In Congestion Games, Taxes Achieve Optimal Approximation

DARIO PACCAGNAN, Imperial College London, UK
MARTIN GAIRING, University of Liverpool, UK

We consider the problem of minimizing social cost in atomic congestion games and show, perhaps surprisingly, that efficiently computed taxation mechanisms yield the *same* performance achievable by the best polynomial time algorithm, even when the latter has full control over the players' actions. It follows that no other tractable approach geared at incentivizing desirable system behavior can improve upon this result, regardless of whether it is based on taxations, coordination mechanisms, information provision, or any other principle. In short: Judiciously chosen taxes achieve optimal approximation.

Three technical contributions underpin this conclusion. First, we show that computing the minimum social cost is NP-hard to approximate within a given factor depending solely on the admissible resource costs. Second, we design a tractable taxation mechanism whose efficiency (price of anarchy) matches this hardness factor, and thus is optimal. As these results extend to coarse correlated equilibria, any no-regret algorithm inherits the same performances, allowing us to devise polynomial time algorithms with optimal approximation.

To appear at the 22nd ACM Conference on Economics & Computation (EC'21).
Authors' e-mail addresses: `d.paccagnan@imperial.ac.uk`, `m.gairing@liverpool.ac.uk`

The authors would like to thank Rahul Chandan, Tim Roughgarden and Rahul Savani for useful comments and suggestions.

1 INTRODUCTION

In many of today's networked systems, e.g., road-traffic networks, the overall performance greatly depends on the interaction between the users' individual behaviour and the underlying infrastructure. A major issue emerging in these settings is the performance degradation that arises when users act with their sole individual interests in mind [49]. A prototypical example is road-traffic routing: When drivers choose routes that minimize their own travel time, the aggregate congestion could be much higher compared to that of a centrally-imposed routing.

While improved performances could be attained if a coordinator was able to dictate the choices of each user, imposing such control is often infeasible or impossible, with traffic routing providing just one illustration. Hence, different approaches including coordination mechanisms [20], Stackelberg strategies [26], taxation mechanisms [12], information provision [7], and alternative methods for sharing resource costs [28] have been proposed as *indirect interventions* to influence the resulting outcome. Amongst these, taxation mechanisms have attracted significant attention, as witnessed by the growing literature on the topic. Unfortunately, in spite of the scientific interest, it remains largely unclear, if, and to what extent, the use of indirect interventions reduces the best achievable system performance, thus prompting a natural question:

> Is there any performance degradation incurred when moving from centrally imposed decision making to the use of indirect interventions, such as taxation mechanisms?

Here, we answer this question in relation to the well-studied class of atomic congestion games, commonly utilized to model a variety of resource allocation problems including traffic routing [51], machine scheduling [32], sensor allocation [45, 46], and minimum power routing [2]. Within this context, we show, perhaps surprisingly, that *no* performance degradation arises if taxation mechanisms are judiciously designed. Specifically, we derive tractable taxation mechanisms ensuring that any equilibrium outcome yields the *same* performance achievable by the best centralized polynomial time algorithm, even when the latter has full control over the users' decisions. The upshot of our contribution can be summarized as follows:

> *In congestion games, judiciously designed taxes achieve optimal approximation, and no other tractable intervention, whether based on coordination mechanisms, information provision, or any other principle can improve upon this result.*

Our manuscript contains three contributions that lead to this conclusion. First, we show that computing the minimum social cost in congestion games is NP-hard to approximate within an explicitly given factor depending solely on the class of admissible resource costs. Second, we show how to design tractable taxation mechanisms, through the solution of a convex optimization problem, ensuring that any corresponding equilibrium outcome has an efficiency (price of anarchy) matching the hardness factor previously derived. While this result applies to pure/mixed Nash equilibria, it also extends to correlated/coarse correlated equilibria, and thus to any online learning algorithm where players update their actions and achieve low regret, in the same spirit of the "total price of anarchy" pioneered by Blum et al. [10]. We build upon this observation to derive polynomial time algorithms achieving the optimal approximation factor.

Comparison with existing results. Our work connects with, and generalizes a number of existing results, in addition to closing different open questions, as we briefly highlight next. We refer to Section 1.3 for a more detailed literature review.

The problem of determining computational lower bounds for minimizing the social cost in atomic congestion game has been initially studied by Meyers and Schulz [43]. Since then, a number of works have pursued a similar line of research [13, 48, 52], though no tight bounds were known, even for linear congestion games. Our work provides the best possible inapproximability results, completely settling the hardness question for congestion games with general underlying resource costs.

The study of taxation mechanisms has recently received growing attention, especially within the literature of congestion games. Nevertheless, prior to this work, a general methodology to design optimal taxation mechanisms (i.e., mechanisms minimizing the price of anarchy) has been unavailable, with [9, 12] providing interesting results limitedly to polynomial congestion games. Extending the work of Caragiannis et al. [12], Bilò and Vinci [9] propose taxes whose efficiency can be quantified a priori conjecturing that their design is optimal, albeit no computational lower bounds are given. Our work resolves the problem of determining optimal taxes for the broader class of congestion games with non-decreasing and semi-convex resource costs, proving the conjecture of [9] as a special case. Recently, Paccagnan et al. [44] designed optimal *local* taxation mechanisms, i.e., optimal taxation mechanisms whose tax levied on each resource must rely solely on the local properties of that resource. Interestingly, our work shows that the performance of optimal local mechanisms almost matches that achievable by the best polynomial time algorithm. Roughgarden [52] studies how lower bounds on the price of anarchy can be derived from computational lower bounds. For congestion games with optimal taxes, we show that such an approach does provide *tight* bounds.

The study of approximation algorithms for minimizing the social cost in congestion games has been motivated by scheduling and routing problems, e.g., [2, 5]. Makarychev and Sviridenko [41] provide the best known approximation algorithm, based on a natural linear programming relaxation and randomized rounding. While their result applies to the more general class of optimization problems with a "diseconomy of scale", the algorithms we propose here enjoy an equal or strictly better approximation ratio, and can not be further improved, owing to the matching hardness result presented.

1.1 Congestion games and taxation mechanisms

Congestion games were introduced in a landmark paper by Rosenthal approximately 50 years ago [50]. Since then, they have found applications in numerous fields, including network design [3], machine scheduling [55], vehicle-target assignment [17], wireless data networks [56]. Most notably, congestion games are utilized to model selfish routing on a traffic network [51]. In a congestion game we are given a set of players $\{1, \ldots, N\}$, and a set of resources $\mathcal{R}$. Each player can choose a subset of the set of resources which she intends to use. We list all feasible choices for player i in the set $\mathcal{A}_i \subseteq 2^{\mathcal{R}}$. The cost for using each resource $r \in \mathcal{R}$ depends only on the total number of players concurrently selecting that resource, and is denoted with $\ell_r : \mathbb{N} \to \mathbb{R}_{>0}$. Once all players have made a choice $a_i \in \mathcal{A}_i$, each player incurs a cost obtained by summing the costs of all resources she selected. Finally, the social cost represents the sum of the resource costs incurred by all players

$$\mathrm{SC}(a) = \sum_{i=1}^{N} \sum_{r \in a_i} \ell_r(|a|_r), \tag{1}$$

where $|a|_r$ denotes the number of players selecting resource r in allocation $a = (a_1, \ldots, a_n)$. We denote with $\mathcal{G}$ the set of all congestion games where all resource costs $\{\ell_r\}_{r\in\mathcal{R}}$ belong to a given set of cost functions $\mathscr{L}$. Given an instance $G = (N, \mathcal{R}, \{\mathcal{A}_i\}_{i=1}^N, \{\ell_r\}_{r\in\mathcal{R}})$ of congestion game, we denote with `MinSC` the problem of globally minimizing the social cost in (1).

Taxation mechanisms. As self-interested decision making often deteriorates the system performance, taxation mechanisms have been proposed to ameliorate this issue. Formally, a taxation mechanism $T : G \times r \to \tau_r$ associates an instance G, and a resource $r \in \mathcal{R}$ to a taxation function $\tau_r : \mathbb{N} \to \mathbb{R}_{\geq 0}$. Note that each taxation function τ_r is congestion-dependent, that is, it associates the number of players in resource r to the corresponding tax. As a consequence, each player i experiences a cost factoring both the cost associated to the chosen resources, and the tax, i.e.,

$$C_i(a) = \sum_{r \in a_i} [\ell_r(|a|_r) + \tau_r(|a|_r)].$$

As typical in the literature, we measure the performance of a given taxation mechanism T using the ratio between the social cost incurred at the worst-performing outcome, and the minimum social cost. Given the self-interested nature of the players, an outcome is commonly described by any of the following classical equilibrium notions: pure or mixed Nash equilibria, coarse or coarse correlated equilibria.[1] When considering pure Nash equilibria, for example, the performance of a taxation mechanism T is gauged using the notion of *price of anarchy* [39], i.e.,

$$\mathrm{PoA}(T) = \sup_{G \in \mathcal{G}} \frac{\mathtt{NeCost}(G,T)}{\mathtt{MinCost}(G)}, \tag{2}$$

where $\mathtt{MinCost}(G) = \min_{a \in \mathcal{A}} SC(a)$ is the minimum social cost for instance G, and $\mathtt{NeCost}(G,T)$ denotes the highest social cost at a Nash equilibrium obtained when employing the mechanism T on the game G. By definition, $\mathrm{PoA}(T) \geq 1$ and the lower the price of anarchy, the better performance T guarantees. While it is possible to define the notion of price of anarchy for each and every equilibrium class mentioned, we do not pursue this direction, as we will show that all these metrics coincide within our setting. Thus, we will simply use $\mathrm{PoA}(T)$ to refer to the efficiency values of *any* and *all* these equilibrium classes. Finally, we observe that, while taxation mechanisms influence the players' perceived cost, they do not impact the expression of the social cost, which is still of the form in (1).

1.2 Our contributions

The resounding message contained in this work can be summarized as follows: In congestion games, optimally designed taxation mechanisms can be tractably computed, and achieve the same performance as the best centralized polynomial time algorithm. Three technical contributions substantiate this claim, and are further discussed in the ensuing paragraphs:

i) We prove a tight NP-hardness result for approximating the minimum social cost;
ii) We design a tractable taxation mechanism whose price of anarchy matches the hardness factor;
iii) We obtain a polynomial time algorithm with the best possible approximation by combining the previous results with existing algorithms (e.g., no-regret dynamics).

All our results hold for the widely studied case of non-decreasing semi-convex resource costs.[2] Results ii) and iii) extend to network congestion games, as we discuss in the conclusions.

Inapproximability of minimum social cost. Our first contribution is concerned with determining tight inapproximability results for the problem of minimizing social cost in congestion games. Our hardness result applies already in the setup where all resources feature the same cost.

Theorem 1. *In congestion games where all resources feature the same cost $b : \mathbb{N} \to \mathbb{R}_{>0}$ and b is non-decreasing semi-convex,* MinSC *is* NP*-hard to approximate within any factor smaller than*

$$\rho_b = \sup_{x \in \mathbb{N}} \frac{\mathbb{E}_{P \sim \mathrm{Poi}(x)}[Pb(P)]}{xb(x)}, \tag{3}$$

where we define $b(0) = 0$. If $\rho_b = \infty$, then MinSC *is* NP*-hard to approximate within any finite factor.*[3]

Naturally, Theorem 1 applies directly to richer classes of congestion games, whereby resource costs can differ. For example, if the cost on each resource can be constructed by non-negative linear combination of given functions $\{b_1, \dots, b_m\}$, then MinSC is NP-hard to approximate within any factor smaller than $\max_{j \in \{1,\dots,m\}} \rho_{b_j}$, i.e., smaller than that produced by the worst function.

[1]It is worth observing that each class of equilibria appearing in this list is a superset of the previous [53]. Therefore, since pure Nash equilibria are guaranteed to exist even in congestion games where taxes are used (they are, in fact, potential games), all mentioned equilibrium's sets are non-empty and thus the notion of *price of anarchy* introduced in (2) well defined.
[2]$f : \mathbb{N} \to \mathbb{R}_{>0}$ is semi-convex if $xf(x)$ is convex, i.e., $(x+1)f(x+1) - xf(x) \geq xf(x) - (x-1)f(x-1)$ $\forall x \in \mathbb{N}, x \geq 2$.
[3]Throughout the manuscript, $\mathrm{Poi}(x)$ denotes a Poisson distribution with parameter x.

As a special case, we obtain hardness results for the thoroughly studied class of polynomial congestion games with maximum degree d, corresponding to non-negative linear combinations of $\{1, x, \ldots, x^d\}$. In this case, the highest degree monomial x^d determines the worst factor, which reduces to the $(d + 1)$'st Bell number (see Section 3.4), as summarized in the next statement.

Corollary 1. *In congestion games with resource costs obtained by non-negative combinations of* $\{1, \ldots, x^d\}$, `MinSC` *is* NP*-hard to approximate within a factor smaller than the* $(d + 1)$*'st Bell number.*

Taxes achieve optimal approximation. Our second contribution provides a technique to efficiently design taxation mechanisms matching the hardness result. The following statement is a succinct variant of that in Theorem 3 where we also give an expression for the taxation mechanism.

