Finite Models for a Spatial Logic with Discrete and Topological Path Operators

Sven Linker
Lancaster University in Leipzig, Germany

Fabio Papacchini
University of Liverpool, UK

Michele Sevegnani
University of Glasgow, UK

Abstract

This paper analyses models of a spatial logic with path operators based on the class of neighbourhood spaces, also called pretopological or closure spaces, a generalisation of topological spaces. For this purpose, we distinguish two dimensions: the type of spaces on which models are built, and the type of allowed paths. For the spaces, we investigate general neighbourhood spaces and the subclass of quasi-discrete spaces, which closely resemble graphs. For the paths, we analyse the cases of quasi-discrete paths, which consist of an enumeration of points, and topological paths, based on the unit interval. We show that the logic admits finite models over quasi-discrete spaces, both with quasi-discrete and topological paths. Finally, we prove that for general neighbourhood spaces, the logic does not have the finite model property, either for quasi-discrete or topological paths.

2012 ACM Subject Classification Theory of computation → Modal and temporal logics; Mathematics of computing → Topology

Keywords and phrases spatial logic, topology, finite models

Digital Object Identifier 10.4230/LIPIcs.MFCS.2021.82

Funding
Fabio Papacchini: partly supported by the EPSRC through grants EP/R026084 and EP/R026173.
Michele Sevegnani: supported by the EPSRC under PETRAS SRF grant MAGIC (EP/S035362/1).

1 Introduction

The safe and correct operation of systems in a wide range of application domains is increasingly dependent on spatial reasoning to evaluate the structure of space and how space might evolve over time. Examples include target counting in wireless sensor networks [19, 2], cyber-physical systems [22], transport systems [9], structural analysis [17], and medical imaging [6]. *Neighbourood spaces*, also known as closure or pretopological spaces [23, 14], have emerged as a popular formalism in these scenarios due to their ability to natively represent topological spaces but also simple graphs and simple directed graphs. In this paper, we focus on *SLCS*, a modal logic introduced by Ciancia et al. [11] for the specification and verification of spatial properties over neighbourhood spaces. This logic features a closure modality $\mathcal{N}$ (near) and path modalities $\mathcal{R}$ (reachable from) and $\mathcal{P}$ (propagates to). While model checking algorithms and software support have been developed, the model theory of this logic is still not well understood. In particular, it is not known what kind of spaces can be expressed by various classes of formulas. Answering this question is complicated by how the near modality interacts with the path modalities which is substantially different from the modality interactions in discrete modal logic.

We make the following research contributions:

1. we show that SLCS does not admit finite models on general neighbourhood spaces;

2. we prove that there are formulas that are only satisfiable on infinite models even when restricting to either quasi-discrete paths (similar to paths on graphs) or standard topological paths;
3. we define a finite model construction using filtration arguments for models with quasi-discrete underlying spaces and quasi-discrete or topological paths.

Related Work

The analysis of SLCS is increasingly gaining traction both in Theoretical Computer Science and Topology.

In recent work [18], we presented bisimulations for SLCS formulas using path operators that show the equivalence of formulas between bisimilar models. Ciancia et al. [12] used co-algebraic methods to present bisimulations over quasi-discrete models that are well-matched (i.e., they characterise the class of quasi-discrete models), but did not extend this result to arbitrary spaces. Importantly, the authors restricted the set of SLCS formulas to omit path operators. Castelnovo and Miculan [7] defined a categorical semantics for various fragments of SLCS using hyperdoctrines with paths and investigated how to extend the logic to other spaces with closure operators, such as probabilistic automata.

Rieser [20] used the unit interval to define and analyse a homotopy theory for closure spaces, that is, how paths can be transformed into one another. Bubenik and Milićević [5] further investigated how different generalisations of the unit interval yield different path objects. None of these definitions is immediately applicable to SLCS paths, which are much more concrete.

2 Neighbourhood Spaces

In this section we recall the notions of neighbourhood spaces and some related results from general topology we will use in this paper. Our main reference is [23]. For additional general results on these topics and for the proofs of the results reported here, we refer the reader to this source.

▶ **Definition 1** (Filter). *Given a set X, a* filter *F on X is a subset of $\mathbb{P}(X)$, such that F is closed under intersections, whenever $Y \in F$ and $Y \subseteq Z$, then also $Z \in F$, and finally $\emptyset \notin F$.*

▶ **Definition 2** (Neighbourhood Space). *Let X be a set, and let $\eta\colon X \to \mathbb{P}(\mathbb{P}(X))$ be a function from X to the set of filters on it, where every $\eta(x)$ is such that $x \in \bigcap_{N \in \eta(x)} N$. We call η a* neighbourhood system *on X, and $\mathcal{X} = (X, \eta)$ a* neighbourhood space. *For every set $A \subseteq X$, we have the (unique)* interior *and* closure *operators defined as follows.*

$$\mathcal{I}_\eta(A) = \{x \in A \mid A \in \eta(x)\} \qquad \mathcal{C}_\eta(A) = \{x \in X \mid \forall N \in \eta(x)\colon A \cap N \neq \emptyset\}$$

An element $x \in X$ has a minimal neighbourhood *if there exists $N \in \eta(x)$ such that $N \subseteq N'$ for any neighbourhood $N' \in \eta(x)$. We use $N_{min}(x)$ to refer to the minimal neighbourhood of x. If each element $x \in X$ has a minimal neighbourhood, then we call $\mathcal{X}$* quasi-discrete. *Finally, if for every element $x \in X$ and any neighbourhood $N \in \eta(x)$, there is a neighbourhood $M \in \eta(x)$, such that for every $y \in M$, we have also that $N \in \eta(y)$, then $\mathcal{X}$ is* topological.

Neighbourhood spaces as we introduced them are exactly the *pretopological spaces* as defined by Choquet [8] and the *closure spaces* introduced by Čech [23], as shown by Kent and Min [16].[1] Furthermore, a topological neighbourhood space is just a topological space as usual.

▶ **Definition 3** (Connectedness ([23] 20.B.1)). *Let $\mathcal{X} = (X, \eta)$ be a neighbourhood space. Two subsets U and V of X are* semi-separated, *if $\mathcal{C}(U) \cap V = U \cap \mathcal{C}(V) = \emptyset$. A subset U of $\mathcal{X}$ is* connected, *if it is not the union of two non-empty, semi-separated sets. The space $\mathcal{X}$ is connected, if X is connected.*

We also introduce a special kind of connected neighbourhood space, endowed with a linear order.

▶ **Definition 4** (Index Space). *If (I, η) is a connected neighbourhood space and $\leq \subseteq I \times I$ a linear order on I with the bottom element $0 \in I$, then we call $\mathcal{I} = (I, \eta, \leq, 0)$ an* index space.

In the following sections, we will often use the concept of continuous functions. Generally, we will use the notation $f[A]$ for the image of a set $A \subseteq X$ under a function $f : X \to Y$.

▶ **Definition 5** (Continuous Function ([23] 16 A.4)). *Let $\mathcal{X}_i = (X_i, \eta_i)$ for $i \in \{1, 2\}$ be two neighbourhood spaces. A function $f : X_1 \to X_2$ is* continuous at x_1, *if for every $N_2 \in \eta_2(f(x_1))$, there is an $N_1 \in \eta_1(x_1)$ such that $f[N_1] \subseteq N_2$. Equivalently, for every $Y \subseteq X_1$, if $x_1 \in \mathcal{C}_1(Y)$, then $f(x_1) \in \mathcal{C}_2(f[Y])$. If f is continuous at every $x_1 \in X_1$, we simply say that f is continuous. We will also write $f : \mathcal{X}_1 \to \mathcal{X}_2$.*

Observe that this coincides with the well-known definition of continuous functions on topological spaces.

▶ **Definition 6** (Path). *For an index space $\mathcal{I}$ and a neighbourhood space $\mathcal{X}$, a continuous function $p : \mathcal{I} \to \mathcal{X}$ is an $\mathcal{I}$-*path *on $\mathcal{X}$. If $p(0) = x$, we will also write $p : x \rightsquigarrow \infty$ to denote* a path starting in *x.*

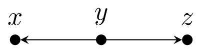
Two typical index spaces are $\mathcal{I}_{\mathbb{R}} = ([0, 1], \eta_{\mathbb{R}}, \leq, 0)$, the unit interval with the standard topology based on open intervals, and $\mathcal{I}_{\mathbb{N}} = (\mathbb{N}, \eta_{\mathbb{N}}, \leq, 0)$, where $\eta_{\mathbb{N}}$ is given by the quasi-discrete neighbourhood system induced by the successor relation. That is, the minimal neighbourhood of each point n is given by $\{n, n + 1\}$. We call $\mathcal{I}_{\mathbb{R}}$-paths *topological paths* and $\mathcal{I}_{\mathbb{N}}$-paths *quasi-discrete paths.*

▶ **Definition 7** (Separation and Distinguishability). *Let $\mathcal{X} = (X, \eta)$ be a neighbourhood space and $x, y \in X$ be two distinct points of $\mathcal{X}$. If $\eta(x) \neq \eta(y)$, we say that x and y are* distinguishable *in $\mathcal{X}$. If there is both an $N \in \eta(x)$ such that $y \notin N$ and an $M \in \eta(y)$ such that $x \notin M$, then we call x and y T_1-separated. Equivalently, in terms of closures, two distinct points x and y are distinguishable, if $x \notin \mathcal{C}(\{y\})$ or $y \notin \mathcal{C}(\{x\})$. They are T_1-separated, if $(\{x\} \cap \mathcal{C}(\{y\})) \cup (\mathcal{C}(\{x\}) \cap \{y\}) = \emptyset$.*

The space $\mathcal{X}$ is a symmetric space *(or R_0-space), if every two distinguishable points are T_1-separated.*

The following lemma implies that quasi-discrete paths that visit a non-quasi discrete point on a symmetric space cannot get back into "quasi-discrete territory".

