Optimal regulators for a class of nonlinear stochastic systems

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Abstract

We consider a class of nonlinear stochastic systems with *square-root* nonlinearities appearing in the diffusion terms. The optimal control problems with *indefinite* quadratic criteria in both finite and infinite horizon are formulated and solved in an explicit closed-form. It turns out that *all* optimal controls are of an *affine* state-feedback form, despite the fact that the system is nonlinear. We use the method of completion of squares and *new* types of Riccati differential and algebraic equations to find the solutions. An application to the problem of optimal investment in a market with a stochastic interest rate is given.

Keywords: Optimal control; Nonlinear stochastic systems; Square-root nonlinearity; Optimal investment.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given complete probability space and $(W(t), t \geq 0)$ a one dimensional standard Brownian motion defined on this space. The filtration $(\mathcal{F}_W(t), t \geq 0)$ is defined as the augmentation of $\sigma\{W(s) : 0 \leq s \leq t\}$ by all the $\mathbb{P}$-null sets of $\mathcal{F}$. Consider the linear stochastic control system:

$$\begin{cases} dx_\ell(t) = [A_\ell(t)x_\ell(t) + B_\ell(t)u_\ell(t)]dt + [C_\ell(t)x_\ell(t) + D_\ell(t)u_\ell(t)]dW(t), \quad t \geq 0, \\ x_\ell(0) \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

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for some suitable deterministic matrix-valued functions $A_\ell(\cdot), B_\ell(\cdot), C_\ell(\cdot), D_\ell(\cdot)$ and an $\mathcal{F}_W(t)$-adapted control process $u_\ell(\cdot)$ such that (1.1) has a unique strong solution. The stochastic *linear-quadratic (LQ)* control problem is the optimal control problem of minimizing the quadratic cost functional

$$\mathbb{E}\left\{\int_0^T [x_\ell'(t)Q_\ell(t)x_\ell(t) + u_\ell'(t)R_\ell(t)u_\ell(t)]dt + x_\ell'(T)S_\ell x_\ell(T)\right\},$$

subject to (1.1). The LQ control problem has been studied extensively since its introduction by [18] for deterministic systems (see, e.g., [1], [11]). One of the first solutions to the stochastic LQ problem with multiplicative noise and deterministic coefficients was given in [37], [38] (see, e.g., [41], [12], for a texbook account). The case with random coefficients was considered by [3], which initiated the study of backward stochastic differential equations, and has been studied further in, for example, [28], [35], [36]. The case with a fixed final state was solved in [14], with state-dependent weights in [15], and for the mean-field setting see, for example, [39], [40]. A typical assumption in the LQ control problems is that the coefficients of the cost functional have certain definiteness properties. This assumption can be weakened further, and it leads to indefinite LQ control, see, for example, [6], [25], [7], [32], [33], [31], [30], [24], [16], [23], [26], [27].

In this paper we introduce a certain *nonlinear* generalisation to the stochastic LQ control problem that *retains* the explicit closed-form solvability. Let $(W_1(t), t \geq 0)$ and $(W_2(t), t \geq 0)$ be m-dimensional and η-dimensional standard Brownian motions, respectively, defined on our given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that $W(\cdot)$, $W_1(\cdot)$ and $W_2(\cdot)$ are mutually independent. The filtration $(\mathcal{F}(t), t \geq 0)$ is defined as the augmentation of $\sigma\{(W(s), W_1(s), W_2(s)) : 0 \leq s \leq t\}$ by all the $\mathbb{P}$-null sets of $\mathcal{F}$. Instead of the linear system (1.1), we consider the following nonlinear stochastic system with *square-root* nonlinearities in the diffusion terms:

$$
\begin{cases}
dx_1(t) = [G_1(t)x_1(t) + H_1(t)]dt + \Gamma_1(x_1(t),t)dW_1(t), \\
dx_2(t) = [A_1(t)x_1(t) + A_2(t)x_2(t) + B_1(t)u(t)]dt \\
\qquad\qquad +[C_1(t)x_1(t) + C_2(t)x_2(t) + D_1(t)u(t)]dW(t) \\
\qquad\qquad +\Gamma_2(x_1(t),x_2(t),u(t),t)dW_2(t), \\
x_1(0) = x_{10} \in \mathbb{R}^m_+, \quad x_2(0) = x_{20} \in \mathbb{R}^n.
\end{cases} \tag{1.2}
$$

Thus, the linear system (1.1) has been expanded to include additional diffusion terms $\Gamma_1(x_1(t),t)$ and $\Gamma_2(x_1(t),x_2(t),u(t),t)$ which we assume to be of square-root nonlinear form as follows:

$$
\begin{aligned}
\Gamma_1(x_1(t),t) &:= \mathrm{diag}(\sqrt{x_{11}(t)}, \sqrt{x_{12}(t)}, \ldots, \sqrt{x_{1m}(t)}), && (1.3)\\
\Gamma_2(x_1(t),x_2(t),u(t),t) &:= E(t)\mathrm{diag}(\sqrt{\phi_1}, \sqrt{\phi_2}, \ldots, \sqrt{\phi_\eta}), && (1.4)
\end{aligned}
$$

where $\mathrm{diag}(\cdot)$ denotes the diagonal matrix; $x_{1i}(\cdot)$, $i = 1, ..., m$, are the elements of $x_1(\cdot)$; and

$$
\phi_k := x_2(t)'Q_k(t)x_2(t) + u(t)'R_k(t)u(t) + x_1(t)'L_k(t) + Z_k(t), \quad k = 1, 2, \ldots, \eta.
$$

Denoting by $L^\infty(0,T;\mathbb{R}^{n\times m})$ the set of uniformly bounded matrices of dimension $n \times m$ (and $L^\infty(0,T;\mathcal{S}^{n\times m})$ if the matrix is also symmetric) our assumptions on the coefficients of (1.2) are:

$$
\begin{aligned}
&H_1(\cdot), L_k(\cdot) \in L^\infty(0,T;\mathbb{R}^m), \quad A_1(\cdot), C_1(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times m}),\\
&A_2(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n}), C_2(\cdot) \in L^\infty(0,T;\mathbb{R}^{m\times m}),\\
&B_1(\cdot), D_1(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n_u}), \quad E(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times \eta}),\\
&Q_k(\cdot) \in L^\infty(0,T;\mathcal{S}^n), \quad R_k(\cdot) \in L^\infty(0,T;\mathcal{S}^{n_u}),\\
&\mathrm{diag}(g_1(\cdot), ..., g_m(\cdot)) =: G_1(\cdot) \in L^\infty(0,T;\mathbb{R}^{m\times m}).
\end{aligned}
$$

We further assume that $Q_k(t) \geq 0$, $R_k(t) \geq 0$, $Z_k(t) > 0$, and the components of $L_k(t)$ are non-negative for $k = 1, 2, ..., \eta$. The system (1.2) can be written in the following more compact form:

$$
\begin{cases}
dx(t) = [A(t)x(t) + B(t)u(t) + H(t)]dt + [C(t)x(t) + D(t)u(t)]dW(t) \\
\qquad\qquad +\theta_1(x(t),u(t),t)dW_1(t) + \theta_2(x(t),u(t),t)dW_2(t), \\
x'(0) \in \mathbb{R}^{1\times m}_+ \times \mathbb{R}^{1\times n},
\end{cases} \tag{1.5}
$$

where $x'(\cdot) := [x_1'(\cdot), x_2'(\cdot)]$ and

$$A(t) := \begin{bmatrix} G_1(t) & 0 \\ A_1(t) & A_2(t) \end{bmatrix}, \quad B(t) := \begin{bmatrix} 0 \\ B_1(t) \end{bmatrix}, \quad C(t) := \begin{bmatrix} 0 & 0 \\ C_1(t) & C_2(t) \end{bmatrix},$$
$$D(t) := \begin{bmatrix} 0 \\ D_1(t) \end{bmatrix}, H(t) := \begin{bmatrix} H_1(t) \\ 0 \end{bmatrix}, \quad \theta_1(x(t), t) := \begin{bmatrix} \Gamma_1(x_1(t), t) \\ 0 \end{bmatrix},$$
$$\theta_2(x(t), u(t), t) := \begin{bmatrix} 0 \\ \Gamma_2(x_1(t), x_2(t), u(t), t) \end{bmatrix}.$$

