Testing for Asymmetric Comovements*

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ABSTRACT

This paper aims to provide nonparametric tests for asymmetric comovements between random variables. We consider the popular Cramér-von Mises and Kolmogorov-Smirnov test statistics based on the distance between positive and negative joint conditional exceedance distribution functions. These tests can capture both linear and nonlinear dependence in the data and do not require selecting kernel functions and bandwidths. We derive the asymptotic distributions of the tests and establish the validity of a block multiplier-type bootstrap that one can use in finite-sample settings. We also show that these tests are consistent for any fixed alternative and have non-trivial power for detecting local alternatives converging to the null at the parametric rate. Monte Carlo simulations and a real financial data analysis illustrate satisfactory performance of the proposed tests.

Keywords: Asymmetric comovements; block multiplier bootstrap; Cramér-von Mises and Kolmogorov-Smirnov tests; distribution function; exceedance; financial markets.

Journal of Economic Literature classification: C12; C14; C15.

1 Introduction

The knowledge of the symmetric or asymmetric nature of the comovements between asset returns is essential for optimal portfolio allocation and risk management. In a recent paper, Dahlquist, Farago, and Tédongap (2017) show that, when coupled with generalized disappointment aversion preferences, asymmetric comovements between asset returns yield qualitatively different optimal portfolios from those of the standard model, in which the asymmetry is ignored; see also Ang and Chen (2002) and Patton (2004). Furthermore, Tsafack (2009) shows that in the presence of asymmetric comovements, an optimal portfolio which is based on a multivariate symmetric comovements assumption will lead to an underestimation of portfolio Value-at-Risk and Expected Shortfall. Several statistical procedures have been developed to test symmetric comovements between returns. However, most of the proposed tests were built based on the coefficient of exceedance correlation, thus they are expected to have low power for detecting full and non-linear dependence, since by construction this coefficient only measures linear dependence and cannot capture the full dependence structure that can be due to the dependence in higher order moments. In this paper, we develop model-free tests for symmetric comovements which accounts for nonlinear and full dependence in the data. Our ultimate goal is to derive tests that are fully nonparametric, but converge at the parametric rate and, in particular, do not require selecting bandwidths and kernel functions.

Early studies have tested the symmetric comovements between asset returns in the context of parametric models; see Longin and Solnik (2001), Ang and Bekaert (2002), Ang and Chen (2002), Patton (2004) and references therein. For a pair of variables (X_t, Y_t) [hereafter both X_t and Y_t are standardized to have zero mean and unit variance], these parametric tests are based on the classical Pearson-type exceedance correlations: $\rho^+(c)$ =corr$(X_t, Y_t | X_t > c, Y_t > c)$ and $\rho^-(c)$ =corr$(X_t, Y_t | X_t < -c, Y_t < -c)$, for a given exceedance level $c \geq 0$.[1] They test the null hypothesis

$$H_0^L : \rho^+(c) = \rho^-(c)$$

against the alternative hypothesis

$$H_1^L : \rho^+(c) \neq \rho^-(c).$$

The above tests, however, focus on only linear types of asymmetric comovements and cannot reflect vast asymmetric structures beyond the second moments. Modelling asymmetric exchange rate dependence, Patton (2006) shows that using linear correlation is not sufficient to describe the dependence between exchange rates, and consequently this might have a negative effect (or low degree of protection) on a hedging strategy constructed using linear correlation.

[1] Observe that the standardization of the data (X_t and Y_t in the current paper), which is quite common in the literature of symmetry testing for financial data, might induce some estimation effect; see, e.g., Chen (2016) among others. However, the examination of this effect is left for future research.

Moreover, few recent papers have proposed alternative tests for symmetric comovements; see Hong, Tu, and Zhou (2007), Jiang, Wu, and Zhou (2018a), and Jiang, Maasoumi, Pan, and Wu (2018b). Hong et al. (2007) derive a test for symmetry in correlation that does not require the specification of a statistical model for the data. However, their test only captures linear dependence and is unable to detect asymmetric comovements beyond the second moment. This has motivated Jiang et al. (2018a) to introduce an entropy measure that quantifies asymmetry in the joint distribution of an individual stock return and market return. Their measure is used to build a model-free test to determine if asymmetric comovements exist in the stock markets. Their test statistic is a bivariate normalized version of the entropy measure originally proposed by Granger, Maasoumi, and Racine (2004). The calculation of their test statistic is based on a nonparametric kernel smoothing method that depends on the selection of kernel function and bandwidths. Jiang et al. (2018b) suggest a modified mutual information measure constructed from the perspective of the whole return distribution rather than the first two moments. Based on this measure, they develop a model-free test for asymmetric comovements. However, their test again is based on a smoothing method that requires the selection of kernel function and bandwidth parameters. As we know, the performance (size and power) of most nonparametric tests depends on the bandwidth parameter. In practice, a bad choice of this parameter can negatively affect the size and power of the tests, which might lead to misleading empirical results. To overcome this issue, in the following we provide nonparametric tests of asymmetric comovements that do not rely on bandwidth parameter. Perhaps most importantly, unlike the latter two papers, we are testing asymmetric comovements rather than testing the bivariate symmetry of the underlying distribution for (X_t, Y_t); namely, Jiang et al. (2018a, b) test against asymmetric comovements by testing bivariate symmetry, which is a more restrictive testing problem.

To motivate our hypotheses of interest, we introduce two concepts. Positive exceedance distribution of (X_t, Y_t) at an exceedance level $c \geq 0$ is defined as the conditional joint distribution function of (X_t, Y_t) when both exceed c:

$$F^{+}(x, y; c) := \Pr(X_t \leq x, Y_t \leq y | X_t > c, Y_t > c) \equiv \int_c^x \int_c^y f(\bar{x}, \bar{y})\, d\bar{x}\, d\bar{y}/p^{+}(c),$$

for all $x > c$ and $y > c$, where $f(x, y)$ is the joint probability density function of (X_t, Y_t) and $p^{+}(c) := \Pr(X_t > c, Y_t > c) \equiv \int_c^\infty \int_c^\infty f(\bar{x}, \bar{y})\, d\bar{x}\, d\bar{y}$, i.e. the probability of (X_t, Y_t) falling into the subspaces $(c, \infty) \times (c, \infty)$. Similarly, negative exceedance distribution of (X_t, Y_t) at c is the conditional joint distribution function of $(-X_t, -Y_t)$ when both exceed c:

$$F^{-}(x, y; c) := \Pr(-X_t \leq x, -Y_t \leq y | -X_t > c, -Y_t > c) \equiv \int_{-x}^{-c} \int_{-y}^{-c} f(\bar{x}, \bar{y})\, d\bar{x}\, d\bar{y}/p^{-}(c),$$

for all $x > c$ and $y > c$, where $p^{-}(c) := \Pr(-X_t > c, -Y_t > c) \equiv \int_{-\infty}^{-c} \int_{-\infty}^{-c} f(\bar{x}, \bar{y})\, d\bar{x}\, d\bar{y}$ be the probability of $(-X_t, -Y_t)$ falling into the subspaces $(c, \infty) \times (c, \infty)$. Applying standard change of variables, one can see that $F^-(x,y;c) = \int_c^x \int_c^y f(-\bar{x},-\bar{y})\,d\bar{x}\,d\bar{y}/p^-(c)$. Note that $p^+(c)$ is not necessarily equal to $p^-(c)$. Under bivariate symmetry of (X_t, Y_t), $p^+(c) = p^-(c)$ holds for every c. In addition, $p^+(c) + p^-(c)$ is smaller than one as only the first and third quadrants are considered.

For a given exceedance level $c \geq 0$, in this paper we test the null hypothesis

$$H_0 : F^+(x,y;c) = F^-(x,y;c), \quad \text{for all } x > c \text{ and } y > c, \tag{1}$$

against the alternative hypothesis

$$H_1 : F^+(x,y;c) \neq F^-(x,y;c), \quad \text{for some } x > c \text{ and } y > c. \tag{2}$$

Our test statistics are of Cramér-von Mises and Kolmogorov-Smirnov -type, which are based on the suitable distances between $F^+(x,y;c)$ and $F^-(x,y;c)$. For example, when $c = 0$, we are essentially testing whether the joint distribution function of positive returns is the same as that of negative returns. If the null hypothesis is rejected, there must exist some characteristics stemming from asymmetric exceedance distributions, whether those characteristics are due to asymmetric correlations or any other higher order moments. In this broad sense, our tests of asymmetric comovements are robust to nonlinear dependence displayed in the abundance of financial data.

Obviously, testing H_0 against H_1 is only testing a partial feature of the joint distribution function of (X_t, Y_t), especially the distributional features in its first and third quadrants, while those features in the second and fourth quadrants are not informative of our testing problem. This distinguishes our testing problem from the existing ones. For instance, a conceptually related but a very different problem is testing the so-called bivariate reflected (or central) symmetry, which deals with the whole shape of the bivariate distribution, i.e. $f(x,y) = f(-x,-y)$ for all $(x,y) \in \mathbb{R}^2$ or equivalently testing $F(x,y) = \overline{F}(-x,-y)$ for all $(x,y) \in \mathbb{R}^2$, with $F(x,y) := \Pr(X_t \leq x, Y_t \leq y) \equiv \int_{-\infty}^x \int_{-\infty}^y f(\bar{x},\bar{y})\,d\bar{x}\,d\bar{y}$ denoting the bivariate CDF and $\overline{F}(x,y) = 1 - F(x,\infty) - F(\infty,y) + F(x,y)$ denoting the associated bivariate survival function, which has been well studied in the literature; see e.g. Ghosh and Ruymgaart (1992), Aki (1993), Neuhaus and Zhu (1998), Henze et al. (2003) and Bouzebda and Cherfi (2012) among others. As discussed before, Jiang et al. (2018a, b) also test bivariate symmetry to test H_0 against H_1.

Our tests are able to capture both linear and nonlinear dependence in the data and, unlike the existing nonparametric tests, they do not require nonparametric smoothing. We derive the asymptotic distributions of our nonparametric tests and establish the asymptotic validity of a block multiplier-type bootstrap that one can use in finite-sample settings. We also show that our tests are consistent for any fixed alternative and they have non-trivial power for detecting local alternatives converging to the null at the parametric rate $\sqrt{T}$, with T the sample size. A Monte Carlo simulation study reveals that the bootstrap-based tests have reasonable finite-sample size and power for a variety of data generating processes and different sample sizes.

We also provide an empirical application where our tests are used to test for the asymmetric comovements between the S&P 500 daily return and the daily returns on 29 individual stocks. The results show that the symmetric comovements hypothesis is rejected for the majority of stocks under consideration for commonly used exceedance levels. Finally, inspired by Deng (2016), we also consider a second empirical study for testing the asymmetric comovments among five major stock market indices, which are reported in the online Appendix C. Our results suggest that there is a great tendency of asymmetric comovements among the five stock markets.

The rest of the paper is organized as follows. Section 2 presents the general theoretical framework which underlies the null and alternative hypotheses of interest and the test statistics. In Section 3, we provide the asymptotic distributions of our nonparametric tests and study their consistency and local power properties. In Section 4, we establish the validity of the block multiplier bootstrap to implement the proposed tests. Section 5 presents a Monte Carlo simulation exercise to investigate the finite-sample properties of the tests. Section 6 is devoted to an empirical application and the conclusions relating to the results are given in Section 7. Proofs of the theoretical results and the auxiliary testing procedure used for producing the results discussed in Section 2 can be found in a separate companion supplemental Appendix, which is available online. All the empirical results and additional simulation results can also be found in the online **Appendix**.

