Testing for Asymmetric Comovements

Online Appendix

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ABSTRACT

This online appendix contains all the proofs of the theoretical results stated in the main paper titled as "Testing for Asymmetric Comovements" [**Appendix A**]. It also provides the asymptotic properties of estimators for $p^+(c)$ and $p^-(c)$ and the details of how to test $p^+(c) = p^-(c)$ vs. $p^+(c) \neq p^-(c)$ using these estimators [**Appendix B**]. It reports the tables of the empirical results analyzed and discussed in Sections 2 and 6 of the main paper [**Appendix C**]. It also includes empirical results for the analysis of asymmetric comomvents among 5 major financial market indices. Finally, additional simulation results, regarding the different block length choice with $L = 2$ and 4, different degrees of dependence in the data generating process with the autoregressive parameter $\rho = 0$ (i.e., the i.i.d. case), 0.8 and 0.9 in the AR(1) process, and the test results of two popular parametric tests for asymmetric comovements (i.e., the C_ρ and J_ρ tests), are reported [**Appendix D**].

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A Appendix: Proofs of main results

This appendix provides all the proofs of the main theoretical results developed in Sections 3 and 4 of the main paper.

Proof of Lemma 1: Under the null hypothesis H_0, in order to prove that the $\sqrt{T}$-scaled infeasible process $\sqrt{T}\tilde{S}_T(x,y)$ converges weakly to the zero mean Gaussian process $S_\infty(x,y)$ with the long-run covariance kernel given by $\mathbb{K}(\cdot,\cdot)$, we first note that under H_0,

$$\begin{aligned}\sqrt{T}\tilde{S}_T(x,y) =&\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left[1(c<X_t\le x)\,1(c<Y_t\le y)\,p^- - 1(c<-X_t\le x)\,1(c<-Y_t\le y)\,p^+\right]\\ =&p^-\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\{1(c<X_t\le x)\,1(c<Y_t\le y)-\mathbb{E}[1(c<X_1\le x)\,1(c<Y_1\le y)]\}\\ &-p^+\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\{1(c<-X_t\le x)\,1(c<-Y_t\le y)-\mathbb{E}[1(c<-X_1\le x)\,1(c<-Y_1\le y)]\}\\ \equiv&p^-\alpha_T(x,y)-p^+\beta_T(x,y), \end{aligned}\tag{1}$$

where $\alpha_T(x,y)$ and $\beta_T(x,y)$ are two empirical processes indexed by $(x,y)\in\mathbb{R}^2$.

It suffices to show that $\alpha_T(\cdot,\cdot)\Rightarrow\alpha_\infty(\cdot,\cdot)$, where $\alpha_\infty(\cdot,\cdot)$ is a zero mean Gaussian process with long-run covariance kernel

$$\mathbb{E}[\eta_1(x,y)\,\eta_1(x_1,y_1)]+\sum_{i=1}^{\infty}\{\mathbb{E}[\eta_1(x,y)\,\eta_{1+i}(x_1,y_1)]+\mathbb{E}[\eta_{1+i}(x,y)\,\eta_1(x_1,y_1)]\},$$

where $\eta_t(x,y)\equiv 1(c<X_t\le x)\,1(c<Y_t\le y)-\mathbb{E}[1(c<X_t\le x)\,1(c<Y_t\le y)]$. To prove the weak convergence of $\alpha_T(\cdot,\cdot)$, define the pseudometric ρ_d on $\mathbb{R}^2$:

$$\rho_d((x,y),(x_1,y_1))\equiv\left\{\mathbb{E}\left|\eta_t(x,y)-\eta_t(x_1,y_1)\right|^2\right\}^{1/2}.$$

By Theorem 10.2 of Pollard (1990), this follows if we can prove **(i)** the total boundedness of a pseudometric space $(\mathbb{R}^2,\rho_d)$; **(ii)** the stochastic equicontinuity of $\{\alpha_T(\cdot,\cdot):T\ge1\}$; and **(iii)** the finite dimensional (fidi) convergence.