Theorem 2. *Consider the class of congestion games with resource costs obtained by non-negative combinations of functions* $b_1, \ldots, b_m$*, each positive, non-decreasing, semi-convex in* $\mathbb{N}$*. For any given* $\varepsilon > 0$*, it is possible to efficiently compute a taxation mechanism whose price of anarchy is upper bounded by* $\max_j \rho_{b_j} + \varepsilon$*. The result holds for pure/mixed Nash and correlated/coarse correlated equilibria.*

The extension to coarse correlated equilibria is significant as it gives performance bounds that apply not only to Nash equilibria, but also whenever players revise their action and achieve low regret. This imposes much weaker assumptions on both the game and its participants' behaviour [53]. Whilst the theorem applies to a broad class of resource costs, when specialized to polynomial congestion games of degree d, it allows to efficiently design taxation mechanisms whose price of anarchy equals the $(d + 1)$'st Bell number plus ε, for any arbitrarily small choice of $\varepsilon > 0$.

Tight polynomial algorithms. Since the result in Theorem 2 holds also for correlated/coarse correlated equilibria, we can leverage existing polynomial time algorithms to compute such equilibria, and inherit an approximation ratio matching the corresponding price of anarchy. The following statement summarizes this result, while the ensuing discussion provides two possible approaches to do so. We remark that Corollary 2 is a direct consequence of Theorem 2 and polynomial computability of correlated equilibria [34]. The fact that the resulting approximation factor we obtain always matches or strictly improves upon that of [41] is shown in Section 4.1.

Corollary 2. *Consider congestion games with resource costs obtained by non-negative combinations of functions* $b_1, \ldots, b_m$*, each positive, non-decreasing, semi-convex. For any* $\varepsilon > 0$*, there exists a polynomial time algorithm to compute an allocation with a cost lower than* $(\max_j \rho_{b_j} + \varepsilon) \cdot \min_{a \in \mathcal{A}} \mathrm{SC}(a)$*.*

The approximation ratio presented in Corollary 2 can be achieved, for example, as follows. Given a desired tolerance $\varepsilon > 0$, we design a taxation mechanisms ensuring a price of anarchy of $\max_j \rho_{b_j} + \varepsilon/2$, which can be done in polynomial time thanks to Theorem 2. We use such taxation mechanisms, and compute an exact correlated equilibrium in polynomial time leveraging the result of Xin Jiang and Leyton-Brown [34], who propose a variation of Papadimitriou and Roughgarden's Ellipsoid Against Hope algorithm [47]. Remarkably, [34] guarantees that the resulting correlated equilibrium has polynomial-size support, i.e., it places non-zero probability only over a polynomial number of pure strategy profiles. Hence, we compute a correlated equilibrium and enumerate all pure strategy profiles in its support, identifying with a^* that with lowest cost. Since the price of anarchy bounds of Theorem 2 hold for correlated equilibria, the pure strategy profile a^* inherits a matching (or better) approximation ratio.

Alternatively, one can employ the same taxation mechanism as in the above, and let players simultaneously revise their action for t rounds employing a no-regret algorithm. Well-known families of such algorithms include multiplicative-weights, and Follow the Perturbed/Regularized Leader whereby the average regret decays to an arbitrary ε in a polynomial number of rounds, see [14, 36].

We then focus on the pure strategy profile a_t^* with the lowest social cost encountered during the t rounds of any such algorithm t. Owing to [53, Thm 3.3], its approximation ratio is upper bounded by the corresponding price of anarchy plus an error term that goes to zero with the same rate of the average regret. Waiting for polynomially many rounds suffices to reduce the error as desired.

1.3 Further related work

As our work provides tight computational lower bounds, optimal taxation mechanisms, and polynomial time algorithms with the best possible approximation, we review the relevant literature connected with these three areas in the ensuing paragraphs.

Computational lower bounds. The study of computational lower bounds for minimizing the social cost in congestion games has been pioneered by Meyers and Schulz [43], though remarkable precursors include Chakrabarty et al. [15], as well as Blumrosen and Dobzinski [11] who considered a notably different model whereby player-specific cost functions are utilized. Relative to the classical model of congestion games with convex non-decreasing resource costs, Meyers and Schulz show that minimizing the social cost is strongly NP-hard, and it is hard to approximate within any finite factor, unless P=NP. Identical results are shown for network congestion games. Whilst this might feel as a contradiction of our results, it is worth noting that their analysis allows for resource costs to be adversarily selected amongst any convex non-decreasing function. On the contrary, our result can be thought of as a refined version of theirs, whereby the computational lower bound we derive is parametrized by the class of admissible resource costs. Naturally, we recover their inapproximability result when resource costs can be arbitrarily selected amongst convex non-decreasing functions.[4]

Motivated by the possibility to translate computational lower bounds to lower bounds on the price of anarchy, Roughgarden [52] also studied this problem. Relative to polynomial resource costs of maximum degree d and non-negative coefficients, he showed that minimizing the social cost is inapproximable within any factor smaller than $(\beta d)^{d/2}$, for some constant $\beta > 0$. In this setting, even without taxes, equilibria with much better performances are guaranteed to exist. In particular, the *price of stability* (measuring the quality of the best-performing equilibrium) is known to grow only linearly with the degree d [18]. While coordinating the players to one such *good* equilibrium is highly desirable, our hardness result implies that this cannot be achieved in polynomial time.

Spurred by the inapproximability results of Meyers and Schulz, and Roughgarden, a number of works have focused on restricting the allowable class of problems: Del Pia et al. [48] consider totally unimodular congestion games (and generalizations thereof) and show NP-hardness for the asymmetric case; Castiglioni et al. [13] show NP-hardness even in singleton congestion games with affine resource costs. Similar questions have been explored for online versions of the problem by Klimm et. al. [38], and for congestion games with positive externalities by de Keijzer and Schäfer [21].

Taxation mechanisms. Different approaches, such as coordination mechanisms [20], Stackelberg strategies [26], taxation mechanisms [12], signalling [7], cost-sharing strategies [28], and many more, have been proposed to cope with the performance degradation associated to selfish decision making. Amongst them, taxation mechanism have attracted significant attention thanks to their ability to indirectly influence the resulting system performance. While the study of taxation mechanisms in road-traffic networks was initiated by Pigou [49], who utilized a continuous flow model, the design of taxation mechanisms in (atomic) congestion games was pioneered much more recently by Caragiannis et al. [12]. In this respect, [12] and many of the subsequent works, build on the solid theoretical ground developed in the years subsequent to the definition of the price of

[4]For example MinSC is NP-hard to approximate within any finite factor when $b(x) = e^x$. This is an immediate consequence of Theorem 1 and the fact that $\rho_b = \infty$ for exponentially increasing resource costs.

anarchy [39], including exact knowledge of the price of anarchy in congestion games with linear [4, 19] and polynomial [1] resource costs, the advent of the smoothness framework [53], as well as primal dual approaches [8, 16].

While these results provide us with a strong theory to quantify the price of anarchy, prior to this work, the design of optimal taxation mechanism (i.e., taxation mechanisms minimizing the price of anarchy) has been an open question even when restricting attention to linear resource costs. Most notably, Caragiannis et al. [12] considers linear congestion games and designs taxation mechanisms achieving a price of anarchy of 2 for mixed Nash equilibria. More recently, Bilò and Vinci [9] extend their results to polynomial congestion games of degree d achieving a price of anarchy (for coarse correlated equilibria) equal to the $(d + 1)$'st Bell number. Our work resolves the problem of designing optimal taxation mechanisms for congestion games with semi-convex non-decreasing resource costs, and, as a special case, shows that the mechanisms proposed for linear [12] and polynomial [9] resource costs are optimal, as conjectured by the authors.

Perhaps closest in spirit to our work, is the recent result by Paccagnan et al. [44], whereby the authors leverage a tractable linear programming formulation to design optimal taxation mechanisms that utilize solely *local* information. Naturally, the corresponding values of the optimal price of anarchy they achieve are inferior to ours (here, we design optimal taxation mechanisms without any restrictions on what type of information we use), though the efficiency values derived in [44] are remarkably close to the optimal values obtained here. For example, for affine (resp. quadratic) congestion games, they achieve an optimal price of anarchy of 2.012 (resp. 5.101), to be compared with a value of 2 (resp. 5). This suggests that restricting the attention to taxation mechanisms utilizing solely local information is sufficient to match almost exactly the performance of the best polynomial time algorithm.

We conclude observing that similar questions have been considered for variants of the classical setup studied here. For example, [27] focuses on symmetric network congestion games, [29, 31, 33] focus on taxing a subset of the resources. Finally, we remark that optimal taxation mechanisms can be easily derived for *non-atomic* congestion games, where players have only an infinitesimal impact on the congestion. In this setting, it is known that marginal cost taxes incentivize optimal behaviour [49]. On the contrary, in the atomic regime the same marginal cost taxes do not improve - and instead significantly deteriorate - the resulting system efficiency [44].

Approximation algorithms. A number of polynomial time algorithms have been proposed for approximating the minimum social cost in congestion games and their network counterpart as discussed in [2, 30, 41] and references therein. The best known approximation is due to Makarychev and Sviridenko [41] who use randomization to round the solution of a natural linear programming relaxation. They provide a general expression for the resulting approximation factor as a function of the allowable resource costs. Related works have also considered modifications of the classical setup: Harks et al. [30] provide approximation algorithms for polymatroid congestion games, whereas Kumar et al. [40] considers scheduling problems on unrelated machines. While the result of [41] hold for the more general class of optmization problems with "diseconomy of scale", the approximation ratios we obtain here always match or strictly improve upon that of [41].

2 TECHNIQUES AND HIGH LEVEL IDEAS

Two technical contributions are at the core of this manuscript: a tight inapproximability result for minimizing the social cost, and a methodology to design taxation mechanisms whose efficiency matches the hardness factor. Armed with these results, the design of efficient algorithms with optimal approximation falls from the polynomial computability of correlated equilibria, as anticipated.

NP*-hardness of approximation.* Our computational lower bounds are shown reducing the problem of minimizing the social cost in congestion games from the classical *label cover* problem [25] (more specifically from the gap variant of this problem). A central tool we utilize in our reduction is a gadget called *partitioning system*, extending that originally defined by Feige in [25]. Conceptually, the expression of the hardness factor (3) falls from the very definition of such object, which is duplicated multiple times and properly arranged in our construction. Partitioning systems have been recently studied also by Dudycz et al. [23] and Barman et al. [6] in the context of approval voting, and for a generalization of the maximum coverage problem. We are not aware of applications of these gadgets to cost minimization problems. Finally, we observe (at least informally) that a sightly better approximation factor and matching hardness result can also be obtained if the number of agents N is finite and taken into account, whereby the Poisson distribution appearing in (3) is replaced by the binomial distribution $\mathrm{Bin}(N, x/N)$.

Taxes achieve optimal approximation. We derive taxes matching the hardness factor leveraging two chief ingredients: a parametrized class of taxation mechanisms satisfying a key recursion, and a suitably defined convex optimization program. The convex optimization problem we consider corresponds to a modification of the original `MinSC`, whereby we relax the integrality constraints and replace the cost $x\ell_r(x)$ produced by each resource with $\mathbb{E}_{P\sim\mathrm{Poi}(x)}[P\ell_r(P)]$, see (9). The solution vector of this program (which is provably convex) is used as set of parameters for the class of mechanisms previously defined. The performance bound on the price of anarchy is finally shown through a smoothness-like approach, where we leverage both the expression of the mechanisms and the solution vector of the convex program. We observe that, for the case of polynomial resource costs, convex programs have been used before to design suitable taxation mechanisms [9]. Our result, instead, applies to general semi-convex non-decreasing resource costs.

3 NP-HARDNESS OF APPROXIMATION

In this section we prove Theorem 1, i.e., we show that approximating the minimum social cost below the factor ρ_b defined in (3) is NP-hard already for congestion games where all resource costs are identical to b. The proof is based on a reduction from `GapLabelCover`, where we make use of a generalization of Feige's partitioning system [25]. We proceed as follows: In Section 3.1 we introduce `GapLabelCover` and, independently, the partitioning system. In Section 3.2 we present the reduction and in Section 3.3 prove Theorem 1. Section 3.4 shows that $\max_j \rho_{b_j}$ reduces to the $(d{+}1)$'st Bell number when resource costs are polynomials of maximum degree d, as claimed in Corollary 1. Throughout, we use $[m]$ to denote the set $\{1,\dots,m\}$, and $\mathbb{N}_0$ for the set of natural numbers including zero.

3.1 Background tools

We start by introducing `GapLabelCover`, a commonly utilized NP-hard problem to obtain tight inapproximability results. We employ the weak-value formulation of the problem, implicitly used in Feige [25] and also defined in Dudycz et al. [23].