[1] To be exact, Kent and Min's definition of neighbourhood spaces is more general than ours, as they do not require the neighbourhood systems to be filters. In fact, they show that a neighbourhood space where each neighbourhood system is a filter constitutes a pretopological space.



(a) Quasi-discrete space $\mathcal{X}$

$$p(i) = \begin{cases} x & i < \frac{1}{2} \\ y & i = \frac{1}{2} \\ z & i > \frac{1}{2} \end{cases}$$

(b) A valid topological path over $\mathcal{X}$

Figure 1 Example of a topological path on a quasi-discrete space.

▶ **Lemma 8.** *Let $\mathcal{Q} = (Q, \eta_Q)$ be a quasi-discrete space and $\mathcal{X} = (X, \eta)$ be a non-quasi-discrete, but symmetric space. Furthermore let $x \in X$ be a point that does not have a minimal neighbourhood. Any continuous function $f : \mathcal{Q} \to \mathcal{X}$ that visits x at some point q can only visit points that are indistinguishable from x at any $q' \in N_{min}(q)$. In terms of closures, this is equivalent to the following condition: if $q \in \mathcal{C}(\{q'\})$, then $f(q')$ is indistinguishable from x.*

Proof. Let $f : \mathcal{Q} \to \mathcal{X}$ be a continuous function with $f(q) = x$ and for some $q' \in N_{min}(q)$, we have $f(q') = y$ where x and y are distinguishable. Hence, there is an $N \in \eta(x)$ such that $y \notin N$. However, for any $M \in \eta_Q(q)$, we have that $N_{min}(q) \subseteq M$, which of course means also $q' \in M$. But $f(q') \notin N$, so $f[M] \not\subseteq N$. So f is not continuous at q, which contradicts the assumption on f. ◀

We will often refer to the fact that quasi-discrete spaces closely resemble graphs: we can consider the points in the minimal neighbourhood of a point x to be connected to x by an edge. The following example provides a better understanding of the difference in behaviour of topological and quasi-discrete paths over quasi-discrete neighbourhood spaces.

▶ **Example 9.** Consider the quasi-discrete neighbourhood space $\mathcal{X}$ in Fig. 1a. Any path p defined over $\mathcal{I}_{\mathbb{N}}$ is such that for any $i \in \mathcal{I}_{\mathbb{N}}$, if $p(i) = x$ or $p(i) = z$, then $p(j) = p(i)$ for any $j \geq i$. However, path p defined in Fig. 1b is a valid path when considering topological paths.

3 Spatial Logic for Neighbourhood Spaces

In this section, we briefly recall SLCS on general neighbourhood spaces. To that end, we first present spatial models based on neighbourhood spaces and then present the syntax and semantics of SLCS.

▶ **Definition 10** (Neighbourhood Model). *Let $\mathcal{X} = (X, \eta)$ be a neighbourhood space, $\mathcal{I}$ an index space, AP a countable set of propositional atoms, and let $\nu : X \to \mathbb{P}(AP)$ be a valuation. Then $\mathcal{M} = (\mathcal{X}, \mathcal{I}, \nu)$ is a* neighbourhood model over $\mathcal{I}$-paths. *We will also write $\mathcal{M} = (X, \eta, \nu)$ to denote neighbourhood models, if the index space is clear from the context.*

We lift all suitable definitions from Sect. 2 to neighbourhood models in the obvious ways. For example, we will speak of continuous functions between the underlying spaces of two models as continuous functions between the models.

We will often use the special case of models with quasi-discrete spaces over quasi-discrete paths, since such models are graph-like models with standard paths on graphs.

▶ **Definition 11** (Purely Quasi-Discrete Models). *Let $\mathcal{X}$ be a quasi-discrete neighbourhood space. A model $\mathcal{M} = (\mathcal{X}, \mathcal{I}_{\mathbb{N}}, \nu)$ over quasi-discrete paths is a* purely quasi-discrete *neighbourhood model.*

▶ **Definition 12** (Syntax of SLCS).

$$\varphi, \psi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid \mathcal{N}\,\varphi \mid \varphi\,\mathcal{R}\,\psi \mid \varphi\,\mathcal{P}\,\psi$$

$\mathcal{N}$ is read as near*, $\mathcal{R}$ is read as* reachable from*, and $\mathcal{P}$ is read as* propagates to*.*

The intuition behind the modalities is as follows. A point satisfies $\mathcal{N}\,\varphi$, if it is contained in the closure of the set of points satisfying φ. Hence, even if it does not satisfy φ itself, it is close to a point that does. A point x satisfies $\varphi\,\mathcal{R}\,\psi$ if there is a point y satisfying ψ such that x is reachable from y via a path where every point on this path between x and y satisfies φ. Propagation is in a sense the converse modality, i.e., if there is a point y satisfying ψ such that there is a path starting in x and reaching y at some index, and all points in between satisfy φ, then x satisfies $\varphi\,\mathcal{P}\,\psi$. This intuition is formalised in the following semantics.

▶ **Definition 13** (Path Semantics of SLCS). *Let $\mathcal{M} = ((X, \eta), \mathcal{I}, \nu)$ be a neighbourhood model and $x \in X$. The path semantics of SLCS with respect to $\mathcal{M}$ are defined inductively as follows.*

$$\begin{aligned}
&\mathcal{M}, x \models p && \text{iff } p \in \nu(x)\\
&\mathcal{M}, x \models \neg\varphi && \text{iff not } \mathcal{M}, x \models \varphi\\
&\mathcal{M}, x \models \varphi \wedge \psi && \text{iff } \mathcal{M}, x \models \varphi \text{ and } \mathcal{M}, x \models \psi\\
&\mathcal{M}, x \models \mathcal{N}\,\varphi && \text{iff } x \in \mathcal{C}(\{y \mid \mathcal{M}, y \models \varphi\})\\
&\mathcal{M}, x \models \varphi\,\mathcal{R}\,\psi && \text{iff there is } p \colon y \rightsquigarrow \infty \text{ and } n \text{ such that } p(n) = x \text{ and } \mathcal{M}, y \models \psi\\
& && \quad\text{and for all } 0 < i < n \colon \mathcal{M}, p(i) \models \varphi\\
&\mathcal{M}, x \models \varphi\,\mathcal{P}\,\psi && \text{iff there is } p \colon x \rightsquigarrow \infty \text{ and } n \text{ such that } \mathcal{M}, p(n) \models \psi\\
& && \quad\text{and } \forall i \colon 0 < i < n \implies \mathcal{M}, p(i) \models \varphi
\end{aligned}$$

In addition to the defined Boolean operators, we also allow for the other common derivable connectives. Specifically, $\varphi \vee \psi = \neg(\neg\varphi \wedge \neg\psi)$, $\top = \varphi \vee \neg\varphi$, $\bot = \neg\top$, $\varphi \rightarrow \psi = \neg\varphi \vee \psi$, and $\varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. For a class of models $\mathfrak{M}$, we say that φ is *valid* in $\mathfrak{M}$ if, and only if, $\mathcal{M}, x \models \varphi$ for every $\mathcal{M} = ((X, \eta), \mathcal{I}, \nu) \in \mathfrak{M}$ and $x \in X$.

▶ **Definition 14** (Relative Equivalence). *Let Σ be a subformula closed set of SLCS formulas, $\mathcal{M}$ a neighbourhood model, and $x, y \in \mathcal{M}$ be two points of $\mathcal{M}$. Then x and y are* equivalent relative to Σ *iff they satisfy the same formulas in Σ, i.e., $x \simeq_\Sigma y$ iff $\{\varphi \in \Sigma \mid \mathcal{M}, x \models \varphi\} = \{\varphi \in \Sigma \mid \mathcal{M}, y \models \varphi\}$. This is an equivalence relation, and we will denote the equivalence classes of x by $[x]_\Sigma$ and $[x]$, if Σ is clear from the context.*

The following lemmas present properties of formulas on different classes of models. We start with the most familiar class: purely quasi-discrete models. On these models, we have a clear connection between the near modality and the propagate path operator.