We first establish conditions **(i)** and **(ii)**. First of all, noting that

$$\begin{aligned}\eta_t(x,y)-\eta_t(x_1,y_1)=&[1(c<X_t\le x)\,1(c<Y_t\le y)-1(c<X_t\le x_1)\,1(c<Y_t\le y_1)]\\ &-p^+\left[F^+(x,y)-F^+(x_1,y_1)\right]\\ \equiv&\eta_{t1}(x,y,x_1,y_1)-p^+\left[F^+(x,y)-F^+(x_1,y_1)\right].\end{aligned}$$

For the part $\eta_{t1}(x,y,x_1,y_1)$, it is easy to note that the class of functions

$$\left\{1(c<X_t\le x)\,1(c<Y_t\le y):(x,y)\in\mathbb{R}^2\right\}$$

is a type IV class with index 2 [see Andrews (1994, p. 2278)] that satisfies the L_2-continuity condition. To see this, letting $\zeta < 1$ and noting that $\forall(x,y) \in \mathbb{R}^2$ and for every $\zeta_1 > 0$ and $\zeta > 0$ such that $\sqrt{\zeta_1^2 + \zeta_2^2} \leq \zeta$, we have

$$
\begin{aligned}
&\mathbb{E}\left\{\sup_{(x_1,y_1)\in\mathbb{R}^2:|x_1-x|<\zeta_1,|y_1-y|<\zeta_2} |\eta_{t1}(x,y,x_1,y_1)|^2\right\} \\
=&\mathbb{E}\left\{\sup_{(x_1,y_1)\in\mathbb{R}^2:|x_1-x|<\zeta_1,|y_1-y|<\zeta_2} \left|1\left(X_t \leq x\right)1\left(Y_t \leq y\right) - 1\left(X_t \leq x_1\right)1\left(Y_t \leq y_1\right)\right|^2\right\} \\
\leq&2\mathbb{E}\left\{\sup_{x_1\in\mathbb{R}:|x_1-x|<\zeta_1} \left|1\left(X_t \leq x\right) - 1\left(X_t \leq x_1\right)\right|^2\right\} + 2\mathbb{E}\left\{\sup_{y_1\in\mathbb{R}:|y_1-y|<\zeta_2} \left|1\left(Y_t \leq y\right) - 1\left(Y_t \leq y_1\right)\right|^2\right\} \\
=&2\mathbb{E}\left\{\sup_{x_1\in\mathbb{R}:|x_1-x|<\zeta_1} \left|1\left(X_t \leq x\right) - 1\left(X_t \leq x_1\right)\right|\right\} + 2\mathbb{E}\left\{\sup_{y_1\in\mathbb{R}:|y_1-y|<\zeta_2} \left|1\left(Y_t \leq y\right) - 1\left(Y_t \leq y_1\right)\right|\right\} \\
=&2\mathbb{E}\left[1\left(x-\zeta_1 \leq X_t \leq x+\zeta_1\right)\right] + 2\mathbb{E}\left[\left(y-\zeta_2 \leq Y_t \leq y+\zeta_2\right)\right] \\
=&2\Pr\left(x-\zeta_1 \leq X_t \leq x+\zeta_1\right) + 2\Pr\left(y-\zeta_2 \leq Y_t \leq y+\zeta_2\right) \\
\leq&C(\zeta_1+\zeta_2) \\
\leq&C\zeta
\end{aligned}
$$