The motivation for considering system (1.5) is twofold: it is a *rare* case of a nonlinear stochastic control system for which we can solve an optimal control problem in an explicit closed form, and it has the potential for applications as illustrated in section 4. The equation for the state $x_1(\cdot)$ is a multi-dimensional version of the well-known Cox-Ingersoll-Ross (CIR) model, which has wide applicability in modelling interest rates (see, for example, [9], [34], [4]). The nonlinearities in the state $x_2(\cdot)$ without the control variables and the linear dependence on the state $x_1(\cdot)$, are also known and the resulting equation admits an explicit solution in some cases (see, for example, [21]), and have been used to model nonlinear stochastic uncertainties in the control system model (see [5], [17]). However, the stochastic control system (1.5) where the states $x_1(\cdot)$ and $x_2(\cdot)$ are coupled, and the nonlinear diffusion parts of $x_2(\cdot)$ contain nonlinearly the control variables, appears to not have been considered before (for example, it does not fall within the nonlinear stochastic control systems considered in [8], [20]).

We associate with (1.5) the following *quadratic-linear* functional:

$$\begin{aligned} J(u(\cdot)) \;\; := \;\; & \mathbb{E}\Bigg\{ \int_0^T \left(\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}' \begin{bmatrix} Q(t) & L(t) \\ L(t)' & R(t) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + x(t)' L_d(t) + u(t)' L_e(t) \right) dt \\ & + x(T)' \bar{H} x(T) + \bar{L}_c' x(T) \Bigg\}, \end{aligned}$$

where

$$R(\cdot) \in L^\infty(0, T; \mathcal{S}^{n_u}), \quad Q(\cdot) \in L^\infty(0, T; \mathcal{S}^{m+n}), \quad L(\cdot) \in L^\infty(0, T; \mathbb{R}^{(m+n)\times n_u}),$$
$$L_d(\cdot) \in L^\infty(0, T; \mathbb{R}^{m+n}), \quad L_e(\cdot) \in L^\infty(0, T; \mathbb{R}^{n_u}), \quad \bar{H} \in \mathcal{S}^{m+n}, \quad \bar{L}_c \in \mathbb{R}^{m+n}.$$

Our aim is in finding the solution to the following optimal control problem:

$$\begin{cases} \min\limits_{u(\cdot)\in\mathcal{A}} J(u(\cdot)) \\ \\ \text{s.t. } (1.5), \end{cases} \tag{1.6}$$

where $\mathcal{A}$ is a suitable admissible set of controls to be defined precisely later.

It is thus clear that the problem (1.6) is an optimal stochastic control problem with a *general* quadratic-linear cost functional, the weighting matrices of which we permit to be *indefinite* in general, and the control system dynamics are nonlinear with a multiplicative noise. This nonlinear generalization of the stochastic LQ control problem is thus a new. The solutions to such optimal control problems in an explicit closed-form, while desirable from the applications point of view, are scarce. Indeed, due to the square-root nonlinearity of the problem, the usual Lipschitz type conditions for the existence of the solution, which are essential in defining the admissible set of controls, do not apply directly. Moreover, due to the nonlinearity of the system, it might be expected that the solution could be a nonlinear control law. If one is to consider an infinite horizon version of the problem (1.6), then an additional problem is the *stability* of the system, which is again a difficult problem when nonlinear systems are concerned.

In this paper, we make the following contributions to this problem:

a) In section 2 we find all solutions to the problem (1.6) in an explicit closed-form. In order to achieve this, we first introduce a new type of a Riccati differential equation with algebraic equality and inequality constraints which have additional new terms. The introduction of this equation was necessitated by the nonlinearities in the system dynamics, and these do not appear in the classical LQ control problem. Moreover, a certain more involved version of the completion of squares method is employed in deriving the solutions. In turns out that although the system dynamics are nonlinear, all solutions to the control problem (1.6) are of an affine state-feedback form. We further show that under such controls the equations of the system dynamics have unique strong solutions.

b) In section 3 we formulate an infinite horizon version of the problem

(1.6) by using an infinite-horizon *average* cost functional. All solutions to such an optimal control problem are derived as well in an explicit closed-form. Moreover, we also address the stability problem of the system under the optimal controls. Due to the quadratic-linear nature of the cost functional and the nonlinearities in the system, the known stability results for linear or nonlinear stochastic systems could not be applied, and we thus had to prove this result directly.

c) In section 4 we consider one application of our results. There we consider an optimal investment problem in a market with nonlinear stochastic interest rate. This problem, in its original formulation, is not of the type (1.6). However, after a particular reformulation, we show that it turns out to be an example of the optimal control problem (1.6). We have thus obtained the solution to such a fundamental problem of financial engineering as an application of our general results.

Before we proceed further, recall the following useful facts for the Moore-Penrose generalised inverses (see, for example, [29], [30]).

Lemma 1. *Let a matrix $M \in \mathbb{R}^{m\times n}$ be given. Then there exists a unique matrix $M^\dagger \in \mathbb{R}^{n\times m}$ such that:*

$$\begin{cases} MM^\dagger M = M, & M^\dagger M M^\dagger = M^\dagger, \\ (MM^\dagger)' = MM^\dagger, & (M^\dagger M)' = M^\dagger M, \end{cases} \tag{1.7}$$

where the matrix $M^\dagger$ is called the Moore-Penrose pseudo inverse of M.

Lemma 2. *For a symmetric matrix S, the following hold:*

(i) $S^\dagger = (S^\dagger)'$,

(ii) $SS^\dagger = S^\dagger S$,

(iii) $S \geq 0$ *if and only if* $S^\dagger \geq 0$.

Lemma 3. *If L, M, and N are given matrices, then the matrix equation*

$$LXM = N, \tag{1.8}$$

has a solution X if and only if

$$LL^{\dagger}NM^{\dagger}M = N. \tag{1.9}$$

In this case, any solution to (1.8) can be represented as

$$X = L^{\dagger}NM^{\dagger} + S - L^{\dagger}LSMM^{\dagger},$$

for some S of appropriate dimensions.

2. Optimal regulator in finite horizon

In this section we give the solution to the problem (1.6). The set of all solutions is found in an explicit closed form through certain ordinary differential equations with algebraic constraints. We begin first by considering these equations and their solvability, which also enables us to define the admissible set $\mathcal{A}$ of controls. Throughout the paper, when no confusion arises, we omit the argument t for notational simplicity.

Let $M := [0' \quad I]'$ be an $(m+n)\times n$ matrix with the identity matrix I being of dimension n. Using the unit vectors e_k of dimension η, we define $N_k(t) := ME(t)e_k$, $\forall\ t \geq 0$, $k = 1, 2, ..., \eta$. The following is our Riccati differential equation:

$$\begin{cases} \dot{P} + PA + A'P + C'PC + \sum_{k=1}^{\eta} N_k'PN_kMQ_kM' + Q \\ \quad - (C'PD + PB + L)(D'PD + \sum_{k=1}^{\eta} N_k'PN_kR_k + R)^{\dagger}(D'PC + B'P + L') = 0, \\ P(T) = \bar{H}, \\ (D'PD + \sum_{k=1}^{\eta} N_k'PN_kR_k + R)(D'PD + \sum_{k=1}^{\eta} N_k'PN_kR_k \\ \quad + R)^{\dagger}(D'PC + B'P + L') - (D'PC + B'P + L') = 0, \\ D'PD + \sum_{k=1}^{\eta} N_k'PN_kR_k + R \geq 0, \qquad \text{a.e.} \quad t \in [0, T]. \end{cases} \tag{2.1}$$

This equation has great similarity with the Riccati differential equation of indefinite LQ control problem of [30], but it is more general as it has the additional terms $\sum_{k=1}^{\eta} N_k' P N_k M Q_k M'$ and $\sum_{k=1}^{\eta} N_k' P N_k R_k$, which are due to the nonlinearities ϕ_k.