2 Testing framework

Consider two random variables of interest X and Y, which are defined in the probability space $(\Omega, \mathcal{A}, \mathcal{P})$. Let $\left\{(X_t, Y_t)' \in \mathbb{R}^2 : \; t = 0, \pm 1 \pm 2, \ldots\right\}$ be a strictly stationary and ergodic bivariate time series process from (X, Y). Denote by $f(x, y)$ and $F(x, y)$ the joint probability density function (PDF) and the cumulative distribution function (CDF) of $(X_t, Y_t)'$, respectively.

Before we outline our main testing framework and introduce our test statistics, we note that whenever $p^+(c) \neq p^-(c)$ for the given c, the probabilities $p^+(c)$ and $p^-(c)$ will play a role that cannot be ignored, in sharp contrast to the testing problem which involves testing the whole shape of $F(x, y)$. To check the appropriateness of the assumption $p^+(c) = p^-(c)$, in Section 6 we use 30 time series (stock prices) to estimate $p^+(c)$ and $p^-(c)$ and test if they are equal for various c's. The results are reported in Table 1 of the online Appendix C. The latter indicates that it is inappropriate to assume that $p^+(c) = p^-(c)$ when we build a test for the symmetric comovements hypothesis in (1). Hence, a valid test should be based on the comparison between the conditional versions $F^+(x, y; c)$ and $F^-(x, y; c)$, which provides a good motivation for the use of the tests that we propose in this paper. For more details, the reader is referred to Section 6.

To test the null hypothesis H_0 in (1) against the alternative hypothesis in (2), we use the following characterization. Since $\mathbb{E}\left[1(c<X\leq x)1(c<Y\leq y)\right]=\int_c^x\int_c^y f(\bar{x},\bar{y})\,d\bar{x}\,d\bar{y}$ and $\mathbb{E}\left[1(c<-X\leq x)1(c<-Y\leq y)\right]=\int_{-x}^{-c}\int_{-y}^{-c} f(\bar{x},\bar{y})\,d\bar{x}\,d\bar{y}$, the null H_0 can be rewritten as:

$$H_0: S(x,y;c)=0, \quad \text{for all } x>c \text{ and } y>c,$$

where

$$\begin{aligned} S(x,y;c) &= \left(F^+(x,y;c)-F^-(x,y;c)\right)p^+(c)p^-(c) \\ &= \mathbb{E}\left[1(c<X\leq x)1(c<Y\leq y)\right]p^-(c)-\mathbb{E}\left[1(c<-X\leq x)1(c<-Y\leq y)\right]p^+(c), \end{aligned} \tag{3}$$

with $1(A)$ an indicator function defined on event A, which takes value one when event A is true and zero otherwise. The above characterization helps avoid the random denominators issue, which can improve the finite-sample performance of our test statistics.

We call the methodology based on the above characterization the *integrated approach* (also known as the *global approach*) because it uses integrated (or cumulative) measures of dependence. Note that $S(x,y;c)$ is a fully nonparametric dependence measure that can quantify potential departures from symmetric co-movements regardless whether the underlying dependence structure among the data is of linear or nonlinear nature. The tests that we consider in this paper will be constructed based on the nonparametric deviance measure $S(x,y;c)$, which, unlike the majority of the literature, is completely model-free since it does not impose any restrictions on the parametric form of the joint cumulative distribution function of $(X_t,Y_t)'$. In addition, tests based on the *integrated approach* have an important advantage to practitioners as the estimation of $S(x,y;c)$ does not depend on any user-chosen parameters such as bandwidths and kernels. On the contrary, tests constructed using the density-based measure, which we call the *local approach*, have to select bandwidth parameters, a task not trivial and often without clear guidance.

Without causing much confusion, in the following we suppress the dependence of $p^+(c)$, $p^-(c)$ and $S(x,y;c)$ on c by using the notations p^+, p^- and $S(x,y)$, respectively. Similar simplification applies for the associated estimators. Nevertheless, one should always bear in mind that probabilities p^+ and p^- are generally different for different exceedance level c, see Table 1 of the online **Appendix C**. Consequently, $S(x,y)$ is a different measure for a different exceedance level c.

To construct a feasible testing procedure for our problem, we first need to consistently estimate $S(x,y)$. In view of a sample $\left\{(X_t,Y_t)':1\leq t\leq T\right\}$ of size T, the nonparametric measure $S(x,y)$ can be consistently estimated by replacing the unknown expectations in (3) by their associated sample analogues:

$$S_T(x,y)=\frac{1}{T}\sum_{t=1}^{T}\left[1(c<X_t\leq x)1(c<Y_t\leq y)\hat{p}^- - 1(c<-X_t\leq x)1(c<-Y_t\leq y)\hat{p}^+\right], \tag{4}$$

where $\hat{p}^+ := T^{-1}\sum_{t=1}^{T}1(X_t>c)1(Y_t>c)$ and $\hat{p}^- := T^{-1}\sum_{t=1}^{T}1(X_t<-c)1(Y_t<-c)$ are $\sqrt{T}$−consistent estimators of p^+ and p^-, respectively; see the online **Appendix B**. Note that our stochastic process $S_T(x,y)$

- commonly referred as empirical distribution function - is simply the difference between two bivariate empirical processes (with one left truncated and the other right truncated), which are scaled by the estimated probabilities $\hat{p}^-$ and $\hat{p}^+$, respectively.

Note also that, when there is no truncation, the process $S_T(x, y)$ in (4) reduces to the following standard bivariate empirical process (empirical distribution function):

$$\frac{1}{T}\sum_{t=1}^{T}\left[1(X_t \leq x)1(Y_t \leq y) - 1(X_t \geq -x)1(Y_t \geq -y)\right], \tag{5}$$

which can be employed to test for the bivariate reflected symmetry of distribution function $F(x, y)$, e.g., testing $\overline{F}(-x, -y) = F(x, y)$ for all $(x, y) \in \mathbb{R}^2$, with $\overline{F}(x, y) := 1 - F(x, \infty) - F(\infty, y) + F(x, y)$ the bivariate survival function associated with $F(x, y)$. However, the process (5) is not designed to test (1) against (2). In fact, it may be expected that test statistics based on (5) would suffer some power loss because of ignoring completely the probabilities p^+ and p^-, whose estimation may help enhance the testing power.

The stochastic process $S_T(x, y)$ will be the main ingredient of our test statistics. Large deviations of it from zero indicate a sign of H_0 not being true. On one hand, when the null hypothesis of symmetric comovements (1) holds, $S_T(x, y)$ is expected to be close to zero for all $(x, y) \in (c, \infty) \times (c, \infty)$, and under some mild regularity conditions stated in the next section, $\sqrt{T}S_T(x, y)$ can be shown to converge weakly to a zero mean Gaussian process with certain covariance structure. On the other hand, if the underlying data generating process $(X_t, Y_t)'$ does possess a dependence structure with asymmetric comovements as in (2), $S_T(x, y)$ will be different from zero for a positive Lebesgue measure of $(x, y) \in (c, \infty) \times (c, \infty)$, and consequently $\sqrt{T}S_T(x, y)$ diverges to infinity for some values of (x, y). This distinctive behavior of $\sqrt{T}S_T(x, y)$ under the null and alternative hypotheses forms the logic of our testing procedure and guarantees the consistency of our proposed tests.

The test statistics that we consider are based on proper continuous functionals of the $\sqrt{T}-$rescaled stochastic process $\sqrt{T}S_T(x, y)$, say $\varphi\left(\sqrt{T}S_T\right)$ for some continuous and even functional $\varphi(\cdot)$. We consider two of the most popular functionals: Cramér-von Mises (CvM) and Kolmogorov-Smirnov (KS) functionals. Test statistics that are based on other functionals are also possible. The CvM-type test statistic is given by:

$$CvM_T = \int_c^{\infty}\int_c^{\infty}\left(\sqrt{T}S_T\left(x, y\right)\right)^2 \Psi\left(dx, dy\right) = \int_{\mathbb{R}}\int_{\mathbb{R}}\left(\sqrt{T}S_T\left(x, y\right)1(x > c)1(y > c)\right)^2 \Psi\left(dx, dy\right),$$

where $\Psi\left(x, y\right)$ is a (potentially random) integrating function. For example, $\Psi\left(x, y\right)$ can be any bivariate distribution function which is absolutely continuous with respect to the Lebesgue measure. Hereafter, we focus our attention on the following CvM test statistic:

$$CvM_T = \int_{\mathbb{R}}\int_{\mathbb{R}}\left(\sqrt{T}S_T\left(x, y\right)1(x > c)1(y > c)\right)^2 \hat{F}_T\left(dx, dy\right) = \sum_{t=1}^{T}\left(S_T\left(X_t, Y_t\right)1(X_t > c)1(Y_t > c)\right)^2, \tag{6}$$

with $\hat{F}_T\left(x, y\right) = T^{-1}\sum_{t=1}^{T}1\left(X_t \leq x\right)1\left(Y_t \leq y\right)$ the empirical distribution function (EDF) of $\left\{(X_t, Y_t)'\right\}_{t=1}^{T}$.

The KS-type test statistic is given by:

$$KS_T = \sup_{x>c,y>c} \left| \sqrt{T} S_T(x,y) \right| = \sup_{(x,y)\in\mathbb{R}^2} \left| \sqrt{T} S_T(x,y) 1(x>c) 1(y>c) \right|.$$

Note that the KS_T statistic takes the supremum over all points (x,y) in the subspace $(c,\infty)\times(c,\infty)$. It is more computationally demanding than CvM_T. To implement KS_T test in practice, we suggest to compute it by taking the maximum over points (x,y) in the sample space $\left\{(X_t,Y_t)'\right\}_{t=1}^T$, i.e.,

$$KS_T = \max_{1\le t\le T} \left| \sqrt{T} S_T\left(X_t,Y_t\right) 1(X_t>c) 1(Y_t>c) \right|. \tag{7}$$

The above form is easier to maximize and less time consuming, though it might suffer some power loss. Furthermore, notice that the proposed test statistics CvM_T and KS_T can be computed as follows:

$$CvM_T = \tfrac{1}{T^2} \textstyle\sum_{t=1}^T \left[\sum_{s=1}^T \left[1(c<X_s\le X_t)1(c<Y_s\le Y_t)\hat{p}^- - 1(c<-X_s\le X_t)1(c<-Y_s\le Y_t)\hat{p}^+\right]\right]^2,$$

$$KS_T = \tfrac{1}{\sqrt{T}} \max_{t=1,\cdots,T} \left| \textstyle\sum_{s=1}^T \left[1(c<X_s\le X_t)1(c<Y_s\le Y_t)\hat{p}^- - 1(c<-X_s\le X_t)1(c<-Y_s\le Y_t)\hat{p}^+\right]\right|, \tag{8}$$

where $\hat{p}^+$ and $\hat{p}^-$ were defined previously.

The test statistics CvM_T and KS_T allow for both linear and nonlinear dependence structure in the data and do not require any local smoothing like the density-based approaches. As we will show in the following sections, the proposed tests are consistent against the fixed alternative H_1 in (2) and have non-trivial power in detecting sequences of local alternatives converging to the null at the parametric rate $T^{-1/2}$. In particular, these tests are designed to work for time series data.