for each $\zeta > 0$, where the first step follows immediately by noting that if $X_t \leq c$ or $Y_t \leq c$, the expectation is zero, the second step follows from $1\left(X_t \leq x\right)1\left(Y_t \leq y\right) - 1\left(X_t \leq x_1\right)1\left(Y_t \leq y_1\right) = [1\left(X_t \leq x\right) - 1\left(X_t \leq x_1\right)]1\left(Y_t \leq y\right) + [1\left(Y_t \leq y\right) - 1\left(Y_t \leq y_1\right)]1\left(X_t \leq x_1\right)$ and $|a+b|^2 \leq 2|a|^2 + 2|b|^2$, the third step follows from the fact that $1\left(X_t \leq x\right) - 1\left(X_t \leq x_1\right)$ as well as $1\left(Y_t \leq y\right) - 1\left(Y_t \leq y_1\right)$ can only take values $-1, 0, 1$, the fourth step follows because of $\sup_{x_1\in\mathbb{R}:|x_1-x|<\zeta_1}\left|1\left(X_t \leq x\right) - 1\left(X_t \leq x_1\right)\right| = 1(x-\zeta_1 \leq X_t \leq x+\zeta_1)$ and $\sup_{y_1\in\mathbb{R}:|y_1-y|<\zeta_2}\left|1\left(Y_t \leq y\right) - 1\left(Y_t \leq y_1\right)\right| = 1(y-\zeta_2 \leq Y_t \leq y+\zeta_2)$, the second to last step follows from **Assumption A.1 (ii)**. Consequently,

$$
\begin{aligned}
&\mathbb{E}\left\{\sup_{(x_1,y_1)\in\mathbb{R}^2:|x_1-x|<\zeta_1,|y_1-y|<\zeta_2} |\eta_t(x,y) - \eta_t(x_1,y_1)|^2\right\} \\
\leq&2\mathbb{E}\left\{\sup_{(x_1,y_1)\in\mathbb{R}^2:|x_1-x|<\zeta_1,|y_1-y|<\zeta_2} |\eta_{t1}(x,y,x_1,y_1)|^2\right\} \\
&+ 2(p^+)^2 \sup_{(x_1,y_1)\in\mathbb{R}^2:|x_1-x|<\zeta_1,|y_1-y|<\zeta_2} \left|F^+(x,y) - F^+(x_1,y_1)\right|^2 \\
\leq&C\zeta + C\left(\zeta_1^2 + \zeta_2^2\right) \leq C\zeta.
\end{aligned}
$$

Therefore, the class of functions

$$
\mathcal{M} = \left\{1\left(c < X_t \leq x\right)1\left(c < Y_t \leq y\right) - \mathbb{E}\left[1\left(c < X_t \leq x\right)1\left(c < Y_t \leq y\right)\right] : (x,y) \in \mathbb{R}^2\right\}
$$

is a class of uniformly bounded functions satisfying the L_2-continuity. Note that L_2-continuity implies that the bracketing number satisfies

$$
N\left(\epsilon, \mathcal{M}, \|\cdot\|_{L_2(\mathbb{P})}\right) \leq C\left(\frac{1}{\epsilon}\right)^2,
$$

which in conjunction with **Assumption A.1 (i)** implies that

$$\int_0^1 \epsilon^{-\frac{\gamma}{2+\gamma}} N\left(\epsilon, \mathcal{M}, \|\cdot\|_{L_2(\mathbb{P})}\right)^{\frac{1}{4}} d\epsilon \leq C \int_0^1 \epsilon^{-\frac{\gamma}{2+\gamma}-\frac{1}{2}} d\epsilon < \infty,$$

for $0 < \gamma < 2$. It follows that conditions **(i)-(ii)** are satisfied by Theorem 2.2 of Andrews and Pollard (1994).

The fidi convergence required in condition **(iii)** holds by the Cramér-Wold device and a central limit theorem for bounded random variables under strong mixing conditions. See Corollary 5.1 in Hall and Heyde (1980, p. 132).