▶ **Lemma 15.** *On all purely quasi-discrete neighbourhood models $\mathcal{M} = (\mathcal{X}, \mathcal{I}_\mathbb{N}, \nu)$ we have that $\mathcal{M}, x \models \mathcal{N}\,\varphi$ iff $\mathcal{M}, x \models \varphi \vee \bot\,\mathcal{P}\,\varphi$.*

Proof. If $\mathcal{M}, x \models \varphi$, the equivalence is clear. Otherwise, assume $\mathcal{M}, x \models \bot\,\mathcal{P}\,\varphi$. This means that there is a point y and a path $p \colon x \rightsquigarrow \infty$ such that $p(1) = y$ and $\mathcal{M}, y \models \varphi$. Since p is continuous, this means that there is a neighbourhood N of 0 such that $p[N] \subseteq N_{min}(x)$. Since every neighbourhood of 0 contains 1, this means $y \in N_{min}(x)$, and so $\mathcal{M}, x \models \mathcal{N}\,\varphi$. The other direction is similar. ◀

If we consider quasi-discrete models over topological paths, this connection is less clear. The main reason for this is that over topological graphs, $\bot\,\mathcal{P}\,\varphi$ is equivalent to φ, which is easy to prove. However, we can still establish a bit less obvious connection between the modalities.

▶ **Lemma 16.** *On quasi-discrete models over topological paths, $(a \wedge \mathcal{N}(b \wedge \neg a)) \rightarrow \mathcal{N}(\neg a \wedge (b\,\mathcal{P}\,a))$ is valid.*

Proof. Let $\mathcal{M} = (\mathcal{X}, \mathcal{I}_{\mathbb{R}}, \nu)$ with $\mathcal{X} = (X, \eta)$ be a quasi-discrete model and let $x \in X$ such that $\mathcal{M}, x \models a \wedge \mathcal{N}(b \wedge \neg a)$. That is, $x \models a$ and $x \in \mathcal{C}(\{y \mid \mathcal{M}, y \models b \wedge \neg a\})$. Since $\mathcal{X}$ is quasi-discrete, this means that there is a $y \in N_{min}(x)$ such that $\mathcal{M}, y \models b \wedge \neg a$. Then, the path $p\colon \mathcal{I}_{\mathbb{R}} \rightarrow \mathcal{X}$ with $p(i) = y$ for $i < 1$ and $p(i) = x$ for $i = 1$ is a witness for $\mathcal{M}, y \models b\,\mathcal{P}\,a$. This function is indeed continuous: Consider $N \in \eta(p(i))$. If $i < 1$, we can always choose an $N_i \in \eta_{\mathcal{I}}(i)$ such that $\forall j \in N_i$ we have $j < 1$, since $\mathcal{I}$ has arbitrarily small neighbourhoods, which means $p[N_i] = \{y\} \subseteq N$. If $i = 1$, we have for any neighbourhood $N_i \in \eta_{\mathcal{I}}(i)$, that is $p[N_i] \subseteq \{x, y\} \subseteq N_{min}(x) \subseteq N$. Furthermore, $p(0) = y$, and for $n = 1$, we have $p(n) = x$, and for all $0 < i < n$, $\mathcal{M}, p(i) \models b$. Since $y \in N_{min}(x)$, we have that $\mathcal{M}, x \models \mathcal{N}(\neg a \wedge (b\,\mathcal{P}\,a))$. ◀

Furthermore, on any kind of model over topological paths, we get that the reachable and propagate modalities are equivalent. Intuitively, this is clear, since for topological paths, there is no inherent direction on the index space, in contrast to the quasi-discrete index space, where the successor relation is directed.

▶ **Lemma 17.** *On any neighbourhood model over topological paths $\mathcal{M} = (\mathcal{X}, \mathcal{I}_{\mathbb{R}}, \nu)$ we have that $\mathcal{M}, x \models \varphi\,\mathcal{P}\,\psi$ iff $\mathcal{M}, x \models \varphi\,\mathcal{R}\,\psi$.*

Proof. Let $\mathcal{M} = ((X, \eta), \mathcal{I}_{\mathbb{R}}, \nu)$ be a neighbourhood model over topological paths, and $x \in X$ a point of $\mathcal{M}$ such that $\mathcal{M}, x \models \varphi\,\mathcal{P}\,\psi$. So there is a path $p\colon \mathcal{I}_{\mathbb{R}} \rightarrow \mathcal{M}$ and $n \in [0, 1]$, such that $p(0) = x$, $p(n) = y$ and $\mathcal{M}, y \models \psi$, and $\forall k\colon 0 < k < n$, we have $\mathcal{M}, p(k) \models \varphi$. Since p is topological, we can assume without loss of generality that $n = 1$. Now the path p' defined by $p'(i) = p(1 - i)$ is a witness for $\mathcal{M}, x \models \varphi\,\mathcal{R}\,\psi$. Indeed, let $N \in \eta(p'(i))$ be a neighbourhood of $p'(i)$. By definition of p', we have $p'(i) = p(1 - i)$. We know that p is continuous at $1 - i$, so there is a neighbourhood $N' \in \eta_i(1 - i)$ such that $p[N'] \subseteq N$. But, we also have that $N^i = \{j \mid 1 - j \in N'\}$ is a neighbourhood of i and, since $p'(j) = p(1 - j)$, we have that $p'[N^i] \subseteq N$ as well. So, p' is continuous. Furthermore, $p'(0) = p(1)$, so $\mathcal{M}, p'(0) \models \psi$, $p'(1) = x$, and for all k with $0 < k < 1$, we have $\mathcal{M}, p'(k) \models \varphi$, by definition of p'. The other direction is similar. ◀

4 No Finite Model Property for Arbitrary Neighbourhood Spaces

In this section, we prove that SLCS does not have the finite model property if we consider the class of all neighbourhood models. That is, we show that there exist SLCS formulas that are satisfiable only over models $\mathcal{M} = ((X, \eta), \mathcal{I}, \nu)$ where X is not finite. Our first observation is that there are satisfiable formulas that are not satisfiable on purely quasi-discrete models.

▶ **Lemma 18.** *There exist SLCS satisfiable formulas that are not satisfiable on any finite model over quasi-discrete paths.*

Proof. Consider model $\mathcal{M} = ((\mathbb{R}, \eta_{\mathbb{R}}), \mathcal{I}_{\mathbb{R}}, \nu)$ in Fig. 2. It follows that $\mathcal{M}, 1 \models \mathcal{N}\,a \wedge \neg a \wedge \neg(\bot\,\mathcal{P}\,a)$. By Lemma 15, this formula is a contradiction on purely quasi-discrete models. Finally, since every finite space is quasi-discrete, the lemma holds. ◀

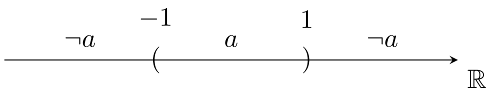


Figure 2 Model $\mathcal{M} = ((\mathbb{R}, \eta_{\mathbb{R}}), \mathcal{I}_{\mathbb{R}}, \nu)$ such that $\mathcal{M}, 1 \models \mathcal{N}\, a \wedge \neg a \wedge \neg(\bot\, \mathcal{P}\, a)$.

There are two key differences between the model in Fig. 2 and purely quasi-discrete models: the type of underlying space, and the type of paths allowed. So, we now restrict both of these dimensions one after the other. First, we show that SLCS does not admit finite models over topological paths, if we consider the full set of neighbourhood spaces, by constructing a counterexample based on the result of Lemma 16.

▶ **Lemma 19.** *There exist SLCS formulas that are satisfiable on models with topological paths, but not on any finite model with topological paths.*

Proof. We construct a topological model $\mathcal{M} = (\mathcal{X}, \mathcal{I}_{\mathbb{R}}, \nu)$ that contains a point satisfying $a \wedge \mathcal{N}(b \wedge \neg a) \wedge \neg \mathcal{N}(\neg a \wedge (b\, \mathcal{P}\, a))$. For the topological space, we use the *topologists sine curve.* For that purpose, let $S = \{(r, \sin \frac{1}{r}) \mid 0 < r \leq 1\}$. The space is then defined by $\mathcal{X} = (X, \eta)$, where $X = \{(0,0)\} \cup S$, and η is the neighbourhood system induced by treating this set as a subset of the Euclidean plane $\mathbb{R}^2$. That is, $N \in \eta(x)$ if there is an open ball of some radius r around x, i.e., some $B_r = \{y \mid \|x - y\| < r\}$, where $\| \cdot \|$ is the Euclidean distance, such that $N \supseteq B_r \cap X$. We set the valuation ν by $\nu((0,0)) = \{a\}$ and $\nu(x) = \{b\}$ for $x \neq (0,0)$.