Assumption 1. *Equation (2.1) has a unique solution.*

While there are no general sufficient conditions that ensure this assumption holds, which is also the case for the simpler equation of LQ control in [30], there is one important case for which we have such conditions. If $D'PD + \sum_{k=1}^{\eta} N_k' P N_k R_k + R > 0$, equation (2.1) reduces to:

$$\begin{cases} \dot{P} + PA + A'P + C'PC + \sum_{k=1}^{\eta} N_k' P N_k M Q_k M' + Q \\ \quad - (C'PD + PB + L)(D'PD + \sum_{k=1}^{\eta} N_k' P N_k R_k + R)^{-1}(D'PC + B'P + L') = 0, \\ P(T) = \bar{H}, \\ D'PD + \sum_{k=1}^{\eta} N_k' P N_k R_k + R > 0, \quad \text{a.e.} \qquad t \in [0,T], \end{cases} \tag{2.2}$$

Due to the terms $\sum_{k=1}^{\eta} N_k' P N_k M Q_k M'$ and $\sum_{k=1}^{\eta} N_k' P N_k R_k$, we cannot apply known results to show the solvability of (2.2), and thus need to rewrite them in a more convenient form as follows. Let $\bar{Q}_k := M Q_k M'$ and $\Lambda := E'M'PME$. Since the matrices Q_k are assumed nonnegative, then so are matrices $\bar{Q}_k$ and can thus be written as $\bar{Q}_k = \bar{Q}_k^{1/2} \bar{Q}_k^{1/2}$. Denoting by Λ_{kk} the kk'th element of Λ, which is of course $\Lambda_{kk} = e_k' \Lambda e_k$, we have:

$$\sum_{k=1}^{\eta} N_k' P N_k M Q_k M' = \sum_{k=1}^{\eta} \bar{Q}_k^{1/2} \Lambda_{kk} \bar{Q}_k^{1/2}. \tag{2.3}$$

Let $\xi_{k\tau}$ be a $\eta \times (m+n)$-dimensional matrix in which the $k\tau$'th element is 1, whereas all other elements are 0. The matrix $\Lambda_{kk} I$ can be written as

$$\Lambda_{kk} I = \sum_{\tau=1}^{m+n} \xi_{k\tau}' \Lambda \xi_{k\tau},$$

which when substituted in (2.3) gives

$$\begin{aligned}\sum_{k=1}^{\eta} N_k' P N_k M Q_k M' &= \sum_{k=1}^{\eta} \bar{Q}_k^{1/2} \left(\sum_{\tau=1}^{m+n} \xi_{k\tau}' \Lambda \xi_{k\tau} \right) \bar{Q}_k^{1/2} \\ &= \sum_{k=1}^{\eta} \sum_{\tau=1}^{m+n} \bar{Q}_k^{1/2} \xi_{k\tau}' E' M' P M E \xi_{k\tau} \bar{Q}_k^{1/2}.\end{aligned}$$

The term $\sum_{k=1}^{\eta} N_k' P N_k R_k$ can be rewritten in a similar form. The solvability of (2.2) now follows from the known results on Riccati equations, such as [7], [31], under different assumptions on Q, R, and $\bar{H}$ (e.g. if $Q \geq 0$, $R > 0$, $\bar{H} \geq 0$, then (2.2) has a unique solution).

We now introduce another differential equation. Let ϵ_a be a unit vector of dimension m, $b_a := [\epsilon_a' \quad 0]$ an $(m+n)$-dimensional vector, and $\tilde{M}' := [I \quad 0]$ an $m \times (m+n)$-dimensional matrix. Consider the equation:

$$\begin{cases} \dot{L}_c + A' L_c + 2PH + \sum_{a=1}^{m} b_a' P_{aa} + \sum_{k=1}^{\eta} N_k' P N_k \tilde{M} L_k + L_d - (C'PD \\ \quad + PB + L)(D'PD + \sum_{k=1}^{\eta} N_k' P N_k R_k + R)^{\dagger}(L_e + B' L_c) = 0, \quad t \in [0, T], \\ \\ (D'PD + \sum_{k=1}^{\eta} N_k' P N_k R_k + R)(D'PD + \sum_{k=1}^{\eta} N_k' P N_k R_k \\ \quad + R)^{\dagger}(L_e + B' L_c) - (L_e + B' L_c) = 0, \qquad \text{a.e.} \quad t \in [0, T], \\ \\ L_c(T) = \bar{L}_c, \end{cases} \tag{2.4}$$

where P_{aa} denotes the aa'th element of P. This is thus a linear differential equation with a certain equality constraint.

Assumption 2. *Equation (2.4) has a unique solution.*

Similarly to the case of Riccati equation, if $D'PD + \sum_{k=1}^{\eta} N_k' P N_k R_k + R >$

0, equation (2.4) reduces to:

$$\begin{cases} \dot{L}_c + A'L_c + 2PH + \sum_{a=1}^{m} b_a' P_{aa} + \sum_{k=1}^{\eta} N_k' P N_k \tilde{M} L_k + L_d - L_e - B'L_c = 0, \quad t \in [0,T], \\ L_c(T) = \bar{L}_c, \end{cases}$$

which has a unique solution due to its linearity and our assumptions on the coefficients.

We can now define the admissible set of controls $\mathcal{A}$. Let $L^0_{\mathcal{F}}(0,T;\mathbb{R}^n)$ and $L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$ denote the set $\mathcal{F}(t)$-adapted processes and square-integrable processes, respectively, of dimension n. Also let:

$$\begin{aligned} \rho(x(t),u(t),t) &:= (2x'P + L_c)(Cx + Du), \\ \rho_1(x(t),u(t),t) &:= (2x'P + L_c)\theta_1(x(t),u(t),t), \\ \rho_2(x(t),u(t),t) &:= (2x'P + L_c)\theta_2(x(t),u(t),t). \end{aligned}$$

These terms appear as diffusion coefficients of the differential of $x'(t)P(t)x(t)+x'(t)L_c(t)$. We consider only the following set of control processes:

$$\begin{aligned} \mathcal{A} \;:=\; & \{u(\cdot) \in L^0_{\mathcal{F}}(0,T;\mathbb{R}^{n_u}) : \text{equation (1.5) has a unique strong solution on } [0,T], \\ & \rho(x(t),u(t),t) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}),\ \rho_1(x(t),u(t),t) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{1\times m}), \\ & \rho_2(x(t),u(t),t) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{1\times \eta})\}. \end{aligned}$$