We reject the null hypothesis H_0 whenever the statistic CvM_T or KS_T exceeds some "large" values. Unfortunately, the determination of these "large" values is not straightforward, since the limiting null distributions of CvM_T and KS_T depend on unknown features of the underlying data generating process, e.g., the serial dependence structure. Thus, obtaining the critical values from these limiting distributions is usually hard, if not impossible at all. Nevertheless, these critical values can be well approximated by an easy-to-implement block multiplier-type bootstrap that we will describe and validate in Section 4. The essence of the proposed bootstrap is to mimic the first order behavior of $S_T(x,y)$ in (4) in the time series context.

3 Asymptotic theory

This section aims to study the asymptotic properties of the feasible stochastic process $S_T(x,y)$ in (4) under the null, fixed alternative and a sequence of local alternatives. Consequently, the asymptotic properties of the test statistics CvM_T and KS_T are established. Before doing so, we consider the following regularity condition on the bivariate time series process $(X_t,Y_t)'$.

Assumption A.1: (i) $\{Z_t = (X_t, Y_t)' : t = 0, \pm 1, \pm 2, \ldots\}$ is a strictly stationary strong mixing bivariate time series process with mixing coefficients $\alpha(j)$ such that $\sum_{j=0}^{\infty} j^2 \alpha(j)^{\frac{\gamma}{4+\gamma}} < \infty$ for some $0 < \gamma < 2$; (ii) both distributions of X_t and Y_t have bounded densities with respect to the Lebesgue measure.

Assumption A.1 is a condition on the data generating process $\{Z_t\}$ and restricts the process $\{Z_t\}$ to be strictly stationary strong mixing with mixing rates decaying sufficiently fast. The condition $0 < \gamma < 2$ is used to prove the stochastic equicontinuity of a certain empirical process. The assumption is quite weak and allows for very general serial dependence structures in the data. It is satisfied by many processes and in particular by the GARCH-type processes that are suitable for financial returns, see e.g. Carrasco and Chen (2002). As discussed in the sections below, the martingale difference sequence (MDS) assumption for some appropriate functions of Z_t given the information set at time $t-1$ is not assumed to establish our main theoretical results.[2]

In the sequel, we provide the asymptotic theory for the stochastic process $\sqrt{T} S_T(x, y)$ for a given exceedance level $c \geq 0$, thus of the test statistics CvM_T and KS_T that are based on $\sqrt{T} S_T(x, y)$.

3.1 Asymptotic null distribution

For the construction of the feasible process $S_T(x, y)$ in (4), the unknown probabilities p^+ and p^- have been replaced with $\sqrt{T}-$consistent estimators $\hat{p}^+$ and $\hat{p}^-$. This turns out to have non-negligible effects on the asymptotic behavior of the test statistics CvM_T and KS_T that are based on $S_T(x, y)$. To illustrate the effects of replacing p^+ and p^- with the associated estimators $\hat{p}^+$ and $\hat{p}^-$, we introduce the following infeasible stochastic process:

$$\tilde{S}_T(x, y) = \frac{1}{T} \sum_{t=1}^{T} \left[1(c < X_t \leq x) 1(c < Y_t \leq y) p^- - 1(c < -X_t \leq x) 1(c < -Y_t \leq y) p^+ \right].$$

Naturally, if both p^+ and p^- were known, the infeasible process $\tilde{S}_T(x, y)$ could be used to detect any departures from symmetric comovements, whether the underlying dependence structure embedded in $(X_t, Y_t)'$ is of linear or nonlinear nature. In this sense, the estimator $\tilde{S}_T(x, y)$ is a fully model-free measure, gauging the degrees of asymmetric comovements between X_t and Y_t.

Let

$$\varepsilon_t(x, y) := \varepsilon_t(x, y; c) \equiv 1(c < X_t \leq x) 1(c < Y_t \leq y) p^- - 1(c < -X_t \leq x) 1(c < -Y_t \leq y) p^+.$$

The following lemma shows that $\sqrt{T} \tilde{S}_T(x, y) \equiv T^{-1/2} \sum_{t=1}^{T} \varepsilon_t(x, y)$ converges weakly to a zero mean Gaussian process with a certain long-run covariance kernel under the null hypothesis H_0. We let "$\Rightarrow$" denote weak convergence, see Definition 1.3.3 and Chapter 1.6 in van der Vaart and Wellner (1996). This lemma is

[2] Specifically, an MDS assumption for both directions $1(c < Z_t \leq z)$ and $1(c < -Z_t \leq z)$ with $z = (x, y)'$, i.e., the no predictability assumption in both directions.

proved by an application of the finite-dimensional (fidi) distributions of $\sqrt{T}\tilde{S}_T$ and the uniform tightness of this process [see the proof of Lemma 1 in the online **Appendix A**].

Lemma 1 *Suppose Assumption* ***A.1*** *is satisfied. Then under the null hypothesis* H_0 *of symmetric comovements in (1),*

$$\sqrt{T}\tilde{S}_T(x,y) \Rightarrow S_\infty(x,y), \ \text{as } T \to \infty,$$

where $S_\infty(x,y)$ *is a zero mean Gaussian process with long-run covariance kernel*

$$\begin{aligned}\mathbb{K}\left((x,y),(x',y')\right) :=& \mathbb{E}\left[S_\infty(x,y)S_\infty(x',y')\right] \\ \equiv& \mathbb{E}\left[\varepsilon_1(x,y)\varepsilon_1(x',y')\right] + \sum_{i=1}^{\infty}\{\mathbb{E}\left[\varepsilon_1(x,y)\varepsilon_{1+i}(x',y')\right] + \mathbb{E}\left[\varepsilon_{1+i}(x,y)\varepsilon_1(x',y')\right]\}.\end{aligned}$$

Lemma 1 indicates that the long-run covariance kernel of the asymptotic null process $\{S_\infty(x,y) : x \in (c,\infty), y \in (c,\infty)\}$ depends on the serial dependence structure in the data $\{(X_t,Y_t)'\}$, signified by the covariance terms $\sum_{i=1}^{\infty}\{\mathbb{E}[\varepsilon_1(x,y)\varepsilon_{1+i}(x',y')]+\mathbb{E}[\varepsilon_{1+i}(x,y)\varepsilon_1(x',y')]\}$. These terms can be further simplified under certain circumstances. Let $\mathcal{F}_{t-1}$ be the sigma algebra generated by the past of $Z_t = (X_t,Y_t)'$, i.e., $\mathcal{F}_{t-1} = \sigma\left(Z'_{t-1}, Z'_{t-2}, \ldots\right)$. Now, if $\{\varepsilon_t(x,y), \mathcal{F}_t\}$ is an MDS such that $\mathbb{E}[\varepsilon_t(x,y)|\mathcal{F}_{t-1}] = 0$ *a.s.* [for instance, when $\mathbb{E}[1(c < Z_t \le z)|\mathcal{F}_{t-1}] = \mathbb{E}[1(c < Z_t \le z)] \equiv F^+(x,y)p^+$ and $\mathbb{E}[1(c < -Z_t \le z)|\mathcal{F}_{t-1}] = \mathbb{E}[1(c < -Z_t \le z)] \equiv F^-(x,y)p^-$ for all $z = (x,y)'$, i.e., there is no predictability in both directions, or when $\{Z_t\}_{t=1}^n$ is simply an independent sequence], we have under H_0 that

$$\begin{aligned}\mathbb{K}\left((x,y),(x',y')\right) &= \mathbb{E}\left[\varepsilon_1(x,y)\varepsilon_1(x',y')\right] \\ &= \left(F^+\left(x\wedge x', y\wedge y'\right)p^- + F^-\left(x\wedge x', y\wedge y'\right)p^+\right)p^+p^- \\ &\overset{H_0}{=} p^+p^-\left(p^+ + p^-\right)F^+\left(x'\wedge x', y\wedge y'\right),\end{aligned}$$

with $a \wedge b \equiv \min\{a,b\}$, and $F^+(x,y)$ and $F^-(x,y)$ are respectively the positive and negative exceedance distribution functions we defined before. Consequently, in the special case of $\{Z_t\}$ being an MDS process, we can normalize $\sqrt{T}\tilde{S}_T(x,y)$ by $p^+p^-(p^+ + p^-)F^+(x,y)$ [or equivalently by $p^+p^-(p^+ + p^-)F^-(x,y)$] and make the associated asymptotic null process $\{S_\infty(x,y) : x \in (c,\infty), y \in (c,\infty)\}$ asymptotically pivotal. But we emphasize that in this paper we will not require any MDS assumption for $\{Z_t\}$ and our results hold under very general serial dependence structure in $\{Z_t\}$.

With the assistance of $\tilde{S}_T(x,y)$, we proceed to study $S_T(x,y)$ in (4). Due to the presence of so-called "parameter estimation uncertainty", which is caused by estimating p^+ and p^-, the asymptotic null distribution of $\sqrt{T}S_T(x,y)$ is generally different from that of $\sqrt{T}\tilde{S}_T(x,y)$ stated in Lemma 1. The next theorem establishes that $S_T(x,y)$ can be decomposed into a summation of the infeasible process $\tilde{S}_T(x,y)$ plus a non-negligible (stochastic) drift term that captures properly the estimation uncertainty because of using $\hat{p}^+$

and $\hat{p}^-$. As a result, the asymptotic null distribution of $\sqrt{T}S_T(x,y)$ can be readily obtained using Lemma 1 and the following asymptotic representation [see the proof of Theorem 1 in the online **Appendix A**].

Theorem 1 *Suppose Assumption **A.1** is satisfied. Then under the null hypothesis H_0 of symmetric co-movements in (1),*

$$S_T(x,y) = \tilde{S}_T(x,y) - F^+(x,y)\tilde{R}_T + o_p\left(\frac{1}{\sqrt{T}}\right), \text{ as } T \to \infty, \tag{9}$$

uniformly in $(x,y) \in (c,\infty)^2$, where $\tilde{R}_T$ is a zero mean random variable such that

$$\begin{aligned}\tilde{R}_T :=& \frac{1}{T}\sum_{t=1}^{T}\varepsilon_t(\infty,\infty)\\ \equiv& \frac{1}{T}\sum_{t=1}^{T}\left[1(X_t>c)1(Y_t>c)p^- - 1(X_t<-c)1(Y_t<-c)p^+\right].\end{aligned}$$

Remark 1: *Recall that $p^+ = \mathbb{E}[1(X_t>c)1(Y_t>c)]$ and $p^- = \mathbb{E}[1(X_t<-c)1(Y_t<-c)]$. According to the central limit theorem (CLT) for strictly stationary ergodic sequences [see Hall and Heyde (1980)], the random variable $\tilde{R}_T$ satisfies*

$$\sqrt{T}\tilde{R}_T \xrightarrow{d} N\left(0,\sigma^2\right),$$

where

$$\begin{aligned}\sigma^2 :=& \mathbb{E}\left[\varepsilon_1^2(\infty,\infty)\right] + 2\sum_{i=1}^{\infty}\mathbb{E}\left[\varepsilon_1(\infty,\infty)\varepsilon_{1+i}(\infty,\infty)\right]\\ \equiv& p^+p^-\left(p^+ + p^-\right) + 2\sum_{i=1}^{\infty}\mathbb{E}\left[\varepsilon_1(\infty,\infty)\varepsilon_{1+i}(\infty,\infty)\right].\end{aligned} \tag{10}$$