Now, every neighbourhood of $(0,0)$ contains a value from S, and thus $\mathcal{M}, (0,0) \models a \wedge \mathcal{N}(b \wedge \neg a)$. Furthermore, it is well known [21] that in this space, $(0,0)$ is not path-connected to S, which means that no path starting in any point $s \in S$ can reach $(0,0)$. This implies, that no point $s \in S$ satisfies $b\, \mathcal{P}\, a$, since there is no path that ever reaches a point that satisfies a. So, no point on the model satisfies $\neg a \wedge (b\, \mathcal{P}\, a)$. In particular, this means that $\mathcal{M}, (0,0) \models \neg \mathcal{N}(\neg a \wedge (b\, \mathcal{P}\, a))$. So, we have $\mathcal{M}, (0,0) \models a \wedge \mathcal{N}(b \wedge \neg a) \wedge \neg \mathcal{N}(\neg a \wedge (b\, \mathcal{P}\, a))$. But this formula is not satisfiable on any quasi-discrete model with topological paths, according to Lemma 16. Since finite models are quasi-discrete, SLCS does not generally admit finite models over topological paths. ◀

Finally, even when considering only quasi-discrete paths, there are SLCS formulas which are not satisfiable on finite models.

▶ **Lemma 20.** *There exist SLCS formulas that are satisfiable on models with quasi-discrete paths, but not on any finite model with quasi-discrete paths.*

Proof. Let X be an infinite, uncountable set and let $\mathcal{X} = (X', \eta)$ be the double pointed countable complement topology over X (see [21]). For this definition, let $\mathcal{Y}$ be the set of all subsets of X, such that for every $Y \in \mathcal{Y}$, either $Y = \emptyset$, or the complement of Y is countable. X' is constructed from X by "doubling" all points, i.e., $X' = \{x' \mid x \in X\} \cup X$, where each x' is a new, distinct, element to the x it is constructed from. Then, let $\mathcal{Y}'$ be the doubling of every set in $\mathcal{Y}$ in a similar way, and η be defined by $\eta(x) = \{N \mid \exists Y \in \mathcal{Y}' \colon Y \subseteq N \wedge x \in Y\}$. Note that this definition implies that for any y and its doubled point y', we have $\eta(y) = \eta(y')$. Define $\mathcal{M} = (\mathcal{X}, \mathcal{I}_{\mathbb{N}}, \nu)$ by letting $x, x' \in X'$ be a designated pair of points in X' and ν be given by $\nu(y) = \{a\}$, if $y \in \{x, x'\}$ and $\nu(y) = \{b\}$ otherwise.

Now consider any neighbourhood $N \in \eta(x)$. There is always some $y \in N$ that is different from x and x', since otherwise the complement of N would be uncountable. Hence, every neighbourhood N contains some element y with $\mathcal{M}, y \models b$, which implies $\mathcal{M}, x \models \mathcal{N}\, b$.

However, since the underlying space of $\mathcal{M}$ is symmetric, by Lemma 8, any quasi-discrete path starting in x may only visit x or x', which both do not satisfy b. Hence $\mathcal{M}, x \not\models \bot \, \mathcal{P} \, b$. So, $\mathcal{N} \, b \wedge \neg(\bot \, \mathcal{P} \, b)$ is satisfiable on this model. But no finite model can satisfy this formula, since it is necessarily purely quasi-discrete. ◀

5 Finite Model Property for Quasi-Discrete Spaces

In this section, we prove that SLCS admits finite models if we restrict the class of models to quasi-discrete models. That is, the models correspond to directed graphs. Our approach is similar to standard approaches in modal logic [4]. In particular, we use filtrations with respect to a subformula closed set Σ for both types of models. Since topological paths and quasi-discrete paths behave very differently, we further distinguish the class into models over quasi-discrete paths and over topological paths.

5.1 Quasi-Discrete Spaces with Quasi-Discrete Paths

In this subsection, we prove that SLCS has the finite model property on purely quasi-discrete neighbourhood models. That is, the paths are similar to typical paths on graph structures.

The following lemma allow us to transfer information about the satisfaction of the path operators to other points.

▶ **Lemma 21.** *Let $\mathcal{M}$ be a purely quasi-discrete neighbourhood model and $x, y \in \mathcal{M}$ two points such that $y \in N_{min}(x)$. Then the following hold.*

1. *If $\mathcal{M}, y \models \varphi$ and $\mathcal{M}, y \models \varphi \, \mathcal{P} \, \psi$, then also $\mathcal{M}, x \models \varphi \, \mathcal{P} \, \psi$.*
2. *If $\mathcal{M}, x \models \varphi \, \mathcal{R} \, \psi$ and $\mathcal{M}, x \models \varphi$, then also $\mathcal{M}, y \models \varphi \, \mathcal{R} \, \psi$.*

Proof. We only prove the first statement as the second is similar.

From $\mathcal{M}, y \models \varphi \, \mathcal{P} \, \psi$ we know that there is a path $p \colon \mathcal{I} \to \mathcal{M}$ with $p(0) = y$ and an index $n \in \mathcal{I}$ such that $\mathcal{M}, p(n) \models \psi$ and for all $0 < i < n$, we have $\mathcal{M}, p(i) \models \varphi$. Now consider the continuous function $p_x \colon \mathcal{I} \to \mathcal{M}$ given by $p_x(0) = x$ and $p_x(i+1) = p(i)$. Then p_x is indeed a path, since $\mathcal{M}$ is quasi-discrete and $y \in N_{min}(x)$. Also, we have $\mathcal{M}, p_x(n+1) \models \psi$ and, since $\mathcal{M}, y \models \varphi$, for all $0 < i < n+1$, we have $\mathcal{M}, p_x(i) \models \varphi$. Hence $\mathcal{M}, x \models \varphi \, \mathcal{P} \, \psi$. ◀

We now define filtrations for purely quasi-discrete models. Most parts of this definition are standard, when we consider $\mathcal{N}$ similar to an existential modality. For the two path operators, we added additional properties that allow us to transfer information about the existence of paths from the filtration back to the original model.

▶ **Definition 22** (Filtration). *Let Σ be a subformula closed set of SLCS formulas, and $\mathcal{M} = (X, \eta, \nu)$ a purely quasi-discrete neighbourhood model. We call a purely quasi-discrete neighbourhood model $\mathcal{M}_f = (X_f, \eta_f, \nu_f)$ a* filtration *of $\mathcal{M}$* through *Σ, if it satisfies the following conditions:*

1. *$X_f = \{[x]_\Sigma \mid x \in X\}$*
2. *if $y \in N_{min}(x)$, then $[y] \in N_{min}([x])$*
3. *if $[y] \in N_{min}([x])$, then for each $\mathcal{N} \varphi \in \Sigma$, we have that if $\mathcal{M}, y \models \varphi$, then $\mathcal{M}, x \models \mathcal{N} \varphi$*
4. *if there is a sequence $[x_0] \dots [x_n]$ with $[x_{i+1}] \in N_{min}([x_i])$ for all $0 \leq i < n$, then for every $\varphi \, \mathcal{P} \, \psi \in \Sigma$, we have that whenever $\mathcal{M}, x_i \models \varphi$ for each $0 < i < n$ and $\mathcal{M}, x_n \models \psi$, then also $\mathcal{M}, x_0 \models \varphi \, \mathcal{P} \, \psi$*
5. *if there is a sequence $[x_0] \dots [x_n]$ with $[x_{i+1}] \in N_{min}([x_i])$ for all $0 \leq i < n$, then for every $\varphi \, \mathcal{R} \, \psi \in \Sigma$, we have that whenever $\mathcal{M}, x_i \models \varphi$ for each $0 < i < n$ and $\mathcal{M}, x_0 \models \psi$, then also $\mathcal{M}, x_n \models \varphi \, \mathcal{R} \, \psi$*

6. $\nu_f([x]) = \{p \in AP \mid \mathcal{M}, x \models p\}$

As usual, satisfiability of formulas in Σ is preserved between a model and its filtration through Σ. So our filtration is properly defined.

▶ **Lemma 23.** *Let $\mathcal{M}_f$ be a filtration of $\mathcal{M}$ through Σ. Then for all $\varphi \in \Sigma$, we have $\mathcal{M}, x \models \varphi$ iff $\mathcal{M}_f, [x] \models \varphi$.*

Proof. We proceed by induction on the structure of formulas. The base case for atomic propositions is immediate by Def. 22. The cases for the boolean operators are standard.

The case for $\varphi = \mathcal{N}\psi$ is similar to standard modal logic [4]: we have $\mathcal{M}, x \models \mathcal{N}\psi$ iff $x \in \mathcal{C}(\{y \mid \mathcal{M}, y \models \psi\})$ which by definition of the closure is equivalent to $\forall N \in \eta(x)\colon N \cap \{y \mid \mathcal{M}, y \models \psi\} \neq \emptyset$. On quasi-discrete models, this is equivalent to $\exists y \in N_{min}(x)\colon \mathcal{M}, y \models \psi$. By property 2 of filtrations and the induction hypothesis, this implies $\exists [y] \in N_{min}([x])\colon \mathcal{M}_f, [y] \models \psi$. Applying similar equivalences as before, we get that $\mathcal{M}_f, [x] \models \mathcal{N}\psi$. Conversely, assume we have $\mathcal{M}_f, [x] \models \mathcal{N}\psi$. With the same reasoning as above, this is equivalent to $\exists [y] \in N_{min}([x])\colon \mathcal{M}_f, [y] \models \psi$. By the induction hypothesis, we get $\mathcal{M}, y \models \psi$, and from property 3 of filtrations, we have $\mathcal{M}, x \models \mathcal{N}\psi$.