We are left to demonstrate that the sample covariance kernel converges to that of the limiting Gaussian process $\alpha_\infty(\cdot,\cdot)$. By the Davydov inequality [see e.g., Bosq (1998)],

$$\begin{aligned}
|\mathbb{E}\left[\alpha_T(x,y)\alpha_T(x_1,y_1)\right]| =& \left|\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\mathbb{E}\left[\eta_t(x,y)\eta_s(x_1,y_1)\right]\right| \\
\leq& \frac{16}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\alpha\left(|t-s|\right) \\
\leq& 16\sum_{j=0}^{\infty}\alpha(j) < \infty,
\end{aligned}$$

where the last step is implied by the strong mixing condition in **Assumption A.1 (i)**. It follows that $\mathbb{E}\left[\alpha_T(x,y)\alpha_T(x_1,y_1)\right]$ is absolutely convergent, and

$$\mathbb{E}\left[\alpha_T(x,y)\alpha_T(x_1,y_1)\right] \to \mathbb{E}\left[\eta_1(x,y)\eta_1(x_1,y_1)\right] + \sum_{i=1}^{\infty}\{\mathbb{E}\left[\eta_1(x,y)\eta_{1+i}(x_1,y_1)\right] + \mathbb{E}\left[\eta_{1+i}(x,y)\eta_1(x_1,y_1)\right]\}.$$

This completes the proof of $\alpha_T(\cdot,\cdot) \Rightarrow \alpha_\infty(\cdot,\cdot)$. In the same manner, we can prove $\beta_T(\cdot,\cdot) \Rightarrow \beta_\infty(\cdot,\cdot)$. Combining the results of $\alpha_T(\cdot,\cdot)$ and $\beta_T(\cdot,\cdot)$, by Pollard (1990, Section 10), we have

$$\Lambda_T(\cdot,\cdot) \equiv (\alpha_T(\cdot,\cdot), \beta_T(\cdot,\cdot))' \Rightarrow \Lambda_\infty(\cdot,\cdot), \tag{2}$$

where $\Lambda_\infty(\cdot,\cdot)$ is a mean zero Gaussian process with certain covariance kernel $\mathbb{K}_\Lambda$. Finally, the results (1) and (2) together complete the proof of **Lemma 1**. ■

Proof of Theorem 1: First of all, observe that the following straightforward decomposition holds:

$$\begin{aligned}
\sqrt{T}S_T(x,y) =& \hat{p}^+\hat{p}^-\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left[\frac{1(c<X_t\leq x)1(c<Y_t\leq y)}{\hat{p}^+} - \frac{1(c<-X_t\leq x)1(c<-Y_t\leq y)}{\hat{p}^-}\right] \\
=& \hat{p}^+\hat{p}^-\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left[\frac{1(c<X_t\leq x)1(c<Y_t\leq y)}{p^+} - \frac{1(c<-X_t\leq x)1(c<-Y_t\leq y)}{p^-}\right] \\
&-\hat{p}^+\hat{p}^-\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left[\frac{1(c<X_t\leq x)1(c<Y_t\leq y)}{\hat{p}^+}\frac{\hat{p}^+-p^+}{p^+}\right] \\
&+\hat{p}^+\hat{p}^-\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left[\frac{1(c<-X_t\leq x)1(c<-Y_t\leq y)}{\hat{p}^-}\frac{\hat{p}^--p^-}{p^-}\right] \\
:=& A_T - B_T + C_T.
\end{aligned} \tag{3}$$

We only have to deal with terms A_T, B_T and C_T separately.

We write

$$M_T(x,y) = \frac{1}{T}\sum_{t=1}^{T}\left[\frac{1(c < X_t \le x)1(c < Y_t \le y)}{p^+} - \frac{1(c < -X_t \le x)1(c < -Y_t \le y)}{p^-}\right].$$

Note that $M_T(x,y) = \frac{1}{p^+\sqrt{T}}\alpha_T(x,y) - \frac{1}{p^-\sqrt{T}}\beta_T(x,y)$ under the null hypothesis. Therefore, the weak convergence results $\alpha_T(\cdot,\cdot) \Rightarrow \alpha_\infty(\cdot,\cdot)$ and $\beta_T(\cdot,\cdot) \Rightarrow \beta_\infty(\cdot,\cdot)$ established in the proof of **Lemma 1** immediately imply that

$$\sup_{(x,y)}\left|\frac{1}{T}\sum_{t=1}^{T} 1\left(c < X_t \le x\right) 1\left(c < Y_t \le y\right) - \mathbb{E}\left[1\left(c < X_1 \le x\right) 1\left(c < Y_1 \le y\right)\right]\right| \xrightarrow{p} 0, \tag{4}$$

$$\sup_{(x,y)}\left|\frac{1}{T}\sum_{t=1}^{T} 1\left(c < -X_t \le x\right) 1\left(c < -Y_t \le y\right) - \mathbb{E}\left[1\left(c < -X_1 \le x\right) 1\left(c < -Y_1 \le y\right)\right]\right| \xrightarrow{p} 0. \tag{5}$$