A `LabelCover` instance is described by a tuple $(L, R, E, h, [\alpha], [\beta], \{\pi_e\}_{e\in E})$, where

- L and R are sets of left and right vertices of a bi-regular bipartite graph with edge set E and right degree h (i.e., the degree of all vertices in R equals h),
- $[\alpha]$ and $[\beta]$ represent left and right alphabets, and
- for every edge $e \in E$, a constraint function $\pi_e : [\alpha] \to [\beta]$ maps left labels to right labels.

Given a left labeling $\mathcal{L} : L \to [\alpha]$, i.e., a map that associates a left label to every left vertex, we say that a right vertex $u \in R$ is

- *strongly satisfied* if for every pair of neighbors $v, v' \in L$ of u it is $\pi_{(v,u)}(\mathcal{L}(v)) = \pi_{(v',u)}(\mathcal{L}(v'))$;

- *weakly satisfied* if there exist two distinct neighbors $v, v' \in L$ of u, s.t. $\pi_{(v,u)}(\mathcal{L}(v)) = \pi_{(v',u)}(\mathcal{L}(v'))$.

For any $\delta > 0, h \in \mathbb{N}$ let $\texttt{GapLabelCover}(\delta, h)$ denote the following problem: Given a LabelCover instance $(L, R, E, h, [\alpha], [\beta], \{\pi_e\}_{e\in E})$, distinguish between

YES: there exists a labeling that strongly satisfies all right vertices;

NO: no labeling weakly satisfies more than a fraction δ of the right vertices.

When the right alphabet is sufficiently large, GapLabelCover is NP-hard, as recalled next.

Proposition 1 ([23, 25]). *$\forall\delta > 0$, $h \in \mathbb{N}$, $h \geq 2$, and β sufficiently large (depending on δ, h), $\texttt{GapLabelCover}(\delta, h)$ is NP-hard.*

Similarly to Barman et al. [6] and Dudycz et al. [23], we generalise a combinatorial object introduced by Feige [25], called *partitioning system*, which we also equip with a cost function. Given a ground set of elements $[n]$, integers β, h, k such that $kn/h \in \mathbb{N}$, $\beta \geq h \geq k$, a cost functions $c : \mathbb{N} \to \mathbb{R}_{>0}$, and $\eta > 0$, a partitioning system with parameters (n, β, h, k, η) is a collection of partitions $\mathcal{P}_1, \ldots, \mathcal{P}_\beta$ of $[n]$ such that:

P1) Every partition $\mathcal{P}_j$ is a collection of subsets $P_{j,1}, \ldots, P_{j,h} \subseteq [n]$ each with kn/h elements and such that each element from $[n]$ is selected by k sets in $P_{j,1}, \ldots, P_{j,h}$. Observe that, for any $\mathcal{P}_j$ we have $|\mathcal{P}_j| = h$, and the above implies

$$\sum_{r\in[n]} c(|\mathcal{P}_j|_r) = c(k)n,$$

where $|\mathcal{P}_j|_r$ denotes the number of sets in the collection $\mathcal{P}_j$ to which element r belongs, and we extended the definition of the function c to include $c(0) = 0$, to ease the notation.

P2) for any $B \subseteq [\beta]$ with $|B| = h$ and for any function $i : [\beta] \to [h]$, let $Q = \{P_{j,i(j)}, j \in B\}$. It is

$$\sum_{r\in[n]} c(|Q|_r) \geq \left(\mathbb{E}_{X\sim\mathrm{Bin}(h,k/h)}[c(X)] - \eta\right) n.$$

To gain some intuition on properties P1 and P2, we provide a graphical representation of a partitioning system in Fig. 1, whereby each box contains kn/h elements from the ground set $[n]$. Property 1 asserts that every time we select and entire row, we are guaranteed to cover every element in $[n]$ precisely k times, Property 2 asserts that every time we select one and only one set from each row, for a total of h rows, many sets cover the same elements so that the resulting cost is high.

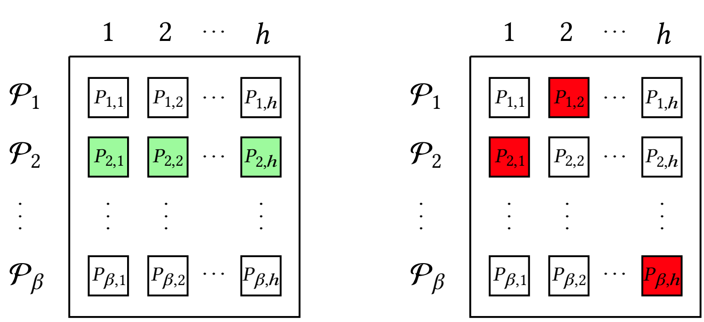

Fig. 1. A partitioning system with parameters (n, β, h, k, η). Each box contains kn/h elements from the ground set $[n]$. Property P1 ensures that selecting an entire row results in low cost (left, in green). Property P2 ensures that selecting one and only one box per each row, for a total of h rows, results in a high cost (right, in red).

At this stage, we recall that the probability mass function of the binomial distribution $\mathrm{Bin}(h, k/h)$ converges pointwise for fixed $k \geq h$ to the probability mass function of the Poisson distribution

Poi(k) as h grows large [24]. Hence, we informally observe that, when $\rho_b < \infty$,

$$\frac{\sum_{r\in Q} c(|Q|_r)}{\sum_{r\in\mathcal{P}_j} c(|\mathcal{P}_j|_r)} = \frac{\mathbb{E}_{X\sim\mathrm{Bin}(h,k/h)}[c(X)] - \eta}{c(k)} \xrightarrow{h\to\infty} \frac{\mathbb{E}_{X\sim\mathrm{Poi}(k)}[c(X)] - \eta}{c(k)},$$

see Lemma 3 in Appendix A.1.1 for a proof. If we choose $c(x) = xb(x)$, let $h \to \infty$ and consider the worst case over k, this ratio precisely matches the inapproximability result we aim to derive (cfr. the previous expression and ρ_b in (3)). We are thus left to piece these elements together in the ensuing section. Before doing so, we remark that partitioning systems do exist for every choice of $\eta > 0$ as long as n is taken sufficiently large as stated in the next proposition. Its proof follows the same approach of that in [6], and is included in Appendix A.1.2 for completeness. We remark that, when used in the upcoming reduction, we will be able to compute a partitioning systems in a time that is independent on the size of the instance we reduce from.

Proposition 2. *Let $c : \mathbb{N} \to \mathbb{R}_{>0}$ non-decreasing be given. For every choice of $\beta \geq h \geq k$ integers with $kn/h \in \mathbb{N}$, $\eta \in (0, 1)$, and $n \geq \frac{c(k)^2}{2\eta^2}[\log(10)+\beta\log(h+1)]$ a partitioning system with parameters (n, β, h, k, η) and cost function c exists. It can be found in time depending solely on h, n and β.*

3.2 Reduction

We first provide the reduction and prove the hardness result in the case when $\rho_b < \infty$. We consider the case $\rho_b = \infty$ separately at the end of Section 3.3. Starting from the resource cost function b and a fixed $\varepsilon > 0$, we will first construct a partitioning system with parameters (n, β, h, k, η). We then reduce an instance of `LabelCover` $C = (L, R, E, h, [\alpha], [\beta], \{\pi_e\}_{e\in E})$ to an instance of congestion game $G = (N, \mathcal{R}, \{\mathcal{A}_i\}_{i=1}^N, \{\ell_r\}_{r\in\mathcal{R}})$ with identical resource cost. The idea is to define G by creating a copy of the partitioning system for every right vertex and use that to define the players' allocations $\mathcal{A}_i$.

Formally, given b and $\varepsilon > 0$, let k be the maximizer[5] of (3), c the cost function defined from b as in $c(x) = xb(x)$ for all $x \in \mathbb{N}$ and $c(0) = 0$. We choose $\eta < \min\{\varepsilon c(k)/4, 1\}$, $\delta \leq \varepsilon/(2\rho_b)$, and $h \geq k$ such that $|\mathbb{E}_{X\sim\mathrm{Bin}(h,k/h)}[Xb(X)] - \mathbb{E}_{X\sim\mathrm{Poi}(k)}[Xb(X)]| \leq \varepsilon c(k)/4$ (which exists thanks to the convergence result in Lemma 3 in Appendix A.1.1). Consequently, we fix β large enough to ensure hardness in Proposition 1, and choose n so that $kn/h \in \mathbb{N}$ and n is large enough to have existence of the partitioning system from Proposition 2. Owing to the same proposition, since h and n, β are now fixed, a partitioning system can be computed in time independent of the size of the `LabelCover` instance.

Now we take an instance of `LabelCover` $C = (L, R, E, h, [\alpha], [\beta], \{\pi_e\}_{e\in E})$ where h, β are defined above and thus Proposition 1 (NP-hardness) holds. For each right vertex $u \in R$ we use the local partitioning system with parameters (n, β, h, k, η). We refer to the resources in the partitioning system corresponding to the right vertex $u \in R$ with $\{1^u, \ldots, n^u\}$. Similarly we use $\mathcal{P}_j^u = \{P_{j,1}^u, \ldots, P_{j,h}^u\}$ for the local partitions. The congestion game $G = (N, \mathcal{R}, \{\mathcal{A}_i\}_{i=1}^N, \{\ell_r\}_{r\in\mathcal{R}})$ is defined as follows

- each left vertex corresponds to a player, so that the number of players is $N = |L|$;
- the ground set of resources is the union of the resources introduced by each local partitioning system on every right vertex, i.e., $\mathcal{R} = \cup_{u\in R}\{1^u, \ldots, n^u\}$;
- each resource cost is equal to b, i.e., $\ell_r(x) = b(x)$ for all $r \in \mathcal{R}$, $x \in \mathbb{N}$;
- as each left vertex $v \in L$ corresponds to one and only one player $i \in [N]$, we refer to a left vertex as to $i \in [N]$ instead of as $v \in L$ to ease the notation. For player $i \in [N]$ we construct each pure strategy $a_i \in \mathcal{A}_i$ as follows. We let the left vertex i select a label $l \in [\alpha]$, and correspondingly take the union over all right vertices $u \in \mathcal{N}(i)$ neighbouring with i, of the

[5]For ease of exposition, we show the result when the supremum is attained at some value $k \in \mathbb{N}$. If this is not the case, then the supremum must be achieved at $k \to \infty$. One then fixes $k \in \mathbb{N}$ and proceeds with the reasoning as is. This will give rise to an additional error term $\nu(k)$ with $\lim_{k\to\infty} \nu(k) = 0$ in the ensuing Equation (4), where the right hand side will be replaced by $(\rho_b + \nu(k))c(k)$. Nevertheless one can select k sufficiently large and control such error to a desired accuracy.

resources belonging to the block $P^u_{j,i}$, where $j = \pi_{(i,u)}(l)$. Repeating over all left labels we obtain the strategy set $\mathcal{A}_i$. Formally

$$\mathcal{A}_i = \left\{\cup_{u\in\mathcal{N}(i)} P^u_{j,i}, \quad \text{where} \quad j = \pi_{(i,u)}(l), \quad \forall l \in [\alpha]\right\}.$$

The following figure exemplifies the construction. We conclude remarking that the above procedure implicitly defines a map associating a profile of left labels $(l_1, \ldots, l_N)$ (one per each left vertex) to an allocation $(a_1, \ldots, a_N) \in \mathcal{A}$, and that spanning through all possible choices of $(l_1, \ldots, l_N)$ produces all possible allocations in $\mathcal{A}$. This observation will be useful in proving the hardness result.

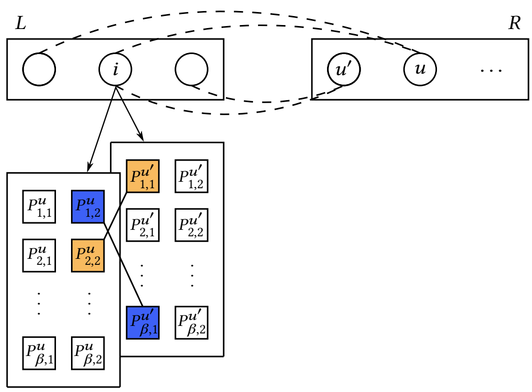

Fig. 2. Given a label cover instance $C = (L, R, E, h, [\alpha], [\beta], \{\pi_e\}_{e\in E})$, our reduction associates every left vertex in L to a player in the game G. Here we exemplify how the action set $\mathcal{A}_i$ is generated for player $i \in L$. To ease the presentation, we consider a left alphabet of size 2 and use blue and orange to identify the left labels. Since i has two right neighbours, u and u', we construct two partitioning systems with ground set of resources $\{1^u, \ldots, n^u\}$ and $\{1^{u'}, \ldots, n^{u'}\}$. Constraints $\pi_{(i,u)}(\texttt{blue}) = 1$, $\pi_{(i,u)}(\texttt{orange}) = 2$ and $\pi_{(i,u')}(\texttt{blue}) = \beta$, $\pi_{(i,u')}(\texttt{orange}) = 1$ are given, and we represent them graphically with the fact that on the left partitioning system, the label blue (resp. orange) is associated to the block in row $1 = \pi_{(i,u)}(\texttt{blue})$ (resp. row $2 = \pi_{(i,u)}(\texttt{orange})$). Similarly for the right partitioning system using $\pi_{(i,u')}$ to determine the row. The set $\mathcal{A}_i$ is readily constructed as $\mathcal{A}_i = \{P^u_{1,2} \cup P^{u'}_{\beta,1}, P^u_{2,2} \cup P^{u'}_{1,1}\}$, where the first (resp. second) allocation corresponds to a blue (resp. orange) left label.