The optimal control problem to be solved (1.6) is now fully defined and we can proceed to its solution. It is convenient to first define the following matrices:

$$
\begin{aligned}
\bar{R}(t) &:= D(t)'P(t)D(t) + \sum_{k=1}^{\eta} N_k(t)'P(t)N_k(t)R_k(t) + R(t), \\
\bar{X}(t) &:= 2(D(t)'P(t)C(t) + B(t)'P(t) + L(t)'), \\
\bar{Y}(t) &:= L_e(t) + B(t)'L_c(t), \\
\bar{S}(t) &:= \dot{P}(t) + P(t)A(t) + A(t)'P(t) + C(t)'P(t)C \\
&\quad + \sum_{k=1}^{\eta} N_k(t)'P(t)N_k(t)MQ_k(t)M' + Q(t) \\
\bar{T}(t) &:= 2P(t)H(t) + \sum_{a=1}^{m} b_a' P_{aa}(t) + \sum_{k=1}^{\eta} N_k(t)'P(t)N_k(t)\tilde{M}L_k(t) \\
&\quad + \dot{L}_c(t) + A(t)'L_c(t) + L_d(t), \\
\bar{Z}(t) &:= \sum_{k=1}^{\eta} N_k(t)'P(t)N_k(t)Z_k(t) + L_c'(t)H(t), \\
\zeta(t) &:= \sum_{k=1}^{\eta} N_k(t)'P(t)N_k(t)Z_k(t) + L_c'(t)H(t) \\
&\quad - \frac{1}{4}[L_e(t) + B(t)'L_c(t)]'[D(t)'P(t)D(t) + \sum_{k=1}^{\eta} N_k(t)'P(t)N_k(t)R_k(t) \\
&\quad + R(t)]^{\dagger}[L_e(t) + B(t)'L_c(t)].
\end{aligned}
$$

If $Y(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n_u \times (m+n)})$ and $z(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n_u})$ are two given processes, we define:

$$
\begin{aligned}
\Psi^1(t) &:= Y(t) - \bar{R}^{\dagger}\bar{R}Y(t), \\
\Psi^2(t) &:= z(t) - \bar{R}^{\dagger}\bar{R}z(t), \\
u^*(t) &:= -\left(\frac{1}{2}\bar{R}^{\dagger}\bar{X} + \Psi^1(t)\right)x(t) - \Psi^2(t) - \frac{1}{2}\bar{R}^{\dagger}\bar{Y}. \qquad (2.5)
\end{aligned}
$$

Theorem 1. *If the processes $Y(\cdot)$ and $z(\cdot)$ are such that $u^*(\cdot) \in \mathcal{A}$, then $u^*(\cdot)$ is a solution to the optimal control problem (1.6). The corresponding optimal cost is*

$$
J(u^*(\cdot)) = x_0'P(0)x(0) + L_c(0)'x_0 + \int_0^T \zeta(t)dt.
$$

Moreover, any optimal control is of the form (2.5).

Before we give the proof to this main result, let us emphasize that the solution to the optimal control problem (1.6) is not unique in general. Indeed, for any pair $Y(\cdot)$ and $z(\cdot)$ that satisfies the requirement of $u^*(\cdot) \in \mathcal{A}$, we obtain one solution. In the special case of $\bar{R} > 0$ the processes $\Psi^1(t)$ and $\Psi^2(t)$ are zero. This means $u^*(\cdot)$ is independent of $Y(\cdot)$ and $z(\cdot)$ and it thus represents the unique optimal control process, which is an affine function of state.

Proof of Theorem 1. For any $u(\cdot) \in \mathcal{A}$ the following holds:

$$\begin{aligned}
\mathbb{E}[x(T)'P(T)x(T)] &= x_0'P(0)x_0 + \mathbb{E}\Bigg[\int_0^T \Big\{x'\dot{P}(t)x + 2x'P(t)[A(t)x + B(t)u + H(t)] \\
&+ [C(t)x + D(t)u]'P(t)[C(t)x + D(t)u] + \mathrm{tr}[\theta_1(x,t)'P(t)\theta_1(x,t)] \\
&+ \mathrm{tr}[\theta_2(x,u,t)'P(t)\theta_2(x,u,t)\Big]\Big\}dt\Bigg].
\end{aligned}$$

The trace terms contain quadratic forms of the nonlinearities θ_1 and θ_2 and can be written as:

$$\begin{aligned}
\mathrm{tr}[\theta_1(x,t)'P(t)\theta_1(x,t)] &= x'\sum_{a=1}^{m} b_a' P_{aa}. \\
\mathrm{tr}[\theta_2(x,u,t)'P(t)\theta_2(x,u,t)] &= x'\left(\sum_{k=1}^{\eta} N_k(t)'P(t)N_k(t)MQ_k(t)M'\right)x \\
&+ u'\left(\sum_{k=1}^{\eta} N_k(t)'P(t)N_k(t)R_k(t)\right)u \\
&+ x'\sum_{k=1}^{\eta} N_k(t)'P(t)N_k(t)\tilde{M}L_k(t) \\
&+ \sum_{k=1}^{\eta} N_k(t)'P(t)N_k(t)Z_k(t).
\end{aligned}$$

We can thus write $\mathbb{E}[x(T)'P(t)x(T)]$ as:

$$\begin{aligned}
&\mathbb{E}[x(T)'P(t)x(T)] \\
= \;& x_0'P(0)x_0 + \mathbb{E}\Bigg[\int_0^T \Big\{x'\Big[\dot{P}(t) + P(t)A(t) + A(t)'P(t) + C(t)'P(t)C(t) \\
&+ \sum_{k=1}^{\eta} N_k(t)'P(t)N_k(t)MQ_k(t)M'\Big]x + 2u'[D(t)'P(t)C(t) + B(t)'P(t)]x \\
&+ u'\left[D(t)'P(t)D(t) + \sum_{k=1}^{\eta} N_k(t)'P(t)N_k(t)R_k(t)\right]u \\
&+ x'\left(2P(t)H(t) + \sum_{a=1}^{m} b_a'P_{aa}(t) + \sum_{k=1}^{\eta} N_k(t)'P(t)N_k(t)\tilde{M}L_k(t)\right) \\
&+ \sum_{k=1}^{\eta} N_k(t)'P(t)N_k(t)Z_k(t)\Big\}dt\Bigg]. \qquad (2.6)
\end{aligned}$$

For any $u(\cdot) \in \mathcal{A}$, the following holds:

$$\mathbb{E}[L_c(T)'x(T)] = L_c(0)'x_0 + \mathbb{E}\int_0^T [(\dot{L}_c' + L_c'A)x + L_c'Bu + L_c'H]dt. \qquad (2.7)$$

Be combining (2.6) and (2.7), we obtain:

$$\mathbb{E}[x(T)'\bar{H}x(T) + \bar{L}'x(T)] = x_0'P(0)y + L_c(0)'x_0 + \mathbb{E}\int_0^T \Theta(t, x(t), u(t))dt,$$

where

$$
\begin{aligned}
& \Theta(t, x(t), u(t)) \\
:= \quad & x'\{[\dot{P}(t) + P(t)A(t) + A(t)'P(t) + C(t)'P(t)C \\
& + \sum_{k=1}^{\eta} N_k(t)'P(t)N_k(t)MQ_k(t)M']\}x \\
& + 2u'[D(t)'P(t)C(t) + B(t)'P(t)]x + u'[D(t)'P(t)D(t) \\
& + \sum_{k=1}^{\eta} N_k(t)'P(t)N_k(t)R_k(t)]u + u'B(t)'L_c(t) \\
& + x'[2P(t)H(t) + \sum_{a=1}^{m} b_a' P_{aa}(t) + \sum_{k=1}^{\eta} N_k(t)'P(t)N_k(t)\tilde{M}L_k(t) \\
& + \dot{L}_c(t) + A(t)'L_c(t)] + \sum_{k=1}^{\eta} N_k(t)'P(t)N_k(t)Z_k(t) + L_c'(t)H(t).
\end{aligned}
$$

The cost functional $J(\cdot)$ can now be written as:

$$
\begin{aligned}
J(u(\cdot)) \quad = \quad & x_0'P(0)x_0 + L_c(0)'x_0 + \mathbb{E}\int_0^T [\Theta(t, x(t), u(t)) + x(t)'Q(t)x(t) \\
+ \quad & 2u(t)'L(t)'x(t) + u(t)'R(t)u(t) + x(t)'L_d(t) + u(t)'L_e(t)]dt
\end{aligned}
\tag{2.8}
$$