Note that the above convergence result for $\sqrt{T}\tilde{R}_T$ holds regardless of whether the null hypothesis H_0 in (1) *holds or not. Under a general serial dependence structure in $(X_t,Y_t)'$, similar to the long-run covariance kernel $\mathbb{K}(\cdot,\cdot)$ given in Theorem 1, the long-run variance σ^2 for the limiting distribution of $\sqrt{T}\tilde{R}_T$ also has a complicated expression due to the presence of covariance terms $2\sum_{i=1}^{\infty}\mathbb{E}\left[\varepsilon_1(\infty,\infty)\varepsilon_{1+i}(\infty,\infty)\right]$, which characterizes the contributions of the serial dependence in the data. Though it is not our main interest, a consistent estimator of σ^2 can be obtained by a block bootstrap [see Lahiri (2003) for an overview of block bootstrap].*

As shown in Theorem 1, the stochastic drift term $F^+(x,y)\sqrt{T}\tilde{R}_T$ in the asymptotic expansion of $\sqrt{T}S_T(x,y)$ represents the undesirable "parameter estimation uncertainty" due to the estimation of p^+ and p^-. Because of this term, the limiting null distribution of $\sqrt{T}S_T(x,y)$ is different from that of $\sqrt{T}\tilde{S}_T(x,y)$. In the following corollary, the limiting distribution of $\sqrt{T}S_T(x,y)$ is not pivotal and it depends in a complex way on the underlying data generating process, especially the serial dependence structure in the data. Consequently, the critical values of CvM_T and KS_T cannot be tabulated and are case-dependent. To overcome these difficulties, in Section 4 we propose an easy-to-implement block multiplier bootstrap procedure to approximate the first order behavior of $\sqrt{T}S_T(x,y)$. The bootstrap procedure fully exploits the asymptotic decomposition obtained in Theorem 1. Note that both the serial dependence structure in the data and the "parameter estimation uncertainty" represented by $F^+(x,y)\sqrt{T}\tilde{R}_T$ cannot be ignored in any valid bootstrap procedure. Thus, the idea behind our procedure is to simulate via the Monte Carlo method the distribution of $\sqrt{T}S_T(x,y)$ by mimicking the representation in Theorem 1, and by doing so it enables us to approximate the associated critical values (or p-values) of the proposed test statistics as accurately as desired. It is important to emphasize again that in this paper we do not have to assume MDS or independence for the bivariate time series sequence $\{Z_t\}$, and hence accommodate for general serial dependence in the data. This aspect is particularly important for empirical applications based on economic or financial data.

The following corollary states the asymptotic null distribution of $\sqrt{T}S_T(x,y)$, which is a straightforward consequence of Lemma 1 and Theorem 1, hence its proof will be omitted.

Corollary 1 *Suppose Assumption* ***A.1*** *is satisfied. Then under the null hypothesis H_0 of symmetric co-movements in (1),*

$$\sqrt{T}S_T(x,y) \Rightarrow \hat{S}_\infty(x,y), \text{ as } T\to\infty,$$

where

$$\hat{S}_\infty(x,y) = S_\infty(x,y) - F^+(x,y)V,$$

with $S_\infty(x,y)$ the same Gaussian process as in Lemma 1 and V is a zero mean normal random variable with variance σ^2 given by (10)*. Moreover, the long-run covariance kernel of the Gaussian process $\hat{S}_\infty(x,y)$ is:*

$$\begin{aligned}\widehat{\mathbb{K}}\left((x,y),(x',y')\right) :=& \mathbb{E}\left[\hat{S}_\infty(x,y)\hat{S}_\infty(x',y')\right]\\ \equiv& \mathbb{K}\left((x,y),(x',y')\right) - F^+(x,y)\mathbb{E}\left[S_\infty(x',y')V\right]\\ &- F^+(x',y')\mathbb{E}\left[S_\infty(x,y)V\right] + F^+(x,y)F^+(x',y')\sigma^2,\end{aligned}$$

where the covariance between $S_\infty(x,y)$ and V is

$$\begin{aligned}\mathbb{E}\left[S_\infty(x,y)V\right] &= p^+p^-\left(F^+(x,y)p^- + F^-(x,y)p^+\right) + \sum_{i=1}^{\infty}\{\mathbb{E}\left[\varepsilon_1(x,y)\varepsilon_{1+i}(\infty,\infty)\right] + \mathbb{E}\left[\varepsilon_{1+i}(x,y)\varepsilon_1(\infty,\infty)\right]\}\\ &\overset{H_0}{=} p^+p^-\left(p^+ + p^-\right)F^+(x,y) + \sum_{i=1}^{\infty}\{\mathbb{E}\left[\varepsilon_1(x,y)\varepsilon_{1+i}(\infty,\infty)\right] + \mathbb{E}\left[\varepsilon_{1+i}(x,y)\varepsilon_1(\infty,\infty)\right]\}.\end{aligned}$$

The next Corollary 2 states the asymptotic null distributions of the test statistics based on a continuous and even functional of $\sqrt{T}S_T(x,y)$, in particular, those of CvM_T and KS_T. These results are an immediate consequence of Corollary 1, hence its proof will be omitted.

Corollary 2 *Suppose Assumptions **A.1** is satisfied. Then under the null hypothesis H_0 of symmetric comovements in (1), for any continuous and even functional $\varphi(\cdot)$, we have*

$$\varphi\left(\sqrt{T}S_T\left(x,y\right)\right) \xrightarrow{d} \varphi\left(\hat{S}_\infty\left(x,y\right)\right),$$

with $\hat{S}_\infty\left(x,y\right)$ the same Gaussian process as in Corollary 1. In particular,

$$CvM_T \xrightarrow{d} \int_c^\infty \int_c^\infty \hat{S}^2_\infty\left(x,y\right) dF\left(x,y\right),$$

$$KS_T \xrightarrow{d} \sup_{(x,y)\in(c,\infty)^2} \left|\hat{S}_\infty\left(x,y\right)\right|.$$

3.2 Consistency and local power

In this section, we study the consistency and local power properties of CvM_T and KS_T tests based on $\sqrt{T}S_T\left(x,y\right)$ against the fixed alternative H_1 and a sequence of local alternatives converging to H_0 at a parametric rate. The following theorem states the consistency of these two tests under H_1 [see the proof of Theorem 2 in the online **Appendix A**].

Theorem 2 *Suppose Assumption **A.1** is satisfied. Then under the alternative hypothesis H_1 of asymmetric comovements in (2),*

$$\frac{CvM_T}{T} \xrightarrow{p} \int_c^\infty \int_c^\infty \left(\left(F^+(x,y) - F^-(x,y)\right)p^+p^-\right)^2 dF(x,y) > 0,$$

$$\frac{KS_T}{\sqrt{T}} \xrightarrow{p} \sup_{(x,y)\in(c,\infty)^2} \left|\left(F^+(x,y) - F^-(x,y)\right)p^+p^-\right| > 0.$$

As CvM_T and KS_T always diverge to infinity under H_1, it is important to study their behavior under some local alternatives. We now examine the asymptotic behavior of the stochastic process $\sqrt{T}S_T(x,y)$, consequently of the tests CvM_T and KS_T, under a sequence of local alternatives that converge to the null H_0 at the parametric rate $\sqrt{T}$, which is the fastest rate possible known in the context of testing symmetric comovements. Specifically, we focus on the following local alternatives:

$$H_{1T} : F_T^+(x,y) = F_T^-(x,y) + \frac{\Delta(x,y)}{\sqrt{T}}, \text{for all} x > c \text{ and } y > c, \tag{11}$$

where the deterministic function $\Delta(x,y)$ is not a constant function of (x,y) and may take different forms with different c's. We assume that $\Delta(x,y) \neq 0$ for (x,y) in a set of positive measure on $(c,\infty)\times(c,\infty)$. In addition, it satisfies $\lim_{x\to c^+, y\to c^+} \Delta(x,y) = 0$ and $\lim_{x\to\infty, y\to\infty} \Delta(x,y) = 0$ such that $F_T^+(x,y)$ and $F_T^-(x,y)$ are properly defined for each $T \geq 1$. Note that the drift term $\frac{\Delta(x,y)}{\sqrt{T}}$ characterizes the degree of departure (i.e., the degree of asymmetric comovements) of $F_T^+(x,y) - F_T^-(x,y)$ from zero (i.e., when the null hypothesis of symmetric comovements holds true). Specifically, $\Delta(x,y)$ denotes the direction of departure from the null, while $\frac{1}{\sqrt{T}}$ is the speed at which the departure vanishes to zero as $T \to \infty$. Here, $\frac{1}{\sqrt{T}}$ is the fastest possible rate for a test to detect local alternatives like (11). Note also that the subscript T in both $F_T^+(x,y)$ and $F_T^-(x,y)$ signifies the dependence of exceedance distribution functions on sample size T in the setting of local alternatives.

The following Theorem provides the asymptotic distribution of $\sqrt{T}S_T(x,y)$ under the local alternatives H_{1T} in (11) [see the proof of Theorem 3 in the online **Appendix A**].

Theorem 3 *Suppose Assumption **A.1** is satisfied. Then under the sequence of local alternatives H_{1T} in (11), we have*

$$\sqrt{T}S_T(x,y) \Rightarrow \hat{S}_\infty^1(x,y), \text{ as } T \to \infty,$$

where

$$\hat{S}_\infty^1(x,y) = \hat{S}_\infty(x,y) + p^+p^-\Delta(x,y),$$

with $\hat{S}_\infty(x,y)$ the same Gaussian process as in Corollary 1 and $\Delta(x,y)$ the deterministic shift function defined in (11).

By Theorem 3, under the local alternatives H_{1T} in (11), the asymptotic null process $\hat{S}_\infty(x,y)$ is non-trivially shifted by $p^+p^-\Delta(x,y)$, which guarantees the non-trivial local power of our tests CvM_T and KS_T in detecting $\sqrt{T}-$local alternatives provided $\Delta(x,y) \neq 0$ for (x,y) in a set of positive measure on $(c,\infty)^2$. Note that the deterministic shift function $\Delta(x,y)$ in H_{1T} captures the directions of departure from the null of symmetric comovements. Obviously, when directions $\Delta(x,y) = 0$ almost everywhere (a.e.), the asymptotic result in Theorem 3 simply reduces to that of Corollary 1. Whenever $\Delta(x,y) \neq 0$ for at lease some (x,y), our tests CvM_T and KS_T will converge to different limits rather than those given in Corollary 2, delivering non-trivial local power.

4 Block multiplier bootstrap

The result in Corollary 2 shows that the asymptotic null distributions of the test statistics $\varphi\left(\sqrt{T}S_T\right)$ based on any continuous and even functional $\varphi(\cdot)$, such as CvM_T and KS_T, are not asymptotically pivotal, depending on the underlying data generating processes in a complicated manner. Consequently, critical values from the asymptotic distributions are not readily available. In particular, the long-run covariance kernel $\widehat{\mathbb{K}}(\cdot,\cdot)$ of $\hat{S}_\infty(x,y)$ in Corollary 1 depends on the serial dependence structure in the data as well as unknown quantities attributed to the parameter estimation effect, which complicates greatly the implementation of the test statistics using asymptotic null distributions.

