaré supergravity theory after gauge fixing the extra superconformal symmetries. Despite z^A parameterising the physical degrees of freedom of the theory, we instead will work with X^I and keep track of the gauge fixing conditions which we must impose while finding our solutions, where we should understand our Lagrangian as given by

$$\mathbf{e}_4^{-1}\mathcal{L} = -\frac{1}{2\kappa_4^2}R - \frac{1}{\kappa_4^2}g_{I\bar{J}}\partial_\mu X^I \partial^\mu \bar{X}^{\bar{J}} + \frac{1}{4}\mathcal{I}_{IJ}F^I_{\mu\nu}F^{J|\mu\nu} + \frac{1}{4}\mathcal{R}_{IJ}F^I_{\mu\nu}\tilde{F}^{J|\mu\nu}.$$

One of the primary benefits is that in this form, the field equations are invariant under symplectic transformations. This is discussed in more detail in Section 4.2.

The matrix couplings[5] $(g_{I\bar{J}}, \mathcal{I}_{IJ}, \mathcal{R}_{IJ})$ and hence the dynamics of the Lagrangian, are entirely determined by a holomorphic function known as the prepotential. We denote the prepotential by $F(X^I)$, which is homogenous of degree two. Note that although the same Latin letter is used for the prepotential and the gauge field strength, the presence of the spacetime indices (and context) should help distinguish them. Throughout the thesis, we have been referring to Chapter 6, stating that we would be finding planar symmetric solutions to the STU model. We can consider the STU model through making a choice for the prepotential [104]

$$F(X) = \frac{X^1 X^2 X^3}{X^0}. \tag{4.1.5}$$

The matter content of the STU model is that of $\mathcal{N} = 2$ supergravity coupled to three vector multiplets. As a result, there will be three complex scalar fields (which are sometimes referred to by the letters s, t, u) and (3+1) gauge fields. In Appendix E, we use this prepotential to find the exact forms of the coupling matrices $g_{A\bar{B}}$ and $\mathcal{I}_{IJ}$ as functions of the physical scalars z^A.

We now offer some formulae which help us build an understanding of the couplings in (4.1.4) in terms of the complex scalars X^I and the prepotential $F(X^0, \ldots X^{n_V})$. We will use the notation

$$F = F(X^0, \ldots X^{n_V}), \quad F_I = \frac{\partial F}{\partial X^I}, \quad F_{IJ} = \frac{\partial F^2}{\partial X^I X^I},$$

for the derivatives of the prepotential.

From the superconformal perspective, the kinetic term for the complex scalars comes with a coupling term $N_{IJ} = -i(F_{IJ} - \bar{F}_{IJ})$, which can be understood as a Kähler metric on the target manifold parameterised with our complex holomorphic scalars as coordinates. We can then write the scalar metric as the second derivative of the Kähler potential

$$N_{IJ} = \frac{\partial^2 K}{\partial X^I \partial \bar{X}^{\bar{J}}} \qquad K(X, \bar{X}) = i(X^I \bar{F}_I - F_I X^I). \tag{4.1.6}$$

[5] The coupling of the physical scalars $g_{A\bar{B}}$ is distinct to, but can be computed from $g_{I\bar{J}}$. This relation is given by (4.1.10) and the calculation is performed explicitly in Appendix E for the case of the STU model.

We can write the Kähler potential in terms of the scalar metric and complex scalars

$$\begin{aligned} i(X^I\bar{F}_I - F_I X^I) &= i(X^I\bar{F}_{IJ}\bar{X}^{\bar{J}} - X^J F_{IJ} X^I), \\ &= iX^I\bar{X}^{\bar{J}}(\bar{F}_{IJ} - F_{IJ}), \\ &= -i^2 X^I\bar{X}^{\bar{J}} N_{IJ}, \\ K &= XN\bar{X}, \end{aligned}$$

where we use the notation $XN\bar{X} = X^I N_{IJ}\bar{X}^{\bar{J}}$.

The scalar metric $g_{A\bar{B}}$ of the associated Poincaré supergravity theory has a related, but different Kähler potential given by

$$\mathcal{K}(X,\bar{X}) = -\log K \quad \Rightarrow \quad e^{-\mathcal{K}} = i(X^I\bar{F}_I - F_I X^I). \tag{4.1.7}$$

Setting the D-gauge of the theory is equivalent to fixing the value of the potential, commonly the choice is made such that $e^{-\mathcal{K}} = 1$.

To derive the form of the scalar couplings $g_{A\bar{B}}$, we first take the second derivative of the new potential

$$g = \frac{\partial^2\mathcal{K}}{\partial X^I\bar{X}^{\bar{J}}} dX^I \otimes d\bar{X}^{\bar{J}}, \tag{4.1.8}$$

which is a rank-two tensor field. This is not the metric for the holomorphic coordinates X^I as it has a two-dimensional kernel with null vectors:

$$X^I g_{I\bar{J}} = 0, \qquad g_{I\bar{J}}\bar{X}^{\bar{J}} = 0.$$

This means that two real degrees of freedom are non-propagating. As we showed above, we can recover the independent propagating degrees of freedom through considering ratios of the holomorphic coordinates. This allows us to understand this as a metric for the scalars z^A, despite being derived from X^I. We can explicitly calculate the components of the scalar couplings in terms of the complex scalars X^I and the scalar metric

$$g_{I\bar{J}} = -\frac{N_{IJ}}{XN\bar{X}} + \frac{(N\bar{X})_I(XN)_J}{(XN\bar{X})^2}, \tag{4.1.9}$$

where we use the notation such that $(NX)_I = N_{IJ}X^J$. The coupling for the scalar fields is defined from this matrix

$$g_{A\bar{B}} = g_{I\bar{J}}\frac{\partial X^I}{\partial z^A}\frac{\partial\bar{X}^{\bar{J}}}{\partial\bar{z}^{\bar{B}}}. \tag{4.1.10}$$

The couplings of the gauge fields are derived in [101], which can be expressed in terms of the prepotential by [100]

$$\mathcal{N}_{IJ} = \mathcal{R}_{IJ} + i\mathcal{I}_{IJ} = \bar{F}_{IJ} + i\frac{(XN)_I(XN)_J}{XNX}, \tag{4.1.11}$$

where in our conventions $\mathcal{I}_{IJ}$ is negative-definite. We note here that the field strengths F^I fit into the conformal, but not the Poincaré vector multiplets. We choose to work with F^I, as (F^I, G_I) is a symplectic vector. We will expand on this in Section 4.2. One can obtain the gauge fields of the Poincaré supergravity theory from a linear combination of those from the superconformal theory.

Gauged supergravity We take a brief aside to mention the fairly confusing terminology of *gauged supergravity*. We discussed above that to obtain supergravity from rigid supersymmetry, the super Poincaré group is gauged making the symmetries local, promoting partial derivatives to covariant derivatives. The new gauge fields are packaged into the supergravity multiplet and from the Lagrangian perspective, we find the appearance of the Einstein-Hilbert term. This (ungauged) supergravity theory has no *additional* gauge symmetries and so the matter is uncharged and all vector fields are Abelian. In contrast, *gauged* supergravity has additional gauge symmetries, which appear in the form of charged matter fields or non-Abelian vector fields. These additional gauge symmetries introduce a scalar potential. In this sense, all supergravity theories are gauged, just some more than others. In this thesis, we will only be interested in solutions to ungauged supergravity theories, but will commonly refer to gauged supergravity when discussing the extension of our work or tangential research.

4.1.3 *Supersymmetric black holes*

In Section 4.1.2, we mentioned that the bosonic content of $\mathcal{N} = 2$ pure supergravity in four dimensions matched that of Einstein-Maxwell theory, which we studied in Section 3.3. As the truncation of fermions is a consistent truncation, we know that the Reissner-Nordström solution is a solution to supergravity. We now justify the claim that the extremal Reissner-Nordström solution is supersymmetric following a discussion in [49].

Let us first understand what we mean by a supersymmetric solution, and why this is interesting from the perspective of looking at black hole solutions. As an analogy, let us consider the role of Killing vector fields when finding solutions to Einstein's equations. We understand that Killing vector fields generate the symmetries of a spacetime. In Section 2.3.2.1, we saw that when we imposed the maximum number of Killing vector fields to be present, the solution to Einstein's equations was almost totally determined. On the flip side, if we assume no symmetry properties and look for some generic solution of Einstein's equations, solving the field equations becomes an incredibly difficult problem. For the black hole solutions we considered, such as the Schwarzschild solution and the Reissner-Nordström solution, we have that they are both static and spherically symmetric and so have four Killing vectors (time-translation and spacial rotations). From the point of view of the field equations, the symmetries which appeared in our ansatz played a role in allowing us to find exact solutions. However, we also can learn about the solutions from the symmetries which are broken. For the Schwarzschild solution, spacial translations are broken, but the underlying theory itself is translation invariant. We can then understand all black holes related by some spacelike translation as being equivalent in that they have the same energy [49].

There is a supersymmetric analogue to this, where we say a solution is supersymmetric if it is invariant under supersymmetric transformations. We say a field configuration Φ_0 is supersymmetric if

$$\delta_{\epsilon(x)} \Phi\Big|_{\Phi_0} = 0. \tag{4.1.12}$$

The variation is performed on all fields Φ and evaluated in the field configuration Φ_0. The fermionic parameter $\epsilon(x)$ depends on the spacetime coordinates and is known as a Killing spinor. The

Killing spinor equations (4.1.12) are first order equations and are generically easier to solve than the second order Einstein's equations. By imposing that the solution is supersymmetric, we impose restrictions on the field equations which ultimately reduce the complexity of solving Einstein's equations in an analogous way as to how our Killing vector fields reduced the solution space. A maximally supersymmetric solution should have $4\mathcal{N}$ Killing spinors and solutions which break certain supersymmetries have some fractional number of Killing spinors. For spherically symmetric, finite mass solutions of supergravity, we would expect the solutions to be BPS states. Solutions that preserve one-half of the supersymmetries (solutions with $2\mathcal{N}$ Killing spinors) are then said to be $\frac{1}{2}$-BPS solutions.

Returning to the extremal Reissner-Nordström solution, we understand it as embedded into $\mathcal{N} = 2$ pure supergravity and so there is a maximum of eight Killing spinors. As both of the gravitini are truncated out, the supersymmetry variation of the gauge field and graviton are trivial as the background is bosonic. The remaining conditions come from the gravitini variations:

$$\delta_\epsilon \psi^A_\mu \stackrel{!}{=} 0,$$

which are the only variations which impose conditions on the bosonic fields. It can be shown that by inserting in the field content of the extremal solution, there are precisely four Killing spinors. As such, we can understand the extremal Reissner-Nordström solutions as preserving one-half of the supersymmetries [101]. Another perspective of seeing the extremal solutions as BPS solutions comes from the analogy of the extremal limit $M = |Q|$ to the BPS limit of $M = |Z|$. We note here that although BPS solutions are always extremal solutions, extremal solutions need not always be BPS solutions. A common first example of this is the Kerr-Newman solution which describes a spinning, charged black hole. Here, we find that the extremal solution is not BPS, and the recovery of a BPS state requires 'over rotating' the black hole, producing a spacetime with a naked singularity [105].

We can also understand the near-horizon geometry of the extremal solution from a new perspective thanks to supersymmetry. At the asymptotic distance, the spacetime is four-dimensional Minkowski, which has eight Killing spinors for $\mathcal{N} = 2$ supergravity. At the horizon, we find the Bertotti-Robinson solution $AdS_2 \times S^2$, which is a product of maximally symmetric spaces and which also has eight Killing spinors. We then see that the extreme Reissner-Nordström solution interpolates between two vacua of $\mathcal{N} = 2$ between the asymptotic limit and the horizon. We will see a similar behaviour for the $\frac{1}{2}$-BPS state solutions in Section 4.5.3.

In this thesis, we are concerned with finding non-extremal solutions, and so the Killing spinor equations will not be appropriate for our research. However, in Chapter 6 we will take our four-dimensional solutions and uplift them into higher dimensions. From this higher-dimensional perspective, we will see that we can interpret the extremal limit of our solutions as intersecting brane configurations. In Section 4.5.3 and Section 4.5.4, we will give an overview of the construction of these intersecting brane solutions, which are BPS configurations. Furthermore, in Section 7.2, we find that for our class of non-extremal solutions in six dimensions, it is possible to restrict the integration constants in such a way that a BPS solution is recovered, despite the fact we never enforce supersymmetry while solving the equations of motion.

4.2 ELECTROMAGNETIC DUALITY

In this section we take a brief detour to discuss electromagnetic duality and its generalisation, which appears for $\mathcal{N} = 2$ vector multiplets. This will be important to understand when studying the thermodynamic properties of the STU model in Chapter 6, as we make use of an electric-magnetic duality frame where the magnetic charges $\mathcal{P}^A$, $A = 1, 2, 3$ are replaced by electric charges $\tilde{\mathcal{Q}}_A$. Additionally, find that from the perspective of the duality transformations, we are led to a clear picture of computing the conserved charges of the gauge fields.

4.2.1 *Maxwell theory*

To explain the idea, we first consider a theory with a single Abelian vector field A_μ with field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and a curved spacetime Maxwell type action[6]

$$S[A] = \int -\frac{1}{2} F \wedge \star F \,. \tag{4.2.1}$$

We begin our discussion writing down the field equations

$$\nabla_\mu F^{\mu\nu} = 0, \qquad \epsilon_{\mu\nu\rho\sigma} \partial^\nu F^{\rho\sigma} = 0.$$

The first are the Euler-Lagrange equations which can be found though varying the Lagrangian. The second is the Bianchi identity, which is an integrability condition to ensure the presence of the gauge potential A_μ, and doesn't depend on the spacetime metric.

The electromagnetic duality is a symmetry of the field equations that interchanges the electric and magnetic fields. The symmetry can be made apparent through writing down the equations of motion in terms of differential forms

$$d \star F = 0, \qquad dF = 0. \tag{4.2.2}$$

Considering the Hodge-star, we can define a dual gauge field by

$$\tilde{F} = \star F,$$

replacing $F \to \tilde{F}$ into the equations of motion, we find

$$d \star \tilde{F} = d \star^2 F = dF = 0, \qquad d\tilde{F} = d \star F = 0,$$

where we have used that $\star^2 = -1$ for a two-form in four-dimensional Minkowski spacetime. We see that swapping $F \leftrightarrow \tilde{F}$ will leave these two equations invariant, however, their interpretation will change. The Euler-Lagrange equations become the Bianchi identity for the dual field, ensuring that $\tilde{F} = d\tilde{A}$, and the Bianchi identity becomes the new Euler-Lagrange equations for an action of the form

$$S[\tilde{A}] = \int -\frac{1}{2} \tilde{F} \wedge \star \tilde{F} \,. \tag{4.2.3}$$

The electromagnetic duality is more general than the interchange of $F \leftrightarrow \star F$, and can instead be understood as the invariance of the equations of motion under the mapping from F to some

[6] For a review on the form notation used here, see Appendix A.

linear combination of F and $\star F$. Another way of thinking about this is that we are free to make some uniform rotation under the action of $\mathrm{Sp}(2,\mathbb{R}) \simeq \mathrm{SL}(2,\mathbb{R})$ of the vector $(F, \star F)$ [101].[7] We note here explicitly that the electromagnetic duality is *not* a symmetry of the Lagrangian. If we make the substitution $F \to \tilde{F}$ into (4.2.1), we obtain

$$\begin{aligned} S[A] &= \int -\frac{1}{2} F \wedge \star F, \\ &\to \int -\frac{1}{2}\tilde{F} \wedge \star \tilde{F} = \int -\frac{1}{2} \star F \wedge \star^2 F = \int \frac{1}{2} F \wedge \star F, \end{aligned}$$

which we see produces a sign error. The reason that this substitution doesn't work is that F is not a fundamental field; the action (4.2.1) is a function of the gauge potential A, and not the gauge field F. If we want to correctly dualise this action, we must first promote the Bianchi identity to an equation of motion by including it into the action with a Lagrange multiplier. This process is known as *Hodge duality* and in Appendix C, we offer a general discussion of this process for a p-form in a D-dimensional spacetime.

We begin by including a three-form λ to act as a Lagrange multiplier

$$S[F] = \int -\frac{1}{2} F \wedge \star F - dF \wedge \star \lambda. \tag{4.2.4}$$

This allows the Bianchi identity to become an equation of motion for the Lagrange multiplier

$$\begin{aligned} S[\star\lambda + \star\delta\lambda; F] &= S[\star\lambda; F] + \int -dF \wedge \star\delta\lambda, \\ 0 &= \int -dF \wedge \star \delta\lambda, \quad \Rightarrow \quad dF = 0. \end{aligned}$$

Varying the action with respect to the field strength, we find a modified equation of motion

$$\begin{aligned} S[\star\lambda; F + \delta F] &= S[\star\lambda, F] + \int -\delta F \wedge \star F - d(\delta F) \wedge \star\lambda, \\ &= S[\star\lambda, F] + \int -\delta F \wedge \star F + (\delta F) \wedge d \star \lambda, \\ &= S[\star\lambda, F] + \int \delta F \wedge [- \star F + d \star \lambda], \\ &\Rightarrow \star F = d(\star\lambda). \end{aligned}$$

Collecting these together, we write down the equations of motion for the first order action as

$$dF = 0, \qquad \star F = d(\star\lambda),$$

where we note that the original equations of motion still hold as $d^2 = 0$. In this form, we can perform the dualisation by making the identification

$$\begin{aligned} &\tilde{F} = \star F, \\ &\tilde{A} = \star\lambda \quad \Rightarrow \quad \tilde{F} = d\tilde{A}. \end{aligned} \tag{4.2.5}$$

If we substitute these dual fields the action (4.2.4), the resulting action is of the form

$$S[\tilde{A}] = \int -\frac{1}{2}\tilde{F} \wedge \star \tilde{F} \qquad \tilde{F} = d\tilde{A}, \tag{4.2.6}$$

[7] From the point of view of the field equations, we can actually further generalise this so the symmetries are generated by $\mathrm{GL}(2,\mathbb{R})$. Fixing the normalisation of the gauge field reduces this to $\mathrm{SL}(2,\mathbb{R}) \simeq \mathrm{Sp}(2,\mathbb{R})$. Once we consider the preservation of the charge quantisation condition, there is a further restriction to $\mathrm{Sp}(2,\mathbb{Z})$.

which preserves the duality in the equations of motion, but now also has the correct sign in the kinetic term when compared to (4.2.3).

The electromagnetic duality also helps us define our conserved charges. Writing our field equations as Bianchi identities:

$$d\tilde{F} = 0, \quad dF = 0, \tag{4.2.7}$$

we can apply Stoke's theorem to obtain values for the electric and magnetic charges[8]

$$\mathcal{Q} = \int_X \tilde{F} = \int_X \star F, \qquad \mathcal{P} = \int_X F, \tag{4.2.8}$$

Here we leave the codimension-two manifold generic and understand that for point-like charges the surface X has the topology of a sphere and for solutions with planar symmetry, we take X to be a plane. Note that for planar solutions, the integral over the plane is divergent and so we must instead consider charge densities. The equations of motion and Bianchi identities, which are valid outside charges, tell us that both F and $\tilde{F}$ are closed. This allows one to deform the integration surfaces X continuously, as long as one avoids moving them through the charges. This is not a problem for the solutions we consider, where the charges are located at singularities, which are not included within the domain of our parameters. Often it is convenient to evaluate the charges in a limit where X is evaluated at infinity, and this is in particular how charges are computed throughout this thesis.

The dualisation procedure can be used to replace magnetic charges by electric charges. This can be convenient since in a fixed duality frame electric charges are Noether charges and can couple minimally to the gauge field, whereas magnetic charges are topological and do not have local couplings to the gauge field. For black hole thermodynamics, and in particular, the Euclidean action formalism introduced in Section 3.6 and used in Chapter 8, we find it convenient to replace magnetic charges by electric charges. The dual charges are found by replacing F by $\tilde{F}$. Using that

$$\tilde{\tilde{F}} = \star\tilde{F} = \star^2 F = -F,$$

we find

$$\begin{aligned} \tilde{\mathcal{Q}} &= \int_X \tilde{\tilde{F}} = -\int_X F = -\mathcal{P}, & (4.2.9)\\ \tilde{\mathcal{P}} &= \int_X \tilde{F} = \mathcal{Q}. & (4.2.10) \end{aligned}$$

and we note that the transformation $(\mathcal{Q}, \mathcal{P}) \to (-\mathcal{P}, \mathcal{Q})$ is symplectic.

Symplectic transformations As a brief review, a symplectic vector space (V, ω) is built from an even-dimensional vector space V and the symplectic form ω, which is a non-degenerate, skew-symmetric bilinear form. Picking a basis for V, we can write ω as a matrix, which is commonly picked to be of the form

$$\Omega = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}.$$

[8] Note that the normalisation here matches that of the action (4.2.1) and will differ in factors from the equations used for the charges of Reissner-Nordström solution in Section 3.3.3, and also the STU model calculations performed in Section 8.4.1. Unfortunately, different models discussed within the thesis have different natural normalisations, so this mismatch is somewhat unavoidable.

A symplectic transformation is a linear map $L : V \to V$ such that the symplectic form is preserved

$$\omega(Lu, Lv) = \omega(u, v).$$

In a basis, this is the same as the requirement that a matrix M obeys $M^T \Omega M = \Omega$, and a matrix which obeys this relationship is said to be a *symplectic matrix*. The symplectic group $\mathrm{Sp}(2n) \subset \mathrm{GL}(2n)$ is the group of symplectic matrices. Later, when discussing the c-map in Section 4.4, we will show the real formulation of special geometry allows us to write our $\mathcal{N} = 2$ supergravity theory in a symplectically covariant manner. For more details, we refer to [106].

4.2.2 $\mathcal{N} = 2$ *vector multiplets*

Let us now consider the Lagrangian of $n_V + 1$ Abelian gauge fields

$$\mathbf{e}_4^{-1}\mathcal{L} = \frac{1}{4}\mathcal{I}_{IJ}F^I_{\mu\nu}F^{J|\mu\nu} + \frac{1}{4}\mathcal{R}_{IJ}F^I_{\mu\nu} \star F^{J|\mu\nu}. \tag{4.2.11}$$

which appears as the gauge field contribution from the n_V vector multiplets in the bosonic field content of $\mathcal{N} = 2$ supergravity (4.1.4). The coupling matrices $\mathcal{R}_{IJ}$ and $\mathcal{I}_{IJ}$ depend on the complex scalar fields X^I and are completely determined by the form of the prepotential.

In order to discuss the underlying symplectic duality, let us first decompose the gauge fields into their selfdual and anti-selfdual components

$$F^I_{\mu\nu} = iF^{+|I}_{\mu\nu} + F^{-|I}_{\mu\nu}, \qquad \star F^I_{\mu\nu} = -iF^{+|I}_{\mu\nu} + iF^{-|I}_{\mu\nu},$$

which are related by

$$\star F^{+|I}_{\mu\nu} = -iF^{+|I}_{\mu\nu}, \qquad \star F^{-|I}_{\mu\nu} = iF^{-|I}_{\mu\nu},$$

and each piece can be expressed in terms of the gauge field and its Hodge dual

$$F^{\pm|I}_{\mu\nu} = \frac{1}{2}\left(F^I_{\mu\nu} \pm i \star F^I_{\mu\nu}\right).$$

Using the coupling (4.1.11) we can rewrite the above action in the following form [101]

$$\mathbf{e}_4^{-1}\mathcal{L} = \frac{i}{2}\left(F^{+|I}\mathcal{N}_{IJ}F^{+|J} - F^{-|I}\bar{\mathcal{N}}_{IJ}F^{-|J}\right),$$

where we do not include the spacetime indices for cosmetic reasons.

The field equations of this Lagrangian are given by

$$d\left(\mathcal{N}_{IJ} \star F^{+|J} - \bar{\mathcal{N}}_{IJ} \star F^{-|J}\right) = 0, \qquad dF^I = 0.$$

Decomposing the Bianchi identities into selfdual and anti-selfdual components, we can write it into the form

$$d\left(\star F^{+|I} - \star F^{-|I}\right) = 0.$$

As we did in the Maxwell example, we can now define a dual field to make the field equations symmetric

$$G^{\pm}_{I|\mu\nu} = \mathcal{N}_{IJ}F^{\pm|J}_{\mu\nu},$$

such that the field equations of the theory are in the form

$$d\left(\star G^{+|I} - \star G^{-|I}\right) = 0, \qquad d\left(\star F^{+|I} - \star F^{-|I}\right) = 0.$$

As with the Maxwell example, the field equations are not simply invariant on the interchange $G_I^\pm \leftrightarrow F^{\pm\,|I}$ but for the whole duality rotation [101]

$$\begin{pmatrix} F^{\pm|I} \\ G_J^\pm \end{pmatrix} \to \begin{pmatrix} U^I{}_K & Z^{IL} \\ W_{JK} & V_J{}^L \end{pmatrix} \begin{pmatrix} F^{\pm|K} \\ G_L^\pm \end{pmatrix} = \begin{pmatrix} \tilde{F}^{\pm|I} \\ \tilde{G}_J^\pm \end{pmatrix}.$$

From the point of view of the field equations, we can understand the rotation as an element

$$\mathcal{O} = \begin{pmatrix} U^I{}_K & Z^{IL} \\ W_{JK} & V_J{}^L \end{pmatrix}, \qquad \mathcal{O} \in \mathrm{GL}(2n_V + 2, \mathbb{R}).$$

However, in order to ensure that the Euler-Lagrange equations descend from some Lagrangian after rotation, the group is restricted to $\mathrm{Sp}(2n_V + 2, \mathbb{R})$.[9] We can realise this restriction by looking at how the gauge coupling transforms under action of $\mathcal{O}$ and the restrictions we must place to preserve its symmetry. The gauge coupling transforms fractionally linearly [107]

$$\mathcal{N}_{IJ} \to (V_I{}^K \mathcal{N}_{KL} + W_{IL})[(U + Z\mathcal{N})^{-1}]^L{}_J.$$

Imposing that $\mathcal{N}_{IJ} = \mathcal{N}_{JI}$ restricts the sub-matrices

$$\begin{aligned} U^T W - W^T U &= 0, \qquad U^T V - W^T Z = \mathbb{1}_n, \\ Z^T V - V^T Z &= 0, \end{aligned}$$

which are precisely the relations which define an element of the symplectic group.

We note that this transformation acts continuously, but once charge quantisation is taken into account it is broken to a discrete subgroup $\mathrm{Sp}(2n_V + 2, \mathbb{Z})$. As the vector is covariant under symplectic transformations, we say that $(F^{\pm|I} G_J^\pm)^T$ is a *symplectic vector*. Within the context of $\mathcal{N} = 2$ supergravity, there is another symplectic vector of the form $(X^I, F_I)^T$, where to be precise, we remind the reader $F_I = \partial_I F$ is the derivative of the prepotential. This duality for the complex scalar and (the derivative of) the prepotential leads to the equivalence of various theories built from prepotentials of different forms [107]. For more details on how various pieces of the $\mathcal{N} = 2$ Lagrangian transform under the action of the symplectic group, we refer to [108]. A totally comprehensive review of these duality rotations within the context of $\mathcal{N} = 2$ supergravity can be found in [53].

Let us now consider this duality with a concrete example which we will use in later calculations for the thermodynamics in Chapter 8. In the planar solutions of the STU model we derive in Chapter 6, we make restrictions on the field configurations to allow for exact solutions to the differential equations. As a consequence, we find that the coupling matrices are restricted by $\mathcal{R}_{IJ} = 0$ and the remaining coupling matrix $\mathcal{I}_{IJ}$ is diagonal. Let us now consider the duality discussed above in this restricted case. Under these assumptions, the Lagrangian (4.2.11) simplifies into the form

$$\mathbf{e}_4^{-1}\mathcal{L} = +\frac{1}{4}\left(\mathcal{I}_{00} F^0_{\mu\nu} F^{0|\mu\nu} + \sum_{A=1}^{3} \mathcal{I}_{AA} F^A_{\mu\nu} F^{A|\mu\nu}\right) + \cdots, \tag{4.2.12}$$

[9] To be more precise, preservation of a Lagrangian formulation reduces from $\mathrm{GL}(2n_V + 2, \mathbb{R}) \to \mathrm{Sp}(2n_V + 2, \mathbb{R}) \times \mathbb{R}^{2n_V+2}$ and we fix the scalings by setting the normalisation of the gauge fields.

where we remind the reader that the STU model is built from $n_V = 3$ vector multiplets. This amounts to $(3+1)$ copies of the type of vector field Lagrangian we have considered in the Maxwell example, but now the gauge field appears with a coupling which is spacetime dependent.

The symplectic transformation in this special case becomes an exchange of the electric and magnetic fields while additionally inverting the coupling. To demonstrate this, let us return to the single Abelian vector field example

$$S[A] = \int -\frac{1}{2g^2} F \wedge \star F \,, \tag{4.2.13}$$

but now we include an explicit coupling g which is allowed to depend on the spacetime coordinates. Terms like this commonly appear in the study of black holes in supergravity, where the gauge field is coupled to a dilaton such that the coupling is $g^2 = e^{\alpha\phi}$ [109, 110]. For our case when considering the $\mathcal{N} = 2$ vector multiplets, we would take $g^{-2} = -\mathcal{I}_{II}$, where the additional minus sign appears as $\mathcal{I}_{IJ}$ is negative-definite. The field equations of (4.2.13) take the form

$$d\left(\frac{1}{g^2} \star F\right) = 0, \qquad dF = 0. \tag{4.2.14}$$

As such, we are prompted to define the dualised gauge fields in the following way:

$$\tilde{F} = \frac{1}{g^2} \star F.$$

Substituting F as a function of $\tilde{F}$ into the field equations with the relationship

$$F = -g^2 \star \tilde{F},$$

gives the following set of field equations for the dual field

$$d\left(\frac{1}{g^2} \star F\right) = -d\left(\star^2 \tilde{F}\right) = d\tilde{F} = 0,$$
$$dF = d\left(-g^2 \star \tilde{F}\right) = 0.$$

We can make these equations appear more symmetric with the original pair though introducing a dual coupling $g^{-1} = \tilde{g}$ such that the field equations can be written as

$$dF = 0 \quad \Leftrightarrow \quad d\left(\frac{1}{\tilde{g}^2} \star \tilde{F}\right), \qquad d\left(\frac{1}{g^2} \star F\right) = 0 \quad \Leftrightarrow \quad d\tilde{F} = 0.$$

We can thus understand this dualisation procedure as exchanging the Euler-Lagrange equations and the Bianchi identity with the additional constraint that the gauge coupling is inverted. For the dual field, the Bianchi identity ensures that $\tilde{F} = d\tilde{A}$ and the new Euler-Lagrange equations exist as the equations of motion for a dual action

$$S[\tilde{A}] = \int -\frac{1}{2\tilde{g}^2} \tilde{F} \wedge \star \tilde{F} \,. \tag{4.2.15}$$

The process of dualising the action (4.2.13) into (4.2.15) follows the steps outlined for the Maxwell theory, where one must promote the Bianchi identity to an equation of motion through introducing a Lagrange multiplier. This is carried out in full detail in Appendix C for a p-form kinetic term in D dimensions. Applying this procedure to (4.2.12), where it is understood that F_A are magnetically charged, we can write down the Hodge dualised action that is purely electric

$$e_4^{-1}\mathcal{L} = +\frac{1}{4\kappa_4^2}\left(\mathcal{I}_{00}F^0_{\mu\nu}F^{0|\mu\nu} + \sum_{A=1}^{3}\mathcal{I}^{AA}\tilde{F}_{A|\mu\nu}\tilde{F}_A^{\mu\nu}\right) + \cdots,$$

where we draw attention to the inversion of the couplings $\mathcal{I}_{AA} \to \mathcal{I}^{AA}$.

Again, considering the Bianchi identities of the gauge field and its dual, we can compute the conserved charges. The electric and magnetic charges are given by

$$\mathcal{Q} = \int_X \tilde{F} = \int_X \frac{1}{g^2} \star F\,, \qquad \mathcal{P} = \int_X F\,. \tag{4.2.16}$$

where we now see that the electric charge appears with the coupling. Returning to our concrete example, before dualisation we have the electric and magnetic charges: $(\mathcal{Q}_0, \mathcal{P}^A)$. We can map these to be purely electric charges: $(\mathcal{Q}_0, \tilde{\mathcal{Q}}_A)$, where $\tilde{\mathcal{Q}}_A = -\mathcal{P}^A$ with the above duality map. We can compute the value of the conserved charges with the following integrals

$$\mathcal{Q}_0 = \int_X \tilde{F}_0 = \int_X \star\left(-\mathcal{I}_{00}F^0\right)\,, \tag{4.2.17}$$

$$\tilde{\mathcal{Q}}_A = -\int_X F^A = \int_X \star\left(-\mathcal{I}^{AA}\tilde{F}_A\right)\,. \tag{4.2.18}$$

4.3 DIMENSIONAL REDUCTION

In this section, we review the procedure of dimensional reduction. We will use dimensional reduction as a tool at several points within this thesis, using it to better understand the structure of our supergravity solutions, and as a method to simplify the equations of motion while finding solutions to gravitational systems. In Section 4.3.1, we give a brief discussion of the history of dimensional reduction to allow a physical perspective of the subsequent calculations. In Section 4.3.2, we give a prescription of how to dimensionally reduce a generic $(D+1)$-dimensional Lagrangian, which is suitable for the application of the various models we consider. Finally, in Section 4.3.3 we give a worked example of the dimensional reduction of the STU model from six to four dimensions, following [111]. Later, while discussing the c-map in Section 4.4, we will see a second example of dimensional reduction as we compactify $\mathcal{N} = 2$ supergravity coupled to n_V vector multiplets from four to three dimensions, following [31, 112].

4.3.1 *Kaluza-Klein reduction*

Dimensional reduction first appeared when Kaluza [113] suggested to Einstein that one could unify electromagnetism with gravity by considering a five-dimensional spacetime. In essence, Kaluza's suggestion was to study a five-dimensional metric where the electromagnetic gauge potential appears as components of the metric. By ensuring that this metric also contained a four-dimensional subspace that described the physical spacetime we experience, Kaluza aimed to geometrically capture both electromagnetism and gravity within a single tensor. Years later, Klein noticed that if this extra fifth dimension was small and compact, this five-dimensional spacetime effectively reduces to four-dimensional gravity coupled to electromagnetism [114].

In essence, the Kaluza-Klein reduction assumes a cylindrical condition, where the higher-dimensional spacetime is assumed to have the form $M_5 = S^1 \times M_4$, with one compact coordinate wrapped into a tight circle

$$x^0 \simeq x^0 + 2\pi L.$$

This periodicity allows us to interpret this compactification coordinate as an angular coordinate, and we understand L as the radius of circle we compactify over.

For the following discussion, we will assume we are working with a spacetime of dimension $(D+1)$, with coordinates $x^{\hat{\mu}}$, where $\hat{\mu} = \{0, \ldots, D\}$, and our compact coordinate is x^0. Let us study how this identification effects a scalar field $\Phi(x^{\hat{\mu}})$. We can make the Fourier expansion for the field

$$\Phi(x^{\hat{\mu}}) = \Phi(x^0, x^\mu) = \sum_n \phi_n(x^\mu) e^{inx^0/L}.$$

In this form, we find the so-called Kaluza-Klein tower of an infinite number of D-dimensional fields $\phi_n(x^\mu)$, with masses $\frac{|n|}{L}$. Klein's 'cylinder condition' is to truncate out every mode except the massless mode ϕ_0, by assuming that the radius of the circle is very very small. This truncation happens as when the radius shrinks, the masses of the massive modes grow. As the massive modes then exist at energy levels far above the effective field theories we compactify over, they are safely ignored. It is this procedure of keeping only the massless modes in a Fourier expansion which we define as the *Kaluza-Klein reduction*. In Section 4.3.2, this procedure is carried out for a general Lagrangian containing the Einstein-Hilbert and gauge field contribution.

Reducing pure gravity from five to four dimensions produces a Lagrangian of the following form

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{2}R - \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{\sqrt{3}\phi} F_{\mu\nu} F^{\mu\nu} \right),$$

which seems to be an incredible result (see Section 4.3.2 for the reduction formula). Both gravity and electromagnetism come falling out of a five dimensional theory, with the field strength $F = dV$ behaving as we would expect a U(1) gauge field to [115]. Although mathematically elegant, the physical motivation of describing gravity and electromagnetism together using a fifth, compact dimension was questioned. Firstly, although the field content is the same, the equations of motion in five dimensions include the further constraint [116]

$$\hat{R}_{00} = 0 \quad \Rightarrow \quad F_{\mu\nu} F^{\mu\nu} = 0.$$

It is only after setting this that we obtain the equations of motion we saw in Equation (2.3.1) through considering $R_{0\mu} = 0$ and $R_{\mu\nu} = 0$. We cannot say Einstein-Maxwell theory is a consistent truncation of five-dimensional pure gravity without ignoring part of the five-dimensional equations of motion. We must also ask where the motivation for this additional dimension comes from. The cylindrical condition imposes that our physical fields are independent of this extra coordinate, and so it appears in Kaluza-Klein theory as a spectator or mathematical oddity.

These criticisms resulted in Kaluza-Klein compactification not being considered seriously as a unification tool or a description of 'real-world' physics. However, years later within the context of higher-dimensional supergravity and the various superstring theories in ten dimensions, dimensional compactification came back into focus. We will discuss this with more context in Section 4.5.

In our work, we will not be concerned with phenomenological questions, but instead we will use dimensional compactification as a tool to solve equations of motion in Chapter 6 and as a way to understand our four-dimensional solutions as brane configuration in Chapter 7. The fact remains though, that if we hope to see a stringy understanding of quantum gravity, somewhere along the way we'll need to find a consistent way to tie up all the extra dimensions.

As a closing remark, in this thesis we will only be concerned with reductions over the circle S^1 or tori T^n (which is n repeated reductions over S^1). However, the compact, internal manifold can have a much richer structure. We will not be employing any of these generalisations, but include suitable references for further reading. Performing a Kaluza-Klein reduction over the sphere, one obtains a reduced theory containing Yang-Mills gauge fields [117]. In the context of supergravity, compact manifolds of special holonomy are reduced over to break supersymmetry, see for example Calabi-Yau reductions [118, 119, 120] which produce $\mathcal{N} = 2$ supergravity solutions in four dimensions, reducing the total number of supercharges by three-quarters. Rather than compactifying over manifolds, another method to break supersymmetry comes from reducing over orbifolds [121], or orientifolds [122]. The Scherk-Schwarz reduction [123] is a generalisation of the Kaluza-Klein reduction where the assumption that the lower-dimensional fields are independent of the internal coordinates is relaxed. One application for the Scherk-Schwarz reduction is to produced gauged supergravity in lower dimensions [124, 125].

4.3.2 *Dimensional reduction formulae*

In this section, we give the result of the Kaluza-Klein reduction of a generic model containing contributions from an Einstein-Hilbert term and a gauge field term. The results will be stated without proof, and are included as a reference for when we perform reductions within the thesis. A wealth of good references exist with explicit computations such as [112, 115, 117, 126].

The first step in the reduction is to make the Kaluza-Klein ansatz for the metric tensor, suitable for the reduction over the circle

$$ds^2_{D+1} = -\epsilon e^{2\beta\phi}(dx^0 + V_\mu dx^\mu)^2 + e^{-2\alpha\phi} ds^2_D, \tag{4.3.1}$$

where x_0 is the compact direction, ϕ is the Kaluza-Klein scalar and V_μ is the Kaluza-Klein vector. The constants, α and β, are set by requiring that the Einstein-Hilbert part of the action reduces in the so-called 'Einstein-frame', in which the Ricci scalar has no coupling with the Kaluza-Klein scalar, and the kinetic term for the Kaluza-Klein scalar has a factor of $-\frac{1}{2}$.[10] This is done through picking

$$\alpha^2 = \frac{1}{2(D-1)(D-2)}, \quad \beta = (D-2)\alpha.$$

[10] A common technique is to set $\alpha = 0$ and $\beta = 1$ to simplify the equations during the reduction. The Einstein-frame can then be recovered after a conformal transformation. An alternative choice is called the 'string frame', where $\alpha, \ \beta$ are picked so the Ricci scalar has a factor of $e^{-2\phi}$.

The constant ϵ is used to allow us to express the reduction over x^0 of arbitrary signature, where

$$\epsilon = \begin{cases} -1 \text{ if } x^0 \text{ is spacelike,} \\ \;\;1 \text{ if } x^0 \text{ is timelike,} \end{cases}$$

such that the signature of the $(D+1)$-dimensional metric is

$$\{-\epsilon \underbrace{-\ldots-}_{t\text{-times}} + \ldots +\}.$$

This allows us to consider the Kaluza-Klein reduction over both timelike and spacelike dimensions. Generally, the coordinate is assumed to be spacelike and we reduce from one Minkowski theory to another. However, reducing over timelike coordinates and considering the resulting Euclidean theory has a wealth of applications too. One that we focus on in particular is the timelike reduction of $\mathcal{N} = 2$ supergravity from four to three dimensions in the context of the c-map, which is discussed properly in Section 4.4, and used in Chapter 6 to obtain non-extremal, planar symmetric solutions to the STU model. More generally, for a review of special geometry and Euclidean supergravity see [32, 100, 127, 128].

In the following, we will denote $(D+1)$-dimensional fields with a hat, and the $(D+1)$-dimensional indices count from $\hat{\mu} = \{0, \ldots, D\}$, whereas the unhatted will not include the compact dimension: $\mu = \{1, \ldots, D\}$. When written in form notation, we will use the Hodge-star operator, which is also dimension-dependent; however, we do not include hats and instead understand the Hodge operator from context.

We begin with a generic $(D+1)$-dimensional Lagrangian containing the Einstein-Hilbert term, and a gauge contribution from a p-form

$$\hat{S} = \hat{S}_{EH} + \hat{S}_{F^2} = \int d^{D+1}x\sqrt{-\hat{g}}\left[-\frac{1}{2}\hat{R} - \frac{1}{2p!}\hat{F}_{\hat{\mu}_1\ldots\hat{\mu}_p}\,\hat{F}^{\hat{\mu}_1\ldots\hat{\mu}_p}\right].$$

Using the following identities

$$\star A \wedge B = \frac{1}{p!}A_{\mu_1\ldots\mu_p}B^{\mu_1\ldots\mu_p}\star 1, \qquad \star 1 = \sqrt{-\hat{g}}\,d^{D+1}x,$$

for some p-forms A, B, we can write Lagrangian using differential forms

$$\hat{S} = \int_{\mathcal{M}_{D+1}} -\star\frac{1}{2}\hat{R} - \frac{1}{2}\hat{F}_p \wedge \star\hat{F}_p,$$

for more discussion on form notation, see Appendix A. Note that the p-form is assumed to be exact: $F = dA_{p-1}$ for some gauge potential A_{p-1}.

Commonly we consider Lagrangians which also contain a scalar kinetic term, however, the Kaluza-Klein reduction of this term is trivial. We replace a derivative over $(D+1)$ dimensions with the derivative over the remaining D dimensions

$$\partial_{\hat{\mu}}\varphi\partial^{\hat{\mu}}\varphi \to \partial_\mu\varphi\partial^\mu\varphi.$$

Note that in form notation, this is $d\varphi \wedge \star d\varphi \to d\varphi \wedge \star d\varphi$.

We can break the reduction down into two parts. Starting with the Einstein-Hilbert term, it is found that the Ricci scalar reduces as

$$\begin{aligned}\hat{S}_{EH} &= \int_{\mathcal{M}_{D+1}} -\frac{1}{2}\star\hat{R} \\ &\to \int_{\mathcal{M}_D}\left[-\frac{1}{2}\star R - \frac{1}{2}d\phi\wedge\star d\phi + \frac{1}{2}\epsilon e^{\left(\frac{2D-2}{D-2}\right)^{\frac{1}{2}}\phi} dV\wedge\star dV\right].\end{aligned} \tag{4.3.2}$$

We see that from the $(D+1)$-dimensional Ricci scalar, the resulting D-dimensional spacetime has contributions from a D-dimensional Ricci scalar, as well as a scalar kinetic term from the Kaluza-Klein scalar and a gauge field contribution from the two-form dV, due to the Kaluza-Klein vector.

The reduction of a p-form gauge field contribution is given by

$$\begin{aligned}\hat{S}_{F^2} &= \int_{\mathcal{M}_{D+1}} -\frac{1}{2}F_p\wedge\star F_p \\ &\to \int_{\mathcal{M}_D} e^{(2p-D)\alpha\phi}\left[-e^{-(\beta+2\alpha)\phi}\frac{1}{2}F_{p-1}\wedge\star F_{p-1} - \epsilon e^{\beta\phi}\frac{1}{2}(F_p - V\wedge F_{p-1})\wedge\star(F_p - V\wedge F_{p-1})\right],\end{aligned}$$

where we do not substitute in the values for α, β to make this easier on the eye. We can simplify this slightly, by defining a new gauge potential

$$B_{p-1} = A_{p-1} - V\wedge A_{p-2}, \qquad G_p = dB_{p-1},$$

allowing us to write down the Kaluza-Klein reduction of a p-form as

$$S_{F_p^2} = \int_{\mathcal{M}_D}\left[-\epsilon e^{(2p-2)\alpha\phi}\frac{1}{2}G_p\wedge\star G_p - e^{2(p-D)\alpha\phi}\frac{1}{2}F_{p-1}\wedge\star F_{p-1}\right]. \tag{4.3.3}$$

In short, the Kaluza-Klein reduction of a p-form in $(D+1)$ dimensions produces an action in D dimensions including both a p-form and a $(p-1)$-form. In the special case where we reduce a two-form gauge field with potential $\hat{A}_{\hat{\mu}}$, we obtain a gauge field term for A_μ, and a scalar kinetic term for A_0.

One-piece we do not consider here in full generality is the dimensional reduction of topological terms arising from the gauge fields. The form of the topological term (and hence its reduction) is dependent on the values of both p and D and so is hard to discuss generically. However, we will reduce a four-dimensional topological term built from a two-form in Section 4.4, which we now outline.

When $(D+1)$ is even and the gauge field present is a $(p+1)$-form, and $p+1 = \frac{D+1}{2}$ is also even, there can be a topological term

$$\hat{S}_{\text{top}} = \int \frac{1}{2}d\hat{A}_p\wedge d\hat{A}_p.$$

Decomposing the gauge field $d\hat{A}_p$, we can write it as

$$d\hat{A}_p = dA_p + dx^0\wedge dA_{p-1}.$$

Expanding this relation in the topological term

$$\begin{aligned}\hat{S}_{\text{top}} &= \int \frac{1}{2}(dA_p + dx^0 \wedge dA_{p-1}) \wedge (dA_p + dx^0 \wedge dA_{p-1}),\\ &= \frac{1}{2}\int dA_p \wedge dx^0 \wedge dA_{p-1} + dx^0 \wedge dA_{p-1} \wedge dA_p,\\ &= \frac{1}{2}\int (-)^{p+1} dx^0 \wedge dA_p \wedge dA_{p-1} + (-)^{p(p+1)} dx^0 \wedge dA_p \wedge dA_{p-1},\\ &= \int dx^0 \wedge dA_p \wedge dA_{p-1}.\end{aligned}$$

Integrating we obtain the reduced term in D dimensions

$$S_{\text{top}} = \int dA_p \wedge dA_{p-1}.$$

The case we consider in the c-map in Section 4.4 is when $D = 3$ and $p = 1$.

Written with Lorentz indices expanded, we can collect the Ricci scalar and gauge kinetic term computations to write a D-dimensional action after reduction. Here we make the choice $p = 2$, as this will be the most common choice throughout the thesis

$$\begin{aligned}S = \int d^D x\sqrt{-g}\Bigg[&-\frac{1}{2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{4}\epsilon e^{\left(\frac{2D-2}{D-2}\right)^{\frac{1}{2}}\phi} V_{\mu\nu}V^{\mu\nu}\\ &+ \epsilon e^{\left(\frac{2(n-2)}{n-1}\right)^{\frac{1}{2}}\phi}\frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{4}e^{\left(\frac{2}{(D-2)(D-1)}\right)^{\frac{1}{2}}\phi} G_{\mu\nu}G^{\mu\nu}\Bigg].\end{aligned}$$

where $V_{\mu\nu}$ is the field strength from the Kaluza-Klein vector, and $\chi = A_0$ is a scalar field, where we decompose the gauge potential as $\hat{A} = \chi dx^0 + A_\mu dx^\mu$. The explicit form of the new gauge field is

$$G_{\mu\nu} = F_{\mu\nu} - 2V_{[\mu}\partial_{\nu]}\chi.$$

4.3.3 *Dimensional reduction of $\mathcal{N} = 1$, 6D supergravity*

As a concrete example of Kaluza-Klein compactification, we now perform the reduction of $\mathcal{N} = 1$, six-dimensional supergravity coupled to tensor multiplets over $T^2 = S^1 \times S^1$ [129, 104]. This reduction produces a four-dimensional theory with the $\mathcal{N} = 2$ supergravity multiplet, and three vector multiplets. As discussed, this is the STU model which is the focus of Chapter 6.

4.3.3.1 *Reduction from six to five dimensions*

The action for $\mathcal{N} = 1$, six-dimensional supergravity coupled to one tensor multiplet is [104]

$$\hat{S}_{6D} = \int d^6x\sqrt{-\hat{g}}\left(-\hat{R} - \frac{1}{2}\partial^{\hat{\mu}}\phi\partial_{\hat{\nu}}\phi - \frac{1}{12}e^{-\sqrt{2}\phi}H_{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}}\right). \tag{4.3.4}$$

Where ϕ is the dilaton, and $\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}$ is a three-form field strength $\hat{H} = d\hat{B}$. To perform the reduction, we make the ansatz

$$ds_6^2 = e^{2\beta\sigma}(dz_6 + \tilde{\mathbb{A}}_1)^2 + e^{-2\alpha\sigma}ds_5^2, \tag{4.3.5}$$

where we reduce over the compact, spacelike coordinate z_6, the Kaluza-Klein scalar is represented by σ, to differentiate it from the dilaton appearing in (4.3.4), and the Kaluza-Klein vector is labeled $\tilde{\mathbb{A}}_1 = (\tilde{\mathbb{A}}_1)_\mu dx^\mu$.

Referring to the above equations, setting $D = 5$ in (4.3.2), the Ricci scalar and dilaton terms reduce to

$$S_{EH+\phi^2} = \int d^5x\sqrt{-g}\left(-R - \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{4}e^{\frac{4}{\sqrt{6}}\sigma}(\tilde{\mathbb{F}}_1)^2 - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi\right),$$

where we use $\tilde{\mathbb{F}}_1 = d\tilde{\mathbb{A}}_1$, and we will use $F^2 = F_{\mu\nu}F^{\mu\nu}$ throughout this reduction to reduce the number of indices in our expressions.

The gauge term can be computed with (4.3.3), setting $D = 5$ and $p = 3$ to find

$$S_{H^2} = \int d^5x\sqrt{-g}e^{-\sqrt{2}\phi}\left(-\frac{1}{12}e^{-\frac{2}{\sqrt{6}}\sigma}\mathbb{H}^2 - \frac{1}{4}e^{\frac{2}{\sqrt{6}}\sigma}(\mathbb{F}_2)^2\right),$$

where we denote the $(p-1)$-form $\tilde{\mathbb{F}}_2 = d\tilde{\mathbb{A}}_2$, and the p-form as $\mathbb{H} = H - \tilde{\mathbb{F}}_2 \wedge \tilde{\mathbb{A}}_1$. Before reducing this five-dimensional solution, we can do some book-keeping to simplify the above expression. As we are in five dimensions, a three-form is Hodge-dual to a two-form. Performing the Hodge dualisation explained in Section 4.2, we can find that

$$\tilde{\mathbb{F}}_3 = e^{-\frac{2}{\sqrt{6}}\sigma-\sqrt{\phi}}\mathbb{H}.$$

Inserting this into the above action we find we can write

$$S_{H^2} = \int d^5x\sqrt{-g}\left(-\frac{1}{4}e^{\frac{2}{\sqrt{6}}\sigma+\sqrt{2}\phi}(\tilde{\mathbb{F}}_3)^2 - \frac{1}{4}e^{\frac{2}{\sqrt{6}}\sigma-\sqrt{2}\phi}(\mathbb{F}_2)^2\right)$$

We can further simplify our terms by reparameterising the scalar fields ϕ, σ, defining three new scalar fields

$$h_1 = e^{2\sigma/\sqrt{6}}, \qquad h_2 = e^{\phi/\sqrt{2}-\sigma/\sqrt{6}}, \qquad h_3 = e^{-\phi/\sqrt{2}-\sigma/\sqrt{6}},$$

which are subject to the constraint $h_1h_2h_3 = 1$, maintaining the degrees of freedom captured by the scalar fields. We compute the derivatives of the scalar fields as

$$\partial_\mu h_1 = \frac{2\partial_\mu\sigma}{\sqrt{6}}h_1, \qquad \partial_\mu h_2 = \left(\frac{\partial_\mu\phi}{\sqrt{2}} - \frac{\partial_\mu\sigma}{\sqrt{6}}\right)h_2, \qquad \partial_\mu h_3 = \left(-\frac{\partial_\mu\phi}{\sqrt{2}} - \frac{\partial_\mu\sigma}{\sqrt{6}}\right)h_3.$$

Squaring and summing, we see that we can write the kinetic term for the scalar fields ϕ, σ in terms of h_i:

$$(\partial\phi)^2 + (\partial\sigma)^2 = \sum_{i=1}^{3}(h_i)^{-2}(\partial h_i)^2,$$

which in form notation is given by

$$\star\, d\phi \wedge d\phi + \star d\sigma \wedge d\sigma = \sum_{i=1}^{3}\frac{\star dh^i \wedge dh^i}{(h^i)^2}. \tag{4.3.6}$$

Written using the scalars h_i, the five-dimensional action simplifies into the form

$$\begin{aligned} S_{5D} &= \int d^5x\sqrt{-g}\left[-R - \sum_{i=1}^{3}\frac{1}{2h_i^2}\left(\partial_\mu h_i\partial^\mu h_i + \frac{1}{2}(\tilde{\mathbb{F}}_i)^2\right)\right], \\ &= \int_{M_5} -\star R - \frac{1}{2}\sum_{i=1}^{3}\frac{1}{h_i^2}\left(dh_i \wedge \star dh_i + \tilde{\mathbb{F}}_i \wedge \star\tilde{\mathbb{F}}_i\right). \end{aligned}$$

As a consistency check, we can compare this action with the one derived in [100] and see that the only difference is an overall scaling of the gauge fields

$$\mathbb{F}_i = \frac{1}{\sqrt{3}}\mathcal{F}_i \qquad \mathbb{H} = -\frac{1}{\sqrt{3}(h^3)^2}\star_5 \mathcal{F}_3, \tag{4.3.7}$$

where we use $\mathcal{F}_i$ for the notation of the gauge fields appearing in Equation (6.2) of [100].

4.3.3.2 *Reduction from five to four dimensions*

We are now in a position to reduce the five-dimensional action to four dimensions. We re-write the action, where we now include hats on the five-dimensional fields, ready to reduce to four dimensions

$$\hat{S}_{5D} = \int d^5x\sqrt{-\hat{g}}\Bigg[-\hat{R} - \frac{1}{2}\partial^{\hat{\mu}}\phi\partial_{\hat{\nu}}\phi - \frac{1}{2}\partial^{\hat{\mu}}\sigma\partial_{\hat{\nu}}\sigma \\ - \frac{1}{4}e^{-4\sigma/\sqrt{6}}(\hat{\tilde{\mathbb{F}}}_1)^2 + \frac{1}{4}e^{-2\phi/\sqrt{2}+2\sigma/\sqrt{6}}(\hat{\tilde{\mathbb{F}}}_2)^2 + \frac{1}{4}e^{2\phi/\sqrt{2}+2\sigma/\sqrt{6}}(\hat{\tilde{\mathbb{F}}}_3)^2\Bigg].$$

We begin through making the ansatz

$$ds_4^2 = e^{2\beta\lambda}(dz_5 + \tilde{A}_4)^2 + e^{-2\alpha\lambda}ds_4^2, \tag{4.3.8}$$

where we are reducing over a spacelike coordinate dz_5, the Kaluza-Klein scalar is denoted with a λ to keep it distinct from the previous ones, and the Kaluza-Klein vector is denoted $\tilde{A}_4$.

Using (4.3.2) with $D = 4$, we find that the Einstein-Hilbert reduces to

$$S_{EH} = \int d^4x\sqrt{-g}\left(-R - \frac{1}{2}\partial_\mu\lambda\partial^\mu\lambda - \frac{1}{4}e^{\sqrt{3}\lambda}\tilde{F}_4\right),$$

where we write the field strength of the Kaluza-Klein vector as $\tilde{F}_4 = d\tilde{A}_4$. We can consider all three gauge field reductions simultaneously, and using (4.3.3), with $D = 4$ and $p = 2$ we find that

$$\hat{\tilde{\mathbb{F}}}_i \wedge \star\hat{\tilde{\mathbb{F}}}_i \to e^{\frac{\lambda}{\sqrt{3}}}\tilde{F}_i \wedge \star\tilde{F}_i + e^{-\frac{2\lambda}{\sqrt{3}}}d\chi_i \wedge \star d\chi_i, \qquad \tilde{F}_i = d\tilde{A}_i,$$

where the five-dimensional vectors $\mathbb{A}_i$ have been reduced to the pair of four-dimensional vectors $\tilde{A}_i$ and a scalars χ_i.

Before we combine all these terms together, it is helpful to make the following redefinition of the scalars

$$\varphi_1 = -\frac{2}{\sqrt{6}}\sigma - \frac{1}{\sqrt{3}}\lambda, \quad \varphi_2 = -\frac{1}{\sqrt{2}}\phi + \frac{1}{\sqrt{6}}\sigma - \frac{1}{\sqrt{3}}\lambda, \quad \varphi_3 = \frac{1}{\sqrt{2}}\phi + \frac{1}{\sqrt{6}}\sigma - \frac{1}{\sqrt{3}}\lambda,$$

which, after doing so allows us to write the Lagrangian in the following form:

$$S_{4D} = \int d^4x\sqrt{-g}\left[R - \frac{1}{2}(d\varphi_i \wedge \star d\varphi_i + e^{2\varphi_i}d\chi_i \wedge \star d\chi_i) - \frac{1}{2}e^{-\varphi_1-\varphi_2-\varphi_3}\left(\tilde{F}_4 \wedge \star\tilde{F}_0 + e^{2\varphi_i}\tilde{F}_i \wedge \star\tilde{F}_i\right)\right].$$

We will return to this reduction process in Chapter 7.1 when we perform the uplift of a four-dimensional solution to five, and then six dimensions. During these calculations, we will work under an additional assumption which we call the 'purely imaginary condition'. This is equivalent to the condition $\chi_i = 0$,[11] and so under these conditions, the four-dimensional action further simplifies into the form

$$S_{4D} = \int d^4x\sqrt{-g}\left(-R - \frac{1}{2}(d\varphi_i \wedge \star d\varphi_i) - \frac{1}{2}e^{-\varphi_1-\varphi_2-\varphi_3}\left(\tilde{F}_4 \wedge \star\tilde{F}_4 + e^{2\varphi_i}\tilde{F}_i \wedge \star\tilde{F}_i\right)\right). \tag{4.3.9}$$

[11] These scalar fields are sometimes referred to as axions, and the 'purely imaginary condition', is sometimes also known as the 'axion free condition'

A note on transgression terms

Before we conclude this section, we make a note on an omission we have made during our calculations. In the Hodge dualisation of the three-form in five dimensions, an additional topological term appears in the five-dimensional action:

$$S_{\text{top}} = \int d^5x\, \tilde{\mathbb{F}}_2 \wedge \tilde{\mathbb{F}}_3 \wedge \tilde{\mathbb{A}}_1. \tag{4.3.10}$$

The appearance of topological terms after Hodge dualisation is explained in the general case in Appendix C. When performing the dimensional reduction from five to four dimensions, if the topological term (4.3.10) is also reduced, the gauge fields in the four-dimensional theory are modified by the so-called transgression terms. For the reduction of a p-form, all transgression terms are proportional to the resulting $(p-1)$-form.

For this current reduction, the modification of the gauge field terms is given precisely in [130, 131], where they also include the resulting four-dimensional topological term

$$S_{\text{top}} = \int d^4x\, \chi_1 \left(F_1 \wedge F_4 + F_2 \wedge F_3\right),$$

which also arises from the reduction of (4.3.10). The reason we do not include the topological terms in the computations of this section is that we are interested in the results of this calculation in the application of Chapter 6, where we demand that $\chi_i = 0$. As all the modifications to the two-forms in four dimensions are proportional to χ_i (as well as the four-dimensional topological term itself), we effectively set to zero all contributions from the dimensional reduction of the topological term. Thus, we find that it is sufficient to ignore them in favour of computational clarity in this example.

4.4 THE C-MAP

In this section, we use a Kaluza-Klein reduction to dimensionally reduce $\mathcal{N} = 2$ supergravity coupled to n_V vector multiplets in four dimensions, producing a theory described by $n_H = n_V + 1$ hypermultiplets coupled to $\mathcal{N} = 2$ supergravity in three dimensions [132]. This mapping between the target spaces of the four and three-dimensional theories is known as the *c-map*. In particular, we use that in three dimensions it is possible to dualise vector fields into scalar fields, leading to a simplification of the equations of motion for the solutions considered in Section 6.2.

We follow the work of [31, 32], which developed the c-map through introducing a new formulation in terms of special real coordinates and the corresponding Hesse potential; the development of which led to a series of publications [28, 31, 33, 34] on the construction of non-extremal, stationary solutions in theories of four-dimensional $\mathcal{N} = 2$ vector multiplets coupled to gauged and ungauged supergravity. It was this work that we continued in [39] which led to the cosmological solutions of $\mathcal{N} = 2$ supergravity we present in Chapter 6.

4.4.1 *Dimensional reduction*

We begin with the four-dimensional Lagrangian for $\mathcal{N} = 2$ supergravity coupled to n_V vector multiplets, described in Section 4.1. We note that this Lagrangian can be obtained from the reduction of type II theories over a Calabi-Yau manifold [119], or heterotic theories compactified over $K3 \times T^2$ [120, 133]. We rewrite the Lagrangian with our four-dimensional field content, which will be denoted by a hat, as will the spacetime indices: $\hat{\mu} \in \{0,1,2,3\}$.

$$\mathbf{e}_4^{-1}\mathcal{L}_4 = -\frac{1}{2}\hat{R}_4 - g_{A\bar{B}}(z,\bar{z})\partial_{\hat{\mu}}z^A\partial^{\hat{\mu}}\bar{z}^{\bar{B}} + \frac{1}{4}\mathcal{I}_{IJ}(z,\bar{z})\hat{F}^I_{\hat{\mu}\hat{\nu}}\hat{F}^{J|\hat{\mu}\hat{\nu}} + \frac{1}{4}\mathcal{R}_{IJ}(z,\bar{z})\hat{F}^I_{\hat{\mu}\hat{\nu}}\tilde{\hat{F}}^{J|\hat{\mu}\hat{\nu}}. \tag{4.4.1}$$

Note that, in our conventions, $g_{A\bar{B}}$ is positive-definite and $\mathcal{I}_{IJ}$ is negative-definite.

Rather than working with the scalar fields z^A, we choose to work with special coordinates X^I, which introduces symplectic covariance of the field equations. As explained in Section 4.1.2, the coordinates X^I live in a larger ambient space which introduces gauge freedoms that will need to be fixed. To return back to the physical hypersurface, parameterised by z^A, we must fix both the dilatations and the U(1) phase transformations [33].

The dilatations can be broken through imposing the D-gauge

$$-i(X^I\bar{F}_I - F_I\bar{X}^I) = 1. \tag{4.4.2}$$

Fixing the U(1) phase transformation necessarily breaks the symplectic covariance, and so we choose to leave U(1) symmetry of the special coordinates until we have solved the equations of motion. For more details on this we refer to [31, 112].

In Chapter 6, the gauge condition is fixed while imposing the 'purely imaginary' field configuration, which are a set of conditions for the physical fields. The presence of the U(1) gauge symmetry means that when counting the independent degrees of freedom, it is important that we realise that one of the relations within the set is not a condition of physical fields, but rather a gauge fixing. Any condition which is not U(1) invariant would be suitable, however, conventionally and within this thesis we identify the condition

$$\mathrm{Im}(X^0) = 0,$$

as the gauge fixing relation. The 'purely imaginary' condition will be discussed in more detail in Section 6.1.2 from the context of finding solutions to the equations of motion.

The map from the generic (inhomogenous) coordinates z^A to homogenous coordinates X^I in the Lagrangian is induced by the following transformation

$$\bar{g}_{A\bar{B}}(z,\bar{z})\partial_{\hat{\mu}}z^A\partial^{\hat{\mu}}\bar{z}^{\bar{B}} \to g_{I\bar{J}}(X,\bar{X})\partial_{\hat{\mu}}X^I\partial^{\hat{\mu}}\bar{X}^{\bar{J}}. \tag{4.4.3}$$

where we have used that $g_{I\bar{J}}$ has a two-dimensional kernel, so that the number of independent degrees of freedom remains the same.[12] Rewriting the Lagrangian (4.4.1) in terms of the special coordinates

$$\mathbf{e}_4^{-1}\mathcal{L}_4 = -\frac{1}{2}\hat{R}_4 - g_{I\bar{J}}(X,\bar{X})\partial_{\hat{\mu}}X^I\partial^{\hat{\mu}}\bar{X}^{\bar{J}} + \frac{1}{4}\mathcal{I}_{IJ}\hat{F}^I_{\hat{\mu}\hat{\nu}}\hat{F}^{J|\hat{\mu}\hat{\nu}} + \frac{1}{4}\mathcal{R}_{IJ}\hat{F}^I_{\hat{\mu}\hat{\nu}}\tilde{\hat{F}}^{J|\hat{\mu}\hat{\nu}}. \tag{4.4.4}$$

Although we will only need the reduction over a timelike circle in Section 6.1.1, in this section, we follow the work of [31] and allow for the dimensional reduction of (4.4.1) over either a timelike or spacelike direction. We do this as explained in Section 4.3.2, by introducing factors of ϵ.