By the completion of square we obtain the following for the integrand:

$$
\begin{aligned}
& \Theta(t, x, u) + x'Q(t)x + 2u'L(t)'x + u'R(t)u + x'L_d(t) + u'L_e(t) \\
= \quad & u'\bar{R}u + u'\bar{X}x + u'\bar{Y} + x'\bar{S}x + x'\bar{T} + \bar{Z} \\
= \quad & \left[u + \frac{1}{2}\bar{R}^{\dagger}\bar{X}x + \frac{1}{2}\bar{R}^{\dagger}\bar{Y}\right]' \bar{R} \left[u + \frac{1}{2}\bar{R}^{\dagger}\bar{X}x + \frac{1}{2}\bar{R}^{\dagger}\bar{Y}\right] - \frac{1}{4}x'\bar{X}'\bar{R}^{\dagger}\bar{X}x \\
& - \frac{1}{2}x'\bar{X}'\bar{R}^{\dagger}\bar{Y} - \frac{1}{4}\bar{Y}\bar{R}^{\dagger}\bar{Y} + x'\bar{S}x + x'\bar{T} + \bar{Z}.
\end{aligned}
$$

By Lemma 1, Lemma 2-(ii), and Assumption 1 we have: $\bar{R}\Psi^{\gamma}(t) = \bar{R}^{\dagger}\Psi^{\gamma}(t) = 0$, and $\bar{X}'\Psi^{\gamma}(t) = 0$, where $\gamma = 1, 2$. Hence, the integrand (2.9) can be written as:

$$
\begin{aligned}
& \Theta(t, x, u) + x'Q(t)x + 2u'L(t)'x + u'R(t)u + x'L_d(t) + u'L_e(t) \\
= \quad & \left[u + \left(\frac{1}{2}\bar{R}^{\dagger}\bar{X} + \Psi^1(t)\right)x + \Psi^2(t) + \frac{1}{2}\bar{R}^{\dagger}\bar{Y}\right]' \bar{R} \left[u + \left(\frac{1}{2}\bar{R}^{\dagger}\bar{X} + \Psi^1(t)\right)x\right. \\
& \left. + \Psi^2(t) + \frac{1}{2}\bar{R}^{\dagger}\bar{Y}\right] + x'\left(\bar{S} - \frac{1}{4}\bar{X}'\bar{R}^{\dagger}\bar{X}\right)x + x'\left(\bar{T} - \frac{1}{2}\bar{X}'\bar{R}^{\dagger}\bar{Y}\right) + \bar{Z} - \frac{1}{4}\bar{Y}'\bar{R}^{\dagger}\bar{Y}.
\end{aligned}
$$

Due to Assumption 1 and Assumption 2 we have $\bar{S} - \frac{1}{4}\bar{X}'\bar{R}^\dagger\bar{X} = 0$ and $\bar{T} - \frac{1}{2}\bar{X}'\bar{R}^\dagger\bar{Y} = 0$. Thus for any $u(\cdot) \in \mathcal{A}$ the cost functional becomes:

$$
\begin{aligned}
J(u(\cdot)) &= x_0'P(0)x_0 + L_c(0)'x_0 + \int_0^T \zeta(t)dt \\
&\quad + \mathbb{E}\int_0^T \left[u + \left(\frac{1}{2}\bar{R}^\dagger\bar{X} + \Psi^1(t)\right)x + \Psi^2(t) + \frac{1}{2}\bar{R}^\dagger\bar{Y}\right]' \\
&\quad \times \bar{R}\left[u + \left(\frac{1}{2}\bar{R}^\dagger\bar{X} + \Psi^1(t)\right)x + \Psi^2(t) + \frac{1}{2}\bar{R}^\dagger\bar{Y}\right]dt \qquad (2.9) \\
&\geq x_0'P(0)x_0 + L_c(0)'x_0 + \int_0^T \zeta(t)dt.
\end{aligned}
$$

If $u(t) = u^*(t)$ for *a.e.* $t \in [0,T]$ *a.s.*, this lower bound is achieved. This proves that $u^*(\cdot)$ is an optimal control.

We now show that any optimal control process $u_o(\cdot) \in \mathcal{A}$ *must* be of the form (2.5) for some $Y(\cdot)$ and $z(\cdot)$. As the representation (2.9) holds for any admissible control process, it implies that it is necessary to have:

$$
\bar{R}^{1/2}\left[u_o + \left(\frac{1}{2}\bar{R}^\dagger\bar{X} + \Psi^1(t)\right)x + \Psi^2(t) + \frac{1}{2}\bar{R}^\dagger\bar{Y}\right] = 0.
$$

This further implies (after multiplication from the left by $\bar{R}^{1/2}$):

$$
\begin{aligned}
\bar{R}\left[u_o + \left(\frac{1}{2}\bar{R}^\dagger\bar{X} + \Psi^1(t)\right)x + \Psi^2(t) + \frac{1}{2}\bar{R}^\dagger\bar{Y}\right] &= 0, \\
\bar{R}u_o + \frac{1}{2}\bar{R}\bar{R}^\dagger(\bar{X}x + \bar{Y}) &= 0.
\end{aligned}
$$

From Lemma 3, it is clear that this last equation with u_o as the unknown is of the type (1.8), it satisfies condition (1.9) and thus has a solution, and all solutions are given as:

$$
u_o = -\frac{1}{2}\bar{R}^\dagger(\bar{X}x + \bar{Y}) + z - \bar{R}\bar{R}^\dagger z,
$$

which indeed has the form (2.5). $\square$

Theorem 1 provides an explicit analytical solution to the considered stochastic optimal control problem. It is notable that the optimal controls turn out to be of affine linear state-feedback form. In order to apply the previous theorem, one needs to check if $u^*(\cdot) \in \mathcal{A}$ for the particular $Y(\cdot)$ and $z(\cdot)$, i.e. one needs to check if equation (1.5) has a unique strong solution with sufficient integrability under such a control.The following result gives a sufficient condition for the admissibility of $u^*(\cdot)$.

Theorem 2. *If $\bar{R}^\dagger(\cdot)$ and $Y(\cdot)$ are uniformly bounded, then $u^*(\cdot) \in \mathcal{A}$.*

Proof. The state equation $x_1(\cdot)$ is a set of decoupled CIR stochastic differential equations. Each of such equations has a unique *positive* solution with any degree of integrability (see, for example, [9], [34]). Therefore, under the control $u^*(\cdot)$, the equation for the state $x_2(\cdot)$ is a stochastic differential equation with the process $x_2(\cdot)$ as the only unknown, i.e. the state $x_1(\cdot)$ appears only as a known process in such an equation. Moreover, such an equation, due to the process $Y(\cdot)$ is with a random generator (see, for example, section 1.6.4 of [41]). We now focus in showing that such an equation has a unique strong solution with the required integrability. According to Theorem 1.6.16 of [41]), it is sufficient to show that the drift and the diffusion terms in the equation of $x_2(\cdot)$ satisfy a uniform Lipschitz condition. Since the drift term in that equation is linear, and $Y(\cdot)$ is assumed bounded, this condition is satisfied. In order to show that the same holds for the diffusion term, and since the matrix Γ_2 is diagonal, it is sufficient to consider one of the terms $\sqrt{\phi_k}$. We now show that these are indeed uniformly Lipschitz, which then implies that $u^*(\cdot) \in \mathcal{A}$ by Theorem 1.6.16 of [41]). Let us define first the following:

$$
\begin{aligned}
K_1(t) &:= -\left(\frac{1}{2}\bar{R}^\dagger(t)\bar{X}(t) + \Psi^1(t)\right), \quad t \in [0,T], \\
K_2(t) &:= -\Psi^2(t) - \frac{1}{2}\bar{R}^\dagger(t)\bar{Y}(t), \quad t \in [0,T], \\
\tilde{Q}_k(t) &:= \begin{bmatrix} 0 & 0 \\ 0 & Q_k(t) \end{bmatrix}, \quad t \in [0,T], \\
\tilde{Z}_k(t) &:= x_1'(t)L_k(t) + Z_k(t), \quad t \in [0,T], \\
\psi_k(t,y) &:= \left\{ y'\tilde{Q}_k y + [K_1(t)y + K_2(t)]'R_k(t)[K_1(t)y + K_2(t)] + \tilde{Z}_k(t) \right\}^{1/2}, \\
&\qquad t \in [0,T], \quad y \in \mathbb{R}^{m+n}.
\end{aligned}
$$

Due to the assumptions on $\bar{R}^\dagger(\cdot)$ and $Y(\cdot)$, the process $K_1(\cdot)$ is also uniformly bounded. Moreover, under control $u^*(\cdot)$, it holds that $\sqrt{\phi_k} = \psi_k(t, x(t))$.