To overcome the above issues, we propose a block multiplier-type bootstrap procedure to simulate the critical values (or p-values) of the tests. In order for the proposed procedure to work, it has to preserve the serial dependence structure in the data and at the same time address the “parameter estimation uncertainty” stated in Theorem 1. In doing so, the proposed bootstrap procedure has several appealing theoretical and empirical properties such as it is easy to implement and it does not require the computation of new estimates at each bootstrap replication. In addition, apart from the number of bootstrap replications and the block length, our procedure does not involve user-chosen tuning parameters such as kernel function and bandwidth parameter required for testing methods based on standard kernel density estimators.

Formally speaking, in the light of the asymptotic representation of $S_T(x,y)$ in terms of $\tilde{S}_T(x,y)$ and a drift term, which we stated in Theorem 1, we propose to approximate the asymptotic behavior of $S_T(x,y)$ using its block multiplier bootstrap version. Let $L \equiv L(T)$ denote the block length, which increases slowly as sample size T increases. We introduce

$$S_T^*(x,y) \equiv \frac{1}{T}\sum_{t=1}^{T-L+1} \xi_t \sum_{s=t}^{t+L-1}\left[\hat{\varepsilon}_s(x,y) - \hat{F}^+(x,y)\hat{\varepsilon}_s(\infty,\infty)\right], \tag{12}$$

where

$$\hat{\varepsilon}_t(x,y) = 1(c < X_t \le x)\,1(c < Y_t \le y)\,\hat{p}^- - 1(c < -X_t \le x)\,1(c < -Y_t \le y)\,\hat{p}^+,$$

$$\hat{\varepsilon}_t(\infty,\infty) = 1(X_t > c)\,1(Y_t > c)\,\hat{p}^- - 1(X_t < -c)\,1(Y_t < -c)\,\hat{p}^+,$$

and

$$\hat{F}^+(x,y) = \frac{1}{T}\sum_{t=1}^{T} 1(c < X_t \le x)1(c < Y_t \le y)/\hat{p}^+$$

is the consistent estimator of $F^+(x,y)$. Here, $\{\xi_t\}_{t=1}^{T-L+1}$ is a sequence of independent and identically distributed (i.i.d.) random variables with mean zero and variance $1/L$. In addition, the sequence $\{\xi_t\}_{t=1}^{T-L+1}$ is independent of the original sample $\left\{(X_t,Y_t)'\right\}_{t=1}^T$.

In the following we state formally the assumptions on block length L and multiplier ξ_t.

Assumption A.2: (i) $\{\xi_t\}_{t=1}^{T-L+1}$ are i.i.d. and independent of the process $\{(X_t,Y_t)'\}$; (ii) $\mathbb{E}(\xi_t) = 0$, $\mathbb{E}(\xi_t^2) = 1/L$, and $\mathbb{E}(\xi_t^4) = O(1/L^2)$; (iii) As $T \to \infty$, $L \to \infty$ and $L/T^{1/2} \to 0$.

Following Inoue (2001) and Su and White (2012), $\{\xi_t\}_{t=1}^{T-L+1}$ can be chosen to be a sequence of i.i.d. random variables from $N(0,1/L)$.

Now, once $\sqrt{T}S_T^*(x,y)$ is calculated, the block multiplier bootstrapped version of the test statistics $\varphi(\sqrt{T}S_T(x,y))$ (e.g. CvM_T and KS_T) is simply given by $\varphi(\sqrt{T}S_T^*(x,y))$ (e.g. CvM_T^* and KS_T^*), and we then repeat the procedure many times to simulate the critical values or p-values. For example, for the test statistic CvM_T, the associated block multiplier bootstrap procedure can be easily implemented through the following four steps:

Step 1: Calculate CvM_T based on $\sqrt{T}S_T(x,y)$ given in (4) using the original sample $\left\{(X_t,Y_t)'\right\}_{i=1}^T$;

Step 2: For a given block length L, generate ξ_t, for $t = 1,\ldots,T-L+1$, and calculate the bootstrapped test statistic CvM_T^* based on $\sqrt{T}S_T^*(x,y)$ given in (12);

Step 3: Repeat Step **2** B times so that we get a sequence of bootstrapped test statistics $\left\{CvM_{T,j}^*\right\}_{j=1}^B$, where $CvM_{T,j}^*$ denotes the value of the bootstrapped test statistic computed from the $j-$th bootstrap;

Step 4: For any given significance level α, compute the $(1-\alpha)-$th sample quantile of $\left\{CvM_{T,j}^*\right\}_{j=1}^B$, say $CvM_T^{*\alpha}$. We reject the null hypothesis at significance level α if $CvM_T > CvM_T^{*\alpha}$.

Note that instead of using the bootstrapped critical value $CvM_T^{*\alpha}$ in **Step 4**, we can compute the following bootstrapped p-value, p^*, say,

$$p^* = \frac{1}{B}\sum_{j=1}^{B} 1\left(CvM_{T,j}^* \geq CvM_T\right).$$

For the given significance level α, we then reject the null hypothesis if $p^* < \alpha$. The implementation based on a block multiplier bootstrap for the test statistic KS_T can be done following similar steps.

The next theorem establishes the asymptotic validity of the above proposed bootstrap procedure for implementing CvM_T or KS_T [see the proof of Theorem 4 in the online **Appendix A**].

Theorem 4 *Suppose Assumptions **A.1-A.2** are satisfied. Then under the null hypothesis H_0 of symmetric comovements in (1), or under the sequence of local alternatives H_{1T} in* (11),

$$\sqrt{T}S_T^*\left(x,y\right) \underset{*}{\overset{p}{\Rightarrow}} \hat{S}_\infty\left(x,y\right),$$

where $\hat{S}_\infty\left(x,y\right)$ is the same Gaussian process as in Corollary 1, and "$\underset{*}{\overset{p}{\Rightarrow}}$" denotes the weak convergence in probability under the bootstrap law, i.e., conditional on the original sample $\left\{(X_t,Y_t)'\right\}_{t=1}^T$. Additionally, for any continuous and even functional $\varphi(\cdot)$, we have $\varphi\left(\sqrt{T}S_T^*\left(x,y\right)\right) \underset{*}{\overset{d}{\to}} \varphi\left(\hat{S}_\infty\left(x,y\right)\right)$ in probability under the bootstrap law.

Under the bootstrap law, Theorem 4 states that the block multiplier bootstrapped process $\sqrt{T}S_T^*\left(x,y\right)$ converges weakly to $\hat{S}_\infty\left(x,y\right)$, which is the same asymptotic null distribution of $\sqrt{T}S_T\left(x,y\right)$ as in Corollary 1, hence it justifies the asymptotic validity of our bootstrap procedure. Thereafter, since under the alternative hypothesis the test statistics based on $\sqrt{T}S_T\left(x,y\right)$ like CvM_T and KS_T diverge to positive infinity while the associated bootstrapped test statistics based on $\sqrt{T}S_T^*\left(x,y\right)$ still converge in distribution (hence are bounded), the test statistics will exceed the critical values and reject the alternative. That is, our tests coupled with the block multiplier bootstrap are consistent. Similarly, our tests have non-trivial power against the specified sequence of local alternatives in (11).

Remark 2: *If $\left\{\varepsilon_t\left(x,y\right), \mathcal{F}_t\right\}$ is an MDS [e.g., when $\left\{Z_t = (X_t,Y_t)'\right\}$ is simply an independent sequence], the long-run covariance kernel of $\hat{S}_\infty$ is simplified greatly and we no longer need to mimic the serial dependence* *structure in the data. As a result, we can just set block length $L = 1$ and the proposed block multiplier bootstrap will then reduce to a "naive" multiplier bootstrap:*

$$\check{S}_T^* (x, y) = \frac{1}{T} \sum_{t=1}^{T} \xi_t \left[\hat{\varepsilon}_t (x, y) - \hat{F}^+(x, y) \hat{\varepsilon}_t (\infty, \infty) \right].$$

In the context of i.i.d. observations, variants in the spirits of $\check{S}_T^*(x, y)$ have been studied in many different testing problems.

*We also note that in the special case of MDS, instead of using $N(0, 1)$ for $\{\xi_t\}$ as is commonly used in the block multiplier bootstrap, a two-point distribution for $\{\xi_t\}$ is often preferred. Following the suggestion of Mammen (1993), the latter distribution is chosen to be an i.i.d Bernoulli random variable with probability masses $Pr\big(\xi_t = (1 + \sqrt{5})/2\big) = (\sqrt{5} - 1)/2\sqrt{5}$ and $Pr\big(\xi_t = (1 - \sqrt{5})/2\big) = (\sqrt{5} + 1)/2\sqrt{5}$. Another popular choice of $\{\xi_t\}$ is such that $Pr(\xi_t = 1) = Pr(\xi_t = -1) = 1/2$; see Liu (1988) and de Jong (1996) among others. Tables 17 and 18 in the online **Appendix D** demonstrate the finite sample properties of CvM_T and KS_T using the two-point distribution $Pr(\xi_t = 1) = Pr(\xi_t = -1) = 1/2$ when there is no dependence in the data. For independent data, our unreported simulation results indicate that the finite sample performance (both empirical size and empirical power) of our tests does not seem to change when the other choice of $\{\xi_t\}$ is used.*

One can expect that discarding completely the serial dependence structure in the data, test statistics implemented through the above "naive" multiplier bootstrap $\sqrt{T}\check{S}_T^* (x, y)$ (although it has taken into account the parameter estimation uncertainty) will not work. As a matter of fact, conditional on the original sample $\left\{(X_t, Y_t)'\right\}_{t=1}^T$, it can be shown that

$$\begin{aligned}
&\mathbb{E}\left[T\check{S}_T^*(x, y)\check{S}_T^*(x', y') \middle| \left\{(X_t, Y_t)'\right\}_{t=1}^T \right] \\
=&\frac{1}{T} \sum_{t=1}^{T} \left[\hat{\varepsilon}_t (x, y) - \hat{F}^+(x, y)\hat{\varepsilon}_t (\infty, \infty) \right] \left[\hat{\varepsilon}_t \left(x', y'\right) - \hat{F}^+(x', y')\hat{\varepsilon}_t (\infty, \infty) \right] \\
=&\frac{1}{T} \sum_{t=1}^{T} \left[\varepsilon_t (x, y) - F^+(x, y)\varepsilon_t (\infty, \infty) \right] \left[\varepsilon_t \left(x', y'\right) - F^+(x', y')\varepsilon_t (\infty, \infty) \right] + o_p(1) \\
=&\mathbb{E}\left\{ \left[\varepsilon_1 (x, y) - F^+(x, y)\varepsilon_1 (\infty, \infty) \right] \left[\varepsilon_1 \left(x', y'\right) - F^+(x', y')\varepsilon_1 (\infty, \infty) \right] \right\} + o_p(1) \\
\overset{H_0}{\to}& p^+p^- \left(p^+ + p^-\right) \left(F^+ \left(x \wedge x', y \wedge y'\right) - F^+(x, y)F^+(x', y')\right) \text{ in probability.}
\end{aligned}$$

Consequently, $\sqrt{T}\check{S}_T^* (x, y)$ does not converge to $\hat{S}_\infty (x, y)$ under the null hypothesis H_0, as the covariance kernel of the asymptotic null process of $\sqrt{T}\check{S}_T^* (x, y)$ is different from the correct covariance kernel $\widehat{\mathbb{K}}(\cdot, \cdot)$ of $\hat{S}_\infty(x, y)$ as given in Corollary 1. In fact, $p^+p^- (p^+ + p^-) (F^+ (x \wedge x', y \wedge y') - F^+(x, y)F^+(x', y'))$ will be the covariance kernel of $\hat{S}_\infty(x, y)$ only if the MDS assumption holds. Thus, test statistics implemented through $\sqrt{T}\check{S}_T^* (x, y)$ can only be used when MDS holds.