As a result, we have

$$\begin{aligned}\sup_{(x,y)}|M_T(x,y)| \le& \frac{1}{p^+}\sup_{(x,y)}\left|\frac{1}{T}\sum_{t=1}^{T} 1\left(c < X_t \le x\right) 1\left(c < Y_t \le y\right) - \mathbb{E}\left[1\left(c < X_1 \le x\right) 1\left(c < Y_1 \le y\right)\right]\right| \\ &+ \frac{1}{p^-}\sup_{(x,y)}\left|\frac{1}{T}\sum_{t=1}^{T} 1\left(c < -X_t \le x\right) 1\left(c < -Y_t \le y\right) - \mathbb{E}\left[1\left(c < -X_1 \le x\right) 1\left(c < -Y_1 \le y\right)\right]\right| \\ &\xrightarrow{p} 0.\end{aligned}$$

That is, $M_T(x,y) = o_p(1)$ uniformly in (x,y) under the null.

For the first term A_T, we have

$$\begin{aligned}A_T =& \hat{p}^+\hat{p}^-\sqrt{T}M_T(x,y) \\ =& p^+p^-\sqrt{T}M_T(x,y) + \sqrt{T}(\hat{p}^+ - p^+)p^-M_T(x,y) + \sqrt{T}(\hat{p}^- - p^-)p^+M_T(x,y) \\ &+ \frac{1}{\sqrt{T}}\sqrt{T}(\hat{p}^+ - p^+)\sqrt{T}(\hat{p}^- - p^-)M_T(x,y) \\ =& \sqrt{T}\tilde{S}_T(x,y) + o_p(1),\end{aligned}$$

where we have used the following results: $M_T(x,y) = o_p(1)$ uniformly in (x,y), and $\sqrt{T}(\hat{p}^+ - p^+) = O_p(1)$ and $\sqrt{T}(\hat{p}^- - p^-) = O_p(1)$ which are established in the online **Appendix B**.

For the second term B_T, we have

$$\begin{aligned}
B_T =& \frac{\hat{p}^-}{p^+}\left(\hat{p}^+ - p^+\right)\frac{1}{\sqrt{T}}\sum_{t=1}^{T} 1(c < X_t \le x)1(c < Y_t \le y) \\
=& \frac{p^-}{p^+}\sqrt{T}\left(\hat{p}^+ - p^+\right)\frac{1}{T}\sum_{t=1}^{T} 1(c < X_t \le x)1(c < Y_t \le y) \\
&+ \frac{1}{\sqrt{T}p^+}\sqrt{T}\left(\hat{p}^+ - p^+\right)\sqrt{T}\left(\hat{p}^- - p^-\right)\frac{1}{T}\sum_{t=1}^{T} 1(c < X_t \le x)1(c < Y_t \le y) \\
=& \frac{p^-}{p^+}\sqrt{T}\left(\hat{p}^+ - p^+\right)\mathbb{E}\left[1(c < X_t \le x)1(c < Y_t \le y)\right] + o_p(1) + O_p\left(\frac{1}{\sqrt{T}}\right) \\
=& \sqrt{T}\left(\hat{p}^+ - p^+\right)F^+(x,y)p^- + o_p(1),
\end{aligned}$$

with the second to last equality follows from (4), and the facts that $\sqrt{T}(\hat{p}^+ - p^+) = O_p(1)$ and $\sqrt{T}(\hat{p}^- - p^-) = O_p(1)$, which are established in the online **Appendix B**. Thereafter, by noting that

$$\sqrt{T}\left(\hat{p}^+ - p^+\right) = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(1(X_t > c)1(Y_t > c) - p^+\right),$$

we have

$$B_T = F^+(x,y)p^-\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(1(X_t > c)1(Y_t > c) - p^+\right) + o_p(1)$$

uniformly in (x, y).