3.3 Proof of the result

As anticipated, we first prove the result for the case of $\rho_b < \infty$. At the end of this section, we turn the attention to $\rho_b = \infty$. For any given instance of LabelCover $C = (L, R, E, h, [\alpha], [\beta], \{\pi_e\}_{e\in E})$, resource cost b, and $\varepsilon > 0$, we consider an instance of congestion game $G = (N, \mathcal{R}, \{\mathcal{A}_i\}_{i=1}^N, \{\ell_r\}_{r\in\mathcal{R}})$ constructed as in the previous section. We will now show that

- *completeness (Section 3.3.1)*: If the instance C is a YES, then $\min_{a\in\mathcal{A}} SC(a) \leq n|R|c(k)$,
- *soundness (Section 3.3.2)*: If the instance C is a NO, then $\min_{a\in\mathcal{A}} SC(a) > (\rho_b - \varepsilon)n|R|c(k)$.

An algorithm solving MinSC with an approximation ratio smaller than $\rho_b - \varepsilon$ will be able to distinguish between YES/NO of an NP-hard promise problem. This will conclude the proof.

3.3.1 Completeness. We intend to show that if C is a YES, then $\min_{a\in\mathcal{A}} \mathrm{SC}(a) \leq n|R|c(k)$. This follows readily. In fact, if C is a YES, there exists a labeling that strongly satisfies all right vertices. This implies that there exists an allocation $a^* \in \mathcal{A}$ whereby, for any given right vertex, all neighbouring left vertices (players) have selected blocks belonging to an entire row of the corresponding partitioning system. Thanks to property P1 the cost of the allocation a^* on every local partition is equal to $nc(k)$. Since the total cost is additive over the local partitions, we obtain the result

$$SC(a^*) = n|R|c(k) \implies \min_{a\in\mathcal{A}} SC(a) \leq n|R|c(k).$$

3.3.2 Soundness. We intend to show that if C is a NO, then $\min_{a\in\mathcal{A}} \mathrm{SC}(a) \geq (\rho_b - \varepsilon)n|R|c(k)$ which is equivalent to showing $\mathrm{SC}(a) \geq (\rho_b - \varepsilon)n|R|c(k)$ for all $a \in \mathcal{A}$. Towards this goal, we build upon the last observation presented in Section 3.2, i.e., the fact that our construction associates each profile of left labels to an allocation, and that spanning through all possible choices of $(l_1, \dots, l_N)$ produces all possible allocations in $\mathcal{A}$. Hence, it suffices to prove the desires property by considering all possible combinations of profiles $(l_1, \dots, l_N)$ and the corresponding induced cost, instead of considering all $a \in \mathcal{A}$. Since C is a NO instance, for any possible choice of $(l_1, \dots, l_N)$, no more than δ fraction of the right vertices are weakly satisfied. Owing to property P2, each partitioning system corresponding to a non weakly satisfied right vertex has a cost larger than $\left(\mathbb{E}_{X\sim\mathrm{Bin}(h,k/h)}[c(X)] - \eta\right) n$. Thus, since at least $(1-\delta)|R|$ right vertices are not weakly satisfied, it is

$$\mathrm{SC}(a) \geq \left(\mathbb{E}_{X\sim\mathrm{Bin}(h,k/h)}[c(X)] - \eta\right)(1-\delta)n|R|, \qquad \forall a \in \mathcal{A}.$$

We conclude with some cosmetic manipulation. In particular, we recall that the binomial distribution converges to the Poisson distribution when the number of trials grows large and the success probability of each trial goes to zero [24]. In our settings, this corresponds to the fact that the probability mass function of $\mathrm{Bin}(h, k/h)$ converges pointwise for fixed k to that of $\mathrm{Poi}(k)$ as $h \to \infty$.

Since $\rho_b < \infty$, we observe that

$$\lim_{h\to\infty} \mathbb{E}_{X\sim\mathrm{Bin}(h,k/h)}[c(X)] = \mathbb{E}_{X\sim\mathrm{Poi}(k)}[c(X)] = \rho_b c(k), \tag{4}$$

where equality holds thanks to Lemma 3 in Appendix A.1.1. This implies the existence of a function $\theta(h)$ with $\theta(h) \to 0$ as $h \to \infty$ allowing to control the error, and for which $\mathbb{E}_{X\sim\mathrm{Bin}(h,k/h)}[c(X)] \geq \rho_b c(k) - \theta(h)$. In other words the LHS can be made arbitrarily close to $\rho_b c(k)$ by selecting h sufficiently large. Thanks to the choice of h, it is $\theta(h) \leq \varepsilon c(k)/4$. Hence,

$$\begin{aligned}
\mathrm{SC}(a) &\geq (\rho_b c(k) - \theta(h) - \eta)(1-\delta)n|R| \\
&= \left[\rho_b - \frac{\theta(h)}{c(k)} - \frac{\eta}{c(k)} - \left(\rho_b - \frac{\theta(h)}{c(k)} - \frac{\eta}{c(k)}\right)\delta\right] n|R|c(k) \\
&\geq \left[\rho_b - \frac{\theta(h)}{c(k)} - \frac{\eta}{c(k)} - \rho_b\delta\right] n|R|c(k) \\
&> \left[\rho_b - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} - \frac{\varepsilon}{2}\right] n|R|c(k) = (\rho_b - \varepsilon)n|R|c_k.
\end{aligned} \tag{5}$$

The last inequality holds by the choice of parameters, ensuring that $\frac{\theta(h)}{c(k)} \leq \frac{\varepsilon}{4}$, $\frac{\eta}{c(k)} < \frac{\varepsilon}{4}$, $\rho_b\delta \leq \frac{\varepsilon}{2}$.

Case of $\rho_b = \infty$. As anticipated, we treat the case of unbounded ρ_b separately. Towards this goal, we follow the same reduction of section Section 3.2, with minor modification on the choice of parameters. We replace ρ_b with a fixed (and conceptually large) constant M. Since $\rho_b = \infty$, we note that $\mathbb{E}_{X\sim\mathrm{Poi}(k)}[c(X)]/c(k)$ is unbounded at some k (possibly infinity). Since the probability mass functions of $\mathrm{Bin}(h, k/h)$ and $\mathrm{Poi}(k)$ converge, we can choose the pair h and k so that $\mathbb{E}_{X\sim\mathrm{Bin}(h,k/h)}[c(X)] \geq Mc(k)$. Finally, we set $\delta \leq \varepsilon/(2M)$. One then follows the same proof as in the case of bounded ρ_b, whereby (5) is replaced with $\mathrm{SC}(a) \geq (M - \eta/c(k))(1-\delta)n|R|c(k) \geq (M - \frac{\eta}{c(k)} - M\delta)n|R|c(k) \geq (M - \varepsilon/4 - \varepsilon/2)n|R|c(k) > (M - \varepsilon)n|R|c(k)$. Since M can be taken to be arbitrarily large, the problem is NP-hard to approximate within any finite ratio. □

3.4 Hardness factor ρ_b for polynomial resource cost

Corollary 1 claims that, when resource costs are obtained by non-negative combinations of polynomials of maximum degree $d > 0$, MinSC is hard to approximate within any factor smaller than the $(d+1)$'st Bell number. Note that this is a direct consequence of Theorem 1, which we can apply since for $d \geq 0$, each monomial x^d is positive, non-decreasing, semi-convex for $x \in \mathbb{N}$. We are only left to calculate the value of ρ_b for each x^d. We begin with the case of $b(x) = x^d$ and $d \in \mathbb{N}_0$ as the derivation is straightforward and illustrative. Using the definition of ρ_b in (3) it is

$$\rho_b = \sup_{x \in \mathbb{N}} \frac{\mathbb{E}_{P \sim \mathrm{Poi}(x)}[P^{d+1}]}{x^{d+1}} = \sup_{x \in \mathbb{N}} \sum_{i=0}^{d+1} x^{i-(d+1)} \left\{ \begin{matrix} d+1 \\ i \end{matrix} \right\} = \sum_{i=0}^{d+1} \left\{ \begin{matrix} d+1 \\ i \end{matrix} \right\} = \mathcal{B}(d+1).$$

In the second equality we used the fact that the $(d+1)$'st moment of the Poisson distribution $\mathrm{Poi}(x)$ equals $\sum_{i=0}^{d+1} x^i \left\{ \begin{smallmatrix} d+1 \\ i \end{smallmatrix} \right\}$, where $\left\{ \begin{smallmatrix} d+1 \\ i \end{smallmatrix} \right\}$ is a Stirling number of the second kind [42, p. 63]. The third equality holds because each function $x^{i-(d+1)}$ is non-increasing, owing to $i \leq (d+1)$, and thus the supremum is attained at $x = 1$. The last one is due to the definition of the $(d+1)$'st Bell number, which we denote with $\mathcal{B}(d+1)$, see [42, Eq. 1.2].

One can repeat a similar reasoning also when $d \geq 0$ is not integer and show that the expression inside the supremum is non-increasing in $x \geq 1$, e.g., by computing its derivatives. If follows that the supremum is attained at $x = 1$, and the definition of ρ_b gives

$$\rho_b = \frac{1}{e} \sum_{i=0}^{\infty} \frac{i^{d+1}}{i!}. \tag{6}$$

The latter expression is sometimes referred to as the fractional Bell number. Note that using Dobiński's formula [42, Eq. 1.25], one recovers the Bell number $\mathcal{B}(d+1)$ when $d \in \mathbb{N}_0$. Observing that the expression in (6) is increasing in d, one concludes that, when multiple monomials are utilized, the value $\max_j \rho_{b_j}$ is attained by the monomial with highest degree.

4 TAXES ACHIEVE OPTIMAL APPROXIMATION

In this section we show how to compute a taxation mechanism whose price of anarchy matches the hardness factor. Since taxation mechanisms can be utilized to derive polynomial time algorithms with an approximation factor matching the price of anarchy (Section 1.2), in the ensuing Section 4.1 we compare the optimal price of anarchy with the best known polynomial approximation of [41].

We start by introducing a parameterised family of taxation mechanisms for which we will later provide (efficiently computable) parameters that achieve the desired result. Our taxation mechanisms take as input a congestion game G, where all resource costs can be obtained by a non-negative combination of functions $b_1, \dots, b_m$, each positive, non-decreasing, semi-convex in $\mathbb{N}$. For each $b_j : \mathbb{N} \to \mathbb{R}_{>0}$ given, we extend its definition to $b_j : \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ by setting $b(0) = 0$. This is without loss of generality and merely needed to ease the notation. Let $p_j : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be defined as

$$p_j(v) = \mathbb{E}_{P \sim \mathrm{Poi}(v)}[P b_j(P)] = \left(\sum_{i=0}^{\infty} i b_j(i) \frac{v^i}{i!} \right) e^{-v}. \tag{7}$$

Definition 1 (Parameterised Taxation Mechanisms). *Given a parameter vector $(v_r)_{r \in \mathcal{R}}$ with $v_r \in \mathbb{R}_{\geq 0}$ and a set of resources $\mathcal{R}$ with cost function $\ell_r(x) = \sum_{j=1}^{m} \alpha_j^r b_j(x)$ for $r \in \mathcal{R}$, define a parameterised taxation function $\tau_r : \mathbb{N}_0 \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, with $\tau_r(x) = \tau_r(x, v_r) = \sum_{j=1}^{m} \alpha_j^r [f_j(x, v_r) -$*

$b_j(x)]$ *where* $f_j(x,0)=b_j(x)$, $f_j(0,v)=0$, *and*

$$f_j(x,v)=\frac{(x-1)!}{v^x}\sum_{i=0}^{x-1}\frac{p_j(v)-ib_j(i)}{i!}v^i,\qquad x\in\mathbb{N}_0, v\in\mathbb{R}_{>0}. \tag{8}$$

The following lemmas provide three important properties of the above taxation mechanisms. Lemma 1 ensures that taxes are non-negative and that modified resource costs after applying taxes are non-decreasing. Lemma 2 shows that the f_j's defined in Definition 1 satisfy a recursion which will be crucial later on. The proof of both lemmas can be found in Appendices A.2.2 and A.2.3.

Lemma 1. *For all* $v\in\mathbb{R}_{\geq 0}$, *the taxation mechanism introduced in Definition 1 satisfies: (a)* $f_j(x+1,v)\geq f_j(x,v)$ *for all* $x\in\mathbb{N}_0$ *and* $j\in[m]$, *and (b)* $\tau_r(x,v)\geq 0$ *for all* $x\in\mathbb{N}_0$ *and all* $r\in\mathcal{R}$.