[12] The two-dimensional kernel descending from the scale and U(1) invariance of the superconformal theory.

We impose an ansatz on our four-dimensional solution such that the metric can be decomposed into the appropriate form to allow the standard Kaluza-Klein dimensional reduction

$$ds_4^2 = -\epsilon e^{\phi}(dy + V_\mu dx^\mu)^2 + e^{-\phi} ds_3^2. \tag{4.4.5}$$

As we are in $(D+1) = 4$ dimensions, $\alpha = \beta = \frac{1}{2}$, and ϕ, V_μ are the Kaluza-Klein scalar and vector respectively.

Under this assumption, the gauge fields $\hat{A}^I_{\hat{\mu}}$ decompose into

$$\hat{A}^I = \zeta^I dy + (A^I_\mu - \zeta^I V_\mu) dx^\mu,$$

where $\hat{A}^I_y = \zeta^I$, and we include $\zeta^I V_\mu$ to eliminate 'naked' vector potentials, allowing us to write everything in terms of gauge invariant field strengths [32].

Performing the reduction of (4.4.1) using the rules described in Section 4.3.2, the three-dimensional Lagrangian is given by

$$\begin{aligned}\mathbf{e}_3^{-1}\mathcal{L}_3 = &\frac{1}{2}\left(-R_3 - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{4}\epsilon e^{2\phi}V^{\mu\nu}V_{\mu\nu}\right) - g_{I\bar{J}}\partial_\mu X^I\partial^\mu\bar{X}^{\bar{J}} \\ &+ \frac{1}{4}e^{\phi}\mathcal{I}_{IJ}(F^I_{\mu\nu} + \zeta^I V_{\mu\nu})(F^{J|\mu\nu} + \zeta^J V^{\mu\nu}) \\ &- \frac{1}{2}\epsilon e^{-\phi}\mathcal{I}_{IJ}\partial_\mu\zeta^I\partial^\mu\zeta^J - \frac{1}{2}\epsilon\varepsilon^{\mu\nu\rho}\mathcal{R}_{IJ}(F^I_{\mu\nu} + \zeta^I V_{\mu\nu})\partial_\rho\zeta^J.\end{aligned} \tag{4.4.6}$$

The first line comes from the reduction of the Einstein-Hilbert and scalar kinetic terms, the remainder of the Lagrangian comes from the reduction of the vector fields.

A great benefit of working in three dimensions, is that we are able to simplify the Lagrangian through the dualisation of the vector fields (A^I, V) into scalar fields $(\tilde{\zeta}_I, \tilde{\phi})$. We can write down the dual Lagrangian by including a Lagrange multiplier term, given by

$$\mathbf{e}_3^{-1}\mathcal{L}_{\text{Lm}} = \frac{1}{2}\epsilon\varepsilon^{\mu\nu\rho}\left(F^I_{\mu\nu}\partial_\rho\tilde{\zeta}_I - V_{\mu\nu}\partial_\rho\left(\tilde{\phi} - \frac{1}{2}\zeta^I\tilde{\zeta}_I\right)\right). \tag{4.4.7}$$

We can then eliminate the vector terms through computing their equations of motion from the new Lagrangian $\tilde{\mathcal{L}}_3 = \mathcal{L}_3 + \mathcal{L}_{\text{Lm}}$. The resulting computation allows us to write the vector terms as functions of the scalars

$$\begin{aligned}V_{\mu\nu} &= 2e^{-2\phi}\varepsilon_{\mu\nu\rho}\left(\partial^\rho\tilde{\phi} + \frac{1}{2}(\zeta^I\partial^\rho\tilde{\zeta}_I - \tilde{\zeta}_I\partial^\rho\zeta^I)\right), \\ F^I_{\mu\nu} &= \lambda e^{-\phi}\mathcal{I}^{IJ}\epsilon\varepsilon_{\mu\nu\rho}\left[\partial^\rho\tilde{\zeta}_J - \mathcal{R}_{JK}\partial^\rho\zeta^K\right] - \zeta^I V_{\mu\nu}.\end{aligned} \tag{4.4.8}$$

These can be substituted back into $\tilde{\mathcal{L}}_3$ to obtain

$$\begin{aligned}\mathcal{L}_3 = &-\frac{1}{2}R_3 - \frac{1}{4}\partial_\mu\phi\partial^\mu\phi - g_{I\bar{J}}\partial_\mu X^I\partial^\mu\bar{X}^{\bar{J}} \\ &- e^{-2\phi}\left[\partial^\rho\tilde{\phi} + \frac{1}{2}\left(\zeta^I\partial^\rho\tilde{\zeta}_I - \tilde{\zeta}_I\partial^\rho\zeta^I\right)\right]\left[\partial_\rho\tilde{\phi} + \frac{1}{2}\left(\zeta^I\partial_\rho\tilde{\zeta}_I - \tilde{\zeta}_I\partial_\rho\zeta^I\right)\right] \\ &- \frac{\epsilon}{2}e^{-\phi}\left[\mathcal{I}_{IJ}\partial_\mu\zeta^I\partial^\mu\zeta^J + \mathcal{I}^{IJ}\left(\partial^\rho\tilde{\zeta}_I - \mathcal{R}_{IM}\partial^\rho\zeta^M\right)\left(\partial_\rho\tilde{\zeta}_J - \mathcal{R}_{JN}\partial_\rho\zeta^N\right)\right],\end{aligned} \tag{4.4.9}$$

where we have dropped the tilde over $\mathcal{L}_3$ as we will no longer work with the previous Lagrangian. We relegate some additional computational details into Appendix D.

This Lagrangian is of the standard form often found in the literature [132], however, in [31] it is explained that inspired by the r-map[13] [100], it is favourable to perform a field redefinition through scaling the complex coordinates

$$Y^I := e^{\phi/2} X^I. \tag{4.4.10}$$

In these new variables, the D-gauge becomes an expression relating the Kaluza-Klein scalar to our new scalar fields

$$-i(Y^I \bar{F}_{\bar{I}} - F_I \bar{Y}^{\bar{I}}) = e^{\phi}. \tag{4.4.11}$$

The matrix $g_{I\bar{J}}$ is homogenous of degree two and so transforms as

$$g_{I\bar{J}}(X,\bar{X}) = e^{\phi} g_{I\bar{J}}(Y,\bar{Y}). \tag{4.4.12}$$

The coupling matrices $\mathcal{R}_{IJ}$ and $\mathcal{I}_{IJ}$ are homogenous of degree zero and so

$$\mathcal{I}_{IJ}(X,\bar{X}) = \mathcal{I}_{IJ}(Y,\bar{Y}), \quad \mathcal{R}_{IJ}(X,\bar{X}) = \mathcal{R}_{IJ}(Y,\bar{Y}). \tag{4.4.13}$$

With all of this taken into account, the Lagrangian (4.4.9) is now of the form

$$\begin{aligned}
\mathbf{e}_3^{-1}\mathcal{L}_3 = &-\frac{1}{2}R_3 - g_{I\bar{J}}(Y,\bar{Y})\partial_\mu Y^I \partial^\mu \bar{Y}^J - \frac{1}{4}\partial_\mu\phi\partial^\mu\phi \\
&- e^{-2\phi}\left[\partial^\rho\tilde{\phi} + \frac{1}{2}\left(\zeta^I\partial^\rho\tilde{\zeta}_I - \tilde{\zeta}_I\partial^\rho\zeta^I\right)\right]\left[\partial_\rho\tilde{\phi} + \frac{1}{2}\left(\zeta^I\partial_\rho\tilde{\zeta}_I - \tilde{\zeta}_I\partial_\rho\zeta^I\right)\right] \\
&- \frac{\epsilon}{2}e^{-\phi}\left[\mathcal{I}_{IJ}\partial_\mu\zeta^I\partial^\mu\zeta^J + \mathcal{I}^{IJ}\left(\partial^\rho\tilde{\zeta}_I - \mathcal{R}_{IM}\partial^\rho\zeta^M\right)\left(\partial_\rho\tilde{\zeta}_J - \mathcal{R}_{JN}\partial_\rho\zeta^N\right)\right].
\end{aligned} \tag{4.4.14}$$

4.4.2 *Special real formulation*

We will now write the Lagrangian (4.4.14) using special real coordinates. These calculations follow [31]. We decompose the complex coordinate Y^I and the derivative of the prepotential as

$$\begin{aligned}
Y^I &= x^I + iu^I(x,y), \\
F_I &= y_I + iv_I(x,y),
\end{aligned} \tag{4.4.15}$$

and collect both x^I and y_I to form the special real coordinates

$$q^a := \begin{pmatrix} x^I \\ y_I \end{pmatrix}. \tag{4.4.16}$$

With the real scalars q^a, we can write the *Hessian metric*

$$H_{ab} = \frac{\partial^2 H}{\partial q^a \partial q^b},$$

where $H(q^a)$ is the *Hesse potential* and we can relate the Hesse metric to the Kähler metric by the relationship

$$g = \mathrm{Re}\left(N_{IJ} dY^I \otimes d\bar{Y}^{\bar{J}}\right) = H_{ab} q^a \otimes q^b.$$

[13] The r-map can be understood in analogy to the c-map, where instead a five-dimensional supergravity theory coupled to n_v vector multiplets is dimensionally reduced to four dimensions.

The Hesse potential itself is related to the holomorphic prepotential by a Legendre transformation replacing u^I with y_I as independent variables [134]:

$$\begin{aligned} H(x,y) &= 2\mathrm{Im}F(x,y) - 2y_I u^I(x,y), \\ &= \frac{i}{2}\left(Y^I\bar{F}_I - F_I\bar{Y}^I\right) = -\frac{1}{2}\phi. \end{aligned}$$

The D-gauge written in terms of the rescaled real scalar fields takes the simple form of

$$-2H(q^a) = e^{\phi}.$$

We now briefly discuss two additional metric tensors which appear in the following computations. In Section 4.1.2 we wrote down an additional Kähler potential by taking the logarithm of the Kähler potential of the corresponding superconformal theory. In a similar way, we define a new Hesse potential by

$$\tilde{H} := -\frac{1}{2}\log(-2H), \qquad \tilde{H}_{ab} := \frac{\partial\tilde{H}}{\partial q^a\partial q^b}.$$

The tensor $\tilde{H}_{ab}$ is a non-degenerate rank two tensor field and as such can be interpreted as a new Hessian metric. The inverse metric $\tilde{H}^{ab}$ has a corresponding Hesse potential with a flipped sign:

$$\tilde{H}^{ab} := \frac{\partial(-\tilde{H})}{\partial q_a\partial q_b},$$

where the dual coordinate q_a has been defined as:

$$q_a := \tilde{H}_a := \frac{\partial\tilde{H}}{\partial q^a} = -\frac{H_a}{2H} = -\frac{1}{H}\begin{pmatrix} v_I \\ -u^I \end{pmatrix}. \tag{4.4.17}$$

This definition implies the following relations

$$q_a = -\tilde{H}_{ab}q^b \Rightarrow q^a = -\tilde{H}^{ab}q_b, \qquad \partial_\mu q_a = \tilde{H}_{ab}\partial_\mu q^b, \tag{4.4.18}$$

where we have used that $\tilde{H}_a$ is homogeneous of degree -1 and then an application of the chain rule in the second relation. We can use $\tilde{H}_{ab}$ to lower the metric index in $\partial_\mu\hat{q}^a$ to obtain a covector field

$$\partial_\mu\hat{q}_a := \tilde{H}_{ab}\partial\hat{q}^b. \tag{4.4.19}$$

The second additional metric comes from considering the vector field coupling. We can express the vector kinetic matrix $\mathcal{N}_{IJ}$ in terms of a new, real matrix $\hat{H}_{ab}$. In [135] its form is found to be

$$\hat{H}_{ab} := \begin{pmatrix} \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \end{pmatrix}. \tag{4.4.20}$$

This matrix appears as the coupling matrix of the vector kinetic terms after dimensional reduction, and we find that this particular expression simplifies our work and is preferable to rewriting $\mathcal{R}_{IJ}$ and $\mathcal{I}_{IJ}$ themselves.

These three metrics can be related through the expressions which are proved in [31]

$$\begin{aligned} \tilde{H}_{ab} &= -\frac{1}{2H}\left(H_{ab} - \frac{H_aH_b}{H}\right), \\ &= \frac{1}{H}\hat{H}_{ab} - \frac{2}{H^2}(\Omega_{ac}q^c)(\Omega_{bd}q^d). \end{aligned} \tag{4.4.21}$$

We now turn to writing the Lagrangian (4.4.14) in terms of our special real coordinates. We outline some steps and refer to [31, 112] for an in-depth discussion. The Kaluza-Klein scalar can be written in terms of the Hesse potential

$$e^{\phi} = -i(Y^I \bar{F}_{\bar{I}} - F_I \bar{Y}^{\bar{I}}) = -2H, \tag{4.4.22}$$

and the scalar kinetic term is found to be [31]

$$g_{I\bar{J}} dY^I d\bar{Y}^J = \left[-\frac{1}{2H} H_{ab} + \frac{1}{4H^2} H_a H_b + \frac{1}{H^2} (\Omega_{ac} q^c)(\Omega_{bd} q^d)\right] dq^a dq^b. \tag{4.4.23}$$

Using that

$$\phi = \log(-2H) \Rightarrow \frac{\partial \phi}{\partial H} = \frac{1}{H}, \tag{4.4.24}$$

and using the chain rule

$$\partial_\mu \phi = \frac{\partial \phi}{\partial H} \frac{\partial H}{\partial q^a} \partial_\mu q^a = \frac{1}{H} H_a \partial_\mu q^a, \tag{4.4.25}$$

we can rewrite the kinetic term for the Kaluza-Klein scalar field as

$$\frac{1}{4} \partial_\mu \phi \partial^\mu \phi = \left[\frac{1}{4H^2} H_a H_b\right] \partial_\mu q^a \partial^\mu q^b. \tag{4.4.26}$$

We are now left to rewrite the terms arising from the gauge fields. We can do this elegantly by defining the scalar field

$$\hat{q}^a := \frac{1}{2} \begin{pmatrix} \zeta^I \\ \tilde{\zeta}_I \end{pmatrix}, \tag{4.4.27}$$

which can be related to the physical field strengths by

$$\partial_\mu \zeta^I = \hat{F}^I_{\mu 0}, \qquad \partial_\mu \tilde{\zeta}_I = \hat{G}_{I|\mu 0}. \tag{4.4.28}$$

With a bit of algebraic work, we are able to rewrite gauge field terms as

$$\begin{aligned} &-\frac{\epsilon}{2} e^{-\phi} \left[\mathcal{I}_{IJ} \partial_\mu \zeta^I \partial^\mu \zeta^J + \mathcal{I}^{IJ} (\partial_\mu \tilde{\zeta}_I - \mathcal{R}_{IK} \partial_\mu \zeta^K)(\partial^\mu \tilde{\zeta}_J - \mathcal{R}_{JL} \partial^\mu \zeta^L)\right] \\ &= \frac{\epsilon}{2} \hat{H}_{ab} \partial_\mu \hat{q}^a \partial^\mu \hat{q}^b. \end{aligned} \tag{4.4.29}$$

We can then relate the above to the Hesse potential using the relation (4.4.21). When this is done we obtain

$$\frac{\epsilon}{H} \hat{H}_{ab} \partial_\mu \hat{q}^a \partial^\mu \hat{q}^b = \epsilon \left[-\frac{1}{2H} H_{ab} + \frac{1}{2H^2} H_a H_b + \frac{2}{H^2} (\Omega_{ac} q^c)(\Omega_{bd} q^d)\right] \partial_\mu \hat{q}^a \partial^\mu \hat{q}^b. \tag{4.4.30}$$

Piecing together equations (4.4.23), (4.4.26) and (4.4.30) we can write down the three-dimensional Lagrangian in terms of the special real coordinates and the Hesse potential

$$\begin{aligned} \mathbf{e}_3^{-1} \mathcal{L}_3 = &-\frac{1}{2} R_3 - \left[-\frac{1}{2H} H_{ab} + \frac{1}{2H^2} H_a H_b\right] \left(\partial_\mu q^a \partial^\mu q^b - \epsilon \partial_\mu \hat{q}^a \partial^\mu \hat{q}^b\right) \\ &- \frac{1}{H^2} \left(q^a \Omega_{ab} \partial_\mu q^b\right)^2 + \epsilon \frac{2}{H^2} \left(q^a \Omega_{ab} \partial_\mu \hat{q}^b\right)^2 \\ &- \frac{1}{4H^2} \left(\partial_\mu \tilde{\phi} + 2\hat{q}^a \Omega_{ab} \partial_\mu \hat{q}^b\right)^2. \end{aligned} \tag{4.4.31}$$

Lastly we simplify this by reintroducing the function $\tilde{H}$ and its corresponding metric $\tilde{H}_{ab}$ [31]

$$\begin{aligned}\mathbf{e}_3^{-1}\mathcal{L}_3 = &-\frac{1}{2}R_3 - \tilde{H}_{ab}\left(\partial_\mu q^a\partial^\mu q^b - \epsilon\partial_\mu\hat{q}^a\partial^\mu\hat{q}^b\right)\\ &-\frac{1}{H^2}\left(q^a\Omega_{ab}\partial_\mu q^b\right)^2 + \epsilon\frac{2}{H^2}\left(q^a\Omega_{ab}\partial_\mu\hat{q}^b\right)^2\\ &-\frac{1}{4H^2}\left(\partial_\mu\tilde{\phi} + 2\hat{q}^a\Omega_{ab}\partial_\mu\hat{q}^b\right)^2.\end{aligned} \tag{4.4.32}$$

4.5 SUPERGRAVITY IN HIGHER DIMENSIONS

In this section, we give a limited introduction to supergravity in higher dimensions, effective to describe p-branes and their relationship to black hole solutions. This is a vast topic which deserves much more space than we give to it, and so we begin with a few references for a reader interested in the topic. A comprehensive overview is given in [136], which covers everything and more of what we hope to describe. Marolf has a chapter in [137] which is focused on how p-branes give rise to black hole solutions and is particularly readable. Duff has some TASI lecture notes on brane solutions from the perspective of AdS/CFT research [138]. Additional TASI lectures by Peet [139] and a review from ICTP by Stelle [140] are based on BPS solutions and black holes from the perspective of string theory. Duff's article 'Formally known as strings' offers a historical perspective on the history of higher-dimensional supergravity and its relationship to M-theory [141].

This section first introduces supergravity in eleven dimensions and its relationship to string theory and M-theory. We then discuss the fundamental objects known as p-branes, which generalise the notion of the charged particle in four dimensions. We conclude the section by sketching how one can build black hole solutions which have ten and eleven dimension interpretations as intersecting brane configurations.

4.5.1 *Eleven-dimensional supergravity*

In Section 4.1.1, supergravity was introduced as the local description of supersymmetry, which itself was motivated as the extension of the Poincaré symmetry through the introduction of a *fermionic* generator describing a symmetry transformation between fermions and bosons. In this context, it made sense to discuss supersymmetry in four dimensions, but we can consider the supersymmetry algebras in arbitrary dimension. Nahm proved in 1977 that if we wish to maintain that the highest spin states have spin-2, then the spacetime dimension is restricted to $D \leq 11$ [95]. As a very rough sketch, we understand this from our discussions in four dimensions. In four dimensions, each spinor has four components and maintaining that the helicity $|\lambda| \leq 2$ forced at most 32 supercharges, setting $\mathcal{N} = 8$. In eleven dimensions, a single spinor has 32 components, and so maximal supersymmetry is given by $\mathcal{N} = 1$. If we were to try and raise the number of dimensions, we would find our spinors had at least 64 components (for $D = 12$), and so $D > 11$ is incompatible with our assumptions of highest spin states.

More surprisingly, in 1978, Cremmer, Julia and Scherk proved that there is a unique eleven-dimensional supergravity theory described by the supergravity multiplet containing the graviton G_{MN}, the graviphoton ψ_M and a three-form gauge potential $\mathcal{A}_{MNP}$, where the spacetime indices run from $M, N \in \{0, \ldots 10\}$. In this initial context, eleven-dimensional supergravity was thought of as a way to derive four-dimensional supergravity from dimensional reduction. It was this eleven-dimensional theory which bought new attention to the Kaluza-Klein reduction when the community started to try and find ways to wrap up the seven dimensions. In 1981, Witten produced a paper stating that the minimum number of dimensions to compactify over to get a standard-like model in four dimensions was seven [142]. This seemed to only further motivate that eleven-dimensional supergravity was the 'sweet-spot' for a unified theory.[14]

While supergravity was being worked on, Veneziano's dual model theory to describe the strong interaction [18] had wildly changed direction after being realised as a model of the relativistic string with the graviton appearing in the massless spectrum of the closed string. In 1984, the so-called 'first superstring revolution' began when Green and Schwarz developed the type I superstring with the gauge group SO(32) [20]; a theory free from gauge and gravitational anomalies. String theory became the new hope for a unification theory and for a few years, eleven-dimensional supergravity took a back seat. By 1985, there were five consistent string theories: the type I string already mentioned, two heterotic strings with gauge groups of either $E_8 \times E_8$ or SO(32) [143, 144] and two theories of closed strings called type IIA and type IIB [19]. It was argued that one could break the maximal supersymmetry of the string in ten dimensions by reducing the heterotic string over a Calabi-Yau three fold, producing a theory with $\mathcal{N} = 1$ supersymmetry in four dimensions [145]. The new question was that if string theory was supposedly a 'theory of everything', why were there five different theories. There was also the question of the non-perturbative realisation for the superstring, and other phenomenological questions such as how to make the choice of which Calabi-Yau to reduce over, and how can we understand the smallness of the cosmological constant [138].

Eleven-dimensional supergravity made a surprising return to the limelight after Witten introduced a new, non-perturbative theory known as M-theory [22]. By appealing to a set of stringy dualities, it was put forward that one overarching theory in eleven dimensions could describe each of the five string theories as various points in a moduli space, with eleven-dimensional supergravity appearing as the low energy limit.

$$\left.\begin{array}{l} E_8 \times E_8 \text{ Heterotic string} \\ SO(32) \text{ Heterotic string} \\ SO(32) \text{ Type I string} \\ \text{Type IIA string} \\ \text{Type IIB string} \\ \text{Eleven-dimensional supergravity} \end{array}\right\} \text{M-theory}$$

For the remainder of this section, we will focus on the low energy, bosonic field content of eleven-dimensional supergravity and its relationship to type IIA/IIB string theory, motivated that eleven-

[14] This upper and lower bound for $D = 11$ from very different perspectives is described by Duff as what appears 'to this day seems to be merely a gigantic coincidence' [138].

dimensional supergravity isn't only interesting in isolation but also through how M-theory makes contact with all consistent string theories.

The low-energy action for the bosonic fields of eleven-dimensional supergravity is given by

$$S_{11D} = \frac{1}{2\kappa_{11}^2} \int -R \star 1 - \frac{1}{2} \star \mathcal{F} \wedge \mathcal{F} - \frac{1}{6} \mathcal{F} \wedge \mathcal{F} \wedge \mathcal{A}. \tag{4.5.1}$$

Where $\mathcal{A}$ is the three-form gauge potential with the corresponding field strength $\mathcal{F} = d\mathcal{A}$ and $2\kappa_{11}^2 = 16\pi G_{11}$ is the gravitational coupling. The fermionic completion of this action is given in [146]. Although eleven dimensions is obviously something we're less used to, from the point of view of the action it's not that different to the Einstein-Maxwell action (3.3.1), in which gravity is coupled to some two-form gauge field. This similarity will come up again when we consider p-branes shortly. Note that there is no scale dependent coupling, *i.e.* there is no dilaton.

Let us now consider the low energy, bosonic field content of the type II strings, which we will refer to as type IIA/IIB supergravity. The matter content of type IIA/IIB supergravity comes from two sectors known as the Neveu–Schwarz (NS-NS) sector and the Ramond (RR) sector, which get their names from the boundary conditions used from the string theory perspective. The NS-NS sectors of the type IIA and type IIB string theory are the same, whereas the RR sector is different for each. The bosonic content of the NS-NS sector is given by the graviton $g_{\mu\nu}$, a two-form potential $B_{\mu\nu}$, known as the Kalb-Ramond field, and the dilaton ϕ. We can write the action as [22]

$$S_{\text{NS-NS}} = \frac{1}{2\kappa_{10}^2} \int e^{-2\phi} \left(-R \star 1 + 4 d\phi \wedge \star d\phi - \frac{1}{2} H \wedge \star H \right), \qquad H = dB.$$

where we note that this is written in the *string-frame* where the Ricci scalar appears in the action coupled to the dilaton. This can be removed through a conformal transformation to obtain the action in the Einstein frame [17]

$$S_{\text{NS-NS}} = \frac{1}{2\kappa_{10}^2} \int \left(-R \star 1 - \frac{1}{2} d\phi \wedge \star d\phi - \frac{1}{2} e^{-\phi} H \wedge \star H \right).$$

The RR sector is built from additional p-form field strengths. For the type IIA string, the gauge potentials are odd, with a one-form A_1 and three-form A_3. We can write down the low energy bosonic action for the RR sector [22]

$$S_{\text{IIA}\,|\,\text{RR}} = -\frac{1}{2\kappa_{10}^2} \int F_2 \wedge \star F_2 + \tilde{F}_4 \wedge \star \tilde{F}_4 - \frac{1}{4} F_4 \wedge F_4 \wedge B,$$

where the last term is a topological Chern-Simons term and the field strengths are given by

$$F_2 = dA_1, \qquad \tilde{F}_4 = dA_3 + A_1 \wedge H + B \wedge F_4 \wedge F_4,$$
$$F_4 = dA_3.$$

In contrast, the IIB string has a RR sector built from even-form gauge potentials: A_0 a scalar, A_2 a two-form gauge potential and A_4 a four-form gauge potential, which has a self dual field strength F_5. The self-duality of F_5 prohibits a totally satisfactory Lagrangian, but if we are happy to impose this as an additional constraint, one can write [147]

$$S_{\text{IIB}\,|\,\text{RR}} = -\frac{1}{2\kappa_{10}^2} \int F_1 \wedge \star F_1 + \tilde{F}_3 \wedge \star \tilde{F}_3 + \frac{1}{2} \tilde{F}_5 \wedge \star \tilde{F}_5 + A_4 \wedge H \wedge F_3,$$

where the various field strengths are given by

$$\begin{aligned} F_1 &= dA_0, & \tilde{F}_3 &= F_3 - A_0 H, \\ F_3 &= dA_2, & \tilde{F}_5 &= F_5 - \frac{1}{2}A_2 \wedge H + \frac{1}{2}B \wedge F_3, \\ F_5 &= dA_4, & \tilde{F}_5 &= \star\tilde{F}_5, \end{aligned}$$

where we note the last piece of the above is the self-duality constraint $\tilde{F}_5 = \star\tilde{F}_5$, imposed by hand.

For the following work, we will not need the exact form of the actions, but we have collected them all together as many resources include only one or two. What we will be focused on is studying the fundamental objects of each theory by looking at how charged objects couple to the p-forms.

As a closing remark, we comment on how these theories are related to one another. At the low energy level, we can build a duality map through dimensional compactification. Beginning with the eleven-dimensional theory, reducing (4.5.1) over a compact, spacelike circle, one obtains the action for type IIA supergravity. It was first conjectured by Townsend in [148] that the full type IIA string theory can be obtained by the dimensional compactification of the supermembrane in eleven dimensions. This is an example of the symmetry known as *S-duality*, where the dilaton in the strong coupling limit behaves like an additional dimension [149]. Another S-duality appearing within string theory is the $\mathrm{SL}(2,\mathbb{Z})$ invariance of type IIB string theory for theories related by the flipping of the string coupling g_S[150].

The type IIA and type IIB supergravity theories can be shown to be dual through compactifying them both down to nine dimensions, where one can map the field content into each other. This duality is also seen at the level of string theory, which manifests as the stringy symmetry known as *T-duality*, which is full equivalence at the level of the string partition function. Roughly, T-duality is the statement that type IIA (IIB) compactified over a circle of radius R is equivalent to type IIB (IIA) compactified over a circle of radius R^{-1}. In the process of this duality, the momentum and winding modes of the string are interchanged.

4.5.2 *p-branes*

Given the actions of eleven-dimensional supergravity and the low energy descriptions of the type IIA/IIB strings, we can consider the fundamental objects of the various theories. The notion of a p-brane comes naturally as a generalisation of the charged particles of Maxwell theory, which is where we begin.

Maxwell theory contains a one-form potential A which enters the action as a two-form field strength $F = dA$. Let us consider some charged particle that couples to the gauge potential in the following way

$$S_e = \mathcal{Q}\int A = \mathcal{Q}\int d\tau \left[A_\mu \frac{dx^\mu}{d\tau}\right].$$

The charge of the particle is found from the now familiar formula using Gauss' law:

$$\mathcal{Q} = \int_{S^2} \star F.$$

As we saw in Section 4.2, the electromagnetic duality allows one to define a dual gauge potential $\tilde{A}$ and we can consider the magnetic particle (magnetic monopole) that couples to $\tilde{A}$ analogously to the electric particle

$$S_m = \mathcal{P}\int \tilde{A} = \mathcal{P}\int d\tau \left[\tilde{A}_\mu \frac{dx^\mu}{d\tau}\right],$$

where the charge of the particle is found by integrating over the flux of the field

$$\mathcal{P} = \int_{S^2} F.$$

We note that point-like magnetic charges have not been measured experimentally, but the symmetry we see here suggests that they exist, but with masses far beyond the reach of current experiments [151].

Let us now consider an identical system, but allow our Maxwell theory to be defined in D-dimensions. We have a one-form potential which will couple to a particle. However, to surround the particle in D dimensions, we do not integrate over a S^2 but rather over S^{D-2} such that the electric charge is found from

$$\mathcal{Q} = \int_{S^{D-2}} \star F,$$

where we note that $\star F$ will be a $(D-2)$-form. Let us now consider the magnetic dual, which will have a charge equal to the integral over the flux of the field

$$\mathcal{P} = \int_{S^2} F.$$

We can no longer interpret this as a magnetic *particle*, but rather as an object that extends into space. Working in D dimensions, the sphere S^2 doesn't necessarily surround a point-like particle, but rather an extended object spanning $(D-4)$ spacelike dimensions. These extended objects are called p-branes. As some examples, we can think of a particle as a 0-brane and a string as a 1-brane. Taking the example of $D = 10$, we see that a particle that electrically couples to a one-form will have a magnetically dual 6-brane.

Let us now allow our discussion to be totally general, where our theory contains some gauge potential A_{p+1} which is a $(p+1)$-form with a $F_{p+2} = dA_{p+1}$ field strength. By considering the charges of this gauge potential and its magnetic dual, we can understand the extended objects that couple to it. Our $(p+1)$-form gauge potential couples electrically to the world volume of a p-brane with $n = p+1$ spacetime dimensions.[15] We can write the action of the brane coupling to the $(p+1)$-form as

$$S = \mathcal{Q}\int A_{p+1},$$

where the electric charge of the p-brane is computed from

$$\mathcal{Q} = \int_{S^{D-p-2}} \star F_{p+2}.$$

[15]This note is mainly for a student reading this getting used to the numbers. A particle has no extension in space, but moves through time. As such, we think of the particle as a 0-brane which has a worldvolume of one spacetime dimension. Generally we describe some theory with a gauge field which is an $(p+2)$-form, which has an $(p+1)$ gauge potential that couples to an extended object whose worldvolume has $(p+1)$ spacetime dimensions, which we call a p-brane. From this, the quick statement is given an $n-$form field strength (this is what appears in the action usually), one has a $(n-2)$-brane that couples electrically to its gauge potential.

Similarly, the magnetic dual has a charge given by the integral over the flux

$$\mathcal{P} = \int_{S^{p+2}} F_{p+2}.$$

This charge can be understood as being sourced from a $(D-p-4)$-brane. Another way to see this is to consider the charge from the dual field with

$$\tilde{\mathcal{Q}} = \int_{S^{D-p-2}} \star\tilde{F}_{D-p-2}.$$

which is the electric charge of a $(D-p-2)$-form associated to a $(D-p-4)$-brane.

With this general rule, we can write down the extended objects of eleven-dimensional supergravity as well as type IIA and IIB supergravity. Note that these are not only objects in supergravity but also in M-theory and type IIA/IIB string theory [152].

For eleven-dimensional supergravity, we have only one gauge potential, which is the three-form $\mathcal{A}$, with the corresponding 2-brane known as the M2 brane. This plays the role of the electron in four-dimensional Maxwell theory, coupling electrically to $\mathcal{A}$. The magnetic dual of the M2 brane is the $(11-2-4=5)$ 5-brane known as the M5 brane. We will discuss these in more detail in the next section, from the context of BPS solutions of supergravity.

In the type IIA/IIB supergravities, there are the NS-NS and RR sectors. In the NS-NS sector, we have the two-form B which sources the electric 1-brane, known as the F1 string and its magnetic dual the NS5 brane. The F1 string is known as the *fundamental string* and is the string of perturbative string theory. For type IIA, the RR gauge potentials are the one-form A_1 and the three-form A_3 which have the electric D0 particle and the D2 membrane coupling to them. Their magnetic duals are known as the D6 brane and D4 brane, respectively. Finally, the RR sector of the type IIB string has a zero-form A_0, two-form A_2 and four-form A_4. The two-form couples to the D1 string and its magnetic dual is the D5 brane. The electric source of the zero-form is sometimes called the D(-1) brane; when studied in Euclidean theories, we can consider this as an instanton solution in space and Euclidean time [151]. Its magnetic dual is the D7 brane. Finally the four-form couples both electrically and magnetically to the D3 brane as a result of the self-duality of the five-form $F_5 = dA_4 = \star F_5$. This is summarised in Table 4.3. Here we have used words such as string and membrane for the lower dimensional branes to help with visualisation, but often all these extended objects are simply referred to as branes, e.g. the D1 brane or D0 brane.

4.5.3 *Extremal brane solutions*

We now discuss p-branes from the perspective of solutions to the field equations. In particular, we are going to discuss the extremal solutions which preserve one half of the supersymmetry. As such, these extremal solutions are sometimes referred to as BPS solutions, or more accurately $\frac{1}{2}$–BPS solutions. Solutions are found by making an ansatz for the p-brane with the symmetry of $\mathrm{ISO}(1,p) \times \mathrm{SO}(D-p-1)$, with the corresponding line element

$$ds^2 = e^{2A(r)}dx_{\parallel}^2 + e^{2B(r)}dx_{\perp}^2.$$

We denote the coordinates parallel to the brane with $dx_{\parallel}^2 = \eta_{\mu\nu}dy^{\mu}dy^{\nu}$, accounting for the symmetry group $\mathrm{ISO}(1,p)$, and the directions transverse to the brane as $dx_{\perp}^2 = \delta_{mn}dx^m dx^n$, accounting for the symmetries of the sphere. The variable r is the isotropic coordinate corresponding to

Electric Brane	D = 11	Type IIA	Type IIB	Magnetic Dual
D(-1) Instanton	—	—	A_0	D7 Brane
D0 Particle	—	A_1	—	D6 Brane
F1 String	—	B	B	NS5 Brane
D1 String	—	—	A_2	D5 Brane
M2 Membrane	$\mathcal{A}$	—	—	M5 Brane
D2 Membrane	—	A_3	—	D4 Brane
D3 Brane	—	—	A_4	D3 Brane

Table 4.3: Summary of the electric p-branes and their magnetic duals [147]

the transverse space: $r = \sqrt{x^m x_m}$. For the following discussion, we limit $p \leq 6$ to allow our solutions to be asymptotically flat. The inappropriate fall off of the so-called 'large branes' is a common feature for brane solutions in arbitrary dimension where there are less than three transverse directions to the source, *e.g.* black holes in three dimensions, or black strings in four dimensions [49].

We will not solve the field equations themselves, but rather write down the line elements with proper referencing so we can discuss their form. In [140, 138], it is explained how to find brane solutions from the field equations in good detail. For our purposes, we wish to have these extremal p-brane solutions as building blocks for the next section, in which we look at the intersection of branes and their relationship to black hole solutions in lower dimensions.

Let us begin by looking at the M2 and M5 brane solutions of eleven-dimensional supergravity. These are two of the four basic solutions of eleven-dimensional supergravity, with the other two being the PP-wave and the Kaluza-Klein monopole where the gauge field $\mathcal{A} = 0$. As these are also solutions in ten dimensions (where both the field strength and dilaton are assumed to be zero), we will cover them at the conclusion of this section.

Let us begin with the solution of the electric M2 brane [153], which is described by

$$ds^2 = H_2^{-\frac{2}{3}} dx_{\|}^2 + H_2^{\frac{1}{3}} dx_{\perp}^2, \qquad \mathcal{A}_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho}\, H_2^{-1}.$$

where H_2 is a harmonic function with respect to the transverse coordinates

$$\triangle^{\perp} H_2 = 0.$$

One particular choice we can make for the harmonic function corresponds to the single-centred solution

$$H_2(r) = 1 + \frac{Q_2}{r^6},$$

where Q_2 is related to the charge of the M2 brane. More generally, we write the harmonic function of a single centred solution with respect to the transverse coordinates of a p-brane solution as

$$H_p(r) = 1 + \frac{Q_p}{r^{D-p-3}}. \tag{4.5.2}$$

Here r is understood to be the isotropic coordinate as defined before, with the source located at $r \to 0$ and Q_p is related to the charge of the brane.

The global structure of the M2 brane can be understood in analogy to the extremal Reissner-Nordström solution of Einstein-Maxwell theory studied in Section 3.3. There is a Killing horizon located for $r \to 0$ and crossing the horizon, we find that the timelike Killing vector remains timelike and the singularity is therefore a timelike singularity. The conformal diagram is that of the extremal Reissner-Nordström (see Figure 3.8) but with each point on the diagram representing $\mathbb{R}^7 \times S^2$ rather than S^2. We also see that just as the extremal Reissner-Nordström solution interpolated between $\mathcal{M}_4$ in the asymptotic limit to the Bertotti-Robinson solution $AdS_2 \times S^2$ near the horizon, the M2 brane interpolates between $\mathcal{M}_{11}$ in the asymptotic limit, to the near horizon geometry $AdS_4 \times S^7$.

Let us now consider the M5 brane, which has a solution given by [154]

$$ds^2 = H_5^{-\frac{1}{3}} dx_\parallel^2 + H_5^{\frac{2}{3}} dx_\perp^2, \qquad \mathcal{F}_{\mu\nu\rho\sigma} = 3Q_5 \varepsilon_{\mu\nu\rho\sigma}, \qquad H_5 = 1 + \frac{Q_5}{r^3}.$$

As with the M2 brane, we have a horizon located at $r \to 0$, and we find that the near horizon geometry is given by $AdS_7 \times S^4$. However, unlike the M2 brane, there is no spacetime singularity at all, and instead we find a copy of the spacetime on either side of the horizon [155].

We might be worried that this solution is a counterexample for the Penrose singularity theorem, as we seem to have a trapping horizon bounding a region which contains no singularity. However, as the trapping horizon of the M5 brane solution is non-compact, a key assumption in Penrose's argument is broken, and so we should not expect the theorem to hold [137].

The D-brane solutions of type IIA and type IIB supergravity can be written down in a generic form, where specific solutions are obtained by setting the value of p with the following string-frame line element [49]

$$\begin{aligned} ds^2 &= H_p^{-\frac{1}{2}} dx_\parallel^2 + H_p^{\frac{1}{2}} dx_\perp^2, \qquad e^\phi = H_p^{\frac{3-p}{4}}, \\ A_p &= \begin{cases} \frac{1}{H_p} dt \wedge dx^1 \wedge \ldots \wedge dx^p & p \leq 3, \\ \star \left(\frac{1}{H_p} dt \wedge dx^1 \wedge \ldots \wedge dx^p \right) & p \geq 3, \end{cases} \end{aligned} \tag{4.5.3}$$

where H_p is set by (4.5.2). The presence of the dilaton can often lead to singular behaviour when dimensionally reducing the brane, and care must be taken to stabilise the scalar fields so they do not diverge on the horizon. We will postpone further discussion of the solutions until the next section, where we construct a four-dimensional black hole from an arrangement of intersecting branes from IIB supergravity.

The final two solutions are not brane solutions but are instead key parts of the construction of intersecting brane configurations. They come from solutions where the gauge field (and the dilaton for ten-dimensional theories) vanishes. As such, these can be considered as solutions of general relativity in higher dimensions. They are known as the PP-wave, or null particle solutions, and the Taub-NUT or Kaluza-Klein monopole solutions.

The PP-wave, also sometimes called the Aichelburg-Sexl [156] solution, can be found by infinitely boosting the Schwarzschild solution while scaling the mass M such that the total energy

E is finite, and was first discussed by [157]. We can think of the PP-wave as being the solution describing the gravitational field of a null particle, and it is parameterised with a harmonic function H_K with a line element given by

$$ds^2 = -dt^2 + dz^2 + (H_K - 1)(dt - dz)^2 + dx_\perp^2,$$

where the perpendicular space has $(D-2)$ spacelike dimensions. The PP-wave solution can be superimposed along the direction of a brane such that we understand momentum modes moving in a direction along the brane. From the point of view of dimensional compactification, reducing over a direction with a PP-wave introduces an electric charge in the lower-dimensional theory, with the identification of the compact coordinate playing the role of charge quantisation.

The final solution we consider here is known as the Taub-NUT or Kaluza-Klein monopole solution, first considered as a solution in general relativity [50, 158], and then later interpreted in five dimensions, where it could be understood as a magnetic monopole solution after Kaluza-Klein compactification [159], as well as an eleven-dimensional solution [160].

The line element of the Taub-NUT solution is given by

$$\begin{aligned} ds^2 &= -dt^2 + d\vec{y}^2 + H_{KK} dx^2 + H_{KK}^{-1}\left(d\theta + A_i dx^i\right)^2, \\ H &= \phi = 0, \qquad \vec{\nabla} B = \vec{\nabla} \times \vec{A}, \end{aligned}$$

and like the other solutions, is parameterised by a single harmonic function, which for this solution is a harmonic in the three-dimensional coordinates x^i. The spacetime coordinates: $-dt^2 + d\vec{y}^2$ cover $(D-4)$ dimensions and we can think of the spacetime having a decomposition as $\mathcal{M}^{D-4} \times K_4$ where K_4 is the Taub-NUT space with a line element

$$ds_{TN}^2 = H_{KK}^{-1}\left(d\theta + A_i dx^i\right)^2, \qquad H_{KK} = 1 + \frac{K}{|r|}.$$

We see that this naturally lends itself to compactification, with θ as the compactification dimension. It can be shown that like the BPS branes we discussed, the PP-wave and Taub-NUT solutions are also $\frac{1}{2}$-BPS solution [140].

4.5.4 *Black holes and intersecting branes*

In this section, we outline the process of describing supersymmetric black hole solutions in lower dimensions as the dimensional compactification of the brane solutions given in the previous section. Given a p-brane solution in D dimensions, we have two options for dimensional compactification. Reducing over a coordinate that runs parallel to the brane, we would obtain a $(p-1)$-brane in $(D-1)$ dimensions. This procedure is known as a double reduction or a wrapping. As the coordinates running parallel to the brane are isometries of the solution, this is always valid. The alternative reduction choice is to dimensionally reduce over a coordinate transverse to the brane. This produces a solution describing a p-brane in $(D-1)$ dimensions. However, generically the reduction direction will not be an isometry of the spacetime, as the harmonic functions that describe brane solutions are dependent on the isotropic coordinates built from the transverse directions. To perform the reduction, an isometry direction is established through smearing out the brane in a periodic array along the compactification direction. This generalises the harmonic function from a single source to that of a multi-centred solution. The branes can be placed arbitrarily within the array, with the gravitational and electromagnetic repulsions cancelling out due to the *no-force* property of the BPS branes. To avoid going too far off-topic for the thesis, we do not detail this further but refer to the lecture notes [49] which perform calculations and give examples on this topic. The upshot of this procedure is that the harmonic function then depends on all transverse directions except the reduction direction, which is then an isometry and a Kaluza-Klein reduction can be performed. These smeared brane configurations are often referred to as delocalised branes where the symmetry of the source is no longer spherical, but cylindrical.

Constructing black holes from brane configurations is particularly interesting as from the higher-dimensional perspective, where we understand the microscopic origins of the M-branes and D-branes from M-/string theory. We can calculate the statistical entropy of the branes in higher dimensions and then compare it with the Bekenstein-Hawking area law of the black hole solutions in lower dimensions, opening a window into the quantum description of black hole thermodynamics [24, 161, 162]. However, in this thesis we are interested in these black hole solutions from a reversed perspective, where we will show that the four-dimensional solutions we derive in Chapter 6 can be described as higher-dimensional solutions through a process of dimensional lifting. In Chapter 7, we find the explicit solutions in ten and eleven dimensions which we can understand within the context of the brane solutions discussed in this section. As such, we include this section to give context for later discussions, and will not make further comments on entropy counting of brane configurations.

The main obstruction when finding black holes from the dimensional compactification of p-branes is their singular behaviour after reduction. For a generic p-brane solution, the result after compactification is degenerate, with a horizon of zero area leaving behind a null singularity in the lower-dimensional solution. This is a common issue for solutions with scalar fields which take singular values within the spacetime. When we solve our equations of motion in Chapter 6, we will have to carefully pick our integration constants such that our scalar fields do not diverge on the horizon. From the context of dimensional reduction, singular behaviour signifies that our solutions do not make sense from a lower-dimensional perspective.

In order to produce black hole solutions in lower dimensions, we then need a way to control the behaviour of the scalar fields. We can then understand the problem of finding regular solutions as the problem of 'stabilising the moduli' [49]. One way of doing this is to have the scalar functions given as ratios of harmonic functions, which can be achieved through introducing multiple p-branes into the spacetime which intersect along common directions. For the remainder of this section, we will discuss how to write down solutions of intersecting branes and give an example of the intersection of a D1 brane and D5 brane which produces a non-singular solution in four dimensions.

From the single brane solutions discussed in the previous section, it is possible to write down a set of generic rules to construct intersecting brane configurations that preserve some fraction of the supersymmetry. The steps we now offer are taken from an unpublished book by Klaus Behrndt [163], but the solutions we build will be given with published resources. The key part to understand is that the harmonic functions are solutions to Laplace's equation in the number of relative transverse directions to the whole configuration and so their fall off in terms of the isotropic coordinate depends on the brane system and not the constituent branes.

We will focus on solutions of intersecting branes that preserve some fraction of supersymmetry, which requires the number of relative transverse dimensions to be

$$n = 4k, \qquad k \in \mathbb{Z},$$

which for $D \leq 11$ leaves us with $n = 0, 4, 8$. When $k = 1$, the fraction of supersymmetry preserved is one-quarter, which is what we will focus on in this discussion. When $n = 0$, we consider parallel branes, which include the stacked brane configurations which are commonly studied due to their relationship to the AdS/CFT correspondence [14, 15, 161] which arguably has been the most celebrated and productive area of string theory and black hole physics for the past twenty-five years. For the case of $n = 8$, to obtain spherical black holes from dimensional reduction requires reducing over a non-flat space, and the resulting solutions are asymptotic to (anti)-de Sitter rather than Minkowski [163].

Let us now consider the rules for generating intersecting solutions. Considering the intersection of a p-brane and q-brane, the metric components multiply as

$$g_{\mu\nu}^{(p\times q)} = g_{\mu\nu}^{(p)} g_{\mu\nu}^{(q)},$$

where we note that there's no summation over indices within this rule. The dilaton, if present, and gauge fields combine as

$$\phi^{(p\times q)} = \phi^{(p)} + \phi^{(q)}, \qquad F^{(p\times q)} = F^{(p)} + F^{(q)},$$

where gauge fields will only combine when $p = q$, otherwise they are independent. For solutions with three or more intersecting branes, these rules described must be true for all pairs of branes.

As a ten-dimensional example, lets construct the D1-D5 brane configuration [161] from the general solutions of D-branes in (4.5.3) and the above rules

$$\begin{aligned} ds^2_{1\times 5} &= \frac{1}{\sqrt{H_1 H_5}}\left[-dt^2 + dz^2\right] + \sqrt{\frac{H_1}{H_5}}\left[dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2\right] + \sqrt{H_1 H_5}dx_\perp^2, \\ e^\phi &= \sqrt{\frac{H_1}{H_5}}, \qquad H_i = 1 + \frac{Q_i}{r^2}, \end{aligned} \tag{4.5.4}$$

where we notice that sending $Q_1 \to 0$ or $Q_5 \to 0$ will recover the solution for the D5 or D1 brane solution respectively, and we suppress the form of the gauge potentials which are independent from each other as they are different rank forms.

Another example we can study is the triple intersection of M5 branes [164] which has a line element given by

$$\begin{aligned} ds^2_{5\times 5\times 5} = (H_1 H_2 H_3)^{-\frac{1}{3}}\big[&- dt^2 + dz^2 + H_1 H_2 H_3 d\vec{x}^2 \\ &+ H_1(dy_1^2 + dy_2^2) + H_2(dy_3^2 + dy_4^2) + H_3(dy_5^2 + dy_6^2)\big], \end{aligned} \tag{4.5.5}$$

where we now just use counting indices for the three M5 branes which have harmonics H_i.

We see that by intersecting the D1 and D5 brane, the coupling e^{ϕ} approaches a constant in the limit $r \to 0$. We can think of performing a dimensional reduction by wrapping over the coordinates y_i in the solution (4.5.4) to obtain the six-dimensional solution [163]

$$ds_6^2 = \frac{1}{\sqrt{H_1 H_5}}\left[-dt^2 + dz^2\right] + \sqrt{H_1 H_5}\left(ds^2 + r^2 d\Omega_3^2\right), \qquad e^{-2\phi} = 1, \qquad e^{2\sigma} = \sqrt{\frac{H_1}{H_5}}.$$

The ten-dimensional dilaton is balanced by the determinant of the internal space spanned by the y_i coordinates and becomes a constant. The Kaluza-Klein scalar σ parametrises the volume of the four torus. The wrapped D1-D5 solution is a dyonic string, with an electric charge sourced by the D1 brane, and magnetic charge from the D5 brane. To obtain a black hole solution (a 0-brane), we must reduce over the coordinate spanned by the string z. Performing this reduction, the metric written in the Einstein frame can be found to be [49]

$$ds_5^2 = -\left(H_1 H_5\right)^{-\frac{2}{3}} dt^2 + \left(H_1 H_5\right)^{\frac{1}{3}}\left(dr^2 + r^2 d\Omega_3^2\right).$$

However, in the limit of $r \to 0$, the area of the event horizon tends towards zero. We can understand this singularity being sourced by $g_{zz} \to 0$ in the limit of $r \to 0$ from the six-dimensional perspective. We can interpret the vanishing of the area as the black string being in the ground state, and to stabilise the solution, we can include a PP-wave along the common intersection of the D1-D5 system, exciting the solution. Including a PP-wave along the string gives us a ten-dimensional solution [161]

$$\begin{aligned} ds^2_{1\times 5\times PP} = {} & \frac{1}{\sqrt{H_1 H_5}}\left[-dt^2 + dz^2 H_K (dz - dt)^2\right] \\ & + \sqrt{\frac{H_1}{H_5}}\left[dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2\right] + \sqrt{H_1 H_5} dx_\perp^2. \end{aligned} \tag{4.5.6}$$

Reducing this over the five-torus we obtain a five-dimensional solution built from three charges [49]

$$ds_5^2 = -\left(H_1 H_5 H_K\right)^{-\frac{2}{3}} dt^2 + \left(H_1 H_5 H_K\right)^{\frac{1}{3}}\left(dr^2 + r^2 d\Omega_3^2\right), \qquad H_i = 1 + \frac{Q_i}{r^2}.$$

This solution has a horizon of finite area, and the near horizon geometry is given by $AdS_2 \times S^3$ [163]. In the special limit where all charges are equal, the solution simplifies to the Tangherlini solution [165]

$$ds_5^2 = -H^{-2} dt^2 + H\left(dr^2 + r^2 d\Omega_3^2\right), \qquad H = 1 + \frac{Q}{r^2},$$

which can be thought of as the five-dimensional equivalent to the four-dimensional extremal Reissner-Nordström solution.

Finally, if we wish to obtain a four-dimensional black hole, we will have to perform one last reduction. However, the remaining spacelike coordinates are not generally isometries of the spacetime and so we will have to smear the solution before we can reduce it. The extended brane becomes a string, and our harmonic functions are restricted to depend only on three of the four coordinates in the relative transverse space $dx_\perp^2$. In the limit of $r \to 0$, we see that $\left(H_1 H_5 H_K\right)^{\frac{1}{3}}$ will diverge, and so following how we stabilised the previous reduction with the introduction of the PP-wave, we must add an additional charge before reducing to four dimensions. This can be achieved through assuming that the relative transverse directions cover the Taub-NUT space, rather than Minkowski. From a ten-dimensional perspective, we consider a D1-D5 brane intersection with a PP-wave and a four-dimensional space with a Kaluza-Klein monopole [163]

$$\begin{aligned} ds^2_{1\times5\times PP\times TN} = &\frac{1}{\sqrt{H_1H_5}}\left[-dt^2+dz^2H_K(dz-dt)^2\right] \\ &+\sqrt{\frac{H_1}{H_5}}\left[dy_1^2+dy_2^2+dy_3^2+dy_4^2\right]+\sqrt{H_1H_5}\left[\frac{1}{H_{KK}}(dx_4+\vec{A}d\vec{x})^2+H_{KK}d\vec{x}^2\right]. \end{aligned} \tag{4.5.7}$$

This can now be reduced over a six-torus to obtain a four-dimensional solution with four gauge fields [163]

$$\begin{aligned} ds^2 &= -\left(H_KH_1H_{KK}H_5\right)^{-\frac{1}{2}}dt^2+\left(H_KH_1H_{KK}H_5\right)^{\frac{1}{2}}\left(dr^2+r^2d\Omega_2^2\right), \\ F^1 &= d\left(H_1^{-1}\right)\wedge dt, \qquad F^3=(\varepsilon_{\mu\nu\rho}\partial_\rho H_5)dx^\mu\wedge dx^\nu, \\ F^2 &= d\left(H_K^{-1}\right)\wedge dt, \qquad F^4=(\varepsilon_{\mu\nu\rho}\partial_\rho H_{KK})dx^\mu\wedge dx^\nu \\ e^{2\sigma_1} &= \frac{H_K}{H_1}, \qquad e^{2\sigma_2}=\frac{H_5}{H_{KK}}, \qquad e^{-2\phi}=\sqrt{\frac{H_KH_5}{H_1H_{KK}}}, \end{aligned} \tag{4.5.8}$$

where we have two electric charges descending from the D1 brane and the PP wave, and two magnetic charges, descending from the D5 brane and the Kaluza-Klein monopole. The scalars σ_i correspond to the radius of the internal torus from the reduction from six to four dimensions. The area of the horizon is finite, and the near horizon geometry is found to be $AdS_2\times S^2$. This solution can be thought of as a generalisation of the extremal Reissner-Nordström solution with four distinct gauge fields.

We will not go through the details, but the same story can be played for the triple intersection of M5 branes (4.5.5). By wrapping over the internal space, one obtains a five-dimensional solution with three magnetic charges corresponding to the three M5 branes. Obtaining a black hole in four dimensions by compactifying over the z coordinate requires a stabilisation charge, which like the D1-D5 solution, is achieved through including PP-wave along common intersection [164]

$$\begin{aligned} ds^2_{5\times5\times5\times PP} =&(H_1H_2H_3)^{-\frac{1}{3}}\left[dudv+H_Kdu^2+H_1H_2H_3d\vec{x}^2\right. \\ &\left.+H_1(dy_1^2+dy_2^2)+H_2(dy_3^2+dy_4^2)+H_3(dy_5^2+dy_6^2)\right]. \end{aligned} \tag{4.5.9}$$

This can be reduced over a T^7 to obtain a line element in four dimensions [166]

$$ds_4^2=-\left(H_KH_1H_2H_3\right)^{-\frac{1}{2}}dt^2+\left(H_KH_1H_2H_3\right)^{\frac{1}{2}}\left(dr^2+r^2d\Omega_2^2\right), \tag{4.5.10}$$

which again will have four distinct gauge fields and scalar fields corresponding to the volume of the internal torus produced though compactification. These four-dimensional, extremal solutions obtained from the compactification of intersecting BPS solutions in ten and eleven dimensions are related to extremal solutions of the STU model [166]. In Chapter 7, we will show that when taking the extremal limit of the higher-dimensional, non-extremal solutions we will encounter these intersections again.

Part II

COSMOLOGICAL KILLING HORIZONS AND THEIR THERMODYNAMICS

5

COSMOLOGICAL SOLUTIONS OF EINSTEIN-MAXWELL THEORY

In this chapter, we begin our research into planar symmetric solutions by considering Einstein-Maxwell theory as a toy model. We find that the qualitative behaviour for the planar symmetric solutions of the STU model, which are the focus of Chapter 6, is also found for the simpler, Einstein-Maxwell model. This gives us the great opportunity to discuss some general properties of our cosmological solutions from the perspective of a model we are already familiar with.

The structure of this chapter is as follows. In Section 5.1, we solve the equations of motion, imposing that our solution should be planar symmetric and static. In Section 5.2, the causal structure of the solution is studied. In particular, we find that the static ansatz leads to a spacetime region of finite size, where the transverse coordinate is bounded between a singularity and the Killing horizon. Using Eddington-Finkelstein coordinates, we analytically continue through the horizon into a second region in which the metric is time-dependent. As this patch of the spacetime contains the asymptotic region, we call this region the exterior and refer to our solutions as *cosmological solutions*. We study the geodesic motion for null and timelike curves and show that with the exception of transverse null curves, all geodesics are effectively repelled by the singularity. In the static patch of spacetime, we compute the conserved charges and offer a discussion on possible mass-like parameters. In Section 5.3, we consider the global properties of a generalisation of these solutions. We do this in such a way that the discussion covers the solutions of both the Einstein-Maxwell theory and the STU model. For this generalised discussion, we construct Kruskal-like coordinates, and from this, we draw a Penrose-Carter diagram. We then classify the horizons of our cosmological solutions following the work in Section 3.1.5, and find that our horizons are past and future *inner horizons*. We conclude this chapter with a discussion of the extremal limit in Section 5.4. We find that for extremal, planar symmetric solutions of Einstein-Maxwell, the spacetime contains a naked singularity.

5.1 PLANAR SYMMETRIC SOLUTIONS

We begin this chapter finding planar symmetric, static solutions to the Einstein-Maxwell theory (3.3.1), whose action we repeat for convenience

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}(-R - F^2).$$

The most general metric ansatz we can make which matches our conditions is given by

$$ds^2 = -e^{2F(r)}dt^2 + e^{2H(r)}dr^2 + Y^2(r)(dx^2 + dy^2),$$

where the coordinates take values in

$$x, y, t \in \mathbb{R} \qquad r \in [0, \infty).$$

Our equations of motion are Einstein's equations:

$$R_{\mu\nu} = -8\pi T_{\mu\nu},$$

where we have used that the stress-energy tensor is traceless, and Maxwell's equations:

$$\nabla_\mu F^{\mu\nu} = 0, \qquad \nabla_{[\mu} F_{\nu\rho]} = 0. \tag{5.1.1}$$

Unlike the Reissner-Nordström solution studied in Section 3.3, we have explicitly assumed our spacetime is static. The main motivation for this is that the following equations simplify when the coordinate dependence is only on the transverse coordinate r. We will also assume staticity when solving the equations of motion for planar symmetric solutions of the STU model, and so this allows a similar starting point for both chapters.

As in Section 3.3, we first compute the stress-energy tensor through considering the form of the gauge fields. Matching our metric ansatz, we will assume the electric and magnetic fields spatially only depend on the transverse direction. We can write them down

$$E_r = F_{tr} = \alpha(t,r), \qquad B_r = \frac{2g_{rr}}{e^{(F+H)}r^2}F_{xy} \quad \Rightarrow \quad F_{xy} = r^2\beta(t,r),$$

we have chosen the extra factor of r^2 in the magnetic field for convenience. From (3.3.2), we can find the form of the stress energy tensor. First calculating,

$$F^2 = 2(F_{tr}F_{tr}g^{tt}g^{rr} + F_{xy}F_{xy}g^{xx}g^{yy}) = 2\left(\beta^2 - \alpha^2 e^{-2(F+H)}\right),$$

we find the non-zero components as

$$\begin{aligned}
T_{tt} &= \frac{1}{8\pi}\left(\alpha^2 e^{-2H} + \beta^2 e^{2F}\right), \\
T_{rr} &= -\frac{1}{8\pi}\left(\alpha^2 e^{-2F} + \beta^2 e^{2H}\right), \\
T_{xx} = T_{yy} &= \frac{r^2}{8\pi}\left(\alpha^2 e^{-2(H+F)} + \beta^2\right).
\end{aligned}$$

To solve Einstein's equations, we need the components of the Ricci tensor. Using (2.1.4), the non-zero Christoffel symbols are found to be

$$\begin{aligned}
&\Gamma^t{}_{tr} = \partial_r F, && \Gamma^r{}_{tt} = e^{2(F-H)}\partial_r F, && \Gamma^r{}_{xx} = \Gamma^r{}_{yy} = -e^{-2H}Y\partial_r Y, \\
&\Gamma^r{}_{rr} = \partial_r H, && \Gamma^x{}_{rx} = \Gamma^r{}_{ry} = \frac{1}{Y}\partial_r Y.
\end{aligned}$$

Substituting these values into (2.2.3), we obtain the Ricci tensor, all non-zero components are included below

$$
\begin{aligned}
R_{tt} &= e^{2(F-H)}\left\{\frac{\partial F}{\partial r}\left(\frac{\partial H}{\partial r}-\frac{\partial F}{\partial r}-\frac{2}{Y}\frac{\partial Y}{\partial r}\right)-\frac{\partial^2 F}{\partial r^2}\right\},\\
R_{rr} &= \frac{\partial F}{\partial r}\left(\frac{\partial F}{\partial r}-\frac{\partial H}{\partial r}\right)+\frac{\partial^2 F}{\partial r^2}-\frac{2}{Y}\left(\frac{\partial H}{\partial r}\frac{\partial Y}{\partial r}-\frac{\partial^2 Y}{\partial r^2}\right),\\
R_{xx} = R_{yy} &= e^{-2H}\left\{Y\left(\frac{\partial Y}{\partial r}\left(\frac{\partial F}{\partial r}-\frac{\partial H}{\partial r}\right)+\frac{\partial^2 Y}{\partial r^2}\right)+\left(\frac{\partial Y}{\partial r}\right)^2\right\}.
\end{aligned}
\tag{5.1.2}
$$

As with the spherically symmetric case, by combining the following terms

$$
\begin{aligned}
0 &= T_{rr}+e^{2(H-F)}T_{tt} = e^{2(H-F)}R_{tt}+R_{rr},\\
&= \frac{2}{Y}\left\{\frac{\partial Y}{\partial r}\left(\frac{\partial F}{\partial r}+\frac{\partial H}{\partial r}\right)-\frac{\partial^2 Y}{\partial r^2}\right\},
\end{aligned}
$$

we obtain a differential equation, which can be solved by picking

$$
F(r) = -H(r), \qquad \frac{\partial^2 Y}{\partial r^2} = 0, \quad \Rightarrow \quad Y(r) = Ar + B.
$$

The integration constants A, B can be absorbed through coordinate transformation: $\{t,r,x,y\} = \{t, r-B, A^{-1}x, A^{-1}y\}$. After making these choices, our line element is in the form

$$
ds^2 = -e^{2F(r)}dt^2 + e^{-2F(r)}dr^2 + r^2(dx^2+dy^2). \tag{5.1.3}
$$

From Maxwell's equations, we have that

$$
\begin{aligned}
\nabla_t(r^2F_{tr}) = 0 \quad &\Rightarrow \quad \alpha(t,r) = \alpha(r),\\
\nabla_r(r^2F_{tr}) = 0 \quad &\Rightarrow \quad \alpha(r) = -\frac{Q}{r^2},
\end{aligned}
$$

Similarly, the Bianchi identity gives:

$$
\begin{aligned}
\partial_t F_{xy} = 0 \quad &\Rightarrow \quad \beta(t,r) = \beta(r),\\
\partial_r F_{xy} = 0 \quad &\Rightarrow \quad \beta(r) = -\frac{P}{r^2}.
\end{aligned}
$$

The integration constants Q, P are set using Gauss' law, matching with our method Section 3.3. However, unlike the Reissner-Nordström solution, the codimension-two manifold has an infinite surface area, and so conserved charges associated with planar symmetric solutions will be divergent. To work with this, we consider charge densities per unit coordinate area. Alternatively, we could consider these the total charges contained in a two-torus after compactification of the plane.

With the exact form of the gauge field contribution, we can look at xx-component of Einstein's equations

$$
R_{xx} = -8\pi T_{xx} = -r^2\left(\frac{Q^2}{r^4}+\frac{P^2}{r^4}\right).
$$

The Ricci tensor component can be simplified from the form in (5.1.2) into a total derivative

$$
R_{xx} = \partial_r\left(re^{2F(r)}\right),
$$

to obtain the differential equation

$$\partial_r\left(re^{2F(r)}\right) = -\frac{Q^2+P^2}{r^2},$$

which can be integrated to

$$e^{2F(r)} = \frac{C}{r} + \frac{Q^2+P^2}{r^2}. \tag{5.1.4}$$

To ensure the presence of a Killing horizon for the timelike Killing vector $k^\mu = (\partial_t, 0, 0, 0)$, we pick our integration constant $C < 0$ such that $e^{2F(r)}$ has a zero for $r > 0$. We will later find a curvature singularity for $r = 0$ and so allowing $C > 0$, would remove the horizon and the resulting spacetime would contain a naked singularity. For comparison to the Reissner-Nordström solution, we can write $C = -2M$, for $M > 0$.

