

# Remarks on Galois rational covers

A.V.Pukhlikov

In this note we improve the theorem on Galois rational covers  $X \dashrightarrow V$  for primitive Fano varieties  $V$ , recently proven by the author, in the two directions: we extend to the maximum the class of Galois groups  $G$ , for which the proof works, and relax the conditions that must be satisfied by the variety  $V$  — the divisorial canonicity alone is sufficient.

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**1. Primitive Fano varieties.** All varieties considered in this note are defined over the field of complex numbers. Recall that a projective variety  $V$  of dimension  $M \geq 3$  is a *primitive Fano variety*, if it is factorial, has at most terminal singularities, and its anti-canonical class  $(-K_V)$  is ample and generates the Picard group,  $\text{Pic } V = \mathbb{Z}K_V$ .

A primitive Fano variety  $V$  is *divisorially canonical*, if for every effective divisor  $D \sim -nK_V$ , where  $n \geq 1$ , the pair  $(V, \frac{1}{n}D)$  is canonical: for every exceptional divisor  $E$  over  $V$  the inequality

$$\text{ord}_E D \leq n \cdot a(E),$$

which is opposite to the Noether-Fano inequality, holds.

The divisorial canonicity is a very strong property. For many families of higher-dimensional Fano varieties (including, for instance, hypersurfaces of degree  $M + 1$  in  $\mathbb{P}^{M+1}$  for  $M \geq 5$ ) it is known that a Zariski general variety in these families is divisorially canonical. The list of those families is given at the end of the paper.

**2. Galois rational covers.** Fix a divisorially canonical primitive Fano variety  $V$ . A rational map  $X \dashrightarrow V$  of a finite degree, where  $X$  is some projective variety, is called a *Galois rational cover*, if the corresponding field extension  $\mathbb{C}(V) \subset \mathbb{C}(X)$  is a Galois extension. In [1] it was shown that if the variety  $V$  (in addition to the condition of divisorial canonicity) satisfies the following two technical conditions:

(\*1) for every anti-canonical divisor  $R \in |-K_V|$ , every prime number  $p \geq 2$  and every, possibly reducible, closed subset  $Y \subset V$  of codimension  $\geq 2$  there is a non-singular curve  $N \subset V$ , such that

$$p \nmid (N \cdot K_V),$$

$N \cap Y = \emptyset$  and  $N$  meets  $R$  transversally at non-singular points,

(\*2) for every, possibly reducible, closed subset  $Y \subset V$  of codimension  $\geq 2$  there is a non-singular rational curve  $N \subset V$ , such that  $N \cap Y = \emptyset$ ,

then there are no Galois rational covers  $X \dashrightarrow V$  of degree  $\geq 2$  with an abelian Galois group  $G$ , where  $X$  is a rationally connected variety.

If the variety  $V$  is non-singular, then the condition (\*2) holds automatically, since  $V$  is rationally connected, see [2, Chapter II]. For a non-singular hypersurface  $V \subset \mathbb{P}^{M+1}$  of degree  $M + 1$  the condition (\*1) is easy to check, see [1, Sec. 3]. It is not hard to check this condition for non-singular Fano complete intersections in the projective space, too; however, we will show below that the conditions (\*1) and (\*2) are in fact unnecessary and can be dropped.

**3. The main result.** The aim of this note is to improve the theorem, shown in [1], in the two directions: firstly, to extend the class of Galois groups  $G$ , for which the proof given in [1] works to the maximum (answering a question of Yu. G. Prokhorov), and, secondly, to show that the conditions (\*1) and (\*2) are not needed. For a group  $G$  its commutant is denoted by the symbol  $[G, G]$ . If the equality  $G = [G, G]$  holds, then the group  $G$  is said to be perfect. The following claim is true.

**Theorem 1.** *For a divisorially canonical primitive Fano variety  $V$  there are no Galois rational covers  $X \dashrightarrow V$ , the Galois group  $G$  of which is not perfect ( $G \neq [G, G]$ ), where  $X$  is a rationally connected variety.*

**Proof.** We will show that the theorem stated above follows from the proof of the main result of [1] with minimal additional arguments. First of all, let us consider the question, for what class of groups the proof given in [1] works. Assume that  $X \dashrightarrow V$  is a Galois rational cover with some, not necessarily abelian, Galois group  $G$ . If  $G_1 \triangleleft G$  is a proper non-trivial normal subgroup, then the rational map  $X \dashrightarrow V$  is the composition of the rational maps

$$X \dashrightarrow X_1 \dashrightarrow V,$$

where  $\mathbb{C}(X_1) \subset \mathbb{C}(X)$  is a Galois extension with the Galois group  $G_1$ , and  $\mathbb{C}(V) \subset \mathbb{C}(X_1)$  is a Galois extension with the group  $G/G_1$ . Since the image of a rationally connected variety is rationally connected, the quotient group  $G/G_1$  is abelian if and only if  $G_1 \supset [G, G]$ , and, finally, in every finite abelian group there is a subgroup, the quotient group by which is a cyclic group of a prime order, Theorem 1 is equivalent to the following claim.