Lemma 2. *For all* $j\in[m]$, $v\in\mathbb{R}_{\geq 0}$, *and* $x\in\mathbb{N}_0$, *we have* $xb_j(x)-xf_j(x,v)+vf_j(x+1,v)=p_j(v)$.

For ease of presentation, we let $s_i=|\mathcal{A}_i|$, and refer to the k-th action available to player i as to $a_{i,k}$, $k\in[s_i]$. We use $\Delta(s)$ to denote the s-th dimensional simplex. The taxation mechanism that optimises the price of anarchy makes use of the following convex program

$$\begin{aligned}
\min \quad & \sum_{r\in\mathcal{R}}\sum_{j=1}^{m}\alpha_j^r\, p_j(v_r) \\
\text{subject to} \quad & v_r=\sum_{i=1}^{N}\sum_{k\in[s_i]\,:\,r\in a_{i,k}} y_{i,k} \qquad \text{for all } r\in\mathcal{R}, \\
& y_i\in\Delta(s_i) \qquad\qquad\qquad\quad \text{for all } i\in[N].
\end{aligned} \tag{9}$$

We are now ready to state our main result of this section, which is an extension of Theorem 2. We state the result when $\max_j \rho_{b_j}<\infty$, else the problem is inapproximable as seen in Theorem 1.

Theorem 3. *Consider a congestion game G with resource costs obtained by non-negative combination of functions* $b_1,\dots,b_m$, *each positive, non-decreasing, semi-convex in* $\mathbb{N}$. *Let* $\max_j\rho_{b_j}<\infty$ *and denote with* $(\bar{y}_i)_{i\in[N]}$, $(\bar{v}_r)_{r\in\mathcal{R}}$ *a solution of the convex program* (9).

- *The taxation mechanism introduced in Definition 1 with parameter vector* $(\bar{v}_r)_{r\in\mathcal{R}}$ *has a price of anarchy no-higher than* $\max_j\rho_{b_j}$.
- *Moreover, for any choice of* $\varepsilon>0$ *one can design in polynomial time, through the approximate solution of* (9), *a taxation mechanism whose price of anarchy is no-higher than* $\max_j\rho_{b_j}+\varepsilon$.

Proof. Given a congestion game G, we consider the corresponding program (9). Let us verify that (9) is indeed convex. Since the constraints are linear and the objective function is a (non-negative) linear combination of univariate functions p_j it suffices to show that each $p_j(v)$ is convex in v. This holds true as p_j is defined in (7) as the expectation of a convex function over a Poisson distributed random variable. For completeness we provide a proof of this fact in Lemma 4 in Appendix A.2.1.

Let $(\bar{y}_i)_{i\in[N]}$, $(\bar{v}_r)_{r\in\mathcal{R}}$ be an optimal solution of the convex program (9) and consider the taxation mechanism from Definition 1 with parameter vector $(\bar{v}_r)_{r\in\mathcal{R}}$.

To complete the proof we will use a smoothness approach, with a minor modification: Instead of comparing an action profile a (e.g., an equilibrium allocation) with another action profile a' (e.g., an optimal allocation), we will compare an action profile a against a mixed profile $y=(y_1,\dots,y_N)\in\Delta(s_1)\times\cdots\times\Delta(s_N)$. Specifically, we will choose the mixed profile $\bar{y}$ solving (9), and show that

$$\sum_{i=1}^{N}\sum_{k=1}^{s_i}\bar{y}_{i,k}[\bar{C}_i(a)-\bar{C}_i(a'_{i,k},a_{-i})]\geq \mathrm{SC}(a)-\rho_b\mathrm{SC}(a^{\mathrm{opt}}),\qquad \forall a\in\mathcal{A}. \tag{10}$$

where $\bar{C}_i(a)$ denotes the modified cost function $\bar{C}_i(a) = \sum_{r\in a_i} \bar{\ell}_r(|a|_r) = \sum_{r\in a_i} \sum_{j=1}^m \alpha_j^r f_j(|a|_r, \bar{v}_r)$, and $a'_{i,k}$ the k-th action available to agent i. Once (10) is shown, the desired bound on the price of anarchy follows readily for pure Nash equilibria and more generally extends all the way to coarse correlated equilibria [53]. In the former case, substituting the profile a with any pure Nash equilibrium a^{ne}, and summing the equilibrium conditions $0 \geq \bar{C}_i(a^{\text{ne}}) - \bar{C}_i(a'_{i,k}, a^{\text{ne}}_{-i})$, one obtains $0 \geq \sum_{i=1}^N \sum_{k=1}^{s_i} \bar{y}_{i,k}[\bar{C}_i(a^{\text{ne}}) - \bar{C}_i(a'_{i,k}, a^{\text{ne}}_{-i})]$, so that

$$0 \geq \sum_{i=1}^{N} \sum_{k=1}^{s_i} \bar{y}_{i,k}[\bar{C}_i(a^{\text{ne}}) - \bar{C}_i(a'_{i,k}, a^{\text{ne}}_{-i})] \geq \text{SC}(a^{\text{ne}}) - \rho_b \text{SC}(a^{\text{opt}}),$$

from which one concludes. Since (10) holds for all $a \in \mathcal{A}$, the same bound on the price of anarchy holds for the much broader class of coarse correlated equilibrium. To see this, let σ be any coarse correlated equilibrium over $\mathcal{A}_1 \times \cdots \times \mathcal{A}_N$, and consider the expected value of (10). Due to linearity of the expectation and the definition of coarse correlated equilibria, we have $0 \geq \mathbb{E}_{a\sim\sigma}\left[\sum_{i=1}^N \sum_{k=1}^{s_i} \bar{y}_{i,k}[\bar{C}_i(a) - \bar{C}_i(a'_{i,k}, a_{-i})]\right]$, from which one concludes.

We are thus left to prove the smoothness condition (10). Given an optimal allocation $a^{\text{opt}} \in \arg\min_{a\in\mathcal{A}} \text{SC}(a)$, let $v_r^{\text{opt}} = |a^{\text{opt}}|_r$. Inequality (10) follows from

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{k=1}^{s_i} \bar{y}_{i,k}[\bar{C}_i(a) - \bar{C}_i(a'_{i,k}, a_{-i})] &= \sum_{i=1}^{N} \bar{C}_i(a) - \sum_{i=1}^{N}\sum_{k=1}^{s_i} \bar{y}_{i,k}\bar{C}_i(a'_{i,k}, a_{-i}) \\
(\bar{\ell}_r \text{ is non-decreasing by Lemma 1(a)}) \quad &\geq \sum_{i=1}^{N} \bar{C}_i(a) - \sum_{i=1}^{N}\sum_{k=1}^{s_i} \bar{y}_{i,k} \sum_{r\in a'_{i,k}} \bar{\ell}_r(|a|_r + 1) \\
(\text{changing order of summation}) \quad &= \sum_{r\in a} \left[|a|_r \bar{\ell}_r(|a|_r) - \bar{v}_r \bar{\ell}_r(|a|_r + 1)\right] \\
(\text{substitute } \bar{\ell}_r(|a|_r) = \sum_{j=1}^{m} \alpha_j^r f_j(|a|_r, \bar{v}_r)) \quad &= \sum_{r\in a}\sum_{j=1}^{m} \alpha_j^r \left[|a|_r f_j(|a|_r, \bar{v}_r) - \bar{v}_r f_j(|a|_r + 1, \bar{v}_r)\right] \\
(\text{recursion in Lemma 2}) \quad &= \sum_{r\in a}\sum_{j=1}^{m} \alpha_j^r \left[c_j(|a|_r) - p_j(\bar{v}_r)\right] \\
(\bar{v}_r \text{ optimal solution of (9)}) \quad &\geq \sum_{r\in a}\sum_{j=1}^{m} \alpha_j^r [c_j(|a|_r) - p_j(v_r^{\text{opt}})] \\
(\text{by def. of } \rho_b\text{, it is } p_j(v_r^{\text{opt}}) \leq \rho_b c_j(v_r^{\text{opt}})) \quad &\geq \sum_{r\in a}\sum_{j=1}^{m} \alpha_j^r [c_j(|a|_r) - \rho_b c_j(v_r^{\text{opt}})] \\
&= \text{SC}(a) - \rho_b \text{SC}(a^{\text{opt}}).
\end{aligned} \tag{11}$$

Since (9) is a convex program with polynomially many decision variables and constraints, it can be solved to arbitrary precision in polynomial time. The argument in Eq. (11) will go through with a minor change on the fifth line, where one pays a multiplicative factor $1 + \delta$ with $\delta > 0$. Correspondingly, we obtain a price of anarchy of $(1 + \delta)\rho_b$ in place of ρ_b. Selecting δ sufficiently small, one obtains a price of anarchy of $(1 + \delta)\rho_b \leq \rho_b + \varepsilon$ for any choice of $\varepsilon > 0$. □

We conclude observing that the expression of $p_j(v)$ in (7) can be computed analytically for many commonly studied classes of resource costs. Nevertheless, if this is not the case, one can approximate $p_j(v)$ to arbitrary precision by truncating the sum at a finite value. One can then verify the same properties shown in Lemmas 1 and 2 and apply a reasoning identical to that in (11). An additional multiplicative error will arise in (11), although this can be made as small as desired.

4.1 Comparison with existing approximations

A strength of the approach followed thus far is that optimally designed taxes can be used to derive polynomial time algorithms matching the hardness factor. This can be done relying on existing algorithms, e.g., no-regret dynamics, as discussed in Section 1.2. For this reason, we now turn the attention to comparing the optimal price of anarchy of Theorem 3 with the best known approximation ratio of Makarychev and Sviridenko [41]. Specifically, when all resource costs are identical to b, [41] gives a randomized algorithm with an approximation ratio of $\mu_b = \sup_{x\in\mathbb{R}_{>0}} \mathbb{E}_{P\sim\mathrm{Poi}(1)}[(xP)b(xP)]/(xb(x))$. While their result applies to the broader class of optimization problems with a "diseconomy of scale", the approximation ratio in Corollary 2 always matches or strictly improves upon theirs. This follows from

$$\rho_b = \sup_{x\in\mathbb{N}} \frac{\mathbb{E}_{P\sim\mathrm{Poi}(x)}[Pb(P)]}{xb(x)} \leq \sup_{x\in\mathbb{N}} \frac{\mathbb{E}_{P\sim\mathrm{Poi}(1)}[(xP)b(xP)]}{xb(x)} \leq \mu_b.$$

The last inequality follows trivially by replacing $\mathbb{N}$ with $\mathbb{R}_{>0}$ and using the definition of μ_b, while the first inequality can be shown using the notion of convex ordering between distributions.[6] An example where the approximation ratios coincide is given by $b(x) = x^d$, in which case $\rho_b = \mu_b$ equal the $(d+1)$'st Bell number, while an instance where the inequality is strict is provided by $b(x) = x+1$, in which case $\rho_b = \sup_{x\in\mathbb{N}} (x+2)/(x+1) = 3/2 < 2 = \sup_{x\in\mathbb{R}_{>0}} (2x+1)/(x+1) = \mu_b$.

5 DISCUSSION AND CONCLUSIONS

Interventions, such as those based on taxes, information provision, and other principles, are a commonly utilized approach to improve the system welfare when direct control over the individuals is infeasible or impossible. Whilst thoroughly studied, it is often unclear whether such indirect forms of control are unavoidably associated with reduced performances as compared to that achievable with full/dictatorial control over the individuals.

Focusing on congestion games, this paper shows that *no* such performance degradation arises. On the contrary, judiciously designed interventions can be efficiently computed and achieve the same performance of the best centralized polynomial time algorithm. We achieve this result by providing a tight computational lower bound for the problem of minimizing the system cost in congestion games, and by designing suitable taxation mechanisms. We thus obtain polynomial time algorithms based on taxes matching the hardness factor.

There remain many opportunities for further work on interventions in congestion games. One important research direction is to shed further light on the interplay between (more general) interventions and the achievable performances. It is interesting to understand whether other approaches based on, e.g., information provisioning, or cost-sharing mechanisms are equally powerful. In other words, does the positive result obtained here hold for other classes of interventions?