Now consider $\varphi = \psi \mathcal{P} \chi$. If $\mathcal{M}, x \models \psi \mathcal{P} \chi$, this is equivalent to the existence of a path $p\colon x \rightsquigarrow \infty$ and a n and $\mathcal{M}, p(n) \models \chi$ as well as $\forall i\colon 0 < i < n$, we have $\mathcal{M}, p(i) \models \psi$. That is, there is a sequence $x_0, \dots, x_n$ such that $x_0 = x$ and $x_{i+1} \in N_{min}(x_i)$ for all $i < n$. By property 2, we have $[x_{i+1}] \in N_{min}([x_i])$ for all $i < n$, and by the induction hypothesis, $\mathcal{M}_f, [x_n] \models \chi$ and for all $0 < i < n$, we get $\mathcal{M}_f, [x_i] \models \psi$, That is, $\mathcal{M}_f, [x] \models \psi \mathcal{P} \chi$. Conversely, assume $\mathcal{M}_f, [x] \models \psi \mathcal{P} \chi$. Then there is a sequence $[x_0], \dots, [x_n]$ such that $[x_{i+1}] \in N_{min}([x_i])$ for all $0 \leq i < n$, and $\mathcal{M}_f, [x_n] \models \chi$, as well as for all $0 < i < n$, we get $\mathcal{M}_f, [x_i] \models \psi$. By the induction hypothesis, we get $\mathcal{M}, x_n \models \chi$ and $\mathcal{M}, x_i \models \psi$ for every $0 < i < n$. Hence, by property 4, and since $x_0 \simeq x$, we have $\mathcal{M}, x \models \psi \mathcal{P} \chi$.

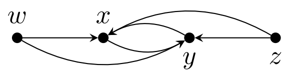
The case for $\psi \mathcal{R} \chi$ is similar, by using property 5. ◀

Finally, we prove that there is always a filtration through Σ for any given purely quasi-discrete model. This definition corresponds to the usual definition of smallest filtration [4].

▶ **Lemma 24.** *Let Σ be a subformula closed set of formulas and $\mathcal{M}$ a purely quasi-discrete model. Furthermore, let X_Σ be the set of equivalence classes of $\simeq_\Sigma$, ν_Σ be defined as in Def. 22 (6), and $\eta_s([x]) = \langle \{[y] \mid \exists y', x'\colon y' \in [y] \wedge x' \in [x] \wedge y \in N_{min}(x)\} \rangle$ for each $[x] \in X_\Sigma$. Then the model $(X_\Sigma, \eta_s, \nu_\Sigma)$ is a filtration of $\mathcal{M}$ through Σ.*

Proof. Properties 1, 2 and 6 are immediate. So now assume that $[y] \in N_{min}([x])$ and let $\mathcal{N}\varphi \in \Sigma$ such that $\mathcal{M}, y \models \varphi$. Then by definition of η_s, there are $x' \in [x]$ and $y' \in [y]$ such that $y' \in N_{min}(x')$. Since $y \simeq_\Sigma y'$, we have $\mathcal{M}, y' \models \varphi$, and due to $y' \in N_{min}(x')$, this implies $x' \in \mathcal{C}(\{y \mid \mathcal{M}, y \models \varphi\})$, which means $\mathcal{M}, x' \models \mathcal{N}\varphi$. Since $x \simeq_\Sigma x'$, this implies $\mathcal{M}, x \models \mathcal{N}\varphi$. Hence property 3 holds.

For proving property 4, we proceed by induction on the length of sequence $[x_0] \dots [x_n]$. For the base case, we have $\mathcal{M}, x_0 \models \psi$, which implies $\mathcal{M}, x_0 \models \varphi \mathcal{P} \psi$. So, assuming the property holds for suited sequences of length up to n, consider a sequence $[x_0] \dots [x_n]$ such that the conditions of the property are satisfied. In particular, $[x_1] \dots [x_n]$ is a sequence, where $[x_{i+1}] \in N_{min}([x_i])$, and for all $1 < i < n$ we have $\mathcal{M}, x_i \models \varphi$ and $\mathcal{M}, x_n \models \psi$. Hence, by the induction hypothesis, $\mathcal{M}, x_1 \models \varphi \mathcal{P} \psi$. Furthermore, by assumption on the sequence, we get $\mathcal{M}, x_1 \models \varphi$. Now, by the definition of η_s, we know that there are $x'_0 \in [x_0]$ and $x'_1 \in [x_1]$ such that $x'_1 \in N_{min}(x'_0)$, and since $x_1 \simeq x'_1$, both $\mathcal{M}, x'_1 \models \varphi$ as well as $\mathcal{M}, x'_1 \models \varphi \mathcal{P} \psi$ hold. Hence, by Lemma 21 (1), we have $\mathcal{M}, x'_0 \models \varphi \mathcal{P} \psi$, and since $x_0 \simeq x'_0$, also $\mathcal{M}, x_0 \models \varphi \mathcal{P} \psi$.



(a) Quasi-discrete model $\mathcal{M}$

$$p(i) = \begin{cases} w & i \leq \frac{1}{2} \\ x & \frac{1}{2} < i < 1 \wedge i \in \mathbb{Q} \\ y & \frac{1}{2} < i < 1 \wedge i \in \mathbb{R} \setminus \mathbb{Q} \\ z & i = 1 \end{cases}$$

(b) Path with uncountably many changes

$$p'(i) = \begin{cases} w & i \leq \frac{1}{2} \\ x & \frac{1}{2} < i < 1 \\ z & i = 1 \end{cases}$$

(c) Simplified path

Figure 3 Example of path simplification.

Property 5 can be proven similarly to the previous case, but using Lemma 21 (2). ◀

From the definition of filtration and Lemmas 23 and 24, where X_Σ is finite as the set of subformulas of a formula is finite, we obtain our first finite model property result.

▶ **Theorem 25.** *If φ is a SLCS formula that is satisfiable on a purely quasi-discrete neighbourhood model, then φ is satisifiable on a finite purely quasi-discrete neighbourhood model.*

5.2 Quasi-Discrete Spaces with Topological Paths

In this section, we prove that SLCS also admits finite models for the class of quasi-discrete models over topological paths. This case is interesting, since topological paths behave very differently from quasi-discrete paths. For example, topological paths are not required to comply with the direction of the edges of the underlying graph.

▶ **Example 26.** Consider the model in Fig. 3a. We can define a topological path p as in Fig. 3b. This function is indeed continuous. For $i < \frac{1}{2}$, the function is continuous, since it is constant. At $i = \frac{1}{2}$, we have that for the minimal neighbourhood $N_{min}(w) = \{w, x, y\}$, we can always find a neighbourhood N' of $\frac{1}{2}$ that does not contain 1, and so $p[N'] \subseteq N_{min}(w)$. If $\frac{1}{2} < i < 1$, then $N_{min}(p(i)) = \{x, y\}$, and we can choose any neighbourhood $N' \in \eta(i)$ that does not contain values less than $\frac{1}{2}$ and greater or equal to 1 to show continuity. At 1, the function is continuous for similar reasons as at $\frac{1}{2}$. So the function is a path.

However, path p contains many "superfluous detours" in the set $\{x, y\}$. A simpler path would be path p' in Fig. 3c, or a variation in which p' maps to y instead of x. This path only visits points that were visited by p as well, but omits these detours.

The following Lemma formalises the intuition explained in Example 26. We will use it to normalise the paths used as witnesses for the satisfaction of the propagate modality when we prove the existence of filtrations.

▶ Remark 27. From this point onward, we will use the following slight abuse of notation. For two indices $r, s \in [0, 1]$, we write $p[r, s] = \{p(i) \mid r < i < s\}$ to denote the values of a path p on the open interval between r and s. If $p[r, s]$ is a singleton (i.e., p is constant on the interval (r, s)), we will also treat $p[r, s]$ as a single value, to avoid unnecessary parentheses.

▶ **Lemma 28** (Path Simplification). *Let $\mathcal{M} = ((X, \eta), \mathcal{I}_\mathbb{R}, \nu)$ a neighbourhood model, where (X, η) is a quasi-discrete space, and let $p\colon [0, 1] \to X$ be a path on $\mathcal{M}$ such that p has a finite image. Then there is a path p' and a sequence of indices $i_0, \ldots, i_n$ with $i_0 = 0$, $i_n = 1$ and $i_r < i_{r+1}$ for all $r < n$, such that*

1. *$p'(i) = p(i)$ for all the indices in the sequence,*
2. *p' is constant on each open interval (i_r, i_{r+1}),*

3. $p'[i_r, i_{r+1}] \neq p'[i_s, i_{s+1}]$ *for* $r \neq s$,
4. *if* $p'(i_{r+1}) \neq p'[i_r, i_{r+1}]$, *then* $p'[i_r, i_{r+1}] \in N_{min}(p'(i_{r+1}))$,
5. *if* $p'(i_r) \neq p'[i_r, i_{r+1}]$, *then* $p'[i_r, i_{r+1}] \in N_{min}(p'(i_r))$,
6. *if* $p(i) \neq p'(i)$, *then there are* $r, s \in [0, 1]$ *and* $y \in X$ *with* $r < i < s$ *such that* $p(r) = p(s) = y$ *and* $p'(r) = p'(s) = y$.