**Theorem 2.** *For a divisorially canonical primitive Fano variety  $V$  and every prime number  $p \geq 2$  there are no Galois rational covers  $X \dashrightarrow V$ , the Galois group of which is a cyclic group of order  $p$ , where  $X$  is a rationally connected variety.*

Now let us show that the proof of the main result of [1] gives Theorem 2 without using the conditions (1\*) and (2\*).

**4. Cyclic covers of the variety  $V$ .** Fix a prime number  $p \geq 2$  and a cyclic

cover  $\sigma: X \dashrightarrow V$  of order  $p$ , where  $X$  is a rationally connected variety, assuming that such covers exist. We may assume that  $X$  is a non-singular projective variety and  $\sigma$  is a morphism. In [1, Propositions 1,2] the following objects are constructed:

— a birational morphism  $\varphi: V^+ \rightarrow V$ , which is a composition of blow ups with non-singular centres, where  $V^+$  is a non-singular projective variety with the Picard group

$$\text{Pic } V^+ = \mathbb{Z}H \oplus \bigoplus_{i \in I} \mathbb{Z}E_i,$$

where  $H = -K_V$  is the anti-canonical class of the variety  $V$ , the ample generator of the group  $\text{Pic } V$  (we omit the pull back symbol  $\varphi^*$ ), and  $E_i$ ,  $i \in I$ , are all  $\varphi$ -exceptional prime divisors on  $V^+$ ,

— a non-singular quasi-projective variety  $U_X$ , a birational morphism  $\varphi_X: U_X \rightarrow X$  and a Zariski open subset  $U \subset V^+$ , such that

(i) the rational map

$$\varphi^{-1} \circ \sigma \circ \varphi_X: U_X \dashrightarrow V^+$$

extends to a morphism  $\sigma_U: U_X \rightarrow V^+$ , the image of which is  $U$ ,

(ii) the inequality

$$\text{codim}((V^+ \setminus U) \subset V^+) \geq 2$$

holds,

(iii) the map  $\sigma_U: U_X \rightarrow U$  is a cyclic cover of order  $p$ , branched over a non-singular hypersurface  $W \subset U$ .

Let  $\overline{W}$  be the closure of the effective divisor  $W$  in  $V^+$ . Then

$$\overline{W} \sim nH + \sum_{i \in I} \zeta_i E_i$$

for some  $n \in \mathbb{Z}_+$  and  $\zeta_i \in \mathbb{Z}$ . In [1, Sec. 3] it was shown (and this is the key step), that

$$n \in \{0, 1\}.$$

It is in order to exclude these two options that the conditions (\*2) (if  $n = 0$ ) and (\*1) (if  $n = 1$ ) were needed. However, we will show that these two cases are easily excluded by means of the explicit constructions in [1, Sec. 5]. This would complete the proof of Theorem 2, which implies Theorem 1.

**5. The explicit construction of a cyclic cover.** Since the variety  $V^+$  is obtained from  $V$  by means of a sequence of blow ups, and the codimension of the complement  $V^+ \setminus U$  is at least 2, there is an open subset  $U^+ \subset U$ , the image  $U_V = \varphi(U^+) \subset V$  of which on  $V$  is an open subset, and moreover,

$$\text{codim}((V \setminus U_V) \subset V) \geq 2$$

and the map  $\varphi|_{U^+}: U^+ \rightarrow U_V$  is an isomorphism. In order to construct the subset  $U^+$ , one should simply remove from  $U$  all closed subsets  $E_i \cap U$ ,  $i \in I$ : their image