Similarly, for the last term C_T in (3), we can show that

$$C_T = F^-(x,y)p^+\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(1(X_t < -c)1(Y_t < -c) - p^-\right) + o_p(1),$$

uniformly in (x, y).

Finally, under the null hypothesis $H_0 : F^+(x, y) = F^-(x, y)$, combing the above results for A_T, B_T and C_T, we obtain

$$\begin{aligned}
\sqrt{T}S_T(x,y) =& \sqrt{T}\tilde{S}_T(x,y) - F^+(x,y)\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left[1(X_t > c)1(Y_t > c)p^- - 1(X_t < -c)1(Y_t < -c)p^+\right] + o_p(1) \\
\equiv& \sqrt{T}\tilde{S}_T(x,y) - F^+(x,y)\sqrt{T}\tilde{R}_T + o_p(1),
\end{aligned}$$

uniformly in (x, y), which concludes the proof of **Theorem 1**. ■

Proof of Theorem 2: Using the results in (4) and (5), under the alternative hypothesis H_1 in Equation (2) of the main paper, for a set with positive Lebesgue measure of (x, y), we can immediately obtain

$$\begin{aligned}
S_T(x,y) =& \hat{p}^-\frac{1}{T}\sum_{t=1}^{T} 1(c < X_t \le x)1(c < Y_t \le y) - \hat{p}^+\frac{1}{T}\sum_{t=1}^{T} 1(c < -X_t \le x)1(c < -Y_t \le y) \\
=& p^-\mathbb{E}\left[1(c < X_t \le x)1(c < Y_t \le y)\right] - p^+\mathbb{E}\left[1(c < -X_t \le x)1(c < -Y_t \le y)\right] + o_p(1) \\
=& \left(F^+(x,y) - F^-(x,y)\right)p^+p^- + o_p(1) \\
\xrightarrow{p}& \left(F^+(x,y) - F^-(x,y)\right)p^+p^- \ne 0,
\end{aligned} \tag{6}$$

where the second step follows easily because $\hat{p}^+ = p^+ + o_p(1)$ and $\hat{p}^- = p^- + o_p(1)$. In addition, it is immediate to see that

$$\sup_{(x,y)} |S_T(x,y)| \le \hat{p}^- + \hat{p}^+ \le 1, \tag{7}$$

due to the facts that $0 \le \hat{p}^- \le 1$ and $0 \le \hat{p}^+ \le 1$.

Thus, we have

$$\begin{aligned}
\frac{CvM_T}{T} &= \int_{\mathbb{R}^2} \left(S_T(x,y)\, 1(x>c)1(y>c)\right)^2 d\hat{F}(x,y) \\
&= \int_{\mathbb{R}^2} \left(S_T(x,y)\, 1(x>c)1(y>c)\right)^2 dF(x,y) \\
&\quad + \int_{\mathbb{R}^2} \left(S_T(x,y)\, 1(x>c)1(y>c)\right)^2 d\left(\hat{F}(x,y) - F(x,y)\right) \\
&\xrightarrow{p} \int_{\mathbb{R}^2} \left(\left(F^+(x,y) - F^-(x,y)\right) p^+ p^- 1(x>c)1(y>c)\right)^2 dF(x,y) > 0,
\end{aligned}$$

where we have used (6) and (7) as well as the fact that $\sup_{(x,y)} |\hat{F}(x,y) - F(x,y)| = o_p(1)$.