Proof. Let $\mathcal{M}$ and p be as required, let $x \in X$ be a point in the space, and $0 \leq s \leq 1$ an index. We indicate by $\mathrm{sI}(p, x, s)$ the smallest subinterval I of $[s, 1]$ such that $\forall i \in [s, 1] \setminus I$ it holds that $p(i) \neq x$. Let a be the infimum (resp., supremum) of $\mathrm{sI}(p, x, s)$, then it follows that $\forall N \in \eta(a)$ there exists an $i \in N \cap \mathrm{sI}(p, x, s)$ such that $p(i) = x$.

We now construct the sequence of indices $i_0, \ldots, i_n$ and the path p'. We set $i_0 = 0$, $p'(0) = p(0)$, and then proceed as follows starting from $\mathrm{sI}(p, p(0), i_0)$.

Consider an index i_k, a point $x \in X$, and let a be the supremum of $\mathrm{sI}(p, x, i_k)$. We set $p'(i) = x$ for all $i_k < i < a$, we set $p'(a) = p(a)$, and

1. if $a \notin \mathrm{sI}(p, x, i_k)$, we set $i_{k+1} = a$, and then proceed with $\mathrm{sI}(p, p(a), i_{k+1})$;
2. otherwise (i.e., $a \in \mathrm{sI}(p, x, i_k)$), we need to find a possible way to proceed with the path following the index a. That is, we need to find the right point and index for the function sI. Let $S = \{y \in N_{min}(p(a)) \mid \forall N \in \eta(a) \colon y \in p[N \cap [a, 1]]\} \setminus \{p(a)\}$. Observe that $S \neq \emptyset$ as p is a continuous function on X, and any point in S is a good candidate for the continuation of the construction. Now we need to understand whether or not to move from the index i_k to the index i_{k+1}. If $i_k = a$, then we proceed by choosing any of the $y \in S$ and considering $\mathrm{sI}(p, y, i_k)$. Otherwise, we proceed by choosing any of the $y \in S$, setting $i_{k+1} = a$, and considering $\mathrm{sI}(p, y, i_{k+1})$.

Since p has a finite image, the process above terminates when $i_k = 1$.

Now let p' be the path constructed as above. Properties 1, 2 and 3 are immediate results of the construction of p'. Let us show that property 4 holds, and consider the case where $p'(i_{r+1}) \neq p'[i_r, i_{r+1}]$. By construction we know that i_{r+1} is the supremum of $\mathrm{sI}(p, x, i_r)$, which means that $\forall N \in \eta(i_{r+1}) \exists i \in N \cap (i_r, i_{r+1})$ with $p(i) = x = p'[i_r, i_{r+1}]$. By continuity of p it must hold that $\exists N' \in \eta(i_{r+1})$ such that $p[N'] \subseteq N_{min}(p(i_{r+1}))$. As $p'[i_r, i_{r+1}] \in p[N']$, then $p'[i_r, i_{r+1}] \in N_{min}(p'(i_{r+1}))$. Property 5 follows immediately from point 2 above since we select y among the elements in the minimal neighbourhood. Finally we consider property 6. Let i be an index such that $p(i) \neq p'(i)$. By property 1, we know that i cannot be any of the indices in the resulting sequence. Let i_k and i_{k+1} be the two indices in the resulting sequence such that $i_k < i < i_{k+1}$. By definition of $\mathrm{sI}(p, p'(i), i_k)$, there must exist two indices r and s such that $p(r) = p(s) = p'(i)$, and $i_k \leq r < i < s \leq i_{k+1}$. By property 2 $p'[i_k, i_{k+1}] = p'(i)$, and the property holds. ◀

Similarly to the case with quasi-discrete paths, the following lemma allow us to transfer information about the satisfaction of the path operator to neighbouring points.

▶ **Lemma 29.** *Let* $\mathcal{M}$ *be a quasi-discrete neighbourhood model over topological paths and* $x, y \in \mathcal{M}$ *two points. Then the following hold.*

1. *If* $y \in N_{min}(x)$, $\mathcal{M}, y \models \varphi$ *and* $\mathcal{M}, y \models \varphi \,\mathcal{P}\, \psi$, *then also* $\mathcal{M}, x \models \varphi \,\mathcal{P}\, \psi$.
2. *If* $x \in N_{min}(y)$, $\mathcal{M}, x \models \varphi$, $\mathcal{M}, y \models \varphi$ *and* $\mathcal{M}, y \models \varphi \,\mathcal{P}\, \psi$, *then also* $\mathcal{M}, x \models \varphi \,\mathcal{P}\, \psi$.

Proof. Case (1): Let p and n be witnesses for $\mathcal{M}, y \models \varphi \,\mathcal{P}\, \psi$. There are two cases to consider. In the first case, p stays on y for an infinite number of indices. That is, the initial segment of p is not a singleton. Then we can define p' by $p'(0) = x$ and $p'(i) = p(i)$ for $i > 0$. Since p is continuous p' is continuous for every $i > 0$. For $i = 0$, we can take any neighbourhood $N \in \eta_{\mathbb{R}}(0)$ that only extends into the initial segment of p, where $p(j) = y$ for any $i \in N$

with $i \neq 0$. Then $p'[N] \subseteq N_{min}(x)$. So p' is also continuous at 0, and since $\mathcal{M}, y \models \varphi$, it is a witness for $\mathcal{M}, x \models \varphi \,\mathcal{P}\, \psi$. In the other case, p stays on y for the single index 0, and then moves to some point z. Then we define p' by $p'(0) = x$, $p'(i) = y$ for $0 < i \leq \frac{1}{2}$ and $p'(i) = p(2i-1)$ for $i > \frac{1}{2}$. Similar to the case above, p' is continuous at 0. Since the constant path is continuous, p' is continuous at $0 < i < \frac{1}{2}$. And since p is continuous at $2i - 1$, p' is continuous at i for $i \geq \frac{1}{2}$. Furthermore, with $n' = \frac{1}{2}(n+1)$, p' is a witness for $\mathcal{M}, x \models \varphi \,\mathcal{P}\, \psi$.

Case (2): By assumption on y, there is a path $p \colon \mathbb{R} \to \mathcal{M}$ and a value n, such that $p(0) = y$, $\mathcal{M}, p(n) \models \psi$ and for all i with $0 < i < n$, we have $\mathcal{M}, p(i) \models \varphi$. Using this path, we can construct the path p' by setting $p'(i) = x$ if $i < \frac{1}{2}$ and $p'(i) = p(2i - 1)$ for $i \geq \frac{1}{2}$. This function is continuous, and thus a path. Furthermore, we have $\mathcal{M}, p'(n+1) \models \psi$, and of course for all i with $0 < i < \frac{1}{2}(n+1)$ we have $\mathcal{M}, p'(i) \models \varphi$. So this path is a witness for $\mathcal{M}, x \models \varphi \,\mathcal{P}\, \psi$. ◀

We now proceed with the definition of filtrations for quasi-discrete models over topological paths. As can be expected, the definition differs from Def. 22 only in the treatment of paths. Instead of explicitly enumerating the equivalence classes on a path, we only assume the existence of a path on the filtration, and then transfer the satisfaction back to the original model. Furthermore, we do not need to consider the reachability path operator, since it is equivalent to the propagate modality, by Lemma 17.

▶ **Definition 30** (Filtration with Topological Paths)**.** *Let Σ be a subformula closed set of SLCS formulas, and $\mathcal{M} = ((X, \eta), \mathcal{I}_{\mathbb{R}}, \nu)$ a neighbourhood model, where (X, η) is a quasi-discrete space. We call the neighbourhood model $\mathcal{M}_f = ((X_f, \eta_f), \mathcal{I}_{\mathbb{R}}, \nu_f)$ a* filtration *of $\mathcal{M}$ over* topological paths *through Σ, if it satisfies the following conditions:*
1. $X_f = \{[x]_\Sigma \mid x \in X\}$
2. *if $y \in N_{min}(x)$, then $[y] \in N_{min}([x])$*
3. *if $[y] \in N_{min}([x])$, then for each $\mathcal{N}\varphi \in \Sigma$, we have that if $\mathcal{M}, y \models \varphi$, then $\mathcal{M}, x \models \mathcal{N}\varphi$*
4. *if $\pi \colon [0,1] \to X_f$ is a path on $\mathcal{M}_f$ where $\pi(i) = [x_i]$, then for every $\varphi \,\mathcal{P}\, \psi \in \Sigma$, we have that whenever $\mathcal{M}, x_i \models \varphi$ for each $0 < i < n$ and $\mathcal{M}, x_n \models \psi$, then also $\mathcal{M}, x_0 \models \varphi \,\mathcal{P}\, \psi$*
5. $\nu_f([x]) = \{p \in AP \mid \mathcal{M}, x \models p\}$

As in the purely quasi-discrete case, satisfaction of all formulas in the subformula closed set Σ is preserved on filtrations through Σ.