In summary, we have found a class of planar symmetric solutions to the Einstein-Maxwell theory, with a line element

$$\begin{aligned} ds^2 &= -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(dx^2+dy^2), \qquad & f(r) &= \left(-\frac{2M}{r} + \frac{e^2}{r^2}\right), \\ e &= \sqrt{Q^2+P^2}, & F &= -\frac{Q}{r^2}dt\wedge dr + Pdx\wedge dy, \end{aligned} \tag{5.1.5}$$

parameterised by the integration constants (M, Q, P). However, unlike the Reissner-Nordström solution, we cannot compare this line element to Newtonian physics by taking the weak field approximation. As a result, we cannot at this point derive a physical interpretation of the integration constant M. In the following sections, we will study this line element and its properties and attempt to find a way to properly interpret M from both a gravitational and thermodynamic perspective.

5.2 CAUSAL STRUCTURE

Studying the metric, we again are interested in the domain of validity of the coordinate system. Like the spherically symmetric solution, the Ricci scalar is zero, but we can compute the Kretschmann scalar[1] from the line element and we find

$$K = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{4\left(3e^2-2Mr\right)^2}{r^8}.$$

We see for $r = 0$, as with the spherically symmetric solutions, the Kretschmann scalar diverges indicating the presence of a singularity. Hypersurfaces of constant r will have normal covectors

$$n_\mu = (0, dr, 0, 0), \qquad n^2 = -f(r).$$

This shows that the surface for constant $r_h = e^2/2M$ will be a null hypersurface. The Killing vector $k^\mu = (\partial_t, 0, 0, 0)$ has a vanishing norm at r_h, and so we understand this null hypersurface as a Killing horizon.

During our integration, we explicitly set $M > 0$ to ensure the presence of a Killing horizon. We see that for this patch of spacetime, the function $f(r) > 0$, when the spacelike coordinate

[1] We used Mathematica for this computation, but SageMath [167] or even pen and paper (and patience) will work.

$0 < r < r_h$. This is in stark contrast to the spherically symmetric solutions, in which we solved the equations of motion and found ourselves in static regions of spacetime for $r_h < r < \infty$. We find that for these planar solutions to Einstein-Maxwell, imposing that the spacetime should be static, produces a finite static region containing a timelike singularity. We can understand this region as similar to the static region behind the Cauchy horizon $r = r_-$ from the Reissner-Nordström solution discussed previously (see regions II and IV in Figure 3.7).

We note here that we cannot simply say the 'asymptotic' region is found when taking the limit to $r \to \infty$. We will see in Section 6.3, that this naive limit for the planar solutions of the STU model is just some point in spacetime, which a null curve reaches in finite affine parameter. For the spherically symmetric solutions we discussed, the 'asymptotic limit' is defined by the limit in which the Minkowski solution is recovered. As this is not possible for our planar solutions, we instead define the asymptotic limit as the point which takes an infinite affine parameter for a null geodesic to reach. To perform this calculation, we will first need to perform a coordinate transformation to study the spacetime for $r > r_h$.

5.2.1 *Eddington-Finkelstein coordinates*

To better understand this solution, let us first make an Eddington-Finkelstein coordinate transformation. We do this in the now familiar way, defining a tortoise coordinate $r_\star$ and the ingoing and outgoing null coordinates v, u:

$$dr_\star = \frac{dr}{f(r)}, \qquad v = t + r_\star, \quad u = t - r_\star,$$

using ingoing coordinates $\{v, r, x, y\}$, we can write the line element

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2 d\vec{X}^2, \quad f(r) = \left(-\frac{2M}{r} + \frac{e^2}{r^2}\right). \tag{5.2.1}$$

Notice that now the coordinate range for the transverse coordinate covers all of $0 < r < \infty$. Crossing the horizon for $r = r_h$, the Killing vector ∂_v is spacelike for $r > r_h$. In this new region of spacetime, the transverse coordinate is timelike.

5.2.2 *Geodesic motion*

Using (5.2.1), we can study the geodesic motion within our spacetime for the full range of r. We begin by writing down the Lagrangian of our system

$$s = -f(r)\left(\frac{dv}{d\lambda}\right)^2 + 2\frac{dv}{d\lambda}\frac{dr}{d\lambda} + r^2\left(\frac{dx}{d\lambda}\right)^2 + r^2\left(\frac{dy}{d\lambda}\right)^2, \tag{5.2.2}$$

where $s = \{1, 0, -1\}$ for spacelike, null and timelike geodesics respectively and λ is our affine parameter. We compute the constants of motion

$$E = -k \cdot u = f(r)\frac{dv}{d\lambda} - \frac{dr}{d\lambda}, \qquad a = r^2\frac{dx}{d\lambda}, \qquad b = r^2\frac{dy}{d\lambda},$$

which allows us to rewrite the above equation into the form

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - V(r), \qquad V(r) = f(r)\left(-s + \frac{a^2 + b^2}{r^2}\right).$$

We have written this in a suggestive way, allowing us to view this as the equation of motion for a particle with mass $m = 2$, where we write $V(r)$ to make explicit the interpretation of this function as an effective potential.

First, let us check that $r \to \infty$ is a suitable limit to describe as the asymptotic region. We consider null, transverse geodesics in which $s = a = b = 0 \Rightarrow V(r) = 0$, and the above equation simplifies to

$$\frac{dr}{d\lambda} = \pm E \quad \Rightarrow \quad \lambda = \pm\frac{1}{E}\int dr = \pm\frac{1}{E} r\Big|_{r_0}^{\infty} .$$

We see that it will take infinite affine parameter to reach $r \to \infty$ from some finite point $r_0 > 0$, confirming that $r \to \infty$ is our asymptotic region. We will discuss the asymptotic structure further in Section 5.2.4.

With the above interpretation as the equation of motion of some massive particle, we can study the form of the potential. The domain of validity for the equation of motion is restricted by the inequality

$$V(r) \leq E^2.$$

The point at which $V(r_0) = E^2$ is interpreted as the classical turning point of the particle's trajectory. Studying the potential $V(r)$ for the domain of r in the static region, we can look at the paths of causal geodesics.

In Figure 6.4, we plot $V(r)$ and see that for the static region of our spacetime, the potential is everywhere positive and therefore repulsive. When decreasing r from the horizon towards the singularity, the potential monotonically increases until it diverges in the limit of the singularity. As such, we are guaranteed a unique solution for $V(r_0) = E^2$ within the static region, and hence the existence of a classical turning point.

There is one exception to this: the case when $s = a = b = 0$, specific to transverse null geodesics where the potential is everywhere zero. As a result, our spacetime is not geodesically complete, as transverse null rays will reach the singularity in a finite proper time.

We conclude that for non-zero potentials, a particle will arrive from $\mathcal{J}^-$ and necessarily fall through the horizon at $r = r_h$. The particle will then continue towards the singularity to a minimum distance from the singularity at r_0. At this point, it will be reflected and then continue off through the Killing horizon into a second dynamic spacetime, towards $\mathcal{J}^+$. The only causal geodesics which do not follow these trajectories are those for which $V(r) = 0$. These are precisely the transverse null geodesics which fall through the horizon from $\mathcal{J}^-$ and straight into the singularity. We can understand this turning point due to the repulsive potential generated by the singularity at $r = 0$.

Proper acceleration

Seeing that all timelike geodesics reach a classical turning point before reaching the singularity, it is interesting also to study the acceleration of massive particles at rest within the static patch of the spacetime.

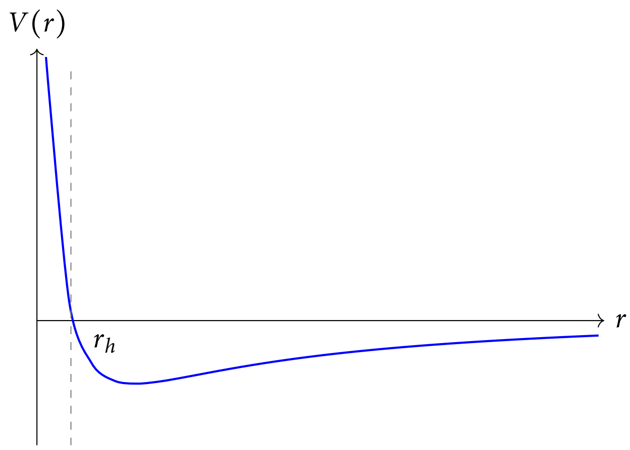


Figure 5.1: Behaviour of the effective 'potential' as a function of r for the set of causal geodesics excluding null transverse geodesics, for which $V = 0$. We see that in the static region, for $r < r_h$, the potential is repulsive.

Particles at rest follow orbits of the stationary Killing vector field k^μ with a proper velocity defined by

$$u^\mu = \frac{k^\mu}{\sqrt{-k^2}},$$

where the normalisation has been chosen such that $u^2 = -1$. From this, the proper acceleration can be found

$$A^\mu = u^\nu \nabla_\nu u^\mu = \frac{1}{2}\partial^\mu \log(-k^2). \tag{5.2.3}$$

From the metric (5.1.5), we can compute the norm of the Killing vector: $k^2 = -f(r)$ to find that

$$A^\mu = \frac{1}{2} g^{\mu\nu}\partial_\nu \log\left(f(r)\right). \tag{5.2.4}$$

The only non-zero component of the acceleration is in the transverse direction, and we find

$$A^r = \frac{1}{2} f(r)\partial_r \log\left(f(r)\right) = \frac{1}{2}\partial_r f(r).$$

Taking the derivative

$$A^r = \frac{2M}{r^2} - \frac{2e^2}{r^3}. \tag{5.2.5}$$

As the static region is bounded for $0 < r < r_h$, we see that $A^r < 0$ throughout the static patch of the solution. As such, a particle at rest always experiences a force repelling it from the singularity, which matches with our interpretation of a timelike geodesic experiencing repulsion. We remark that the behaviour of geodesics and Killing orbits in our planar symmetric solution is qualitatively the same as for the interior region of the non-extremal Reissner-Nordström solution behind the Cauchy horizon for $r = r_-$ (regions II and IV in Figure 3.7). Moreover, in both cases, this static interior region is bounded by horizons both in the past and in the future to regions where the Killing vector field becomes spacelike.

5.2.3 *Conserved charges*

Electromagnetic charges

From the line element (5.1.5), we can compute the conserved charges associated to the Maxwell gauge field. Using the relationship (3.3.13) we find

$$\begin{aligned}\mathcal{Q} &= \frac{1}{4\pi}\int_{\mathbb{R}^2}\star F,\\ &= \frac{1}{4\pi}\int_{\mathbb{R}^2}\sqrt{-g}\epsilon_{trxy}g^{tt}g^{rr}F_{tr}dx\wedge dy,\\ &= \frac{1}{4\pi}\int_{\mathbb{R}^2}Q dx\wedge dy.\end{aligned}$$

The integral over the plane $\mathbb{R}^2$ is divergent. For simplicity in our computations, we parameterise it as

$$\mathcal{Q} = \frac{Q\omega}{4\pi}, \qquad \omega = \int_{\mathbb{R}^2} dx\wedge dy. \tag{5.2.6}$$

For the following discussion, we allow ω to remain explicit in our expressions. However, for the solutions of the STU model presented in Chapter 6, where there are many more integration constants, we instead consider charge densities by setting $\omega = 1$. Alternatively, we could compactify the plane as a torus, and compute a finite charge by integrating over T^2 instead. The conserved magnetic charge is computed in a similar way

$$\begin{aligned}\mathcal{P} &= \frac{1}{4\pi}\int_{\mathbb{R}^2}F,\\ &= \frac{1}{4\pi}\int_{\mathbb{R}^2}\sqrt{-g}F_{xy}dx\wedge dy,\\ &= \frac{P\omega}{4\pi}.\end{aligned}$$

Mass

As discussed in Section 3.4, computing the mass in general relativity requires some extra structure. When the solution is static and asymptotically flat, we have a wealth of options which all agree. The Komar energy requires only that the solution is static, but to interpret this as a mass-like quantity, one must have a suitable asymptotic fall off for the solution. The alternative computation comes from the Brown-York mass, which can be defined quasi-locally, but suffers from the necessity to pick the correct background to normalise the result.

For the planar symmetric solution of Einstein-Maxwell theory, our static region is of finite size. As such, we cannot take an asymptotic limit in the static patch. However, we can still look at mass-like quantities which are position-dependent, following the work of [168] in which similar solutions were analysed. In this section, we will compute both a position-dependent Komar mass following Section 3.4.1 and a Brown-York quasi-local mass following Section 3.4.2.

The Komar energy is found from (3.4.1), repeated here

$$E_K = -\frac{1}{8\pi}\int_{\mathbb{R}^2}\star dk.$$

In the static region, where $r < r_h$, the timelike Killing vector field is $k^\mu = \partial/\partial t$. We can compute the exterior derivative and its Hodge-star

$$dk = \partial_r f(r) dt \wedge dr, \qquad \star dk = -\sqrt{-g}\partial_r f(r) dx \wedge dy,$$

and so the energy is given by

$$\begin{aligned} E_K &= \frac{1}{8\pi}\int_{\mathbb{R}^2} r^2 \partial_r \left(-\frac{2M}{r} + \frac{e^2}{r^2}\right) dx \wedge dy, \\ &= \frac{\omega}{4\pi}\left(M - \frac{e^2}{r}\right). \end{aligned}$$

Looking at the above result, it is tempting to say that the Komar mass is found by taking the limit $r \to \infty$

$$M_K = \lim_{r\to\infty} E_K = \frac{\omega M}{4\pi}, \tag{5.2.7}$$

which when compared to the charges above, looks like the expected value compared to the Reissner-Nordström solution, and the various factors of 4π could be removed by allowing $\omega_{S^2} = 4\pi$ to match the integral over the sphere. However, taking this limit, we would we find ourselves in a region where k^μ is not a timelike Killing vector field, and the original arguments we make for this to be a mass-like parameter are gone. As a result, we have no physical motivation to understand $\lim_{r\to\infty} E_K$ as a mass-like parameter. We could alternatively think of this as the Noether charge associated to spacelike isometries, and so a form of momentum. However, we do not follow this line of reasoning much further than noting it as an interesting perspective.

In Section 8.3, we will use the Euclidean action formalism to derive the thermodynamic internal energy for this solution, and we find that interestingly this calculation matches the naive limit taken above. However, for now, we remain within the static region and save further discussion for later. Notice that for $r < r_h$, the quantity E_K is negative. From this, we can relate the realisation of the position-dependent mass-like parameter being negative to the 'repulsive' behaviour of the singularity. However, performing a similar computation in the static region behind the Cauchy horizon for the Reissner-Nordström solution produces a negative, position-dependent mass. As such, we do not conclude that these static solutions are 'negative mass solutions' but rather realise one might say the Reissner-Nordström solution has an effective negative mass in a particular region.

We can also compute the Brown-York energy using (3.4.3), repeated here

$$E_{BY} = -\frac{1}{8\pi}\int_{\mathbb{R}^2} \sqrt{\sigma}(\mathtt{k} - \mathtt{k}_0).$$

Looking at the trace of the extrinsic curvature, we see that like the Reissner-Nordström solution, it is of the form

$$\mathtt{k} = \sigma^{\mu\nu}\mathtt{k}_{\mu\nu} = \frac{2\sqrt{f(r)}}{r},$$

Ignoring the normalisation term $\mathtt{k}_0$, we can write down the Brown-York energy as

$$\begin{aligned} E_{BY} &= -\frac{1}{4\pi}\int_{\mathbb{R}^2} r\left(-\frac{2M}{r} + \frac{e^2}{r^2}\right)^{\frac{1}{2}}, \\ &= -\frac{\omega}{4\pi} r\left(-\frac{2M}{r} + \frac{e^2}{r^2}\right)^{\frac{1}{2}}. \end{aligned}$$

For the static domain, we have $r < r_h$ and hence $E_{BY} < 0$. However, unlike the Komar energy, taking the limit of $r \to \infty$, we find that M_{BY} is imaginary and so not well defined. This is not too surprising, as we only have a good description for E_{BY} when the spacetime is stationary.

Interestingly, we note a slight variation of this story by instead considering the quasi-local definition for the mass developed by Katz, Lynden, Bell and Israel [69]. The Katz-Lynden-Bell-Israel energy differs from the Brown-York quasi-local energy through the introduction of the lapse function $N = \sqrt{f(r)}$ into the expression

$$E_{KLBI} = -\frac{1}{8\pi}\int_{\mathbb{R}^2} \sqrt{\sigma} N(\mathtt{k} - \mathtt{k}_0).$$

In this form, we can compute the energy and find that it is given by

$$\begin{aligned} E_{KLBI} &= -\frac{1}{8\pi}\int_{\mathbb{R}^2} r^2 \frac{2f(r)}{r}, \\ &= -\frac{\omega}{4\pi}\left(-2M + \frac{e^2}{r}\right), \end{aligned}$$

and taking the limit, evaluating the boundary at $r \to \infty$, we find that the energy is real and finite

$$M_{KLBI} = \lim_{r\to\infty} E_{KLBI} = \frac{M\omega}{4\pi}.$$

However, just like taking the limit for the Komar mass, this moves us into a region where the spacetime is no longer stationary, and the interpretation of the limit as a mass-like quantity is no longer valid. Notice also that like the Komar energy, and the Brown-York energy the Katz-Lynden-Bell-Israel energy is everywhere negative for $r < r_h$.

We note here that we have not included the background contribution associated with $\mathtt{k}_0$. In standard treatments, $\mathtt{k}_0$ is computed by considering the background geometry of the solution. The planar symmetry of our solution means we have no maximally symmetric spacetime asymptotically, and so no natural background geometry to pick. Alternatively, $\mathtt{k}_0$ can be introduced as a counter-term to remove divergences in the energy when considering the asymptotic limit. However, as we have seen, the asymptotic contribution for our solution is finite and so we have no natural choice for a counter-term. We will comment on this again with slightly more detail in Chapter 6 in the context of the planar solution of the STU model. To have confidence in a derived mass-like parameter, we wait until Section 8.3 where we have the additional structure of a thermodynamic potential to calculate the internal energy, which we derive from the thermodynamic partition function obtained through a modified implementation of the Euclidean action formalism.

5.2.4 *The dynamic region*

From the line element (5.2.1), we can study the spacetime for which $r > r_h$. Reversing the coordinate transformation, we can write a line element using the coordinates $\{t, r, x, y\}$, valid for $r > r_h$. However, in this region, we have $f(r) < 0$, and the coordinates t, r are spacelike and timelike respectively. For a clearer discussion for the remainder of this chapter, we relabel the coordinates $t \leftrightarrow r$, such that we use the symbol t for the timelike coordinate. In doing this, we are labelling our coordinates with regard to the external region of the spacetime. In this form, we have a line element given by

$$ds^2 = -\frac{dt^2}{\tilde{f}(t)} + \tilde{f}(t)dr^2 + t^2 d\vec{X}^2, \quad \tilde{f}(t) = \left(\frac{2M}{t} - \frac{e^2}{t^2}\right), \tag{5.2.8}$$

where $\tilde{f}(x) = -f(x)$ and the Killing horizon is located for $t = t_h$ such that $\tilde{f}(t_h) = 0$. The line element is valid for $t_h < t < \infty$.

We see from (5.2.8) that the only coordinate dependence of the line element is from the timelike coordinate t, and so we understand this as a time-dependent, or *cosmological* solution. The singularity for $t = 0$ is hidden behind a Killing horizon. The horizon itself is a place in time, like the Cauchy horizon for $r = r_-$ in the Reissner-Nordström solution, which all timelike and null geodesics will inevitably cross. We see that from the static ansatz we began with, the derived planar symmetric solution of the Einstein-Maxwell theory is a cosmological solution containing a Killing horizon. For the remainder of the discussion, we drop the tilde on the function $f(t)$.

We can evaluate the surface gravity of the Killing horizon using the Kodama-Hayward formulation (3.5.4) to find

$$\kappa = -\frac{1}{2}\partial_t f(t_h) = -\frac{4M^3}{e^4}, \tag{5.2.9}$$

which we see is negative. In Section 5.3.1, we will study the Kruskal coordinates and the expansions of null geodesics. This will allow for the classification of the trapping horizons for cosmological, planar symmetric solutions we discuss.

Evaluating (5.2.8) in the asymptotic limit of $t \to \infty$, we obtain the line element

$$ds^2 = -\frac{t}{2M}dt^2 + \frac{2M}{t}dr^2 + t^2 d\vec{X}^2 , \tag{5.2.10}$$

which is the 'positive mass' version of the planar (type-D) A-III vacuum solution of pure Einstein gravity [169]. Introducing a new timelike coordinate $t \propto \tau^{2/3}$ and absorbing numerical factors by rescaling (r, x, y), we can write the line element into the form

$$ds^2 = -d\tau^2 + \tau^{-2/3}dr^2 + \tau^{4/3}d\vec{X}^2 , \tag{5.2.11}$$

which belongs to the class of type-D Kasner solutions [51]. These are the simplest homogeneous but anisotropic vacuum cosmological solutions of pure Einstein gravity. The A-III/Kasner solution is defined for $0 < t, \tau < \infty$ and describes a universe starting in a big bang at $t = \tau = 0$, then expanding in the (x, y)-directions while contracting in the transverse direction r. The Penrose-Carter diagram for the Kasner solution is drawn in Figure 5.2. Its time-reversed version, given by

$$\begin{aligned} ds^2 &= \frac{t}{2M}dt^2 - \frac{2M}{t}dr^2 + t^2 d\vec{X}^2 , \qquad -\infty < t < 0 , \\ &= -d\tau^2 + \tau^{-2/3}dr^2 + \tau^{4/3}d\vec{X}^2 , \qquad -\infty < \tau < 0 , \end{aligned}$$

describes a universe which contracts in the (x, y) directions, expands transversally, and ends in a big crunch at $t = \tau = 0$. We then see that the planar solutions describes a bouncing cosmology which interpolates between a contracting and an expanding Kasner cosmology. The presence of the Killing horizon removes the spacelike big crunch and big bang singularities at $t = \tau = 0$

and instead introduces an intermediate region containing two timelike singularities. These singularities can be interpreted as sources, and later in Chapter 7, we can embed Einstein-Maxwell theory into the STU-supergravity and subsequently into string theory, allowing these source to be identified as brane configurations.

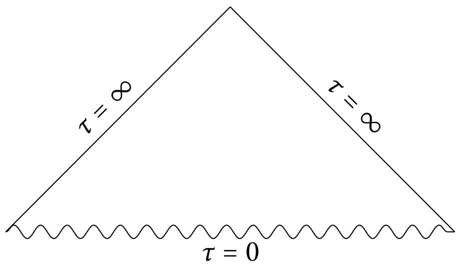


Figure 5.2: Penrose-Carter diagram for the type-D Kasner solution

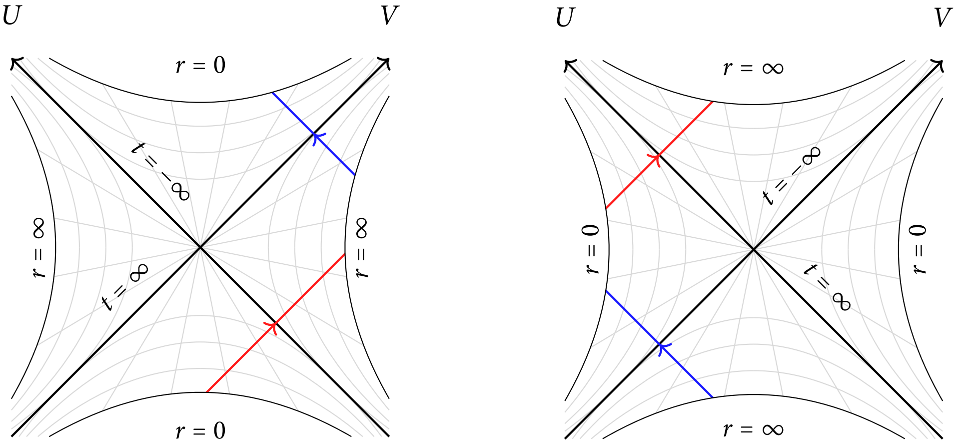
5.3 GLOBAL STRUCTURE OF PLANAR COSMOLOGICAL SOLUTIONS

In this section, we perform Kruskal-like coordinate transformations which allow us to construct the maximal analytic extensions of the planar symmetric solutions of Einstein-Maxwell. In fact, we will take this opportunity to discuss time-dependent metrics of the form (5.2.8), but leave the form of the function $f(t)$ general. This will allow us to reuse the results from this discussion when we consider the planar symmetric solutions of the STU model, as well as the de Sitter solution in static coordinates, which we introduced in Section 2.3.2.1 and will revisit as an example in a thermodynamic computation in Section 8.2.

Obtaining Kruskal-like coordinate systems for these solutions will allow us to understand their global causal structure and to identify the type of all horizons using the classification of trapping horizons reviewed in Section 3.1.5. We will find throughout this section only very small departures from the calculations we performed in Section 3.1.5 for the Schwarzschild solution, and so we begin by comparing black hole solutions with a single Killing horizon (*e.g.* the Schwarzschild solution) to cosmological solutions with a single Killing horizon (*e.g.* the planar symmetric solutions to Einstein-Maxwell).

The solutions we discuss fall into two categories which are distinguished by the causal relationship between their interior and exterior regions. For all solutions, we call regions *exterior* if transverse/radial null geodesics reach a horizon in one direction, but can be extended to infinite affine parameter in the other direction. In terms of our standard transverse/radial coordinate, the asymptotic region is at $r \to \infty$. In contrast, regions are called *interior* regions if transverse/radial null geodesics terminate at a curvature singularity in one direction and reach a horizon in the other. We refer to solutions with a static exterior region as *black hole solutions* and those with a time-dependent exteriors as *cosmological solutions*.

To properly compare horizons between these solutions, we must set an orientation. We will consider the horizons crossed by future-directed null geodesics which pass from the exterior to the interior region. We pick this convention as this is the horizon one considers for ingoing null curves in the Schwarzschild solution, which is crossed by geodesics which start from $\mathscr{J}^-$ and



(a) Black hole solutions

(b) Cosmological solutions

Figure 5.3: Kruskal diagrams for black hole and cosmological solutions. Surfaces of constant r are hyperbola and surfaces of constant t are straight lines. Also included are the ingoing (blue) and outgoing (red) null geodesics which are future-pointing.

terminate at the singularity. We will use the static line elements to define ingoing and outgoing null geodesics, fixing the global time orientation of the maximally extended spacetime. Fixing the orientation is important as the extension contains two isometric static regions, where the line element takes the same form in terms of coordinates (t, r), but where the timelike Killing vector field ∂_t is future-directed in one region, but past-directed in the other (where future-directed is defined globally by picking one of the patches to fix the time orientation).

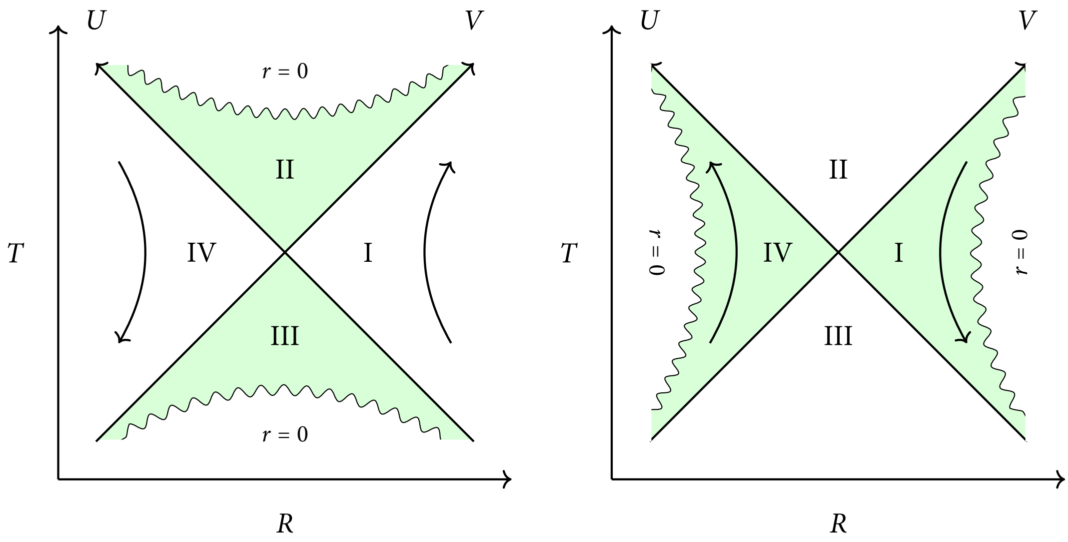
Starting from a 'standard static patch', chosen to fix the definition of ingoing/outgoing and hence the direction of time, we define Kruskal coordinates and obtain a maximally extended spacetime containing four distinct regions. By computing the expansions of null geodesic congruences for each region, we can identify the form of the horizons separating them. Within the context of the thermodynamics we consider in Chapter 8, we consider future horizons, where the exterior region can causally influence the interior, but not vice versa. The horizons between such regions are *future outer horizons* for black hole solutions and *future inner horizons* for cosmological solutions. For the thermodynamic formalism based on the Euclidean action that we use in Section 8.1, we assume that temperature and surface gravity are related according to [57, 58], that is, they are proportional. We then find that black holes have positive temperature, while (contracting) cosmologies have negative temperature.

5.3.1 *Kruskal coordinates*

We begin by considering the static patch of cosmological solutions which have a line element of the form

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\vec{X}^2, \tag{5.3.1}$$

as we saw in Equation (5.1.5). Later, in Section 6.4, we will find the STU model has metric of the same form and in Section 8.2, we will write down the de Sitter solution in static coordinates which is only different in that the two-dimensional line element $d\vec{X}^2$ is instead the metric on S^2:



(a) Black hole solutions (b) Cosmological solutions

Figure 5.4: Comparison of the Kruskal diagrams for black hole and cosmological solutions. Shaded regions correspond to the interior regions, curved lines show the direction of the Killing vector field in the static patches of the spacetime.

$d\Omega^2$. Note that the function $f(r)$ is positive for $r_{\text{sing}} < r < r_h$, and r_{sing} is the position of the singularity. The domain of r is such that this is the *interior* region of the solution. We assume that $f(r)$ has a simple zero and therefore changes sign at $r = r_h$. Since the Killing vector field ∂_t becomes spacelike for $r > r_h$, the outside region is dynamical. We assume that $f(r)$ is negative for $r_h < r < \infty$, with $r \to \infty$ at infinite distance. Thus the horizon at $r = r_h$ is a cosmological horizon. Under the conditions we have imposed on $f(r)$, the surface gravity is *negative*

$$\kappa = \frac{1}{2}\partial_r f(r)\bigg|_{r=r_h} < 0.$$

As we keep the function general, it will be useful later to use its Taylor expansion near the horizon:

$$f(r) = 2\kappa(r - r_h) + \mathcal{O}((r - r_h)^2)\ . \qquad (5.3.2)$$

First defining the tortoise coordinate

$$r_\star = \int f(r)^{-1} dr, \qquad 0 < r_\star < \infty\ ,$$

we introduce future-pointing null coordinates

$$v = t + r_\star, \qquad u = t - r_\star, \qquad -\infty < u, v < \infty\ , \quad r(v,u) > r_{\text{sing}}\ .$$

Changing to null coordinates, the line element takes the form

$$ds^2 = -f(r)dvdu + r^2 d\vec{X}^2\ ,$$

where r is an implicitly defined function of $r(u,v)$. Note that v is future- and outward-pointing while u is future- and inward-pointing in the interior region. This is the same assignment as in black hole solutions considered in Section 3.1.5. With coordinates fixed in this way, we can clearly see what is the difference compared to the static patch of the black hole solutions. Since r points in the opposite direction, the roles of interior and exterior are exchanged, where interior means $r < r_h$. While v points outwards in both cases, it points away from the horizon for the black hole solutions, but towards the horizon for the cosmological ones. This makes it natural to define Kruskal coordinates such that the standard static region, which we use to fix the overall time orientation, is Region IV, rather than Region I.

With the region set, we can construct Kruskal like coordinates. We start with the static line element, rewritten using null coordinates (u, v), where v is future-pointing and outward-pointing, while u is future-pointing and inward-pointing, relative to the local coordinates (t, r). This fixes the definitions of the expansion $\theta_\pm$, and the direction of physical time. We define global null Kruskal coordinates (U, V) such that they point in the same direction as (u, v):

$$\begin{aligned} V &= -e^{\kappa v} \qquad & -\infty < V < 0 \qquad &\Leftrightarrow \quad -\infty < v < \infty \, , \\ U &= e^{-\kappa u} \qquad & 0 < U < \infty \qquad &\Leftrightarrow \quad -\infty < u < \infty \, , \end{aligned}$$

where the factors of the surface gravity have been included to make manifest that the metric is regular at $r = r_h$ where $f(r)$ has its zero. We have also used that $\kappa < 0$. The standard static patch is Region IV, and it is illustrative to compare the black hole case and the cosmological case in Figure 5.3 and Figure 5.4.

The line element in Kruskal coordinates is given by

$$ds^2 = -\frac{f(r)e^{-2\kappa r_\star}}{\kappa^2} dV dU + r^2 d\vec{X}^2 \, .$$

We can show this is regular on the horizon using the Taylor expansion (5.3.2)

$$-2\kappa r_\star = \int \left(-\frac{2\kappa}{2\kappa(r - r_h)} + \mathcal{O}(r - r_h) \right) dr = -\log(r_h - r) + \mathcal{O}((r - r_h)^2) \, ,$$

where we note that the choice $\log(r_h - r)$ is made as in the static patch we have $r < r_h$. Putting this together, we obtain

$$-f(r)e^{-2\kappa r_\star} = -2\kappa(r - r_h)e^{-\log(r_h - r)} = 2\kappa + \mathcal{O}((r - r_h)^2) \, ,$$

such that on the horizon, the line element is given by

$$ds^2 = \frac{2}{\kappa} dV dU + r_h^2 d\vec{X}^2 \, .$$

In this form, we see that we can extend the Kruskal null coordinates such that

$$-\infty < U, V < \infty \, ,$$

subject to the constraint that $r(U, V) > r_{\text{sing}}$, where we implicitly write

$$UV = -e^{2\kappa r_\star} \, , \qquad \frac{V}{U} = -e^{2\kappa t} \, .$$

The direction of various coordinates in the respective regions is shown in Figure 5.5.

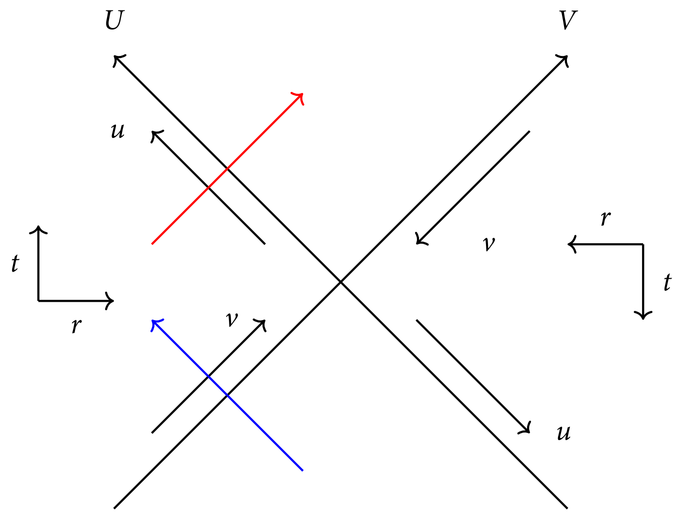


Figure 5.5: Flow of coordinates in the static regions of the Kruskal diagram for cosmological solutions. The red arrow denotes future-directed outgoing null geodesics, the blue arrow denotes future-directed ingoing null geodesics

5.3.2 *Classification of horizons*

Before we begin, let us precalculate a few useful formulae relating our static coordinates to our Kruskal coordinates

$$dr = \frac{f(r)}{VU}\frac{1}{2\kappa}\left(VdU + UdV\right) \,, \tag{5.3.3}$$

$$dt = -\frac{1}{VU}\frac{1}{2\kappa}\left(VdU - UdV\right) \,. \tag{5.3.4}$$

To calculate the expansions, we start with the Killing vector field which we can write down in static coordinates as $k = \frac{\partial}{\partial t}$. Using the metric tensor, we write down the covector field $k^\flat = -f(r)dt$. From the results above (5.3.4), we can rewrite the Killing covector field in terms of the Kruskal-like coordinates

$$k^\flat = -f(r)dt = \frac{f(r)}{VU}\frac{1}{2\kappa}\left(VdU - UdV\right),$$

with the corresponding vector given by

$$k = \kappa\left(-U\frac{\partial}{\partial U} + V\frac{\partial}{\partial V}\right).$$

We now write down the geodesics which are future-pointing within region IV, the standard static region according to our conventions. We pick the normalisation of the covectors to ensure that they are future pointing and pick an overall normalisation for clean results

$$\ell_+^\flat = \frac{1}{\kappa}dU, \qquad \ell_-^\flat = \frac{1}{\kappa}dV \,, \tag{5.3.5}$$

or as vectors

$$\ell_+ = 2\kappa\frac{VU}{f(r)}\left(\frac{\partial}{\partial V}\right), \qquad \ell_- = 2\kappa\frac{VU}{f(r)}\left(\frac{\partial}{\partial U}\right) \,.$$

We can double check that the normals are future-pointing by computing the inner product of these with the Killing vector field

$$k \cdot \ell_+ = -U, \qquad k \cdot \ell_- = V\,,$$

which we see obeys $k \cdot \ell_\pm < 0$ in region IV, where $U > 0$ and $V < 0$.

Their expansions are calculated following Section 3.1.3, using (3.1.4)

$$\theta_\pm = \nabla_\mu \ell^\mu_\pm = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\ell^\mu_\pm\right), \qquad \sqrt{-g} = -\frac{f(r)\sqrt{h}}{VU}\frac{r^2}{2\kappa^2}\,,$$

where we use that $h = \det\left(r^2 d\vec{X}^2\right)$ for either the two-sphere or the two-plane depending on the symmetry of the solution. This results in

$$\theta_+ = -\frac{4\kappa}{r}\frac{VU}{f(r)}\frac{\partial r}{\partial V}, \qquad \theta_- = -\frac{4\kappa}{r}\frac{VU}{f(r)}\frac{\partial r}{\partial U}.$$

To calculate the sign, we use (5.3.3) to find that

$$\frac{\partial r}{\partial V} = \frac{f(r)}{VU}\frac{U}{2\kappa}, \qquad \frac{\partial r}{\partial U} = \frac{f(r)}{VU}\frac{V}{2\kappa},$$

such that the expression for the expansions simplifies to

$$\theta_+ = \frac{2U}{r}, \qquad \theta_- = \frac{2V}{r}.$$

While this already determines the types of all horizons in virtue of knowing the signs for all four regions, we also calculate the Lie derivative at the horizon explicitly. We find that

$$\mathcal{L}_{\ell_-}\theta_+ = 2\kappa\frac{VU}{f(r)}\partial_-\left(\frac{2U}{r}\right) = \frac{4\kappa}{r}\left(\frac{VU}{f(r)}\right) - \frac{2VU}{r^2},$$

and on the horizon, we find

$$\mathcal{L}_{\ell_-}\theta_+\bigg|_{r=r_h} = \frac{4\kappa}{r_h}\cdot\frac{1}{2\kappa} = \frac{2}{r_h}, \qquad \mathcal{L}_{\ell_+}\theta_-\bigg|_{r=r_h} = \frac{2}{r_h}.$$

Let us look at the left part of the Kruskal diagram, that is, regions III, IV and II (see Figure 5.4b). The physics of the sequence III, I, IV is equivalent but parametrised differently since ∂_t is past-pointing in region I. In region IV of the Kruskal diagram, we have $U > 0$ and $V < 0$. On the horizon given by $V = 0$, we have:

$$\theta_+ > 0, \quad \theta_- = 0, \quad \mathcal{L}_{\ell_+}\theta_- > 0,$$

which shows that this is a *past inner horizon*. For the horizon set by $U = 0$, we have that

$$\theta_+ = 0, \quad \theta_- < 0, \quad \mathcal{L}_{\ell_-}\theta_+ > 0,$$

which is a *future inner horizon*. The expansions for all four regions is illustrated in Figure 5.6.

When considering future-directed causal geodesics which pass through a horizon from the exterior to the interior, we must consider the future inner horizon between region III and region IV. For a future inner horizon, we have that $T_H \propto \kappa < 0$. Later, in Chapter 8, we will see that this matches with the geometric interpretation of the temperature when we study the first law for the cosmological solutions of this thesis. Overall, the sequence III, IV (or I), II, describes a cosmic bounce, since the solution is a contracting cosmology in III and an expanding cosmology in II.

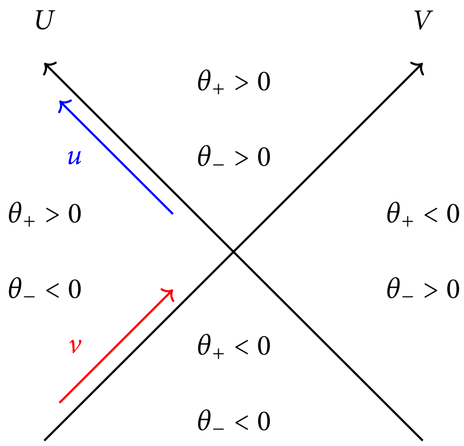


Figure 5.6: Signs of the expansions $\theta_\pm$ in the four quadrants of the Kruskal diagram for a cosmological solution where $\kappa < 0$.

5.3.3 *Penrose-Carter diagram*

From the form of the Kruskal diagram given in Figure 5.3b, and the Penrose-Carter diagram for the Kasner solution in Figure 5.2, we can draw the Penrose-Carter diagram for the planar solutions of Einstein-Maxwell. In fact, we will see that this diagram is suitable also for the class of planar symmetric solutions we discuss in Chapter 6, but more on that later.

The diagram has four regions. Regions II and III are the dynamic regions of the spacetime, with the Kasner spacetime located in the asymptotic limit $t \to \infty$. A Killing horizon is located at a point in time: t_h. All timelike and null geodesics will cross the horizon. Regions I and IV are the static regions, which have a finite size, bounded between $0 < t < t_h$. All causal geodesics, with the exception of null, transverse geodesics, travel to a classical turning point $t_0 > 0$, after which the curve moves with increasing t, until it crosses the horizon again. Null transverse geodesics are terminated with finite length once they reach the singularity. After crossing the horizon, all geodesics inevitably travel to a Kasner-like solution for $t \to \infty$.

5.4 EXTREMAL LIMIT

Let us briefly discuss the extremal limit for this class of planar solutions. We found in (5.2.9) that the surface gravity is negative and equal to

$$\kappa = -\frac{4M^3}{e^4}.$$

In the spherically symmetric solutions, we found that $\kappa \to 0$ when we allowed the mass and charge to be equal: $M \to e$. For this solution, we see that in this limit the surface gravity will remain finite. Instead, by taking the limit $M \to 0$, the surface gravity vanishes and the line element for this solution is given by

$$ds^2 = \frac{t^2}{e^2}dt^2 - \frac{e^2}{t^2}dr^2 + t^2\left(dx^2 + dy^2\right).$$

The resulting spacetime has t, r spacelike and timelike respectively, and so we see the extremal limit produces a static solution with a naked, timelike singularity located at $t = 0$.

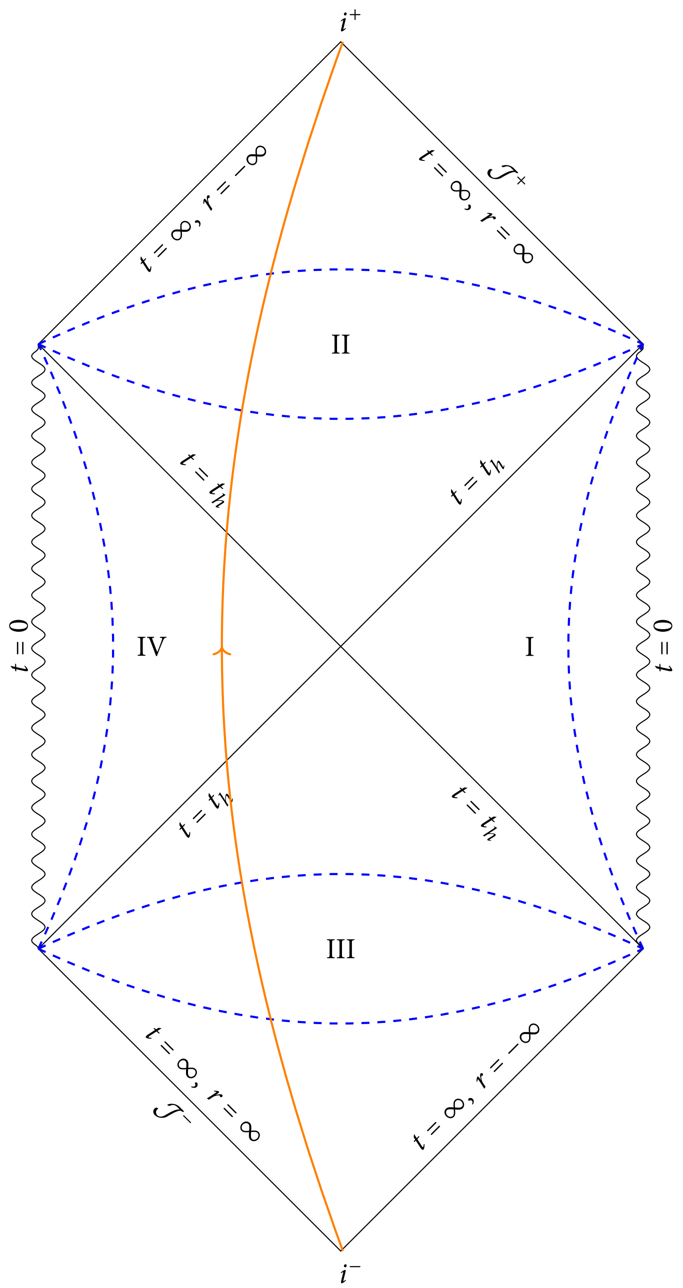


Figure 5.7: Penrose-Carter diagram for our planar symmetric, cosmological solutions. Starting in region III, we have a cosmological spacetime with a horizon located at a finite point in time; any observer must necessarily fall through the horizon. Passing through the horizon, the spacetime is static (region IV) with an avoidable (repulsive) naked singularity located at a point in space. Massive particles at rest experience negative acceleration and will leave the static region into a second dynamic spacetime, region II. An example of a complete timelike geodesic is given in orange, spacelike hypersurfaces of constant time are given in blue.

For the non-extremal solution, the horizon was located for

$$t_h = \frac{e^2}{2M},$$

and so we can understand the limit of $M \to 0$ as pushing the horizon location off to infinity. This results in changing the causal structure such that the dynamic region is lost and only the static patch of spacetime remains, with the singularity left exposed. In terms of catchy phrases for a presentation, one could say that the extremal limit undresses the solution.

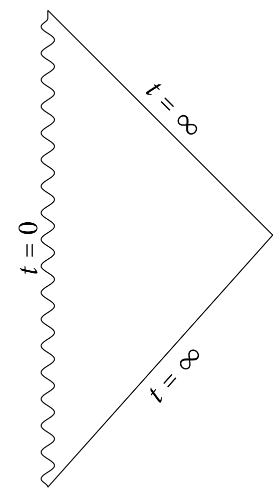


Figure 5.8: Penrose-Carter diagram for extremal planar symmetric solution of the Einstein-Maxwell theory.

Relabelling coordinates such that $t \leftrightarrow r$ and absorbing the constant e, we can write the extremal line element into the form

$$ds^2 = -\frac{dt^2}{r^2} + r^2\left(dr^2 + dx^2 + dy^2\right).$$

5.5 DISCUSSION

In this chapter, we began our study on planar symmetric solutions of general relativity by considering Einstein-Maxwell theory. This is a particularly interesting starting point, as not only does it allow us to directly compare our solutions with the well known Reissner-Nordström solutions, we will also see in Section 6.6, that we recover the Einstein-Maxwell Lagrangian through enforcing that the scalar fields are constant for the $\mathcal{N} = 2$ supergravity solutions we consider in the next chapter. This allows for a comparison between the simpler solution discussed in this chapter to the more complicated supergravity solution which follows.

When solving the equations of motion, we assumed that the spacetime was both static and planar symmetric. We saw in Equation (5.1.4) that the resulting spacetime metric contained either a Killing horizon or a naked singularity, depending on our choice of sign for the integration constant C. Unlike spherically symmetric solutions, we cannot compare our asymptotic geometry to the Newtonian limit and as such we set the sign by hand, ensuring the presence of a Killing horizon by picking $C < 0$. This is beneficial in that it avoids violating cosmic censorship, but it is also in our interest as we wish to better understand the structure of planar symmetric Killing horizons.

The surprising result of these planar symmetric solutions is that despite enforcing that the solution was static, the resulting spacetime metric describes a finite region of spacetime in which the transverse coordinate is bounded between a curvature singularity and the Killing horizon. Analytically continuing through the horizon, we reach a new patch of spacetime which is time-dependent with an asymptotic geometry which is that of the type-D Kasner spacetime. We refer to this region as *exterior* as it contains the asymptotic region, and so we understand the planar symmetric solution as a cosmological solution, and the Killing horizon as a cosmological horizon.

Studying the static region, we saw that the spacetime is geodesically complete for timelike geodesics, which instead of reaching the singularity, reach some minimum distance before being repelled. The only geodesics which reach the singularity in finite affine parameter are null, transverse geodesics. The repulsive interpretation of the singularity within the static region is further justified as massive particles at rest experience a negative proper acceleration, and the position-dependent mass quantities are negative. We can compute the surface gravity of the horizon, which we find is negative and vanishes in the limit $M \to 0$. In the extremal limit, the location of the horizon is pushed off to infinity and the resulting spacetime is everywhere static, with a naked singularity. Considering the Penrose-Carter diagram in Figure 5.7, we can think of this extremal limit as setting the volumes of the dynamic regions to zero as the horizon is moved off to asymptotic infinity.

The main result of this chapter is the generalised discussion of the causal structure of cosmological solutions containing a single horizon. By considering the static patch of the cosmological solutions for line elements which are of the form

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + g(r)d\vec{X}^2,$$

where $d\vec{X}^2$ can be either the metric for the two sphere, or the plane and $f(r)$ has one zero, we can write down the line element in terms of Kruskal-like coordinates. This allows us to then find the Penrose-Carter diagram and the classification of the trapping horizons for the solutions in this chapter, but also for the planar symmetric solutions of the STU model, and for the de Sitter solution, which is used as an example in Section 8.2. We will come back to this solution in Chapter 8 when we consider the thermodynamics of the Killing horizon, and it will appear again in our concluding remarks, when in Section 9.1, we discuss an ongoing project that attempts to classify a larger set of dual solutions, of which the planar symmetric solutions of Einstein-Maxwell are included.

6

COSMOLOGICAL SOLUTIONS TO $\mathcal{N}=2$ SUPERGRAVITY

This chapter contains work first published in [39]. The original aim of this paper was to continue the previous work [28, 31, 33, 34] on the construction of non-extremal, stationary solutions in theories of four-dimensional $\mathcal{N}=2$ vector multiplets coupled to gauged and ungauged supergravity, and to make contact in the extremal limit with the classification of supersymmetric near-horizon geometries. It was this original research idea that developed into the study of planar symmetric, cosmological solutions, that then became the basis of this thesis.

At a technical level, the work of [39] was to extend the work of [28] by allowing the solutions to have multiple charges. In the original work, the solutions were derived from U(1) gauged $\mathcal{N}=2$ supergravity supported by a single electric charge. Solutions were found using the same method we will apply, in which the real formulation of special geometry was used to obtain analytic solutions after the reduction over the timelike coordinate. The solutions were derived using a static, planar symmetric ansatz and were called 'Nernst branes' as they obeyed the strict third law of black hole mechanics: in the extremal limit, the area (density) of the black hole vanished. It was found that the solutions interpolated between two different hyperscale violating Lifshitz geometries between the horizon and the asymptotic limit. In the extremal limit, the Nernst solutions reduced to the solutions of [29, 30]. The solutions had singular behaviour: approaching the horizon, an observer would experience infinite tidal forces and in the asymptotic limit, it was found that the physical scalar fields diverge. In a second paper [34], the Nernst solution was realised as a boosted AdS Schwarzschild black brane in five dimensions and the divergence of the singularities when taking the asymptotic limit was resolved.

In our research [39], we showed that it was possible to solve the equations of motion using the same machinery of [28] for solutions supported by two, three or four charges under Abelian gauge fields. We found that when following the solution generating technique, increasing the number of charges supported by the brane required us to reduce the number of gauge parameters and restrict the form of the prepotential. For the case of four charges, all gauge parameters are required to be set to zero, and so the model we consider simplifies from gauged to ungauged supergravity.

It was found that for the solutions with more than one charge, the strict third law no longer held. The solution with two charges was a black hole solution, which had a single Killing horizon with finite area density in the extremal limit. In the asymptotic limit, the two-charge solution was conformally flat — in contrast to the Lifshitz geometry of the Nernst solution, which was conformally AdS_4. The three-charge ansatz led to solutions for a class of gauged vector multiplet theories, including the gauged STU model, while the four-charge system was a solution to the ungauged STU model. The three- and four-charge solutions had stranger behaviour in the extremal limit. It was found that when the surface gravity was taken to be zero, the area density diverged. Furthermore, as in the planar solutions of the Einstein-Maxwell theory discussed in Section 5.2, the static patch of the three- and four-charge solution interpolated between a curvature singularity and a Killing horizon. By analytical continuation through the horizon, we obtained time-dependent regions which were asymptotic to Kasner-like solutions in the infinite past and infinite future for the three-charge solution and true-Kasner asymptotically for the four-charge solution. These solutions can be understood as 'inside-out' compared to the causal structures of non-extremal black hole and black brane solutions, and should be interpreted as cosmological spacetimes.

Our family of solutions overlaps with previously found solutions of Einstein-Maxwell-Dilaton theories and truncations of supergravity theories in the literature, which in particular display the same Penrose-Carter diagram [168, 170, 171]. We also find that the Penrose-Carter diagram is of the form of the diagram given in Figure 5.7 from our discussion of the planar solutions of Einstein Maxwell theory. Studying the three- and four-charge solutions, it was found that the qualitative behaviour of their causal structure was similar enough to discuss simultaneously. It is because of this that in this thesis we choose to focus only on the ungauged, four charge solutions found in [39], and leave a reference to the paper for further discussion of the cosmological solution of the gauged, three-charge solutions.

The structure of this chapter is as follows. In Section 6.1 we give sufficient background to understand the already established work of [31, 33, 28, 34] and relate this back to our discussion from Section 4.1.2 and Section 4.4. In Section 6.2, the three-dimensional equations of motion are solved, producing a Euclidean instanton solution. In Section 6.3 this is uplifted to produce a four-dimensional solution, where regularity of the Killing horizon is imposed, allowing us to reduce the number of free integration constants. In Section 6.5, the four-dimensional solution is carefully considered, and we find a qualitative similarity in structure when compared to the planar symmetric solutions of the Einstein-Maxwell theory. In Section 6.6, we will show that by taking the special limit in which all scalar fields of the solution are constant, the four-charge solution reduces to the planar symmetric solutions to Einstein-Maxwell theory.

6.1 BACKGROUND

In this section, we overview the approach developed in [31, 33, 28, 34] to construct non-extremal, stationary solutions of $\mathcal{N} = 2$, $D = 4$ supergravity coupled to n_V vector multiplets. During this chapter, we will use the following tools to obtain our static, planar symmetric, four-dimensional solutions:

(i) The dimensional reduction over time to obtain an effective three-dimensional Euclidean theory.

(ii) The real, rather than the more commonly used complex formulation of the special geometry of $\mathcal{N} = 2$ vector multiplets, which is based on a Hesse potential rather than a prepotential.

(iii) A set of conditions which decouples the field equations and allows us to integrate them elementarily; this requires us to impose restrictions on the admissible prepotential/Hesse potential, and to consistently truncate the field content to a subset of 'purely imaginary' (PI) configurations.

Our initial work which led to the results of this thesis began with only small modifications from previous research papers, in particular [28], where planar symmetry was also considered. In this section, we summarise the essential points of each of these steps to a level in which our research was carried out. For further, more technical details, we refer [31] for a discussion of the dimensional reduction beyond what has been covered in Section 4.4 and [33, 28, 39] for more details on the generation of solutions of $\mathcal{N} = 2$ (un)gauged supergravity extending to models more generic than the STU model we consider in this chapter.

Our starting point is the bosonic content of the general two-derivative Lagrangian for n_V vector multiplets coupled to $\mathcal{N} = 2$ Poincaré supergravity introduced in Section 4.1.2. The Lagrangian is given by

$$\mathbf{e}_4^{-1}\mathcal{L} = -\frac{1}{2}R_4 - g_{I\bar{J}}\partial_\mu X^I \partial^\mu \bar{X}^{\bar{J}} + \frac{1}{4}\mathcal{I}_{IJ}F^I_{\mu\nu}F^{J|\mu\nu} + \frac{1}{4}\mathcal{R}_{IJ}F^I_{\mu\nu}\tilde{F}^{J|\mu\nu}, \tag{6.1.1}$$

where $\mu \in \{0, 1, 2, 3\}$ are the spacetime indices, $\mathbf{e}_4$ is the vierbein, R_4 is the Ricci scalar, $F^I_{\mu\nu}$ are the Abelian field strengths, $I, J \in \{0, 1, \ldots, n_V\}$. In our conventions the tilde represents the Hodge-dual

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} . \tag{6.1.2}$$

As discussed in Section 4.1.2, all data in the Lagrangian is encoded by a *prepotential* $F = F(X^I)$, which is a holomorphic function, homogeneous of degree two in the complex scalar fields X^I. By working with the scalar fields of the superconformal theory, we're able to package together $(X^I, F_I)^T$, which is a symplectic vector where $F_I = \partial_I F(X^I)$. We must remember that by working with X^I, we introduce redundant degrees of freedom and must impose the D-gauge and fix the U(1) symmetry to work with the independent degrees of freedom of the Poincaré supergravity theory. The physical scalars of the Poincaré theory are found from ratios of the superconformal scalars

$$z^A = \frac{X^A}{X^0}, \quad \bar{z}^A = \frac{\bar{X}^A}{\bar{X}^0}, \qquad A \in \{1, \ldots, n_V\}. \tag{6.1.3}$$

Working with the scalars X^I has the advantage of formally balancing the number of scalar and vector fields. As a reminder: in Section 4.1.2, we discussed the derivation of Poincaré supergravity by gauge fixing the superconformal theory and that to construct the supergravity multiplet, we needed to include an additional vector multiplet to recover the graviphoton in the supergravity multiplet.

As introduced in Section 4.2, the most important feature of $\mathcal{N} = 2$ vector multiplets is that the field equations (though not the Lagrangian) are invariant under the action of the symplectic group $\mathrm{Sp}(2n_V + 2, \mathbb{R})$, which acts linearly on the field strengths $(F^I_{\mu\nu}, G_{I|\mu\nu})^T$. The symplectic transformations generalise the electric-magnetic duality of the source-free Maxwell equations, and contain stringy symmetries, such as T-duality and S-duality, if the theory under consideration arises as a low-energy effective theory from string theory. By supersymmetry, the full set of field equations is symplectically invariant, with the scalars described by the symplectic vectors $(X^I, F_I)^T$.

Preserving manifest symplectic covariance is vital in our solution generating method, in that it allows a simplification of the equations of motion such that they are solved exactly. However, there is a drawback to working with the complex scalar fields X^I as they do not form a symplectic vector by themselves. Similarly, the couplings $g_{I\bar{J}}$, $\mathcal{I}_{IJ}$ and $\mathcal{R}_{IJ}$ do not transform as tensors under the symplectic group, but have a more complicated behaviour. To resolve these transformation issues, we instead work with the special real formulation, with real scalar fields q^a, $a \in \{1, \ldots, 2n_V + 2\}$, related to the complex scalars X^I by

$$(q^a) = \begin{pmatrix} x^I \\ y_J \end{pmatrix} = \mathrm{Re} \begin{pmatrix} X^I \\ F_I \end{pmatrix}.$$

These real scalar fields have the benefit of transforming as a vector under symplectic transformations. In this formulation, all couplings are encoded in a real function $H = H(q^a)$, called the Hesse potential, which is homogeneous of degree two, and which up to a factor, is the Legendre transform of the imaginary part of the prepotential F (see [31] for further details). It is helpful to think of the complex and real formulations of special geometry as being related to one another in the same way as the Lagrangian and Hamiltonian formulations of mechanics, see for example [172] and [53] for a detailed, comprehensive discussion.

6.1.1 *Dimensional reduction*

As summarised above, we generate four-dimensional solutions through finding Euclidean instanton solutions in three dimensions, which we then uplift back into four dimensions. We accomplish this via a Kaluza-Klein reduction over a timelike coordinate, following the formulae given in Section 4.3.2. The general discussion of this reduction was given in Section 4.4. We begin choosing a Kaluza-Klein ansatz for our four-dimensional spacetime. Following [126], we choose to additionally impose that our four-dimensional solutions are static, and therefore decompose the four-dimensional spacetime metric as

$$ds_4^2 = -e^{\phi} dt^2 + e^{-\phi} ds_3^2 \,, \tag{6.1.4}$$

where ϕ and all matter fields are assumed to be independent of time t. The field ϕ is the Kaluza-Klein scalar and there is no Kaluza-Klein vector since the assumption that field configuration is static[1] sets $V_\mu = 0$. As discussed in Section 4.1.2, by performing a field redefinition of the complex scalar fields, the Kaluza-Klein scalar ϕ is absorbed

$$Y^I := e^{\phi/2} X^I \,. \tag{6.1.5}$$

[1] As a reminder, staticity assumes that the metric is both stationary (time-independent) and hypersurface-orthogonal (no time-space cross-terms)

We do not introduce a new symbol for the corresponding real scalars q^a, which, are subject to the same rescaling, such that from now on we understand the real scalars are written as

$$(q^a) = \begin{pmatrix} x^I \\ y_J \end{pmatrix} = \mathrm{Re}\begin{pmatrix} Y^I \\ F_I(Y) \end{pmatrix}.$$

The advantage of this field redefinition is that the Kaluza-Klein scalar is now on the same footing as the four-dimensional scalar fields. When needed, the Kaluza-Klein scalar can be extracted as $e^\phi = -2H$, where H is the Hesse potential considered as a function of the rescaled real scalars q^a. Upon dimensional reduction, the $(n_V + 1)$ four-dimensional vector fields split into scalars ζ^I and three-dimensional vector fields which can be dualised into a second set of scalars $\tilde{\zeta}_I$. These $2n_V + 2$ scalars can be combined into the symplectic vector $\hat{q}^a = \frac{1}{2}(\zeta^I, \tilde{\zeta}_I)^T$. We refrain from giving the explicit relations between the various fields, and refer the interested reader to [31, 33, 28]. What matters is that all dynamical degrees of freedom are now encoded in the $(4n_V + 4)$ real scalars $q^a, \hat{q}^a$, which form two symplectic vectors and the four-dimensional fields can be recovered with known formula from the literature.

Following the reduction outlined in Section 4.4, the resulting three-dimensional Lagrangian is given by (4.4.32), repeated here for convenience

$$\begin{aligned}\mathbf{e}_3^{-1}\mathcal{L}_3 = &-\frac{1}{2}R_3 - \tilde{H}_{ab}\left(\partial_\mu q^a \partial^\mu q^b - \partial_\mu \hat{q}^a \partial^\mu \hat{q}^b\right) \\ &- \frac{1}{H^2}\left(q^a \Omega_{ab} \partial_\mu q^b\right)^2 + \frac{2}{H^2}\left(q^a \Omega_{ab} \partial_\mu \hat{q}^b\right)^2.\end{aligned}$$

where H is the Hesse potential and

$$\Omega_{ab} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix},$$

are the components of the symplectic form $\Omega = \frac{1}{2}\Omega_{ab} dq^a \wedge dq^b$.

Note that in the above equation, the final line from (4.4.32) is missing, which is a result of the static assumption removing the Kaluza-Klein vector. The equations of motion for the Lagrangian (4.4.32) are derived in [31], but are not included here as the resulting simplification made after the following discussion will allow a more concise derivation of the equations of motion we will consider.

6.1.2 *Restricted field configurations*

In order to obtain solutions of the field equations by decoupling and elementary integration, we make two further assumptions.

(i) We will need to know the Hesse potential explicitly, but models are naturally defined in terms of their prepotential, *e.g.* in the context of Calabi-Yau compactifications of string theory. Since the Hesse potential is obtained by a Legendre transformation of the (imaginary part of the) prepotential, it cannot be computed in closed form for a generic prepotential. We will therefore restrict the form of the prepotential in such a way that we can obtain the Hesse potential explicitly.

(ii) We want the field equations to decouple. This is achieved by imposing a block structure, where the scalar fields within each block are proportional to each other, and where equations within a block do not couple to equations in other blocks. Such block structures appear if we consistently truncate out part of the scalar fields.

We remark that the two types of conditions we impose are not independent. We only need to know a Hesse potential for the subset of fields which are not truncated out consistently. The more fields we truncate out, the larger the class of prepotentials admissible. In [39], as we increased the number of charges, the fields kept in the effective three-dimensional theory became more and more restricted. In [28], it was shown that this restriction must also be applied to the gauging parameters of the scalar potential and that this required factorisation of the admissible prepotentials. In this thesis, we look at the extreme limit of this, in which we allowed the maximal number of distinct charges using this method, and in doing so, set all gauging parameters to zero, fixing the form of the prepotential by requiring it was fully factored.