There are also a number of open questions arising from our work and possible refinements thereof. An interesting direction is that of considering the variant of *network congestion games*, whereby each strategy set $\mathcal{A}_i$ is implicitly given as the set of paths connecting an origin/destination node in an underlying graph. On its own, this more succinct representation of the strategy space would only increase the computational complexity, but on the other hand the graph also imposes

[6]To see this, we leverage the result in [54, Thm 3.A.36] with $X_i = Y \sim x\mathrm{Poi}(1)$, $a_i = 1/x$, and $i = 1, \dots, x \in \mathbb{N}$. Theorem 3.A.36 in [54] applies ensuring that $\sum_{i=1}^{x} a_i X_i \leq_{\mathrm{cx}} Y$, which reads as $\sum_{i=1}^{x} P \leq_{\mathrm{cx}} xP$, where $P \sim \mathrm{Poi}(1)$. Here $\leq_{\mathrm{cx}}$ denotes the convex ordering between distributions [54]. Recalling that $\sum_{i=1}^{x} P \sim \mathrm{Poi}(x)$, it is $\mathrm{Poi}(x) \leq_{\mathrm{cx}} x\mathrm{Poi}(1)$ which implies $\mathbb{E}_{P\sim\mathrm{Poi}(x)} Pb(P) \leq \mathbb{E}_{P\sim\mathrm{Poi}(1)} (xP)b(xP)$, as $xb(x)$ is convex.

more structure. The results in Theorem 3 (design of optimal taxes) extend to network congestion games by replacing the constraint set in the convex program in (9) with the set of feasible flows on the graph. Similarly, Corollary 2 (polynomial time algorithms) also extends provided that one utilizes no-regret algorithms, such as Follow the Perturbed Leader, that do not require explicit description of all possible paths (which might be exponential in the size of the graph). On the contrary, our reduction in the proof of Theorem 1 (inapproximability of minimum cost) induces a general congestion game. The existence of a reduction to network congestion games remains open.

REFERENCES

[1] Sebastian Aland, Dominic Dumrauf, Martin Gairing, Burkhard Monien, and Florian Schoppmann. 2011. Exact price of anarchy for polynomial congestion games. *SIAM J. Comput.* 40, 5 (2011), 1211–1233.

[2] Matthew Andrews, Antonio Fernández Anta, Lisa Zhang, and Wenbo Zhao. 2012. Routing for power minimization in the speed scaling Model. *IEEE/ACM Transactions on Networking* 20, 1 (2012), 285–294.

[3] Elliot Anshelevich, Anirban Dasgupta, Jon Kleinberg, Eva Tardos, Tom Wexler, and Tim Roughgarden. 2008. The price of stability for network design with fair cost allocation. *SIAM J. Comput.* 38, 4 (2008), 1602–1623.

[4] Baruch Awerbuch, Yossi Azar, and Amir Epstein. 2013. The price of routing unsplittable flow. *SIAM J. Comput.* 42, 1 (2013), 160–177.

[5] Yossi Azar and Amir Epstein. 2005. Convex programming for scheduling unrelated parallel machines. In *Proceedings of the 37th Annual ACM Symposium on Theory of Computing, Baltimore, MD, USA, May 22-24, 2005*, Harold N. Gabow and Ronald Fagin (Eds.). ACM, 331–337.

[6] Siddharth Barman, Omar Fawzi, and Paul Fermé. 2021. Tight Approximation Guarantees for Concave Coverage Problems. In *38th International Symposium on Theoretical Aspects of Computer Science, STACS 2021, March 16-19, 2021, Saarbrücken, Germany (Virtual Conference) (LIPIcs, Vol. 187)*, Markus Bläser and Benjamin Monmege (Eds.). 9:1–9:17.

[7] Umang Bhaskar, Yu Cheng, Young Kun Ko, and Chaitanya Swamy. 2016. Hardness results for signaling in bayesian zero-sum and network routing Games. In *Proceedings of the 2016 ACM Conference on Economics and Computation, EC '16, Maastricht, The Netherlands, July 24-28, 2016*, Vincent Conitzer, Dirk Bergemann, and Yiling Chen (Eds.). ACM, 479–496.

[8] Vittorio Bilò. 2018. A unifying tool for bounding the quality of non-cooperative solutions in weighted congestion games. *Theory of Computing Systems* 62, 5 (2018), 1288–1317.

[9] Vittorio Bilò and Cosimo Vinci. 2019. Dynamic taxes for polynomial congestion games. *ACM Transactions on Economics and Computation* 7, 3 (2019), 15:1–15:36.

[10] Avrim Blum, Mohammad Taghi Hajiaghayi, Katrina Ligett, and Aaron Roth. 2008. Regret minimization and the price of total anarchy. In *Proceedings of the 40th Annual ACM Symposium on Theory of Computing, Victoria, British Columbia, Canada, May 17-20, 2008*, Cynthia Dwork (Ed.). ACM, 373–382.

[11] Liad Blumrosen and Shahar Dobzinski. 2006. Welfare maximization in congestion games. In *Proceedings 7th ACM Conference on Electronic Commerce (EC-2006), Ann Arbor, Michigan, USA, June 11-15, 2006*, Joan Feigenbaum, John C.-I. Chuang, and David M. Pennock (Eds.). ACM, 52–61.

[12] Ioannis Caragiannis, Christos Kaklamanis, and Panagiotis Kanellopoulos. 2010. Taxes for linear atomic congestion games. *ACM Transactions on Algorithms* 7, 1 (2010), 13:1–13:31.

[13] Matteo Castiglioni, Andrea Celli, Alberto Marchesi, and Nicola Gatti. 2020. Signaling in bayesian network congestion games: the subtle power of symmetry. *CoRR* abs/2002.05190 (2020). arXiv:2002.05190

[14] Nicolò Cesa-Bianchi and Gábor Lugosi. 2006. *Prediction, learning, and games.* Cambridge University Press.

[15] Deeparnab Chakrabarty, Aranyak Mehta, and Viswanath Nagarajan. 2005. Fairness and optimality in congestion games. In *Proceedings 6th ACM Conference on Electronic Commerce (EC-2005), Vancouver, BC, Canada, June 5-8, 2005*, John Riedl, Michael J. Kearns, and Michael K. Reiter (Eds.). ACM, 52–57.

[16] Rahul Chandan, Dario Paccagnan, and Jason R Marden. 2019. Optimal mechanisms for distributed resource-allocation. *arXiv preprint arXiv:1911.07823* (2019).

[17] Rahul Chandan, Dario Paccagnan, and Jason R. Marden. 2021. Tractable Mechanisms for Computing Near-Optimal Utility Functions. In *Proceedings of the 20th International Conference on Autonomous Agents and MultiAgent Systems* (Virtual Event, United Kingdom) *(AAMAS '21)*. 306–313.

[18] George Christodoulou and Martin Gairing. 2016. Price of stability in polynomial congestion games. *ACM Transactions on Economics and Computation* 4, 2 (2016), 10:1–10:17.

[19] George Christodoulou and Elias Koutsoupias. 2005. The price of anarchy of finite congestion games. In *Proceedings of the 37th Annual ACM Symposium on Theory of Computing, Baltimore, MD, USA, May 22-24, 2005*, Harold N. Gabow and Ronald Fagin (Eds.). ACM, 67–73.

[20] George Christodoulou, Elias Koutsoupias, and Akash Nanavati. 2009. Coordination mechanisms. *Theoretical Computer Science* 410, 36 (2009), 3327–3336.

[21] Bart de Keijzer and Guido Schäfer. 2012. Finding social optima in congestion games with positive externalities. In *Algorithms - ESA 2012 - 20th Annual European Symposium, Ljubljana, Slovenia, September 10-12, 2012. Proceedings (Lecture Notes in Computer Science, Vol. 7501)*, Leah Epstein and Paolo Ferragina (Eds.). Springer, 395–406.

[22] Devdatt P. Dubhashi and Desh Ranjan. 1998. Balls and bins: A study in negative dependence. *Random Structures and Algorithms* 13, 2 (1998), 99–124.

[23] Szymon Dudycz, Pasin Manurangsi, Jan Marcinkowski, and Krzysztof Sornat. 2020. Tight approximation for proportional approval voting. In *Proceedings of the Twenty-Ninth International Joint Conference on Artificial Intelligence, IJCAI 2020*, Christian Bessiere (Ed.). ijcai.org, 276–282.

[24] Rick Durrett. 2010. *Probability: Theory and Examples, 4th Edition.* Cambridge University Press.
[25] Uriel Feige. 1998. A Threshold of $\ln n$ for approximating set Cover. *Journal of the ACM* 45, 4 (1998), 634–652.
[26] Dimitris Fotakis. 2010. Stackelberg strategies for atomic congestion games. *Theory of Computing Systems* 47, 1 (2010), 218–249.
[27] Dimitris Fotakis and Paul G Spirakis. 2008. Cost-balancing tolls for atomic network congestion games. *Internet Mathematics* 5, 4 (2008), 343–363.
[28] Martin Gairing, Kostas Kollias, and Grammateia Kotsialou. 2020. Existence and efficiency of equilibria for cost-sharing in generalized weighted congestion games. *ACM Transactions on Economics and Computation* 8, 2 (2020), 11:1–11:28.
[29] Tobias Harks, Ingo Kleinert, Max Klimm, and Rolf H. Möhring. 2015. Computing network tolls with support constraints. *Networks* 65, 3 (2015), 262–285.
[30] Tobias Harks, Tim Oosterwijk, and Tjark Vredeveld. 2016. A logarithmic approximation for polymatroid congestion games. *Operations Research Letters* 44, 6 (2016), 712–717.
[31] Martin Hoefer, Lars Olbrich, and Alexander Skopalik. 2008. Taxing subnetworks. In *Internet and Network Economics, 4th International Workshop, WINE 2008, Shanghai, China, December 17-20, 2008. Proceedings (Lecture Notes in Computer Science, Vol. 5385)*, Christos H. Papadimitriou and Shuzhong Zhang (Eds.). Springer, 286–294.
[32] Nicole Immorlica, Li (Erran) Li, Vahab S. Mirrokni, and Andreas S. Schulz. 2009. Coordination mechanisms for selfish scheduling. *Theoretical Computer Science* 410, 17 (2009), 1589–1598.
[33] Tomas Jelinek, Marcus Klaas, and Guido Schäfer. 2014. Computing optimal tolls with arc restrictions and heterogeneous players. In *31st International Symposium on Theoretical Aspects of Computer Science (STACS 2014), STACS 2014, March 5-8, 2014, Lyon, France (LIPIcs, Vol. 25)*, Ernst W. Mayr and Natacha Portier (Eds.). Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 433–444.
[34] Albert Xin Jiang and Kevin Leyton-Brown. 2015. Polynomial-time computation of exact correlated equilibrium in compact games. *Games and Economic Behavior* 91 (2015), 347–359.
[35] Kumar Joag-Dev and Frank Proschan. 1983. Negative association of random variables with applications. *The Annals of Statistics* (1983), 286–295.
[36] Adam T. Kalai and Santosh S. Vempala. 2005. Efficient algorithms for online decision problems. *J. Comput. System Sci.* 71, 3 (2005), 291–307.
[37] Achim Klenke and Lutz Mattner. 2010. Stochastic ordering of classical discrete distributions. *Advances in Applied probability* 42, 2 (2010), 392–410.
[38] Max Klimm, Daniel Schmand, and Andreas Tönnis. 2019. The online best reply algorithm for resource allocation problems. In *Algorithmic Game Theory - 12th International Symposium, SAGT 2019, Athens, Greece, September 30 - October 3, 2019, Proceedings (Lecture Notes in Computer Science, Vol. 11801)*, Dimitris Fotakis and Evangelos Markakis (Eds.). Springer, 200–215.
[39] Elias Koutsoupias and Christos Papadimitriou. 1999. Worst-case equilibria. In *Annual Symposium on Theoretical Aspects of Computer Science*. Springer, 404–413.
[40] V. S. Anil Kumar, Madhav V. Marathe, Srinivasan Parthasarathy, and Aravind Srinivasan. 2009. A unified approach to scheduling on unrelated parallel machines. *J. ACM* 56, 5 (2009), 28:1–28:31.
[41] Konstantin Makarychev and Maxim Sviridenko. 2018. Solving optimization problems with diseconomies of scale via decoupling. *Journal of the ACM* 65, 6 (2018), 42:1–42:27.
[42] Toufik Mansour and Matthias Schork. 2015. *Commutation relations, normal ordering, and Stirling numbers.* CRC Press.
[43] Carol A. Meyers and Andreas S. Schulz. 2012. The complexity of welfare maximization in congestion games. *Networks* 59, 2 (2012), 252–260.
[44] Dario Paccagnan, Rahul Chandan, Bryce L. Ferguson, and Jason R. Marden. 2021. Optimal Taxes in Atomic Congestion Games. *ACM Transactions on Economics and Computation* 9, 3, Article 19 (2021), 33 pages.
[45] Dario Paccagnan, Rahul Chandan, and Jason R Marden. 2020. Utility Design for Distributed Resource Allocation–Part I: Characterizing and Optimizing the Exact Price of Anarchy. *IEEE Trans. Automat. Control* 65, 11 (2020), 4616–4631.
[46] Dario Paccagnan and Jason R Marden. 2021. Utility Design for Distributed Resource Allocation–Part II: Applications to Submodular, Covering, and Supermodular Problems. *IEEE Transactions on Automatic Control (Early Access)* (2021).
[47] Christos H. Papadimitriou and Tim Roughgarden. 2008. Computing correlated equilibria in multi-player games. *Journal of the ACM* 55, 3 (2008), 14:1–14:29.
[48] Alberto Del Pia, Michael Ferris, and Carla Michini. 2017. Totally unimodular congestion games. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2017, Barcelona, Spain, Hotel Porta Fira, January 16-19*, Philip N. Klein (Ed.). SIAM, 577–588.
[49] Arthur C Pigou. 1920. *The Economics of Welfare.* Macmillan.
[50] Robert W Rosenthal. 1973. A class of games possessing pure-strategy Nash equilibria. *International Journal of Game Theory* 2, 1 (1973), 65–67.
[51] Tim Roughgarden. 2005. *Selfish routing and the price of anarchy.* MIT Press.