▶ **Lemma 31.** *Let $\mathcal{M}_f$ be a filtration of the quasi-discrete model $\mathcal{M}$ over topological paths through Σ. Then for all $\varphi \in \Sigma$, we have $\mathcal{M}, x \models \varphi$ iff $\mathcal{M}_f, [x] \models \varphi$.*

Proof. We proceed by induction on the structure of formulas. The base case for atomic propositions is immediate by Def. 30. The cases for the boolean operators are standard and the case for $\varphi = \mathcal{N}\psi$ is exactly as for Lemma 23.

Now consider $\varphi = \psi \,\mathcal{P}\, \chi$. If $\mathcal{M}, x \models \psi \,\mathcal{P}\, \chi$, this is equivalent to the existence of a path $p \colon x \rightsquigarrow \infty$ and a n and $\mathcal{M}, p(n) \models \chi$ as well as $\forall i \colon 0 < i < n$, we have $\mathcal{M}, p(i) \models \psi$. Observe that for any j and k such that $p(k) \in N_{min}(p(j))$, we have $[p(k)] \in N_{min}([p(j)])$ by property 2. Furthermore, for any j, we know that there is a $N \in \eta(j)$ such that $p[N] \subseteq N_{min}(p(j))$ by continuity of p. So, these two facts together imply that $\forall k \in N$, we have $[p(k)] \in N_{min}([p(j)])$. Hence we can define $\pi \colon [0,1] \to X_f$ by $\pi(i) = [p(i)]$ and then have that π is a path on $\mathcal{M}_f$ such that $\pi(0) = [x]$. Furthermore, by the induction hypothesis, for all i with $0 < i < n$, we have $\mathcal{M}_f, \pi(i) \models \psi$ and $\mathcal{M}_f, \pi(n) \models \chi$. This of course means $\mathcal{M}_f, [x] \models \psi \,\mathcal{P}\, \chi$.

Conversely, assume $\mathcal{M}_f, [x] \models \psi \,\mathcal{P}\, \chi$. Then there is a path $\pi \colon [0,1] \to X_f$ such that $\pi(0) = [x]$, for all i with $0 < i < n$ we have $\mathcal{M}_f, \pi(i) \models \psi$ and $\mathcal{M}_f, \pi(n) \models \chi$. Let

$\pi(i) = [x_i]$, then we get by the induction hypothesis that $\mathcal{M}, x_i \models \psi$ for all i with $0 < i < n$ and $\mathcal{M}, x_n \models \chi$. By property 4 we get $\mathcal{M}, x_0 \models \psi \,\mathcal{P}\, \chi$ and by $x \simeq x_0$, we get $\mathcal{M}, x \models \psi \,\mathcal{P}\, \chi$.

The case for $\varphi = \psi \,\mathcal{R}\, \chi$ is immediate by Lemma 17 and the previous case. ◀

The main part left in this section is to show that filtrations exist. This is more complicated than in the purely quasi-discrete case, due to the different behaviour of topological paths. However, if we restrict ourselves to *finite* sets Σ, then we can normalise the paths on the filtration according to Lemma 28, and use these simpler paths to establish satisfaction of the path modalities on the original model. Since we are only interested in filtrations through the set of subformulas induced by a single formula, this suffices for our purpose.

▶ **Lemma 32.** *Let Σ be a finite subformula closed set of formulas and $\mathcal{M}$ a quasi-discrete model over topological paths. Furthermore, let X_Σ be the set of equivalence classes of $\simeq_\Sigma$, ν_Σ be defined as in Def. 30 (5), and $\eta_s([x]) = \langle \{[y] \mid \exists y', x' \colon y' \in [y] \wedge x' \in [x] \wedge y \in N_{min}(x)\} \rangle$ for each $[x] \in X_\Sigma$. Then the model $\mathcal{M}_\Sigma = ((X_\Sigma, \eta_s), \mathcal{I}_\mathbb{R}, \nu_\Sigma)$ is a filtration of $\mathcal{M}$ over topological paths through Σ.*

Proof. First observe that $\mathcal{M}_\Sigma$ is indeed a quasi-discrete neighbourhood model over topological paths, since the underlying space of $\mathcal{M}_\Sigma$ is finite, and any finite neighbourhood space is quasi-discrete. We focus only on proving property 4 as all the others are already proved in Lemma 24.

Let $\pi \colon [0, 1] \to X_f$ be a path as required. If $n = 0$ so that $\mathcal{M}, x_n \models \psi$, this means $\mathcal{M}, x_0 \models \psi$, and so trivially $\mathcal{M}, x_0 \models \varphi \,\mathcal{P}\, \psi$. So, without loss of generality, we assume $n = 1$. With $\pi(i) = [x_i]$, we have $\mathcal{M}, x_i \models \varphi$ for $0 < i < 1$ and $\mathcal{M}, x_1 \models \psi$. Since the set of equivalence classes is finite, we can use Lemma 28 to get a path $\sigma \colon [0, 1] \to X_f$, with $x_0 \in \sigma(0)$ and $x_1 \in \sigma(1)$. Furthermore, the properties of σ in Lemma 28 ensure that for all $0 < i < 1$, if $\sigma(i) = [x'_i]$, then $\mathcal{M}, x'_i \models \varphi$.

Now, let $S = \{[z] \mid \exists i \colon \sigma(i) = [z]\}$ be the image of σ. Since S is finite, we define an order on S by setting $[z_i] < [z_j]$ iff there exist s and t with $s < t$ such that $\sigma(s) = [z_i]$ and $\sigma(t) = [z_j]$. By Lemma 28 and since the index space is totally ordered, this order is well-defined. So, in the following we will denote S by the sequence $[z_0], [z_1], \ldots, [z_r]$.

We proceed to prove that $\mathcal{M}, x_0 \models \varphi \,\mathcal{P}\, \psi$ by induction on then length r of this sequence. If $r = 0$, then $[z_0] = [x_1]$. Since $z_0 \simeq x_0 \simeq x_1$ and $\mathcal{M}, x_1 \models \psi$, we get $\mathcal{M}, x_0 \models \psi$, and thus $\mathcal{M}, x_0 \models \varphi \,\mathcal{P}\, \psi$.

Assume that the property holds for all such sequences for a length up to r, and consider $[z_0], [z_1], [z_2], \ldots, [z_r], [z_{r+1}]$. First, we can see that since σ is a path, the sequence $[z_1], [z_2], \ldots, [z_r], [z_{r+1}]$ also induces a path that satisfies the precondition of the property. So, we get by the induction hypothesis $\mathcal{M}, z_1 \models \varphi \,\mathcal{P}\, \psi$. We now need to examine the relation between $[z_0]$ and $[z_1]$. To that end, we first consider the preimages of both classes: $I_0 = \{i \mid \sigma(i) = [z_0]\}$ and $I_1 = \{i \mid \sigma(i) = [z_1]\}$. Furthermore, let j be the supremum of I_0. Recall that by Lemma 28, we have a sequence of indices $i_0, i_1, \ldots$ that partitions the interval $[0, 1]$ according to the values of σ. Now there are two possibilities for the relation between $[z_0]$ and $[z_1]$ according to σ.

1. If $i \in I_0$, then either $i = i_0 = 0$, or $i = i_1$. In the first case, $[z_0] = \sigma(i_0) \neq \sigma[i_0, i_1] = [z_1]$, and so $[z_1] \in N_{min}([z_0])$ by Lemma 28 (5). In the other case, we have $[z_1] = \sigma[i_1, i_2]$, and so $[z_0] = \sigma(i_1) \neq \sigma[i_1, i_2] = [z_1]$. Again, by Lemma 28 (5), we have $[z_1] \in N_{min}([z_0])$.
   By construction of $\mathcal{M}_f$ there are $y_0, y_1 \in \mathcal{M}$ such that $y_1 \in N_{min}(y_0)$ and $y_0 \in [z_0]$ and $y_1 \in [z_1]$. By assumption, we have $\mathcal{M}, x_0 \models \varphi$ as well, so by $x_0 \simeq z_0 \simeq y_0$, we get $\mathcal{M}, y_0 \models \varphi$ and $\mathcal{M}, y_1 \models \varphi \,\mathcal{P}\, \psi$. Then we have $\mathcal{M}, y_0 \models \varphi \,\mathcal{P}\, \psi$ from Lemma 29 (1) and thus $\mathcal{M}, x_0 \models \varphi \,\mathcal{P}\, \psi$.