5 Monte Carlo simulations

We conduct a set of Monte Carlo simulations to investigate the finite sample performance of the tests proposed in the previous sections. Our primary interest is to assess their empirical size and power using a variety of data generating processes (DGPs) under different sample sizes and different nominal levels. In our simulations, we use the closed-form expressions in (8) to compute the test statistics CvM_T and KS_T. The associated block multiplier bootstrapped versions of CvM_T and KS_T, denoted as CvM_T^* and KS_T^*, can also be computed using similar expressions.

As discussed in Section 2, $p^+ = p^-$ seems to be unrealistic in practice, thus it is not appropriate to impose this restrictive assumption in any valid testing procedure. To study the adverse effects of this restriction, in the following we use Monte Carlo simulations to examine the size and power properties of the test statistics that impose $p^+ = p^-$; hereafter denoted as $\widetilde{CvM}_T$ and $\widetilde{KS}_T$. The latter are computed using (8) but without including $\hat{p}^-$ and $\hat{p}^+$. We also implement $\widetilde{CvM}_T$ and $\widetilde{KS}_T$ using the block multiplier bootstrap described in Section 4. The main difference with the implementation of CvM_T and KS_T is that there is no need to address the parameter estimation uncertainty. On the one hand, we expect that $\widetilde{CvM}_T$ and $\widetilde{KS}_T$ will deliver reasonable size results under the null hypothesis only when the restriction is valid. On the other hand, they might demonstrate lower power results under the alternative hypothesis when $p^+ \neq p^-$.

For simplicity of exposition, we shall focus on the case of exceedance level $c = 0$. However, the theoretical results developed in Sections 3 and 4 hold for any specified exceedance level. Simulation results for values $c = 0.5$, 1 and 1.5 are available upon request.

Now, for generating $\{X_t\}_{t=1}^T$ and $\{Y_t\}_{t=1}^T$ with different sample size T, we consider the following two GARCH(1,1) processes:

$$X_t = \alpha_0 + \sigma_{1t}\varepsilon_{1t}, \text{ with } \sigma_{1t}^2 = \alpha_1 + \alpha_2\sigma_{1,t-1}^2 + \alpha_3\varepsilon_{1,t-1}^2,$$

$$Y_t = \beta_0 + \sigma_{2t}\varepsilon_{2t}, \text{ with } \sigma_{2t}^2 = \beta_1 + \beta_2\sigma_{2,t-1}^2 + \beta_3\varepsilon_{2,t-1}^2,$$

where $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)' = (0.795, 2.400, 0.827, 0.090)'$, and $\beta = (\beta_0, \beta_1, \beta_2, \beta_3)' = (0.562, 1.139, 0.844, 0.107)'$. The values of α and β were obtained using real data on S&P 500 daily returns and individual stock returns. The innovations ε_{1t} and ε_{2t} are assumed to be i.i.d. with zero mean and unit variance. We simulate the dependence structures in $(X_t, Y_t)'$ through those of $(\varepsilon_{1t}, \varepsilon_{2t})'$. To achieve this, we introduce the following bivariate Gaussian and Clayton copulas:

$$C_{norm}(u, v; \rho) = \Phi_\rho\left(\Phi^{-1}(u), \Phi^{-1}(v)\right) \quad \text{and} \quad C_{clay}(u, v; \tau) = \left(u^{-\tau} + v^{-\tau} - 1\right)^{-\frac{1}{\tau}},$$

where $\rho \in (-1, 1)$ is the correlation coefficient between the marginal normal distributions, τ is a positive parameter that governs the dependence in the Clayton copula [higher τ indicates stronger asymmetric comovements], Φ^{-1} is the inverse of the standard normal cumulative distribution function, and Φ_ρ is the standard bivariate normal distribution function with correlation coefficient ρ. On the one hand, the Gaussian copula generates symmetric comovements, thus it will serve to illustrate the size of our tests. On the other hand, the Clayton copula does not impose symmetric comovements on the variables, thus it will serve to illustrate the power of our tests. In particular, to investigate the empirical size and power performance of the tests CvM_T and KS_T as well as those of the tests $\widetilde{CvM}_T$ and $\widetilde{KS}_T$, we simulate bivariate data $(X_t, Y_t)'$ via $(\varepsilon_{1t}, \varepsilon_{2t})'$ using the following mixed Gaussian-Clayton copula:

$$(\varepsilon_{1t}, \varepsilon_{2t})' \sim C_{mix}(u, v; \rho, \tau, \kappa) := \kappa C_{norm}(u, v; \rho) + (1-\kappa) C_{clay}(u, v; \tau), \tag{13}$$

where κ and $1-\kappa$ are the weights on the Gaussian and Clayton copulas, respectively. The mixed copula in (13) nests both the Gaussian and Clayton copulas as special cases. For $\kappa = 1$, the mixed copula reduces to a Gaussian copula, and for $\kappa = 0$ it reduces to a Clayton copula. Furthermore, the degree of asymmetric comovements is a decreasing function of κ; namely, the smaller κ is the stronger the asymmetric comovements. DGPs with different values of κ, ρ, and τ are reported in Table 1 at the end of this paper.

The values of ρ and τ in Table 1 were obtained using real data on financial returns. In this table, the first DGP (DGP S1) is used to investigate the size property of the tests, since in this DGP the null hypothesis of symmetric comovements (Gaussian copula) is satisfied. However, in DGP P1 to DGP P5 the null hypothesis is not satisfied, and therefore they serve to illustrate the power of the proposed tests. Furthermore, five sample sizes $T = 240$, 600, 1600, 3200, and 4800 are considered. We use 1000 simulations to compute the empirical size and power. For each simulation we use $B = 200$ bootstrap replications. We consider three nominal sizes 1%, 5%, and 10% to assess the approximation of the proposed block multiplier bootstrap for different nominal levels. Thereafter, as the bootstrap procedure requires to choose a block length L, we use $L = l \times [n^{1/4}]$, with $l = 1$, 2, and 4 to assess the sensitivity of our tests to L, where $[a]$ denotes the smallest integer greater or equal to a. Since the simulation results are fairly insensitive to the choice of l, here we only report results for $l = 1$ to save space. Results for $l = 2$ and 4 are reported in Tables 13–16 in the online **Appendix D**. Lastly, we have also reported simulation results for $\{X_t\}$ and $\{Y_t\}$ following AR(1) processes: $X_t = 0.5 + 0.5X_{t-1} + \varepsilon_{1t}$ and $Y_t = 0.5 + 0.5Y_{t-1} + \varepsilon_{2t}$, where the innovations $(\varepsilon_{1t}, \varepsilon_{2t})' \sim C_{mix}(u, v; \rho, \tau, \kappa)$ in (13). To understand the effect of degrees of dependence in the data on the proposed tests, two others cases for the autoregressive parameter being 0.8 and 0.9 are also investigated. Since the highly dependent structure may violate our required assumption and the associated test results are relatively less informative, they are only reported in Tables 19–24 in the online **Appendix D**. Nevertheless, it is worthwhile to remark that a larger autoregressive parameter can reduce significantly the testing power of the proposed tests due to stronger dependence in the data. This type of phenomena is not completely unexpected as stronger dependence intuitively leads to a smaller effective sample size and thus contains much less information about the underlying data generating mechanism.

Tables 2 to 5 at the end of this paper report the empirical sizes and powers of block multiplier bootstrap-based test statistics CvM_T, KS_T, $\widetilde{CvM}_T$ and $\widetilde{KS}_T$ using the DGPs in Table 1 under the GARCH(1,1) process. Firstly, the empirical sizes seem to suggest that all these tests are somewhat conservative for the three nominal levels under consideration; see column "DGP S1" in Tables 2–5. Simulation results for the i.i.d. DGPs reported in Tables 17 and 18 in the online **Appendix D** indicate that the proposed CvM_T and KS_T tests implemented through the classical multiplier bootstrap procedure discussed in **Remark 2** (i.e., the block length $L = 1$) can preserve the sizes very accurately, which suggests that serial dependence in the data affects adversely the sizes of the tests, making them conservative. It is also interesting to develop an alternative bootstrap procedure to deliver better sizes, which we leave for future's study. Secondly, we see that the tests do reasonably well in terms of power, especially for the DGPs with greater deviations from the null hypothesis and for larger sample sizes. Thereafter, comparing the results from the four tables, we see that Kolmogorov-Smirnov-type tests are slightly more powerful than Cramér-von Mises-type tests. Recall that the smaller κ is the higher the degree of asymmetry. As a result, a pattern of increasing powers from DGP P1 to DGP P5 for all four tests is observed. Not surprisingly, as DGP P1 is extremely close to the null hypothesis, it is the hardest one to detect. Thirdly, the results show that the test statistics CvM_T and KS_T, which have accounted for the parameter estimation effect, are much more powerful against DGPs P1-P5 than $\widetilde{CvM}_T$ and $\widetilde{KS}_T$. The latter two tests ignore p^+ and p^-, the estimation of which helps to enhance power substantially. Tables 6 and 7 report the test results under the AR(1) process with autoregressive parameter being 0.5. Similar patterns as to the GARCH(1,1) case are observed; for example, KS_T is slightly more powerful than CvM_T for all three significance levels.

Though the test statistics $\widetilde{CvM}_T$ and $\widetilde{KS}_T$ can control sizes in our considered setting (because under DGP S1 $p^+ = p^-$ holds, thus they can deliver acceptable size results, though also undersized), they are not recommended as they are much less powerful for small and medium sample sizes in comparison with the proposed tests CvM_T and KS_T. Even when the sample size reaches 4800 - for nominal size 5% - $\widetilde{CvM}_T$ and $\widetilde{KS}_T$ only attain powers of around 0.21 and 0.25 for DGP P2, respectively, while the powers are around 0.69 and 0.81 for CvM_T and KS_T, respectively. Furthermore, as discussed before, it is possible to design DGPs such that $p^+ \neq p^-$ under the null hypothesis, see e.g. columns $c = 0$, $c = 1$ and $c = 1.5$ of Table 2 of the online **Appendix C**. In this case, the tests $\widetilde{CvM}_T$ and $\widetilde{KS}_T$ might not be able to control sizes and have a tendency towards overrejection, while our proposed tests CvM_T and KS_T could still deliver acceptable size results and lead to reliable conclusions.

Finally, for a comparison, in Tables 25 and 26 of the online Appendix D, we report the empirical size and power of two popular alternative tests, i.e., the J_ρ test of Hong et al. (2007) and the C_ρ test of Chen (2016). Nevertheless, given the nonparametric nature of our tests, we think that this comparison would not provide more information than what is already well known in the literature. On the one hand, it is widely accepted that parametric tests will be most powerful when the models are correctly specified, but they are threatened by misspecifications issues. On the other hand, nonparametric tests are not affected by the misspecifications of the assumed models, but they would be less powerful than parametric tests if the latter models are correctly specified.