[52] Tim Roughgarden. 2014. Barriers to Near-Optimal Equilibria. In *55th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2014, Philadelphia, PA, USA, October 18-21, 2014*. IEEE Computer Society, 71–80.

[53] Tim Roughgarden. 2015. Intrinsic robustness of the price of anarchy. *Journal of the ACM* 62, 5 (2015), 32:1–32:42.

[54] Moshe Shaked and J George Shanthikumar. 2007. *Stochastic orders*. Springer Science & Business Media.

[55] Subhash Suri, Csaba D Tóth, and Yunhong Zhou. 2007. Selfish load balancing and atomic congestion games. *Algorithmica* 47, 1 (2007), 79–96.

[56] Cem Tekin, Mingyan Liu, Richard Southwell, Jianwei Huang, and Sahand Haji Ali Ahmad. 2012. Atomic congestion games on graphs and their applications in networking. *IEEE/ACM Transactions on Networking* 20, 5 (2012), 1541–1552.

A APPENDIX

Throughout the appendix we extend the definition of resource costs $b_j : \mathbb{N} \to \mathbb{R}_{>0}$ to $b_j : \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ by setting $b(0) = 0$. This is without loss of generality, and only used to ease the notation.

A.1 Additional material for Section 3

A.1.1 Binomial expected value converges to Poisson expected value.

Lemma 3. *Let $b : \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ non-decreasing, semi-convex, and assume $\rho_b < \infty$. Then $\forall k \in \mathbb{N}$*

$$\lim_{h\to\infty} \mathbb{E}_{X\sim Bin(h,k/h)}[Xb(X)] = \mathbb{E}_{X\sim Poi(k)}[Xb(X)].$$

Proof. We begin noting that the limiting operation is delicate since we do not want to assume boundedness of $c(x) = xb(x)$ for $x \to \infty$, as this would disqualify interesting cases such as that of polynomials. Since $\rho_b < \infty$ by assumption, then $\mathbb{E}_{X\sim \mathrm{Poi}(k)}[c(X)]/c(k) < \infty$ for any $k \in \mathbb{N}$, and thus $\mathbb{E}_{X\sim \mathrm{Poi}(k)}[c(X)] < \infty$ too. Fix k, and let $f_h(i)$ and $f(i)$ denote the probability mass function corresponding to $\mathrm{Bin}(h, k/h)$ and $\mathrm{Poi}(k)$, and recall that $f_h(i) \to f(i)$ for all i. The result follows from

$$\mathbb{E}_{X\sim \mathrm{Poi}(k)}[c(X)] = \sum_{i=0}^{\infty} \lim_{h\to\infty} f_h(i)c(i) \leq \lim_{h\to\infty} \sum_{i=0}^{\infty} f_h(i)c(i) = \lim_{h\to\infty} \mathbb{E}_{X\sim \mathrm{Bin}(h,k/h)}[c(X)] \leq \mathbb{E}_{X\sim \mathrm{Poi}(k)}[c(X)],$$

where the first equality holds by definition of expected value and by replacing $f(i) = \lim_{h\to\infty} f_h(i)$. The following inequality holds by Fatou's lemma and existence of the limit (which hold as this is a series with non-negative terms). As a result, we can interchange the limiting operation with the infinite sum. The next equality is due to the definition of expected value. The last inequality is a consequence of the fact that c is a convex function and the Poisson distribution $\mathrm{Poi}(k)$ dominates the binomial distribution $\mathrm{Bin}(h, k/h)$ in the sense of the convex ordering [54], ensuring that $\mathbb{E}_{X\sim \mathrm{Bin}(h,k/h)}[c(X)] \leq \mathbb{E}_{X\sim \mathrm{Poi}(k)}[c(X)]$ for all h. One way to see this is to utilize the fact that the ratio between the probability mass functions $f_h(i)/f(i)$ is unimodal as shown in [37, Sec 2.7], and that f_h and f are not ordered by the usual stochastic order, thus concluding thanks to [54, Thm. 3.A.53]. Alternatively, one can compare the two expectations directly. □

A.1.2 Proof of Proposition 2.

Proof. Existence of partitioning system is proved through a probabilistic approach similarly to that in [25]. The idea is to construct each $\mathcal{P}_i$ independently from a uniform distribution. More formally, each element in $[n]$ gets assigned to k of the $P_{j,i}$ uniformly at random. This ensures that, by construction, each element in $[n]$ appears in exactly k different sets of $\mathcal{P}_j$. Thus the first property of the partitioning system holds trivially. Let $c(x) = xb(x)$. In order to prove the second property, we fix $B \subseteq [\beta]$ with $|B| = h$ and correspondingly consider $Q = \{P_{j,i(j)}, j \in B\}$. We intend to show that with high probability $\sum_{r\in Q} c(|Q|_r) \geq \left(\mathbb{E}_{X\sim \mathrm{Bin}(h,k/h)}[c(X)] - \eta\right) n$ holds. This would imply that, among all possible ways of constructing $\{\mathcal{P}_i\}_{i=1}^r$ there exists at least one that satisfies the property. To prove that $\sum_{r\in Q} c(|Q|_r) \geq \left(\mathbb{E}_{X\sim \mathrm{Bin}(h,k/h)}[c(X)] - \eta\right) n$ holds with high probability, we compute the expected cost that arises from the probabilistic choice outlined above for $\{\mathcal{P}_i\}_{i=1}^r$. In particular

$$\mathbb{E}[c(|Q|_r)] = \mathbb{E}_{X\sim \mathrm{Bin}(h,k/h)}[c(X_r)],$$

since the number of sets in which each resource appears is given by the random variable $X_r = \sum_{j\in B} \mathbb{1}_{r\in P_{j,i(j)}} \sim \mathrm{Bin}(h, k/h)$, owing to the fact that $\mathbb{1}_{r\in P_{j,i(j)}} \sim \mathrm{Ber}(k/h)$ are independent (here $\mathbb{1}$ denotes the indicator function). We then use Chernoff-Hoeffding bound on the total cost $\sum_{r\in[n]} c(|Q|_r)$, where each term is bounded by $0 \leq c(|Q|_r) \leq c(h)$ owing to the non-decreasingness and non-negativity of c. [7] Thus, with probability smaller or equal to $2e^{-2n\eta^2/(c(h))^2}$, it is $|\sum_{r\in[n]} c(|Q|_r) - \mathbb{E}_{X\sim\text{Bin}(h,k/h)}[c(X)]| \geq \eta n$. Since there are $\binom{\beta}{h} \cdot h^h \leq (1+h)^\beta$ possible choices for B and Q, a union bound guarantees that with probability higher than $1 - 2(1+h)^\beta \cdot e^{-2n\eta^2/(c(h))^2}$, the cost of all B, Q satisfies $|\sum_{r\in[n]} c(|Q|_r) - \mathbb{E}_{X\sim\text{Bin}(h,k/h)}[c(X)]| < \eta n$. With the specific choice of n as in the statement, we are guaranteed this property with a probability of at least 4/5. This shows that a partitioning system always exists. One such object can be computed by simple enumeration over all possible choices, which are only a function of h and n and β. □

A.2 Additional material for Section 4

A.2.1 Convexity of Poisson expected value.

LEMMA 4. *Let $b : \mathbb{N} \to \mathbb{R}_{>0}$ be non-decreasing, semi-convex, and $p : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be*

$$p(v) = \mathbb{E}_{P\sim Poi(v)}[Pb(P)] = \left(\sum_{i=0}^{\infty} ib(i)\frac{v^i}{i!}\right)e^{-v}.$$

Assume ρ_b defined in (3) *is finite. Then, p is convex and differentiable infinitely many times in $\mathbb{R}_{\geq 0}$.*

PROOF. We start by showing that if $\rho_b < \infty$, then $p(v)$ is well defined, i.e., the series converges to a finite value for any choice of $v \in \mathbb{R}_{\geq 0}$ (as standard, $\mathbb{R}$ does not include infinity). To see this, let $c(v) = vb(b)$ and observe that $\rho_b < \infty$ implies $p(v)/c(v) < \infty$ for all fixed $v \in \mathbb{N}$ thanks to the definition of ρ_b, so that $p(v) < \infty$ since $c(v) < \infty$ for finite v. Therefore also $\sum_{i=0}^{\infty} c(i)\frac{v^i}{i!} < \infty$ for all $v \in \mathbb{N}$. Since $\sum_{i=0}^{\infty} c(i)\frac{v^i}{i!}$ is increasing in $v \geq 0$, boundedness over the naturals, immediately implies boundedness of the same expression over the non-negative reals. Exploiting the fact that also e^{-v} is bounded, we have shown that $p(v)$ is well defined, and converges to a finite value for any choice of $v \in \mathbb{R}_{\geq 0}$. As a result p is differentiable infinitely many times in its domain, since it is the product of a convergent power series, and of e^{-v}. We can therefore prove convexity by computing the second order derivative and verifying that it is non-negative. The first derivative reads as

$$\begin{aligned}
p'(v) &= -e^{-v}\left(\sum_{i=0}^{\infty} c(i)\frac{v^i}{i!}\right) + e^{-v}\left(\sum_{i=0}^{\infty} c(i)\frac{iv^{i-1}}{i!}\right) \\
&= -e^{-v}\left(\sum_{i=0}^{\infty} c(i)\frac{v^i}{i!}\right) + e^{-v}\left(\sum_{i=0}^{\infty} c(i+1)\frac{v^i}{i!}\right) \\
&= e^{-v}\sum_{i=0}^{\infty} \frac{v^i}{i!}\Delta c(i),
\end{aligned}$$

where we defined $\Delta c(i) = c(i+1) - c(i)$. Following an identical approach the second derivative is

$$p''(v) = e^{-v}\sum_{i=0}^{\infty} \frac{v^i}{i!}[\Delta c(i+1) - \Delta c(i)].$$

[7]Observe that the random variables $\{c(|Q|_r)\}_{r\in[n]} = \{c(X_r)\}_{r\in[n]}$ are negatively associated, which is enough to conclude thanks to [22] and $\eta \in (0,1)$. Since $c(X_r) = c(\sum_{j\in B} X_{r,j})$, where $X_{r,j} = \mathbb{1}_{r\in P_{j,i(j)}}$, is non-decreasing in $\{X_{r,j}\}_{j\in[\beta]}$, negative association of $\{c(X_r)\}_{r\in[n]}$ can be shown by proving negative association of $\{X_{r,j}\}_{r\in[n],j\in[\beta]}$ [35, Property 6]. For fixed $j \in [\beta]$, the variables $\{X_{r,j}\}_{r\in[n]}$ are negatively associated as they are a permutation distribution of $(0,\ldots,0,1,\ldots,1)$ with $n - kn/h$ zeros and kn/h ones [35, Def. 2.10 and Thm. 2.11]. Owing to this, and thanks to [35, Property 7] negative association of $\{X_{r,j}\}_{r\in[n],j\in[\beta]}$ follows from the above, and from the fact that $\{X_{r,j}\}_{r\in[n]}$ are mutually independent.

The summand corresponding to $i = 0$ reads as $c(2) - 2c(1) = 2(b(2) - b(1)) > 0$ since b non-decreasing. The summands corresponding to $i \geq 1$ are non-negative as semi-convexity of b implies convexity of c. Hence, $p''(v) \geq 0$ for all $v \in \mathbb{R}_{\geq 0}$ as desired. □

A.2.2 Proof of Lemma 1.

Lemma 1 consists of two parts. We prove part (a) in Lemma 5 and part (b) in the ensuing Lemma 6.