For the single-charged Nernst brane solutions [28] of gauged supergravity, it is sufficient to restrict the prepotential to be of the so-called very special form

$$F(Y) = \frac{f(X^1, \dots X^n)}{X^0}, \tag{6.1.6}$$

where f is a homogeneous polynomial of degree three. This condition is equivalent to imposing that the vector multiplet theory can be lifted to five dimensions. In [39], we found that increasing the number of charges required factorising the polynomial $f(X^1, \dots X^n)$ such that the prepotentials were of the form

$$\begin{gathered} F_2(X) = \frac{X^1 f_2(X^i, \dots, X^n)}{X^0}\,, \quad F_3(X) = \frac{X^1 X^2 f_3(X^i, \dots, X^n)}{X^0}\,, \\ F_4(X) = \frac{X^1 X^2 X^3}{X^0}\,, \end{gathered} \tag{6.1.7}$$

where the subscripts count the number of charges present in the theory, and f_2 and f_3 are homogenous polynomial of degree two and one respectively. While $F_2(X)$ is still the generic form for a compactification of the heterotic string on $K3 \times T^2$ at string tree level, $F_4(X)$ is the well known STU model [104], which is also the minimal example for a prepotential of the form $F_3(X)$. While more general models can be defined and solved for by relaxing the condition that f_3 is a polynomial, we do not know how such models could be embedded into string theory and so restricted ourselves to the polynomial case. We have included this discussion of the lower charge solutions as an illustration of how we came to the STU model for the four charge case, but we will not further discuss the form their solutions, which are explained in full detail in [39].

Next we specify the consistent truncation of the scalar fields $q^a, \hat{q}^a$ that we impose to achieve decoupling. In [33], the truncated field configurations were called 'purely imaginary' (PI) because the corresponding four-dimensional physical scalars z^A are purely imaginary, and in terms of real scalar fields, the PI conditions require that

$$y_0 = u^0 = 0, \qquad x^A = 0, \qquad A \in \{1, \dots, n_V\}.$$

This type of condition is also known as 'axion-free', as in our parametrisation, the real parts of z^A have an axion-like shift symmetry for prepotentials of the very special form. Remembering back to the gauge fixing conditions we must impose on the complex scalars X^I, we can identify one of these conditions not as a restriction on the physical scalars, but instead as the condition to fix the U(1) gauge symmetry. Any of these conditions is appropriate for this gauge fixing, but it is standard to think of $\mathrm{Im}\,Y^0 = 0 \Rightarrow u^0 = 0$ to be the gauge fixing condition.

In terms of three-dimensional scalars, the PI condition takes the form

$$(q^a)\big|_{\mathrm{PI}} = (x^0, \dots, 0; 0, y_1, y_2, \dots, y_n). \tag{6.1.8}$$

This is extended to the scalars $\hat{q}^a$ which correspond to four-dimensional gauge fields by

$$(\partial_\mu \hat{q}^a)\big|_{\mathrm{PI}} = \frac{1}{2}(\partial_\mu \zeta^0, \dots, 0; 0, \partial_\mu \zeta_1, \partial_\mu \zeta_2, \dots, \partial_\mu \zeta_n). \tag{6.1.9}$$

We remark that the PI condition maintains that the solution we obtain is a consistent truncation of the full theory. A point of view we do not deeply discuss in this thesis is realising the scalar fields as harmonic maps to a target space geometry. Understanding the scalar fields as coordinates on the target manifold, making a field restriction can be interpreted as picking out some sub-manifold from the target space. From this perspective, the PI condition as a consistent truncation reflects the existence of a distinguished totally geodesic[2] $(2n_V + 2)$-dimensional sub-manifold of the $(4n_V + 4)$-dimensional scalar manifold realised from the three-dimensional effective theory obtained from dimensional reduction.

For notational convenience, we adjust the assignment of indices $a, b, \dots$ to the scalar fields $q^a, \hat{q}^a$ such that the non-constant scalars correspond to indices $a \in \{1, \dots, n_V + 1\}$. We can further simplify the equations of motion through some simple manipulations. All terms in the three-dimensional Lagrangian and field equations which do not involve the scalar potential either involve the constant anti-symmetric matrix Ω_{ab} or the Hesse potential H and its derivatives. Looking at (6.1.8) and (6.1.9), we find that

$$q^a \Omega_{ab} \partial^\mu q^b = q^a \Omega_{ab} \partial^\mu \hat{q}^b = 0.$$

Moreover, we replace the scalar fields q^a by their duals $q_a = \tilde{H}_a = -\tilde{H}_{ab} q^b$, where we used that $\tilde{H}_{ab}$ are homogeneous functions of degree -2. While in general we cannot lower indices on $\hat{q}^a$ in the same way,[3] we can lower the indices after differentiation: $\partial_\mu \hat{q}_a := \tilde{H}_{ab} \partial_\mu \hat{q}^b$ [33]. As the fields $\hat{q}^a$ are essentially the four-dimensional gauge potentials, which only enter into the field equations through their derivatives, this is sufficient for rewriting all field equations with indices a in the lower position. Putting this all together, under the PI condition, the effective three-dimensional Lagrangian is simply

$$\mathbf{e}_3^{-1} \mathcal{L}_3 = -\frac{1}{2} R_3 - \tilde{H}^{ab} \big(\partial_\mu q_a \partial^\mu q_b - \partial_\mu \hat{q}_a \partial^\mu \hat{q}_b \big). \tag{6.1.10}$$

[2] Here, totally geodesic means that geodesics on the subspace are geodesics in the full space.

[3] We do not require the functions $\hat{q}_a$ to be well defined on the scalar manifold, and in particular, we cannot find coordinates by computing $\tilde{H}_{ab}\hat{q}^b$ unless $\tilde{H}_{ab}$ is itself constant. This is because the above coordinate could not be consistent with $\dot{\hat{q}}_a = \tilde{H}_{ab}\dot{\hat{q}}^b$. However, $\hat{q}^a$ are well defined functions on the scalar manifold, and from this $\dot{\hat{q}}^a$ and $\dot{\hat{q}}_a$ are well defined vector and covector fields respectively.

6.2 EUCLIDEAN INSTANTON SOLUTIONS

We are now in the position to formulate the problem that we will solve. Starting from the Lagrangian (6.1.1), we impose that the four-dimensional metric (6.1.4) is static, the PI truncation (6.1.8, 6.1.9) and finally that the three-dimensional metric has planar symmetry

$$ds_3^2 = e^{4\psi} d\tau^2 + e^{2\psi}(dx^2 + dy^2). \tag{6.2.1}$$

All fields, including the unknown function ψ, depend only on the overall transverse coordinate τ in this brane-like ansatz.

The resulting equations of motion can be derived from (6.1.10), alternatively, the restrictions discussed can be applied to the general three-dimensional equations for timelike dimensional reduction derived in [31] by imposing our field restrictions. Either way, the resulting equations of motion are given by

$$\nabla^2 \hat{q}_a = 0, \tag{6.2.2}$$

$$\nabla^2 q_a + \frac{1}{2}\partial_a \tilde{H}^{bc}(\partial_\mu q_b \partial^\mu q_c - \partial_\mu \hat{q}_b \partial^\mu \hat{q}_c) = 0, \tag{6.2.3}$$

$$-\frac{1}{2}R_{(3)\mu\nu} - \tilde{H}^{ab}(\partial_\mu q_a \partial_\nu q_b - \partial_\mu \hat{q}_a \partial_\nu \hat{q}_b) = 0. \tag{6.2.4}$$

The first line gives the equations of motion for for the scalars $\hat{q}^a$, which correspond to the four-dimensional vector field equations. The second line give the equations for the scalars q^a, which encode the four-dimensional scalars z^A and the Kaluza-Klein scalar ϕ. The last line are the three-dimensional Einstein equations which determines the three-dimensional warp factor ψ.

To solve Einstein's equations we use that the non-zero components of the Ricci tensor are found to be

$$R_{\tau\tau} = 2\ddot{\psi} - 2\dot{\psi}^2, \quad R_{xx} = R_{yy} = e^{-2\psi}\ddot{\psi},$$

where we use a dot to denote differentiation by τ. The equations (6.2.4) then reduce to the following form for $\mu, \nu \neq \tau$

$$\frac{1}{2}e^{-4\psi}\ddot{\psi} = 0, \tag{6.2.5}$$

and for $\mu, \nu = \tau$

$$\tilde{H}^{ab}\left(\dot{q}_a \dot{q}_b - \dot{\hat{q}}_a \dot{\hat{q}}_b\right) = \dot{\psi}^2 - \tfrac{1}{2}\ddot{\psi}, \tag{6.2.6}$$

where we have substituted in (6.2.5) to reduce this condition. We remark that (6.2.6) is called the Hamiltonian constraint [33, 28].

We are interested in deriving solutions carrying charge under four gauge fields. The requires enforcing that the equations of motion for all q_a decouple, which in turn requires that the prepotential takes the form

$$F(X) = \frac{X^1 X^2 X^3}{X^0}, \tag{6.2.7}$$

which is the well-known prepotential of the STU model.

From the STU prepotential, we can compute the Hesse potential (4.4.2)

$$H = -\frac{1}{4}(-q_0 q_1 q_2 q_3)^{-\frac{1}{2}}, \tag{6.2.8}$$

and as we know the exact form of the Hesse potential, we can now completely solve for the metric, which is diagonal with elements given by

$$\tilde{H}^{aa} = \frac{1}{4q_a^2} .$$

We note that after making the PI restriction we have reduced from four complex to four real variables.

We start by solving the equations of motion for $\hat{q}_a$. Since all fields are assumed to only depend on τ, Equation (6.2.2) reduces to

$$\ddot{\hat{q}}_a = 0. \tag{6.2.9}$$

Integrating up we obtain

$$\dot{\hat{q}}_a = K_a. \tag{6.2.10}$$

The non-vanishing constants K_a are proportional to the electric charge Q_0 and magnetic charges P^A of the four gauge fields in this solution[4]

$$\dot{\hat{q}}_a = K_a = (-Q_0, 0, 0, 0; 0, P^1, P^2, P^3). \tag{6.2.11}$$

Now turning to the scalar fields q_a, studying (6.2.3) we find that they completely decouple from each other, and we obtain four differential equations

$$\ddot{q}_a - \frac{\dot{q}_a^2 - K_a^2}{q_a} = 0. \tag{6.2.12}$$

This can be integrated to obtain the form for the scalar fields

$$q_a = \pm\frac{K_a}{B_a}\sinh\left(B_a\tau + B_a\frac{h_a}{|K_a|}\right), \tag{6.2.13}$$

where the integration constants have been collected together as

$$B_a = (B_0, B_1, B_2, B_3), \qquad h_a = (h_0, h^1, h^2, h^3).$$

Without loss of generality we set $B_a \geq 0$. To avoid curvature singularities associated with zeros of the fields q_a, we further require that $\mathrm{sign}(h_0) = \mathrm{sign}(Q_0)$ and $\mathrm{sign}(h^A) = \mathrm{sign}(P^A)$. This ensures that there are no zeros for the domain $0 \leq \tau < \infty$.

Finally, we turn to Einstein's equations (6.2.4). From (6.2.5) we have the simple result

$$-\frac{1}{2}e^{-4\psi}\ddot{\psi} = 0 \quad \Rightarrow \quad \ddot{\psi} = 0.$$

The Hamiltonian constraint (6.2.6) allows us to find

$$\frac{1}{4q_a^2}\left(\dot{q}_a^2 - \dot{\hat{q}}_a\right) = \dot{\psi}^2 - \frac{1}{2}\ddot{\psi} = \frac{B_0^2 + B_1^2 + B_2^2 + B_3^2}{4} , \tag{6.2.14}$$

Combining these results together, we can find a solution for the warp factor

$$\dot{\psi} = \pm\frac{\sqrt{\sum_i B_i^2}}{2} \quad \Rightarrow \quad \psi = \pm\frac{\sqrt{\sum_i B_i^2}}{2}\tau + a_0.$$

[4] The minus sign in front of Q_0 reflects that K_a transforms as a covector, and not as a vector, under symplectic transformations.

As it appears in an exponential in the line element (6.2.1), we additionally calculate

$$e^{-4\psi} = e^{-4a_0} e^{\pm 2\sqrt{\sum_i B_i^2}\tau} . \tag{6.2.15}$$

To summarise, we have found the following planar symmetric solution to the time-reduced STU model:

$$\begin{aligned} \dot{q}_a &= K_a, \\ q_a &= \pm \frac{K_a}{B_a} \sinh\left(B_a \tau + B_a \frac{h_a}{K_a} \right), \\ e^{-4\psi} &= e^{-4a_0} e^{\pm 2\sqrt{\sum_i B_i^2}\tau} . \end{aligned} \tag{6.2.16}$$

6.3 FOUR-DIMENSIONAL PLANAR SOLUTIONS

In this section, we study the three-dimensional instanton solutions derived as candidates for black hole solutions in four dimensions. Part of this process involves imposing regularity conditions on the matter content through placing conditions on the integration constants.

As we mentioned in the reduction of the brane configurations in Section 4.5.4, to have well behaved solutions, the scalar fields must be regular on the horizon. From the point of view of $\mathcal{N} = 2$ supergravity coupled to vector multiplets, this condition can be traced back to ensuring certain properties of the prepotential. In our case, when the prepotential is of the very special form (6.1.6), we require that the polynomial function $f(X^1, \dots X^n)$ have no zeros or poles. For the STU model, where $f(X) = X^1 X^2 X^3$, this translates to the simple condition that the scalars z^A do not diverge on the horizon. When considering multi-charged solutions [39], the argument must be made for more generic prepotentials.

We can explicitly write down the four-dimensional, static, planar symmetric line element using (6.1.4) and (6.2.1)

$$ds_4^2 = -e^{\phi} dt^2 + e^{-\phi + 4\psi} d\tau^2 + e^{-\phi + 2\psi}(dx^2 + dy^2), \tag{6.3.1}$$

where the Kaluza-Klein scalar is found from the Hesse potential via (6.2.8)

$$e^{\phi} = -2H = \frac{1}{2} (-q_0 q_1 q_2 q_3)^{-\frac{1}{2}} .$$

We can perform the lift from the three-dimensional fields into four dimensions using the dimensional reduction formulae found originally in [31, 33, 28].

The four-dimensional physical scalars are determined by the three-dimensional scalars through the relations [33]

$$Y^0 = -\frac{1}{4q_0}, \qquad Y^A = -\frac{i}{2} e^{\phi} q_A, \tag{6.3.2}$$

where Y^I are rescaled scalar fields defined in (6.1.5). This yields the form for the physical scalars via (6.1.3)

$$z^A = -i\left(-\frac{q_0 q_A^2}{q_1 q_2 q_3} \right)^{\frac{1}{2}} . \tag{6.3.3}$$

The four-dimensional gauge fields are calculated using $\hat{q}_a$ through the relation

$$\hat{q}^a = \frac{1}{2}\begin{pmatrix}\zeta^I \\ \tilde{\zeta}_I\end{pmatrix}, \tag{6.3.4}$$

where as displayed in (6.1.1), ζ^I are the components of the gauge fields along the reduction dimension and $\tilde{\zeta}_I$ are the Hodge duals of the three-dimensional vectors. Their relation to the four-dimensional gauge fields can be calculated from [33]

$$\partial_\mu \zeta^I := F^I_{\mu t}, \qquad \partial_\mu \tilde{\zeta}_I := G_{I|\mu t}, \tag{6.3.5}$$

where we remind the reader that

$$G_{I|\mu\nu} := \mathcal{R}_{IJ} F^J_{\mu\nu} - \mathcal{I}_{IJ} \tilde{F}^J_{\mu\nu}, \tag{6.3.6}$$

is the dual field strength.

We begin by looking for the location of a Killing horizon. As the solution is static, we have the timelike Killing vector field $k^\mu = \partial_t$ and we can compute its norm $k^2 = e^\phi$. Evaluating this in the limit of $\tau \to \infty$, we find that the norm of the Killing vector falls off as

$$k^2 = \exp\left(-\frac{B_0\tau}{2} - \frac{B_1\tau}{2} - \frac{B_2\tau}{2} - \frac{B_3\tau}{2}\right) \xrightarrow[\tau\to\infty]{} 0\,,$$

where we use that $B_i > 0$. We understand that there is a Killing horizon located in the four-dimensional spacetime for $\tau \to \infty$. We can now take the same limit for the scalar fields (6.3.3) and ensure that they are finite on the horizon. Inserting in our result for the scalar fields q_a, we find the tau dependence of the physical scalars on the horizon goes as:

$$\begin{aligned}
z^1\big|_{\tau\to\infty} &\sim \exp\left[\left(B_0 + B_1 - B_2 - B_3\right)\tau\right],\\
z^2\big|_{\tau\to\infty} &\sim \exp\left[\left(B_0 + B_2 - B_1 - B_3\right)\tau\right],\\
z^A\big|_{\tau\to\infty} &\sim \exp\left[\left(B_0 + B_3 - B_2 - B_1\right)\tau\right].
\end{aligned}$$

Imposing that these fields are finite in the limit $\tau \to \infty$ requires that the following three conditions hold:

$$\begin{aligned}
B_0 + B_1 - B_2 - B_3 &= 0,\\
B_0 + B_2 - B_1 - B_3 &= 0,\\
B_0 + B_3 - B_2 - B_1 &= 0.
\end{aligned}$$

This is satisfied when the integration constants obey $B_0 = B_1 = B_2 = B_3 = B$. The integration constant a_0, which appears in the warp factor, can be removed by a suitable shift in the τ coordinate and after performing the shift, we see the that the warp factor simplifies, up to a sign

$$e^{-4\psi} = e^{\pm 4B\tau}\,.$$

We can set this sign through our final condition, looking at the area density of the horizon. The area of the horizon is given by integral

$$A = \int dxdy e^{-\phi+2\psi}\Big|_{\tau\to\infty}. \tag{6.3.7}$$

As with the planar symmetric solutions of the Einstein-Maxwell theory, our x and y coordinates are not compact, and so the area diverges, reflecting the planar symmetry of our ansatz. To obtain finite quantities we could identify x, y periodically, but as with our discussion in Chapter 5, we prefer to work with densities instead and take ratios relative to the coordinate area. As there are many more functions and integration constants to keep track of in this chapter, we formally set

$$\omega = \int_{\mathbb{R}^2} dx \wedge dy = 1\,,$$

rather than keep ω explicit in our computations. With this understood, we find that the area density of the horizon is found to be

$$e^{-\phi+2\psi}\bigg|_{\tau\to\infty} \sim \exp\left(\frac{4B\tau}{2} \mp 2B\tau\right). \tag{6.3.8}$$

We see that this is finite when we take the upper sign from the square root, thus setting $\sqrt{\sum_i B_i^2} = 2B > 0$.

In Chapter 7, when uplifting the four-charge solution to higher dimensions, and in Chapter 8, when considering the thermodynamics of these solutions, we will need the explicit form of the gauge fields. As we assume all three-dimensional components depend only on the coordinate τ, the non-zero components are found from (6.3.4-6.3.6)

$$\begin{aligned} F^0_{\tau t} &= (\dot{A}^0)_t = 2\dot{\tilde{q}}^0 = 2\tilde{H}^{00}\dot{\tilde{q}}_0 = -\frac{Q_0}{2q_0^2(\tau)}, \\ \tilde{F}_{A|\tau t} &= (\dot{\tilde{A}}_A)_t = \frac{P^A}{2q_A^2(\tau)}, \end{aligned} \tag{6.3.9}$$

where the dot references differentiation by the parameter τ.

In summary, the explicit four-dimensional solution takes the following form:

$$\begin{aligned} \dot{\tilde{q}}_a &= K_a, \\ q_a &= \pm\frac{K_a}{B}\sinh\left(B\tau + B\frac{h_a}{|K_a|}\right), \\ e^{\phi} &= \frac{1}{2}(-q_0q_1q_2q_3)^{-\frac{1}{2}}\,, \\ e^{-4\psi} &= e^{4B\tau}. \end{aligned} \tag{6.3.10}$$

A note on integration constants

Imposing the reality conditions such that the area of the horizon is finite and that the scalar fields take finite values on the horizon has reduced the total number on free integration constants from 13 to 9. This reduction from reality conditions is related to the question of whether there is an analogue or generalisation of the attractor mechanism for non-extremal solutions.

The attractor mechanism [173, 174, 25, 175] forces scalar field to attain unique values, determined by the charges at the event horizon of static extremal BPS black holes.[5] This mechanism reduces the number of integration constants in the second order scalar field equations by a factor

[5] For non-BPS extremal black holes, the horizon values of some scalar fields may remain un-fixed, as long as the variation of these values does not change the black hole entropy [176].

of one-half, since only the asymptotic values of the scalars at infinity remain integration constants that can be chosen arbitrarily.

When constructing solutions using the Killing spinor equations, or more generally, by imposing that the scalar field equations reduce to first order gradient flow equations, this reduction is automatic. When solving the second order field equations directly, the reduction in the number of integration constants enters when imposing that the scalars should take regular values at the horizon, rather than displaying run-away behaviour. Interestingly, this link between regularity and the reduction of the number of integration constants also exists for non-extremal solutions, which we see not only in this case, but for all solutions considered in [39], thus also including solutions of gauged supergravity.

While naively we could have expected to obtain solutions with two integration constants per scalar field, there is only one, corresponding to the field's value at infinity, and one additional constant, which corresponds to the non-extremality parameter. While the values of the scalars at the horizon are not exclusively determined by the charges, they are still fixed and completely determined by the other integration constants. Similar observations were made in [33, 177] for non-extremal five- and four-dimensional black holes, and in [28] for Nernst branes. In [177] this behaviour was named the 'dressed attractor mechanism', since the horizon values of the scalars are given by the same expressions as in the extremal limit, with the charges replaced by dressed charges which depend on the other integration constants. These observations are consistent with the idea of 'hot attractors', which was advocated in [178, 179, 180], and support the idea that the attractor mechanism is relevant, in a modified form, for non-extremal solutions.

6.4 COSMOLOGICAL SOLUTIONS OF THE STU MODEL

In this section, we study the geometry of the planar symmetric solutions derived above. This is accomplished through a series of coordinate transformations that will build a global description of the solution. We will find that qualitatively, this planar symmetric solution has the same structure as the planar symmetric solutions of the Einstein-Maxwell theory. In Section 6.6, we show that through fine-tuning the integration constants, we can recover the Einstein-Maxwell solutions as a consistent truncation of the solutions discussed here.

For the solutions which came before the multi-charged ones discussed in [39], the transverse coordinate took values in $\tau \in [0, \infty)$. We have seen that $\tau \to \infty$ is the location of the horizon, and for the Nernst solutions [28], $\tau \to 0$ was the asymptotic limit, which inspired the coordinate change

$$e^{-2B\tau} = 1 - \frac{2B}{\rho} =: W(\rho). \tag{6.4.1}$$

We understand $\rho = 2B$ as the location of the horizon, and $\rho \to \infty$ as being the location of asymptotic infinity.[6]

[6] As with the Einstein-Maxwell solution, our best definition for an asymptotic region for planar symmetric solutions comes from the behaviour of the null geodesics of the solution; instead of looking for some maximally symmetric geometry associated with the vacuum solution, we look for the region which we can extend null geodesics to infinite affine parameter.

What we show below is that on making this coordinate transformation for the four-charge solution, $\rho = \infty$ is reached by transverse null geodesics in finite affine parameter and therefore cannot be interpreted as the asymptotic region. Introducing a new transverse coordinate ζ, which is defined by $\rho = \zeta^{-1}$, we identify the appropriate asymptotic region as $\zeta \to \infty$ ($\rho = 0$).

Written in terms of the new transverse coordinate ζ, we start to understand the structure of the solution. The locus $\rho \to \infty \Leftrightarrow \zeta \to 0$ is of no particular significance and is a hypersurface within the static region of the spacetime. The Killing horizon is located at $\zeta = (2B)^{-1}$ and we find that within static region $\zeta < (2B)^{-1}$, there is a curvature singularity $\zeta = \zeta_s \leq 0$. As a result, we find the transverse coordinate has finite extension within static region, bounded between the Killing horizon and a timelike curvature singularity. This region can be understood like the region behind the Cauchy horizon in the Reissner-Nordstrom solution, or the static region of the planar symmetric solutions to Einstein-Maxwell theory.

By analytic continuation of the solution, we can extended to the region $(2B)^{-1} < \zeta < \infty$, where the coordinate ζ becomes timelike and the limit $\zeta \to \infty$ is at infinite (timelike) distance. We will interpret region I, $\zeta_s < \zeta < (2B)^{-1}$ as the interior region, and region II, $(2B)^{-1} < \zeta < \infty$ as the exterior region. A summary of this structure is given as an illustration in Figure 6.1. As we have an exterior region which is time-dependent, we refer this class of solutions as cosmological.

6.4.1 *Finding asymptotic infinity*

We beginning by applying the coordinate change (6.4.1) to the fields found in (6.3.10) to obtain a line element for our solution in terms of the coordinates $\{t, \rho, x, y\}$, where $\rho \in (2B, \infty)$.

Beginning with the scalar fields q_0, we find that

$$q_0 = \mp \frac{\mathcal{H}_0}{W^{1/2}}, \qquad q_A = \pm \frac{\mathcal{H}_A}{W^{1/2}}, \tag{6.4.2}$$

where we have defined the harmonic functions

$$\mathcal{H}_a(\rho) := |K_a| \left[\frac{1}{B} \sinh\left(\frac{Bh_a}{|K_a|} \right) + \frac{e^{-\frac{Bh_a}{|K_a|}}}{\rho} \right], \quad a = 0, \ldots, 3.$$

The warp factor is found to be

$$e^{-4\psi} = W(\rho)^{-2},$$

and the Kaluza-Klein scalar transforms as

$$e^{\phi} = \frac{1}{2}(-q_0 q_1 q_2 q_3)^{-\frac{1}{2}} = \frac{W(\rho)}{\mathcal{H}(\rho)}, \tag{6.4.3}$$

where we have defined a new function which is the product of our harmonic functions:

$$\mathcal{H}(\rho) := 2\sqrt{\mathcal{H}_0 \mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_3}. \tag{6.4.4}$$

With the expressions for the Kaluza-Klein function and the warp factor, we can write down the line element with our new transverse coordinate

$$ds^2 = -\frac{W}{\mathcal{H}} dt^2 + \frac{\mathcal{H}}{W} \frac{d\rho^2}{\rho^4} + \mathcal{H}(dx^2 + dy^2). \tag{6.4.5}$$

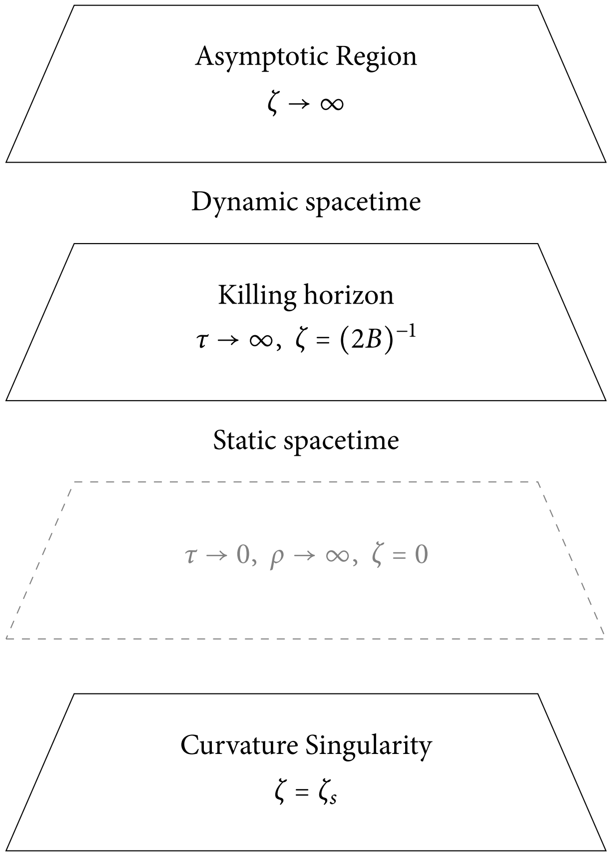


Figure 6.1: Diagram of the spacetime regions. When starting from the 3D solution, a static patch of the spacetime is found, parametrised by τ for $\zeta \in [0, (2B)^{-1}]$. We can extend this spacetime to a singularity where the Kretschmann invariant becomes infinite. Analytically continuing our parameter through the horizon to $\zeta > (2B)^{-1}$ we obtain a time-dependent geometry.

We are now interested in looking for where our asymptotic region is. We do this in an identical procedure to how we studied the geodesic motion in Section 5.2.

The Lagrangian (energy functional) for transverse geodesics in the metric (6.4.5) is

$$\mathcal{L} = -\frac{W}{\mathcal{H}}\dot{t}^2 + \frac{\mathcal{H}}{W\rho^4}\dot{\rho}^2,$$

where the dot represents differentiation with respect to an affine parameter λ. Null geodesics satisfy $\mathcal{L} = 0$. The constant of motion associated to the timelike Killing vector k^μ is

$$E = -k \cdot u = \frac{W\dot{t}}{\mathcal{H}},$$

which can be rearranged to give a differential equation. Upon integration, we obtain an expression for the affine parameter

$$\dot{\rho} = \pm\sqrt{\rho^4 E^2}, \qquad \lambda = \pm \int \frac{d\rho}{E\rho^2}. \tag{6.4.6}$$

This shows that light signals sent from $\rho > 2B$ reach $\rho = \infty$ in finite affine parameter, whereas $\rho \to 0$ is reached in infinite affine parameter. Therefore, $\rho \to 0$ should be interpreted as being at infinite distance and $\rho < 2B$ as the exterior region, while $\rho > 2B$ is the interior region.

Given this observation, we introduce the new transverse coordinate $\zeta = \rho^{-1}$ so that infinity is now at $\zeta \to \infty$. It is important to note that in order to reach the limit of $\zeta \to \infty$ we must cross the Killing horizon located at $\zeta = (2B)^{-1} =: \alpha^{-1}$ into a new, 'exterior' region. In the exterior, ζ is a timelike coordinate and as the line element depends explicitly on ζ, the exterior is non-stationary and as such the solution is interpreted as cosmological.

We also note that for causal information coming from an asymptotic distance, the Killing horizon is a cosmological horizon which is located at a point in time $\zeta = \alpha^{-1}$, and so will be necessarily crossed for all causal geodesics.[7] Once the horizon has been crossed, the timelike singularity can be avoided, and geodesics may leave the static patch into a second dynamic patch of spacetime. This statement is justified with calculations later in Section 6.5.

Following our coordinate transformation, the metric can be written in the following form

$$ds^2 = -\frac{W(\zeta)}{\mathcal{H}(\zeta)}d\eta^2 + \frac{\mathcal{H}(\zeta)}{W(\zeta)}d\zeta^2 + \mathcal{H}(\zeta)(dx^2 + dy^2). \tag{6.4.7}$$

Note that we have relabelled the coordinate t, which is interpreted as time in the interior, as η. This is because t becomes a spacelike coordinate in the exterior patch of spacetime, so we instead use the neutral notation (η, ζ) instead of (t, ρ). After the coordinate change, the metric functions are

$$\begin{aligned} W(\zeta) &= 1 - \alpha\zeta, \\ \mathcal{H}_a(\zeta) &= |K_a|\left[\frac{2}{\alpha}\sinh\left(\frac{\alpha h_0}{2|K_a|}\right) + e^{-\frac{\alpha h_0}{2|K_a|}}\zeta\right]. \end{aligned}$$

We will refer to these harmonics in a condensed format by redefining our integration constants such that

$$\mathcal{H}_a = (\beta_a + \gamma_a\zeta)\,,$$

where

$$\beta_a = \frac{2|K_a|}{\alpha}\sinh\left(\frac{\alpha h_0}{2|K_a|}\right), \qquad \gamma_a = |K_a|\exp\left(-\frac{2h_a}{2|K_a|}\right).$$

We refer to these linear functions as harmonics from the perspective of the intersecting brane configurations we introduced in Section 4.5.4. We will see in Chapter 7 that these solutions can be considered as smeared configurations with one transverse dimension, for which a harmonic function is a linear function.

Finally, we rewrite the gauge fields (6.3.9) which will be particularly important when we consider the dimensional uplift of the solution in Chapter 7 and in our thermodynamic computations in Chapter 8

$$F^0_{\zeta\eta} = (\dot{A}^0)_\eta = -\frac{Q_0}{2\mathcal{H}_0^2}, \qquad \tilde{F}_{A|\zeta\eta} = (\dot{\tilde{A}}_A)_\eta = \frac{P^A}{2\mathcal{H}_A^2}. \tag{6.4.8}$$

6.4.2 *The dynamic region*

With an appropriate set of coordinates, let us now demonstrate that the metric can be analytically continued to $\zeta > \alpha^{-1}$. We make an intermediate coordinate transformation to advanced

[7] This mandatory crossing can be understood in the same way as all causal geodesics reaching the singularity for the Schwarzschild solution once the horizon has been crossed.

Eddington-Finkelstein coordinates

$$v = \eta + \zeta_\star, \qquad d\zeta_\star = \frac{\mathcal{H}}{W} d\zeta,$$

where we have introduced the tortoise coordinate $\zeta_\star$ such that the metric can be written in the form

$$ds^2 = -\frac{W}{\mathcal{H}} dv^2 + 2 d\zeta dv + \mathcal{H}(dx^2 + dy^2).$$

This shows that the metric has no singularity at $\zeta = \alpha^{-1}$ and so we can analytically continue the coordinate ζ to $\zeta > \alpha^{-1}$, and then reverse the coordinate transformation to obtain the metric for the dynamic patch of the spacetime

$$ds^2 = -\frac{W(\zeta)}{\mathcal{H}(\zeta)} d\eta^2 + \frac{\mathcal{H}(\zeta)}{W(\zeta)} d\zeta^2 + \mathcal{H}(\zeta)(dx^2 + dy^2),$$

for $\zeta > \alpha^{-1}$. Note that $W(\zeta)$ is an everywhere negative function within the domain of the dynamic patch of the spacetime. To have a clearer picture of our spacetime, we define a new, always positive function within this domain: $\mathcal{W}(\zeta) := \alpha\zeta - 1$. Using this, we can write down the metric for $\zeta > \alpha^{-1}$ where it is immediately obvious that the coordinate ζ is timelike.

The exterior (region II) is the cosmological region where ζ is timelike and the metric is time-dependent. The interior (region I) is static, for which $\zeta < \alpha^{-1}$ is spacelike and η is timelike. As we will see below, the spacetime ends at a timelike singularity located at ζ_s, where ζ_s is the first zero of $\mathcal{H}(\zeta)$. The respective line elements of these two regions are

$$\begin{aligned} ds_I^2 &= -\frac{W(\zeta)}{\mathcal{H}(\zeta)} d\eta^2 + \frac{\mathcal{H}(\zeta)}{W(\zeta)} d\zeta^2 + \mathcal{H}(\zeta)(dx^2 + dy^2), \\ ds_{II}^2 &= -\frac{\mathcal{H}(\zeta)}{\mathcal{W}(\zeta)} d\zeta^2 + \frac{\mathcal{W}(\zeta)}{\mathcal{H}(\zeta)} d\eta^2 + \mathcal{H}(\zeta)(dx^2 + dy^2). \end{aligned} \tag{6.4.9}$$

Having found coordinates suitable for describing both regions of our solution, we now start to analyse its properties. Using (6.3.3) we obtain the following expressions for the physical scalars:

$$z^1 = -i\left(\frac{\mathcal{H}_0\mathcal{H}_1}{\mathcal{H}_2\mathcal{H}_3}\right)^{\frac{1}{2}}, \quad z^2 = -i\left(\frac{\mathcal{H}_0\mathcal{H}_2}{\mathcal{H}_1\mathcal{H}_3}\right)^{\frac{1}{2}}, \quad z^3 = -i\left(\frac{\mathcal{H}_0\mathcal{H}_3}{\mathcal{H}_1\mathcal{H}_2}\right)^{\frac{1}{2}}. \tag{6.4.10}$$

We can study the asymptotic behaviour of the scalars by taking the limit $\zeta \to \infty$ and we find

$$\mathcal{H}_a \simeq |K_a| e^{-\frac{\alpha h_a}{2|K_a|}} \zeta,$$

such that the scalars all tend to a constant value, as all $\mathcal{H}_a$ depend on ζ in the same way

$$\lim_{\zeta\to\infty} z^A = -i\left(\frac{\gamma_0\gamma_A^2}{\gamma_1\gamma_2\gamma_3}\right)^{\frac{1}{2}}.$$

We can look for the presence of curvature singularities through computing the curvature scalars corresponding to the metric (6.4.9). Unlike the previous solutions discussed so far, the Ricci scalar is non-zero, and is found to be

$$R = \frac{W\left(\mathcal{H}'^2 - 2\mathcal{H}\mathcal{H}''\right)}{2\mathcal{H}^3}, \tag{6.4.11}$$

where we denote a derivative with respect to ζ with a prime. The Kretschmann scalar, $K = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$, is given by

$$\begin{aligned}K = &\frac{3W^2\mathcal{H}''^2}{\mathcal{H}^4} - \frac{2W\mathcal{H}'\mathcal{H}''\left(4W\mathcal{H}' + 3\alpha\mathcal{H}\right)}{\mathcal{H}^5} \\ &+ \frac{\mathcal{H}'^2(27W^2\mathcal{H}'^2 + 44\alpha W\mathcal{H}\mathcal{H}' + 20\alpha^2\mathcal{H}^2)}{4\mathcal{H}^6}.\end{aligned} \tag{6.4.12}$$

We find that both have singular behaviour for the limit of $\mathcal{H}(\zeta) \to 0$. As $\mathcal{H}(\zeta)$ is a polynomial of degree four which factorises into four linear polynomials, it will in general have four distinct zeros at $\zeta = \gamma_a\beta_a^{-1}$. The boundary of the spacetime domain is given by the largest of these zeros, or the 'first zero of $\mathcal{H}(\zeta)$', which we denote ζ_s. We remark that $\zeta_s \leq 0$ for all values of the integration constants. Without loss of generality, we assume that the first zero of $\mathcal{H}$ will be for $\mathcal{H}_0 = 0$, such that the singularity will occur at

$$\zeta_s = \frac{1 - e^{\frac{\alpha h_0}{Q_0}}}{\alpha} = -\frac{\beta_0}{\gamma_0}\,. \tag{6.4.13}$$

6.4.3 *Near horizon geometry*

We can study the near horizon geometry of this solution through making a coordinate transformation

$$\chi^2 = \zeta - \alpha^{-1}, \qquad d\zeta^2 = 4\chi^2 d\chi^2. \tag{6.4.14}$$

The horizon values of the harmonic functions are finite

$$\mathcal{H}_a(\alpha^{-1}) = \frac{|K_a|}{\alpha}\exp\left(\frac{\alpha h_0}{2|K_a|}\right),$$

and after the coordinate change, we find

$$d\zeta^2 = 4\chi^2 dr^2, \quad \mathcal{W} = \alpha\chi^2, \quad \mathcal{H} = \frac{2Z\mathcal{E}}{\alpha^2},$$

where we have defined

$$Z := \sqrt{Q_0P^1P^2P^3}, \quad \mathcal{E} := \exp\left[\frac{\alpha}{4}\left(\frac{h_0}{Q_0} + \frac{h^1}{P^1} + \frac{h^2}{P^2} + \frac{h^3}{P^3}\right)\right].$$

We now substitute these expressions into our metric to obtain the near-horizon line element

$$ds^2 = -\frac{\alpha^3\chi^2}{2Z\mathcal{E}}d\eta^2 + \frac{8Z\mathcal{E}}{\alpha^3}d\chi^2 + \frac{2Z\mathcal{E}}{\alpha^2}(dx^2 + dy^2). \tag{6.4.15}$$

With the near horizon approximation, we can study the Hawking temperature (up to a sign) via Wick-rotation of the timelike coordinate η. First we perform an additional coordinate change

$$dR^2 = \left(\frac{8Z\mathcal{E}}{\alpha^3}\right)d\chi^2,$$

to put our line element into the form in which we can compare it with the Rindler line element [35]. Performing the Wick-rotation $\eta \to -i\eta_E$, we ensure that there is no conical singularity by making the identification $\tau \simeq \tau + \beta$, where β can be understood as the inverse temperature $\beta = T_H^{-1}$. Following this reasoning, we find that

$$2\pi T_H = \pm \frac{\alpha^3}{4Z\mathcal{E}}. \tag{6.4.16}$$

We remember that the sign of the computation is not set through this reasoning, as explained in Section 3.5.2.

6.4.4 *The extremal limit*

We are interested in the extremal limit of the solution in which the surface gravity of the Killing horizon vanishes. We can compute the surface gravity using the Kodama-Hayward formulation (3.5.4), where we additionally have knowledge of the sign, and we find

$$\kappa = -\frac{1}{2}\partial_\zeta \left(\frac{\mathcal{W}}{\mathcal{H}}\right)\bigg|_{\zeta=\alpha^{-1}} = -\frac{1}{2}\frac{\alpha}{\mathcal{H}(\alpha^{-1})}.$$

This allows us to identify that α should be interpreted as the non-extremality parameter, with the extremal limit: $\alpha \to 0$. This is consistent with (6.4.16) where we see that $\kappa = 2\pi T_H$. In Section 6.5, we will obtain a global depiction of the spacetime and from that, we can set the sign of the temperature from the classification of the trapping horizons.

As this solution is a generalisation of the Nernst solution, we are interested in how the entropy density behaves in the extremal limit. We can read off the entropy density of the solution as the prefactor of the line element for the coordinates $\{x, y\}$ and we see

$$S_{BH} = \frac{2Z\mathcal{E}}{\alpha^2}.$$

Surprisingly, unlike the Nernst solution which had vanishing area density, the entropy density diverges in the extremal limit.

We can look at the geometry of the solution in the extremal limit, and we find the metric functions are found to be

$$\alpha \to 0 \quad \Rightarrow \quad \mathcal{W}(\zeta) \to -1, \qquad \beta_a \to h_a \zeta, \qquad \gamma_a \to K_a. \tag{6.4.17}$$

The resulting line element is given by

$$ds^2 = -\mathcal{H}^{-1}(\zeta)d\eta^2 + \mathcal{H}(\zeta)d\zeta^2 + \mathcal{H}(\zeta)(dx^2 + dy^2), \tag{6.4.18}$$

where η, ζ are now everywhere timelike and spacelike respectively, and we understand that the spacetime is static.

Taking the extremal limit has a dramatic effect on the causal structure of the spacetime, similar to what we have already seen when taking the extremal limit for the planar solutions of Einstein-Maxwell in Section 5.4. As the function $\mathcal{W}$ becomes constant we find that the location of the horizon is set by $\mathcal{H}^{-1} \to 0$, which occurs when $\zeta \to \infty$. The horizon location is pushed to $\alpha^{-1} \to \infty$, and the resulting spacetime is everywhere static with a naked singularity. This change in the causal structure is a general feature of the cosmological, planar symmetric solutions we have studied, which was also seen in the three-charge solutions of [39]. Further discussion of the relationship between the causal structure and the extremal limit is left for Section 7.3, where we have the additional perspective of these solutions as intersecting brane solutions in higher-dimensional supergravity.

6.4.5 *The asymptotic limit*

We can probe the geometry of the asymptotic region through taking $\zeta \to \infty$ and we find that

$$\lim_{\zeta\to\infty} \mathcal{H}_a(\zeta) \simeq K_a e^{\frac{\alpha h_a}{2K_a}} \zeta, \qquad \lim_{\zeta\to\infty} \mathcal{H}(\zeta) \simeq 2Z\mathcal{E}\zeta^2, \qquad \lim_{\zeta\to\infty} \mathcal{W}(\zeta) \simeq \alpha\zeta.$$

We use this to write down the asymptotic metric

$$ds^2 = -\frac{2Z\mathcal{E}\zeta}{\alpha} d\zeta^2 + \frac{\alpha}{2Z\mathcal{E}\zeta} d\eta^2 + 2Z\mathcal{E}\zeta^2 (d\bar{x}^2 + d\bar{y}^2), \tag{6.4.19}$$

and with a simple change of coordinates to absorb all of the constants, we find the asymptotic metric is in the form

$$ds^2 = -\bar{\zeta} d\bar{\zeta}^2 + \frac{1}{\bar{\zeta}} d\bar{\eta}^2 + \bar{\zeta}^2 (dx^2 + dy^2). \tag{6.4.20}$$

We find that the asymptotic geometry is the same as for the planar solutions of Einstein-Maxwell, which we understand as the planar Schwarzschild solution (AIII metric) with the mass $M = \frac{1}{2}$ [169]. This matching of asymptotic geometries can be understood as in the asymptotic limit, the scalars fall off to a constant value. As we will show in Section 6.6, the solutions of Einstein-Maxwell theory can be recovered from the STU model by imposing all the scalar fields take constant values.

Making the coordinate transformation

$$\bar{\eta} = \left(\frac{3}{2}\right)^{\frac{1}{3}} z, \qquad \bar{\zeta} = \left(\frac{9}{4}\right)^{\frac{1}{3}} \tau^{\frac{2}{3}}, \qquad (x, y) = \left(\frac{4}{9}\right)^{\frac{1}{3}} (x, y),$$

we can rewrite the asymptotic metric in the form

$$ds^2 = -d\tau^2 + \tau^{2/3} dz^2 + \tau^{4/3}(dx^2 + dy^2),$$

which is the type-D vacuum Kasner solution [51]. The Penrose-Carter diagram for the vacuum Kasner type-D solution was given in Figure (5.2).

6.5 CAUSAL STRUCTURE OF COSMOLOGICAL SOLUTIONS

In this section, we use the general framework we built in Section 5.3 to draw the Kruskal diagram and the Penrose-Carter diagrams for the planar symmetric solutions of the STU model. We can also carry through the computations to classify the horizons and find that they are *inner horizons*. From the conformal diagram, we realise that our planar symmetric solutions have an intersection with a class of cosmological solutions studied in [168, 170] for the case of generalised Einstein-Maxwell-Dilaton theory and the orientifold constructions in [171]. We then consider the static patch in more detail, studying causal geodesics and the worldlines of stationary massive particles, following the methodology introduced in Section 5.2.3. We find that the singularity repels all timelike geodesics and that stationary massive particles experience negative acceleration with respect to the singularity. Finally, we compute various mass-like parameters associated with the static region of the spacetime.

6.5.1 *Kruskal coordinates and the Penrose-Carter diagram*

Studying the line element for region I (6.4.9), in which the transverse coordinate ζ is spacelike, we notice that it is of the form (5.3.1) when we make the identification

$$f(\zeta) = \frac{W(\zeta)}{\mathcal{H}(\zeta)} = \frac{1-\alpha\zeta}{2\sqrt{(\beta_0+\gamma_0\zeta)(\beta_1+\gamma_1\zeta)(\beta_2+\gamma_2\zeta)(\beta_3+\gamma_3\zeta)}}.$$

As we allowed $f(r)$ to be completely general in our construction of Kruskal-like coordinates of our cosmological solutions in Section 5.3.1, we can write down our line element in terms of Kruskal-like coordinates $\{U, V\}$

$$ds^2 = -\frac{f(\zeta)e^{-2\kappa\zeta_\star}}{\kappa^2}dVdU + \mathcal{H}(\zeta)d\vec{X}^2\,.$$

From this, we know that the Kruskal diagram for these solutions will be the Figures 5.3b and 5.4b, which we redraw with explicit labels for each region in Figure 6.2. We can similarly draw the Penrose-Carter diagram for these solutions, which we give in Figure 6.3.

We can continue following the generalised discussion in Section 5.3.2 to classify the horizons in our spacetime. Region IV is that which we understand as having the conventional time orientation. The horizon separating regions III and IV is a *future inner horizon*, which has a temperature proportional to the surface gravity and so is negative. It is this horizon which we consider when studying the thermodynamics of these solutions in Section 8.4. The horizon separating regions IV and II is a *past inner horizon*, which has a temperature with a sign opposite to the surface gravity, and so is positive.

6.5.2 *Geodesic motion*

Following the computations performed in Section 5.2.2, we can study the timelike and null geodesics of the solution through considering the Lagrangian

$$s = \mathcal{L} = -\frac{W(\zeta)}{\mathcal{H}(\zeta)}\dot{\eta}^2 + \frac{\mathcal{H}(\zeta)}{W(\zeta)}\dot{\zeta}^2 + \mathcal{H}(\dot{x}^2+\dot{y}^2), \tag{6.5.1}$$

where $s=0,-1$ for null and timelike geodesics respectively (we will not consider spacelike geodesics for which $s=1$). We calculate the constants of motion from the isometries as

$$E = \frac{W}{\mathcal{H}}\dot{\eta}, \qquad a = \mathcal{H}\dot{x}, \qquad b = \mathcal{H}\dot{y},$$

allowing us to rewrite the Lagrangian as

$$s\frac{W}{\mathcal{H}} = -E^2 + \dot{\zeta}^2 + (a^2+b^2)\frac{W}{\mathcal{H}^2}. \tag{6.5.2}$$

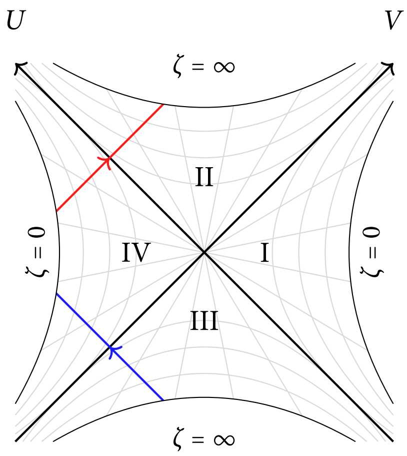


Figure 6.2: Kruskal diagram for the planar symmetric solutions of the STU model, which we can draw from the qualitative similarity of this solution to the general discussion given in Section 5.3. As before, ingoing (outgoing), transverse null geodesics are drawn in blue (red).

Collecting the terms together and separating out an effective potential, we can write the differential equation into the suggestive form

$$\dot{\zeta}^2 = E^2 - V(\zeta), \qquad V(\zeta) = \frac{W}{\mathcal{H}}\left(-s + \frac{a^2+b^2}{\mathcal{H}}\right), \tag{6.5.3}$$

which can be interpreted as the equation of motion for a particle with mass $m = 2$. The domain of validity for the equation of motion is restricted by the inequality

$$V(\zeta) \leq E^2.$$

The point at which $V(\zeta_0) = E$ is interpreted as the classical turning point of the particle's trajectory. Studying the potential $V(\zeta)$ for the domain in which ζ is spacelike, we can look at the motion of causal information along geodesics.

In Figure 6.4 we plot $V(\zeta)$ and see that for region I, in which $\zeta < \alpha^{-1}$, the potential is everywhere positive and therefore considered repulsive. When decreasing ζ from α^{-1} towards ζ_s, we see that the potential monotonically increases until it diverges in the limit of the singularity. As such, we are guaranteed a unique solution for $V(\zeta_0) = E^2$, and hence the existence of a classical turning point. There is one exception to this, the case when $s = a = b = 0$, specific to the motion along transverse null geodesics, where the potential is everywhere zero. In this case, there is no turning point and the transverse, null geodesic reaches $\zeta = \zeta_s$ in finite affine parameter:

$$\lambda = \pm \int \frac{1}{E^2} d\zeta = \pm\frac{1}{E^2}(\zeta_0 - \zeta_s),$$

evaluated from some initial point $\zeta_0 > \zeta_s$. We realise that our spacetime is not geodesically complete.

We can understand this repulsive potential from another perspective through looking at the proper acceleration for a massive particle at rest at some point in region (I/IV) of the spacetime. We consider a massive particle following the orbit of the stationary Killing vector field $k^\mu = \partial_\eta$.

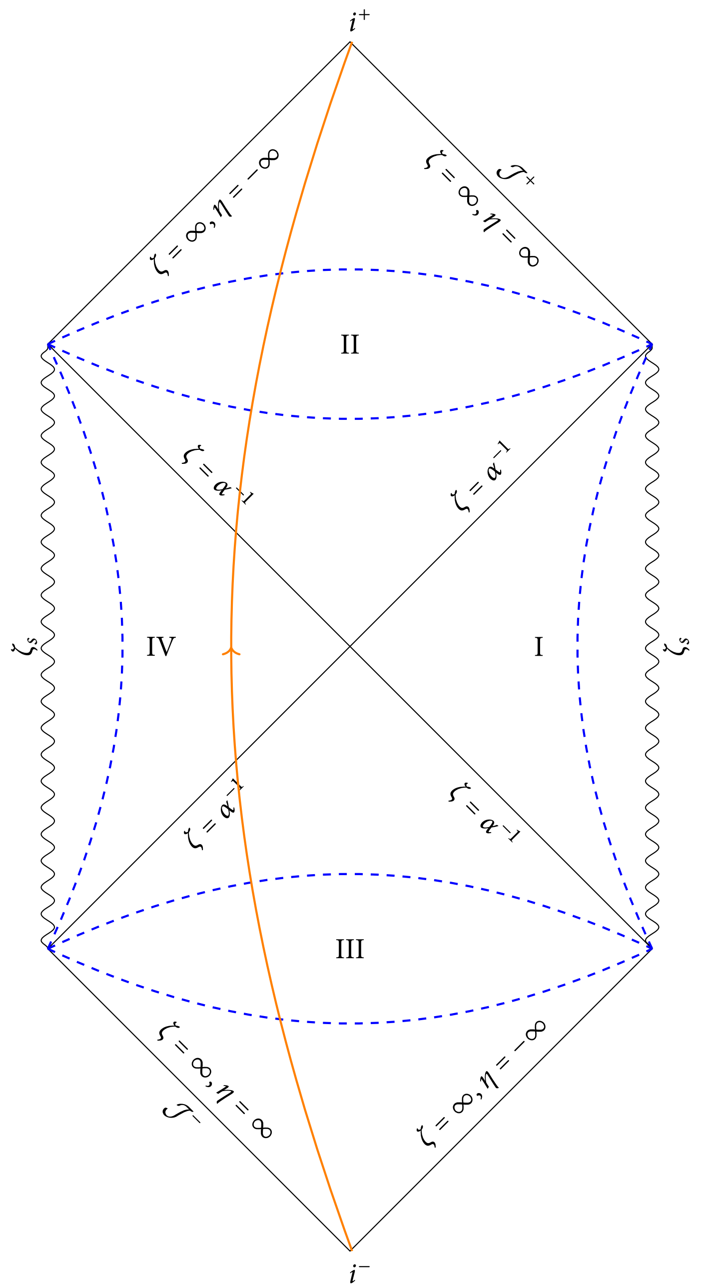


Figure 6.3: Penrose-Carter diagram for the planar cosmological solutions. Starting at $\zeta \to \infty$, we have a cosmological spacetime (III) with a future inner horizon located at a finite point in time; any observer must necessarily fall through the horizon. Passing through the horizon, the spacetime is static (IV) with an avoidable (repulsive) naked singularity located at a point in space. Massive particles at rest experience negative acceleration and will leave the static region through a past inner horizon, into a second dynamic spacetime (I). An example of a complete timelike geodesic is given in orange, spacelike hypersurfaces of constant time are given in blue.

The norm of the Killing vector field is given by

$$k^2 = -\frac{W}{\mathcal{H}},$$

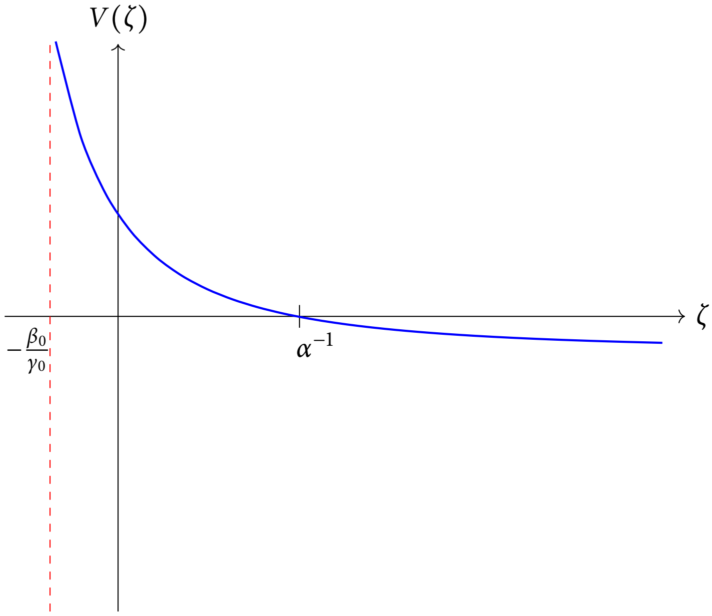


Figure 6.4: Behaviour of the effective potential as a function of ζ for the set of causal geodesics excluding null transverse geodesics, for which $V(\zeta) = 0$.

and the only non-zero component of the proper acceleration (5.2.4) is found to be

$$A^{\zeta} = -\frac{\alpha\mathcal{H} + W\partial_{\zeta}\mathcal{H}}{\mathcal{H}^2}. \tag{6.5.4}$$

As the functions W, $\mathcal{H}$ and $\partial_{\zeta}\mathcal{H}$ are everywhere positive in the static region of spacetime, we see $A^{\zeta} < 0$, and that a particle at rest always experiences a force repelling it from the singularity.

We can understand the behaviour as qualitatively identical the planar Einstein-Maxwell solution discussed in Section 5.2. The metric functions $\mathcal{H}(\zeta)$ and $W(\zeta)$ produce a line element with a more 'complicated' form, but from the point of view of the global structure, we see no difference. This leads us to the same conclusion for the movement of geodesics. For non-zero potentials, a particle will arrive from $\mathcal{J}^-$ and necessarily fall through the horizon at $\zeta = \alpha^{-1}$, which is located at a point in time. The particle will then continue towards the singularity to a minimum distance ζ_0, which is the classical turning point. At this point, the particle will then be reflected by the singularity and continue off through the Killing horizon into a second dynamic spacetime towards $\mathcal{J}^+$. The only causal geodesics which do not follow these trajectories are those for which $V(\zeta) = 0$. These are precisely the transverse null geodesics which fall through the horizon from $\mathcal{J}^-$ and straight into the singularity. In Figure 6.3, an example of a geodesically complete timelike curve is drawn in orange.

6.5.3 *Mass*

We now turn our attention into looking at the mass-like quantities we can compute within the static region of the spacetime. Computation of the conserved electric and magnetic charges of this solution is delayed until Section 8.4. When verifying the first law, we will find it helpful first to dualise our gauge fields, so the solution is purely electric. The computation of conserved charges is better suited within the context of the Euclidean action formalism, which can be done whilst also performing the dualisation, where we compute an exact form for the gauge couplings as functions of ζ. For reference, the charges are given in equations (8.4.8), (8.4.9).

As in Section 5.2.3, we will consider both the Komar formalism, as well as quasi-local Brown-York formalism. For both methods, the mass quantity is extracted from the energy by taking an asymptotic limit. However, both of these formalisms assume the spacetime region to be stationary, and as our static region is finite, taking the asymptotic limit brings us into a region in which our initial assumptions no longer hold. As a result, we cannot interpret the asymptotic evaluation of the energy quantities as mass-like. To remedy this, we do not take the limit and instead find the Komar energy, and quasi-local energy as position-dependent quantities. This methodology is inspired by the computations performed in [168, 170], and mirrors the work already discussed in Section 5.2.

Komar mass

The Komar energy (3.4.1), repeated here,

$$E_K = -\frac{1}{8\pi}\int_{\mathbb{R}^2} \star dk,$$

is computed from our timelike Killing vector k. We again mention our inability to properly normalise this quantity. In spherically symmetric solutions, the Killing vector is normalised such that $k^2 = -1$ in the asymptotic limit, but for these planar symmetric solutions, $k^2 \to 0$ in the limit of $\zeta \to \infty$. Because of this, we leave k^2 unnormalised and study the Komar energy as a position dependent quantity in spacelike ζ.

For the Killing vector: $k^\mu = (\partial_\eta, 0, 0, 0)$ and taking the Hodge dual, with orientation set by $\epsilon_{\eta\zeta xy} = 1$, we find

$$(\star dk)_{\eta\zeta} = -\mathcal{H}\partial_\zeta\left(\frac{W}{\mathcal{H}}\right).$$

The Komar integral is evaluated to

$$E_K = -\frac{1}{8\pi}\int_{\mathbb{R}^2}\left(\alpha + \frac{W}{\mathcal{H}}\partial_\zeta\mathcal{H}\right).$$

Due to the planar symmetry of our solution, this value is divergent when integrating over the plane, so we instead work with the energy density. The resulting position dependent energy density is

$$E_K = -\left(\frac{\alpha}{8\pi} + \frac{\partial_\zeta\mathcal{H}}{8\pi}\frac{W}{\mathcal{H}}\right). \tag{6.5.5}$$

Within the domain of the static region $\mathcal{H}, W, \partial\mathcal{H} > 0$, so as α is positive, the Komar energy will be everywhere negative, regardless of the overall normalisation of k^2. We can play the same game we did in Section 5.2.3 and take the asymptotic limit, despite having no physical motivation. Doing this, we find that it is finite

$$M_K = \frac{\alpha}{8\pi},$$

and that like the planar solutions to Einstein-Maxwell, this asymptotic energy contribution depends only on the non-extremality parameter.

Brown-York mass

Now let us compute the Brown-York energy, using the integral (3.4.3), repeated here

$$E_{BY} = -\frac{1}{8\pi}\int_{\mathbb{R}^2}\sqrt{\sigma}(\mathtt{k}-\mathtt{k}_0), \tag{6.5.6}$$

where $\mathtt{k}$ is the extrinsic curvature of the codimension-two manifold embedded into our spacetime. Comparing our metric (6.4.9) with the ADM decomposition (3.4.4) we identify

$$N^2 = \frac{W}{\mathcal{H}}, \quad V^i = 0, \quad \sigma_{xx} = \sigma_{yy} = \mathcal{H}.$$

The quasi-local energy is then calculated using these quantities together with the trace of the extrinsic curvature

$$\mathtt{k} = \frac{1}{\mathcal{H}}\sqrt{\frac{W}{\mathcal{H}}}\partial_\zeta\mathcal{H}. \tag{6.5.7}$$

As with the Komar energy, we have no natural way to normalise this energy, albeit for a different reason. As discussed in Section 5.2.3, the normalisation for the Brown-York energy comes from picking a suitable background, from which we compute $\mathtt{k}_0$. The most conventional choice, when available, is to use the maximally symmetric space located in the asymptotic limit. In Section 3.4.2, we saw this with the Reissner-Nordström solution, where the divergent energy became finite after we removed the 'infinite energy contribution' from the Minkowski background. Another choice for solutions with electromagnetic charges is to use the extremal solution as the background geometry. The issue with normalisation here is two-fold. Firstly, these normalisations come into play when we choose to set the boundary in the asymptotic region. For these cosmological solutions, this limit brings us into a non-stationary region of the spacetime, thus breaking an initial assumption of the formalism. The second is that even if we accept the limit as a valid concept, neither the background contributions derived from the asymptotic region (6.4.19), or the extremal line element (6.4.18) cancel the divergence as $\zeta \to \infty$ which grows as $\mathcal{O}(\sqrt{\zeta})$. As a result, we study the energy without normalisation, setting $\mathtt{k}_0 = 0$.

Inserting the various pieces into the integral, we find the Brown-York energy

$$\begin{aligned} E_{BY} &= -\frac{1}{8\pi}\int d^2x\sqrt{\sigma}\mathtt{k}, \\ &= -\frac{1}{8\pi}\int d^2x\sqrt{\frac{W}{\mathcal{H}}}\partial_\zeta\mathcal{H}. \end{aligned} \tag{6.5.8}$$

From the point of view of energy density, we calculate

$$E_{BY} = -\frac{\partial_\zeta\mathcal{H}}{8\pi}\sqrt{\frac{W}{\mathcal{H}}}. \tag{6.5.9}$$

This is negative-definite in the static domain due to identical reasoning as for the Komar calculation.

If instead of following the Brown-York mass, we follow the Katz-Lynden-Bell-Israel formalism and include the lapse function, we find that the energy contribution is given by

$$E_{KLBI} = -\frac{1}{8\pi}\int_{\mathbb{R}^2}\sqrt{\sigma}N(\mathtt{k}-\mathtt{k}_0),$$

which for our current solution, gives the energy density

$$E_{KLBI} = -\frac{\partial_\zeta \mathcal{H}}{8\pi}\frac{W}{\mathcal{H}}. \tag{6.5.10}$$

Comparing this with the Brown-York energy, we see introducing the lapse function has no effect on the sign within the static region. However, the Katz-Lynden-Bell-Israel energy does have an interesting property of being finite if we take the (inappropriate) asymptotic limit. Taking the limit of the boundary to $r \to \infty$, we find that the energy is real and finite

$$M_{KLBI} = \lim_{r\to\infty} E_{KLBI} = \frac{\alpha}{4\pi}.$$

and like the Komar mass, depends only on the non-extremality parameter. We will revisit the mass when we understand it as the internal energy of the solution from a thermodynamic perspective. In Section 8.4.3, the mass is computed and we find

$$M = \frac{1}{16\pi}\frac{\alpha}{\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}}.$$

Note that unlike the planar solutions of Einstein-Maxwell, there is a difference between this mass and the Komar mass computed above. This difference comes from a change of coordinates we make in Section 8.4.1, where we impose that the metric over the two-plane falls off as

$$ds_2^2 = \zeta^2(d\bar{x}^2 + d\bar{y}^2),$$

when we take the limit $\zeta \to \infty$. We find that these planar coordinates are related by

$$(x, y) \mapsto 2\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}\,(\bar{x}, \bar{y}),$$

and if we make this coordinate change on the line element for region I in (6.4.9) and recompute the Komar energy, we find

$$E_K = -\frac{1}{16\pi}\frac{1}{\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}}\int_{\mathbb{R}^2}\left(\alpha + \frac{W}{\mathcal{H}}\partial_\zeta\mathcal{H}\right).$$

Again evaluating for $\zeta \to \infty$ we obtain

$$M_K = M = \frac{1}{16\pi}\frac{\alpha}{\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}}. \tag{6.5.11}$$

Showing that these two quantities can be made to be equal through coordinate changes says less about the validity of the solution and more about our inability to know the correct normalisation without some additional information. We see that when considering these position dependent quantities in the static region, the best we can hope for is an understanding of the overall sign of the energy. In Section 8.4, we are able to find an overall normalisation for the Euclidean action by imposing that the thermodynamic derivation for the conserved charges matches Gauss' law. It is this new 'boundary condition', which gives us a foothold to go from, which then leads to the verification for the first law of thermodynamics.