Therefore, it is sufficient to show the uniform Lipschitz property for $\psi_k(\cdot, y)$:

$$\begin{aligned}
& |\psi_k(t,y_1) - \psi_k(t,y_2)| \\
= \; & |\psi_k^2(t,y_1) - \psi_k^2(t,y_2)|[\psi_k(t,y_1) + \psi_k(t,y_2)] \\
= \; & \left|(y_1 - y_2)'\tilde{Q}_k(y_1 + y_2) + [K_1(y_1 - y_2)]'\tilde{R}_k[K_1(y_1 + y_2) + 2K_2]\right| \\
& \times[\psi_k(t,y_1) + \psi_k(t,y_2)] \\
\leq \; & \left[|\tilde{Q}_k^{1/2}||y_1 - y_2||\tilde{Q}_k^{1/2}(y_1 + y_2)| + |K_1||\tilde{R}_k^{1/2}||y_1 - y_2||\tilde{R}_k^{1/2}[K_1(y_1 + y_2) + 2K_2]|\right] \\
& \times[\psi_k(t,y_1) + \psi_k(t,y_2)] \\
\leq \; & K^*|y_1 - y_2|
\end{aligned}$$

for some constant K^*, $t \in [0,T]$, and all $y_1, y_2 \in \mathbb{R}^{m+n}$. □

Notably, checking if equation (1.5) has a unique strong solution with sufficient integrability under $u^*(t)$ in (2.5) is a challenging task. In general, this in not easy even if the system is linear, since in general the process $Y(\cdot)$ can be unbounded and leads to a system with unbounded coefficients (see, e.g., [13] for some sufficient conditions for the solvability in such a case). The problem becomes even more difficult due to the nonlinearity of the system.

3. Infinite horizon regulator

Here we consider an infinite horizon version of the optimal control problem (1.6). As is typical in infinite horizon problems, we assume that all coefficients are constant:

$$\begin{aligned}
& H_1, L_k \in \mathbb{R}^m, \quad A_1, C_1 \in \mathbb{R}^{n\times m}, \quad A_2 \in \mathbb{R}^{n\times n}, \quad C_2 \in \mathbb{R}^{m\times m}, \quad B_1, D_1 \in \mathbb{R}^{n\times n_u}, \\
& E \in \mathbb{R}^{n\times \eta}, \quad Q_k \in \mathcal{S}^n, \quad R_k \in \mathcal{S}^{n_u}, \quad \mathrm{diag}(g_1, ..., g_m) =: G_1 \in \mathbb{R}^{m\times m}.
\end{aligned}$$

We further assume that $Q_k \geq 0$, $R_k \geq 0$, $Z_k > 0$, and the components of L_k are non-negative, for all k. The cost functional is of average type give as:

$$J_\infty(u(\cdot)) := \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left\{\int_0^T \left(\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}' \begin{bmatrix} Q & L \\ L' & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + x(t)'L_d + u(t)'L_e \right) dt \right\},$$

with the following constant coefficients

$$R \in \mathcal{S}^{n_u}, \quad Q \in \mathcal{S}^{m+n}, \quad L \in \mathbb{R}^{(m+n)\times n_u}, \quad L_d \in \mathbb{R}^{m+n}, \quad L_e \in \mathbb{R}^{n_u}.$$

The optimal control problem to be solved is:

$$\begin{cases} \min\limits_{u(\cdot)\in\mathcal{A}_\infty} J_\infty(\cdot) \\ \\ \text{s.t. } (1.5) \end{cases} \tag{3.1}$$

for some suitable admissible set $\mathcal{A}_\infty$. The solution to this problem proceeds similarly to the previous section, however, a further difficulty is the consideration of *stability* under the optimal controls, which places further constraints on the given coefficients. The analog to the differential equation (2.1) is the following Riccati algebraic equation:

$$\begin{cases} PA + A'P + C'PC + \sum\limits_{k=1}^{\eta} N_k' P N_k M Q_k M' + Q \\ \quad - (C'PD + PB + L)(D'PD + \sum\limits_{k=1}^{\eta} N_k' P N_k R_k + R)^{\dagger}(D'PC + B'P + L') = 0, \\ (D'PD + \sum\limits_{k=1}^{\eta} N_k' P N_k R_k + R)(D'PD + \sum\limits_{k=1}^{\eta} N_k' P N_k R_k \\ \quad + R)^{\dagger}(D'PC + B'P + L') - (D'PC + B'P + L') = 0, \\ D'PD + \sum\limits_{k=1}^{\eta} N_k' P N_k R_k + R \geq 0, \end{cases} \tag{3.2}$$

The algebraic equations, different from differential equations, can have more than one solution. We later place further constraints on the solution of (3.2), due to stability requirements, but for the moment we require a solution to exist.

Assumption 3. *There exists a solution $P \in \mathbb{R}^{(n+m)\times(n+m)}$* to equation (3.2).

The analog to the differential equation (2.4) is the following linear algebraic equation:

$$\begin{cases} 2PH + \sum_{a=1}^{m} b_a' P_{aai} + \sum_{k=1}^{\eta} N_k' P N_k \tilde{M} L_k + L_d + A' L_c - (C'PD \\ \quad + PB + L)(D'PD + \sum_{k=1}^{\eta} N_k' P N_k R_k + R)^{\dagger}(L_{ei} + B' L_c) = 0, \\ (D'PD + \sum_{k=1}^{\eta} N_k' P N_k R_k + R)(D'PD + \sum_{k=1}^{\eta} N_k' P N_k R_k \\ \quad + R)^{\dagger}(L_e + B' L_c) - (L_e + B' L_c) = 0, \end{cases} \tag{3.3}$$

Assumption 4. *There exists a solution $L_c \in \mathbb{R}^{m+n}$ to equation (3.3).*

We now define the admissible set of controls $\mathcal{A}_\infty$. Let $L^0_{\mathcal{F}}(0,\infty;\mathbb{R}^n)$ and $L^2_{\mathcal{F}}(0,\infty;\mathbb{R}^n)$ denote the set of real n-dimensional processes that are $\mathcal{F}(t)$-adapted and square-integrable in the interval $[0,T]$ for all $T \in (0,\infty)$, respectively. The following is our stability requirement:

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}[x(T)'Px(T) + x(T)'L_c] = \hat{C} \in \mathbb{R}, \quad \forall x_0' \in \mathbb{R}_+^{1\times m} \times \mathbb{R}^{1\times n}. \tag{3.4}$$