on  $V$  is of codimension  $\geq 2$ . Set

$$U_X^+ = \sigma_U^{-1}(U^+),$$

so that  $\sigma_U^+ : U_X^+ \rightarrow U^+$  (where  $\sigma_U^+$  is obviously the restriction of the morphism  $\sigma_U$  onto  $U_X^+$ ) is the cyclic cover of order  $p$ , branched over a non-singular hypersurface  $W^+ \subset U^+$ . Identifying  $U^+$  and  $U_V$ , we can assume that  $U^+$  is an open subset of the original variety  $V$ . Obviously,

$$U^+ \subset V \setminus \text{Sing } V.$$

Let  $\overline{W^+} \subset V$  be the closure of the effective reduced divisor  $W^+$  in  $V$ . We have:

$$\overline{W^+} \sim nH.$$

For a hypersurface in the projective space the options  $n \in \{0, 1\}$  can be excluded from the purely topological grounds, but we will give an algebro-geometric proof, using only the general properties of the variety  $V$ , based on the explicit construction of a cyclic cover given in [1, Sec. 5]. Let us recall that construction. Since we are interested only what happens over an open subset  $U^+ = U_V \subset V$  with a small complement in  $V$ , we no longer need to consider the variety  $V^+$ .

Arguing as in [1, Sec. 5], we construct the variety  $X_0 \subset V \times \mathbb{P}^1_{(x_0:x_1)}$ , given by the equation

$$a_1 x_1^p - a_0 x_0^p = 0, \tag{1}$$

where  $a_0, a_1 \in H^0(V, \mathcal{O}_V(N))$  are sections without a common divisor of zeros on  $V$ . There is a commutative diagram of maps

$$\begin{array}{ccc} X & \xrightarrow{\beta} & X_0 \\ \sigma \downarrow & & \downarrow \pi \\ V & = & V, \end{array}$$

where the upper horizontal arrow  $\beta$  is a birational map and  $\pi$  is induced by the projection of the direct product  $V \times \mathbb{P}^1$  onto the first factor. Removing from  $U^+$  suitable subsets of codimension  $\geq 2$ , we may assume that the sections  $a_0, a_1$  have no common zeros on  $U^+$ , and the hypersurfaces

$$\{a_0|_{U^+} = 0\} \quad \text{and} \quad \{a_1|_{U^+} = 0\}$$

(in the set-theoretic sense) are non-singular — although possibly reducible. Let  $\mathcal{T}_V$  be the set of all prime divisors on  $V$ , on which one of the sections  $a_0, a_1$  vanishes, so that

$$\bigcup_{T \in \mathcal{T}_V} (T \cap U^+)$$

is a non-singular (possibly reducible) hypersurface. Set for  $T \in \mathcal{T}_V$

$$\mu(T) = \max\{\text{ord}_T a_0, \text{ord}_T a_1\}.$$

(Precisely one of the two integers in the right hand side is positive.) Let us show that the constructions of [1, Sec. 5] imply the following fact.

**Proposition 1.** *The branch hypersurface  $\overline{W^+}$  contains a divisor  $T \in \mathcal{T}_V$  if and only if  $p \nmid \mu(T)$ .*

(Over the complement

$$U^+ \setminus \bigcup_{T \in \mathcal{T}_V} (T \cap U^+)$$

the projection  $\pi$  is not ramified, and the variety  $X_0$  is non-singular; this is obvious from the equation (1).)

**6. Local modifications.** Let us prove Proposition 1, repeating the arguments of [1, Sec. 5] for the open set  $U^+ \subset V$  (from which we can, if necessary, remove closed subsets of codimension  $\geq 2$ ). Set  $\mathcal{X}_1 = U^+ \times \mathbb{P}^1$  and  $X_1 = X_0 \cap \text{pr}_V^{-1}(U^+)$ . Let us construct a sequence of locally-trivial  $\mathbb{P}^1$ -bundles over  $U^+$

$$\mathcal{X}_1 \xleftarrow{\beta_1} \mathcal{X}_2 \xleftarrow{\beta_2} \dots \xleftarrow{\beta_{k-1}} \mathcal{X}_k,$$

with projections  $\pi_i: \mathcal{X}_i \rightarrow U^+$ , in the following way. With respect to some trivialization of the  $\mathbb{P}^1$ -bundle  $\mathcal{X}_i/U^+$  over an open set, intersecting the divisor  $T \in \mathcal{T}_V$ , the hypersurface  $X_i$  — the strict transform of  $X_1$  on  $\mathcal{X}_i$  — is defined by the equation

$$a_{i,1}x_1^p - a_{i,0}x_0^p = 0,$$

where  $(x_0 : x_1)$  are homogeneous coordinates on  $\mathbb{P}^1$  and one of the regular functions, say  $a_{i,1}$ , does not vanish on  $T$ . Assume that