2. Otherwise, we have $i \notin I_0$, and thus $i \in I_1$. Then certainly $i = i_1$, and so $[z_1] = \sigma(i_1) \neq \sigma[i_0, i_1] = [z_0]$. By Lemma 28 (4), we get $[z_0] \in N_{min}([z_1])$. By construction of $\mathcal{M}_f$ there are $y_0, y_1 \in \mathcal{M}$ such that $y_0 \in N_{min}(y_1)$ and $y_0 \in [z_0]$ and $y_1 \in [z_1]$.
However, in this case we also have that $i_1 > 0$, since otherwise $[z_0] = [z_1]$, which contradicts Property 3 of Lemma 28. So there is an $x \in [z_0]$, such that $\mathcal{M}, x \models \varphi$ by the properties of σ. Since $x \simeq y_0$, this means $\mathcal{M}, y_0 \models \varphi$. By assumption on σ, we have $\mathcal{M}, y_1 \models \varphi$ and since $y_1 \simeq z_1$, we also have $\mathcal{M}, y_1 \models \varphi \mathcal{P} \psi$. So, Lemma 29 (2) gives us $\mathcal{M}, y_0 \models \varphi \mathcal{P} \psi$, and with $x_0 \simeq z_0 \simeq y_0$ we can conclude the proof. ◀

The definition of filtrations together with Lemmas 31 and 32 yield the finite model property. Note that we can apply Lemma 32, as the set of subformulas of a formula is finite.

▶ **Theorem 33.** *If φ is a SLCS formula that is satisfiable on a quasi-discrete neighbourhood model over topological paths, then φ is satisifiable on a finite quasi-discrete neighbourhood model over topological paths.*

6 Conclusion

We have shown that SLCS does not have the finite model property over arbitrary neighbourhood models. Furthermore, we have proven that even when restricting to only quasi-discrete paths, there are still formulas that can only be satisfied on infinite models. Finally, we have shown that SLCS has the finite model property over models with underlying quasi-discrete neighbourhood spaces and quasi-discrete or topological paths. These results highlight that the types of spaces allowed have a much stronger impact on the existence of finite models than the types of paths allowed.

Our results are specific to the two types of paths we analysed. While these are the most common ones, it is possible to consider other definitions. Bubenik and Milićević [5] introduced other types of paths over neighbourhood spaces and analysed their properties. For example, they defined an index space based on a finite set $J = \{1, \ldots, m\}$, which is close to the idea of a quasi-discrete space. However, the neighbourhood system on this index space is very different from our setting, since it includes both the predecessor and the successor in the minimal neighbourhood of a point. Several of their other index spaces are even more different. An interesting research direction for future work is to study how these types of paths interact with the operators of SLCS.

A more applied strand of research is to analyse some of the extensions of SLCS. A natural first step would be to consider the temporal extension of SLCS with operators from CTL [10] and prove whether it has the finite model property. This would build upon previous results stating that CTL has the finite model property [15] and the combinations of logics that admit finite models typically also admit finite models [13]. Similarly, interesting future work would be to analyse the extension of SLCS with set-based operators introduced by Ciancia et al. [11], and the metric extensions by Bartocci et al. [1]. Finally, a model-theoretic study of a variant of SLCS presented by Bezhanishvili et al. would be interesting [3]. This variant is defined with a semantics based on polyhedra in continuous spaces, which is in some sense "in between" the class of quasi-discrete, graph-like models, and the class of general, arbitrary neighbourhood spaces.

Our results are a further step towards a comprehensive model theory for SLCS. Understanding how the models of SLCS behave can guide how and where we may apply this logic, as well as its extensions.

References

1. Ezio Bartocci, Luca Bortolussi, Michele Loreti, and Laura Nenzi. Monitoring Mobile and Spatially Distributed Cyber-physical Systems. In *Proceedings of the 15th ACM-IEEE International Conference on Formal Methods and Models for System Design*, MEMOCODE '17, pages 146–155, New York, NY, USA, 2017. ACM. event-place: Vienna, Austria. doi:10.1145/3127041.3127050.
2. Yuliy Baryshnikov and Robert Ghrist. Target enumeration via euler characteristic integrals. *SIAM J. Appl. Math.*, 70(3):825–844, 2009. doi:10.1137/070687293.
3. Nick Bezhanishvili, Vincenzo Ciancia, David Gabelaia, Gianluca Grilletti, Diego Latella, and Mieke Massink. Geometric model checking of continuous space. *CoRR*, abs/2105.06194, 2021. URL: https://arxiv.org/abs/2105.06194, arXiv:2105.06194.
4. Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2001. doi:10.1017/CBO9781107050884.
5. Peter Bubenik and Nikola Milićević. Applied topology using Čech's closure spaces, 2021. arXiv:2104.10206.
6. Fabrizio Banci Buonamici, Gina Belmonte, Vincenzo Ciancia, Diego Latella, and Mieke Massink. Spatial logics and model checking for medical imaging. *Int. J. Softw. Tools Technol. Transf.*, 22(2):195–217, 2020. doi:10.1007/s10009-019-00511-9.
7. Davide Castelnovo and Marino Miculan. Closure hyperdoctrines, with paths. *CoRR*, abs/2007.04213, 2020. arXiv:2007.04213.
8. Gustave Choquet. Convergences. *Annales de l'université de Grenoble. Nouvelle série. Section sciences mathématiques et physiques*, 23:57–112, 1947-1948. URL: http://www.numdam.org/item/AUG_1947-1948__23__57_0/.
9. Vincenzo Ciancia, Stephen Gilmore, Gianluca Grilletti, Diego Latella, Michele Loreti, and Mieke Massink. Spatio-temporal model checking of vehicular movement in public transport systems. *International Journal on Software Tools for Technology Transfer*, 20:289–311, January 2018. doi:10.1007/s10009-018-0483-8.
10. Vincenzo Ciancia, Gianluca Grilletti, Diego Latella, Michele Loreti, and Mieke Massink. An experimental spatio-temporal model checker. In *Software Engineering and Formal Methods. SEFM 2015*, volume 9509 of *Lecture Notes in Computer Science*. Springer, 2015. doi:10.1007/978-3-662-49224-6_24.
11. Vincenzo Ciancia, Diego Latella, Michele Loreti, and Mieke Massink. Model Checking Spatial Logics for Closure Spaces. *Logical Methods in Computer Science*, Volume 12, Issue 4, April 2017. doi:10.2168/LMCS-12(4:2)2016.
12. Vincenzo Ciancia, Diego Latella, Mieke Massink, and Erik de Vink. Towards spatial bisimilarity for closure models: Logical and coalgebraic characterisations, 2020. arXiv:2005.05578.
13. M. E. Coniglio, A. Sernadas, and C. Sernadas. Preservation by fibring of the finite model property. *Journal of Logic and Computation*, 21(2):375–402, April 2011. doi:10.1093/logcom/exq022.
14. Szymon Dolecki and Frédéric Mynard. *Convergence Foundations of Topology*. World Scientific, 2016. doi:10.1142/9012.
15. E. Allen Emerson and Jai Srinivasan. Branching time temporal logic. In J. W. de Bakker, W. P. de Roever, and G. Rozenberg, editors, *Linear Time, Branching Time and Partial Order in Logics and Models for Concurrency*, pages 123–172, Berlin, Heidelberg, 1989. Springer Berlin Heidelberg.
16. D. C. Kent and Won Keun Min. Neighborhood spaces. *International Journal of Mathematics and Mathematical Sciences*, 32(7):387–399, 2002. doi:10.1155/S0161171202202203.
17. Christine Largeron and Stéphane Bonnevay. A pretopological approach for structural analysis. *Information Sciences*, 144(1-4):169–185, 2002.
18. Sven Linker, Fabio Papacchini, and Michele Sevegnani. Analysing spatial properties on neighbourhood spaces. In Javier Esparza and Daniel Král', editors, *45th International*

Symposium on Mathematical Foundations of Computer Science, MFCS 2020, August 24-28, 2020, Prague, Czech Republic, volume 170 of *LIPIcs*, pages 66:1–66:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020. doi:10.4230/LIPIcs.MFCS.2020.66.

19 Sven Linker and Michele Sevegnani. Target counting with presburger constraints and its application in sensor networks. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 475(2231):20190278, 2019. doi:10.1098/rspa.2019.0278.

20 Antonio Rieser. Čech closure spaces: A unified framework for discrete and continuous homotopy. *Topology and its Applications*, 296:107613, 2021. doi:10.1016/j.topol.2021.107613.

21 Lynn Arthur Steen and J. Arthur Seebach, Jr. *Counterexamples in Topology.* Springer-Verlag, New York, 1978. Reprinted by Dover Publications, New York, 1995.

22 Christos Tsigkanos, Timo Kehrer, and Carlo Ghezzi. Modeling and verification of evolving cyber-physical spaces. In Matthias Tichy, Eric Bodden, Marco Kuhrmann, Stefan Wagner, and Jan-Philipp Steghöfer, editors, *Software Engineering und Software Management 2018, Fachtagung des GI-Fachbereichs Softwaretechnik, SE 2018, 5.-9. März 2018, Ulm, Germany*, volume P-279 of *LNI*, pages 113–114. Gesellschaft für Informatik, 2018.

23 Eduard Čech. *Topological spaces.* Academia, Publishing House of the Czechoslovak Academy of Sciences, 1966. Edited by Zdeněk Frolík and Miroslav Katětov.