6 An empirical application

In this section we provide an empirical application where our statistical procedures are used to test the existence of symmetric/asymmetric comovements between stock market returns and individual stock returns. The dataset comes from Yahoo Finance and consists of daily S&P 500 index and 29 individual stocks, with 2517 observations over the period that runs from the 1st of January 2007 to the 31st of December 2016. After excluding Alocoa Inc. (because it has too many missing data in our sampling period), the stocks we consider are the thirty constituents of the Dow Jones Industrial Average index. The 29 individual stocks are: American Express Company (AXP), Boeing Company (BA), Bank of America Corporation (BAC), Caterpillar Inc. (CAT), Cisco Systems, Inc. (CSCO), Chevron Corporation (CVX), E. I. du Pont de Nemours and Company (DD), Walt Disney Company (DIS), General Electric Company (GE), Home Depot, Inc. (HD), Hewlett-Packard Company (HPQ), International Business Machines Corporation (IBM), Intel Corporation (INTC), Johnson & Johnson (JNJ), JPMorgan Chase & Co. (JPM), Coca-Cola Company (KO), McDonald's Corp. (MCD), 3M Company (MMM), Merck & Co. Inc. (MRK), Microsoft Corporation (MSFT), Pfizer Inc. (PFE), Procter & Gamble Co. (PG), AT&T, Inc. (T), Travelers Companies, Inc. (TRV), United-Health Group Incorporated (UNH), United Technologies Corp. (UTX), Verizon Communications Inc. (VZ), Wal-Mart Stores Inc. (WMT), and Exxon Mobil Corporation (XOM). These stocks are the same as those used in Chen (2016), which will help us compare our results with those obtained using the exceedance correlation-based tests of Chen (2016) and Hong, Tu, and Zhou (2007). For each of the above stocks we compute the continuously compounded daily returns by taking the difference between the logarithm of the price at time t and the logarithm of the price at time t-1. A summary of the descriptive statistics of the returns of these stocks can be found in Table 1 of the online **Appendix C**.

Before we outline our main empirical results for testing symmetry between the above stock market return and individual stock returns, we first use the latter data to estimate the probabilities $p^{+}(c)$ and $p^{-}(c)$ and check the appropriateness of the assumption $p^{+}(c) = p^{-}(c)$ for various c's. We calculate the joint probabilities $p^{+}(c)$ and $p^{-}(c)$, when X_t represents the daily S&P 500 return and Y_t is the daily return on each of the above described 29 individual stocks. Asymptotic p-values for testing $H_0^p : p^{+}(c) = p^{-}(c)$ against $H_1^p : p^{+}(c) \neq p^{-}(c)$ are calculated using a simple t-type test statistic; see the detailed description of the testing procedure in the online **Appendix B**. Table 1 of the online **Appendix C** reports the results of estimating $p^+(c)$ and $p^-(c)$ for different values of c and of testing the null H_0^p against the alternative H_1^p. From this, we see that $\hat{p}^+(c)$ is always higher than $\hat{p}^-(c)$ for $c = 0$, $\hat{p}^+(c)$ and $\hat{p}^-(c)$ are statistically equal for $c = 0.5$, and $\hat{p}^+(c)$ is always lower than $\hat{p}^-(c)$ for $c = 1$ and $c = 1.5$. This indicates that it is inappropriate to assume that $p^+(c) = p^-(c)$ when we build a test for the symmetric comovements hypothesis in (1). For $p^-(c) > p^+(c)$, the stocks still tend to move more often with the market when the latter goes down than when it goes up even when $F(x, y) = \overline{F}(-x, -y)$ (or equivalently $f(x, y) = f(-x, -y)$). Thus, the tests that are constructed based on a distance between $F(x, y)$ and $\overline{F}(-x, -y)$ will not reject the symmetric comovements hypothesis, leading to a potential loss of power. Similarly, if not adjusted for $p^+(c)$ and $p^-(c)$, the tests that are based on a distance between $F(x, y)$ and $\overline{F}(-x, -y)$ might suffer from power loss when $p^-(c) < p^+(c)$. Hence, a valid test should be based on the comparison between the conditional versions $F^+(x, y; c)$ and $F^-(x, y; c)$, which provides a good motivation for the use of the proposed tests in practical situations.

We now apply our proposed test statistics CvM_T and KS_T to test for the symmetric comovements hypothesis between S&P 500 daily return and daily return on each of the 29 individual stocks using $B = 1000$ bootstrap replications. We compare our results with those obtained using the exceedance correlation-based (thus parametric) tests of Hong, Tu, and Zhou (2007), say J_ρ, and Chen (2016), say C_ρ. The results using the test statistics $\widetilde{CvM}_T$ and $\widetilde{KS}_T$ are also provided to demonstrate that erroneous conclusions could be reached if p^+ and p^- are ignored.

Empirical results are reported in Tables 3 to 7 of the online **Appendix C** , with the first four tables reporting testing results for a single exceedance level $c = 0$, 0.5, 1, and 1.5, respectively, whereas Table 7 reports results for multiple exceedance levels $c = \{0, 0.5, 1, 1.5\}$. From these tables, particularly for $c = 0$ and $c = 0.5$, our CvM_T and KS_T tests indicate that the symmetric comovements hypothesis should be rejected, at least at 10% significance level, for the majority of stocks. However, for all the exceedance levels considered, the exceedance correlation-based tests J_ρ and C_ρ suggest that the null hypothesis of symmetric comovements cannot be rejected for the majority of stocks under consideration. Thus, following the correlation-based tests the comovements between market and individual returns are not asymmetric, which might have a negative impact on the calculation of optimal portfolio choice and risk management. As we discussed in the Introduction, Dahlquist, Farago, and Tédongap (2017) show that the asymmetric comovements between asset returns yield better optimal portfolio choice compared to the standard model in which the comovements are assumed to be symmetric; see also Ang and Chen (2002) and Patton (2004). Furthermore, Tsafack (2009) shows that ignoring asymmetric comovements leads to a potential underestimation of portfolio Value-at-Risk (VaR) and Expected Shortfall (ES). Lastly, as emphasized before, results from $\widetilde{CvM}_T$ and $\widetilde{KS}_T$ should be interpreted with caution. Specifically, coupled with results from Table 2 of the online **Appendix C**, the seemingly more rejections of $\widetilde{CvM}_T$ and $\widetilde{KS}_T$ than CvM_T and KS_T when $c = 0$ could be due to the overrejection of the former two tests under $p^+ \neq p^-$. On the other hand, the fewer rejections of $\widetilde{CvM}_T$ and $\widetilde{KS}_T$ for the case of $c = 0.5$ might be due to that $p^+ = p^-$ holds, and thus $\widetilde{CvM}_T$ and $\widetilde{KS}_T$ are much less powerful than CvM_T and KS_T. Rejections for the more extreme cases $c = 1$ and $c = 1.5$ are fewer and thus the phenomenon of asymmetric comovements is less pronounced. One potential reason could be because of smaller effective sample sizes at the extremes.

Lastly, inspired by Deng (2016), we also consider a second empirical study for testing the asymmetric comovments among five major stock market indices, which are reported in Tables 8–12 in the online **Appendix C**. Our results suggest that there is a great tendency of asymmetric comovements among the five stock markets. Again, as to the analysis of asymmetric comovements between stock market returns and individual stock returns, our pairwise comparisons may offer some new perspectives on the risk management practices via international market diversification.

7 Conclusions

We proposed nonparametric tests for asymmetric comovements between two random variables, which work for time series data. We considered the popular Cramér-von Mises and Kolmogorov-Smirnov-type test statistics based on the distance between positive and negative joint conditional exceedance distribution functions. The tests are able to capture both linear and nonlinear dependence in the data and do not require nonparametric smoothing and hence do not involve bandwidth parameter. We derived the asymptotic distributions of these tests and established the validity of a block multiplier-type bootstrap which one can use in finite-sample settings. We also showed that the tests are consistent for any fixed alternative and they have non-trivial local power in detecting local alternatives converging to the null at the parametric rate. A Monte Carlo simulation study revealed that the bootstrap-based tests have reasonable finite-sample size and power for a variety of data generating processes and different sample sizes. Finally, we provided an empirical application where the proposed nonparametric tests are used to test the symmetric comovements between the S&P 500 daily return and the daily returns on 29 individual stocks. The results indicated that the symmetric comovements hypothesis is rejected for the majority of stocks under consideration.

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Table 1: Data generating processes

DGPs	Parameter values		
	κ	ρ	τ
DGP S1	1.000	0.951	5.768
DGP P1	0.750	0.951	5.768
DGP P2	0.500	0.951	5.768
DGP P3	0.375	0.951	5.768
DGP P4	0.250	0.951	5.768
DGP P5	0.000	0.951	5.768

Note: This table provides the values of the parameters κ, ρ, and τ of mixed Gaussian-Clayton copula in (13) that we consider in the simulation study to investigate the finite sample properties (e.g. empirical size and power) of nonparametric tests CvM_T and KS_T based on the block multiplier bootstrap procedure as well as those of $\widetilde{CvM}_T$ and $\widetilde{KS}_T$ (imposing $p^+ = p^-$) for testing the null of symmetric comovements in (1) against the alternative in (2).

Table 2: Empirical rejection rates of the block multiplier bootstrap-based Cramér-von Mises test statistic $[CvM_T]$ for $c = 0$ under the GARCH(1,1) process when $L = [T^{1/4}]$

Nominal Sizes	DGPs					
	DGP S1	DGP P1	DGP P2	DGP P3	DGP P4	DGP P5
			$T = 240$			
$\alpha = 1\%$	0.001	0.003	0.004	0.003	0.005	0.013
$\alpha = 5\%$	0.021	0.032	0.035	0.034	0.053	0.064
$\alpha = 10\%$	0.056	0.069	0.088	0.089	0.111	0.153
			$T = 600$			
$\alpha = 1\%$	0.007	0.004	0.014	0.027	0.022	0.074
$\alpha = 5\%$	0.030	0.032	0.063	0.123	0.129	0.239
$\alpha = 10\%$	0.061	0.073	0.131	0.208	0.248	0.382
			$T = 1600$			
$\alpha = 1\%$	0.004	0.015	0.059	0.103	0.188	0.362
$\alpha = 5\%$	0.025	0.060	0.216	0.310	0.479	0.727
$\alpha = 10\%$	0.063	0.132	0.359	0.452	0.648	0.875
			$T = 3200$			
$\alpha = 1\%$	0.002	0.025	0.167	0.366	0.529	0.847
$\alpha = 5\%$	0.025	0.124	0.443	0.708	0.866	0.991
$\alpha = 10\%$	0.055	0.206	0.620	0.847	0.952	1.000
			$T = 4800$			
$\alpha = 1\%$	0.005	0.044	0.333	0.612	0.830	0.992
$\alpha = 5\%$	0.021	0.177	0.689	0.891	0.983	1.000
$\alpha = 10\%$	0.050	0.309	0.839	0.965	0.995	1.000

Note: This table reports the empirical size and power of block multiplier bootstrap-based test statistic CvM_T for testing the null of symmetric comovements in (1) against the alternative in (2) under the DGPs in Table 1 and for $\alpha = 1\%$, 5%, and 10% significance levels. The number of simulations is 1000, the number of bootstrap resamples is $B = 200$, and the block length is $L = [T^{1/4}]$. Here we set the exceedance level to be $c = 0$.