Lemma 5. *Define p_j as in* (7) *and f_j as in* (8), *then $f_j(x+1, v) \geq f_j(x, v)$ for all $x \in \mathbb{N}_0$ and $v \in \mathbb{R}_{\geq 0}$.*

Proof. For simplicity of notation, we drop the subscript j, i.e., we show that $f(x+1, v) \geq f(x, v)$ for all $x \in \mathbb{N}_0$ and $v \in \mathbb{R}_{\geq 0}$. When $v = 0$, then $f(x, 0) = b(x)$, which is non-decreasing as $b(x)$ is so. Thus, in the following we restrict to the case of $v \in \mathbb{R}_{>0}$. When, in addition, $x = 0$ the inequality reduces to $f(1, v) \geq f(0, v) \iff p(v)/v \geq 0$, which holds as $p(v) > 0$ for $v \in \mathbb{R}_{\geq 0}$. We are thus left to consider the case of $x \in \mathbb{N}$ and $v \in \mathbb{R}_{>0}$. Substituting the expression for $f(x+1, v)$ and $f(x, v)$ in the latter inequality and isolating $p(v)$ results in

$$p(v)\left(x\sum_{i=0}^{x}\frac{v^i}{i!} - v\sum_{i=0}^{x-1}\frac{v^i}{i!}\right) \geq \left(x\sum_{i=0}^{x}\frac{c(i)}{i!}v^i - v\sum_{i=0}^{x-1}\frac{c(i)}{i!}v^i\right). \tag{12}$$

The term in brackets on the left hand side can be equivalently written as

$$\sum_{i=0}^{x} x\frac{v^i}{i!} - \sum_{i=0}^{x-1}\frac{v^{i+1}}{i!} = \sum_{i=0}^{x}\frac{x-i}{i!}v^i.$$

Similarly, term in the right hand side brackets, with $c(0) = 0$, is equivalent to

$$\sum_{i=1}^{x} x\frac{c(i)}{i!}v^i - \sum_{i=0}^{x-1}\frac{c(i)}{i!}v^{i+1} = \sum_{i=1}^{x}\frac{xc(i) - ic(i-1)}{i!}v^i.$$

Thus, inequality (12) reduces to

$$p(v) \geq \frac{\sum_{i=1}^{x}\frac{xc(i)-ic(i-1)}{i!}v^i}{\sum_{i=0}^{x}\frac{x-i}{i!}v^i}, \tag{13}$$

whereby we used the fact that the denominator is positive since $x \geq i$, $x \geq 1$. Thus, we require (13) to hold for all $x \in \mathbb{N}$ and $v \in \mathbb{R}_{>0}$. Lemma 7, ensures that the right hand side of the previous inequality is non-decreasing in $x \in \mathbb{N}$, for each fixed $v \in \mathbb{R}_{>0}$. Therefore, (13) holds, if it holds when x is arbitrarily large, that is if

$$p(v) \geq \lim_{x\to\infty}\frac{\sum_{i=0}^{x}\frac{c(i)}{i!}v^i}{\sum_{i=0}^{x}\frac{v^i}{i!}}. \tag{14}$$

Notice, though, that the right hand side in the last expression is precisely equal to $\frac{p(v)e^v}{e^v} = p(v)$, thanks to the definition of $p(v)$ in (7) and to the fact that $\sum_{i=0}^{\infty}\frac{v^i}{i!} = e^v$. Therefore (14) holds (with equality), which completes the proof of the lemma. □

Lemma 6. *The taxation mechanism introduced in Definition 1 is non-negative, i.e., $\tau_r(x) \geq 0$ $\forall x \in \mathbb{N}_0$, $r \in \mathcal{R}$.*

Proof. We will show that $f_j(x, v) \geq b_j(x)$ for any $x \in \mathbb{N}, v \in \mathbb{R}_{\geq 0}, j \in [m]$, as this suffices to conclude. We do so separately for each b_j, and thus drop the index j in the following. The case of

$v = 0$ follows readily, since $f(x, 0) = b(x)$. Similarly, for $x = 0$, it is $f(0, v) = 0 \geq b(0) = 0$. In the remaining cases, we are left to prove

$$f(x,v) = \frac{(x-1)!}{v^k} \sum_{i=0}^{x-1} \frac{p(v) - ib(i)}{i!} v^i \geq b(x) \qquad x \in \mathbb{N}, v \in \mathbb{R}_{>0}.$$

We let $c(x) = xb(x)$ to ease the notation, and solve for $p(v)$. The latter inequality holds if $p(v) \geq (\sum_{i=0}^{x} c(i) \frac{v^i}{i!})/(\sum_{j=0}^{x-1} \frac{v^j}{j!})$ for all $x \in \mathbb{N}$, $v \in \mathbb{R}_{>0}$. We complete the proof showing that the right hand side in the previous expression is non-decreasing in x, so that the desired property holds for any $x \in \mathbb{N}$ if it holds for x arbitrarily large, i.e., if

$$p(v) \geq \lim_{x \to \infty} \frac{\sum_{i=0}^{x} c(i) \frac{v^i}{i!}}{\sum_{j=0}^{x-1} \frac{v^j}{j!}}, \qquad \forall v \in \mathbb{R}_{>0},$$

which is indeed satisfied (with equality), as it reduces to $p(v) \geq (p(v)e^v)/e^v$, owing to the definition of $p(v)$ and $\sum_{j=0}^{\infty} \frac{v^j}{j!} = e^v$. To conclude, we thus need to prove that for all $x \in \mathbb{N}$, $v \in \mathbb{R}_{>0}$, it is

$$\begin{aligned}
\frac{\sum_{i=0}^{x+1} c(i) \frac{v^i}{i!}}{\sum_{j=0}^{x} \frac{v^j}{j!}} \geq \frac{\sum_{i=0}^{x+1} c(i) \frac{v^i}{i!}}{\sum_{j=0}^{x} \frac{v^j}{j!}} &\iff \sum_{i=0}^{x+1} \sum_{j=0}^{x-1} c(i) \frac{v^{i+j}}{i!j!} \geq \sum_{i=0}^{x} \sum_{j=0}^{x} c(i) \frac{v^{i+j}}{i!j!} \\
&\iff \sum_{j=0}^{x-1} c(x+1) \frac{v^{x+1+j}}{(x+1)!j!} \geq \sum_{i=0}^{x} c(i) \frac{v^{i+x}}{i!x!} \\
&\iff \sum_{j=1}^{x} c(x+1) \frac{v^{x+j}}{(x+1)!(j-1)!} \geq \sum_{i=1}^{x} c(i) \frac{v^{i+x}}{i!x!} \\
&\iff \sum_{j=1}^{x} \left[\frac{c(x+1)}{(x+1)!(j-1)!} - \frac{c(j)}{x!j!} \right] v^{j+x} \geq 0 \\
&\iff \sum_{j=1}^{x} [b(x+1) - b(j)] v^{j+x} \geq 0
\end{aligned}$$

which follows from the above chain of implications, and the fact that $b(x)$ is non-decreasing. □

A.2.3 Proof of Lemma 2.

Proof. Using the definition of f_j in (8), observe that

$$\begin{aligned}
vf_j(x+1,v) - xf_j(x,v) &= v\frac{x!}{v^{x+1}} \sum_{i=0}^{x} \frac{p_j(v) - ib_j(i)}{i!} v^i - x\frac{(x-1)!}{v^x} \sum_{i=0}^{x-1} \frac{p_j(v) - ib_j(i)}{i!} v^i \\
&= \frac{x!}{v^x} \frac{p_j(v) - xb_j(x)}{x!} v^x \\
&= p_j(v) - xb_j(x),
\end{aligned}$$

or equivalently $xb_j(x) - xf_j(x,v) + vf_j(x+1,v) = p_j(v)$ as needed. □

A.2.4 Technical Lemma used to prove Lemma 1.

Lemma 7. *Let $c : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be convex, $c(0) = 0$. Then, the function $g : \mathbb{N} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$,*

$$g(x,v) = \frac{\sum_{i=1}^{x} \frac{xc(i) - ic(i-1)}{i!} v^i}{\sum_{i=0}^{x} \frac{x-i}{i!} v^i}$$

is non-decreasing for all $x \in \mathbb{N}$, for any fixed $v \in \mathbb{R}_{>0}$.

Proof. Proving the claim amounts so showing that for all $v \in \mathbb{R}_{>0}, x \in \mathbb{N}$ it is

$$\frac{\sum_{i=1}^{x+1} a(x+1,i)\frac{v^i}{i!}}{\sum_{j=0}^{x+1} b(x+1,j)\frac{v^j}{j!}} \geq \frac{\sum_{i=1}^{x} a(x,i)\frac{v^i}{i!}}{\sum_{j=0}^{x} b(x,j)\frac{v^j}{j!}}, \tag{15}$$

where we let $a(x,i) = xc(i) - ic(i-1)$ and $b(x,j) = x - j$ to ease the notation. Since the denominators on the left and right hand side are positive, and since $b(x+1,x+1) = 0$, (15) is equivalent to

$$\sum_{i=1}^{x}\sum_{j=0}^{x} a(x+1,i)b(x,j)\frac{v^{i+j}}{i!j!} + \sum_{j=0}^{x} b(x,j)a(x+1,x+1)\frac{v^{j+x+1}}{j!(x+1)!} - \sum_{i=1}^{x}\sum_{j=0}^{x} a(x,i)b(x+1,j)\frac{v^{i+j}}{i!j!} \geq 0,$$

which we rewrite

$$\underbrace{\sum_{i=1}^{x}\sum_{j=0}^{x}\left(a(x+1,i)b(x,j) - a(x,i)b(x+1,j)\right)\frac{v^{i+j}}{i!j!}}_{①} + \underbrace{\sum_{j=0}^{x} b(x,j)a(x+1,x+1)\frac{v^{j+x+1}}{j!(x+1)!}}_{②} \geq 0. \tag{16}$$

We now turn our attention to each of the two terms appearing in the previous inequality. In particular, we will show that collecting all the contributions corresponding to the same power of v significantly simplifies the expressions, and allows us to conclude.

We begin with the second term, and substitute the definitions of a, b in $b(x,j)a(x+1,x+1)$ so that

$$② = \sum_{j=0}^{x} \frac{x-j}{j!x!}(c(x+1) - c(x))v^{j+x+1}. \tag{17}$$

We now focus on the first term, and observe that

$$\begin{aligned} a(x+1,i)b(x,j) - a(x,i)b(x+1,j) &= ((x+1)c(i) - ic(i-1))(x-j) - (xc(i) - ic(i-1))(x+1-j) \\ &= -jc(i) + ic(i-1), \end{aligned}$$

where we made use of the definitions of a and b. Hence, we can utilize indices i and $q = i + j$ in place of i, j to rewrite the first term appearing in (16) as

$$\begin{aligned} ① &= \sum_{i=1}^{x}\sum_{j=0}^{x}(ic(i-1) - jc(i))\frac{v^{i+j}}{i!j!} \\ &= \sum_{q=1}^{2x}\sum_{\substack{i\in[x] \\ \text{s.t. } q-x\leq i\leq q}} \frac{ic(i-1) - (q-i)c(i)}{i!(q-i)!}v^q \\ &= \sum_{q=x+1}^{2x}\sum_{i=q-x}^{x} \frac{ic(i-1) - (q-i)c(i)}{i!(q-i)!}v^q + \sum_{q=1}^{x}\sum_{i=1}^{q} \frac{ic(i-1) - (q-i)c(i)}{i!(q-i)!}v^q \\ &= \sum_{q=x+1}^{2x}\sum_{i=q-x}^{x} \frac{ic(i-1) - (q-i)c(i)}{i!(q-i)!}v^q \\ &= \sum_{q=x+1}^{2x} \frac{c(q-x-1) - c(x)}{x!(q-x-1)!}v^q \\ &= \sum_{j=0}^{x-1} \frac{1}{j!x!}(c(j) - c(x))v^{j+x+1} \end{aligned} \tag{18}$$

where the second line is obtained using the fact that q runs from 1 to $2x$, $i \in [x]$ and $j = q - i$ belongs to $0 \leq q - i \leq x$. The third line follows upon distinguishing the case of $x + 1 \leq q \leq 2x$ and $q \in [x]$. The fourth line is due to the fact that the second summand in the third line vanishes, since

$$\sum_{i=1}^{q} \frac{ic(i-1) - (q-i)c(i)}{i!(q-i)!} v^q = \sum_{i=2}^{q} \frac{c(i-1)}{(i-1)!(q-i)!} - \sum_{i=1}^{q-1} \frac{c(i)}{i!(q-i-1)!} = 0.$$

The fifth line follows from

$$\begin{aligned}\sum_{i=q-x}^{x} \frac{ic(i-1) - (q-i)c(i)}{i!(q-i)!} &= \sum_{i=q-x}^{x} \frac{c(i-1)}{(i-1)!(q-i)!} - \sum_{i=q-x}^{x} \frac{c(i)}{i!(q-i-1)!} \\ &= \frac{c(q-x-1)}{(q-x-1)!x!} + \sum_{i=q-x+1}^{x} \frac{c(i-1)}{(i-1)!(q-i)!} - \sum_{i=q-x}^{x-1} \frac{c(i)}{i!(q-i-1)!} - \frac{c(x)}{x!(q-x-i)!} \\ &= \frac{c(q-x-1) - c(x)}{x!(q-x-1)!}\end{aligned}$$

The final line is derived reverting to the original indices i and j.

Thus, in light of (18) and (17), the inequality (15) we need to show reduces to

$$\sum_{j=0}^{x-1} [c(j) - c(x) + (x-j)(c(x+1) - c(x))] \frac{v^{j+x+1}}{j!x!} \geq 0 \quad \forall v \in \mathbb{R}_{>0},\ x \in \mathbb{N}.$$

The summand corresponding to $j = 0$ reads as $xc(x+1) - (x+1)c(x) = x(x+1)(b(x+1) - b(x))$, and it is non-negative since b is non-decreasing. All other summands are non-negative thanks to the convexity of c, that guarantees $c(x+1) - c(x) \geq (c(x) - c(j))/(x-j)$ as $1 \leq j < x$, so that $c(j) - c(x) + (x-j)(c(x+1) - c(x)) \geq 0$. One concludes using these observations and $v > 0$. □