$$\text{ord}_T a_{i,0} \geq p.$$

Then the birational transformation

$$\beta_i: \mathcal{X}_{i+1} \rightarrow \mathcal{X}_i$$

is the composition of the blow up of the subvariety

$$T_i = \pi_i^{-1}(T) \cap X_i$$

and the subsequent contraction of the strict transform of the hypersurface  $\pi_i^{-1}(T) \subset \mathcal{X}_i$ . It is easy to check that locally in a neighborhood of the generic point of the divisor  $T$  the hypersurface  $X_{i+1} \subset \mathcal{X}_{i+1}$  is defined by the equation

$$a_{i+1,1}x_1^p - a_{i+1,0}x_0^p = 0,$$

where  $a_{i+1,1}|_T \neq 0$  and

$$\text{ord}_T a_{i+1,0} = \text{ord}_T a_{i,0} - p,$$

see [1, Sec. 5].

Now setting

$$\mu_i(T) = \max\{\text{ord}_T a_{i,0}, \text{ord}_T a_{i,1}\}$$

for every  $i = 1, \dots, k$  and  $T \in \mathcal{T}_V$ , we get that for every  $i = 1, \dots, k$  there is a precisely one divisor  $T(i) \in \mathcal{T}_V$ , such that

$$\mu_{i+1}(T(i)) = \mu_i(T(i)) - p,$$

and  $\mu_{i+1}(T) = \mu_i(T)$  for all  $T \neq T(i)$ . For the variety  $X_k \subset \mathcal{X}_k$  we have

$$\mu_k(T) \leq p - 1$$

for all  $T \in \mathcal{T}_V$ , and moreover,  $\mu_k(T) \equiv \mu(T) \pmod{p}$ .

Now for each  $T \in \mathcal{T}_V$  there are three options:

(0)  $\mu_k(T) = 0$ , and then the hypersurface  $X_k$  is not ramified over  $T$  and for that reason non-singular over  $T$ , so that  $T \not\subset \overline{W^+}$ ,

(1)  $\mu_k(T) = 1$ , and then the hypersurface  $X_k$  is ramified over  $T$  and non-singular over  $T$ , so that  $T \subset \overline{W^+}$ ,

(2)  $\mu_k(T) \in \{2, \dots, p-1\}$ , and then the variety  $X_k$  has a cuspidal singularity of the type

$$t^p - s^{\mu_k(T)} = 0$$

along the non-singular subvariety  $\pi_k^{-1}(T) \cap X_k$ , in terms of some local coordinates  $t, s$  on the plane; in that case the normalization of the variety  $X_k$  or the obvious sequence of blow ups along non-singular subvarieties, isomorphic to  $T$ , gives a variety, non-singular over  $T$ , that covers  $U^+$  cyclically, and this cyclic cover is ramified over  $T$ , so that here  $T \subset \overline{W^+}$ , too.

Since  $\mu_k(T) \equiv \mu(T) \pmod{p}$ , the proof of Proposition 1 is complete. Q.E.D.

**7. Exclusion of the cases  $n = 0$  and  $n = 1$ .** Let us complete the proof of Theorem 2. Assume that  $n = 0$ , that is to say, the hypersurface  $\overline{W^+}$  is empty. This means that  $\mu(T) \equiv 0 \pmod{p}$  for every  $T \in \mathcal{T}_V$ . It follows that  $p \mid N$  and we can “extract the root” from the sections  $a_0, a_1$  (see Sec. 5): there are sections

$$e_0, e_1 \in H^0(V, \mathcal{O}_V(N/p)),$$

such that  $a_0 = e_0^p$  and  $a_1 = e_1^p$ . But then the equation (1) takes the form

$$e_1^p x_1^p - e_0^p x_0^p = \prod_{i=1}^p (e_1 x_1 - \zeta^i e_0 x_0) = 0,$$

where  $\zeta = \exp(2\pi i/p)$ , that is, the variety  $X_0$  is reducible and is a union of  $p$  irreducible components, covering  $V$  birationally. This is impossible. The contradiction excludes the case  $n = 0$ .