Table 3: Empirical rejection rates of the block multiplier bootstrap-based Kolmogorov-Smirnov test statistic $[KS_T]$ for $c = 0$ under the GARCH(1,1) process when $L = [T^{1/4}]$

Nominal Sizes	DGPs					
	DGP S1	DGP P1	DGP P2	DGP P3	DGP P4	DGP P5
			$T = 240$			
$\alpha = 1\%$	0.002	0.003	0.012	0.006	0.015	0.020
$\alpha = 5\%$	0.016	0.026	0.047	0.039	0.063	0.104
$\alpha = 10\%$	0.055	0.062	0.087	0.102	0.131	0.202
			$T = 600$			
$\alpha = 1\%$	0.005	0.008	0.019	0.036	0.046	0.147
$\alpha = 5\%$	0.031	0.038	0.073	0.171	0.209	0.386
$\alpha = 10\%$	0.061	0.076	0.151	0.267	0.342	0.541
			$T = 1600$			
$\alpha = 1\%$	0.003	0.018	0.093	0.202	0.339	0.676
$\alpha = 5\%$	0.026	0.067	0.281	0.439	0.648	0.902
$\alpha = 10\%$	0.059	0.138	0.421	0.584	0.772	0.961
			$T = 3200$			
$\alpha = 1\%$	0.001	0.042	0.291	0.575	0.795	0.979
$\alpha = 5\%$	0.025	0.140	0.582	0.840	0.960	0.998
$\alpha = 10\%$	0.064	0.240	0.733	0.933	0.983	1.000
			$T = 4800$			
$\alpha = 1\%$	0.004	0.069	0.550	0.813	0.962	0.999
$\alpha = 5\%$	0.023	0.226	0.813	0.955	0.997	1.000
$\alpha = 10\%$	0.055	0.372	0.905	0.993	0.999	1.000

Note: This table reports the empirical size and power of block multiplier bootstrap-based test statistic KS_T for testing the null of symmetric comovements in (1) against the alternative in (2) under the DGPs in Table 1 and for $\alpha = 1\%$, 5%, and 10% significance levels. The number of simulations is 1000, the number of bootstrap resamples is $B = 200$, and the block length is $L = [T^{1/4}]$. Here we set the exceedance level to be $c = 0$.

Table 4: Empirical rejection rates of the block multiplier bootstrap-based Cramér-von Mises test statistic $[\widetilde{CvM}_T]$ for $c = 0$ under the GARCH(1,1) process when $L = [T^{1/4}]$

Nominal Sizes	DGPs					
	DGP S1	DGP P1	DGP P2	DGP P3	DGP P4	DGP P5
			$T = 240$			
$\alpha = 1\%$	0.001	0.004	0.002	0.005	0.001	0.006
$\alpha = 5\%$	0.024	0.026	0.031	0.028	0.041	0.044
$\alpha = 10\%$	0.059	0.060	0.073	0.075	0.081	0.099
			$T = 600$			
$\alpha = 1\%$	0.004	0.005	0.005	0.007	0.007	0.011
$\alpha = 5\%$	0.029	0.030	0.044	0.044	0.047	0.081
$\alpha = 10\%$	0.057	0.066	0.091	0.112	0.114	0.162
			$T = 1600$			
$\alpha = 1\%$	0.006	0.007	0.014	0.031	0.031	0.077
$\alpha = 5\%$	0.030	0.037	0.068	0.111	0.129	0.224
$\alpha = 10\%$	0.068	0.089	0.141	0.205	0.221	0.366
			$T = 3200$			
$\alpha = 1\%$	0.005	0.012	0.031	0.055	0.110	0.205
$\alpha = 5\%$	0.034	0.051	0.117	0.190	0.284	0.495
$\alpha = 10\%$	0.069	0.109	0.226	0.321	0.440	0.681
			$T = 4800$			
$\alpha = 1\%$	0.005	0.009	0.059	0.111	0.167	0.370
$\alpha = 5\%$	0.030	0.049	0.206	0.297	0.446	0.723
$\alpha = 10\%$	0.061	0.120	0.326	0.450	0.618	0.885

Note: This table reports the empirical size and power of block multiplier bootstrap-based test statistic $\widetilde{CvM}_T$ (imposing $p^+ = p^-$) for testing the null of symmetric comovements in (1) against the alternative in (2) under the DGPs in Table 1 and for $\alpha = 1\%$, 5%, and 10% significance levels. The number of simulations is 1000, the number of bootstrap resamples is $B = 200$, and the block length is $L = [T^{1/4}]$. Here we set the exceedance level to be $c = 0$.

Table 5: Empirical rejection rates of the block multiplier bootstrap-based Kolmogorov-Smirnov test statistic $[\widetilde{KS}_T]$ for $c = 0$ under the GARCH(1,1) process when $L = [T^{1/4}]$

Nominal Sizes	DGPs					
	DGP S1	DGP P1	DGP P2	DGP P3	DGP P4	DGP P5
			$T = 240$			
$\alpha = 1\%$	0.001	0.001	0.000	0.000	0.004	0.005
$\alpha = 5\%$	0.011	0.012	0.023	0.020	0.018	0.035
$\alpha = 10\%$	0.037	0.034	0.043	0.056	0.055	0.096
			$T = 600$			
$\alpha = 1\%$	0.003	0.003	0.003	0.004	0.003	0.011
$\alpha = 5\%$	0.012	0.022	0.032	0.036	0.040	0.102
$\alpha = 10\%$	0.047	0.050	0.066	0.088	0.103	0.187
			$T = 1600$			
$\alpha = 1\%$	0.001	0.003	0.010	0.024	0.032	0.109
$\alpha = 5\%$	0.015	0.029	0.064	0.106	0.144	0.332
$\alpha = 10\%$	0.039	0.068	0.149	0.222	0.272	0.497
			$T = 3200$			
$\alpha = 1\%$	0.002	0.004	0.024	0.071	0.141	0.371
$\alpha = 5\%$	0.020	0.040	0.130	0.247	0.411	0.729
$\alpha = 10\%$	0.051	0.099	0.271	0.424	0.583	0.867
			$T = 4800$			
$\alpha = 1\%$	0.002	0.009	0.057	0.162	0.267	0.623
$\alpha = 5\%$	0.015	0.046	0.249	0.415	0.619	0.906
$\alpha = 10\%$	0.041	0.105	0.420	0.597	0.787	0.973

Note: This table reports the empirical size and power of block multiplier bootstrap-based test statistic $\widetilde{KS}_T$ (imposing $p^+ = p^-$) for testing the null of symmetric comovements in (1) against the alternative in (2) under the DGPs in Table 1 and for $\alpha = 1\%$, 5%, and 10% significance levels. The number of simulations is 1000, the number of bootstrap resamples is $B = 200$, and the block length is $L = [T^{1/4}]$. Here we set the exceedance level to be $c = 0$.

Table 6: Empirical rejection rates of the block multiplier bootstrap-based Cramér-von Mises test statistic $[CvM_T]$ for $c = 0$ under the AR(1) process with the autoregressive parameter being 0.5 when $L = [T^{1/4}]$

Nominal Sizes	DGPs					
	DGP S1	DGP P1	DGP P2	DGP P3	DGP P4	DGP P5
			$T = 240$			
$\alpha = 1\%$	0.002	0.004	0.002	0.004	0.004	0.001
$\alpha = 5\%$	0.017	0.015	0.018	0.030	0.034	0.028
$\alpha = 10\%$	0.049	0.030	0.049	0.053	0.068	0.069
			$T = 600$			
$\alpha = 1\%$	0.002	0.003	0.008	0.004	0.010	0.012
$\alpha = 5\%$	0.011	0.015	0.028	0.037	0.061	0.078
$\alpha = 10\%$	0.042	0.040	0.068	0.080	0.138	0.170
			$T = 1600$			
$\alpha = 1\%$	0.002	0.004	0.008	0.025	0.028	0.083
$\alpha = 5\%$	0.008	0.021	0.067	0.124	0.148	0.295
$\alpha = 10\%$	0.016	0.061	0.142	0.217	0.277	0.484
			$T = 3200$			
$\alpha = 1\%$	0.001	0.003	0.025	0.084	0.116	0.305
$\alpha = 5\%$	0.010	0.029	0.164	0.295	0.418	0.693
$\alpha = 10\%$	0.026	0.085	0.295	0.462	0.600	0.861
			$T = 4800$			
$\alpha = 1\%$	0.001	0.008	0.076	0.175	0.283	0.597
$\alpha = 5\%$	0.014	0.063	0.288	0.494	0.670	0.917
$\alpha = 10\%$	0.031	0.117	0.460	0.681	0.856	0.989

Note: This table reports the empirical size and power of block multiplier bootstrap-based test statistic CvM_T for testing the null of symmetric comovements in (1) against the alternative in (2) under the DGPs in Table 1 and for $\alpha = 1\%$, 5%, and 10% significance levels. The number of simulations is 1000, the number of bootstrap resamples is $B = 200$, and the block length is $L = [T^{1/4}]$. Here we set the exceedance level to be $c = 0$.

Table 7: Empirical rejection rates of the block multiplier bootstrap-based Kolmogorov-Smirnov test statistic $[KS_T]$ for $c = 0$ under the AR(1) process with the autoregressive parameter being 0.5 when $L = [T^{1/4}]$

Nominal Sizes	DGPs					
	DGP S1	DGP P1	DGP P2	DGP P3	DGP P4	DGP P5
			$T = 240$			
$\alpha = 1\%$	0.001	0.003	0.001	0.006	0.007	0.007
$\alpha = 5\%$	0.016	0.014	0.021	0.024	0.032	0.035
$\alpha = 10\%$	0.044	0.041	0.043	0.061	0.084	0.104
			$T = 600$			
$\alpha = 1\%$	0.003	0.003	0.009	0.008	0.017	0.026
$\alpha = 5\%$	0.013	0.021	0.042	0.044	0.087	0.151
$\alpha = 10\%$	0.039	0.041	0.078	0.096	0.173	0.247
			$T = 1600$			
$\alpha = 1\%$	0.001	0.004	0.022	0.039	0.078	0.237
$\alpha = 5\%$	0.010	0.028	0.100	0.181	0.265	0.521
$\alpha = 10\%$	0.023	0.060	0.191	0.291	0.412	0.694
			$T = 3200$			
$\alpha = 1\%$	0.001	0.005	0.061	0.161	0.297	0.664
$\alpha = 5\%$	0.010	0.040	0.226	0.434	0.609	0.897
$\alpha = 10\%$	0.028	0.091	0.374	0.629	0.761	0.962
			$T = 4800$			
$\alpha = 1\%$	0.003	0.018	0.149	0.364	0.559	0.907
$\alpha = 5\%$	0.011	0.078	0.422	0.668	0.865	0.997
$\alpha = 10\%$	0.036	0.146	0.575	0.810	0.945	1.000

Note: This table reports the empirical size and power of block multiplier bootstrap-based test statistic KS_T for testing the null of symmetric comovements in (1) against the alternative in (2) under the DGPs in Table 1 and for $\alpha = 1\%$, 5%, and 10% significance levels. The number of simulations is 1000, the number of bootstrap resamples is $B = 200$, and the block length is $L = [T^{1/4}]$. Here we set the exceedance level to be $c = 0$.