6.6 RECOVERING THE PLANAR SOLUTION OF EINSTEIN-MAXWELL THEORY

If we take the limit of setting the physical scalars of the theory to be constant, the geometry of the four-charge solution becomes that of the vacuum solution to the Einstein-Maxwell equations with planar symmetry, studied in Chapter 5. This is expected as the Reissner-Nordström solution is the resulting geometry for the spherically symmetric solution to the STU model with constant physical scalars. This matching of solutions can be understood as the non-extremal generalisation of the double-extremal limit [181].

To reiterate, the physical scalars are given by

$$z^A = -i\mathcal{H}_A\left(\frac{\mathcal{H}_0}{\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3}\right)^{\frac{1}{2}},$$

and we see that they are everywhere constant under the restriction that all the harmonics are equal: $\mathcal{H}_0 = \mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3$. This is achieved by fine-tuning the integration constants such that $Q_0 = P^1 = P^2 = P^3 = K$ and $h_0 = h^1 = h^2 = h^3 = h$.

Understanding that the scalar fields are constant after imposing this matching condition on the integration constants, we can look at how the Lagrangian (6.1.1) changes in this limit. As the scalar fields are constant, the kinetic term for the scalar field vanishes. The gauge coupling $\mathcal{I}_{IJ}$ depends only on the scalar fields, and so is itself a constant. The four gauge fields F^I are all equal and so we can reinterpret the contribution from the $n_V + 1$ gauge fields as a single gauge field contribution. All that remains of (6.1.1) is the Ricci scalar and a single Abelian gauge field and so by a suitable rescaling of F, the action can be brought into the form of the Einstein-Maxwell Lagrangian (3.3.1).

This transition from the STU model to the Einstein-Maxwell theory is also mirrored in our geometry. When we take the above limit for our integration constants, we recover the line element for the solution of Einstein-Maxwell theory with planar symmetry (5.1.5). The metric for the static patch of the spacetime, repeated here

$$ds^2 = -\frac{W(\zeta)}{\mathcal{H}(\zeta)}d\eta^2 + \frac{\mathcal{H}(\zeta)}{W(\zeta)}d\zeta^2 + \mathcal{H}(\zeta)(dx^2 + dy^2), \tag{6.6.1}$$

changes at the level of these functions, which are now given by

$$W(\zeta) = 1 - \alpha\zeta, \qquad \mathcal{H}(\zeta) = 2(\beta + \gamma\zeta)^2,$$

with our integration constants simplified as

$$\alpha = 2B, \quad \beta = \frac{2K}{\alpha}\sinh\left(\frac{\alpha h}{2K}\right), \quad \gamma = K\exp\left(-\frac{\alpha h}{2K}\right), \qquad \alpha, \beta, \gamma \in (0, \infty).$$

The metric written in terms of these new constants for $\zeta < \alpha^{-1}$ is given by

$$ds^2 = -\frac{1 - \alpha\zeta}{2(\beta + \gamma\zeta)^2}d\eta^2 + \frac{2(\beta + \gamma\zeta)^2}{(1 - \alpha\zeta)}d\zeta^2 + 2(\beta + \gamma\zeta)^2(dx^2 + dy^2). \tag{6.6.2}$$

We now show that a suitable change in coordinates maps this line element into our solution (5.1.5), derived in Chapter 5 which we repeat here for comparison

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(dx^2 + dy^2), \qquad f(r) = -\frac{2M}{r} + \frac{Q^2}{r^2}.$$

The equivalence of our solution (6.6.1) and (5.1.5) is found after making the following coordinate transformations

$$2(\beta+\gamma\zeta)^2 = \tilde{r}^2 \Rightarrow \tilde{r} = \sqrt{2}(\beta+\gamma\zeta), \qquad \zeta = \frac{1}{\gamma}\left(\frac{\tilde{r}}{\sqrt{2}} - \beta\right), \qquad d\zeta = \frac{d\tilde{r}}{\sqrt{2}\gamma},$$

we can then rewrite parts of the line element as

$$\frac{1-\alpha\zeta}{2(\beta+\gamma\zeta)^2}d\eta^2 = \left(-\frac{\alpha}{\sqrt{2}\gamma\tilde{r}} + \frac{\alpha\beta+\gamma}{\gamma\tilde{r}^2}\right)d\eta^2,$$
$$\frac{2(\beta+\gamma\zeta)^2}{(\alpha\zeta-1)}d\zeta^2 = \left(-\frac{\alpha}{\sqrt{2}\gamma\tilde{r}} + \frac{\alpha\beta+\gamma}{\gamma\tilde{r}^2}\right)^{-1}\frac{d\tilde{r}^2}{2\gamma^2}.$$

To ensure that the functions preceding the $d\eta^2$ and $d\tilde{r}^2$ are each other's multiplicative inverse we rescale $\tilde{r}$ such that

$$r = \frac{\tilde{r}}{\sqrt{2}\gamma}, \qquad dr = \frac{d\tilde{r}}{\sqrt{2}\gamma}, \qquad \tilde{r} = \gamma\sqrt{2}r.$$

This allows us to write down the metric in the form

$$ds^2 = -f(r)d\eta^2 + \frac{dr^2}{f(r)} + 2\gamma^2 t^2(dx^2+dy^2),$$

where we have defined the function

$$f(r) := -\frac{\alpha}{2\gamma^2}\frac{1}{r} + \frac{\alpha\beta+\gamma}{2\gamma^3}\frac{1}{r^2}.$$

Finally we rescale the x and y coordinates to re-absorb the $2\gamma^2$ factor and rename η to t to arrive at the metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(dx^2+dy^2). \qquad (6.6.3)$$

Looking at the function $f(r)$ and comparing this to (5.1.5) we can relate the integration constants from our solution to the 'mass' and electric charge of the solution.

$$M = \frac{\alpha}{4\gamma^2}, \qquad Q^2 = \frac{\alpha\beta+\gamma}{2\gamma^3}. \qquad (6.6.4)$$

Thinking back to the asymptotic limit of the Komar energy for planar solution Einstein-Maxwell theory (5.2.7) and for the STU model after rescaling (6.5.11)

$$M_{EM} = \frac{M}{4\pi}, \qquad M_{STU} = \frac{1}{16\pi}\frac{\alpha}{\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}} = \frac{\alpha}{16\pi\gamma^2},$$

we see that by making the identification (6.6.4), these mass-like parameters match exactly. Furthermore, these masses are precisely the masses we derive from a thermodynamic perspective in Chapter 8.

6.7 SUMMARY

In this chapter we have derived and studied planar symmetric solutions of $\mathcal{N} = 2$ supergravity in four dimensions. In particular, we have found non-extremal, cosmological solutions of the STU model containing a Killing horizon, which are asymptotic to the type-D Kasner spacetimes. As explained throughout this chapter, the causal structure of this solution matches that of the planar solutions of Einstein-Maxwell theory, so we will not repeat ourselves here. We concluded the chapter by showing that in the limit of ensuring that all the physical scalars z^A are constant, the planar symmetric solution of the STU model becomes that of the Einstein-Maxwell theory. This behaviour is expected as this has already been seen for the case when the solutions are spherically symmetric [181].

The initial motivation for this work was to generalise the Nernst solutions discussed in [28]. As a result, we are biased towards considering the thermodynamics of this solution and its behaviour in the extremal limit. The defining property of the Nernst branes is that the area density of the horizon vanishes in the extremal limit, which has a formal analogy with the strict third law of thermodynamics. However, when taking the extremal limit for these solutions, we instead find that the area density diverges and the resulting spacetime is static with a naked singularity. To better understand this extremal behaviour, we study these solutions in higher dimensions. This is the topic of Chapter 7.

Furthermore, we are interested in whether these solutions have Killing horizons which obey the first law of black hole mechanics. When attempting to find a mass-like parameter, we concluded that there was no natural method to set the overall normalisation of the mass and our value was inherently position dependent. As the first law is a differential relationship, we are unable to verify the first law without additional structure. In Chapter 8, we introduce a novel formulation of the Euclidean action formalism, which when considered together with using Gauss' law as an effective boundary condition allows for the verification for the first law.

As further discussion of this solution is dependent on the results of other chapters of this thesis, we prolong a discussion of the planar solutions of the STU model until further results have been presented.

7

COSMOLOGICAL SOLUTIONS IN HIGHER DIMENSIONS

To better understand the physical origin of the four-charge solution, we turn our attention to finding consistent higher-dimensional embeddings. This is motivated by the success of [34], which offered a new understanding of the Nernst solution [28] by using a five-dimensional embedding. We are further motivated by the work of [171] and comments made by [168] which link cosmological solutions to higher-dimensional theories reduced on orientifolds.

Before continuing, we remark on the work of [182], in which a cosmological solution of the STU model was found. In this work, the authors also began by dimensionally reducing $\mathcal{N} = 2$ vector multiplet supergravity from four to three dimensions. Solutions are then derived through understanding the symmetric spaces associated to the sigma models. In contrast to our work, the authors explicitly search for cosmological rather than static solutions. The problem of uplifting the solution back to four dimensions becomes a problem in representation theory and their resulting four-dimensional spacetime has a general, but quite complicated form. The upshot is that the form of the line element derived has a coordinate structure which only yields a fragment of the solution that we found in Section 6.4. Specifically, their solution does not include the Killing horizon, and the static region and singularity are therefore not discussed. Their work has other virtues though, and several examples of cosmological solutions to $\mathcal{N} = 2$ supergravity are found. In particular, there is an interesting section in which they understand their solutions from a higher-dimensional perspective through a lift from four to ten dimensions on the orientifold $K3 \times T^2/\mathbb{Z}_2$. The explicit relation between their solution and ours is complicated, and we will do not give the mapping.

For our work, we will also study the cosmological solutions in ten and eleven dimensions, but instead of uplifting over the orientifold, we instead consider simpler, toroidal lifts. Despite this simplification, will are still able to relate our cosmological solution to black hole solutions of the STU model and to make contact with six-dimensional BPS solutions. By taking the extremal limit, we will be able to interpret our line elements as intersecting brane solutions from both a string and M-theory perspective.

We note here that in lifting of our solution over tori, we necessarily consider not the reduction of the full string theory, but some consistent truncation. This is because the reduction over the torus does not break any supersymmetry and so the naive reduction would produce $\mathcal{N} = 8$ supergravity in four dimensions. We also note that instead of reducing over an orientifold, we could also consider the decomposition of the fields while reducing over a Calabi-Yau threefold, which would break the right amount of supersymmetry, but do not comment on this further.

This chapter is organised in the following way. First, we rewrite our Lagrangian (6.1.1) in a form that allows a direct comparison with [131]. We then uplift our non-extremal planar solutions of the STU model from four dimensions to five, six, ten and eleven dimensions, expressing our solutions as embedded into truncations of string/M-theory. Upon taking the extremal limit of the four-dimensional solution, we make contact with well-known brane configurations in string/M-theory models. Additional fine-tuning of the four-dimensional electric charges is shown to make the extremal six-dimensional uplift supersymmetric. This is a particularly surprising result as we did not utilise Killing spinor equations and therefore the existence of supersymmetric limit was not guaranteed.

7.1 DIMENSIONAL LIFTING OF THE COSMOLOGICAL STU SOLUTION

In Section 4.3.3, we performed the dimensional reduction of $\mathcal{N}=1$, six-dimensional supergravity coupled to a tensor multiplet, recovering $\mathcal{N}=2$ supergravity coupled to three vector multiplets in four dimensions. This is the STU model, for which we found planar symmetric solutions in Chapter 6. In this section, we reverse this procedure to obtain five and six-dimensional embeddings for the planar symmetric solutions. Furthermore, the five-dimensional solution can be uplifted over T^6 to give an eleven-dimensional line element, which is a solution of a consistent truncation of eleven-dimensional supergravity. Similarly, the six-dimensional solution can be uplifted over T^4, producing a ten-dimensional line element which is a solution to a consistent truncation of string theory.

We will follow the oxidation[1] prescription of [131] to write down consistent truncated string/M-theory Lagrangians and their corresponding metric and gauge field content.

7.1.1 *Rewriting the Lagrangian for uplift*

Our starting point is the Lagrangian (6.1.1), repeated here for reference

$$e_4^{-1}\mathcal{L} = -\frac{1}{2}R - g_{A\bar{B}}\partial_\mu z^A \partial^\mu \bar{z}^{\bar{B}} + \frac{1}{4}\mathcal{I}_{IJ}F^I_{\mu\nu}F^{J|\mu\nu} + \frac{1}{4}\mathcal{R}_{IJ}F^I_{\mu\nu}\tilde{F}^{J|\mu\nu}.$$

To perform the uplift, we first find the exact form of the couplings in terms of our physical scalars z^A. The explicit expressions for the gauge couplings are obtained from the prepotential $F(X)$ using standard special geometry formulae. Full derivations for the couplings are given in appendix E, in which we use the same conventions as [100].

Remembering that we have imposed the 'purely imaginary' condition, the couplings take the form:

$$\mathcal{R}_{IJ} = 0, \qquad \mathcal{I}_{IJ} = \mathrm{diag}\left(-stu, -\frac{tu}{s}, -\frac{su}{t}, -\frac{st}{u}\right), \tag{7.1.1}$$

$$g_{A\bar{B}} = \mathrm{diag}\left(\frac{1}{4s^2}, \frac{1}{4t^2}, \frac{1}{4u^2}\right), \tag{7.1.2}$$

[1] The process of embedding lower-dimensional solutions in higher dimensions is sometimes called *oxidation*, and has nothing to do with chemistry!

where

$$s = -\mathrm{Im}(z^1), \quad t = -\mathrm{Im}(z^2), \quad u = -\mathrm{Im}(z^3).$$

When evaluating the couplings $\mathcal{I}_{IJ}$ on our solution, we find by inserting the value of the scalar fields (6.4.10)

$$\mathcal{I}_{00}^2 = \frac{\mathcal{H}_0^3}{\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3}, \qquad \mathcal{I}_{11}^2 = \frac{\mathcal{H}_0\mathcal{H}_2\mathcal{H}_3}{\mathcal{H}_1^3}, \qquad \mathcal{I}_{22}^2 = \frac{\mathcal{H}_0\mathcal{H}_1\mathcal{H}_3}{\mathcal{H}_2^3}, \qquad \mathcal{I}_{33}^2 = \frac{\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2}{\mathcal{H}_3^3}. \tag{7.1.3}$$

After redefining our scalars

$$s = e^{-\phi_1}, \qquad t = e^{-\phi_2}, \qquad u = e^{-\phi_3}, \tag{7.1.4}$$

the Lagrangian takes the following form

$$e_4^{-1}\mathcal{L} = -\frac{1}{2}R - \frac{1}{4}\partial_\mu\phi_A\partial^\mu\phi_A - \frac{1}{4}e^{-\phi_1-\phi_2-\phi_3}\left[(F^0)^2 + e^{2\phi_A}(F^A)^2\right], \tag{7.1.5}$$

where we sum over $A \in \{1,2,3\}$. Using the STU couplings (7.1.1) and the expressions for the physical scalrs (6.4.10), we can evaluate the scalars ϕ_i on our solution and thus express them as functions of ζ

$$e^{2\phi_1} = \frac{\mathcal{I}_{11}}{\mathcal{I}_{00}} = \frac{\mathcal{H}_2\mathcal{H}_3}{\mathcal{H}_0\mathcal{H}_1}, \qquad e^{2\phi_2} = \frac{\mathcal{I}_{22}}{\mathcal{I}_{00}} = \frac{\mathcal{H}_1\mathcal{H}_3}{\mathcal{H}_0\mathcal{H}_2}, \qquad e^{2\phi_3} = \frac{\mathcal{I}_{33}}{\mathcal{I}_{00}} = \frac{\mathcal{H}_1\mathcal{H}_2}{\mathcal{H}_0\mathcal{H}_3}\,.$$

To embed our solution into higher dimensions, we will use various ansatz given in [131] allowing for us to obtain the results for ten and eleven-dimensional solutions via six and five-dimensional solutions respectively. The relevant truncation of the four-dimensional STU Lagrangian given in [131] is

$$e_4^{-1}\mathcal{L}_4 = -R - \frac{1}{2}\partial_\mu\varphi_i\partial^\mu\varphi_i - \frac{1}{4}e^{-\varphi_1-\varphi_2-\varphi_3}\left[(\mathbb{F}^4)^2 + e^{2\varphi_i}(\tilde{\mathbb{F}}_i)^2\right]. \tag{7.1.6}$$

This is related to our Lagrangian (7.1.5) by an overall factor of 2, together with the following rescaling of the gauge fields and the scalars

$$F^0 = \frac{1}{\sqrt{2}}\mathbb{F}^4, \qquad F^A = \frac{1}{\sqrt{2}}\tilde{\mathbb{F}}_i, \qquad \phi_i = \varphi_i.$$

We will need to keep track of these factors while oxidising, and insert the exact values for the gauge fields into the ansatz of [131].

7.1.2 *Oxidation to five dimensions*

The STU model can be consistently embedded into five dimensions with the bosonic Lagrangian

$$\mathcal{L}_5 = -R \star 1 - \frac{1}{2}h_i^{-2}\left(\star dh_i \wedge dh_i + \star\tilde{\mathbb{F}}_i \wedge \tilde{\mathbb{F}}_i\right) + \tilde{\mathbb{F}}_1 \wedge \tilde{\mathbb{F}}_2 \wedge \tilde{\mathbb{A}}_3, \tag{7.1.7}$$

where the five-dimensional scalars h^i satisfy the constraint $h_1h_2h_3 = 1$. Using the Kaluza-Klein reduction ansatz

$$ds_5 = f^{-1}ds_4^2 + f^2(dz_5 - \mathbb{A}^4)^2\,, \qquad \tilde{\mathbb{A}}_{(5D)i} = \tilde{\mathbb{A}}_i, \tag{7.1.8}$$

we obtain the four-dimensional Lagrangian (7.1.6) when we make the choice $fh_i = e^{-\varphi_i}$. The vector field $\mathbb{A}^4$ is the Kaluza-Klein vector field, while the vector fields $\mathbb{A}_i$ descend from the five-dimensional vector fields.

Introducing new linear combinations σ, φ, λ for the three independent real four-dimensional scalars by

$$\varphi_1 = -\frac{2}{\sqrt{6}}\sigma + \frac{1}{\sqrt{3}}\lambda, \quad \varphi_2 = -\frac{1}{\sqrt{2}}\phi + \frac{1}{\sqrt{6}}\sigma + \frac{1}{\sqrt{3}}\lambda, \quad \varphi_3 = \frac{1}{\sqrt{2}}\phi + \frac{1}{\sqrt{6}}\sigma + \frac{1}{\sqrt{3}}\lambda,$$

the five-dimensional constrained scalars h^i can be expressed in terms of two independent fields

$$h_1 = e^{2\sigma/\sqrt{6}}, \qquad h_2 = e^{\phi/\sqrt{2}-\sigma/\sqrt{6}}, \qquad h_3 = e^{-\phi/\sqrt{2}-\sigma/\sqrt{6}}\,.$$

Combining these two relations we obtain

$$\begin{aligned} h_1 &= \exp\left(-\frac{2\varphi_1}{3} + \frac{\varphi_2}{3} + \frac{\varphi_3}{3}\right) = \left(\frac{\mathcal{I}_{22}\mathcal{I}_{33}}{\mathcal{I}_{11}^2}\right)^{\frac{1}{6}}, \\ h_2 &= \exp\left(\frac{\varphi_1}{3} - \frac{2\varphi_2}{3} + \frac{\varphi_3}{3}\right) = \left(\frac{\mathcal{I}_{11}\mathcal{I}_{33}}{\mathcal{I}_{22}^2}\right)^{\frac{1}{6}}, \\ h_3 &= \exp\left(\frac{\varphi_1}{3} + \frac{\varphi_2}{3} - \frac{2\varphi_3}{3}\right) = \left(\frac{\mathcal{I}_{11}\mathcal{I}_{22}}{\mathcal{I}_{33}^2}\right)^{\frac{1}{6}}, \end{aligned}$$

and expressing the gauge couplings in terms of the harmonic functions $\mathcal{H}_i$,

$$h_i = \frac{\mathcal{H}_i}{(\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3)^{\frac{1}{3}}}\,, \tag{7.1.9}$$

allows us to write down the Kaluza-Klein scalar f in terms of ζ

$$f = e^{-\varphi_i}h_i^{-1} = \left(\frac{\mathcal{I}_{00}^3}{\mathcal{I}_{11}\mathcal{I}_{22}\mathcal{I}_{33}}\right)^{\frac{1}{6}} = \left(\frac{\mathcal{H}_0^3}{\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3}\right)^{1/6}\,. \tag{7.1.10}$$

Five-dimensional metric

Using (7.1.8) together with (7.1.10) and $\mathbb{A}^4 = \sqrt{2}A^0$, as well as collecting common factors, we obtain the following five-dimensional metric for the uplift of our four-charge solution:

$$\begin{aligned} ds_5^2 = (\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3)^{-\frac{1}{3}}\Bigg[&\mathcal{H}_0 dz_5^2 + \frac{\mathcal{W}}{2\mathcal{H}_0}\left(\mathcal{W}\frac{\gamma_0^2}{Q_0^2}+1\right)d\eta^2 - \frac{\mathcal{W}\gamma_0}{\sqrt{2}Q_0}d\eta dz_5 \\ &+ 2\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\left(-\frac{d\zeta^2}{\mathcal{W}} + dx^2 + dy^2\right)\Bigg]. \end{aligned} \tag{7.1.11}$$

Five-dimensional gauge potential

To obtain expressions for the five-dimensional gauge potentials, it is necessary to express all the gauge fields in our solution in terms of electric components. This requires replacing the dual vector potentials $\tilde{A}_A$ by the 'standard' vector potentials A^A. The associated field strength $\tilde{F}_A$ and F^A are related by Hodge duality together with multiplication by inverse gauge coupling matrix

$$F^A = -\mathcal{I}^{AB} \star \tilde{F}_B.$$

Using the form of the gauge potential found in (6.4.8), standard calculations give their form

$$F^A = -\mathcal{I}^{AB} \star \tilde{F}_B = -\frac{p^A}{2\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}} dx \wedge dy. \tag{7.1.12}$$

Integrating and relating to the gauge fields in our ansatz, we obtain the form of the three five-dimensional vector potentials

$$\tilde{\mathbb{A}}_i = \sqrt{2}A^A = \mathfrak{p}_a\left(y dx - x dy\right), \qquad \mathfrak{p}_a = \frac{p^a}{2\sqrt{2\gamma_0\gamma_1\gamma_2\gamma_3}}. \tag{7.1.13}$$

We will return to these gauge fields before uplifting the solution to six dimensions, when it will be necessary to Hodge dualise in five dimensions to obtain a two-form potential.

Extremal limit

We now investigate the effect of the four-dimensional extremal limit defined in Section 6.4.4 for the following higher-dimensional lifts.[2] Just as in the four-dimensional case, the horizon for the five-dimensional solution is pushed out to $\zeta \to \infty$ and the static region takes up the entirety of our spacetime; in other words, the extremal limit results in a solution containing a naked singularity. Simplifying the metric functions using the expressions from (6.4.17) we can write down the five-dimensional line element in the form

$$ds_5^2 = (\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3)^{-\frac{1}{3}}\left[d\eta dz_5 + \mathcal{H}_0 dz_5^2 + \mathcal{H}_1\mathcal{H}_2\mathcal{H}_3(d\zeta^2 + dx^2 + dy^2)\right].$$

We will consider the extremal limit for each of the following uplifts as we further oxidise the STU model.

7.1.3 *Oxidation to eleven dimensions*

To uplift our solution to eleven dimensions, we start with the bosonic part of the eleven-dimensional supergravity Lagrangian

$$\mathcal{L}_{11} = -R \star 1 - \frac{1}{2} \star \mathcal{F} \wedge \mathcal{F} - \frac{1}{6}\mathcal{F} \wedge \mathcal{F} \wedge \mathcal{A},$$

where $\mathcal{A}$ is the three-form such that $\mathcal{F} = d\mathcal{A}$ is the four-form field strength. We can directly embed the five-dimensional STU model into this theory through a Kaluza-Klein reduction on T^6 with the ansatz

$$ds_{11}^2 = ds_5^2 + h_1(dy_1^2 + dy_2^2) + h_2(dy_3^2 + dy_4^2) + h_3(dy_5^2 + dy_6^2),$$

$$\mathcal{A} = \tilde{\mathbb{A}}_1 \wedge dy^1 \wedge dy^2 + \tilde{\mathbb{A}}_2 \wedge dy^3 \wedge dy^4 + \tilde{\mathbb{A}}_3 \wedge dy^5 \wedge dy^6.$$

In a consistent truncation to five-dimensional minimal supergravity, the volume of the torus corresponds to a scalar in a hypermultiplet, while its shape is encoded by scalars in vector multiplets.

[2] We note here that we use the same symbols h_a for the integration constants in (6.4.17) and the constrained five-dimensional scalars, as this allows us to match notation with the literature on five-dimensional solutions. We trust that the reader will infer from context which quantity is meant in a particular expression.

This factorisation imposes the condition $h_1 h_2 h_3 = 1$ on the scalars h_i. By restricting our field content, we can consistently truncate out the hypermultiplets and remain with the five-dimensional STU model with two vector multiplets.

We now combine this $5D/11D$ lift with the previous $4D/5D$ lift. In our four-dimensional solution, we can express the h_i as functions of ζ through the harmonic functions $\mathcal{H}_i$, see (7.1.9). The three-form gauge potential is found directly from the components of (7.1.13). Thus the full line element for the eleven-dimensional lift of the non-extremal planar solution to the four-dimensional STU model is

$$\begin{aligned} ds_{11}^2 = (\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3)^{-\frac{1}{3}}\Bigg[&\mathcal{H}_0 dz_5^2 + \frac{\mathcal{W}}{2\mathcal{H}_0}\left(\mathcal{W}\frac{\gamma_0^2}{Q_0^2}+1\right)d\eta^2 - \frac{\mathcal{W}\gamma_0}{\sqrt{2}Q_0}d\eta dz_5 \\ &+ 2\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\left(-\frac{d\zeta^2}{\mathcal{W}} + dx^2 + dy^2\right) \\ &+ \mathcal{H}_1(dy_1^2+dy_2^2) + \mathcal{H}_2(dy_3^2+dy_4^2) + \mathcal{H}_3(dy_5^2+dy_6^2)\Bigg]. \end{aligned} \tag{7.1.14}$$

Eleven-dimensional extremal limit

By again substituting in the four-dimensional extremal limit (6.4.17) we can write down (7.1.14) in the extremal limit to find

$$\begin{aligned} ds_{11}^2 =& (\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3)^{-\frac{1}{3}}\big[d\eta dz_5 + \mathcal{H}_0 dz_5^2 + \mathcal{H}_1\mathcal{H}_2\mathcal{H}_3(d\zeta^2 + dx^2 + dy^2) \\ &+ \mathcal{H}_1(dy_1^2+dy_2^2) + \mathcal{H}_2(dy_3^2+dy_4^2) + \mathcal{H}_3(dy_5^2+dy_6^2)\big]. \end{aligned} \tag{7.1.15}$$

This extremal solution should remind the reader of the line element (4.5.6) introduced in Section 4.5.4 which described the configuration of three M5 branes intersecting over a string, encoded by $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$, with a PP-wave superimposed along the intersection direction, encoded by $\mathcal{H}_0$ [164]. For solutions with spherical symmetry, which in Section 4.5.4 is enforced by the harmonic functions encoded by H_i, compactification on $T^6 \times S^1$ gave rise to four-charged, BPS black holes. For these solutions, the branes are delocalised along $y^1, \ldots, y^6$ but localised in the remaining three spacelike directions. The four-dimensional line element is given in (4.5.10) and a deeper discussion for these solutions and their relationship to $\mathcal{N} = 2$ supergravity appears in [166]. There is a crucial difference between those solutions and our extremal configuration (7.1.15), which can be seen in the harmonic functions $\mathcal{H}_a$. In our solutions, the M5 branes have in addition been further delocalised in two of the non-compact directions, which upon the reduction over $T^6 \times S^1$, will give rise to planar rather than spherical symmetry which was imposed at the level of the metric ansatz.

7.1.4 *Oxidation to six dimensions*

Lifting the four-dimensional Lagrangian to six dimensions by extending the $4D/5D$ lift requires a tweak of the five-dimensional Lagrangian, namely to Hodge-dualise one of the three vector potentials into a two-form B. This is the reverse process of the computation we performed in Section 4.3.3, where we dualise the three-form so we can reduce all three 2-form field strengths in the same way. The reason we dualise before uplifting is that the six-dimensional supergravity is chiral and both the supergravity multiplet and tensor multiplets contain self-dual or anti-self-dual tensor fields which do not admit a standard Lagrangian description. However, in supergravity coupled to one tensor multiplet (plus vector and hypermultiplets), one self-dual and one anti-self-dual tensor combine into an unconstrained tensor, which allows a standard Lagrangian description. String compactifications to six dimensions are of this type, with the tensor field descending from the ten-dimensional Kalb-Ramond field.

Matching our conventions with the work of [131] we define the three-form from the dualisation of the two-form field strength in five dimensions

$$\tilde{\mathbb{F}}_3 = d\tilde{\mathbb{A}}_3 = -h_1^{-2}h_2^{-2} \star_5 \mathbb{H}. \tag{7.1.16}$$

Making this transformation and substituting into the Lagrangian results in

$$\begin{aligned}\mathcal{L}_5 =& -R\star 1 - \frac{1}{2}h_i^{-2}\star dh_i\wedge dh_i + \frac{1}{2}h_1^{-2}\star\tilde{\mathbb{F}}_1\wedge\tilde{\mathbb{F}}_1\\ &+\frac{1}{2}h_2^{-2}\star\tilde{\mathbb{F}}_2\wedge\tilde{\mathbb{F}}_2 - \frac{1}{2}h_1^{-2}h_2^{-2}\star\mathbb{H}\wedge\mathbb{H}.\end{aligned} \tag{7.1.17}$$

We can now use the results of [131], and first work with the new three-form field strength in five dimensions in terms of ζ.

Dualisation of the five-dimensional gauge field

Taking the Hodge dual of (7.1.16) we find the three-form

$$\star_5\mathbb{H} = -h_1^2h_2^2\tilde{\mathbb{F}}_3\,, \qquad \star_5\star_5\mathbb{H} = -\star_5\left(h_1^2h_2^2\tilde{\mathbb{F}}_3\right), \qquad \mathbb{H} = \star_5\left(h_1^2h_2^2\tilde{\mathbb{F}}_3\right),$$

where we have used that for a k-form ω in n dimensions in the Lorentzian signature $\star\star\omega = (-1)^{k(n-k)+1}\omega$. Substituting in (7.1.12) together with:

$$\sqrt{-g_5} = 2(\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3)^{\frac{2}{3}}, \qquad \epsilon_{\eta\zeta xyz_5} = 1\,,$$
$$h_1^2h_2^2 = h_3^{-2} = (\mathcal{H}_1\mathcal{H}_2)^{\frac{2}{3}}\mathcal{H}_3^{-\frac{4}{3}}, \qquad g^{xx} = g^{yy} = \frac{1}{2}(\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3)^{-\frac{2}{3}},$$

we find that the three-form is

$$\mathbb{H} = -\left(\frac{\mathfrak{p}_3}{\mathcal{H}_3^2}\right)d\eta\wedge d\zeta\wedge dz_5\,.$$

Lift to six dimensions

The six-dimensional Lagrangian is

$$\mathcal{L}_6 = -R\star 1 - \frac{1}{2}\star d\phi\wedge d\phi - \frac{1}{2}e^{-\sqrt{2}\phi}\star H\wedge H,$$

where $H = dB$ is a three-form field strength. The reduction ansatz which reproduces our five-dimensional Lagrangian (7.1.17) is

$$ds_6^2 = e^{\sigma/\sqrt{6}}ds_5^2 + e^{-3\sigma/\sqrt{6}}(dz_6+\tilde{\mathbb{A}}_1)^2\,, \qquad B_{(6D)} = B + \tilde{\mathbb{A}}_2\wedge(dz_6+\tilde{\mathbb{A}}_1),$$

with the field strengths decomposed as

$$H_{(6D)} = \mathbb{H} + \tilde{\mathbb{F}}_2 \wedge (dz_6 + \tilde{\mathbb{A}}_1) \,, \qquad \mathbb{H} = dB - \tilde{\mathbb{A}}_2 \wedge \tilde{\mathbb{F}}_1, \qquad \tilde{\mathbb{F}}_i = d\tilde{\mathbb{A}}_i \,.$$

We see that from our parameterisation of the h_i we can write the six-dimensional Kaluza-Klein scalar as

$$e^{\sigma/\sqrt{6}} = \sqrt{h_1} = \left(\frac{\mathcal{H}_1^3}{\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3}\right)^{\frac{1}{6}} \,.$$

We are now in the position to combine these results to write down the six-dimensional metric for our embedded solution:

$$\begin{aligned} ds_6^2 = (\mathcal{H}_2\mathcal{H}_3)^{-\frac{1}{2}}\Bigg[&\mathcal{H}_0 dz_5^2 + \frac{\mathcal{W}}{2\mathcal{H}_0}\left(\mathcal{W}\frac{y_0^2}{Q_0^2} + 1\right)d\eta^2 - \frac{\mathcal{W}y_0}{\sqrt{2}Q_0}d\eta dz_5 \\ &+ 2\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\left(-\frac{d\zeta^2}{\mathcal{W}} + dx^2 + dy^2\right)\Bigg] + \frac{(\mathcal{H}_2\mathcal{H}_3)^{\frac{1}{2}}}{\mathcal{H}_1}(dz_6 + \tilde{\mathbb{A}}_1)^2 \,, \end{aligned}$$

where the determinant of the metric is

$$\sqrt{-g_6} = 2\mathcal{H}_1\sqrt{\mathcal{H}_2\mathcal{H}_3} \,.$$

The piece containing the gauge field $\mathbb{A}_1$ can be expanded

$$\begin{aligned} (dz_6 + \tilde{\mathbb{A}}_1)^2 &= (dz_6 + (\tilde{\mathbb{A}}_1)_x dx + (\tilde{\mathbb{A}}_1)_y dy)^2 \,, \\ &= (dz_6 + \mathfrak{p}_1(y dx - x dy))^2 \,. \end{aligned}$$

Six-dimensional gauge fields

We now take the gauge fields and express them as a function of the six-dimensional coordinates. We see that for the two remaining one-form potentials, nothing has been changed compared to the lower-dimensional solutions:

$$\tilde{\mathbb{A}}_1 = \sqrt{2}A^1, \qquad \tilde{\mathbb{A}}_2 = \sqrt{2}A^2 \,.$$

The three-form H is found from two pieces

$$H_{(6D)} = \mathbb{H} + \tilde{\mathbb{F}}_2 \wedge (dz_6 + \tilde{\mathbb{A}}_1) \,.$$

This is simplified, as the term

$$\tilde{\mathbb{F}}_2 \wedge \tilde{\mathbb{A}}_1 = 2\mathfrak{p}_2 dx \wedge dy \wedge \mathfrak{p}_1 (y dx - x dy) = 0,$$

is zero due to anti-symmetry. Using the work from the five-dimensional calculations, the six-dimensional three-form field strength is given by:

$$H_{(6D)} = -\left(\frac{\mathfrak{p}_3}{\mathcal{H}_3^2}\right)d\eta \wedge d\zeta \wedge dz_5 - (2\mathfrak{p}_2)dx \wedge dy \wedge dz_6 \,.$$

Six-dimensional extremal limit

Taking the extremal limit (6.4.17), the six-dimensional line element is given by

$$ds_6^2 = \sqrt{\frac{\mathcal{H}_2}{\mathcal{H}_3}}\left[\mathcal{H}_2^{-1}\left(\mathcal{H}_0 dz_5^2 + dz_5 d\eta\right) + \mathcal{H}_3\mathcal{H}_1(d\zeta^2 + dx^2 + dy^2) + \mathcal{H}_3\mathcal{H}_1^{-1}(dz_6 + \tilde{\mathbb{A}}^1)^2\right]. \tag{7.1.18}$$

The three-form in this limit is given by

$$H_{(6D)} = -\left(\frac{\mathfrak{p}_3}{\mathcal{H}_3^2}\right) d\eta \wedge d\zeta \wedge dz_5 - (2\mathfrak{p}_2) dx \wedge dy \wedge dz_6, \qquad \mathfrak{p}_a = \frac{P^a}{2\sqrt{2Q_0 P^1 P^2 P^3}}.$$

7.1.5 *Oxidation to ten dimensions*

The six-dimensional STU model is a consistent truncation of the reduction of type IIB supergravity compactified over T^4. To lift our solution, we only need to include the overall volume of the T^4 as a modulus

$$ds_{10}^2 = ds_6^2 + e^{\phi/\sqrt{2}}(dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2), \qquad \Phi = \frac{\phi}{\sqrt{2}}, \qquad C \equiv B.$$

The expression for the six-dimensional dilaton ϕ in terms of ζ is

$$e^{\sqrt{2}\phi} = \frac{h_2}{h_3} = \left(\frac{\mathcal{I}_{33}}{\mathcal{I}_{22}}\right)^{\frac{1}{2}} \quad \Rightarrow \quad e^{\phi/\sqrt{2}} = \left(\frac{\mathcal{I}_{33}}{\mathcal{I}_{22}}\right)^{\frac{1}{4}} = \sqrt{\frac{\mathcal{H}_2}{\mathcal{H}_3}}.$$

All other data follow straight from the six-dimensional solutions. The ten-dimensional dilaton is given by

$$\Phi = \frac{1}{2}\log\left(\frac{\mathcal{H}_2}{\mathcal{H}_3}\right).$$

The ten-dimensional line element is given by

$$\begin{aligned} ds_{10}^2 = \sqrt{\frac{\mathcal{H}_2}{\mathcal{H}_3}}\Bigg[&\mathcal{H}_0\mathcal{H}_2^{-1}dz_5^2 + \frac{\mathcal{W}}{2\mathcal{H}_0\mathcal{H}_2}\left(\mathcal{W}\frac{y_0^2}{Q_0^2} + 1\right)d\eta^2 - \frac{\mathcal{W}y_0}{\sqrt{2}Q_0\mathcal{H}_2}d\eta dz_5 \\ &+ 2\mathcal{H}_1\mathcal{H}_3\left(-\frac{d\zeta^2}{\mathcal{W}} + dx^2 + dy^2\right) + \frac{\mathcal{H}_3}{\mathcal{H}_1}(dz_6 + \tilde{\mathbb{A}}_1)^2 \\ &+ dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2\Bigg]. \end{aligned}$$

Ten-dimensional extremal limit

Uplifting the extremal six-dimensional solution using the same methods as Section (7.1.5) we find that the line element is

$$\begin{aligned} ds_{10}^2 = \sqrt{\frac{\mathcal{H}_2}{\mathcal{H}_3}}\Bigg[&\mathcal{H}_2^{-1}\left(\mathcal{H}_0 dz_5^2 + dz_5 d\eta\right) + \mathcal{H}_3\mathcal{H}_1(d\zeta^2 + dx^2 + dy^2) \\ &+ \mathcal{H}_3\mathcal{H}_1^{-1}(dz_6^2 + \tilde{\mathbb{A}}^1)^2 + dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2\Bigg]. \end{aligned} \tag{7.1.19}$$

As with the eleven-dimensional uplift, we can recognise this line element as a solution for an intersecting brane configuration. We can compare this with the line element (4.5.7), which is the solution for the D1-D5 brane intersection with a PP-wave superimposed over the intersection direction, and a Taub-NUT space in the relative transverse space. In Section 4.5.4, we showed that upon reduction over a T^6, the solution (4.5.8) gave rise to a four-dimensional black hole solution with a finite area, which interpolated between an $AdS_2 \times S^2$ on the horizon to Minkowski at asymptotic infinity. Here, in the solution (7.1.19) the key difference is again the linear harmonic functions. Where as before the zeros of H_i signalled the location of the brane, the planar symmetry imposed in four dimensions leads to linear harmonic functions $\mathcal{H}_a$, and we understand the brane configuration (7.1.19) as being smeared over two additional dimensions.

7.2 SUPERSYMMETRY IN SIX DIMENSIONS

Supersymmetric solutions of six-dimensional supergravity have been classified in detail. The first classification of supersymmetric solutions in the minimal ungauged six-dimensional theory, with a self-dual three-form, was constructed in [183]. Following on from this, the supersymmetric solutions of six-dimensional U(1), and SU(2) gauged supergravity were classified in [184]. This analysis was done using the spinor bilinears method. Supersymmetric solutions of more general theories coupled to arbitrary vector and tensor multiplets were classified using spinorial geometry methods in [185]; see also [186, 187, 188, 189]. These classifications have been used to find many new examples of solutions, and we shall show that in a certain limit, the six-dimensional solution we have constructed satisfies the necessary and sufficient conditions for supersymmetry.

7.2.1 *Conditions required for supersymmetry*

We now turn our attention to the six-dimensional uplift of our solution and test to see whether there is a configuration of integration constants such that the solution is supersymmetric. In this particular case, the theory of interest is the $U(1)$ gauged supergravity whose supersymmetric solutions were classified in [184], in the special case for which the $U(1)$ gauge parameter is set to zero. The bosonic content of this theory is the metric g, a real three-form G, and a dilaton ϕ. The geometry of these solutions was also considered in [190]. Before considering the six-dimensional uplift in detail, we first summarise the necessary and sufficient conditions on the bosonic fields in order for a generic solution of this theory to be supersymmetric.

The metric for the supersymmetic solutions is given by

$$ds_6^2 = -2H^{-1}(dv+\beta)\left(du+\omega+\frac{1}{2}\mathcal{F}(dv+\beta)\right)+H ds_4^2. \tag{7.2.1}$$

The metric for the four-dimensional base space $\mathcal{B}$ is written as

$$ds_4^2 = h_{mn}dx^m dx^n, \tag{7.2.2}$$

with $\mathcal{F}$ and H as smooth functions and $\beta = \beta_m dx^m$ and $\omega = \omega_m dx^m$ regarded as one-forms on $\mathcal{B}$. The vector $\frac{\partial}{\partial u}$ corresponds to a Killing spinor bilinear and the Killing spinor equations imply that this vector is an isometry, and moreover, that the Lie derivative of the three-form G and the dilaton Φ with respect to $\frac{\partial}{\partial u}$ vanish. However, in general, the metric, the three-form and the dilaton may depend on the v and the x^m coordinates.

Analysis of the algebraic properties of the spinor bilinears through considering the Fierz identities implies that there are three anti-self-dual two-forms on the base $\mathcal{B}$: $J^{(A)}$, $A = 1, 2, 3$, which satisfy the algebra of the imaginary unit quaternions; $\mathcal{B}$ therefore admits an almost hyper-Kähler structure. In addition, the gravitino Killing spinor equations imply that

$$\tilde{d}J^{(A)} = \partial_v\left(\beta \wedge J^{(A)}\right), \tag{7.2.3}$$

where $\tilde{d}$ denotes the exterior derivative restricted to surfaces of constant u and v; and ∂_v denotes the Lie derivative with respect to $\frac{\partial}{\partial v}$. It is also useful to define the differential operator D by

$$D\chi = \tilde{d}\chi - \beta \wedge \partial_v \chi, \tag{7.2.4}$$

where χ is a u-independent differential form on $\mathcal{B}$. Then supersymmetry implies that

$$D\beta = \star_4 D\beta, \tag{7.2.5}$$

where $\star_4$ denotes the Hodge dual on $\mathcal{B}$. This exhausts the conditions on the geometry obtained from the gravitino Killing spinor equation. It remains to consider the conditions on the fluxes.

The Killing spinor equations determine the components of the three-form G as

$$\begin{aligned} e^{\sqrt{2}\Phi}G = &\tfrac{1}{2}\star_4\left(DH + H\partial_v\beta - \sqrt{2}HD\Phi\right) \\ &- \tfrac{1}{2}e^+ \wedge e^- \wedge \left(H^{-1}DH + \partial_v\beta + \sqrt{2}D\Phi\right) \\ &- e^+ \wedge (-H\psi + \tfrac{1}{2}(D\omega)^- - K) + \tfrac{1}{2}H^{-1}e^- \wedge D\beta, \end{aligned} \tag{7.2.6}$$

where K is a self-dual form on the base $\mathcal{B}$, ψ is expressed as

$$\psi = \frac{1}{16}H\epsilon_{ABC}J^{(A)mn}(\partial_v J^{(B)})_{mn}J^{(C)}, \tag{7.2.7}$$

and we have adopted the null basis

$$e^+ = H^{-1}(dv + \beta), \qquad e^- = du + \omega + \frac{1}{2}\mathcal{F}He^+, \qquad e^a = H^{\frac{1}{2}}\tilde{e}^a, \tag{7.2.8}$$

in which the metric is

$$ds_6^2 = -2e^+e^- + \delta_{ab}e^a e^b, \tag{7.2.9}$$

and the basis $\tilde{e}^a = \tilde{e}^a{}_m dx^m$ is a basis for the base $\mathcal{B}$.

On imposing the Bianchi identity $dG = 0$, the following conditions are obtained

$$\begin{aligned} &D\left(H^{-1}e^{\sqrt{2}\Phi}\left(K - H\mathcal{G} - H\psi\right)\right) + \frac{1}{2}\partial_v \star_4\left(D\left(He^{\sqrt{2}\Phi}\right) + He^{\sqrt{2}\Phi}\partial_v\beta\right) \\ &\qquad - H^{-1}e^{\sqrt{2}\Phi}(\partial_v\beta) \wedge \left(K - H\mathcal{G} - H\psi\right) = 0, \end{aligned} \tag{7.2.10}$$

and

$$\begin{aligned} &-D\left(H^{-1}e^{-\sqrt{2}\Phi}\left(K + H\mathcal{G} + H\psi\right)\right) + \frac{1}{2}\partial_v \star_4\left(D\left(He^{-\sqrt{2}\Phi}\right) + He^{-\sqrt{2}\Phi}\partial_v\beta\right) \\ &\qquad + H^{-1}e^{-\sqrt{2}\Phi}(\partial_v\beta) \wedge \left(K + H\mathcal{G} + H\psi\right) = 0, \end{aligned} \tag{7.2.11}$$

where

$$\mathcal{G} = \frac{1}{2H}\Big(\big(D\omega\big)^{+} + \frac{1}{2}\mathcal{F}D\beta\Big), \tag{7.2.12}$$

and $(D\omega)^{\pm}$ denote the self-dual and anti-self dual parts of $D\omega$.

The gauge field equations, $d(e^{2\sqrt{2}\Phi}\star_6 G) = 0$ also imply the following conditions

$$D\star_4\Big(D\big(He^{\sqrt{2}\Phi}\big) + He^{\sqrt{2}\Phi}\partial_v\beta\Big) = 2H^{-1}e^{\sqrt{2}\Phi}\big(K - H\mathcal{G}\big)\wedge D\beta, \tag{7.2.13}$$

and

$$D\star_4\Big(D\big(He^{-\sqrt{2}\Phi}\big) + He^{-\sqrt{2}\Phi}\partial_v\beta\Big) = -2H^{-1}e^{-\sqrt{2}\Phi}\big(K + H\mathcal{G}\big)\wedge D\beta. \tag{7.2.14}$$

As noted in [184], imposing these conditions implies that the dilaton field equation is automatically satisfied, and also all but one component of the Einstein field equations also hold. The remaining ++ component of the Einstein equations must be imposed as an additional condition. On defining

$$L = \partial_v\omega + \frac{1}{2}\mathcal{F}\partial_v\beta - \frac{1}{2}D\mathcal{F}, \tag{7.2.15}$$

this component of the Einstein equation is given by

$$\begin{aligned}\star_4 D\star_4 L &= \frac{1}{2}h^{mn}\partial_v^2\big(Hh_{mn}\big) + \frac{1}{2}\partial_v\big(Hh^{mn}\big)\partial_v\big(Hh_{mn}\big)\\ &- \frac{1}{2}H^{-2}\big(D\omega + \frac{1}{2}\mathcal{F}D\beta\big)^2 - 2L^m(\partial_v\beta)_m + 2H^2(\partial_v\Phi)^2\\ &+ 2H^{-2}\big(K - H\psi + \frac{1}{2}(D\omega)^-\big)^2.\end{aligned} \tag{7.2.16}$$

where we adopt the convention that if X is a two-form on $\mathcal{B}$ then $X^2 = \frac{1}{2}X_{mn}X^{mn}$.

7.2.2 *Matching the solutions*

We now wish to see whether we can satisfy the conditions outlined above for the extremal solution (7.1.18). We begin by taking the $\alpha \to 0$ limit; in four dimensions we can think of this limit as taking the blackening factor to zero and thus, being associated with extremality. In this limit, the resulting metric was found to be (7.1.18) and for convenience, we repeat the full expression for the six-dimensional three-form

$$H_{(6D)} = -\frac{P^3}{2\mathcal{H}_3^2\sqrt{2Q_0P^1P^2P^3}}d\eta\wedge d\zeta\wedge dz_5 - \frac{P^2}{\sqrt{2Q_0P^1P^2P^3}}dx\wedge dy\wedge dz_6. \tag{7.2.17}$$

Comparing our metric with the metric (7.2.1) we extract a four-dimensional base space:

$$ds_6^2 = (\mathcal{H}_2\mathcal{H}_3)^{-\frac{1}{2}}dz_5(d\eta + \mathcal{H}_0 dz_5) + (\mathcal{H}_2\mathcal{H}_3)^{\frac{1}{2}}ds_4^2, \tag{7.2.18}$$

in the form

$$ds_4^2 = \mathcal{H}_1(d\zeta^2 + dx^2 + dy^2) + \mathcal{H}_1^{-1}(dz_6 + \mathbb{A}^1)^2. \tag{7.2.19}$$

Direct comparison to (7.2.1) shows that we should make the following identifications:

$$\beta = \omega = 0, \qquad H = \sqrt{\mathcal{H}_2\mathcal{H}_3}, \qquad \mathcal{F} = \mathcal{H}_0, \qquad dv = dz_5, \qquad 2du = d\eta,$$

with all components of the metric and three-form independent of the v coordinate. The basis vectors are given as:

$$e^+ = (\mathcal{H}_2\mathcal{H}_3)^{-\frac{1}{2}}dz_5, \qquad e^- = \frac{1}{2}d\eta + \frac{1}{2}\mathcal{H}_0 dz_5, \qquad e^a = (\mathcal{H}_1\mathcal{H}_2)^{\frac{1}{4}}\tilde{e}^a_m dx^m.$$

We begin by looking more closely at the base space (7.2.19)

$$ds_4^2 = \mathcal{H}_1(d\zeta^2 + dx^2 + dy^2) + \mathcal{H}_1^{-1}(dz_6 + \mathbb{A}^1)^2,$$

which has a set of basis vectors:

$$\begin{aligned} e^1 &= \mathcal{H}_1^{\frac{1}{2}} d\zeta, & e^2 &= \mathcal{H}_1^{\frac{1}{2}} dx, \\ e^3 &= \mathcal{H}_1^{\frac{1}{2}} dy, & e^4 &= \mathcal{H}_1^{-\frac{1}{2}}(dz_6 + \mathbb{A}^1), \end{aligned}$$

with

$$\mathcal{H}_1 = h_1 + P^1\zeta, \qquad \mathbb{A}^1 = \frac{P^1}{2\sqrt{2Q_0P^1P^2P^3}}\big(ydx - xdy\big). \tag{7.2.20}$$

As the solution is independent of the v coordinate, the condition (7.2.3) implies that the base is hyper-Kähler. In particular, we require that the Ricci scalar of the base must vanish. Computing the Ricci scalar, we find that

$$R_{(4D)} = \frac{P_1^2 - 4\mathfrak{p}_1^2}{2\mathcal{H}_1^3}\ .$$

Solving this for $R_{(4D)} = 0$ imposes the following condition

$$2Q_0P^1P^2P^3 = 1,$$

which we can interpret as a condition for the integration constant

$$Q_0 = \frac{1}{2P^1P^2P^3},$$

and so we see that the supersymmetric limit occurs by fine-tuning the four-dimensional electric charge or alternatively the Kaluza-Klein momentum in five or six dimensions.

Given this fine tuning condition, the base metric is then given by

$$ds_4^2 = \big(h_1 + P^1\zeta\big)\Big(d\zeta^2 + dx^2 + dy^2\Big) + \big(h_1 + P^1\zeta\big)^{-1}\Big(dz^6 + \frac{1}{2}P^1\big(ydx - xdy\big)\Big)^2. \tag{7.2.21}$$

This metric is in the form of the Gibbons-Hawking instanton solution [191, 192]

$$ds_{GH}^2 = U^{-1}(d\tau + \omega)^2 + Ud\vec{x}\cdot d\vec{x},$$

where $\tau = z^6$ is the direction corresponding to the tri-holomorphic isometry $\frac{\partial}{\partial\tau}$ of the hyper-Kähler structure, and $U = h_1 + P^1\zeta$ is a linear harmonic function of the Cartesian coordinates

$\{\zeta, x, y\}$ on $\mathbb{R}^3$, and the one-form $\omega = dz^6 + \frac{1}{2}P^1\big(ydx - xdy\big)$ is a $U(1)$ connection on $\mathbb{R}^3$ which satisfies

$$dU = \star_3 d\omega. \tag{7.2.22}$$

This base space corresponds to a constant density planar distribution of Taub-NUT instantons. Moreover, the conditions imposed on the three-form given in (7.2.6) are consistent with the three-form obtained from the uplift in (7.2.17), on setting $K = 0$ in (7.2.6), and also identifying

$$\Phi = -\frac{1}{2\sqrt{2}} \log\left(\frac{\mathcal{H}_2}{\mathcal{H}_3}\right). \tag{7.2.23}$$

We remark that the dilaton which appears in the classification of [184], which we have denoted by Φ, differs from the dilaton ϕ appearing in previous sections by a scaling

$$\Phi = -\frac{1}{2}\phi = -\frac{1}{2\sqrt{2}} \log\left(\frac{\mathcal{H}_2}{\mathcal{H}_3}\right) \quad \Rightarrow \quad e^{\sqrt{2}\Phi} = \left(\frac{\mathcal{H}_2}{\mathcal{H}_3}\right)^{-\frac{1}{2}}.$$

With these identifications, it is straightforward to match (7.2.6) with (7.2.17), on making use of the identities

$$d\zeta = \mathcal{H}_1^{-\frac{1}{2}} e^1, \qquad \star_4 d\zeta \ = -\mathcal{H}_1^{-\frac{1}{2}} e^2 \wedge e^3 \wedge e^4 = -dx \wedge dy \wedge dz_6. \tag{7.2.24}$$

and

$$e^{-\sqrt{2}\Phi} \star_4 \left(dH - \sqrt{2}Hd\Phi\right) = -P^2 dx \wedge dy \wedge dz_6, \tag{7.2.25}$$

and

$$e^{-\sqrt{2}\Phi} e^+ \wedge e^- \wedge \left(H^{-1}dH + \sqrt{2}d\Phi\right) = \frac{P^3}{2\mathcal{H}_3^2} dz_5 \wedge d\eta \wedge d\zeta. \tag{7.2.26}$$

In addition, the Bianchi identities (7.2.10) and (7.2.11) hold with no further conditions imposed, as all terms are independent of v, and also $K = 0$, $\mathcal{G} = 0$ and $\psi = 0$. The condition $\psi = 0$ follows from (7.2.7), on using the fact that the hyper-complex structures are independent of v.

It remains to consider the gauge field equations (7.2.13) and (7.2.14). The RHS of these equations vanishes identically, as a consequence of the fact that $h = 0$. The remaining content of the gauge field equations is that $He^{\pm\sqrt{2}\Phi}$ be harmonic on the base space. This holds automatically as a consequence of the previously obtained conditions, because $He^{\sqrt{2}\Phi} = \mathcal{H}_3$ and $He^{-\sqrt{2}\Phi} = \mathcal{H}_2$, and ζ is harmonic on the base space as a consequence of (7.2.24). Similarly, the Einstein equation (7.2.16) holds automatically, because all terms on the RHS vanish individually, and also $L = -\frac{1}{2}Q_0 d\zeta$ which is co-closed[3] on the base, again as a consequence of (7.2.24).

7.2.3 *Analysis of the spacetime*

Now we have shown that by fine-tuning our integration constants we can obtain a supersymmetric solution, it is interesting to look at the geometric properties of this spacetime.

Our analysis is focused on the simplified metric

$$ds_6^2 = (\mathcal{H}_2\mathcal{H}_3)^{-\frac{1}{2}} dz_5(d\eta + \mathcal{H}_0 dz_5) + (\mathcal{H}_2\mathcal{H}_3)^{\frac{1}{2}}\left[\mathcal{H}_1(d\zeta^2 + dx^2 + dy^2) + \mathcal{H}_1^{-1}(dz_6 + \mathbb{A}^1)^2\right].$$

[3] A co-closed form satisfies $\delta\omega = 0$, where the co-differential operator: $\delta = (-1)^{n(p-1)+1+t} \star d\star$ is a map from $\Omega^p \to \Omega^{p-1}$.

In the limit of $\zeta \to \infty$ we find that the Riemann tensor falls off as ζ^{-n} for $n \geq 1$. The Ricci curvature of the spacetime is

$$R_{(6D)} = \frac{(h_3 P^2 - h_2 P^3)^2}{4\mathcal{H}_1(\mathcal{H}_2\mathcal{H}_3)^{\frac{5}{2}}},$$

and we notice here that we have the option to pick either a charge $P^{2/3}$ or $h_{2/3}$ value such that the spacetime has zero Ricci scalar:

$$h_3 = \frac{h_2 P^3}{P^2} \quad \Leftrightarrow \quad R_{(6D)} = 0.$$

We can understand this condition by looking back at the harmonic functions

$$\begin{aligned}\mathcal{H}_2\mathcal{H}_3 &= (h_2 + P^2\zeta)(h_3 + P^3\zeta) = \left(h_2 + P^2\zeta\right)\left(\frac{h_2 P^3}{P^2} + P^3\zeta\right) \\ &= P^2 P^3 \left(\zeta + \frac{h_2}{P^2}\right)^2.\end{aligned}$$

and we see that picking the right integration constants we allow the zeros of both $\mathcal{H}_2$ and $\mathcal{H}_3$ to occur simultaneously.

7.3 DISCUSSION

In this chapter, we have taken the non-extremal solutions found in Chapter 6 and through the process of dimensional oxidation, have found solutions to consistent truncations of supergravity in five, six, ten and eleven dimensions. We saw that when taking the four-dimensional extremal limit, that the ten and eleven-dimensional line elements could be understood as describing the intersecting brane configurations of M-theory and IIB string theory introduced in Section 4.5.4. In particular, we recovered solutions which could be interpreted as the triple intersection of M5 branes, with a PP-wave or as the D1-D5 system with a PP-wave and Taub-NUT space. Both of these solutions were argued to give rise to regular black holes in four dimensions with four distinct charges. In the context of planar symmetric solutions of the STU model, we find that planar symmetry causes the additional delocalisation along two directions. In Section 7.2, we performed an investigation of the six-dimensional solution and showed that through fine-tuning of the integration constants, it is possible to recover BPS solutions of six-dimensional supergravity despite having solved the equations of motion in Section 6.2 without imposing the Killing spinor equations, which would preserve some fraction of supersymmetry in our solution. We now conclude this section with a discussion of the planar solutions of the STU model, taking into account various results from the previous three chapters.

Thinking back to the work of the Nernst branes, one of the main focuses of this work has been the behaviour of the solutions in the extremal limit. In Chapter 6, we identified the extremal limit and saw that the area density of the brane solution diverged, rather than vanishing like the Nernst solutions of [28]. This divergence matched the behaviour of the planar symmetric solutions of the Einstein-Maxwell theory discussed in Chapter 5. In fact, the closeness of the planar STU model solutions to the planar solutions of Einstein-Maxwell theory will lead to our discussion of the properties of these solutions through the lens of this simplified model.

As an overview, we have seen that, surprisingly, a method designed to produce static solutions has provided us with a class of cosmological solutions with an extremal limit which produces a spacetime containing a naked singularity. Our procedure of oxidation and the embedding into string theory has not provided by itself any insight into why we obtain cosmological rather than black brane solutions though, as the additional dimensions are purely spectators. Instead, we learn an interesting lesson about the importance of being able to make brane configurations non-extremal. The compactified BPS brane solutions used to obtain four-dimensional BPS black holes have the same causal structure as the extremal Reissner-Nordström solution, which is embedded as a 'double-extreme' limit [181], where all four-dimensional scalars are constant. The essential features of our cosmological solutions can be understood using the charged electro-vac solutions of Einstein-Maxwell theory.

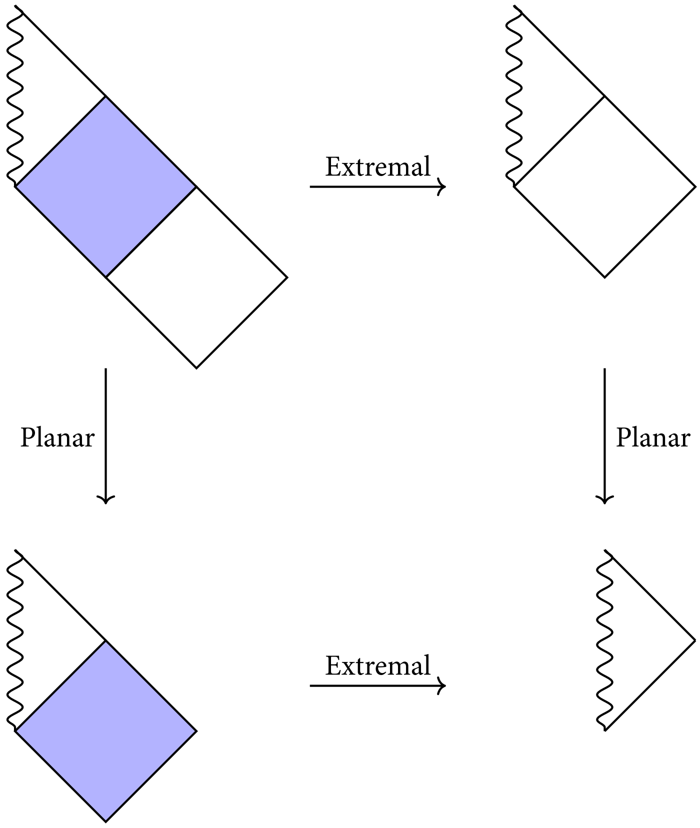


Figure 7.1: Comparison of the conformal diagrams for spherical and planar Reissner-Nordström-like spacetimes. We only display one copy of each type of region. Shaded regions are where the spacetime is dynamical (no timelike Killing vector).

We start with the spherically symmetric extremal Reissner Nordström solution, whose causal structure we have seen is shared by a large class of BPS solutions obtained by compactifying brane configurations. Its maximal analytical extension is a sequence of two types of regions, both static: one containing an asymptotically flat exterior, the other (the interior) containing a timelike singularity which is repulsive to massive neutral particles. In other words, timelike geodesics are infinitely extendable. For our purposes, we focus on just a single pair of such regions, see Figure 7.1 for an illustration. If the solution is made non-extremal, a third type of region occurs, which is dynamical (non-stationary) and located between the two distinct static patches. Let us now consider the effect of replacing spherical by planar symmetry, or, in brane language, of delocalisation of the constituent branes along two non-compact spatial directions. In this case, the solution cannot be asymptotically flat any more. For brane-type solutions, it is a well-known feature that asymptotic flatness requires more than two transverse dimensions: 'large branes' (those with two or less transverse dimensions, like the D7-brane in type IIB) cannot be asymptotically flat. In terms of the causal structure, we lose the static, asymptotically flat patch and remain with a static patch containing the singularity, and a dynamical patch. More precisely, by maximal analytic extension, we end up with two patches of each type, resulting in a conformal diagram which is the same as Schwarzschild rotated by 90 degrees, see Figure (5.7). If we now perform an extremal limit, we also lose the dynamical patch and remain with a static patch containing a singularity. Comparing the four types of conformal diagrams, we see that going from spherical to planar symmetry removes the asymptotically flat region, while the existence of a dynamical patch depends on non-extremality. Viewed from this perspective, the presence of a cosmological patch in our solutions is completely natural, and results from physics already present in Einstein-Maxwell theory. These features are robust under dimensional lifting and persist for the three-charged solution, which is a solution of gauged supergravity and has the same conformal diagram [39]. However, the Nernst brane solutions [28, 34, 29, 30] illustrate that these features do not persist if we modify essential features. Like the three- and four-charge solutions, the single-charged Nernst branes are planar and not asymptotically flat, but they do not share the 'inside-out' feature of a singularity at a finite distance inside the static patch. The essential difference is that Nernst branes require a non-constant scalar fields, and therefore there is no limit in which they become solutions of four-dimensional Einstein-Maxwell theory. Instead, as shown in [34], they lift to boosted AdS-Schwarzschild black brane solutions of five-dimensional AdS gravity.