Similarly to the previous section we define the processes:

$$\begin{aligned} \rho(x(t),u(t),t) &:= (2x'P + L_c)(Cx + Du), \\ \rho_1(x(t),u(t),t) &:= (2x'P + L_c)\theta_1(x(t),u(t),t), \\ \rho_2(x(t),u(t),t) &:= (2x'P + L_c)\theta_2(x(t),u(t),t), \end{aligned}$$

where now P and L_c are solutions to (3.2) and (3.3), respectively. We consider only the following set of control processes:

$$\begin{aligned} \mathcal{A}_\infty := \{ & u(\cdot) \in L^0_{\mathcal{F}}(0,\infty;\mathbb{R}^{n_u}) : \text{equation (1.5) has a unique strong solution on } [0,\infty), \\ & \rho(x(t),u(t),t) \in L^2_{\mathcal{F}}(0,\infty;\mathbb{R}),\ \rho_1(x(t),u(t),t) \in L^2_{\mathcal{F}}(0,\infty;\mathbb{R}^{1\times m}), \\ & \rho_2(x(t),u(t),t) \in L^2_{\mathcal{F}}(0,\infty;\mathbb{R}^{1\times\eta}), \text{ and condition (3.4) holds} \}. \end{aligned}$$

Before we state the solution to the optimal control problem (3.1) we introduce the following convenient notation:

$$
\begin{aligned}
\bar{X} &:= 2(D'PC + B'P + L'), \quad \bar{Y} := L_e + B'L_c, \quad \bar{R} := D'PD + \sum_{k=1}^{\eta} N_k' P N_k R_k + R, \\
\bar{S} &:= PA + A'P + C'PC + \sum_{k=1}^{\eta} N_k' P N_k M Q_k M' + Q, \\
\bar{T} &:= 2PH + \sum_{a=1}^{m} b_a' P_{aa} + \sum_{k=1}^{\eta} N_k' P N_k \tilde{M} L_k + A'L_c + L_d, \\
\bar{Z} &:= \sum_{k=1}^{\eta} N_k' P N_k Z_k + L_c' H. \quad \bar{O} := \bar{Z} - \frac{1}{4}\bar{Y}'\bar{R}^{\dagger}\bar{Y}.
\end{aligned}
$$

If $Y(\cdot) \in L^2_{\mathcal{F}}(0,\infty;\mathbb{R}^{n_u \times (m+n)})$ and $z(\cdot) \in L^2_{\mathcal{F}}(0,\infty;\mathbb{R}^{n_u})$ are two given processes, we define:

$$
\begin{aligned}
\Psi^1(t) &:= Y(t) - \bar{R}^{\dagger}\bar{R}Y(t), \\
\Psi^2(t) &:= z(t) - \bar{R}^{\dagger}\bar{R}z(t), \\
u^*_{\infty}(t) &:= -\left(\frac{1}{2}\bar{R}^{\dagger}\bar{X} + \Psi^1(t)\right)x(t) - \Psi^2(t) - \frac{1}{2}\bar{R}^{\dagger}\bar{Y}.
\end{aligned} \tag{3.5}
$$

Theorem 3. *If the processes $Y(\cdot)$ and $z(\cdot)$ are such that $u^*_{\infty}(\cdot) \in \mathcal{A}_{\infty}$, then $u^*_{\infty}(\cdot)$ is a solution to the optimal control problem (3.1). The corresponding optimal cost is $J_{\infty}(u^*_{\infty}(\cdot)) = -\bar{C} + \bar{O}$. Moreover, any optimal control is of the form (3.5).*

Proof. The proof follows closely that of Theorem 1 and we include it for completeness. Thus, similarly to the proof of Theorem 1, for any $u(\cdot) \in \mathcal{A}_{\infty}$, the following holds:

$$
\mathbb{E}[x(T)'Px(T)] + \mathbb{E}[L_c'x(T)] = x_0'Px_0 + L_c'x_0 + \mathbb{E}\left[\int_0^T \Theta(x(t),u(t))dt\right],
$$

where

$$
\begin{aligned}
&\Theta(x(t),u(t))\\
:= \;& x'\{[PA+A'P+C'PC+\sum_{k=1}^{\eta} N_k'PN_kMQ_kM']\}x+2u'[D'PC+B'P]x\\
&+u'[D'PD+\sum_{k=1}^{\eta} N_k'PN_kR_k]u+u'B'L_c\\
&+x'[2PH+\sum_{a=1}^{m} b_a'P_{aa}+\sum_{k=1}^{\eta} N_k'PN_k\tilde{M}L_k+A'L_c]+\sum_{k=1}^{\eta} N_k'PN_kZ_k+L_c'H.
\end{aligned}
$$

Since all admissible controls satisfy condition (3.4), we have:

$$
0=-\bar{C}+\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T \Theta(x(t),u(t))dt\right],
$$

By adding this to the cost functional and completing the square, we obtain:

$$
\begin{aligned}
J_\infty(u(\cdot)) \;=\;& -\bar{C}+\bar{O}+\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\bigg\{\int_0^T\left[u+\left(\frac{1}{2}\bar{R}^\dagger\bar{X}+\Psi^1\right)x+\frac{1}{2}\bar{R}^\dagger\bar{Y}+\Psi^2\right]'\bar{R}\bigg[u\\
&+\left(\frac{1}{2}\bar{R}^\dagger\bar{X}+\Psi^1\right)x+\frac{1}{2}\bar{R}^\dagger\bar{Y}+\Psi^2\bigg]dt\bigg\}\geq-\bar{C}+\bar{O}.
\end{aligned}
$$

If $u(t)=u^*_\infty(t)$ for *a.e.* $t\in[0,\infty)$ *a.s.*, this lower bound is achieved, which shows that $u^*_\infty(t)$ is an optimal control process. The proof of the claim that any optimal control process is of the form (3.5) proceeds as in the proof of Theorem 1. □

Theorem (3) provides a solution to the optimal control problem stated in (3.1). The problem of deriving conditions that ensure $u^*_\infty(\cdot)\in\mathcal{A}_\infty$ is more involved than in the finite horizon case due to the stability requirement (3.4), which places further constraints on the solution P to the algebraic Riccati equation (3.2). It follows from Theorem 2 that under control $u^*_\infty(t)=\bar{K}_1x(t)+\bar{K}_2$ there exists a unique strong solution to equation (1.5) with any degree of stability. We thus focus on showing that the stability requirement (3.4) is also satisfied.

Let the processes $Y(\cdot)$ and $z(\cdot)$ be constant and define: $\bar{K}_1 := -\Psi^1 - \bar{R}^\dagger \bar{X}/2$, $\bar{K}_2 := -\Psi^2 - \bar{R}^\dagger \bar{Y}/2$, which means $u^*_\infty(t) = \bar{K}_1 x(t) + \bar{K}_2$. We further define the following for notational convenience:

$$\begin{aligned}
\bar{A} &:= PA + A'P + C'PC + \sum_{k=1}^{\eta} N_k' P N_k M Q_k M', \quad \bar{B} := D'PC + B'P, \\
\bar{C} &:= D'PD + \sum_{k=1}^{\eta} N_k' P N_k R_k, \quad \bar{D} := 2PH + \sum_{a=1}^{m} x' b_a' P_{aai} + \sum_{k=1}^{\eta} N_k' P N_k \tilde{M} L_k, \\
\bar{E} &:= \sum_{k=1}^{\eta} N_k' P N_k Z_k, \quad \bar{F} := \bar{A} + \bar{K}_1' \bar{B} + \bar{B}' \bar{K}_1 + \bar{K}_1' \bar{C} \bar{K}_1, \\
\bar{G} &:= 2\bar{B}' \bar{K}_2 + 2\bar{K}_1' \bar{C} \bar{K}_2 + \bar{D}, \quad \bar{L} := \bar{K}_2' \bar{C} \bar{K}_2 + \bar{E}.
\end{aligned}$$

Theorem 4. *If the processes $Y(\cdot)$ and $z(\cdot)$ are constant and such that Assumption 3 and Assumption 4 hold with the following further constraints: (i) the matrix $A + B\bar{K}_1$ has negative eigenvalues; (ii) $P \geq 0$; (iii) $\bar{F} < 0$; (iv) $\bar{L} - \bar{G}' \bar{F}^{-1} \bar{G}/4 < 0$; then $u^*_\infty(\cdot) \in \mathcal{A}_\infty$ with $\hat{C} = 0$.*