Assume that  $n = 1$ . In that case there is a unique divisor  $T^* \in \mathcal{T}_V$ , for which  $p \nmid \mu(T^*)$ , and moreover,  $T^* \sim -K_V$  is a “hyperplane section” (the ample generator

of the Picard group) of the variety  $V$ . Since the sections  $a_0, a_1$  do not vanish simultaneously on any prime divisor, we have, say, that  $a_1|_{T^*} \equiv 0$  and  $a_0|_{T^*} \not\equiv 0$ , and  $p \mid \mu(T)$  for every prime divisor  $T \neq T^*$ ,  $T \in \mathcal{T}_V$ . Therefore, we get:  $\text{ord}_T a_1 \equiv 0 \pmod p$  for  $T \neq T^*$  and  $\text{ord}_{T^*} a_1 \not\equiv 0 \pmod p$ . Since  $a_1$  is a section of the sheaf  $\mathcal{O}_V(N)$ , this implies that

$$p \nmid N.$$

On the other hand,  $\text{ord}_T a_0 \equiv 0 \pmod p$  for all  $T \in \mathcal{T}_V$ . Since  $a_0$  is also a section of the sheaf  $\mathcal{O}_V(N)$ , we get that

$$p \mid N.$$

This contradiction excludes the case  $n = 1$  and completes the proof of Theorems 2 and 1.

**8. Divisorially canonical varieties.** To conclude, we give the list of families of Fano varieties, for a general divisor in which divisorial canonicity is known. In [3] divisorial canonicity is shown for Zariski general smooth hypersurfaces of degree  $M+1$  in  $\mathbb{P}^{M+1}$  for  $M \geq 5$  and (smooth Zariski general) double covers of the projective space  $\mathbb{P}^M$ , branched over a hypersurface of degree  $2M$  for  $M \geq 3$ . In [4] this result was improved: the divisorial canonicity was shown for Zariski general hypersurfaces of degree  $M+1$  in  $\mathbb{P}^{M+1}$ , with at worst quadratic singularities of rank  $\geq 8$ , for  $M \geq 9$ , and moreover, hypersurfaces that do not satisfy the condition of divisorial canonicity form a subset of codimension  $\geq \frac{1}{2}(M-6)(M-5)-5$  in  $\mathbb{P}(H^0(\mathbb{P}^{M+1}, \mathcal{O}_{\mathbb{P}^{M+1}}(M+1)))$ . For the double covers of the space  $\mathbb{P}^M$  in [4] a similar improvement was shown: for  $M \geq 10$  the double space, branched over a Zariski general hypersurface of degree  $2M$  with at worst quadratic singularities of rank  $\geq 4$ , is divisorially canonical, and the branch hypersurfaces, for which the corresponding double cover is not divisorially canonical, form a set of codimension  $\geq \frac{1}{2}(M-4)(M-1)$  in  $\mathbb{P}(H^0(\mathbb{P}^M, \mathcal{O}_{\mathbb{P}^M}(2M)))$ .

For a Zariski general non-singular complete intersection of type

$$d_1 \cdot d_2 \cdot \dots \cdot d_k$$

in  $\mathbb{P}^{M+k}$ , where  $2 \leq d_1 \leq d_2 \leq \dots \leq d_k$  and  $d_1 + \dots + d_k = M + k$ , where the inequality

$$M \geq 2k + 3$$

holds, the divisorial canonicity was shown in [5]. Before that paper, in [6] and [7] the divisorial canonicity was shown for smaller classes of complete intersections of index 1. In [8] the divisorial canonicity was shown for Zariski general smooth Fano double hypersurfaces of index 1 and dimension  $\geq 6$ .

In [9] the divisorial canonicity was established for Fano varieties of index 1 that are  $d$ -sheeted covers of  $\mathbb{P}^M$ , under the assumption that they have at worst quadratic singularities, the rank of which is bounded from below (the bound depends on the dimension  $M$  and the degree of the cover) and satisfy certain additional conditions of general position, and the varieties that are not divisorially canonical form a set, the codimension of which is bounded from below by an integer-valued function of the parameters  $d$  and  $M$ , which grows as  $\frac{1}{2}M^2$  when  $M$  grows.

Finally, for complete intersections of type  $d_1 \cdot d_2$  in  $\mathbb{P}^{M+2}$  the divisorial canonicity was shown for the varieties with at worst quadratic and bi-quadratic singularities, the rank of which is bounded from below, in [10], under the assumption that certain additional conditions of general position are satisfied, and for the codimension of the set of complete intersections that do not satisfy those conditions, an estimate, similar to the estimates above, was obtained.

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Department of Mathematical Sciences,  
The University of Liverpool

*pukh@liverpool.ac.uk*