The close relation of our cosmological solutions to the planar Reissner-Nordström solution also settles the question of whether we need to interpret it as being sourced by negative tension branes. We have found that the local Komar mass is negative in the static patch, which is consistent with the repulsive character of the singularity. However, this feature is also present in the spherical Reissner-Nordström solution, the only difference being that with planar symmetry we lose the asymptotically flat region, and hence the ability to define a 'proper' mass by evaluating the Komar expression at asymptotic infinity. This reflects the general insight, reviewed recently in [193], that the definition of global quantities through conservation laws *à la* Noether requires that general diffeomorphism invariance is 'broken naturally' by the presence of extra structure, such as boundary conditions. That we do not have a static asymptotic region does not provide a good reason to assign negative tension to the sources because locally, the situation is not different from Reissner-Nordström. Moreover, for the cases where we can lift to ten or eleven dimensions, the sources reveal themselves as conventional, positive tension branes.

The only caveat is that our four-dimensional solutions admit other embeddings into string theory, which might change their higher-dimensional interpretation. In particular, it can be shown that the solution found in [182] describes a region of our solution, although in different coordinates, where the existence of a Killing horizon is not obvious. The solution of [182] admits an uplift over the orientifold $K3 \times \mathbb{T}^2/\mathbb{Z}_2$. This alternative embedding, which we have not analysed in detail in this thesis, is interesting because it starts with a compactification which has less than maximal supersymmetry. In contrast, in our uplift we have used toroidal compactifications and start with maximally supersymmetric theories in ten and eleven dimensions. Therefore reduction to a four-dimensional $\mathcal{N} = 2$ theory requires one to truncate the field content after compactification. In [182] the sources are orientifolds, rather than D-branes or M-branes. Some authors [171] have argued that in string theory, orientifolds naturally give rise to cosmological solutions. This is a natural direction to perform additional research, together with the proper dimensional uplift of the three-charged solution of [39], which as a solution of gauged supergravity would need the mechanics of the Sherk-Schwarz reduction mentioned in passing during Section 4.3.1, rather than the simple Kaluza-Klein reduction we have concerned ourselves with within this thesis.

One aspect which we have not investigated in this chapter is the question of whether our solutions are stable. For this we refer to the discussion in [168, 170] which have addressed some aspects of the stability of the horizon. They found that the situation for the first horizon is the same as for the inner horizon of non-extremal Reissner-Nordstrom solution, while for the second horizon no indication for an instability was found.

While our analysis disfavours interpreting the sources of our solutions as negative tension branes, it has been argued that negative branes exist in string theory [194]. In [195] it was shown that when admitting timelike T-duality, the web of string/M-theories contains exotic theories with twisted supersymmetry and negative kinetic energy for some of the fields. Moreover, there exists at least one version of type-II string theory for any possible spacetime signature. According to [194], some of the branes of these exotic theories appear as ‘negative branes’ when viewed from the point of view of a dual theory. This could allow the construction of new, genuinely stringy cosmological solutions, and our formalism could easily be tweaked to study these solutions. We will return to this discussion in the conclusions of this thesis, which with the additional context of the next chapter, offers an interesting project for further work.

8

THE FIRST LAW FOR COSMOLOGICAL SOLUTIONS

The planar solutions of the STU model came from considering the Nernst branes of [28], which were initially motivated by the search for black hole solutions obeying the strict third law of thermodynamics. We found that the Nernst-like behaviour of the solutions was lost as we increased the number of charges supported by the brane solution, but instead we gained a new class of cosmological solutions, where the external region of the spacetime is time-dependent and asymptotic to the Kasner solution.

From the position of already thinking of black hole thermodynamics, it is then natural to the ask what thermodynamic properties these cosmological, planar horizons have. From our classification of trapping horizons in Section 6.5.1, it is possible to compute the temperature of the solution, and we can write down area densities which have an analogy in entropy density. Similarly, we can calculate the conserved charge densities $\mathcal{Q}$ and $\mathcal{P}$ associated to the solutions, together with chemical potentials, however, writing down a meaningful mass parameter has so far escaped us. In Section 5.2.3, and Section 6.5.3, we were able to write down a local expression for mass working in the finite, static region of spacetime, but as we were unable to set an overall normalisation, we cannot verify the first law, which is a differential relationship.

In Section 3.6, we saw that by Wick-rotating the gravitational action, we were able to make a formal equivalence between the gravitational and thermal partition functions. Computing the thermodynamic potential from the thermal partition function allows for the derivation of several thermodynamic parameters through partial derivatives of the free energy, including the internal energy.

The standard simple Wick-rotation discussed in Section 3.6 can be applied for static spacetimes which upon continuation remain real, so that the Euclidean on-shell action can be interpreted as a thermal partition function. The static patches of the planar solutions of Einstein-Maxwell model (Section 5.1) and the STU model (Section 6.4) take the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + g(r)^2(dx^2 + dy^2), \tag{8.0.1}$$

which at first appears suitable for this procedure. However, we also need smooth field configurations to obtain a well-defined and finite Euclidean on-shell action. For the solutions we consider, the static patches have a curvature singularity for some finite value $r = r_{\text{sing}}$ of the transverse coordinate r. We see that although we can Wick-rotate the line element and obtain a real, positive-definite spacetime, there will necessarily be a singularity within the Euclidean section and hence the Euclidean on-shell action is ill-defined. However, as we have seen, these static patches have a horizon at another finite value $r = r_h > r_{\text{sing}}$, and by analytic continuation we reach the external patch of the spacetime, which depends only on the coordinate r, which is timelike for $r > r_h$. After relabelling $r \leftrightarrow t$, we can write the line element of these solutions into the form

$$ds^2 = -\frac{dt^2}{\tilde{f}(t)} + \tilde{f}(t)dr^2 + g(t)^2(dx^2 + dy^2). \tag{8.0.2}$$

Note that the function $\tilde{f}(t)$ has been modified with an additional sign: $\tilde{f}(x) = -f(x)$. This ensures that $\tilde{f}(t)$ is positive-definite within the domain of $t \in (t_h, \infty)$. For the remainder of the discussion, the tilde will be dropped and it is understood that functions f appearing in the line element are positive-definite for each patch, and that the coordinate denoted t is timelike while the coordinate denoted r is spacelike.

These dynamic patches, where $t \in (t_h, \infty)$, more closely resemble the external, static patches of the Reissner-Nordstöm solution we considered in our example in Section 3.3.2. However, in the dynamic patch, the horizontal Killing vector field is spacelike rather than timelike, and the application of the simple Wick-rotation leads to a complex line element and action. To work with this dynamic patch and ensure a real Euclidean section after Wick-rotation, we modify the standard Euclidean method. Before continuing, we note that there are some examples where complex line elements are used in the literature, the canonical example being the Kerr metric [85]. In this case, the generalisation is to admit timelike Killing vector fields which are not hypersurface orthogonal, and the complexification arises from cross terms in the line element. This is different from our case, where the Killing vector field is still hypersurface orthogonal, but spacelike.

This chapter presents work initially completed in [40], which introduced a modification of the Euclidean action formalism though performing a *triple Wick-rotation*. This method in principle can be applied to any metric which has no timelike-spacelike cross-terms, and depends explicitly on time but not on the spatial coordinates. Instead of Wick-rotating the time coordinate, we choose to Wick-rotate all three spacelike coordinates of the line element. As we are working exclusively with line elements of the form (8.0.2), we denote the spatial coordinates (r, x, y) such that the triple Wick-rotation takes the form

$$(r, x, y) \to \pm i(r, x, y), \tag{8.0.3}$$

where we admit either choice of sign. Notice that unlike the standard Wick-rotation, the resulting Euclidean line element will be *negative-definite*, as we work with the mostly-plus conventions. As we saw in Section 3.6, the standard argument for identifying the resulting Euclidean action with a thermodynamic potential depends on the Killing vector being timelike, and thus being related to time translations and energy. In the dynamic outer patch, the Killing vector is spacelike and thus corresponds to spatial translations and momentum. This discrepancy from the standard formalism requires further work to understand fully, but we proceed formally and relate our Euclidean action to a thermodynamic potential, leaving questions about the underlying microscopic theory aside. The 'energy' E is defined as a derivative of this potential, and we prove that its variation δE satisfies a relation which takes the exact form of the first law. As a further consistency check in Section 8.5, we also apply the isolated horizon formalism, which imposes the first law and this way obtains an expression for the energy, and we find that the results of both formalisms agree.

In this chapter, we first discuss a general gravitational action, following the example of the single Wick-rotation in Section 3.6. From this, we write down a general expression for the Euclidean action after the triple Wick-rotation, which is then used in a series of examples. We begin considering the de Sitter solution in Section 8.2. The de Sitter solution, when written in static coordinates, has an observer dependent cosmological horizon, but no singularity. This allows us to study the first law using the Euclidean action formalism for both the standard case in the finite-sized static region, and the triple Wick-rotation for the external dynamic region. Hence, the de Sitter solution is a perfect test case, used as a consistency check for the triple Wick-rotation. Following this, the planar solutions of the Einstein-Maxwell model (Section 8.3) and the STU model (Section 8.4) are analysed using the modified Euclidean action formalism, and we show that for both of these systems, the first law holds.

8.1 TRIPLE WICK-ROTATION

Following Section 3.6, the transformation (8.0.3) is applied to the gravitational action (3.6.1) and the Euclidean action associated with the triple Wick-rotation is calculated. As with the standard Wick-rotation, we calculate the action term-by-term. The bulk contribution transforms as

$$\begin{aligned}-\frac{1}{16\pi}\int_M (R-2\Lambda)\sqrt{-g}d^4x \to &-(\pm i)^3\frac{1}{16\pi}\int_M (R-2\Lambda)\sqrt{-g}d^4x,\\ &= \pm\, i\frac{1}{16\pi}\int_M (R-2\Lambda)\sqrt{-g}d^4x\ .\end{aligned}$$

The GHY-term, as with the single Wick-rotation, transforms with the same sign for $\epsilon = \pm 1$.

1. For surfaces with a timelike unit normal

$$\epsilon = -1, \qquad K \to K, \qquad \sqrt{\gamma}d^3x \to (\pm i)^3\sqrt{\gamma}d^3x.$$

2. For surfaces with a spacelike unit normal,

$$\epsilon = 1, \qquad K \to \mp iK, \qquad \sqrt{\gamma}d^3x \to (\pm i)^2\sqrt{\gamma}d^3x,$$

and we see that for either type of hypersurface, the GHY term transforms under a triple Wick-rotation as

$$+\frac{\epsilon}{8\pi}\int_{\partial M}\sqrt{|\gamma|}(K-K_0)d^3x \to \pm i\frac{1}{8\pi}\int_{\partial M}\sqrt{|\gamma|}(K-K_0)d^3x.$$

Again, as with the standard Wick-rotation, we can write the gauge field contribution as a boundary term as we evaluate the action on shell. Performing the triple Wick-rotation, we find

$$\frac{1}{8\pi}\int_{\partial M}F^{\mu\nu}A_\mu d\Sigma_\nu \to \mp i\frac{1}{8\pi}\int_{\partial M}F^{\mu\nu}A_\mu d\Sigma_\nu,$$

where we have used that $d\Sigma_\mu \to (\pm i)^3 d\Sigma_\mu$, $A_\mu \to \pm iA_\mu$ and $F_{\mu\nu} \to \mp iF_{\mu\nu}$. Piecing this all together, the triple Wick-rotated Euclidean action is given by

$$\begin{aligned}S_E = &\pm\frac{1}{16\pi}\int_M\sqrt{g}(R-2\Lambda)d^4x,\\ &\pm\frac{1}{8\pi}\int_{\partial M}\sqrt{|\gamma|}(K-K_0)d^3x \mp \frac{1}{8\pi}\int_{\partial M}F^{\mu\nu}A_\mu d\Sigma_\nu.\end{aligned} \tag{8.1.1}$$

We then (formally) identify the thermodynamic potential as we do in the standard formulation, evaluating the partition function $\mathcal{Z}$ in a saddle point approximation to obtain

$$\log \mathcal{Z} = -S_E(\beta, \mu) = -\beta\Omega, \tag{8.1.2}$$

where the inverse temperature β and chemical potential μ can be expressed in terms of parameters of the triple-Wick-rotated solution.

8.2 THERMODYNAMICS OF THE DE SITTER SOLUTION

As an introductory example of the implementation of the triple Wick-rotation in spacetimes with dynamic asymptotic regions, we study the de Sitter solution. This example is somewhat simpler than the planar symmetric solutions considered in this thesis, as it is a vacuum solution without a curvature singularity. However, it allows us to demonstrate that the results we obtain using a triple Wick-rotation in the dynamic patch agree with those obtained previously using a single Wick-rotation in the static patch.

8.2.1 *Static patch, single Wick-rotation*

The de Sitter spacetime line element in static coordinates is given by

$$ds^2 = -\left(1 - \frac{r^2}{L^2}\right)dt^2 + \left(1 - \frac{r^2}{L^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \tag{8.2.1}$$

with the cosmological horizon located at $r_h = L$, where L is the de Sitter radius and the domain of our variable is $r \in [0, r_h)$. At $r = r_h$, there is a Killing horizon for the Killing vector field $k^\mu = \partial_t$, which becomes spacelike when we continue to $r > r_h$. Unlike the event horizon for black hole solutions, the cosmological horizon in the de Sitter solution is observer dependent, and is formed by the spacelike separation of events due to the expansion of the spacetime.

Using the Euclidean action formalism, the thermodynamics of de Sitter space can be calculated within the static patch $0 \le r < r_h$ using standard methods. The cosmological constant Λ can be written generally as a function of the de Sitter radius

$$\Lambda = -\frac{(D-1)(D-2)}{2L^2} = -\frac{3}{L^2},$$

where for reference, we first give the relation for general dimension D before setting $D = 4$ for the remainder of our work. Note the non-conventional sign for the cosmological constant. We have decided within this thesis to have the cosmological constant to be proportional to the Ricci scalar, and in our conventions, a manifold with constant positive curvature has $R < 0$. We expand on this in Appendix A.

Under the Wick-rotation $t \to -i\tau$, the line element (8.2.1) maps to the positive-definite line element

$$ds^2 = \left(1 - \frac{r^2}{L^2}\right)d\tau^2 + \left(1 - \frac{r^2}{L^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2. \tag{8.2.2}$$

Entropy & temperature Using the Bekenstein-Hawking area law, the entropy is determined by

$$S_{dS} = \frac{A}{4} = \pi^2 L. \tag{8.2.3}$$

The temperature associated with the horizon is proportional to the Kodama-Hayward surface gravity (3.5.4) , which is found to be $\kappa = -L^{-1}$. Defining our thermodynamic horizon in the usual way, we look for the horizon crossed by future-directed causal geodesics which move from the exterior to the interior of the solution. These are the regions III and IV in Figure 8.1, where the global time orientation is chosen such that the Killing vector field is future-pointing in region III, that is, globally time flows 'upwards' in the diagram. This choice of regions is natural as it has the same causal structure as the part of the extended Schwarzschild spacetime which describes a black hole (regions I and II in the left diagram of Figure 5.4a).

The line element (8.2.1) is in the same form as the cosmological solutions (5.3.1) we consider in Section 5.3.1 and so we see that the horizon separating regions III and IV is a *future inner horizon*, which has a temperature proportional to its surface gravity, thus yielding the Hawking temperature

$$T_H = \frac{\kappa}{2\pi} = -\frac{1}{2\pi L}.$$

We note here that the sign of the temperature is different from the assignments made in other references, including [86, 196], where the Hawking temperature is positive $T_H > 0$. However, according to [196], when the temperature is positive, it is found that entropy is negative. In our case, the sign of the temperature is determined by the type of trapping horizon, but the entropy is always defined by the area law and therefore positive. Note that the expression TdS entering into the first law would be invariant for both approaches.

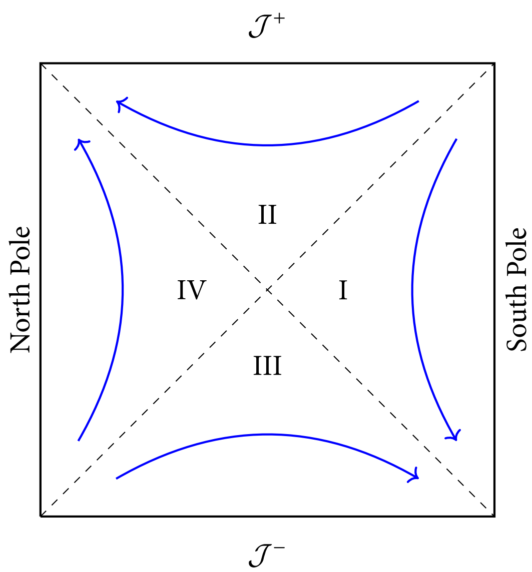


Figure 8.1: Penrose-Carter diagram for the global de Sitter solution. Dashed lines denote the cosmological horizon located for $r = L$ and the North/South poles are identified for $r = 0$. Blue curved arrows denote the flow of the Killing vector field.

Euclidean action Global de Sitter space is a maximally symmetric space of constant positive curvature with topology $\mathbb{R} \times S^3$. Its Kruskal diagram decomposes into four regions, two of which have a timelike Killing vector field and do not intersect the boundary, which is spacelike with topology S^3. If we evaluate the Euclidean action on a static patch, the boundary terms do not contribute and the de Sitter action is completely determined by the bulk terms:

$$S_E = \frac{1}{16\pi} \int_M \sqrt{g}(R - 2\Lambda),$$

where the Ricci curvature is constant: $R = -12L^{-2}$ and the integral over the four-manifold gives

$$S_E = \frac{1}{16\pi} \left(-\frac{6}{L^2}\right) \int_0^\beta d\tau \int_{S^2} \sin\theta d\theta d\phi \int_{r_h}^0 r^2 dr = -\pi L^2.$$

Note the limits on the integration of the radial coordinate r, which have been chosen to run from r_h, the origin of the Euclidean manifold, to the North pole for $r = 0$.[1] As there are no charges in the solution, we work in the canonical ensemble and we have the following relations:

$$\log(Z) = -S_E = -\beta F, \qquad F = E - TS.$$

The de Sitter solution is a maximally symmetric vacuum solution and thus interpreted as a ground state. We therefore choose the natural normalisation $E = 0$. Following from this we obtain

$$S_E = \beta F = -S \qquad \Rightarrow \qquad S = \pi L^2.$$

We see that the thermodynamic entropy matches with (8.2.3) and the first law is satisfied though in a 'degenerate way', as the entropy is constant: $dS = 0 = TdS = dE = d(0) = 0$.

We note that the negativity of the temperature: $\beta < 0$ for this example effects the boundedness of the partition function as the Euclidean action is negative: $S_E < 0$. If instead we had picked a positive temperature, we would return to the case considered within [86], where $S_E > 0$, but the entropy of the solution has a sign difference from the Bekenstein-Hawking area law. We comment on this again in our discussion in Section 8.6.

8.2.2 *Dynamic patch, triple Wick-rotation*

The static patch is not complete and by analytical extension of the coordinate r through the Killing horizon to values $r > r_h$, we obtain a second, dynamical patch, with asymptotic region $r \to \infty$. When crossing the horizon, the function $f(r)$ becomes negative, and we find that the coordinates (t, r) exchange their roles. The timelike coordinate t becomes spacelike, while the spacelike coordinate r becomes timelike. We adopt the convention to relabel coordinates in the dynamic patch so that t is always timelike and r always spacelike.

Then the line element in the dynamic patch is

$$ds^2 = -\left(\frac{t^2}{L^2} - 1\right)^{-1} dt^2 + \left(\frac{t^2}{L^2} - 1\right) dr^2 + t^2 d\Omega_2^2 \,. \qquad (8.2.4)$$

[1] As this might be confusing, let us justify the integration bounds. Although we interpret $r = 0$ as the coordinate origin for static coordinates of the de Sitter solution, this is not the origin for the Wick-rotated Euclidean manifold. When we Wick-rotate, the location of the horizon: $r = r_h$, becomes the origin with the identification $\tau \simeq \tau + \beta$ made to avoid a conical singularity. Our integration limits are then chosen to match the conventions from the origin of the Euclidean space to the boundary and as such we integrate from $r = r_h$ to $r = 0$.

The coordinate domain is $t \in (t_h, \infty)$ where t_h is the Killing horizon located at $t_h = L$. Note that while this cannot be read off from the local form of the line element, we have chosen the continuation from region IV to region III, so that $t \to \infty$ corresponds to past timelike infinity. This is relevant as it determines the sign of the temperature.

Triple Wick-rotation We now perform a triple Wick-rotation where $r \to \pm ir$ and where the sphere S^2 is analytically continued to the hyperbolic plane $\mathcal{H}_2$ by $(\theta, \phi) \to \pm i(\theta, \phi)$. The line element (8.2.4) is mapped to the *negative-definite* line element

$$ds^2 = -\left(\frac{t^2}{L^2} - 1\right)^{-1} dt^2 - \left(\frac{t^2}{L^2} - 1\right) dr^2 - t^2 d\mathcal{H}_2^2\,, \quad d\mathcal{H}_2^2 = d\theta^2 + \sinh^2\theta d\phi^2\,.$$

Temperature & entropy The temperature and entropy associated with the Killing horizon are the same as in the previous calculation. Using the Kodama-Hayward expression (3.5.4), the surface gravity is found to be $\kappa = -L^{-1}$ and for the future inner horizon between regions III and IV, the Hawking temperature is

$$T_H = \frac{\kappa}{2\pi} = -\frac{1}{2\pi L}.$$

The entropy is identical to the static solution and is given by

$$S_{dS} = \frac{A}{4} = \pi^2 L.$$

Euclidean action The dynamical patches of global de Sitter space intersect the boundary, which is spacelike with topology S^3, therefore, we need to take boundary terms into account. After our triple Wick-rotation, the boundary has topology $S^1 \times \mathcal{H}_2$, where the radius of the S^1 is fixed by imposing the absence of a conical singularity.

The Euclidean action for the triple-Wick-rotated system is

$$S_E = \pm\frac{1}{16\pi}\int_M \sqrt{g}(R - 2\Lambda)d^4x \pm \frac{1}{8\pi}\int_{\partial M}\sqrt{\gamma}Kd^3x + \int_{\partial M} L_{ct}[\gamma]d^3x,$$

where a counter term L_{ct} has been included, which we will use to remove divergences from the action. This method is the same as when calculating the Euclidean action for the more commonly considered asymptotically anti-de Sitter spacetimes [197, 198]. We will take our boundary to be at $t \to \infty$, but to properly calculate the counter terms, we first integrate t in the domain $t \in [t_h, \epsilon^{-1})$ and then take the limit of $\epsilon \to 0$. The volume

$$\omega = \int_{\mathcal{H}_2} \sinh\theta d\theta \wedge d\phi,$$

of the hyperbolic plane is divergent. While one option in this situation is to work with densities, we keep ω as a formal constant which corresponds to the parametric volume $\omega_{S^2} = 4\pi$ of the two-sphere in the static patch.

The bulk term of the Euclidean action is

$$S_{\text{Bulk}} = \pm\frac{1}{16\pi}\int_M \sqrt{g}(R - 2\Lambda)d^4x\,,$$

where

$$R = -\frac{12}{L^2}, \qquad \Lambda = -\frac{3}{L^2}, \qquad \sqrt{g} = t^2 \sinh\theta.$$

Putting these into the action and integrating over the manifold we find:

$$\begin{aligned} S_{\text{Bulk}} &= \pm\frac{1}{16\pi}\int_M \sqrt{g}(R-2\Lambda)d^4x, \\ &= \pm\frac{1}{16\pi}\left(-\frac{6}{L^2}\right)\int_0^\beta dr \int_{\mathcal{H}_2} \sinh\theta d\theta d\phi \int_L^{\epsilon^{-1}} dt\, t^2, \\ &= \mp\frac{\beta\omega}{16\pi}\frac{2}{L^2}\left(\frac{1}{\epsilon^3} - L^3\right). \end{aligned}$$

The Gibbons-Hawking-York term

$$S_{\text{GHY}} = \pm\frac{1}{8\pi}\int_{\partial M}\sqrt{-\gamma}Kd^3x,$$

can be calculated in the following way: the normal vector to the boundary for constant t is

$$n^\mu = \left(-\sqrt{f}, 0, 0, 0\right) \quad \Rightarrow \quad n_\mu n^\mu = -1.$$

The trace K of the extrinsic curvature, evaluated on a surface of constant $t = t_0$, can then be computed using (2.2.17):

$$K = \nabla_\mu n^\mu = \frac{3t_0^2 - 2L^2}{t_0 L^2\sqrt{f}}, \qquad \sqrt{-\gamma} = \sqrt{f(t_0)}t_0^2\sinh\theta\,,$$

such that

$$K\sqrt{-\gamma} = \frac{3t_0^3}{L^2} - 2t_0\,.$$

Combining these, we find that the boundary contribution at $t_0 = \epsilon^{-1}$ is:

$$\begin{aligned} S_{GHY} &= \pm\frac{1}{8\pi}\int_{\partial M}\sqrt{-\gamma}Kd^3x, \\ &= \pm\frac{1}{8\pi}\left(\frac{3}{L^2\epsilon^3} - \frac{2}{\epsilon}\right)\int_0^\beta dr\int_{\mathcal{H}_2}\sinh\theta d\theta d\phi, \\ &= \pm\frac{\beta\omega}{8\pi}\left(-\frac{2}{\epsilon} + \frac{3}{L^2\epsilon^3}\right). \end{aligned}$$

The counter term is constructed from the geometric data of the boundary metric:

$$\int_{\partial M} L_{ct}[\gamma]d^3x = \int_{\partial M} d^3x\sqrt{|\gamma|}(c_1 + c_2 R[\gamma]),$$

where $R[\gamma]$ is the Ricci curvature associated to the boundary manifold, and $c_{1,2}$ are renormalisation constants. We can expand out the counter terms in orders of ϵ and find:

$$\sqrt{|\gamma|} = \left(\frac{1}{L\epsilon^3} - \frac{L}{2\epsilon} + \mathcal{O}(\epsilon^1)\right)\sinh\theta,$$

$$R[\gamma]\sqrt{|\gamma|} = \left(-\frac{2}{L\epsilon} + \mathcal{O}(\epsilon^1)\right)\sinh\theta.$$

Comparing terms of order ϵ we find that the counter term is:

$$\int_{\partial M} L_{ct}[\gamma]d^3x = \mp\frac{1}{4\pi L}\int_{\partial M} d^3x\sqrt{|\gamma|}\left(1 + \frac{L^2}{4}R[\gamma]\right).$$

By construction, our action is now finite at the boundary $\epsilon \to 0$ and is of the form:

$$S_E = \pm\frac{\beta\omega}{8\pi}\left(\frac{t_h^3}{L^2}\right) = \mp\frac{2\pi L\omega}{8\pi}\frac{L^3}{L^2} = \mp\frac{\omega L^2}{4}.$$

Picking the sign

$$(r,\theta,\phi) \to +i(r,\theta,\phi),$$

for the triple Wick-rotation, the signs of the Euclidean actions agree for both patches, and the actions only differ by the numerical factors $\omega, \omega_{S^2} = 4\pi$. As these are numbers, which we could eliminate by taking the Euclidean action per coordinate area, the resulting thermodynamics is the same. From the perspective of boundedness, we can pick our triple Wick-rotation such that $S_E > 0$. We will see in the following computations, the first law is recovered regardless of the sign of S_E. Furthermore, we find that the partition function is bounded from below naturally while satisfying other conditions. With this result, we continue to study the first law of thermodynamics for the solutions found in Chapters 5 and Chapters 6.

8.3 PLANAR SOLUTIONS OF EINSTEIN-MAXWELL THEORY

Our next example is the application of the triple Wick-rotation to the planar symmetric solutions of the Einstein-Maxwell equations, studied in Chapter 5. We have seen in Section 6.6 that these solutions are the simplest examples of a class of planar solutions of the STU model, corresponding to the limit where all scalar fields are taken to be constant. We use these solutions as a starting point to study the thermodynamics of the planar symmetric solutions of STU model, as they already show all the qualitative features of the global causal structure of the full class of solutions. We will see from our computations that in this section, that the thermodynamics of planar Einstein-Maxwell solution is simpler than, but representative of, the thermodynamics of planar solutions of the STU model.

Our starting point is the Lorentzian bulk action for Einstein-Maxwell theory (3.3.1), repeated here for convenience

$$S = \frac{1}{16\pi}\int d^4x\,\sqrt{-g}\,\left(-R - F^2\right).$$

In Section 5.1, we found that solving the Einstein-Maxwell equations while imposing planar symmetry and staticity leads to a solution with the line element (5.1.5), repeated here

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(dx^2 + dy^2), \qquad f(r) = -\frac{2M}{r} + \frac{Q^2}{r^2},$$

where we remind the reader that the constant M is taken to be positive to ensure the presence of a horizon. The transverse coordinate r takes values in the interval $0 < r < r_h$, where $r = 0$ is the location of a curvature singularity, while r_h is the location of a Killing horizon, where $f(r_h) = 0$.

We will assume that the solution only carries electric charge, the gauge field is given by

$$F = \left(-\frac{Q}{r}\right)dt \wedge dr\,. \tag{8.3.1}$$

The gauge potential A is found through integration of (8.3.1) together with the standard boundary condition $A(r_h) = 0$:

$$A = \left(-\frac{Q}{r} + \frac{Q}{r_h}\right) dt. \tag{8.3.2}$$

Charge & chemical potential The chemical potential is given by the asymptotic value of the gauge potential [71]; taking this limit for (8.3.2) gives

$$\mu := \lim_{r\to\infty} A_t = \frac{Q}{r_h} = \frac{2M}{Q} .$$

Note that while $r \to \infty$ is outside the static patch $0 < r < r_h$, we will see below that we can analytically extend spacetime to $0 < r < \infty$, so that this limit makes sense. The conserved electric charge was computed using Gauss' law (5.2.6), which was found to be

$$\mathcal{Q} = \lim_{r\to\infty} \frac{1}{4\pi} \int_{\mathbb{R}^2} \star F = \frac{Q\omega}{4\pi}, \qquad \omega = \int_{\mathbb{R}^2} dx \wedge dy .$$

Here ω is the divergent parametric area of the horizon. The factor of 4π is due to the normalisation we have chosen for the gauge field. In our conventions the volume form is defined using the conventional choice $\epsilon_{trxy} = 1$.

8.3.1 *Dynamic patch*

Due to the presence of a curvature singularity at $r = 0$ we cannot apply the standard thermodynamic formalism in the static patch. In Section 5.2.4 we analytically continued our coordinates using advanced Eddington-Finkelstein coordinates at an intermediate step to extend spacetime to the dynamical region $r_h < r < \infty$, where the horizontal Killing vector field became spacelike. As r becomes a timelike coordinate in the dynamic patch, we apply the same convention as in the de Sitter example and relabel the coordinates $(t, r) \to (r, t)$, and redefine f by a minus sign. Then the line element of the dynamic patch takes the form (5.2.8), repeated here

$$ds^2 = -\frac{dt^2}{f(t)} + f(t)dr^2 + t^2(dx^2 + dy^2), \qquad f(t) = \frac{2M}{t} - \frac{Q^2}{t^2}, \qquad t_h = \frac{Q^2}{2M}. \tag{8.3.3}$$

This line element covers region III of Figure 5.7, with $t \to \infty$ corresponding to past timelike infinity. Using advanced Eddington-Finkelstein coordinates, one can show that the Killing horizon between regions III and IV (and I) is an apparent horizon of *future inner type*, consistent with the interpretation as a contracting cosmological solution. We saw this in Section 5.3 from the point of view of Kruskal coordinates and the expansion of null congruences. We could alternatively classify the horizons of the solution using advanced/retarded Eddington-Finkelstein coordinates which are correlated with past/future timelike infinity. For further discussion on this, we refer to Appendix D in [40].

Temperature & entropy To compute the surface gravity and temperature of the future inner horizon, we use the Kodama-Hayward formalism to compute the surface gravity. From Section

5.3, we understand the regions III and IV to be separated by a future inner horizon and so $T_H \propto \kappa$. Applying (3.5.4) to the line element (8.3.3), we obtain

$$\kappa = -\frac{4M^3}{Q^4} \quad \Rightarrow \quad T_H = \frac{\kappa}{2\pi} = -\frac{2M^3}{\pi Q^4}. \tag{8.3.4}$$

The Bekenstein-Hawking area law gives a relation for the entropy of the horizon in terms of the horizon area:

$$S_{BH} = \frac{A}{4} = \frac{\omega t_h^2}{4} = \frac{Q^4 \omega}{16M^2}.$$

As with other extensive quantities, we keep the divergent volume ω as a formal constant rather than using densities.

8.3.2 *Euclidean action*

Our main goal is to show that the future inner horizon satisfies the first law of Killing horizon mechanics, which takes the same form as the first law of thermodynamics. This requires us to identify geometrically defined quantities of the solution with thermodynamic quantities. In standard black hole thermodynamics, the mass M of the black hole is identified with the internal energy of a canonical or grand canonical ensemble. Due to the planar symmetry, and since we are not working in a static patch, we do not have a natural candidate for a mass-like quantity. We will trade this problem for the one of obtaining a well behaved Euclidean action which we interpret as a grand canonical partition function. The mass-like quantity we identify with the internal energy is then obtained using standard thermodynamic relations. The remaining problem in defining the Euclidean action is its normalisation. For solutions which are asymptotic to a 'vacuum', that is to a maximally symmetric spacetime, the normalisation is fixed by including a boundary term such that the Euclidean action is zero when evaluated on the vacuum solution. We do not have this option as our solution is not asymptotic to a maximally symmetric spacetime. Moreover, the GHY-boundary term will turn out to be finite, so there is no natural candidate for a boundary counter term. However, the integral over the two planar directions is divergent, and while we can formally absorb this in a constant ω, we will allow for a finite multiplicative factor $\mathcal{N}$ between the Euclidean action S_E and the grand potential Ω:

$$\beta\Omega = \mathcal{N} S_E\,. \tag{8.3.5}$$

The constant $\mathcal{N}$ parametrises the relative normalisation between thermodynamic and geometric quantities. To fix it, we impose one relation, which we choose to be Gauss' law. That is, we identify the charge $\mathcal{Q}$ defined by the gauge field of our field configuration with the negative derivative of Ω with respect to the chemical potential

$$\frac{\partial\Omega}{\partial\mu} \stackrel{!}{=} -\mathcal{Q}\,. \tag{8.3.6}$$

Once $\mathcal{N}$ has been fixed by this condition, all thermodynamic relations must take their standard form, if our interpretation of $\mathcal{Z} = \exp(-\mathcal{N} S_E)$ as a thermodynamic partition function is correct.

Performing the triple Wick-rotation

$$(r, x, y) \rightarrow \pm i(r, x, y),$$

we obtain the negative-definite line element

$$ds^2 = -f(t)^{-1}dt^2 - f(t)dr^2 - t^2(dx^2 + dy^2). \tag{8.3.7}$$

The Euclidean action is given by

$$\begin{aligned} S_E = &\pm \frac{1}{16\pi}\int_M \sqrt{g}Rd^4x \pm \frac{1}{8\pi}\int_{\partial M}\sqrt{|\gamma|}Kd^3x, \\ &\mp \frac{1}{8\pi}\int_{\partial M} F^{\mu\nu}A_\mu d\Sigma_\nu. \end{aligned}$$

Since the planar Einstein-Maxwell solution has a vanishing Ricci scalar: $R = 0$, the action is completely determined by the boundary terms, which are evaluated in the limit where $t \to \infty$. We do not include a background boundary term K_0, as S_E will turn out to be finite.

The hypersurface $\Sigma = \partial M$ is obtained as the limit of a sequence of slices of the spacetime M for constant time t_0, and has an extrinsic curvature with trace K when considered as an embedded submanifold of M. It can be computed using the formulas reviewed in Section 2.2.6 and we obtain the result

$$K = \frac{3Mt_0 - Q^2}{t_0^2\sqrt{2Mt_0 - Q^2}}, \qquad \sqrt{|\gamma|} = t_0^2\left(\frac{2M}{t_0} - \frac{Q^2}{t_0^2}\right)^{\frac{1}{2}}.$$

Evaluating this in the limit $t_0 \to \infty$ gives

$$S_{GHY} = \pm\frac{1}{8\pi}\int_{\partial M}\sqrt{|\gamma|}K = \pm\frac{3M\beta\omega}{8\pi}.$$

The factor $\beta\omega$ is the parametric volume of the boundary. After Wick-rotation, the coordinate r becomes periodic with period β, in order to avoid a conical singularity at $t = t_h$.[2]

As the gauge potential has only one non zero component, so the boundary term is calculated

$$\mp\frac{1}{8\pi}\int_{\partial M} F^{\mu\nu}A_\mu d\Sigma_\nu = \pm\frac{M\beta\omega}{4\pi}.$$

Together, the GHY-term and the gauge field contribution yield the Euclidean action

$$S_E = \pm 5\frac{M\beta\omega}{8\pi}. \tag{8.3.8}$$

8.3.3 *Verifying the first law*

Formally equating the partition function calculated from the Euclidean action with the negative logarithm of the thermal partition function, $\log(\mathcal{Z}) = -\mathcal{N}S_E = -\beta\Omega$, yields the grand potential

$$\Omega(\beta,\mu) = \frac{\mathcal{N}S_E}{\beta} = \mp 5\mathcal{N}\frac{\beta\mu^4\omega}{(8\pi)^2},$$

which we have written in terms of its natural thermodynamic variables $\beta = T_H^{-1}$ and μ using the relationship

$$M = -\frac{\beta\mu^4}{8\pi}, \qquad \mathcal{Q} = -\frac{\mu^3\beta\omega}{(4\pi)^2}.$$

[2] To be precise, the conical method determines the period up to sign, and we choose β to have the sign determined by the Kodama-Hayward method.

We now apply our normalisation condition (8.3.6): the conserved charge $\mathcal{Q}$ calculated from Gauss' law must match the negative μ-derivative of Ω. Taking the derivative, we find that

$$-\frac{\partial\Omega}{\partial\mu} = \pm 5\mathcal{N}\frac{4\beta\mu^3\omega}{(8\pi)^2} = \pm 5\mathcal{N}\mathcal{Q}, \quad \Rightarrow \quad \mathcal{N} = \mp\frac{1}{5}.$$

From this, the grand potential and partition function are determined to be

$$\Omega(\beta,\mu) = \frac{\beta\mu^4\omega}{(8\pi)^2}, \qquad \log(\mathcal{Z}) = -\frac{\beta^2\mu^4\omega}{(8\pi)^2}\,. \tag{8.3.9}$$

and we see that the partition function is bounded from below.

The free energy $F(\beta,\mathcal{Q})$ is obtained as the Legendre transform of the grand potential

$$F(\beta,\mathcal{Q}) = \Omega - \mu\frac{\partial\Omega}{\partial\mu} = \Omega + \mu\mathcal{Q} = 3\left(-\frac{\pi^2\mathcal{Q}^4}{4\beta\omega}\right)^{\frac{1}{3}}, \tag{8.3.10}$$

where we have used the relation

$$\mu = \left(-\frac{16\pi^2\mathcal{Q}}{\omega\beta}\right)^{1/3},$$

to express the free energy in terms of its natural variables β and $\mathcal{Q}$. From F we can compute the thermodynamic entropy S and check that it matches the Bekenstein-Hawking entropy S_{BH} :

$$S = \beta^2\frac{\partial F}{\partial\beta} = \left(\frac{\pi^2\mathcal{Q}^4\beta^2}{4\omega}\right)^{\frac{1}{3}} = S_{BH}\,. \tag{8.3.11}$$

As a further consistency check, we can also verify that the free energy gives us the correct chemical potential:

$$\frac{\partial F}{\partial\mathcal{Q}} = \left(-\frac{16\pi^2\mathcal{Q}}{\beta\omega}\right)^{1/3} = \mu.$$

The internal energy E, for which we do not have a geometric definition, is computed from the free energy:

$$E = \frac{\partial(F\beta)}{\partial\beta} = \left(-\frac{2\pi^2\mathcal{Q}^4}{\beta\omega}\right)^{1/3} = \frac{M\omega}{4\pi}\,.$$

We observe that E is proportional to the parameter M, and therefore E is positive. Notice that like the de Sitter solution, if we set the area density to $\omega = 4\pi$, we obtain $E = M$. Looking back to (5.2.7), we see that the mass we compute here matches the naive asymptotic limit of the Komar energy. In both the triple Wick-rotation and in the Komar 'mass', we could think of the mass-like parameter being associated to a momentum like quantity, and so it is interesting to see how they match.

Using our previous results we can verify that the thermodynamic variables $E, T, S, \mu, \mathcal{Q}$ satisfy the Smarr relation

$$E = 2TS + \mu\mathcal{Q}\,. \tag{8.3.12}$$

Expressing the internal energy E in terms of its natural variables S and $\mathcal{Q}$, we obtain the equation of state

$$E(S,\mathcal{Q}) = \frac{\pi\mathcal{Q}^2}{(S\omega)^{1/2}}\,.$$

The partial derivates of the internal energy are

$$\frac{\partial E}{\partial S} = -\frac{\pi Q^2}{2S^{3/2}\omega^{1/2}} = \frac{1}{\beta} = T, \qquad \frac{\partial E}{\partial Q} = \frac{2\pi Q}{(S\omega)^{1/2}} = \mu,$$

where both expressions have been simplified by substituting in $S(\beta, Q)$ using (8.3.11). The variation of the internal energy is

$$dE = \frac{\partial E}{\partial S} dS + \frac{\partial E}{\partial Q} dQ = TdS + \mu dQ\,.$$

This relation takes the standard form of the first law of thermodynamics. Note that this works because we have allowed that the temperature is negative. If we had insisted that the temperature is positive, this would have resulted in a non-standard sign for the entropy term.

8.4 PLANAR SOLUTIONS OF THE STU MODEL

We are now in a position to turn to our main application of the triple Wick-rotation; the planar cosmological solutions of the STU model found in Chapter 6, for which we will now verify the first law of thermodynamics. To aid our discussion, we repeat the bosonic Lagrangian for n_V vector multiplets coupled to $\mathcal{N} = 2$ supergravity

$$e_4^{-1}\mathcal{L} = -\frac{1}{2\kappa_4^2}R - \frac{1}{\kappa_4^2}g_{A\bar{B}}\partial_\mu z^A \partial^\mu \bar{z}^{\bar{B}} + \frac{1}{4\kappa_4^2}\mathcal{I}_{IJ}F^I_{\mu\nu}F^{J|\mu\nu} + \frac{1}{4\kappa_4^2}\mathcal{R}_{IJ}F^I_{\mu\nu}\tilde{F}^{J|\mu\nu}, \tag{8.4.1}$$

where compared to (6.1.1), we have restored the four-dimensional gravitational coupling κ_4 by an overall scaling to maintain the form of our solutions. While we used standard supergravity conventions where $\kappa_4^2 = 1$ in Chapter 6, it will be more convenient in the following to use relativist's conventions where $G = 1$ and $\kappa_4^2 = 8\pi$, in order to avoid non-standard numerical factors in thermodynamic relations. We remind the reader that the couplings $g_{A\bar{B}}$, $\mathcal{I}_{IJ}$ and $\mathcal{R}_{IJ}$, where $A, B = 1, \ldots, n$ and $I, J = 0, \ldots, n$ are functions of the scalar fields z^A (6.4.10), and their exact form is derived in Appendix E.

8.4.1 *Dynamic patch*

As in the solutions of Einstein-Maxwell, we begin with the line element in the dynamical patch of the planar symmetric cosmological solution

$$ds^2 = -\frac{\mathcal{H}(\zeta)}{\mathcal{W}(\zeta)}d\zeta^2 + \frac{\mathcal{W}(\zeta)}{\mathcal{H}(\zeta)}d\eta^2 + G(\zeta)(dx^2 + dy^2), \tag{8.4.2}$$

where all functions depend only on the timelike coordinate ζ:

$$\begin{aligned}
\mathcal{W}(\zeta) &= \alpha\zeta - 1,\\
\mathcal{H}_a(\zeta) &= (\beta_a + \gamma_a\zeta),\\
\mathcal{H}(\zeta) &= 2\,(\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3)^{\frac{1}{2}}\,,\\
G(\zeta) &= \zeta^2\left[\left(1+\frac{\beta_0}{\gamma_0\zeta}\right)\left(1+\frac{\beta_1}{\gamma_1\zeta}\right)\left(1+\frac{\beta_2}{\gamma_2\zeta}\right)\left(1+\frac{\beta_3}{\gamma_3\zeta}\right)\right]^{\frac{1}{2}}.
\end{aligned}$$

This line element has been modified compared to (6.4.9), through the rescaling of the planar coordinates $(\bar{x}, \bar{y}) \mapsto (x, y)$ such that the corresponding part of the line element is now

$$\begin{aligned}\mathcal{H}(\zeta)(d\bar{x}^2 + d\bar{y}^2) &= 2\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}\, G(\zeta)(d\bar{x}^2 + d\bar{y}^2),\\ &= G(\zeta)(dx^2 + dy^2)\ .\end{aligned}$$

This rescaling has been chosen such that the asymptotic form of the planar line element is $ds_2^2 = \zeta^2(dx^2 + dy^2)$. In this form, the line element matches the planar Einstein-Maxwell solutions which asymptotically is given by $ds_2^2 = t^2(dx^2 + dy^2)$. We note that as the line element has no functional dependence on the planar coordinates, we could rescale them by an arbitrary function of the thermodynamic data; the choice that we make here appears to be the most natural.

In the following calculations we will rewrite expressions using the integration constants found in Section 6.4. We repeat here the relations between the integration constants found while solving the equations of motion in Section 6.2 and the ones appear in (8.4.2)

$$\beta_a = \frac{2K_a}{\alpha}\sinh\left(\frac{\alpha h_a}{2K_a}\right), \qquad \gamma_a = K_a \exp\left(-\frac{\alpha h_a}{2K_a}\right), \tag{8.4.3}$$

where

$$K_a = \left(Q_0, P^1, P^2, P^3\right),$$

are the four non-zero charges carried by the gauge fields $F^I_{\mu\nu}$.[3] Without loss of generality, we can take the set of integration constants $\{\alpha, K_a, h_a\}$ to be non-negative. We will work with the gauge fields (6.4.8), which are given by

$$F^0_{\zeta\eta} = -\frac{Q_0}{2(\beta_0 + \gamma_0\zeta)^2}, \qquad \tilde{F}_{A|\zeta\eta} = \frac{P^A}{2(\beta_A + \gamma_A\zeta)^2}. \tag{8.4.4}$$

where we explicitly note that we choose to work with $\tilde{F}_{A|\mu\nu}$, which denote the duals of the gauge field $F^A_{\mu\nu}$. The advantage of using the fields $(F^0, \tilde{F}_A)$ instead of (F^0, F^A) is that now all gauge fields and charges appearing in the solution are electric,[4] allowing us to treat all contributions the same thermodynamically. Note that as the gauge couplings are field dependent, dualisation is not just Hodge-star, but additionally involves inverting the couplings. This was discussed in Section 4.2 for the general case of $\mathcal{N} = 2$ supergravity. The precise relation between gauge fields and dual gauge fields is

$$\tilde{F}_A = - \star \mathcal{I}_{AB} F^B \quad \Rightarrow \quad F^A = \star \mathcal{I}^{AB}\tilde{F}_B,$$

where $F^A, \tilde{F}_A$ are the two-forms corresponding to the gauge fields. Note that the coupling matrix $\mathcal{I}_{IJ}$ is invertible, and in our convention is negative-definite.

Temperature In Section 6.5.1, we saw that the Killing horizon located at $\zeta = \zeta_h = \alpha^{-1}$ is a *future inner horizon* and hence $T_H \propto \kappa$. The surface gravity is computed using the Kodama-Hayward

[3] We drop the sign in front of Q_0 within K_a as it allows us to drop the various modulus signs in our integration constants.

[4] This is for computational simplicity. In [88], the authors show how magnetic and electric black hole solutions are equivalent in the semi-classical approach applied here.

formulation (3.5.4) and we find that the temperature is negative, and of the form

$$\begin{aligned} T_H &= -\frac{1}{4\pi}\partial_\zeta\left(\frac{\mathcal{W}(\zeta)}{\mathcal{H}(\zeta)}\right)\bigg|_{\zeta=\alpha^{-1}}, \\ &= -\frac{\alpha^3}{8\pi}\left[(\alpha\beta_0+\gamma_0)(\alpha\beta_1+\gamma_1)(\alpha\beta_2+\gamma_2)(\alpha\beta_3+\gamma_3)\right]^{-\frac{1}{2}}. \end{aligned} \tag{8.4.5}$$

Using the relations above (8.4.3), we can simplify the expression of the temperature to

$$(\alpha\beta_a+\gamma_a) = K_a \exp\left(\frac{\alpha h_a}{2K_a}\right) = \frac{K_a^2}{\gamma_a} \quad\Rightarrow\quad T_H = -\frac{\alpha^3}{8\pi}\frac{\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}}{Q_0P^1P^2P^3}.$$

Entropy Using the Bekenstein-Hawking area law we can compute the entropy density of the solution:

$$\begin{aligned} S_{BH} &= \frac{G(\zeta_h)}{4} = \frac{1}{4\alpha^2}\exp\left[\frac{\alpha}{2}\left(\frac{h_0}{Q_0}+\frac{h_1}{P^1}+\frac{h_2}{P^2}+\frac{h_3}{P^3}\right)\right], \\ &= \frac{1}{4\alpha^2}\frac{Q_0P^1P^2P^3}{\gamma_0\gamma_1\gamma_2\gamma_3}. \end{aligned}$$

Since the planar STU solution has several integration constants, we will suppress the parametric volume ω of the planar directions in this section by setting $\omega = 1$. This can be interpreted as either working with densities of divergent extensive quantities, or as compactifying the planar dimensions on a two-torus with unit area.

Chemical potentials The corresponding gauge potentials are found by integration of the gauge fields (8.4.4), subject to the standard boundary condition $A(\zeta_h) = \tilde{A}(\zeta_h) = 0$:

$$(A^0)_\eta = -\frac{\gamma_0(\alpha\zeta-1)}{2Q_0(\beta_0+\gamma_0\zeta)}, \qquad (\tilde{A}_A)_\eta = \frac{\gamma_A(\alpha\zeta-1)}{2P^A(\beta_A+\gamma_A\zeta)}.$$

We then take the asymptotic limit of the gauge potentials to obtain the chemical potentials

$$\mu^0 := \lim_{\zeta\to\infty} A^0_\eta = -\frac{\alpha}{2Q_0}, \qquad \tilde{\mu}_A := \lim_{\zeta\to\infty} \tilde{A}_{A|\eta} = \frac{\alpha}{2P^A}.$$

Electromagnetic charges As with the Einstein-Maxwell solution, the conserved charges are computed using Gauss' law. However, we need to take into account that the gauge couplings depend on the scalar fields. The gauge field couplings come from $\mathcal{I}_{IJ}$ and were calculated explicitly in (7.1.3)

$$\mathcal{I}_{IJ} = \mathrm{diag}\left(-stu, -\frac{tu}{s}, -\frac{su}{t}, -\frac{st}{u}\right), \qquad \mathcal{I}^{IJ} = \mathrm{diag}\left(-\frac{1}{stu}, -\frac{s}{tu}, -\frac{t}{su}, -\frac{u}{st}\right),$$

where

$$s = -\mathrm{Im}(z^1), \qquad t = -\mathrm{Im}(z^2), \qquad u = -\mathrm{Im}(z^3).$$

Putting in the solution (6.4.10) for the scalar fields z^A we can write these couplings as

$$\begin{aligned} \mathcal{I}_{00} &= -\left(\frac{\mathcal{H}_0^3}{\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3}\right)^{\frac{1}{2}}, & \mathcal{I}_{11} &= -\left(\frac{\mathcal{H}_0\mathcal{H}_2\mathcal{H}_3}{\mathcal{H}_1^3}\right)^{\frac{1}{2}}, \\ \mathcal{I}_{22} &= -\left(\frac{\mathcal{H}_0\mathcal{H}_1\mathcal{H}_3}{\mathcal{H}_2^3}\right)^{\frac{1}{2}}, & \mathcal{I}_{33} &= -\left(\frac{\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2}{\mathcal{H}_3^3}\right)^{\frac{1}{2}}. \end{aligned} \tag{8.4.6}$$

The charge $\mathcal{Q}_0$ carried by the gauge field F^0 is

$$\mathcal{Q}_0 = \lim_{\zeta\to\infty} \frac{1}{8\pi} \int \star(-\mathcal{I}_{00}F^0). \tag{8.4.7}$$

Evaluating (8.4.7) using the expressions for the couplings (8.4.6), and the exact form of the gauge field (8.4.4) we obtain the conserved charge density

$$\mathcal{Q}_0 = -\frac{1}{16\pi}\frac{Q_0}{\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}}. \tag{8.4.8}$$

We use the normalisation $\epsilon_{\eta\zeta xy} = 1$ for the volume form, which is the standard normalisation in the static patch of the solution, where η is timelike and ζ spacelike. Note that the Hodge operator contains a factor of ζ^2, so that when we evaluate the integral in the limit $\zeta \to \infty$ we read out the coefficient of the leading term in the integrand, which is proportional to $1/\zeta^2$. This is the leading behaviour of the field strength F^0, while the coupling $\mathcal{I}_{00}$ approaches a constant.

As mentioned, we have dualised the magnetic field strengths F^A and instead work with their electric duals $\tilde{F}_A$, but we must remember that when we dualise a gauge potential in the Lagrangian the corresponding coupling is inverted. This means the conserved dual electric charges are

$$\tilde{\mathcal{Q}}^A = \lim_{\zeta\to\infty} \frac{1}{8\pi} \int \star(-\mathcal{I}^{AA}\tilde{F}_A),$$

which when evaluated on our solution take the values

$$\tilde{\mathcal{Q}}^A = \frac{1}{16\pi}\frac{P^A}{\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}}. \tag{8.4.9}$$

The dual electric charge $\tilde{\mathcal{Q}}^A$ can be related to the magnetic charge of F^A by $\tilde{\mathcal{Q}}^A = -\mathcal{P}^A$.[5]

8.4.2 *Euclidean action*

Employing the triple Wick-rotation

$$(\eta, x, y) \to \pm i(\eta, x, y),$$

the Euclidean line element has (negative) definite signature and is of the form

$$ds^2 = -\frac{\mathcal{H}(\zeta)}{\mathcal{W}(\zeta)}d\zeta^2 - \frac{\mathcal{W}(\zeta)}{\mathcal{H}(\zeta)}d\eta^2 - G(\zeta)(dx^2 + dy^2). \tag{8.4.10}$$

As we did with the Einstein-Maxwell solution, we evaluate the Euclidean action on-shell, which allows us to write the gauge contributions as boundary terms

$$\begin{aligned} S_E = &\pm \frac{1}{16\pi}\int_M \sqrt{g}\left(R + 2g_{A\bar{B}}\partial_\mu z^A \partial^\mu \bar{z}^{\bar{B}}\right) d^4x, \\ &\pm \frac{1}{8\pi}\int_{\partial M}\sqrt{|\gamma|}\, K d^3x, \\ &\pm \frac{1}{16\pi}\int_{\partial M}(\mathcal{I}_{00}F^{\mu\nu|0})A^0_\mu d\Sigma_\nu \pm \frac{1}{16\pi}\int_{\partial M}(\mathcal{I}^{AA}\tilde{F}^{\mu\nu}_A)\tilde{A}_{\mu|A} d\Sigma_\nu. \end{aligned}$$

We have performed the dualisation procedure such that we work with a purely electric solution.

[5] See Section 4.2 for more information.

Cancellation of bulk terms As in the much simpler case of Einstein-Maxwell theory, the bulk term does not contribute. This is non-trivial since the Ricci scalar does no longer vanish on-shell. However, the gauge field contribution still is a boundary term, and the scalar contribution precisely cancels the gravitational term in the bulk. The trace of Einstein's equation gives

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi T_{\mu\nu} \quad \Rightarrow \quad R = 8\pi T \ .$$

In four dimensions, the gauge fields do not contribute to the trace of the energy-momentum tensor, which is therefore completely determined by the scalars:

$$T = g^{\mu\nu} T_{\mu\nu} = -\frac{2}{8\pi} g_{A\bar{B}} \left(\partial_\mu z^A \partial^\mu \bar{z}^{\bar{B}} \right),$$

which shows that

$$-\frac{1}{2} R = g_{A\bar{B}} \left(\partial_\mu z^A \partial^\mu \bar{z}^{\bar{B}} \right),$$

and therefore the bulk contribution of the solution vanishes. Note that when we set the scalars constant, we recover the electro-vac type solution of Einstein-Maxwell theory considered in the previous section, which is not Ricci flat $R_{\mu\nu} \propto T_{\mu\nu} \neq 0$, but has vanishing Ricci scalar.

Calculation of boundary terms With the bulk terms found vanishing, the Euclidean action for the planar solution of the STU model can be found from the boundary terms. Following the same method as for the planar Einstein-Maxwell solution, the GHY-term is calculated to be

$$\pm \frac{1}{8\pi} \int_{\partial M} \sqrt{|\gamma|} K d^3x = \pm \frac{3}{32\pi} \frac{\alpha\beta}{\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}} \ .$$

The gauge field term is calculated through substituting in the various components and taking the limit of $\zeta \to \infty$, obtaining

$$\pm \frac{1}{16\pi} \int_{\partial M} (\mathcal{I}_{00} F^{\mu\nu|0}) A^0_\mu d\Sigma_\nu = \mp \frac{1}{64\pi} \frac{\alpha\beta}{\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}},$$

and similarly

$$\pm \sum_{A=1}^{3} \frac{1}{16\pi} \int_{\partial M} (\mathcal{I}^{AA} \tilde{F}^{\mu\nu}_A) \tilde{A}_{\mu|A} d\Sigma_\nu = \mp \frac{3}{64\pi} \frac{\alpha\beta}{\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}} \ .$$

Collecting these terms, the Euclidean action is found to be

$$S_E = \pm \frac{1}{32\pi} \frac{\alpha\beta}{\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}} \ .$$

8.4.3 *Verifying the first law*

As in the Einstein-Maxwell case we admit a multiplicative constant $\mathcal{N}$ in the relation between the Euclidean action and the grand potential:

$$\Omega(\beta, \mu^0, \tilde{\mu}_A) = \frac{\mathcal{N} S_E}{\beta} = \pm \frac{\mathcal{N}}{32\pi} \frac{\alpha}{\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}} \ .$$

The constant $\mathcal{N}$ is fixed by imposing that one of the thermodynamic relations takes its standard form. We choose to impose the relation between the μ^0 derivative of Ω and the charge $\mathcal{Q}_0$

$$\left(\frac{\partial\Omega}{\partial\mu^0}\right)_{\beta,\tilde{\mu}_A} \overset{!}{=} -\mathcal{Q}_0 = \frac{1}{16\pi}\frac{Q_0}{\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}}. \tag{8.4.11}$$

To impose this condition, we first need to express the grand potential Ω in terms of its natural variables. This can be done using the relationship

$$\frac{\alpha}{\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}} = \frac{2\beta}{\pi}\mu^0\tilde{\mu}_1\tilde{\mu}_2\tilde{\mu}_3,$$

leading to the expression

$$\Omega(\beta,\mu^0,\tilde{\mu}_A) = \pm\frac{\mathcal{N}}{16\pi^2}\beta\mu^0\tilde{\mu}_1\tilde{\mu}_2\tilde{\mu}_3\,. \tag{8.4.12}$$

Taking the partial derivate we obtain the electric charge from the grand potential

$$\left(\frac{\partial\Omega}{\partial\mu^0}\right)_{\beta,\tilde{\mu}_A} = \pm\frac{\mathcal{N}}{16\pi^2}\beta\tilde{\mu}_1\tilde{\mu}_2\tilde{\mu}_3 = \mp\frac{\mathcal{N}}{16\pi}\frac{Q_0}{\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}}\,.$$

Comparing this with (8.4.11) we find that $\mathcal{N} = \mp 1$. This determines the grand potential to be

$$\Omega(\beta,\mu^0,\tilde{\mu}_A) = -\frac{1}{16\pi^2}\beta\mu^0\tilde{\mu}_1\tilde{\mu}_2\tilde{\mu}_3 = -\frac{1}{8\pi}\frac{\alpha}{4\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}}\,.$$

Note that as in the Einstein-Maxwell case, the grand potential Ω is independent of our choice of sign for the triple Wick-rotation after imposing Gauss' law as a 'boundary condition'. Due to the similarity in form of the gauge fields, is clear that the other derivatives of Ω with respect to chemical potentials give the correct corresponding charges. Computing the partition function, we can see it is bounded from below, just as with the Einstein-Maxwell solution:

$$\log(\mathcal{Z}) = -\beta\Omega = -\frac{1}{4\alpha^2}\frac{Q_0P^1P^2P^3}{\gamma_0\gamma_1\gamma_2\gamma_3}.$$

To obtain the free energy we perform a Legendre transform of the grand potential:

$$\begin{aligned} F(\beta,\mathcal{Q}_0,\tilde{\mathcal{Q}}^A) &= \Omega + \mu^0\mathcal{Q}_0 + \tilde{\mu}_1\tilde{\mathcal{Q}}^1 + \tilde{\mu}_2\tilde{\mathcal{Q}}^2 + \tilde{\mu}_3\tilde{\mathcal{Q}}^3, \\ &= -\frac{1}{8\pi}\frac{\alpha}{4\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}} + \frac{1}{8\pi}\frac{\alpha}{\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}}, \end{aligned}$$

and so the free energy is given by

$$F(\beta,\mathcal{Q}_0,\tilde{\mathcal{Q}}^A) = \frac{1}{8\pi}\frac{3\alpha}{4\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}}. \tag{8.4.13}$$

To express F in terms of its natural thermodynamical variables, we use

$$\beta = -\frac{8\pi}{\alpha^3}\frac{Q_0P^1P^2P^3}{\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}} \quad\Rightarrow\quad \frac{\alpha}{\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}} = \left(\frac{(16\pi)^5\mathcal{Q}_0\tilde{\mathcal{Q}}^1\tilde{\mathcal{Q}}^2\tilde{\mathcal{Q}}^3}{2\beta}\right)^{\frac{1}{3}},$$

and obtain

$$F(\beta,\mathcal{Q}_0,\mathcal{P}^A) = \frac{3}{32\pi}\left(\frac{(16\pi)^5\mathcal{Q}_0\tilde{\mathcal{Q}}^1\tilde{\mathcal{Q}}^2\tilde{\mathcal{Q}}^3}{2\beta}\right)^{\frac{1}{3}}. \tag{8.4.14}$$

We can now verify that all remaining thermodynamic relations take their standard form. First we verify that the Bekenstein-Hawking entropy matches with the thermodynamic definition

$$S = \beta^2 \left(\frac{\partial F}{\partial \beta} \right)_{\mathcal{Q}_0, \tilde{\mathcal{Q}}^A} = \frac{1}{4\alpha^2} \frac{Q_0 P^1 P^2 P^3}{\gamma_0 \gamma_1 \gamma_2 \gamma_3}.$$

A further consistency check comes from ensuring that the chemical potentials that were found from the gauge field satisfy the standard thermodynamic relations for chemical potentials

$$\mu^0 = \left(\frac{\partial F}{\partial \mathcal{Q}_0} \right)_{\beta, \tilde{\mathcal{Q}}^A} = \frac{1}{16\pi \mathcal{Q}_0} \frac{\alpha}{2\sqrt{\gamma_0 \gamma_1 \gamma_2 \gamma_3}} = -\frac{\alpha}{2Q_0},$$

and for the dual gauge fields

$$\tilde{\mu}_A = \left(\frac{\partial F}{\partial \tilde{\mathcal{Q}}^A} \right)_{\beta, \mathcal{Q}_0} = \frac{\alpha}{2P^A},$$

which matches exactly with the chemical potentials found from the asymptotic limit of the vector potentials.

The internal energy of our solution can now be defined by the relation

$$\begin{aligned} E = \left(\frac{\partial (\beta F)}{\partial \beta} \right)_{\mathcal{Q}_0, \tilde{\mathcal{Q}}^A} &= \frac{1}{16\pi} \left(\frac{(16\pi)^5 \mathcal{Q}_0 \tilde{\mathcal{Q}}^1 \tilde{\mathcal{Q}}^2 \tilde{\mathcal{Q}}^3}{2\beta} \right)^{\frac{1}{3}}, \\ &= \frac{1}{16\pi} \frac{\alpha}{\sqrt{\gamma_0 \gamma_1 \gamma_2 \gamma_3}}. \end{aligned} \tag{8.4.15}$$

Next we express the entropy in terms of its natural thermodynamic variables to obtain the equation of state

$$S(E, \mathcal{Q}_0, \tilde{\mathcal{Q}}^A) = -\frac{16 \cdot 4\pi^2 \mathcal{Q}_0 \tilde{\mathcal{Q}}^1 \tilde{\mathcal{Q}}^2 \tilde{\mathcal{Q}}^3}{E^2}.$$

Note that the entropy is positive, due to $\mathcal{Q}_0 < 0$ and $\mathcal{Q}^A > 0$, see (8.4.8) and (8.4.9), bearing in mind that we have chosen Q_0 and P^A to be positive.[6]

We need to verify that the Hawking temperature of our solution satisfies the thermodynamic relation

$$\beta = \frac{1}{T_H} = \left(\frac{\partial S}{\partial E} \right)_{\mathcal{Q}_0, \mathcal{P}^A}.$$

Taking the partial derivate of S with respect to E we find that

$$\left(\frac{\partial S}{\partial E} \right)_{\mathcal{Q}_0 \tilde{\mathcal{Q}}^A} = \frac{16 \cdot 8\pi^2 \mathcal{Q}_0 \tilde{\mathcal{Q}}^1 \tilde{\mathcal{Q}}^2 \tilde{\mathcal{Q}}^3}{E^3}.$$

To compare this with the Hawking temperature we restore the original integration constants:

$$\left(\frac{\partial S}{\partial E} \right)_{\mathcal{Q}_0, \tilde{\mathcal{Q}}^A} = -\frac{8\pi Q_0 P^1 P^2 P^3}{\alpha^3 \sqrt{\gamma_0 \gamma_1 \gamma_2 \gamma_3}} = \beta.$$

Thus the Hawking temperature T_H, calculated from the geometry of the solution agrees with the thermodynamic quantity $T = \partial E / \partial S$.