Proof. Under the control $u^*_\infty(\cdot)$ the following holds:

$$d\mathbb{E}[x(t)] = \{(A + B\bar{K}_1)\mathbb{E}[x(t)] + H\} dt.$$

Since the matrix $A + B\bar{K}_1$ is assumed to have negative eigenvalues, we know that $\mathbb{E}[L_c' x(T)]$ converges to a constant as $T \to \infty$, and thus $\mathbb{E}[L_c' x(T)]/T \to 0$ as $T \to \infty$. For any $0 \leq s \leq T$, due to conditions (iii) and (iv), we also have:

$$\begin{aligned}
& \mathbb{E}[x(T)' P x(T)] \\
= \;& x'(s) P x(s) + \mathbb{E}\left[\int_s^T [x'(t) \bar{F} x(t) + x'(t) \bar{G} + \bar{L}] dt \,\middle|\, \mathcal{F}(s) \right] \\
= \;& x'(s) P x(s) \\
& + \mathbb{E}\left[\int_s^T \left\{ \left[x(t) + \frac{1}{2} \bar{F}^{-1} \bar{G} \right]' \bar{F} \left[x(t) + \frac{1}{2} \bar{F}^{-1} \bar{G} \right] - \frac{1}{4} \bar{G}' \bar{F}^{-1} \bar{G} + \bar{L} \right\} dt \,\middle|\, \mathcal{F}(s) \right] \\
\leq \;& x'(s) P x(s),
\end{aligned}$$

which implies, due to condition (ii), that $\mathbb{E}[x(T)' P x(T)]$ is at most bounded for any $x_0' \in \mathbb{R}_+^{1\times m} \times \mathbb{R}^{1\times n}$, and thus $\mathbb{E}[x(T)' P x(T)]/T \to 0$ as $T \to \infty$. □

4. Application to optimal investment

Here we given an application of the finite-time horizon results to the optimal investment problem with stochastic interest rate. Thus, consider a market of two assets: a bank account $B(\cdot)$ and a stock $S(\cdot)$ the equations of which are

$$\begin{cases} dB(t) = B(t)r(t)dt, \quad t \geq 0, \\ \\ dS(t) = S(t)[\mu(t)dt + \sigma(t)dW(t)], \quad t \geq 0 \\ \\ B(0) = B_0 > 0 \quad \text{and} \quad S(0) = S_0 > 0 \quad \text{are given.} \end{cases} \tag{4.1}$$

Here $r(\cdot)$ is the interest rate of the bank account $B(\cdot)$, whereas $\mu(\cdot)$ and $\sigma(\cdot)$ are the appreciation rate and the volatility of the stock $S(\cdot)$, respectively. In this market we consider an investor endowed with the initial wealth y_0. Let $v_B(t)$ and $v_S(t)$ denote the number of shares that the investor holds in $B(t)$ and $S(t)$, respectively, at time t. Then the investors wealth at time t is $y(t) := v_B(t)B(t) + v_S(t)S(t)$. If $u(t) := v_S(t)S(t)$ denotes the amount of the investor's wealth invested in the stock, then the equation of the *self-financing portfolio* is (see, e.g., [22],[19]):

$$\begin{cases} dy(t) = [r(t)y(t) + (\mu(t) - r(t))u(t)]dt + \sigma(t)u(t)W(t), \quad t \geq 0, \\ \\ y(0) = y_0 > 0. \end{cases} \tag{4.2}$$

Let $\mu(t)$ be a given process and $\sigma(t)$ a deterministic function, whereas for the interest rate $r(t)$ we assume that it follows the Cox-Ingersoll-Ross process (see, e.g.,[34]):

$$\begin{cases} dr(t) = [ar(t) + b]dt + \sqrt{r(t)}dW_{11}(t), \quad t \geq 0, \\ \\ r(0) = r_0 > 0, \end{cases} \tag{4.3}$$

for some constants a and b. Moreover, we assume that $q(t) := \mu(t) - r(t)$ is a deterministic function (note that this is typical assumption in a market with stochastic interest rate, see, e.g. [39]). The problem of *optimal investment* with a *logarithmic utility* is the following optimal control problem:

$$\begin{cases} \max\limits_{u(\cdot)\in\mathcal{A}_\ell} \mathbb{E}[\log(y(T))], \\ \\ \text{s.t. } (4.2), \end{cases} , \tag{4.4}$$

where $\mathcal{A}_\ell$ contains only the control process that ensure (4.2) has a unique positive strong solution. In this formulation, this problem is different from the ones we have considered in the previous sections as both the control system (4.2) and the cost functional $\mathbb{E}[\log(y(T))]$ are different. However, we now show that it can be reformulated as an example of the control problem considered in section 2.

For any $u(\cdot) \in \mathcal{A}_\ell$ it holds that $y(t) > 0$ $a.s.$ $\forall t \in [0, T]$, and in this case the differential of $y_\ell(t) := \log[y(t)]$ is:

$$\begin{cases} dy_\ell(t) = [r(t) + q(t)v(t) - \sigma^2 v^2(t)/2]dt + \sigma(t)v(t)dW(t), \quad t \geq 0, \\ \\ y_\ell(0) = x_0 := \log(y_0), \end{cases} \tag{4.5}$$

where $v(t) := u(t)/y(t)$. In we define the state variables as:

$$\begin{aligned} x_1(t) &:= r(t), \qquad t \geq 0, \\ x_2(t) &:= y_\ell(t) + \int_0^t \frac{1}{2}\sigma^2 v^2(s)ds, \quad t \geq 0, \end{aligned} \tag{4.6}$$

their equations are

$$\begin{cases} dx_1(t) = [ax_1(t) + b]dt + \sqrt{x_1(t)}dW_{11}(t), \quad t \geq 0, \\ \\ dx_2(t) = [x_1(t) + q(t)v(t)]dt + \sigma(t)v(t)dW(t), \quad t \geq 0, \\ \\ x_1(0) = r_0, \quad x_2(0) = \log(y_0), \end{cases} \tag{4.7}$$

which is clearly an example of system (1.2). From (4.6) it follows that the negative value of the logarithmic utility from terminal wealth, i.e. $-\mathbb{E}[\log(y(T))]$, can be written as:

$$J_\ell(v(\cdot)) := \mathbb{E}\left[\int_0^T \frac{1}{2}\sigma^2 v^2(s)ds - x_2(T)\right]. \tag{4.8}$$

Thus, the optimal investment problem (4.4) is equivalent to the optimal control problem of minimising (4.8) subject to (4.7) which is an example of the control problem (1.6), the solution of which is given by Theorem 1.

5. Conclusions

We have introduced a certain nonlinear generalisation of the stochastic LQ control problem, where the controlled system contains square-root nonlinearities. The general case permitting indefinite cost matrices is considered and all optimal controls within a specified admissible set are found in an explicit closed-form through new differential and algebraic Riccati equations. The optimal controls turn out to be of affine linear state-feedback form. An application to the problem of optimal investment in a market with random interest rate is also given. Here we have considered one of the most basic versions of this problem where the system is driven by Brownian motions. This problem has a great potential for further extensions. Formulating it in more general settings (e.g. with random coefficients, other driving noises, with constraints on control variables) is possible and the closed-form solvability is likely to be retained.

For our future research, we would like to extend this work into a more generalized version, in which the system nonlinearities appear both in drift term and diffusion term. A similar nonlinear stochastic system has been reported in [17], in which robust control problems have been studied, and an application to energy Internet system voltage control has been explored. Inspired by this, attention can be drawn to solve optimal control problems based on such more generalized nonlinear stochastic system, and the research outputs can be used in new generation energy management and control system.

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