[6] Note that the sign of the entropy does not change if change the signs of charges. We have just chosen certain charges to be positive or negative in order to avoid carrying around ± signs or to distinguish several cases.

Smarr relation Evaluating the grand potential we find

$$\Omega = E - TS - \mu^0 Q_0 - \tilde{\mu}_A \tilde{Q}^A = -\frac{\alpha}{32\pi\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}} = TS\,,$$

which we can rearrange in the form of a standard Smarr relation

$$E = 2TS + \mu^0 Q_0 + \tilde{\mu}_A \tilde{Q}^A. \tag{8.4.16}$$

First law of thermodynamics We wish to verify the first law:

$$dE = T_H dS + \mu^0 dQ_0 + \tilde{\mu}_1 d\tilde{Q}^1 + \tilde{\mu}_2 d\tilde{Q}^2 + \tilde{\mu}_3 d\tilde{Q}^3.$$

The total differential of E is

$$dE = \left(\frac{\partial E}{\partial S}\right) dS + \left(\frac{\partial E}{\partial Q_0}\right) dQ_0 + \left(\frac{\partial E}{\partial \tilde{Q}^1}\right) d\tilde{Q}^1 + \left(\frac{\partial E}{\partial \tilde{Q}^2}\right) d\tilde{Q}^2 + \left(\frac{\partial E}{\partial \tilde{Q}^3}\right) d\tilde{Q}^3.$$

Having already found that

$$\left(\frac{\partial E}{\partial S}\right) = T_H,$$

we turn our attention to the derivatives with respect to the charges. Using that:

$$E^2 = -\frac{16\cdot 4\pi^2 Q_0 \tilde{Q}^1 \tilde{Q}^2 \tilde{Q}^3}{S},$$

we find

$$\left(\frac{\partial E}{\partial Q_0}\right)_{S,\tilde{Q}^A} = -\frac{1}{2E}\frac{16\cdot 4\pi^2 \tilde{Q}^1 \tilde{Q}^2 \tilde{Q}^3}{S} = -\frac{\alpha}{2Q_0} = \mu^0.$$

Taking derivatives with respect to the magnetic charges we verify

$$\left(\frac{\partial E}{\partial \tilde{Q}^A}\right)_{S,Q_0} = \frac{\alpha}{2P^A} = \tilde{\mu}_A.$$

Hence we see that the first law of thermodynamics holds.

8.5 COMPARISON TO THE ISOLATED HORIZON FORMALISM

We now present an alternative way of studying the first law working entirely on the horizon. This allows us to calculate thermodynamic variables in the static region of the spacetime where the usual definitions of the thermodynamic constants hold. From [199], we find that the first law using variables defined on the horizon is:

$$\delta E_\Delta := \frac{\kappa \delta a_\Delta}{8\pi G} + \mu_a \delta Q^a. \tag{8.5.1}$$

The ambiguity of the energy in the spacetime is fixed by imposing that the infinitesimal energy (mass) is equal to the RHS (8.5.1). For the remainder of this discussion, we set $G = 1$. The subscript Δ denotes variables evaluated on the isolated horizon Δ, which for us is the location of our Killing horizon at $\zeta = \alpha^{-1}$. The contracted a index denotes the multiple charges (in our case, we have four).

There is an important distinction between this work and our work using the Euclidean action formalism. Before, we found that we could derive a mass-like parameter from the free energy, which itself could be computed from the Euclidean action with the additional condition that the thermodynamic charge matched the Gauss' law. We could then vary the mass and verify that it obeyed the first law of thermodynamics. For the isolated horizon formalism, the mass parameter of the solution is defined such that the first law holds. As such, the mass parameter and the first law of black hole mechanics in the isolated horizon can only be considered as self-consistent and does not give a direct way to measure the mass outside of the first law itself. Despite this, we find that the isolated horizon formalism provides a consistency check for our work, which is especially helpful when we consider the issue of normalisation which has followed us throughout this chapter.

In this section, we consider the planar symmetric solutions of the STU model, and then using the mapping described in Section 6.6, we then show that our results are also consistent with the planar solutions of Einstein-Maxwell theory.

8.5.1 *Planar solutions of the STU model*

Before we begin, we make a coordinate change into Eddington-Finkelstein coordinates of the rescaled line element (8.4.2). From there, we identify a Killing vector ℓ which we use to find κ. In a similar way to the previous section, we then determine the electromagnetic terms, but this time evaluated on the horizon rather than for $\zeta \to \infty$.

Eddington-Finkelstein coordinates Beginning with the metric from the dynamic region of the spacetime (8.4.2)

$$ds^2 = -\frac{\mathcal{H}(\zeta)}{\mathcal{W}(\zeta)}d\zeta^2 + \frac{\mathcal{W}(\zeta)}{\mathcal{H}(\zeta)}d\eta^2 + G(\zeta)(dx^2 + dy^2),$$

we make the coordinate change to the outgoing null coordinate u, using a tortoise coordinate $\zeta_\star$

$$\eta = u + \zeta_\star, \qquad d\eta = du + \zeta_\star d\zeta, \qquad \zeta_\star = \frac{\mathcal{H}}{\mathcal{W}},$$

to obtain the Eddington-Finkelstein metric which is well defined for $\zeta = \alpha^{-1}$

$$ds^2 = \frac{\mathcal{W}}{\mathcal{H}}du^2 + 2dud\bar{\zeta} + G(dx^2 + dy^2). \tag{8.5.2}$$

This allows us to identify a suitable null normal vector field

$$\ell = \frac{\partial}{\partial u}.$$

Surface gravity and area term We can reuse our calculation for the surface gravity from (8.4.5) to find

$$\kappa = -\frac{\alpha}{2}\frac{1}{\mathcal{H}(\alpha^{-1})}.$$

From the metric (8.5.2), we can read off the infinitesimal change in the area as:

$$\delta a_\Delta = \delta(G(\alpha^{-1})) = \delta(c\mathcal{H}(\alpha^{-1})) = \mathcal{H}(\alpha^{-1})\delta c + c\delta\mathcal{H}(\alpha^{-1}), \quad c = \frac{1}{2\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}},$$

where we maintain that divergent area contribution from the planar solutions is effectively divided out

$$\omega = \int dx \wedge dy = 1,$$

such that we consider densities, as we did in the Euclidean action formalism.

Putting these together, we find that the first term on the right-hand side of (8.5.1) is given by:

$$\kappa \delta a_\Delta = -\frac{\alpha}{2}\left(\delta c + \frac{c\delta\mathcal{H}(\alpha^{-1})}{\mathcal{H}(\alpha^{-1})}\right).$$

We can simplify this by using the expression

$$\mathcal{H}(\alpha^{-1}) = \frac{4c}{\alpha^2} Q_0 P^1 P^2 P^3,$$

to write down that

$$\begin{aligned}\kappa \delta a_\Delta &= -\frac{1}{2}\alpha\delta c - \frac{1}{2}\alpha c \delta \log(c\alpha^{-2} Q_0 P^1 P^2 P^3),\\ &= -\alpha \delta c + c\delta\alpha - \frac{1}{2}\alpha c\delta\log(Q_0 P^1 P^2 P^3).\end{aligned}$$

Gauge fields and charges From the previous calculation, we found that the gauge field strengths are given by the relations

$$F^0_{\zeta\eta} = -\frac{Q_0}{2(\beta_0 + \gamma_0\zeta)^2}, \qquad \tilde{F}_{A|\zeta\eta} = \frac{P^A}{2(\beta_A + \gamma_A\zeta)^2}.$$

We need to express the gauge field strength in terms of the Eddington-Finkelstein coordinates and then write down the field strength and the corresponding gauge couplings on the horizon. Starting with the gauge field strengths, we see that they are all of the form

$$F = f(\zeta) d\zeta \wedge d\eta.$$

Defining a null basis

$$ds^2 = 2e^+e^- + \delta_{ij}e^i e^j,$$

$$e^+ = du\,, \quad e^- = d\zeta + \frac{\mathcal{W}}{2\mathcal{H}}du\,, \quad e^+\wedge e^- = -d\zeta\wedge du,$$

and we can easily take the Hodge dual

$$F = -f(\zeta)e^+\wedge e^-\,, \qquad \star F = f(\zeta)e^1\wedge e^2 = c\mathcal{H}f(\zeta)dx\wedge dy.$$

This allows us to write down the Hodge duals explicitly

$$\star F^0 = -\frac{Q_0}{2(\beta_0+\gamma_0\zeta)^2}c\mathcal{H}dx\wedge dy\,, \qquad \star\tilde{F}_A = \frac{P^A}{2(\beta_A+\gamma_A\zeta)^2}c\mathcal{H}dx\wedge dy.$$

Evaluated on the horizon, these gauge fields are:

$$\star F^0_\Delta = -\frac{Q_0\alpha^2}{2(\beta_0\alpha+\gamma_0)^2}c\mathcal{H}(\alpha^{-1})dx\wedge dy\,, \qquad \star\tilde{F}_{A|\Delta} = \frac{P^A\alpha^2}{2(\beta_A\alpha+\gamma_A)^2}c\mathcal{H}(\alpha^{-1})dx\wedge dy.$$

The last step is to take the gauge couplings (8.4.6) and evaluate them on the horizon. We find that:

$$\mathcal{I}_{00|\Delta} = -\frac{2(\alpha\beta_0+\gamma_0)^2}{\alpha^2\mathcal{H}(\alpha^{-1})}\,, \qquad \mathcal{I}_{AB|\Delta} = -\delta_{AB}\frac{\alpha^2\mathcal{H}(\alpha^{-1})}{2(\alpha\beta_A+\gamma_A)^2}.$$

We are now in the position to calculate the conserved charges using the integrals

$$\mathcal{Q}_{0|\Delta} = -\frac{1}{8\pi}\int_{\mathbb{R}^2}\star F^0_\Delta \mathcal{I}_{00|\Delta}\,, \qquad \tilde{\mathcal{Q}}^A_\Delta = -\frac{1}{8\pi}\int_{\mathbb{R}^2}\star\tilde{F}_{A|\Delta}\mathcal{I}^{AA}_\Delta,$$

which we can calculate by substituting in the above results to find:

$$\begin{aligned}\mathcal{Q}_{0|\Delta} &= \left(-\frac{Q_0\alpha^2}{2(\alpha\beta_0+\gamma_0)^2}c\mathcal{H}(\alpha^{-1})\right)\cdot\left(\frac{2(\alpha\beta_0+\gamma_0)^2}{\alpha^2\mathcal{H}(\alpha^{-1})}\right) = -\frac{cQ_0}{8\pi},\\ \tilde{\mathcal{Q}}^A_\Delta &= \left(\frac{P^A\alpha^2}{2(\alpha\beta_A+\gamma_A)^2}c\mathcal{H}(\alpha^{-1})\right)\cdot\left(\frac{2(\alpha\beta_A+\gamma_A)^2}{\alpha^2\mathcal{H}(\alpha^{-1})}\right) = \frac{cP^A}{8\pi}.\end{aligned}$$

Gauge potential and chemical potential In the previous section, the gauge fields were found to be:

$$(A^0)_\eta = \frac{Q_0}{2\gamma_0(\beta_0+\gamma_0\zeta)}\,, \qquad (\tilde{A}_A)_\eta = -\frac{P^A}{2\gamma_A(\beta_A+\gamma_A\zeta)}.$$

Re-expressing these in terms of the new Eddington-Finkelstein coordinates, evaluated on the horizon, we find that:

$$\begin{aligned}A^0_\Delta &= \frac{Q_0}{2\gamma_0}\frac{\alpha}{\alpha\beta_0+\gamma_0}\left(du+\frac{\mathcal{H}}{\mathcal{W}}d\zeta\right),\\ \tilde{A}_{A|\Delta} &= -\frac{P^A}{2\gamma_A}\frac{\alpha}{\alpha\beta_A+\gamma_A}\left(du+\frac{\mathcal{H}}{\mathcal{W}}d\zeta\right).\end{aligned}$$

Contracting with the null vector ℓ we find the chemical potentials from the identities (this is justified in [199]):

$$\mu^0 = -\iota_\ell A^0\,, \qquad \tilde{\mu}_A = -\iota_\ell\tilde{A}_A.$$

Simplifying the gauge potential using that we can rearrange our integration constants to obtain

$$\gamma_a(\alpha\beta_a+\gamma_a) = K_a^2,$$

we can write down the chemical potentials

$$\mu^0 = -\frac{\alpha}{2Q_0}\,, \qquad \tilde{\mu}_A = \frac{\alpha}{2P^A}.$$

Now we can write down the second term on the RHS of (8.5.1) by combining this with the conserved charges from the previous expression to find:

$$\begin{aligned}\mu_a\delta\mathcal{Q}^a_\Delta &= \frac{\alpha}{16\pi}\left(\frac{1}{Q_0}\delta(cQ_0)+\frac{1}{P^1}\delta(cP^1)+\frac{1}{P^2}\delta(cP^2)+\frac{1}{P^3}\delta(cP^3)\right)\\ &= \frac{\alpha}{16\pi}\left(4\delta c + c\delta\log\left(Q_0P^1P^2P^3\right)\right)\\ &= \frac{1}{8\pi}\left(2\alpha\delta c+\frac{1}{2}\alpha c\delta\log\left(Q_0P^1P^2P^3\right)\right).\end{aligned}$$

First law of black hole mechanics Using these quantities we are now able to find an expression for the variation of the mass parameter from (8.5.1). Combining our results, we find that:

$$\begin{aligned}\frac{\kappa\delta a_\Delta}{8\pi} + \mu_a \delta \mathcal{Q}^a_\Delta &= \frac{1}{8\pi}\Big[-\alpha\delta c + c\delta\alpha - \frac{1}{2}\alpha c\delta \log(Q_0 P^1 P^2 P^3) \\ &\quad + 2\alpha\delta c + \frac{1}{2}\alpha c \delta \log\left(Q_0 P^1 P^2 P^3\right)\Big], \\ &= \frac{1}{8\pi}\left(\alpha\delta c + c\delta\alpha\right) = \delta\left(\frac{1}{8\pi}\alpha c\right), \\ &= \delta E_\Delta,\end{aligned}$$

and so:

$$E_\Delta = \frac{\alpha c}{8\pi} = \frac{\alpha}{16\pi\sqrt{\gamma_0\gamma_1\gamma_2\gamma_3}},$$

which we can see is identical to the calculations using the Euclidean action formalism (8.4.15).

Smarr relation Our last consistency check comes from the Smarr relation we derived in the Euclidean formalism. Taking each piece and summing together we calculate that:

$$\begin{aligned}\mu_a \mathcal{Q}^a_\Delta + \frac{\kappa a_\Delta}{8\pi} &= \frac{1}{8\pi}\Big[cQ_0 \cdot \frac{\alpha}{2Q_0} + cP^1 \cdot \frac{\alpha}{2P^1} + cP^2 \cdot \frac{\alpha}{2P^2} \\ &\quad + cP^3 \cdot \frac{\alpha}{2P^3} - c\mathcal{H}(\alpha^{-1}) \cdot \frac{\alpha}{2\mathcal{H}(\alpha^{-1})}\Big], \\ &= \frac{\alpha c}{16\pi}(1+1+1+1-1), \\ &= \frac{3}{16\pi}\alpha c = \frac{3E_\Delta}{2},\end{aligned}$$

which again is identical to the relation derived from the quantities obtained using the Euclidean action formalism (8.4.16).

8.5.2 *Planar solutions of Einstein-Maxwell theory*

As we showed in Section 6.6, the Einstein-Maxwell theory is a consistent truncation of the STU model where all four gauge fields are set equal, while the scalar fields are constant. This allowed us to map the solution of the STU model to that of Einstein-Maxwell model by fine tuning the integration constants. We remind the reader that the physical scalar fields (6.4.10) are given by

$$z^A = -i\left(\frac{\mathcal{H}_0\mathcal{H}_A^2}{\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3}\right)^{\frac{1}{2}},$$

and we see that they are everywhere constant under the restriction that $\mathcal{H}_0 = \mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3$. This ensured by setting integration constants equation to each other and consequently the four gauge fields take identical values, and the degrees of freedom contributing to the solution match those of Einstein-Maxwell theory. In the Einstein-Maxwell limit, and the line element becomes

$$ds^2 = -\frac{2(\beta+\gamma\zeta)^2}{(\alpha\zeta-1)}d\zeta^2 + \frac{(\alpha\zeta-1)}{2(\beta+\gamma\zeta)^2}d\eta^2 + \zeta^2\left(1+\frac{\beta}{\gamma\zeta}\right)^2(dx^2+dy^2),$$

and we refer back to Section 6.6 for the form of the integration constants. In Equation (6.6.4), we found how to express the parameters M, Q of the Einstein-Maxwell solution in terms of the constrained integration constants of the STU solution:

$$M = \frac{\alpha}{4\gamma^2}, \qquad Q^2 = \frac{\alpha\beta + \gamma}{2\gamma^3} = \frac{K^2}{2\gamma^4}.$$

where we are using that all the charges are considered as electric and so $P = 0 \Rightarrow e^2 = Q^2$.

We can now study each piece of the above calculation and see how it changes under this mapping, and see that we recover the results from the Euclidean action formalism in Section 8.3. In the Einstein-Maxwell limit each of the relevant pieces simplifies into the form

$$\kappa = -\frac{\alpha^3\gamma^2}{4K^4} = -\frac{4M^3}{Q^4}, \qquad a_\Delta = \frac{K^4}{\gamma^4\alpha^2} = \frac{Q^4}{4M^2},$$

$$\mathcal{Q}_{0|\Delta} = -\tilde{\mathcal{Q}}^A_\Delta = -\frac{K}{16\pi\gamma^2} = -\frac{\sqrt{2}Q}{16\pi},$$

$$\mu^0 = -\tilde{\mu}_A = -\frac{\alpha}{2K} = -\frac{2M}{\sqrt{2}Q},$$

where we have used that $c^{-1} = 2\gamma^2$.

Looking at the variation of the appropriate terms:

$$\kappa\delta a_\Delta = -\frac{4M^3}{q^4}\left(\frac{q^3}{M^2}\delta Q - \frac{Q^4}{2M^3}\delta M\right) = -4\cdot\left(\frac{M}{Q}\delta Q - \frac{1}{2}\delta M\right),$$

$$\mu_a\delta\mathcal{Q}^a = 4\cdot\left(\frac{2M}{\sqrt{2}Q}\frac{\sqrt{2}}{16\pi}\delta Q\right) = \frac{M}{2\pi q}\delta Q,$$

we obtain an expression for the variation of the energy by imposing the first law

$$\delta E_\Delta := \frac{\kappa\delta a_\Delta}{8\pi} + \mu_a\delta\mathcal{Q}^a = \frac{\delta M}{4\pi}.$$

When this is integrated up, we obtain an expression for the internal energy

$$E_\Delta = \frac{M}{4\pi},$$

which matches exactly with the internal energy derived from the free energy, via the Euclidean action formalism in Section 8.3 with the condition that $\omega = 1$.

8.6 DISCUSSION

In this chapter, we have developed a modified version of the Euclidean approach to horizon thermodynamics, which can be applied to a class of cosmological spacetimes whose causal structure is related to black hole solutions by exchanging the role of exterior and interior. In Section 8.1, we introduced the procedure of the triple Wick-rotation and gave how this transformation acts on the gravitational action coupled to an Abelian vector field. From the triple Wick-rotation, we obtain a finite Euclidean on-shell action derived from the exterior region of the cosmological solution, bounded between the horizon and the asymptotic distance. Following the standard formalism introduced in Section 3.6, we take this action and relate it to a grand thermodynamical potential, from which all thermodynamic quantities, the Smarr relation and the first law can be derived. In Section 8.2, we use the example of the de Sitter solution as an initial consistency check, where we first perform the standard Euclidean formalism in the static patch, and then compare this to the result of our modified triple Wick-rotation applied to the dynamic region of the spacetime. We then apply the formalism to the planar solutions of the Einstein-Maxwell theory and the STU model, for which the standard Euclidean formalism cannot be used due to the presence of a singularity within the static region. The triple Wick-rotation allows us to verify the first law for the solutions derived within this thesis. The final section of the chapter offers another consistency check with the quasi-local, isolated horizon formalism which defines the mass from imposing the first law for values computed at the horizon.

For planar solutions of Einstein-Maxwell theory and of the STU model, the formalism allows the definition of a positive mass-like quantity. Remarkably, the formalism works despite that the solution is not asymptotic to a vacuum solution. The results obtained using the triple Wick-rotation are consistent with those derived using the isolated horizon formalism, which is another check of its validity. This positive mass quantity further justifies our conclusions from the previous chapters that these solutions should not be interpreted as negative brane solutions.

More remarkably, the first law of black hole mechanics has been verified for our cosmological solutions despite the lack of physical motivation for the triple Wick-rotation from the perspective of the quantum mechanical path integral. Rather than interpreting the internal energy within the first law as a mass, our physical intuition seems to say that this should be a momentum-like quantity. In Sections 5.2.3 and Section 6.5.3, we saw that the asymptotic limit of the Komar energy was finite and argued that it should be considered as momentum-like as it was generated by a spacelike isometry. From the point of view of the Euclidean action formalism and the path integral, we can think of the rotation in a spacelike coordinate as being associated to a momentum operator, rather than a Hamiltonian. From the thermodynamic perspective, the first law is the statement of conservation of energy and is not associated to a mass or momentum, but rather the internal energy of some thermodynamic system. Thinking back to special relativity, where we can compute the total energy as the norm of the four-momentum, we can try and think about the differences between these solutions and more conventional ones. From the study of static black hole solutions such as the Schwarzschild solution or the Reissner-Nordström solution, the internal energy coming only from the mass seems natural, but for the cosmological models derived within this thesis, the total energy of the system is less clear. If the solution is dynamical, we would not expect all the energy to be stored within a rest mass. What we seem to be able to access with the asymptotic limit of the Komar integral, or from the internal energy from the thermodynamic perspective, is an energy parameter of the solution which can be thought of just as the mass parameter for the Schwarzschild solution which labels an equivalence class of solutions.

Ultimately, the question of understanding the thermodynamics defined using the triple Wick-rotation would be best realised through some underlying microscopic description, which one would hope that the non-standard brane configurations in Section 7.1 may shed light on. It is exciting that the successes of this formalism and this chapter seem to ask more questions than it answers, and is in essence an argument for how the geometric properties of Killing horizons keep reaching out into other areas of physics for deeper understanding.

For the thermodynamic formalism, we used the future inner horizons of the maximally extended solutions. This is natural because these horizons can be crossed by causal geodesics from the outside to the inside, which is the same situation as for black holes. For future inner horizons the surface gravity and temperature are both negative, when computed according to [82, 57, 58], and we have shown that the first law takes its standard form. It is natural to ask what happens if we use the past horizons instead, where causal geodesics cross from the interior to the exterior. This is analogous to asking about the thermodynamics of white holes. For our cosmological solutions the past horizons are past inner horizons. Since the surface gravity is still negative, the expression κdA and hence the 'first law of horizon mechanics' retains its standard form. However, the temperature is now positive and the 'first law of thermodynamics' takes a non-standard form where the sign of the temperature/entropy term is flipped, $TdS \rightarrow -TdS$. This depends, of course, on accepting both the definition of the surface gravity by Kodama-Hayward and the sign of the Hawking temperature being determined by the Parikh-Wilczek tunneling method. This is another option for further work.

Something remarked upon, but not followed in too much detail, is the boundedness of the partition function. We see that applying the triple Wick-rotation for the de Sitter solution, we can set the sign used in the rotation to ensure that $S_E > 0$. For the solutions derived from Einstein-Maxwell theory and the STU model, we saw that our 'boundary conditions' to ensure the matching of computed charges additionally set $S_E > 0$. However, for the de Sitter solution, considering the future inner horizon where $\beta < 0$, we saw that $\mathcal{Z}$ was instead bounded from above when applying the regular Wick-rotation. It could be that by instead considering $t \rightarrow +it$ that this changes, but we leave consideration of the boundedness of $\mathcal{Z}$ with the more general consideration of the stability of these solutions, as our primary focus has been the verification of the first law.

Finally, there is a result obtained in [40] which we now include into our discussion. Considering planar solutions of Einstein-Maxwell where the sign of the gauge coupling is flipped has introduced a curious relation or 'duality' between cosmological and black hole solutions. We will refer to this flipped sign theory as *Einstein-anti-Maxwell*. Taking planar symmetric solutions of the Einstein-anti-Maxwell theory produces black hole solutions asymptotic to the Taub vacuum solution [50]. The exterior of these solutions are static, and the future horizons have positive temperature. The singularity is spacelike, and we can verify the first law of the horizon using standard techniques. Remarkably, the Euclidean on-shell action for these solutions and the Einstein-Maxwell solutions in Section 8.3 are the same, hinting towards a duality.

This duality will be the topic of the closing remarks of this thesis, where we describe how flipping the sign of the Maxwell term exchanges the interior and exterior, spacelike and timelike singularities, and which relates solutions with negative temperature to solutions with negative energy. We think that a promising way to approach all these questions is the embedding into string theory, and a offers a great way to conclude this thesis, sharing some future work and some aspirations for how these results will branch into more areas of theoretical physics.

9

CONCLUSIONS AND OUTLOOK

In our conclusion of this thesis, we present an overview on the natural extensions of our research followed by a discussion of the work completed. In Section 9.1, we outline a particular project which has followed naturally from this thesis, whose work was started in our thermodynamic paper [40], and will be continued in an upcoming publication [200]. We finish the chapter with Section 9.2 with a summary of the work undertaken in the thesis, together with some general remarks on other possible research projects. We hope that this offers a non-technical overview, helping the reader review the motivations and insights that carried us throughout this body of work.

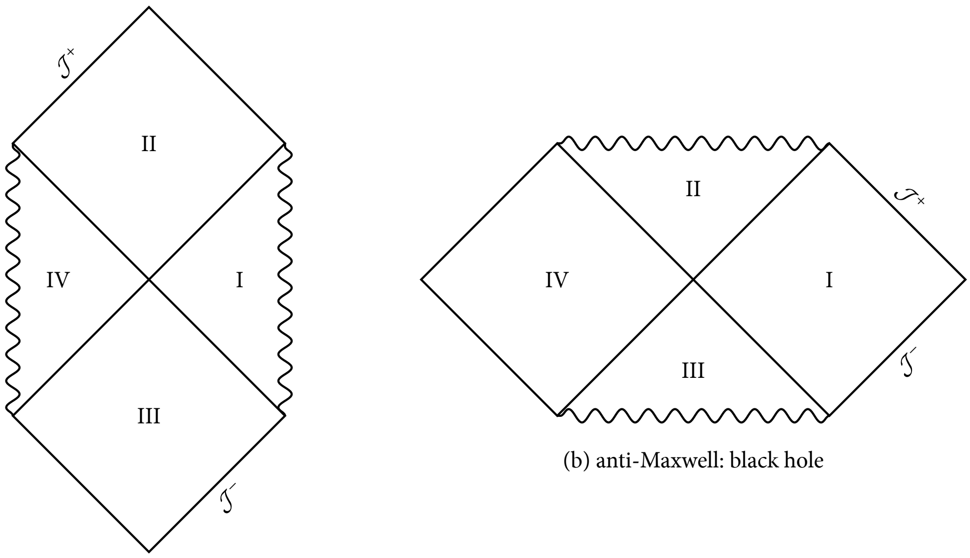
9.1 FURTHER WORK

We begin with a result first presented in [40] where we considered the thermodynamics of a four-dimensional planar symmetric black hole solution, derived from the so-called *Einstein-anti-Maxwell* theory, which differs from the standard Einstein-Maxwell theory through flipping the sign in the gauge field coupling:

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(-R + F^2\right) .$$

When compared to the cosmological solution of Chapter 5, we find the static and dynamic regions of the solution are exchanged such that the first law and other thermodynamic relations can be derived using a conventional Wick-rotation. We will show that this planar black hole solution can be viewed as the 'dual' partner to the planar cosmological solution of Einstein-Maxwell theory. This solution realises the same thermodynamical system as studied in Section 8.3, in the sense that both solutions have the same Euclidean action, and therefore the same grand potential and other thermodynamic potentials. More precisely, the range of some of thermodynamic quantities (temperature, energy) will turn out to be different, suggesting that the two solutions represent different 'phases' of the same system.

In general, fields with flipped sign kinetic terms are referred to as 'phantom' fields [201, 202]. Fields with negative kinetic energy have been discussed in the context of cosmology, because some data suggest that the current expansion of the universe is over-exponential, leading to a 'big-rip' cosmological singularity in finite time. While naively the negative kinetic energy renders the theory unstable, p-form gauge fields with inverted kinetic terms appear in the type II* string theories which are related to type II string theories by timelike T-duality transformations. In these cases, the theory is made consistent through the presence of massive string modes and the related higher gauge symmetries [203]. Gauge fields with flipped kinetic terms also appear in 'Fake-Supergravity' theories [202, 204]. We will return to the type II* theories shortly in the context of generalising the Einstein-anti-Maxwell theory into $\mathcal{N} = 2$ supergravity.



(a) Maxwell: cosmological

(b) anti-Maxwell: black hole

Figure 9.1: Comparison of the Penrose-Carter diagrams of cosmological and black hole solutions. Left side: Planar cosmological solution of Einstein-Maxwell theory. Right side: Planar black bole solution of Einstein-anti-Maxwell theory, same as for the (spherical) Schwarzschild solution of pure Einstein theory.

9.1.1 *Planar solutions of Einstein-anti-Maxwell theory*

We start with an action which simultaneously describes both theories,

$$S = \int d^4x \, \mathbf{e}_4 \left(-\frac{1}{2\kappa_4^2} R + \frac{\varepsilon}{4g^2} F^2 \right),$$

where $\varepsilon = \pm 1$ and $g^2 = 4\pi$ is the gauge coupling. We use vierbein notation, to avoid conflict between the spacetime metric and the gauge coupling. Introducing g is convenient because it allows us to relate both theories by analytic continuation of the coupling constant $g \to ig$ leaving ε fixed. Alternatively, we could relate them by analytic continuation of the gauge fields F, but we prefer to keep F real-valued in both theories. This said, we revert to our standard conventions where $G = 1$, $\kappa_4^2 = 8\pi$ and $g^2 = 4\pi$.

Solving Einstein's equations with a static, planar symmetric ansatz yields a line element of the form:[1]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\vec{X}^2, \quad f(r) = \frac{2C}{r} - \frac{\varepsilon Q^2}{r^2}. \tag{9.1.1}$$

[1] This calculation has an almost identical discussion to the one offered in Section 5.1, where ε appears in the R_{xx} component of the Ricci tensor when equated to the stress energy tensor component T_{xx}. Following this through, we find the charge $e^2 = Q^2$ appearing in the line element carries a factor of ε.

As explained in 5.1, for spherically symmetric solutions, the value of the integration constant C is set by comparing the result in a weak field limit to Newtonian results. In planar symmetric theories, this is not possible as there is no asymptotically flat region. Instead, we choose the sign of C by imposing the existence of a Killing horizon, which implements cosmic censorship by placing the singularity at $r = 0$ behind a horizon. With this in mind, we can write the line element as

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\vec{X}^2, \quad f(r) = \varepsilon\left(\frac{2M}{r} - \frac{Q^2}{r^2}\right), \tag{9.1.2}$$

where the integration constant M is always positive and $C = 2\varepsilon M$. In this form, it is easy to see that the sign of $f(r)$ is set by ε. Namely, when $\varepsilon = -1$, the asymptotic region is dynamic and the static patch for the solution is a finite region of the spacetime, bounded by

$$0 < r < \frac{Q^2}{2M}.$$

Conversely, for $\varepsilon = 1$ the static region is found for coordinate values of

$$r > \frac{Q^2}{2M},$$

such that for Einstein-anti-Maxwell, the asymptotic region of the spacetime is static. Asymptotically this metric is the vacuum Taub solution [50], with a line element

$$ds^2 = -\frac{1}{r}dt^2 + rdr^2 + r^2(dx^2 + dy^2).$$

As the exterior of the solution is static, we can obtain a smooth Euclidean geometry by performing a Wick-rotation of the timelike coordinate t. Unlike the Einstein-Maxwell solution, here we have the standard relation between quantum mechanics and statistical mechanics, which identifies the saddle point approximation of the gravitational functional integral with a thermodynamic potential.

Setting $\varepsilon = 1$, we consider the anti-Maxwell solution and calculate the Euclidean action and corresponding thermodynamic potentials. The line element in the static region is

$$ds^2 = -f(r)^2 dt^2 + \frac{dr^2}{f(r)} + r^2 d\vec{X}^2, \qquad f(r) = \frac{2M}{r} - \frac{Q^2}{r^2}, \quad r_h = \frac{Q^2}{2M}. \tag{9.1.3}$$

Chemical potential The gauge field is in the same form as the solution considering in Chapter 5, which we can write down as

$$F = \left(-\frac{Q}{r}\right) dt \wedge dr, \qquad A = \left(-\frac{Q}{r} + \frac{Q}{r_h}\right) dt, \tag{9.1.4}$$

and by taking the asymptotic limit of the gauge potential, we obtain the chemical potential

$$\mu = \lim_{r\to\infty} A_t = \frac{2M}{Q}\ .$$

Conserved charge The sign flip of the gauge field coupling leads to a sign flip in the conserved charge, which can be understood as relating each theory by performing $g \to -ig$ and using the Equation (4.2.16). Therefore, the conserved electric charge is found to be

$$\mathcal{Q} = -\frac{1}{4\pi}\int_{\partial\Sigma} \star F = -\frac{Q}{4\pi}.$$

Note that we have set $\omega = 1$ for simplicity.

Entropy & Temperature The Einstein-anti-Maxwell solution has an exterior region with a time-like Killing vector which allows the surface gravity to be calculated by the standard method from the Killing vector field

$$\kappa = \frac{4M^3}{Q^4} \quad \Rightarrow \quad \beta = \frac{\pi Q^4}{2M^3}.$$

We note that using the Kodama-Hayward expression (3.5.4), we obtain an identical result. The line element 9.1.3 is of the form of the Schwarzschild solution, which was discussed in Section 3.2. We then realise that the horizons of the solution are outer horizons, and specifically, the horizon separating the static exterior from a dynamic interior for a future-directed causal curve is a *future outer horizon*. In Figure 9.1 these are regions I and II in the conformal diagram on the right-hand side. This being a future outer horizon, we have $\kappa \propto T_H$ and the temperature is positive but has the same magnitude as in the Einstein-Maxwell solution.

The Bekenstein-Hawking area law gives

$$S_{BH} = \frac{r_h^2}{4} = \frac{Q^4}{16M^2}.$$

Euclidean action After the Wick-rotation $t \to -i\tau$ the Euclidean action is given by (3.6.2), with the addition of a sign flip for the gauge field contribution

$$\begin{aligned} S_E = &\frac{1}{16\pi}\int_M \sqrt{g} R d^4x \\ &- \frac{1}{8\pi}\int_{\partial M} \sqrt{|\gamma|} K d^3x + \frac{1}{8\pi}\int_{\partial M} F^{\mu\nu} A_\mu d\Sigma_\nu . \end{aligned}$$

The bulk term does not contribute, since $R = 0$. The GHY-term is

$$-\frac{1}{8\pi}\int_{\partial M} \sqrt{\gamma} K = -\frac{3M\beta}{8\pi}.$$

The gauge field contribution is identical to the one of the Einstein-Maxwell solution

$$\frac{1}{8\pi}\int_{\partial M} F^{\mu\nu} A_\mu d\Sigma_\nu = -\frac{2M\beta}{8\pi},$$

and when these two pieces are taken together, the Euclidean action is found to be

$$S_E = -\frac{5\beta M}{8\pi} ,$$

which is the same as for the Euclidean action of the triple Wick-rotated Einstein-Maxwell action (8.3.8).

As the charge is kept fixed, we associate the grand canonical thermodynamic partition function with the saddle-point approximation of the gravitational partition function:

$$\log \mathcal{Z} = -\mathcal{N} S_E = -\beta\Omega \quad \Rightarrow \quad \Omega(\beta, \mu) = -\frac{5\mathcal{N}\beta\mu^4}{(8\pi)^2} ,$$

where we have expressed Ω in terms of its natural thermodynamical variables. The normalisation constant $\mathcal{N}$ is fixed by imposing the relation

$$\frac{\partial\Omega}{\partial\mu} = -\mathcal{Q} .$$

Since

$$\frac{\partial\Omega}{\partial\mu} = -20\mathcal{N}\frac{\mu^3\beta}{(8\pi)^2}, \quad -\mathcal{Q} = \frac{Q}{4\pi} = \frac{\mu^3\beta}{(4\pi)^2}, \quad \mu = \left(-\frac{(4\pi)^2\mathcal{Q}}{\beta}\right)^{1/3},$$

this fixes $\mathcal{N} = -\frac{1}{5}$, so that the grand potential for the planar Einstein anti-Maxwell solution is

$$\Omega(\beta,\mu) = \frac{\beta\mu^4}{(8\pi)^2}.$$

The free energy is then obtained by a Legendre transformation:

$$F(\beta,\mathcal{Q}) = \Omega + \mu\mathcal{Q} = -3\left(\frac{\mathcal{Q}^4\pi^2}{4\beta}\right)^{\frac{1}{3}}.$$

By taking partial derivatives we can verify that the following two thermodynamic relations take their standard forms:

$$\frac{\partial F}{\partial \mathcal{Q}} = \left(-\frac{(4\pi)^2\mathcal{Q}}{\beta}\right)^{\frac{1}{3}} = \mu, \qquad S = \beta^2\frac{\partial F}{\partial\beta} = \left(\frac{\mathcal{Q}^4\beta^2\pi^2}{4}\right)^{\frac{1}{3}} = S_{BH}.$$

Therefore we are confident in defining the internal energy as

$$E = \frac{\partial(F\beta)}{\partial\beta} = -\frac{M}{4\pi} < 0.$$

We note that E is negative, which reflects that in the Einstein-anti Maxwell theory the vector field has negative kinetic energy. By expressing E in terms of its natural thermodynamic variables, we obtain the following equation of state:

$$E(S,\mathcal{Q}) = -\frac{\pi\mathcal{Q}^2}{\sqrt{S}}.$$

Finally, we compute the total differential of the internal energy,

$$dE = \frac{\partial E}{\partial S}dS + \frac{\partial E}{\partial \mathcal{Q}}\mathcal{Q} = \beta^{-1}dS + \mu d\mathcal{Q},$$

and find that the first law is satisfied. It is interesting to note that the Euclidean action and grand potential, as well as other thermodynamic relations, are the same as for the planar solutions of Einstein-Maxwell theory, except for the range of some of the parameters. For the Einstein-Maxwell solution, the temperature is negative and internal energy is positive, while for the Einstein-anti Maxwell solution, the temperature is positive and internal energy is negative. While both solutions exhibit features indicating instabilities (negative temperature and negative internal energy, respectively), they obey all formal relations of thermodynamics and have the same underlying Euclidean action.

The matching of partition functions for distinct theories is usually a hint towards a deeper underlying duality. In the following section, we comment on a few ideas we have been developing to study this further.

9.1.2 *Project outlook*

The realisation of a duality between two distinct Lorentzian solutions, via the equivalence of their Euclidean actions, has an interesting relation to recent results in $\mathcal{N} = 2$ supersymmetry. In [205], four-dimensional $\mathcal{N} = 2$ supersymmetry algebras have been classified for all possible signatures (t, s), where t is the number to timelike and s the number of spacelike dimensions. A similar discussion was given for the five-dimensional case in [206]. It was found that while the $\mathcal{N} = 2$ supersymmetry algebra is unique in Euclidean signature $(0, 4)$, there are two inequivalent algebras in Minkowski signature $(1, 3)$, namely the standard algebra with compact R-symmetry group $\mathrm{U}(2)$ and a twisted version with R-symmetry group $\mathrm{U}(1, 1)$. The corresponding vector multiplet theories are distinguished by relative signs between various terms in the Lagrangian, including a relative sign between Maxwell and scalar terms. Already in [207] it has been shown that a non-standard $\mathcal{N} = 2$ supergravity theory coupled to vector multiplets with inverted signs for all Maxwell-like terms results from the dimensional reduction of five-dimensional supergravity coupled to vector multiplets with signature $(2, 3)$. This theory reduces to Einstein-anti-Maxwell theory upon truncating out the matter fields and the gravitini. We remark that while vector multiplet theories in signature $(0, 4)$ can likewise be obtained in two ways from five dimensions, the resulting relative signs can be removed by a suitable field redefinition, since the underlying Euclidean supersymmetry algebra is unique up to isomorphism [205]. The situation is summarised in Figure 9.2.

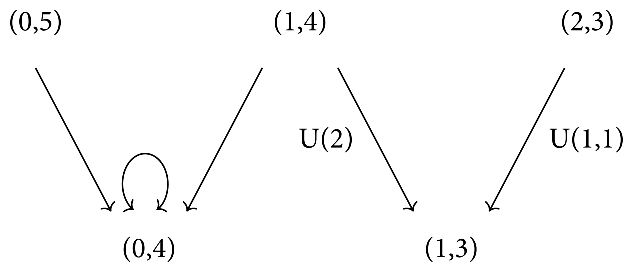


Figure 9.2: This diagram summarises the relations between five-dimensional and four-dimensional vector multiplet theories with spacetime signature (t, s), that is, t timelike and s spacelike dimensions [205]. Two four-dimensional theories in a given signature differ by relative signs between terms in their Lagrangians. In Euclidean signature, these signs can be changed by a suitable field redefinition, and the Euclidean theory is unique. In Minkowski signature there are two non-isomorphic supersymmetry algebras which are distinguished by their R-symmetry groups $\mathrm{U}(2)$ and $\mathrm{U}(1, 1)$, respectively. Therefore the corresponding vector multiplets theories cannot be related by a field redefinition.

The relative sign flips between the Minkowski signature theories are of the same type as those between type II and type II* string theory, which are related to each other by timelike T-duality [203]. By performing a series of timelike and spacelike T-dualities, one can map between four II string theories:

$$\mathrm{IIA} \underset{S_T}{\longleftrightarrow} \mathrm{IIB}^* \underset{S_R}{\longleftrightarrow} \mathrm{IIA}^* \underset{S_T}{\longleftrightarrow} \mathrm{IIB} \underset{S_R}{\longleftrightarrow} \mathrm{IIA}.$$

where we denote a spacelike (timelike) T-duality by S_R (S_T). Moreover, $\mathcal{N} = 2$ supergravity with vector (and hyper) multiplets arises by compactification of type II string theory on Calabi-Yau threefolds. In a future publication, we will present the details of the embedding of the twisted $\mathcal{N} = 2$ supergravity theory into type II* theory, where we find that beginning with one of the four type II theories, we obtain four $\mathcal{N} = 2$ Lagrangians when reducing over the *same* Calabi-Yau manifold:

$$\begin{array}{lcl} \text{IIA} & \longrightarrow & \mathcal{N} = 2\ (N, M) \\ \text{IIB} & \longrightarrow & \mathcal{N} = 2\ (M, N) \\ \text{IIA}^* & \longrightarrow & \mathcal{N} = 2 \text{ twisted } (N, M) \\ \text{IIB}^* & \longrightarrow & \mathcal{N} = 2 \text{ twisted } (M, N) \end{array}$$

Where $(N, M) = (n_V, n_H)$ count the number of vector/hyper multiplets respectively. The values of N, M are set by the specific choice of the Calabi-Yau. Finally, once in four dimensions, we turn to the c-map. Performing a spacelike c-map and then uplifting back to four dimensions, one obtains the well-known result of being able to interchange the number of vector and hyper multiplets. This is equivalent to reducing either IIA or IIB string theory over the same Calabi-Yau manifold. We extend this discussion, showing that by performing successive timelike and spacelike c-maps, one can move between all four (twisted) $\mathcal{N} = 2$ supergravity theories. A summary of these mappings are presented as a cube in Figure 9.3.

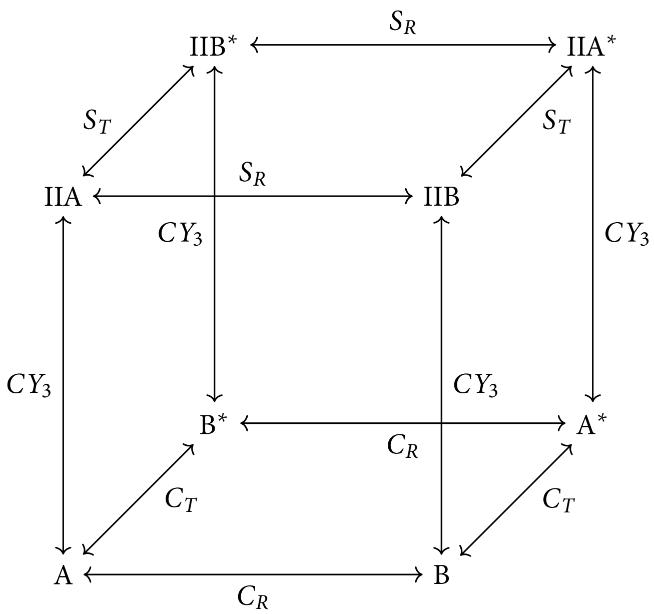


Figure 9.3: On the base of the cube, we have 4 different $\mathcal{N} = 2$ theories. Spacelike and timelike c-maps are denoted by C_R, C_T. Theories A and B have $(n_V, n_H) = (N, M), (M, N)$ respectively. The theories A^*, B^* have the same structure, but they have a non-compact R-symmetry. The twisted theories have Lagrangians with positive-definite gauge couplings. The top of the cube is the T-duality mapping between the type II and type II* theories. The top and base of the cube are mapped to each other by dimensional reduction over a Calabi-Yau threefold.

The research then continues looking at planar symmetric solutions of the twisted supergravity theories and we find a solution to the so-called *anti-STU model* which is a theory of twisted $\mathcal{N} = 2$ supergravity coupled to three vector multiplets where the gauge coupling has a flipped sign. We show that in much the same way we can recover the Einstein-Maxwell theory from the STU model, one can recover the Einstein-anti-Maxwell theory from the anti-STU model. We expect that continuing this research will shed more light onto the thermodynamics of planar solutions and their microscopic interpretation in terms of string theory. We remark that when combining timelike and spacelike T-duality with S-duality, it is possible to change spacetime signature in type II string theory, which provides a second way besides analytical continuation, of relating theories in Euclidean and in Minkowski signature [195]. Solutions in neutral and in general signature have recently found attention in the literature, see for example [208, 209, 210].

However, we must be careful with the distinction between the symmetry between Lagrangians and that of solutions. From a string-theory perspective, we can relate Lagrangians with flipped signs through performing a timelike, then spacelike T-duality. From the four-dimensional perspective, we can map theories to one another by performing consecutive spacelike and timelike c-maps. However, when we solve the equations of motion for these planar symmetric theories, there is an additional restriction made to ensure the presence of the Killing horizon. For the (anti)-Einstein-Maxwell solutions, we saw this as the setting of the sign of C. For the STU model in 6.3, this was achieved through a set of regularisation conditions. Neither of these restrictions can be captured by the above duality transformations, but rather must come from some other perspective.

Let us take the simpler solutions of the (anti)-Einstein-Maxwell solutions. If we imagine that the sign of C is fixed and we perform a duality transformation, we would expect to have a mapping between a solution containing a Killing horizon to a solution containing a naked singularity (and vice-versa). We would only expect to map Killing horizons into each other if we also flipped the sign of C (which we could equally well understand from flipping the sign of the mass M). This behaviour is expected from another perspective from the history of literature of black holes and T-duality transformations (usually studied in the low-energy limit with the Buscher rules). Although the Killing horizon is preserved for spacelike symmetries [211], this is not the case of timelike dualities [212]. Further discussion on this is given in [213], where the authors comment on the mapping between naked singularities and black hole solutions while formulating a manifestly T-dual invariant discussion of the first law of thermodynamics. The duality between naked singularities and black holes is also discussed in [214, 215]. Another instance of mapping between solutions with horizons and naked singularities is discussed in [194] when studying negative brane solutions in string theory.[2]

One idea we wish to follow in more detail is considering planar solutions to each theory not as four distinct solutions where C is constant, but rather as pairs of theories with C permitting both signs: one with a Killing horizon and one with a naked singularity. The conjecture we make is that flipping the sign of C maps solutions within a theory between spacetimes containing a

[2] We became aware of this paper when first studying these cosmological solutions in [39], when the negative-definite mass-parameter of the four-dimensional solution in the static patch seemed to hint that our solutions were negative branes, although this was not followed up after we worked with the extremal limit and studied our solutions as intersecting brane solutions.

Killing horizon or a naked singularity and that T-duality maps solutions with or without a Killing horizon between theories. We also notice that 'flipping the sign' of C could also be understood as the continuation through the singularity. This would allow C to remain fixed, and for the Killing horizon to appear (disappear) by considering $r < 0$. This continuation to glue together regions at singularities has been considered for the Schwarzschild solution [216] where the authors consider the different interpretations of flipping the sign of the gravitational coupling, or continuations through the singularity.

Both from the point of view of thermodynamics and from the one of T-duality, certain spacetime geometries naturally form pairs which share the same underlying Euclidean description. If one takes the Euclidean functional integral as fundamental and allows both the spacetime and the field space to be complex-valued, this will correspond to pairs of complex saddle points of the functional integral which represent dual Minkowksi signature solutions. At this point it is not clear whether the two dual solutions are actually 'the same', that is, gauge equivalent under a chain of string duality transformations, or just have 'the same thermodynamics.' In either case, one could also look for relations to solutions in neutral signature. It will be interesting to further investigate these intriguing relations between geometry, thermodynamics and dualities.

9.2 SUMMARY

In this thesis, we have presented a discussion on four-dimensional planar symmetric solutions and their thermodynamics. Using the c-map and the real formulation of special geometry, we derived a class of non-extremal cosmological solutions of $\mathcal{N} = 2$ supergravity containing a Killing horizon. Our analysis of the extremal limit of these solutions led to the discussion of our four-dimensional solutions embedded into string/M-theory and the recovery of supersymmetric solutions in six dimensions. Validating the first law of thermodynamics led to the development of a novel formulation of the Euclidean action formalism. In this chapter, we briefly discussed future work inspired by the implied duality found from the matching of Euclidean partition functions for distinct theories related by the flipping of the sign for the gauge coupling in the Lagrangians.

Our discussion began considering planar symmetric solutions of Einstein-Maxwell theory in Chapter 5. Following the example of the Reissner-Nordström solution (Section 3.3), we derived the line element by solving Einstein's equations, imposing that the spacetime was planar symmetric and static. To ensure the presence of a Killing horizon, we set the sign of the integration constant: $C = -2M < 0$, and noticed that for $C \geq 0$ the solution contained a naked singularity, violating the cosmic censorship conjecture. Studying the global structure of the planar symmetric solution, we found that the transverse coordinate r had finite extension from the horizon at $r = r_h$ to the timelike curvature singularity at $r = 0$. We understood this static patch of spacetime in analogy to the region behind the Cauchy horizon for the Reissner-Nordström solution. Extending the geometry of the solution by introducing Eddington-Finkelstein coordinates, we then analytically continued through the horizon into a new region of the spacetime in which our transverse coordinate r became timelike, and the coordinate t became spacelike. For $r > r_h$, the spacetime depended only on a timelike coordinate and taking the limit of $r \to \infty$, the asymptotic geometry was understood to be that Kasner type-D vacuum solution. This time-dependent exterior geometry is what led to the naming of these *cosmological* solutions. Computing the surface gravity, we identified $M \to 0$ as the extremal limit of the solution. Applying this limit, we found that the horizon was 'pushed out' to infinity; the dynamic patch of the region shrank to zero size, and the resulting spacetime was everywhere static, with a naked singularity.

We then considered the static patch and studied the geodesics and conserved charges of the solution. It was found that all causal geodesics were repelled by the singularity, with the exception of the null, transverse geodesics which necessarily would reach the singularity in finite affine parameter. We understood this by interpreting the geodesic equation as the equation of motion of a massive particle coupled to a positive (repulsive) potential. Studying the trajectories, the potential ensured the existence of a classical turning point for timelike and non-transverse null geodesics. As a result, these geodesics would never reach the singularity, but would instead be repelled and continue through a Killing horizon. For null, transverse geodesics, the effective potential was zero and hence the geodesic experiences no repulsion. An alternative point of view was then offered, studying the proper acceleration for a static, massive observer. We found that the acceleration for a massive particle following orbits of the timelike Killing vector field was negative, indicating the experience of a repulsive force. Using Gauss' law, we computed the conserved electric charge (density) of the solutions, and we offered a discussion on computing a mass-like parameter for the solution. As the solution was not asymptotically flat, and the asymptotic region not stationary, we considered position-dependent mass-like quantities within the static region using both the Komar energy and quasi-local energies. However, we found no natural normalisation for our calculations, and so we could at most comment on the overall sign, which we found was everywhere negative for all treatments. We also noted that for the Komar energy and the Katz-Lynden-Bell-Israel quasi-local energy, we could take the asymptotic limit and find a finite quantity. Usually taking this limit for the Komar energy is associated with the Komar mass under the assumption that the solution is asymptotically flat and stationary. As the planar symmetric solution had neither of these properties, we could not draw the same conclusion. Instead, as asymptotically the conserved charge is computed from a spacelike isometry, the finite quantity could be thought of as a conserved momentum, generated by translations at asymptotic infinity. Later in the thesis, this conserved charge would appear again as the internal energy of the solution from a thermodynamic perspective.

We concluded our discussion considering the global structure of cosmological solutions, allowing the line element to be generalised to not only cover the planar solutions of Einstein-Maxwell theory but also for the planar solutions of the STU model and the de Sitter solution when written in static coordinates. In this general form, we were able to write down the Kruskal coordinates for the solutions and from them, the corresponding Penrose-Carter diagrams. Following a discussion first offered in [40], we then carefully studied each of the four regions and showed that following our conventions, the spacetime region IV could be understood as having the conventional flow of time (see Figure 5.4b). Using the expansions of null geodesic congruences, we were then able to classify the horizons. We found that the horizon between regions III and IV was a *future inner* horizon and the horizon between regions II and IV was a *past in-*

ner horizon (see Figure 5.6). The classification of the horizons allowed us to set the sign of the temperature from the Kodama-Hayward surface gravity, which we would need when applying the Euclidean action formalism. To match the standard case for the Schwarzschild solution, we would consider the future inner horizon, which is crossed by future-directed, causal geodesics travelling from the external to the internal regions of the solution. For future inner horizons, the Hawking temperature is proportional to the surface gravity, which is negative for the solutions we discussed within this section.

We then turned to study the planar symmetric solutions of the STU model. This research was the initial starting point for the work presented in this thesis, and the previous discussion of the solutions of Einstein-Maxwell theory serves as the minimal example, which can be recovered from the STU model through the restriction that the scalar fields are constant throughout the spacetime. This relationship between the more complex solutions of $\mathcal{N} = 2$ supergravity coupled to vector multiplets and the familiar Einstein-Maxwell theory allowed us to build upon already established ideas, helping to give space for the more involved discussions particular to the solution generating method employed within this chapter.

Our work was motivated through wanting to consistently generalise the Nernst brane solutions of [28]. The Nernst branes are non-extremal solutions of $\mathcal{N} = 2$ gauged supergravity coupled to an arbitrary number of vector multiplets with a single electric charge. Their defining characteristic was that in the extremal limit, the area (density) of the Killing horizon vanished and so could be understood as black hole solutions obeying the strict third law of thermodynamics. Our goal was to look at increasing the number of charges, and in [39], we presented a discussion of dyonic solutions with one electric charge and between one and three magnetic charges. With the interesting thermodynamic properties of the Nernst brane, it is natural to then to think about the thermodynamics of the multi-charged solutions. Analysing the three multi-charges solutions, we found that the solutions no-longer obeyed the Nernst law. However, for the three and four charged solutions, we instead had the surprise of deriving cosmological solutions, despite having imposed staticity while solving the equations of motion. In this thesis, we focused on the four charge solution which was a solution of the well known STU model, and this static ansatz producing a cosmological solution could be understood in analogy to the Einstein-Maxwell solution of the previous chapter.

To obtain the four-dimensional solutions, we built upon a growing history of work which employs the c-map to derive non-extremal solutions to $\mathcal{N} = 2$ supergravity. Starting with the STU model, the solution is assumed to be static, and subsequently dimensionally reduced over a time-like circle to produce a three-dimensional Euclidean theory. In three dimensions, the Hodge-star operator dualises vectors into scalars, and we can consistently write the three-dimensional field content as $2n_V + 2 = 8$ real scalar fields. The real formulation of special geometry allows us to write our theory in a symplectically covariant way, and the integration of the equations of motion can be done exactly after making suitable restrictions on the field content. For our solutions, this was achieved by making the purely imaginary field content restriction, and through integration of the field equations, we wrote down a Euclidean instanton solution in three dimensions. Using previously established relations, we could then uplift the solution back into four dimensions, where the resulting spacetime is necessarily static. Additionally, for our discussion, planar symmetry was imposed for the three-dimensional solution; however, this procedure would work equally well for spherical symmetry and could also be extended to consider hyperbolic symmetry.

The four-dimensional solutions were then regularised by ensuring that the presence of a Killing horizon with finite area density and that the physical scalar fields when evaluated on the horizon were not divergent. Regularisation placed restrictions on the integration constants of the theory, and we noted that this reduction of the number of remaining integration constants had similar behaviour as the restrictions from considering the attractor mechanism for non-extremal solutions. We noted that this would be an interesting avenue for further work through its possible relation to the contemporary work of the hot-attractors [178, 180] . Given the four-dimensional solution, we then studied the null geodesics of the spacetime and identified the asymptotic geometry by identifying the region for which a null geodesic would take infinite affine parameter to reach. Using curvature scalars, we found the location of four curvature singularities. However, as we must terminate the transverse coordinate at physical singularities in our manifold, we only considered the location of the first one, which could be picked without loss of generality through ordering the magnitudes of the integration constants. We noticed that the canonical coordinates used for these solutions were not appropriate, and by making a change of coordinates, it became apparent that the transverse coordinate had finite extension within the static region between the Killing horizon and the curvature singularity. Analytic continuation through the horizon produces a new region of the spacetime which is dynamic, and we understood the planar symmetric solution of the STU model as cosmological, in direct analogy to our previous analysis of the planar symmetric solutions of Einstein-Maxwell.

Following from this similarity, we then showed explicitly that the global structure of these solutions were identical to the planar symmetric solutions of Einstein-Maxwell theory. Asymptotically, the spacetime geometry was that of the Kasner type-D vacuum solution, and taking the extremal limit had the same effect of 'pushing out' the horizon to infinity, leaving behind a static spacetime containing a naked singularity with the area density of the horizon diverging. From the generalised discussion from the previous chapter, we knew that the horizons would be inner horizons and that the horizon crossed by future-directed causal geodesics passing from the exterior to the interior would cross a future inner horizon. Again, this would be important for the later thermodynamics, where the Hawking temperature was proportional to the Kodama-Hayward surface gravity which we found was negative. We studied the geodesic motion within the static region and found that the singularity was repulsive using identical considerations as from the previous chapter. We showed explicitly that for all geodesics (with the exception of null, transverse ones) there existed a classical turning point and hence geodesics could never reach the singularity. This was signalled both by the positive effective potential appearing in the geodesic equations and the negative proper acceleration for static, massive particles. The mass-like parameter has the same story as the Einstein-Maxwell discussion, where only a position-dependent quantity can be sensibly computed from either the Komar or quasi-local formalisms. Without extra information to set a normalisation, we could only conclude that the energy was negative-definite within the static region. Allowing ourselves to play with certain parameters, we again found a constant, conserved quantity associated to taking the asymptotic limit for the Komar energy and the Katz-Lynden-Bell-Israel energy. These quantities, like in the planar solution of Einstein-Maxwell, would appear again when computing the thermodynamic internal energy.

We concluded the chapter by looking at the steps needed to restrain the integration constants such that the Einstein-Maxwell theory was recovered from the STU model. We found that by ensuring all the harmonic functions $\mathcal{H}_a$ were equal, the scalar fields of the theory were constant throughout the spacetime and the resulting solution could be mapped to the solutions of the previous chapter. We noticed that applying this coordinate transformation gave us a way to express the parameter M of the Einstein-Maxwell solution to the integration constants α, γ of the simplified solution of the STU model.

The development of the cosmological solutions of the STU model from the Nersnt solutions immediately offered interesting questions. For the four-dimensional solutions of [28], there were divergences of the scalar fields in the asymptotic limit and infinite tidal forces as one approached the horizon. From a holographic perspective, these were understood as UV and IR divergences of the solution respectively. The asymptotic curvature singularities of the solution were regularised by interpreting the solutions in five dimensions, where the electric charge was understood as a momentum wave in five dimensions [34]. For the Nernst solutions, the defining behaviour was the vanishing area density of the Killing horizon in the extremal limit, but for our solution the extremal limit produced a naked singularity and the rather than vanishing, the entropy diverged. We saw this same behaviour for the three charge solution in [39]. However, for the two charge solution, the entropy was finite in the extremal limit.

To try and understand the naked singularity appearing in the extremal limit, we chose to follow the philosophy of the dimensional uplifting of the Nernst brane and in Chapter 7, we took the four-dimensional solutions and embedded them into five, six, ten and eleven dimensions. We showed that from the ten-dimensional embedding, the four-dimensional extremal limit produced a line element reminiscent of the D1-D5 intersection with a Kaluza-Klein monopole and a PP-wave along the intersection direction. However, unlike the conventional intersection reviewed in our background (Section 4.5.4) which could be understood as a black hole in four dimensions, we found that our initial planar ansatz in four dimensions has the effect of smearing the brane intersection along two additional directions; the planar symmetry delocalised the intersecting brane description. The smearing can be seen at the level of the harmonic functions, which are linear in the transverse coordinate. The story is very similar in eleven dimensions, where the four-dimensional extremal limit produces a line element for the triple intersection of M5 branes with a PP-wave superimposed along the common direction. Again the difference between our solution and the canonical example is the smearing along two additional directions and signalled in the form of the harmonic functions, which are linear in the transverse coordinate.

Equipped with the higher dimensional representations of our solutions, we were able to comment on a supersymmetric solution in six dimensions. Following the work of [183, 184], we were able to interpret the question of the existence of a supersymmetric solution as to whether we could put sufficient restrictions on the integration constants to obtain a four-dimensional hyper-Kähler base space and satisfy specific equations for the three-form field strength. We found that this was possible with a fine-tuning of the charges once the four-dimensional extremal limit had been taken. The resulting geometry had vanishing Ricci scalar, and was supersymmetric. This result was surprising, as at no point in our derivation of the four-dimensional solutions did we require supersymmetry (for example through imposing the Killing spinor equations). Within our background discussion, we mentioned that BPS solutions required taking the extremal limit, but that the extremal limit was not always sufficient to obtain BPS solutions. Here in six-dimensions, we see this again where there is the additional requirement of the balancing of the charges to recover BPS solutions.

In our consideration of our uplifting, we followed the dimensional oxidation from [131], which uplifted over T^n, embedding our lower dimensional solution into a consistent truncation of higher-dimensional supergravity. An immediate question one can ask is how would this discussion change if we uplifted over other manifolds (or their generalisations). In [168], a similar class of cosmological solution were discussed, and they comment that following [171], one could expect solutions of this form to appear from a string theory perspective after the dimensional reduction over an orientifold. In [182], the uplift of a patch of our cosmological solution is given over the orientifold $K3 \times T^2/\mathbb{Z}_2$. However, due to their method of solution generation, the resulting line elements are very complicated. We chose not to follow this line of research in our original paper. However, given that our solutions have the benefit of a fairly simple coordinate description covering the maximal extension of the spacetime, it would be interesting to apply their uplift procedure to our work. Finally, one could repeat the analysis we presented within this thesis for the three-charge solution. However, this would require modifying the uplift by instead considering the Scherk-Schwarz reduction, where the dependence on the reduction dimension would then produce the gauging parameters in the reduced theory.

Although the Nernst behaviour was not maintained while generalising to a higher number of charges, it is natural to continue studying other thermodynamic properties. The focus of our discussion became whether the first law of black hole mechanics could have a thermodynamic interpretation for the cosmological horizons of the planar solutions of the Einstein-Maxwell theory and the STU model. Building from this, it would also be interesting to look at the stability of these solutions, both from a gravitational perspective and through computing the specific heat capacities. Neither of these topics were considered and would be interesting topics for further work.

The core of how to interpret the validity of the first law for our solutions was how to determine an appropriate internal energy. Our previous discussions of the position-dependent mass-like quantities suffered from having no natural normalisation. As the first law is a differential relationship, it was vital to have a well-motivated overall scaling of the energy. The most promising values were the asymptotic values of the energies, but for both the Komar and quasi-local quantities, taking the asymptotic limit pushed us into a region of spacetime which was no longer stationary and the conserved quantity could not be trusted as a mass-like parameter.

We decided to use the Euclidean action formalism, a common method for solutions which are not asymptotically flat (although, usually, the exterior regions of most discussions are still stationary).[3] The cosmological nature of our solutions meant that the usual Wick-rotation of the timelike coordinate would not be suitable. In the static region, the Wick-rotated geometry would include the curvature singularity, and in computing the action, the spacetime geometry must be smooth. For the dynamic region, the Wick-rotation of the timelike coordinate would produce a complex line element. One method, usually employed for rotating solutions where there are timelike/spacelike cross-terms, is to additionally complexify certain integration constants. However, following this reasoning, we would find that our thermodynamic quantities would become complex. Rather than working with complex parameters, we noticed that through Wick-rotating all spacelike coordinates in the dynamic region of the spacetime, we could obtain a smooth Euclidean geometry. We noted that although this method produced a perfectly respectable Euclidean action which we could then use the saddle point-approximation to obtain a gravitational partition function, the original quantum mechanical motivation to interpret this as a thermodynamic partition function no longer held. However, proceeding formally and interpreting the partition function thermodynamically, we were able to continue and verify the first law for our solutions. Understanding how this works, and further researching the relationship between the gravitational and thermodynamic partition functions is another broad and interesting direction for further work.

However, before being able to continue from the partition function to the first law, there was an additional complication on how to properly normalise the Euclidean action. In the standard treatment of the Euclidean action formalism, there is usually some divergent contribution from evaluating the boundary terms in the asymptotic limit. An additional background contribution is included in the action, built from the boundary geometry. For asymptotically flat solutions, this is usually the contribution from the Minkowski background. For other solutions, such as the de Sitter solution we discussed in Section 8.2, the boundary geometry is used to construct a counter term. This renormalisation of the Euclidean action naturally determines the background and overall scaling of the action. For our classes of planar symmetric solutions, the asymptotic contribution of the boundary term is finite, and so there is no natural background contribution to include. This problem was common throughout our analysis of the planar symmetric solutions, where the lack of a symmetric space in the asymptotic limit produced various questions in the normalisation of our thermodynamic quantities. For the temperature, we found no natural normalisation for the norm Killing vector field, leaving an overall scaling left unfixed for the surface gravity. For the entropy and electric charge (density), reparameterisation of the planar coordinates leads again to an ambiguity in the total overall normalisation.

To continue, we realised we would need some other condition to allow for an internal consistency of our model. It is the diverse utility of the partition function that allowed us to establish our missing 'boundary' term. From Gauss' law, we computed the conserved electric charge (density), and from the gauge potential, we could compute the chemical potential through taking the asymptotic limit of the t-component (note that this limit was taken after the potential is properly gauge fixed). We could then take our Euclidean action and relate this to the grand canonical

[3] A common use for the Euclidean action formalism is to derive the partition function for anti-de Sitter solutions in conjunction with the AdS/CFT correspondence.

potential — the natural thermodynamic partition function derived from our theory. From the thermodynamic perspective, we know that varying this potential with respect to the chemical potential is equal to the negative of the thermodynamic charge. Asserting that the charge computed from Gauss' law matched exactly to the thermodynamic charge from the partition function was enough to set the overall numerical normalisation of the partition function.

After setting the normalisation, we could then make a Legendre transform to the grand canonical potential to obtain the free energy. Varying the free energy, we can compute the thermodynamic chemical potential, entropy and internal energy. By construction, the chemical potential matches that from the gauge field, but we additionally find that the entropy matches the Bekenstein-Hawking area law. The internal energy is the focus of this result, and unlike the other parameters, we have no natural parameter to compare with for consistency. However, we do find that this thermodynamic internal energy matches the asymptotic contribution from the Komar energy and the Katz-Lynden-Bell-Israel quasi-local energy. The matching of these quantities is curious and deserves further attention. One thing we notice is that all of these conserved quantities have some natural interpretation as momentum-like parameters. It would be very interesting to follow this and understand how this takes the place of the usual mass which we see in standard treatments of the first law of black hole solutions.

Equipped with the internal energy, it was then a simple job to write down the equation of state and to vary it. We find that the standard form of the first law holds, as well as the integrated Smarr's law. This discussion is the same for both the planar solutions of both the Einstein-Maxwell and STU model. We concluded the chapter by then considering the alternative treatment of the isolated horizon formalism. This method begins by assuming the first law and then using parameters defined on the horizon to obtain a mass after integration. This procedure is then not suitable for verifying the first law, but we do find that through following the method, we find a mass parameter consistent with the internal energy of the Euclidean action formalism. A general and open-ended discussion is the application of the triple Wick-rotation for other cosmological solutions. An immediate option is the three-charge solution of [39], which would also need careful consideration of how to account for the gauging parameters. We note that although we are biased towards this option, the triple Wick-rotation would be a suitable method for studying the thermodynamics of any four-dimensional solution where the exterior region of the spacetime geometry depended only on the timelike coordinate.

To conclude, we have presented a class of non-extremal, cosmological solutions of $\mathcal{N} = 2$ supergravity through using the real formulation of the c-map and the uplift of the resulting three-dimensional Euclidean instanton solutions. We have understood the singular extremal solutions as modified intersecting brane configurations in ten and eleven dimensions and recovered supersymmetric solutions in six dimensions. By modifying the Euclidean action formalism, we were then able to verify the first law of thermodynamics our solutions and stumbled upon an apparent duality where the Euclidean partition functions of distinct Lorentzian theories are the same. We hope that this insight leads to a discussion of the thermodynamics of theories related by T-duality and a discussion of black hole/cosmological solution pairs within the framework of double field theory or generalised geometry.

Part III

APPENDICES

A

NOTATION AND CONVENTIONS

A.1 NOTATION

Throughout this thesis we use that $c = \hbar = 1$ except when these constants are reintroduced to illustrate a point. The gravitational coupling

$$\kappa_4^2 = 8\pi G,$$

will be treated differently in various chapters to closely match the references of the discussion. As a rule of thumb, when considering relativistic problems, we take $G = 1$ such that $\kappa_4^2 = 8\pi$, and for discussions of supergravity we take that $\kappa_4^2 = 1$.

In all cases, we use the Einstein summation convention for repeated indices

$$X^\mu X_\mu = \sum_\mu X^\mu X_\mu,$$

and will use the bracket notation for symmetric and anti-symmetric parts of a tensor

$$X_{(\mu\nu)} = \frac{1}{2}\left(X_{\mu\nu} + X_{\mu\nu}\right), \qquad X_{[\mu\nu]} = \frac{1}{2}\left(X_{\mu\nu} - X_{\mu\nu}\right).$$

We define the Levi-Civita symbol[1] by:

$$\varepsilon_{01\ldots n} = 1,$$

which is antisymmetric in its indices. The Levi-Civita symbol is *not a tensor*. To enable us to raise and lower the indices with the metric, we enhance the symbol ε to be the Levi-Civita tensor ϵ, by including the determinant of the spacetime metric:

$$\epsilon_{\mu_1\ldots\mu_n} = \sqrt{|g|}\varepsilon_{\mu_1\ldots\mu_n}.$$

Now we can raise the indices to find that

$$\epsilon^{\mu_1\ldots\mu_n} = \epsilon_{\nu_1\ldots\nu_n} g^{\mu_1\nu_1}\ldots g^{\mu_n\nu_n} = \sqrt{g}\varepsilon_{\nu_1\ldots\nu_n} g^{\mu_1\nu_1}\ldots g^{\mu_1\nu_1} = \frac{(-)^t}{\sqrt{g}}\varepsilon^{\mu_1\ldots\nu_n},$$

where t counts the number of timelike dimensions. The Levi-Civita tensor can be contracted to obtain the identity

$$\epsilon_{\mu_1\ldots\mu_p\sigma_{p+1}\ldots\sigma_p}\epsilon^{\nu_1\ldots\nu_p\sigma_{p+1}\ldots\sigma_n} = (-)^t p!(n-p)!\delta^{\nu_i}_{\mu_1}\ldots\delta^{\nu_p}_{\mu_p}.$$

[1] Another way of thinking about the Levi-Civita symbol is as the generalised Kronecker delta: $\delta^{1\ldots n}_{\mu_1\ldots\mu_n} = \varepsilon_{01\ldots n}$

The Levi-Civita tensor appears in the description of the volume form. For an n-dimensional manifold, the volume form is

$$\begin{aligned}\mathrm{vol}_n &= \sqrt{|g|}dx^1 \wedge \ldots \wedge dx^n = \frac{1}{n!}\sqrt{|g|}\varepsilon_{\mu_1 \ldots \mu_n} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n}, \\ &= \frac{1}{n!}\epsilon_{\mu_1 \ldots \mu_n} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n}.\end{aligned}$$

We define a p-form by

$$X = \frac{1}{p!} X_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}.$$

The wedge product of p-,q-forms is given generally by

$$(X \wedge Y)_{\mu_1 \ldots \mu_p \nu_1 \ldots \nu_q} = \frac{(p+q)!}{p!q!} X_{[\mu_1 \ldots \mu_p} Y_{\nu_1 \ldots \nu_q]},$$

and it is useful to remember

$$X \wedge Y = (-)^{pq} Y \wedge X. \tag{A.1.1}$$

Taking the exterior derivative of a p-form X gives a $(p+1)$-form:

$$\begin{aligned}&d : \Omega^p \to \Omega^{p-1} \\ &dX = \frac{1}{p!}\partial_\nu X_{\mu_1 \ldots \mu_p} dx^\nu \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}.\end{aligned}$$

The Hodge-star is defined by:

$$\begin{aligned}&\star : \Omega^p \to \Omega^{n-p} \\ &X \wedge \star Y := (X, Y)\ \mathrm{vol}_n,\end{aligned}$$

where the inner product in components is given by

$$(X, Y) = \frac{1}{p!} X_{\mu_1 \ldots \mu_p} Y^{\mu_1 \ldots \mu_p}.$$

From this, we can write down the components of the Hodge dual as

$$\star X = \frac{1}{p!(n-p)!} X_{\mu_1 \ldots \mu_p} \epsilon^{\mu_1 \ldots \mu_p}{}_{\nu_{p+1} \ldots \nu_n} dx^{\nu_{p+1}} \wedge \ldots \wedge dx^{\nu_n}.$$

The double application of the Hodge-star gives back a p-form with

$$\star \star X = (-1)^{p(n-p)+t} X,$$

where n is the dimension of the manifold, and t counts the number of timelike dimensions. Using the Hodge-star, we can write the volume form as

$$\mathrm{vol}_n = \star 1,$$

and we can use this to rewrite a Lagrangian into the language of forms. As an example the Einstein-Maxwell Lagrangian can be written as

$$\int_M \sqrt{-g} d^4x \left(-\frac{1}{2\kappa_4^2} R - \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} \right) \equiv \int_M \left(-\frac{1}{2\kappa_4^2} \star R - \frac{1}{2g^2} F \wedge \star F \right).$$

A.2 SIGN CONVENTIONS

In this thesis, we use a set of sign conventions that was established in the research [39, 40] which are the core of the discussions of this body of work. These conventions were picked to follow the work of [28], which was the starting point for the planar symmetric solutions of $\mathcal{N} = 2$ supergravity and allowed for the comparison and homogenisation of our papers with the preceding ones. We will show below a set of three signs which were highlighted as variable in [43] and use the parametrisation of conventional signs as in [102].

Studying general relativity involves picking conventions for three distinct signs $s_i = \pm 1$, $i = 1, 2, 3$. The first is the overall sign of the Minkowski metric

$$\eta_{ab} = s_1 \mathrm{diag}(- + ++),$$

and decides whether we work with a 'mostly-plus' or 'mostly-minus' signature. The second sign choice comes from the definition of the Riemann tensor:

$$R^{\mu}{}_{\nu\rho\sigma} = s_2 (\partial_\rho \Gamma^{\mu}{}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\tau}{}_{\nu\sigma}\Gamma^{\mu}{}_{\tau\rho} - \Gamma^{\tau}{}_{\nu\rho}\Gamma^{\mu}{}_{\tau\sigma}) ,$$

and the third sign from the Einstein equations

$$s_3 \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = \kappa_4^2 T_{\mu\nu} ,$$

where it is understood that T_{00} is always positive (for normal matter). The signs s_2, s_3 enter into the definitions of the Ricci tensor and Ricci scalar:

$$s_2 s_3 R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}, \qquad R = g^{\mu\nu} R_{\mu\nu}.$$

These three signs enter into a Lagrangian for gravity, vector and scalar fields as:

$$\mathcal{L} = \left(s_1 s_3 \frac{R}{2\kappa_4^2} - s_1 \frac{1}{\kappa_4^2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} \right).$$

Generally, the conventions used in a particular paper can usually be reconstructed using that the kinetic terms are positive. This depends of cause on knowing that the overall sign of the Lagrangian has been fixed accordingly, and that we are not dealing with a non-standard theory with flipped kinetic terms.[2] We also need to assume that the energy-momentum tensor is defined such that T_{00} is positive and therefore:

$$T_{\mu\nu} = -s_1 \frac{2}{\sqrt{-g}} \frac{\delta(\mathcal{L}_m \sqrt{-g})}{\delta g^{\mu\nu}},$$

where $\mathcal{L}_m$ is the matter contribution to the Lagrangian.

In this thesis, we use the same sign conventions as in [39, 40] which in turn where taken over from [28]. This is a parametrisation where the Einstein-Hilbert and scalar term enter with a minus sign:

$$\mathcal{L} = \left(-\frac{R}{2\kappa_4^2} - \frac{1}{\kappa_4^2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} \right) .$$

[2] As an example for non-standard sign conventions, in our conclusions we mention the interesting duality suggested by the matching of Euclidean partition functions for theories which differ in the overall sign for the sign of the gauge field kinetic terms. For more detail, see Section 9.1.

From this we can read off

$$s_1 = 1, \qquad s_3 = -1.$$

Defining the Ricci tensor such that $s_2 s_3 = 1$, consistency determines the overall sign of the Riemann tensor as $s_2 = -1$,

$$R^{\mu}{}_{\nu\rho\sigma} = -(\partial_\rho \Gamma^{\mu}{}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\tau}{}_{\nu\sigma}\Gamma^{\mu}{}_{\tau\rho} - \Gamma^{\tau}{}_{\nu\rho}\Gamma^{\mu}{}_{\tau\sigma}).$$

It follows that Einstein's equations are:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa_4^2 T_{\mu\nu}.$$

With these conventions, a spacelike surface of positive curvature has $\mathrm{sign}(R) = s_1 s_3 = -1$, and so we have the slightly strange understanding that a positively curved space has a negative Ricci scalar! From the perspective of the (anti) de Sitter solutions, we take the action to be of the form

$$S = -\frac{1}{16\pi} \int_M (R - 2\Lambda)\sqrt{-g}\, d^4x$$

such that when solving the equations of motion, the Ricci scalar is proportional to the cosmological constant. This means that for the de Sitter solution we have $\Lambda < 0$ and for the anti-de Sitter solution, we have that $\Lambda > 0$. This is against most conventional research in the area, and descends from the action built from $s_3 = -1$.

Following these conventions through to the Euclidean action, we find that these signs appear as:

$$S = \frac{s_1 s_3}{2\kappa_4^2} \int_M \sqrt{g}(R - 2\Lambda) d^4x + \frac{s_1 s_4' \epsilon}{\kappa_4^2} \int_{\partial M} \sqrt{|\gamma|} K d^3x, \tag{A.2.1}$$

where the fourth sign s_4', which arises from the definition of the second fundamental form, is discussed in Section 2.2.6. Note that s_4' is distinct from s_4 in [102], which is related to the spin connection. Since we only consider bosonic fields within this thesis, this sign is irrelevant for us.

B

THERMODYNAMIC POTENTIALS

In our calculations of the first law of thermodynamics, we use the Euclidean action formalism to derive a gravitational partition function. This is related to thermodynamic partition functions and from these, we obtain thermodynamic potentials. In this appendix, we give some extra space to the thermodynamic potentials we consider and the various parameters we can obtain through computing partial derivatives. We end by re-focusing for the case of our black hole solutions and the quantities we are interested in. The majority of this discussion follows the textbook by Sethna [217].

B.1 EQUATION OF STATE

The entropy $S(E,V,N)$ can be considered as the first of our *thermodynamic potentials* which depends on the energy E, the volume V and the particle number N. In a similar way, we can rewrite this into the form $E(S,V,N)$ and consider the energy as a thermodynamic potential.

From the statistical mechanics perspective, the temperature is defined from the entropy through the relation

$$T := \left.\frac{\partial S}{\partial E}\right|_{V,N}.$$

We can relate this to the other partial derivatives using the identity

$$\left.\frac{\partial S}{\partial E}\right|_{V,N}\left.\frac{\partial E}{\partial V}\right|_{S,N}\left.\frac{\partial V}{\partial S}\right|_{E,N} = -1,$$

which allows us to find the other values of the partial derivatives

$$\frac{P}{T} = \left.\frac{\partial S}{\partial V}\right|_{E,N}, \quad -\frac{\mu}{T} = \left.\frac{\partial S}{\partial N}\right|_{E,V} \quad \Rightarrow \quad \left.\frac{\partial E}{\partial V}\right|_{S,N} = -P, \quad \left.\frac{\partial E}{\partial N}\right|_{S,V} = \mu,$$

where P is the pressure and μ is the chemical potential. These relations can be understood and are expected from the first law of thermodynamics

$$dE = TdS - PdV + \mu dN.$$

In the main body of the thesis, we verify the first law of thermodynamics by computing the energy and writing it in terms of the entropy and electric charge (which takes the place of particle number N for relativistic systems). It is common to refer to the expression $S(E,V,N)$ or $E(S,V,N)$ as the equation of state. Varying this, we can check the above relations and verify that the first law holds. However, for the solutions we consider, we don't have access to the energy itself and we must first derive it from other thermodynamic potentials.

Contact geometry We mention in passing that it can be shown that the first law of thermodynamics is equivalent to the existence of a contact geometry which can be thought of as the odd-dimensional analogue for symplectic geometry. As tempted as we are to continue on this interesting note, digressing within a digression in our appendix seems extravagant. Our intuition is that this geometric picture of the first law (of both thermodynamics and black hole mechanics) might be an interesting way to try and classify how the non-standard work of the triple Wick-rotation was able to have a well defined first law, despite the lack of a quantum mechanical link between partition functions. For references on contact geometry, we offer [218, 219].

B.2 THE CANONICAL ENSEMBLE

To obtain the internal energy of the solution, we work with the canonical ensemble and derive the Helmholtz free energy. The canonical ensemble governs the equilibrium behaviour for a system at fixed temperature, and we can write the probability that a state s has energy E_s using the Boltzmann distribution

$$\rho(s) \propto \exp\left(-\frac{E_s}{k_B T}\right),$$

where k_B is the Boltzmann constant. The partition function appears as the normalisation factor for the probability

$$\rho(s) = \frac{1}{Z}\exp\left(-\frac{E_s}{k_B T}\right),$$

with its explicit form as

$$Z(T, N, V) = \sum_n \exp\left(-\frac{E_n}{k_B T}\right).$$

This probability distribution is the definition of the canonical ensemble, describing systems exchanging energy with the external world at temperature T. The partition function Z does more than just act as a normalisation, but instead can be used to describe the statistical properties of the system. For example, computing the average *internal energy* we find that

$$\langle E\rangle = \sum_n E_n P_n = \frac{1}{Z}\sum_n E_n \exp\left(-\beta E_n\right) = -\frac{\partial \log Z}{\partial \beta}, \tag{B.2.1}$$

where we are using that the inverse temperature is $\beta^{-1} = k_B T$. This internal energy will be the parameter we derive and vary to verify the first law.

Similarly, one can compute the entropy of the solution and find it as a function of thermodynamic variables and the (logarithm of the) partition function

$$S = -k_B \sum_n P_n \log P_n = \frac{\langle E\rangle}{T} + k_B \log Z.$$

From the logarithm of the partition function, we can write down a new thermodynamic potential, hinted by its presence in the above relationships:

$$F(T,N,V) = -k_B T \log Z = \langle E\rangle - TS,$$

which is known as the *Helmholtz free energy*. Its total derivative is related to other thermodynamic parameters by

$$dF = -SdT - PdV + \mu dN,$$

from which we can read off the various values of the partial derivatives

$$\left.\frac{\partial F}{\partial T}\right|_{N,V} = -S, \qquad \left.\frac{\partial F}{\partial V}\right|_{N,T} = -P, \qquad \left.\frac{\partial F}{\partial N}\right|_{V,T} = \mu.$$

The internal energy can be computed given the free energy and the (inverse) temperature through rewriting (B.2.1)

$$\langle E\rangle = \frac{\partial(\beta F)}{\partial\beta}.$$

B.3 GRAND CANONICAL ENSEMBLE

The grand canonical ensemble describes a system which exchanges both energy and particle number. We find that due to the boundary conditions placed on the Euclidean action within the body of the thesis, this is the natural thermodynamic partition function we consider.

When both energy and particle number are allowed to be exchanged, the probability for a state to have energy E_s and particle number N_s is

$$\rho(s) = \frac{1}{\mathcal{Z}}\exp\left(-\frac{E_s - \mu N_s}{k_B T}\right),$$

where the normalisation $\mathcal{Z}$ is the *grand partition function*:

$$\mathcal{Z}(T,\mu,V) = \sum_n \exp\left(-\frac{E_n - \mu N_n}{k_B T}\right).$$

Within this context, we can consider μ to be the energy required to add a particle to the system adiabatically[1] while keeping the $(N+1)$-particle system in equilibrium.

Just as with the canonical ensemble, one can write down a thermodynamic potential from the logarithm to obtain the *grand potential* given by

$$\Omega(T,\mu,V) = -k_B T \log\mathcal{Z} = \langle E\rangle - TS - \mu N.$$

This can be related to the Helmholtz free energy by a Legendre transformation

$$\Omega = F - \mu N,$$

and the total derivative of the grand potential is given by

$$d\Omega = -SdT - PdV - Nd\mu,$$

allowing us to read off the form of the various partial derivatives

$$\left.\frac{\partial\Omega}{\partial T}\right|_{\mu,V} = -S, \qquad \left.\frac{\partial\Omega}{\partial V}\right|_{\mu,T} = -P, \qquad \left.\frac{\partial F}{\partial\mu}\right|_{V,T} = -N.$$

[1] An adiabatic process occurs between a system and its surroundings without changing the mass or temperature.

B.4 BACK TO BLACK HOLES

In the body of the thesis, we are concerned with the thermodynamics of charged black hole solutions. In relativistic thermodynamics, the particle number N is not conserved and therefore it is replaced by conserved charges.

Let us consider the case of a single conserved charge $\mathcal{Q}$. The natural boundary conditions of the Euclidean action fix the electric charge and allow the chemical potential to vary, and as such, the natural thermodynamic partition function we relate to our gravitational calculation is the grand canonical partition function. To obtain the internal energy, we are interested in working with the Helmholtz free energy. One option is to place additional boundary data with a charge projection operator [88] such that the gravitational partition function is related to the canonical partition function. This is the option taken in [85, 88]. We take the alternative method of leaving the boundary data as it is, and instead compute the grand canonical potential from the partition function followed by making a Legendre transformation to obtain the free energy. From the free energy, the internal energy can be computed and thus the equation of state.

We take the volume to be fixed so that the internal energy only depends on entropy and charge, $E = E(S, \mathcal{Q})$. The free energy $F(T, \mathcal{Q}) = E - TS$ and the grand potential $\Omega(T, \mu) = E - TS - \mu\mathcal{Q}$ are related to $E(S, \mathcal{Q})$ by Legendre transformations which exchange the extensive variables $S, \mathcal{Q}$ with the intensive variables temperature $T = \beta^{-1}$ and chemical potential μ, where we have returned to our conventions where $k_B = 1$.

Various partial derivatives can be read off from the total differentials

$$dE = TdS + \mu d\mathcal{Q}\,, \quad dF = -SdT + \mu d\mathcal{Q}\,, \quad d\Omega = -SdT - \mathcal{Q}d\mu. \tag{B.4.1}$$

In particular, we obtain the following relations used in the main text:

$$\mathcal{Q} = -\frac{\partial \Omega}{\partial \mu}, \quad \mu = \frac{\partial F}{\partial \mathcal{Q}}, \quad \beta = \frac{1}{T} = \frac{\partial S}{\partial E}, \quad S = -\frac{\partial F}{\partial T} = \beta^2 \frac{\partial F}{\partial \beta},$$

and

$$\frac{\partial(\beta F)}{\partial \beta} = F + TS = E.$$

These equations are sufficient for the computations and discussions we present in this work. For more information on statistical mechanics, we refer to [217] and we found [88] a particularly thorough and well written resource for a black hole perspective of thermodynamic partition functions.

C

HODGE DUALITY

In this appendix we continue the discussion put forward in Section 4.2 and consider the Hodge dualisation and rewriting of Lagrangians with arbitrary p-forms and spacetime dimension. We begin with the case for constant coupling, and then follow this considering spacetime-dependent couplings. We end our discussion with the generation of topological terms when considering the Hodge dualisation for gauge fields after dimensional reduction. For more details on dimensional reduction, we point back towards Section 4.3. During this discussion, we will assume a spacetime manifold M of dimension d, with t timelike directions and Lagrangians built with the kinetic term for p-form gauge fields. Much of this discussion follows ideas from [220, 103].

Electromagnetic duality is an example of Poincaré duality, which formalises the isomorphism $H^p(M) \to H_{d-p}(M)$[1] on a closed, oriented manifold M of dimension d. Hodge theory allows this duality to be reformulated as an isomorphism between harmonic forms through the use of the Hodge-star.

The Hodge-star is a map from $\Omega^p \to \Omega^{d-p}$ and is defined such that for the p-forms $\alpha,\ \beta \in \Omega^p$, we can express the inner product as

$$(\alpha, \beta) = \int_M \alpha \wedge \star\beta.$$

A harmonic form is a form $\omega \in \Omega^p$ such that both ω and $\star\omega$ are closed:

$$d\omega = 0, \qquad d \star \omega = 0.$$

The Hodge theorem proves that for harmonic forms, the Hodge-star is an isomorphism. We mention these more mathematical details for context, but relegate further details to [221, 222]. A particularly good reference for physicists needing basics in this area is given in the appendix of [151], and in volume two of superstring theory [223].

Maxwell's equations define a harmonic form F by

$$dF = 0, \qquad d \star F = 0,$$

where $F = dA$ is a two-form. Hodge's theorem yields an isomorphism between $F \to \tilde{F} = \star F$. $\tilde{F}$ is a new two-form which also satisfies the equations of motion, with the role of the electric and magnetic components switched. We note that this mapping is a symmetry at the level of the

[1] That is, the p^{th} cohomology group is isomorphic to the $(d-p)^{\text{th}}$ homology group.

field equations, but not at the level of the Lagrangian. We can understand this as the Lagrangian is a functional of the gauge potential A and not the gauge field. As we saw in the main text, to properly dualise the Lagrangian, the gauge field must be promoted to the level of a dynamic field by promoting the Bianchi identity to a field equation using a Lagrange multiplier. For a p-form in an d-dimensional space, we see that if the Hodge-dual is directly substituted into the Lagrangian, a factor of $(-)^t$ is picked up, where t counts the number of minus signs in the metric signature:

$$
\begin{aligned}
\int_M F_p \wedge \star F_p \mapsto \int_M \tilde{F}_p \wedge \star \tilde{F}_p, \\
&= \int_M \star F_p \wedge \star \star F_p, \\
&= (-)^{p(d-p)+t} \int_M \star F_p \wedge F_p, \\
&= (-)^{2p(d-p)+t} \int_M F_p \wedge \star F_p, \\
&= (-)^t \int_M F_p \wedge \star F_p,
\end{aligned}
$$

where we have used that for a p-form

$$\star \star \omega_p = (-)^{p(d-p)+t} \omega. \tag{C.0.1}$$

We see that for Lorentzian theories, where $t = 1$, replacing the p-field strength with its Hodge-dual introduces a sign error.

C.1 HODGE DUALITY FOR P-FORM POTENTIALS

The correct procedure of dualisation is performed as follows. We are able to write the second order action:

$$S[A] = -\frac{1}{2} \int F_p \wedge \star F_p, \qquad F_p = dA_{(p-1)},$$

as a first order action after promoting F_p to be a fundamental field. We do this by including a new term into the action:

$$S[F] = \int -\frac{1}{2} F_p \wedge \star F_p + (-1)^{p+1} dF_p \wedge \star \lambda_{(p+1)}. \tag{C.1.1}$$

This allows the Bianchi identity to become an equation of motion obtained by varying the Lagrange multiplier: a $(d-p-1)$-form. Let us see this explicitly:

$$
\begin{aligned}
S[\star\lambda + \delta \star \lambda; F] &= S[\star\lambda; F] + \int (-1)^{p+1} dF_p \wedge \star \delta\lambda_{(p+1)}, \\
0 &= \int (-1)^{p+1} dF_p \wedge \star \delta\lambda_{(p+1)}, \\
&\Rightarrow dF_p = 0.
\end{aligned}
$$

By varying the action with respect to the field strength we find that the equation of motion is modified into the form

$$
\begin{aligned}
S[\star\lambda; F + \delta F] &= S[\star\lambda, F] + \int -\delta F_p \wedge \star F_p + (-1)^{p+1} d(\delta F_p) \wedge \star \lambda_{(p+1)}, \\
&= S[\star\lambda, F] + \int -\delta F_p \wedge \star F_p + (-1)^{2p+2} (\delta F_p) \wedge d \star \lambda_{(p+1)}, \\
&= S[\star\lambda, F] + \int \delta F_p \wedge \left[- \star F_p + d \star \lambda_{(p+1)} \right], \\
&\Rightarrow \star F_p = d(\star \lambda_{(p+1)}),
\end{aligned}
$$

where in the second line, integrating by parts introduces the usual '–' together with a factor of $(-)^p$ commuting the exterior derivative d past the p-form F_p.

Collecting these together, we write down the equations of motion for the first order action as:

$$\begin{aligned} dF_p &= 0, \\ \star F_p &= d(\star\lambda_{(p+1)}). \end{aligned}$$

Now to dualise the action such that we express our action in terms of $(d-p)$-forms, we make the identification:

$$\begin{aligned} \tilde{F}_{(d-p)} &= \star F_p, \\ \tilde{A}_{(d-p-1)} &= \star\lambda_{(p+1)}, \qquad \tilde{F} = d\tilde{A}, \end{aligned} \tag{C.1.2}$$

and notice that the equations of motion are invariant. If we substitute (C.1.2) into the first order action (C.1.1), we find that our new action is of the form:

$$S[\tilde{A}] = -\frac{1}{2}\int \tilde{F}_{(d-p)} \wedge \star\tilde{F}_{(d-p)}, \qquad \tilde{F}_{(d-p)} = d\tilde{A}_{(d-p-1)},$$

which we see now has the correct sign for a kinetic term.

C.2 FORM FIELDS COUPLED TO DILATON FIELDS

Let us now generalise this discussion to include a spacetime dependent coupling for the gauge field. In this discussion, we use a dilaton coupling which is what appears for gauge field kinetic terms descending from a string theory perspective. Our calculations would is unchanged by using some generic coupling $g(x)^{-2}$, or a coupling matrix $\mathcal{I}_{IJ}(X^I)$ as appears in the $\mathcal{N}=2$ supergravity action coupled to vector multiplets (6.1.1).

The action with a dilaton coupling is given by:[2]

$$S[A,\phi] = -\frac{1}{2}\int e^{-\alpha\phi} F_p \wedge \star F_p, \qquad F_p = dA_{(p-1)},$$

where α is a constant. Again, we want to promote this action into first order form with the addition of a Lagrange multiplier:

$$S[F,\star\lambda,\phi] = \int -\frac{1}{2}e^{-\alpha\phi}F_p \wedge \star F_p + (-1)^{p+1} dF_p \wedge \star\lambda_{(p+1)}. \tag{C.2.1}$$

The equation of motion for the Lagrange multiplier will be no different from the previous section, however, the equation of motion for the field strength F_p is given as:

$$\begin{aligned} S[F+\delta F, \star\lambda, \phi] &= S[F,\star\lambda,\phi] + \int -e^{-\alpha\phi}\delta F_p \wedge \star F_p + (-1)^{p+1} d(\delta F_p) \wedge \star\lambda_{(p+1)}, \\ &= S[\star\lambda, F] + \int -e^{-\alpha\phi}\delta F_p \wedge \star F_p + (-1)^{2p+2}(\delta F_p) \wedge d(\star\lambda_{(p+1)}), \\ &= S[\star\lambda, F] + \int \delta F_p \wedge \left[-e^{-\alpha\phi} \star F_p + d\left(\star\lambda_{(p+1)}\right)\right], \\ &\Rightarrow e^{-\alpha\phi} \star F_p = d(\star\lambda_{(p+1)}). \end{aligned}$$

[2] Here we have suppressed the term in the action for the kinetic term of the scalar field as it is not relevant to the current discussion.

Now in order to make the dualisation we must identify:

$$\tilde{F}_{(D-p)} = e^{-\alpha\phi} \star F_p,$$
$$\tilde{A}_{(D-p-1)} = (\star\lambda_{(p+1)}), \qquad \tilde{F} = d\tilde{A}.$$

We can then substitute these relations into the first action (C.2.1) to obtain a dualised Lagrangian

$$S[\tilde{B}, \phi] = -\frac{1}{2}\int e^{\alpha\phi}\tilde{F}_{(D-p)} \wedge \star\tilde{F}_{(D-p)}, \qquad\qquad \tilde{F}_{(D-p)} = d\tilde{A}_{(D-p-1)},$$

with the equations of motion left invariant. We see from this that the action gives the same equations of motion when we make the dualisation:

$$F \to \tilde{F} = e^{-\alpha\phi} \star F, \qquad \phi \to \tilde{\phi} = -\phi.$$

We notice that the gauge field coupling has been inverted.

C.3 TOPOLOGICAL TERMS AND TRANSGRESSION TERMS

We conclude this appendix with a discussion of the generation of topological terms from the Hodge dualisation of form fields after performing a Kaluza-Klein reduction. The Kaluza-Klein reduction of topological terms produces the so-called *transgression terms* which modify the structure of the form fields in the lower-dimensional theory. Generally, for a full understanding of the lower dimensional field content for consecutive reductions, it is vital for all these terms to be included. In the body of the thesis, a note is made about the appearance of these terms and it is explained for our calculations, the field restrictions we make mean that the transgressions terms are set to zero. Here, we expand on these comments and give general formula for the appearance of topological terms, followed by the example for the case of reducing from six to five dimensions, showing that the dualisation of the three-form in five dimensions introduces the Chern-Simons form into our theory.

We begin with the action for a $(d+1)$-dimensional theory for a p-form potential:

$$S[B] = -\frac{1}{2}\int H_{(p+1)} \wedge \star H_{(p+1)}, \qquad H = dB,$$

and reduce this over an S^1 using the Kaluza-Klein procedure to obtain a $d-$dimensional theory. In Section 6.1.1 we cover this, and using the formula (4.3.3) we can write the reduced action is of the form:

$$\begin{aligned} S[\mathbb{B}, \mathbb{A}, A] = -\frac{1}{2}\int & e^{2(d-p-1)\alpha\phi} d\mathbb{A}_{(p-1)} \wedge \star d\mathbb{A}_{(p-1)} \\ & - e^{-2p\alpha\phi}\left(d\mathbb{B}_{(p)} - d\mathbb{A}_{(p-1)} \wedge A_{(1)}\right) \wedge \star\left(d\mathbb{B}_{(p)} - d\mathbb{A}_{(p-1)} \wedge A_{(1)}\right), \end{aligned} \tag{C.3.1}$$

where ϕ and $A_{(1)}$ are the Kaluza-Klein scalar and vector respectively and our $(d+1)$-dimensional p-form potential B_p has been reduced into the p-form, $\mathbb{B}_p$ and the $(p-1)$-form, $\mathbb{A}_{(p-1)}$.

Upon reduction, it is standard that if $p \geq d/2$, then we should use the Hodge dualisation procedure to rewrite the p-form potential as a $(d-p-2)$-form potential:

$$\mathbb{B}_{(p)} \to \tilde{\mathbb{B}}_{(d-p-2)}.$$

Just as before, we do this by promoting $\mathbb{H} = d\mathbb{B}$ into the fundamental field by using a Lagrange multiplier to enforce the Bianchi identity as field equations. To reduce the noise of this calculation, we will only write down the second line of (C.3.1) together with the new Lagrange multiplier term:

$$\begin{aligned}\tilde{S}[\mathbb{H},\mathbb{A},A] = &-\frac{1}{2}\int e^{-2p\alpha\phi}(\mathbb{H}_{(p+1)} - d\mathbb{A}_{(p-1)}\wedge A_{(1)})\wedge\star(\mathbb{H}_{(p+1)} - d\mathbb{A}_{(p-1)}\wedge A_{(1)})\\ &+\int(-)^p d\mathbb{H}_{(p+1)}\wedge\star\lambda_{(p+2)}.\end{aligned}$$

Integrating by parts we can write this as:

$$\begin{aligned}\tilde{S}[\mathbb{H},\mathbb{A},A] = &-\frac{1}{2}\int e^{-2p\alpha\phi}(\mathbb{H}_{(p+1)} - d\mathbb{A}_{(p-1)}\wedge A_{(1)})\wedge\star(\mathbb{H}_{(p+1)} - d\mathbb{A}_{(p-1)}\wedge A_{(1)})\\ &+\int \mathbb{H}_{(p+1)}\wedge d\star\lambda_{(p+2)}.\end{aligned}$$

Varying with respect to $\mathbb{H}_{(p+1)}$ we obtain the algebraic relation:

$$\begin{aligned}\star(\mathbb{H}_{(p+1)} - d\mathbb{A}_{(p-1)}\wedge A_{(1)}) &= e^{2p\alpha\phi}d\star\lambda_{(p+2)},\\ (\mathbb{H}_{(p+1)} - d\mathbb{A}_{(p-1)}\wedge A_{(1)}) &= (-)^{p(d+p)+d}e^{-2p\alpha\phi}\star d\star\lambda_{(p+2)},\\ \mathbb{H}_{(p+1)} &= (-)^{p(d+p)+d}e^{2p\alpha\phi}\star d\star\lambda_{(p+2)} + d\mathbb{A}_{(p-1)}\wedge A_{(1)}.\end{aligned}$$

Substituting this back into the original action, whilst making the identification for our new dual fields:

$$\tilde{\mathbb{H}}_{(d-p-1)} = e^{-2p\alpha\phi}\star\mathbb{H}_{(p+1)}, \qquad \tilde{\mathbb{B}}_{(d-p-2)} = \star\lambda_{(p+2)},$$

we obtain:

$$\begin{aligned}\tilde{S}[\mathbb{H},\mathbb{A},A] &= (-)^{p(d+p)+d}\int\frac{1}{2}e^{2p\alpha\phi}\tilde{\mathbb{H}}_{(d-p-1)}\wedge\star\tilde{\mathbb{H}}_{(d-p-1)} + \tilde{\mathbb{H}}_{(d-p-1)}\wedge\mathbb{F}_{(p)}\wedge A_{(1)}\\ &\qquad\tilde{\mathbb{H}}_{(d-p-1)} = d\tilde{\mathbb{B}}_{(d-p-2)} \qquad \mathbb{F}_{(p)} = d\mathbb{A}_{(p-1)}.\end{aligned}$$

Including in the piece we dropped off earlier, the full, dimensionally reduced action would look like:

$$\begin{aligned}S[\mathbb{B},\mathbb{A},A] = (-)^{p(d+p)+d}&\int\frac{1}{2}e^{2p\alpha\phi}\tilde{\mathbb{H}}_{(d-p-1)}\wedge\star\tilde{\mathbb{H}}_{(d-p-1)} - \frac{1}{2}e^{2(d-p-1)\alpha\phi}\mathbb{F}_{(p)}\wedge\star\mathbb{F}_{(p)}\\ &+\int\tilde{\mathbb{H}}_{(d-p-1)}\wedge\mathbb{F}_{(p)}\wedge A_{(1)},\end{aligned} \tag{C.3.2}$$

where we understand the last contribution as a topological term and

$$\tilde{\mathbb{H}}_{(d-p-1)} = d\tilde{\mathbb{B}}_{(d-p-2)}, \qquad \mathbb{F}_{(p)} = d\mathbb{A}_{(p-1)}.$$

As an example, let us consider the case where $p = 2$, $d = 5$ and therefore $\alpha^{-1} = \sqrt{24}$ (see Section 6.1.1). We can write the dimensionally reduced, Hodge dualised kinetic term for the original three-form as

$$\begin{aligned}S[\mathbb{B},\mathbb{A},A] = -\frac{1}{2}\int_{\mathcal{M}_6}\mathbb{H}_{(3)}\wedge\star\mathbb{H}_{(3)} \to &-\frac{1}{2}\int_{\mathcal{M}_5}e^{2\phi/\sqrt{6}}\tilde{\mathbb{H}}_{(2)}\wedge\star\tilde{\mathbb{H}}_{(2)} + \int_{\mathcal{M}_5}\tilde{\mathbb{H}}_{(2)}\wedge\mathbb{F}_{(2)}\wedge A_{(1)}\\ &-\frac{1}{2}\int_{\mathcal{M}_5}e^{2\phi/\sqrt{6}}\mathbb{F}_{(2)}\wedge\star\mathbb{F}_{(2)}.\end{aligned} \tag{C.3.3}$$

Lastly, if we were to couple this to a Dilaton field such that the original $(d+1)$-dimensional action was written as:

$$S = -\frac{1}{2}\int_{\mathcal{M}_{d+1}} e^{\beta\lambda} H_{(p+1)} \wedge \star H_{(p+1)},$$

we would find upon reduction that the sign in front of λ would swap for the term dualised, like so:

$$\begin{aligned} S[\mathbb{B},\mathbb{A},A] = (-)^{p(d+p)+d} \int \frac{1}{2} e^{2p\alpha\phi-\beta\lambda} \tilde{\mathbb{H}}_{(d-p-1)} \wedge \star\tilde{\mathbb{H}}_{(d-p-1)} - \frac{1}{2} e^{2(d-p-1)\alpha\phi+\beta\lambda} \mathbb{F}_{(p)} \wedge \star\mathbb{F}_{(p)} \\ + \tilde{\mathbb{H}}_{(d-p-1)} \wedge \mathbb{F}_{(p)} \wedge A_{(1)}, \end{aligned} \tag{C.3.4}$$

and so, in our case, where $\beta = -\sqrt{2},\ d = 5,\ p = 2$ we find that:

$$\begin{aligned} S[\mathbb{B},\mathbb{A},A] = -\int \frac{1}{2} e^{2\phi/\sqrt{6}+\sqrt{2}\lambda} \star \tilde{\mathbb{H}}_{(2)} \wedge \tilde{\mathbb{H}}_{(2)} - \frac{1}{2} e^{2\phi/\sqrt{6}-\sqrt{2}\lambda} \star \mathbb{F}_{(2)} \wedge \mathbb{F}_{(2)} \\ - \int \tilde{\mathbb{H}}_{(2)} \wedge \mathbb{F}_{(2)} \wedge A_{(1)}. \end{aligned} \tag{C.3.5}$$

D

C-MAP CALCULATION DETAILS

In this appendix, we detail some additional steps for the calculations performed in Section 4.4. There are no surprising results, but when learning this, it took some time to compute them all, and so we include this as a resource for future students.

D.1 DUALISING VECTOR FIELDS

The Lagrangian after reduction is given by

$$\begin{aligned}\mathcal{L}_3 = &\frac{1}{2}\left(-R_3 - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{4}\epsilon e^{2\phi}V^{\mu\nu}V_{\mu\nu}\right) - g_{I\bar{J}}\partial_\mu X^I\partial^\mu\bar{X}^{\bar{J}} \\ &+ \frac{1}{4}e^\phi\mathcal{I}_{IJ}(F^I_{\mu\nu} + \zeta^I V_{\mu\nu})(F^{J|\mu\nu} + \zeta^J V^{\mu\nu}) \\ &- \frac{1}{2}\epsilon e^{-\phi}\mathcal{I}_{IJ}\partial_\mu\zeta^I\partial^\mu\zeta^J - \frac{1}{2}\epsilon\mathcal{R}_{IJ}(F^I_{\mu\nu} + \zeta^I V_{\mu\nu})\partial_\rho\zeta^J\varepsilon^{\mu\nu\rho}.\end{aligned} \tag{D.1.1}$$

To dualise this, we include the Lagrange multiplier:

$$\mathcal{L}_{\text{Lm}} = \frac{1}{2}\epsilon\varepsilon^{\mu\nu\rho}\left(F^I_{\mu\nu}\partial_\rho\tilde{\zeta}_I - V_{\mu\nu}\partial_\rho\left(\tilde{\phi} - \frac{1}{2}\zeta^I\tilde{\zeta}_I\right)\right). \tag{D.1.2}$$

Varying with respect to $F^I_{\mu\nu}$

$$\begin{aligned}\mathcal{L}_3[\delta F^I_{\mu\nu}] + \mathcal{L}_{\text{Lm}}[\delta F^I_{\mu\nu}] = &+\frac{1}{2}e^\phi\mathcal{I}_{IJ}\left(F^{J|\mu\nu} + \zeta^J V^{\mu\nu}\right)\delta F^I_{\mu\nu} \\ &- \frac{1}{2}\epsilon\varepsilon^{\mu\nu\rho}\mathcal{R}_{IJ}\partial_\rho\zeta^J\delta F^I_{\mu\nu} + \frac{1}{2}\epsilon\varepsilon^{\mu\nu\rho}\partial_\rho\tilde{\zeta}_I\delta F^I_{\mu\nu} = 0.\end{aligned}$$

Rearranging this we obtain

$$\begin{aligned}-e^\phi\mathcal{I}_{IJ}\left(F^{J|\mu\nu} + \zeta^J V^{\mu\nu}\right) &= \epsilon\varepsilon^{\mu\nu\rho}\left(\partial_\rho\tilde{\zeta}_I - \mathcal{R}_{IJ}\partial_\rho\zeta^J\right) \\ F^{I|\mu\nu} + \zeta^I V^{\mu\nu} &= -e^{-\phi}\mathcal{I}^{IJ}\epsilon\varepsilon^{\mu\nu\rho}\left(\partial_\rho\tilde{\zeta}_J - \mathcal{R}_{JK}\partial_\rho\zeta^K\right).\end{aligned} \tag{D.1.3}$$

Varying with respect to $V_{\mu\nu}$

$$\begin{aligned}\mathcal{L}_3[\delta V_{\mu\nu}] + \mathcal{L}_{\text{Lm}}[\delta V_{\mu\nu}] = &\frac{1}{4}\epsilon e^{2\phi}V^{\mu\nu}\delta V_{\mu\nu} + \frac{1}{2}e^\phi\mathcal{I}_{IJ}\zeta^I\left(F^{J|\mu\nu} + \zeta^J V^{\mu\nu}\right)\delta V_{\mu\nu} \\ &- \frac{1}{2}\epsilon\mathcal{R}_{IJ}\zeta^I\partial_\rho\zeta^J\varepsilon^{\mu\nu\rho}\delta V_{\mu\nu} \\ &- \frac{1}{2}\epsilon\varepsilon^{\mu\nu\rho}\partial_\rho\left(\tilde{\phi} - \frac{1}{2}\zeta^I\tilde{\zeta}_I\right)\delta V_{\mu\nu} = 0.\end{aligned}$$

Rearranging this

$$\frac{1}{2}\epsilon e^{2\phi}V^{\mu\nu} = -e^{\phi}\mathcal{I}_{IJ}\zeta^I\left(F^{J|\mu\nu} + \zeta^J V^{\mu\nu}\right) + \epsilon\mathcal{R}_{IJ}\zeta^I\partial_\rho\zeta^J\varepsilon^{\mu\nu\rho} + \epsilon\varepsilon^{\mu\nu\rho}\partial_\rho\left(\tilde{\phi} - \frac{1}{2}\zeta^I\tilde{\zeta}_I\right).$$

Inserting in (D.1.3) we obtain

$$\begin{aligned}
\frac{1}{2}\epsilon e^{2\phi}V^{\mu\nu} &= \epsilon\varepsilon^{\mu\nu\rho}\zeta^I\left(\partial_\rho\tilde{\zeta}_I - \mathcal{R}_{IJ}\partial_\rho\zeta^J\right) + \epsilon\varepsilon^{\mu\nu\rho}\mathcal{R}_{IJ}\zeta^I\partial_\rho\zeta^J + \epsilon\varepsilon^{\mu\nu\rho}\partial_\rho\left(\tilde{\phi} - \frac{1}{2}\zeta^I\tilde{\zeta}_I\right),\\
&= \varepsilon^{\mu\nu\rho}\epsilon\zeta^I\partial_\rho\tilde{\zeta}_I + \epsilon\varepsilon^{\mu\nu\rho}\partial_\rho\left(\tilde{\phi} - \frac{1}{2}\zeta^I\tilde{\zeta}_I\right),\\
&= \epsilon\varepsilon^{\mu\nu\rho}\left[\zeta^I\partial_\rho\tilde{\zeta}_I + \partial_\rho\tilde{\phi} - \frac{1}{2}\left(\zeta^I\partial_\rho\tilde{\zeta}_I + \tilde{\zeta}_I\partial_\rho\zeta^I\right)\right],\\
&= \epsilon\varepsilon^{\mu\nu\rho}\left[\partial_\rho\tilde{\phi} + \frac{1}{2}\left(\zeta^I\partial_\rho\tilde{\zeta}_I - \tilde{\zeta}_I\partial_\rho\zeta^I\right)\right].
\end{aligned}$$

Rearranging this, we obtain

$$V_{\mu\nu} = 2e^{-2\phi}\varepsilon_{\mu\nu\rho}\left[\partial^\rho\tilde{\phi} + \frac{1}{2}\left(\zeta^I\partial^\rho\tilde{\zeta}_I - \tilde{\zeta}_I\partial^\rho\zeta^I\right)\right].$$

Going back to (D.1.3), we can rearrange this for $F^I_{\mu\nu}$

$$F^I_{\mu\nu} = -e^{-\phi}\mathcal{I}^{IJ}\epsilon\varepsilon_{\mu\nu\rho}\left[\partial^\rho\tilde{\zeta}_J - \mathcal{R}_{JK}\partial^\rho\zeta^K\right] - \zeta^I V_{\mu\nu}.$$

Naming

$$B^I_{\mu\nu} := F^I_{\mu\nu} + \zeta^I V_{\mu\nu} = -e^{-\phi}\mathcal{I}^{IJ}\epsilon\varepsilon_{\mu\nu\rho}\left(\partial^\rho\tilde{\zeta}_J - \mathcal{R}_{JK}\partial^\rho\zeta^K\right).$$

We can write the Lagrangian

$$\begin{aligned}
\mathcal{L}_3 = &-\frac{1}{2}R_3 - \frac{1}{4}\partial_\mu\phi\partial^\mu\phi - g_{I\bar{J}}\partial_\mu X^I\partial^\mu\bar{X}^{\bar{J}}\\
&+ \frac{1}{8}\epsilon e^{2\phi}V_{\mu\nu}V^{\mu\nu} + \frac{1}{4}e^{\phi}\mathcal{I}_{IJ}B^I_{\mu\nu}B^{J|\mu\nu}\\
&- \frac{1}{2}\epsilon e^{-\phi}\mathcal{I}_{IJ}\partial_\mu\zeta^I\partial^\mu\zeta^J\\
&- \frac{1}{2}\epsilon\varepsilon^{\mu\nu\rho}\mathcal{R}_{IJ}B^I_{\mu\nu}\partial_\rho\zeta^J,
\end{aligned}$$

and the Lagrange multiplier as

$$\begin{aligned}
\mathbf{e}_3^{-1}\mathcal{L}_{\mathrm{Lm}} &= \frac{1}{2}\epsilon\varepsilon^{\mu\nu\rho}\left(B^I_{\mu\nu}\partial_\rho\tilde{\zeta}_I - V_{\mu\nu}\zeta^I\partial_\rho\tilde{\zeta}_I - V_{\mu\nu}\partial_\rho\left(\tilde{\phi} - \frac{1}{2}\zeta^I\tilde{\zeta}_I\right)\right),\\
&= \frac{1}{2}\epsilon\varepsilon^{\mu\nu\rho}\left(B^I_{\mu\nu}\partial_\rho\tilde{\zeta}_I - V_{\mu\nu}\left[\partial_\rho\tilde{\phi} + \frac{1}{2}\left(\zeta^I\partial_\rho\tilde{\zeta}_I - \tilde{\zeta}_I\partial_\rho\zeta^I\right)\right]\right),\\
&= \frac{1}{2}\epsilon\varepsilon^{\mu\nu\rho}B^I_{\mu\nu}\partial_\rho\tilde{\zeta}_I - \frac{1}{4}\epsilon e^{2\phi}V_{\mu\nu}V^{\mu\nu}.
\end{aligned}$$

Combining these we obtain

$$\begin{aligned}
\mathcal{L}_3 + \mathcal{L}_{\mathrm{Lm}} = &-\frac{1}{2}R_3 - \frac{1}{4}\partial_\mu\phi\partial^\mu\phi - g_{I\bar{J}}\partial_\mu X^I\partial^\mu\bar{X}^{\bar{J}}\\
&- \frac{1}{8}\epsilon e^{2\phi}V_{\mu\nu}V^{\mu\nu} + \frac{1}{4}e^{\phi}\mathcal{I}_{IJ}B^I_{\mu\nu}B^{J|\mu\nu}\\
&- \frac{1}{2}\epsilon e^{-\phi}\mathcal{I}_{IJ}\partial_\mu\zeta^I\partial^\mu\zeta^J\\
&+ \frac{1}{2}\epsilon\varepsilon^{\mu\nu\rho}B^I_{\mu\nu}\left(\partial_\rho\tilde{\zeta}_I - \mathcal{R}_{IJ}\partial_\rho\zeta^J\right).
\end{aligned}$$

Rearranging the last term, we can simplify this

$$\begin{aligned}\mathcal{L}_3 + \mathcal{L}_{\text{Lm}} = &-\frac{1}{2}R_3 - \frac{1}{4}\partial_\mu\phi\partial^\mu\phi - g_{I\bar{J}}\partial_\mu X^I\partial^\mu\bar{X}^{\bar{J}} \\ &-\frac{1}{8}\epsilon e^{2\phi}V_{\mu\nu}V^{\mu\nu} \\ &-\frac{1}{4}e^{\phi}\mathcal{I}_{IJ}B^I_{\mu\nu}B^{J|\mu\nu} \\ &-\frac{1}{2}\epsilon e^{-\phi}\mathcal{I}_{IJ}\partial_\mu\zeta^I\partial^\mu\zeta^J.\end{aligned}$$

Substituting in the values from the equations of motion we find that

$$\begin{aligned}V^{\mu\nu}V_{\mu\nu} &= \epsilon 8e^{-4\phi}\left[\partial^\rho\tilde{\phi} + \frac{1}{2}\left(\zeta^I\partial^\rho\tilde{\zeta}_I - \tilde{\zeta}_I\partial^\rho\zeta^I\right)\right]\left[\partial_\rho\tilde{\phi} + \frac{1}{2}\left(\zeta^I\partial_\rho\tilde{\zeta}_I - \tilde{\zeta}_I\partial_\rho\zeta^I\right)\right], \\ \mathcal{I}_{IJ}B^I_{\mu\nu}B^{J|\mu\nu} &= 2\epsilon e^{-2\phi}\mathcal{I}^{IJ}\left(\partial^\rho\tilde{\zeta}_I - \mathcal{R}_{IM}\partial^\rho\zeta^M\right)\left(\partial_\rho\tilde{\zeta}_J - \mathcal{R}_{JN}\partial_\rho\zeta^N\right),\end{aligned}$$

where we have used that $\varepsilon_{\mu\nu\rho}\varepsilon^{\mu\nu\kappa} = \epsilon 2!\delta^\kappa_\rho$.

$$\begin{aligned}\mathcal{L}_3 + \mathcal{L}_{\text{Lm}} = &-\frac{1}{2}R_3 - \frac{1}{4}\partial_\mu\phi\partial^\mu\phi - g_{I\bar{J}}\partial_\mu X^I\partial^\mu\bar{X}^{\bar{J}} \\ &- e^{-2\phi}\left[\partial^\rho\tilde{\phi} + \frac{1}{2}\left(\zeta^I\partial^\rho\tilde{\zeta}_I - \tilde{\zeta}_I\partial^\rho\zeta^I\right)\right]\left[\partial_\rho\tilde{\phi} + \frac{1}{2}\left(\zeta^I\partial_\rho\tilde{\zeta}_I - \tilde{\zeta}_I\partial_\rho\zeta^I\right)\right] \\ &- \frac{\epsilon}{2}e^{-\phi}\left[\mathcal{I}_{IJ}\partial_\mu\zeta^I\partial^\mu\zeta^J + \mathcal{I}^{IJ}\left(\partial^\rho\tilde{\zeta}_I - \mathcal{R}_{IM}\partial^\rho\zeta^M\right)\left(\partial_\rho\tilde{\zeta}_J - \mathcal{R}_{JN}\partial_\rho\zeta^N\right)\right].\end{aligned}$$

E

STU SUPERGRAVITY COUPLINGS

In Section 7.1, we prepare our Lagrangian for dimensional uplift through writing it in a form such that we can easily compare our conventions to those used in [131]. To do this, we need the explicit form of the couplings which appear in our Lagrangian. As explained in Section 4.1.2, the $\mathcal{N} = 2$ supergravity theory is completely determined by the prepotential, which then naturally becomes our starting point. The couplings are also used in Section 8.4 when we compute the on-shell Euclidean action of the planar solutions of the STU model.

The STU prepotential (4.1.5), repeated here, is of the form

$$F = \frac{X^1 X^2 X^3}{X^0}.$$

First we compute the derivatives of F with respect to the complex scalars X^I. Taking the first derivative of $F(X)$ we obtain

$$F_I = \left(-\frac{X^1X^2X^3}{(X^0)^2}, \frac{X^2X^3}{X^0}, \frac{X^1X^3}{X^0}, \frac{X^1X^2}{X^0}\right), \tag{E.0.1}$$

and taking one further derivative

$$F_{IJ} = \begin{pmatrix} \frac{2X^1X^2X^3}{(X^0)^3} & -\frac{X^2X^3}{(X^0)^2} & -\frac{X^1X^3}{(X^0)^2} & -\frac{X^1X^2}{(X^0)^2} \\ -\frac{X^2X^3}{(X^0)^2} & 0 & \frac{X^3}{X^0} & \frac{X^2}{X^0} \\ -\frac{X^1X^3}{(X^0)^2} & \frac{X^3}{X^0} & 0 & \frac{X^1}{X^0} \\ -\frac{X^1X^2}{(X^0)^2} & \frac{X^2}{X^0} & \frac{X^1}{X^0} & 0 \end{pmatrix}. \tag{E.0.2}$$

We work with the physical scalars z^A, and introduce the scalars (s, t, u) to clean up our computations

$$z^A = \frac{X^A}{X^0}, \qquad \mathrm{Im}(z^1) = -s, \quad \mathrm{Im}(z^2) = -t, \quad \mathrm{Im}(z^3) = -u. \tag{E.0.3}$$

To remove the spurious degrees of freedom we remember we must gauge fix our complex scalars, in our conventions we pick

$$\mathrm{Im}X^0 = 0, \qquad \mathrm{Re}X^A = 0,$$

and we can pick any of these to be the gauge fixing term. This allows us to relate the complex scalars X^I to the scalars

$$X^0, \qquad X^1 = -isX^0, \qquad X^2 = -itX^0, \qquad X^3 = -iuX^0.$$

From this, we can write down all the coupling matrices of the Lagrangian in terms of the real fields $\{s, t, u, X^0\}$. The prepotential is given by

$$F(X^0, z^A) = istu(X^0)^2, \tag{E.0.4}$$

and its derivatives by

$$F_I = \left(-istuX^0, tuX^0, suX^0, stX^0\right), \qquad F_{IJ} = \begin{pmatrix} 2istu & -tu & -su & -st \\ -tu & 0 & -iu & -it \\ -su & -iu & 0 & -is \\ -st & -it & -is & 0 \end{pmatrix}. \tag{E.0.5}$$

The scalar metric (4.1.6), and its inverse, can then be found from

$$N_{IJ} = 2\mathrm{Im}F_{IJ} = \begin{pmatrix} 4stu & 0 & 0 & 0 \\ 0 & 0 & -2u & -2t \\ 0 & -2u & 0 & -2s \\ 0 & -2t & -2s & 0 \end{pmatrix}, \qquad N^{IJ} = \begin{pmatrix} \frac{1}{4stu} & 0 & 0 & 0 \\ 0 & \frac{s}{4tu} & -\frac{1}{4u} & -\frac{1}{4t} \\ 0 & -\frac{1}{4u} & \frac{t}{4su} & -\frac{1}{4s} \\ 0 & -\frac{1}{4t} & -\frac{1}{4s} & \frac{u}{4st} \end{pmatrix}. \tag{E.0.6}$$

We are interested in finding expressions for (4.1.11) and (4.1.9) which we can do by looking at the following intermediate quantities:

$$\begin{aligned} (NX)_I &= 4X^0\left(stu, -itu, -isu, -ist\right), \\ (N\bar{X})_I &= 4X^0\left(stu, itu, isu, ist\right), \\ (XN\bar{X}) &= K = -8stu(X^0)^2, \\ (XNX) &= 16stu(X^0)^2. \end{aligned}$$

Taking the outer product we find

$$(N\bar{X})_I(NX)_J = 16stu(X^0)^2 \begin{pmatrix} stu & itu & isu & ist \\ -itu & \frac{tu}{s} & u & t \\ -isu & u & \frac{su}{t} & s \\ -ist & t & s & \frac{st}{u} \end{pmatrix},$$

and

$$(NX)_I(NX)_J = 16stu(X^0)^2 \begin{pmatrix} stu & -itu & -isu & -ist \\ -itu & -\frac{tu}{s} & -u & -t \\ -isu & -u & -\frac{su}{t} & -s \\ -ist & -t & -s & -\frac{st}{u} \end{pmatrix}.$$

We are now able to write down the coupling matrices using the above information. From (4.1.11), we find the general gauge coupling to be

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + i\frac{(XN)_I(XN)_J}{XNX} = \begin{pmatrix} -istu & 0 & 0 & 0 \\ 0 & -\frac{itu}{s} & 0 & 0 \\ 0 & 0 & -\frac{isu}{t} & 0 \\ 0 & 0 & 0 & -\frac{ist}{u} \end{pmatrix}. \tag{E.0.7}$$

As we have imposed the purely imaginary condition through our gauge fixing, we have $\mathcal{R}_{IJ} = 0$. The imaginary component, and its inverse is given by

$$\mathcal{I}_{IJ} = \begin{pmatrix} -stu & 0 & 0 & 0 \\ 0 & -\frac{tu}{s} & 0 & 0 \\ 0 & 0 & -\frac{su}{t} & 0 \\ 0 & 0 & 0 & -\frac{st}{u} \end{pmatrix}, \qquad \mathcal{I}^{IJ} = \begin{pmatrix} -\frac{1}{stu} & 0 & 0 & 0 \\ 0 & -\frac{s}{tu} & 0 & 0 \\ 0 & 0 & -\frac{t}{su} & 0 \\ 0 & 0 & 0 & -\frac{u}{st} \end{pmatrix}. \tag{E.0.8}$$

Similarly, from (4.1.9), we find the form for the scalar field coupling

$$g_{I\bar{J}} = -\frac{N_{IJ}}{XN\bar{X}} + \frac{(N\bar{X})_I(XN)_J}{(XN\bar{X})^2} = \frac{1}{(X^0)^2}\begin{pmatrix} \frac{3}{4} & \frac{i}{4s} & \frac{i}{4t} & \frac{i}{4u} \\ -\frac{i}{4s} & \frac{1}{4s^2} & 0 & 0 \\ -\frac{i}{4t} & 0 & \frac{1}{4t^2} & 0 \\ -\frac{i}{4u} & 0 & 0 & \frac{1}{4u^2} \end{pmatrix}. \tag{E.0.9}$$

This is the coupling for the scalar fields X^I, however, we only need the elements of the physical scalar coupling, which is given by (4.1.10), repeated here

$$g_{A\bar{B}} = g_{I\bar{J}}\frac{\partial X^I}{\partial z^A}\frac{\partial \bar{X}^{\bar{J}}}{\partial \bar{z}^B},$$

which when computed is of the simple form

$$g_{A\bar{B}} = \begin{pmatrix} \frac{1}{4s^2} & 0 & 0 \\ 0 & \frac{1}{4t^2} & 0 \\ 0 & 0 & \frac{1}{4u^2} \end{pmatrix}. \tag{E.0.10}$$

We see that both (E.0.8) and (E.0.10) are diagonal, and so there will be no cross terms for our scalar or gauge field kinetic terms. We can begin to start expanding our $4D$ Lagrangian (4.1.4) for the STU model of $\mathcal{N} = 2$ supergravity

$$e_4^{-1}\mathcal{L} = -\frac{1}{2}R - g_{A\bar{B}}\partial_\mu z^A \partial^\mu \bar{z}^{\bar{B}} + \frac{1}{4}\mathcal{I}_{IJ}F^I_{\mu\nu}F^{J|\mu\nu} + \frac{1}{4}\mathcal{R}_{IJ}F^I_{\mu\nu}\tilde{F}^{J|\mu\nu}.$$

Substituting in (E.0.8) and (E.0.10) we can write down an action in the form

$$e_4^{-1}\mathcal{L} = -\frac{1}{2}R - \frac{1}{4}\left(\frac{(\partial s)^2}{s^2} + \frac{(\partial t)^2}{t^2} + \frac{(\partial u)^2}{u^2}\right)$$
$$- \frac{1}{4}\left(stu(F^0)^2 + \frac{tu}{s}(F^1)^2 + \frac{su}{t}(F^2)^2 + \frac{st}{u}(F^3)^2\right).$$

Making a redefintion of our scalars fields

$$s = e^{-\phi_1}, \qquad t = e^{-\phi_2}, \qquad u = e^{-\phi_3},$$

we obtain the Lagrangian

$$e_4^{-1}\mathcal{L} = -\frac{1}{2}R - \frac{1}{4}\partial_\mu\phi_i\partial^\mu\phi_i - \frac{1}{4}e^{-\phi_1-\phi_2-\phi_3}\left[(F^0)^2 + e^{2\phi_A}(F^A)^2\right].$$

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