Similarly, we can show that

$$\begin{aligned}
\frac{KS_T}{\sqrt{T}} &= \sup_{(x,y)\in\mathbb{R}^2} |S_T(x,y)1(x>c)1(y>c)| \\
&\xrightarrow{p} \sup_{(x,y)\in\mathbb{R}^2} \left|\left(F^+(x,y) - F^-(x,y)\right) p^+ p^- 1(x>c)1(y>c)\right|,
\end{aligned}$$

which concludes the proof of **Theorem 2**. ■

Proof of Theorem 3: First of all, under the local alternative hypothesis H_{1T} specified in Equation (11) of the main paper, let us denote an auxiliary process:

$$\tilde{S}_T^1(x,y) = p^+ p^- \frac{1}{T} \sum_{t=1}^{T} \left[\frac{1(c < X_t \le x)1(c < Y_t \le y)}{p^+} - \frac{1(c < -X_t \le x)1(c < -Y_t \le y)}{p^-} - \frac{\Delta(x,y)}{\sqrt{T}} \right], \tag{8}$$

where the summand is a sequence with mean zero under the local alternatives H_{1T}, i.e., $\tilde{S}_T^1(x,y) \xrightarrow{p} p^+p^- \left(F_T^+(x,y) - F_T^-(x,y) - \Delta(x,y)/\sqrt{T}\right) = 0$ under H_{1T}. Furthermore, since as required $\Delta(\infty,\infty) = 0$ to make Equation (11) a valid sequence of local alternatives, we also have

$$\begin{aligned}
\tilde{S}_T^1(\infty,\infty) &= p^+ p^- \frac{1}{T} \sum_{t=1}^{T} \left[\frac{1(X_t > c)1(Y_t > c)}{p^+} - \frac{1(X_t < -c)1(Y_t < -c)}{p^-} - \frac{\Delta(\infty,\infty)}{\sqrt{T}} \right] \\
&= p^+ p^- \frac{1}{T} \sum_{t=1}^{T} \left[\frac{1(X_t > c)1(Y_t > c)}{p^+} - \frac{1(X_t < -c)1(Y_t < -c)}{p^-} \right] \\
&\equiv \tilde{R}_T.
\end{aligned}$$

Note that under H_{1T},

$$
\begin{aligned}
\sqrt{T}S_T(x,y) =& \hat{p}^+\hat{p}^- \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left[\frac{1(c < X_t \le x)1(c < Y_t \le y)}{\hat{p}^+} - \frac{1(c < -X_t \le x)1(c < -Y_t \le y)}{\hat{p}^-} \right] \\
=& \hat{p}^+\hat{p}^- \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left[\frac{1(c < X_t \le x)1(c < Y_t \le y)}{p^+} - \frac{1(c < -X_t \le x)1(c < -Y_t \le y)}{p^-} - \frac{\Delta(x,y)}{\sqrt{T}} \right] \\
& - \hat{p}^+\hat{p}^- \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left[\frac{1(c < X_t \le x)1(c < Y_t \le y)}{\hat{p}^+} \frac{\hat{p}^+ - p^+}{p^+} \right] \\
& + \hat{p}^+\hat{p}^- \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left[\frac{1(c < -X_t \le x)1(c < -Y_t \le y)}{\hat{p}^-} \frac{\hat{p}^- - p^-}{p^-} \right] \\
& + \hat{p}^+\hat{p}^- \Delta(x,y) \\
:=& A'_T - B_T + C_T + \hat{p}^+\hat{p}^- \Delta(x,y).
\end{aligned}
$$

Similar to the analysis of term A_T in the proof of Theorem 1, we can readily show that

$$
\begin{aligned}
A'_T &= \left(\frac{\hat{p}^+ - p^+}{p^+} + 1 \right) \left(\frac{\hat{p}^- - p^-}{p^-} + 1 \right) \sqrt{T} \tilde{S}^1_T(x,y) \\
&= \sqrt{T} \tilde{S}^1_T(x,y) + o_p(1).
\end{aligned}
$$

Collecting the above result for A'_T as well as those results for B_T and C_T (but with $F^+_T(x,y)$ and $F^-_T(x,y)$ replacing $F^+(x,y)$ and $F^-(x,y)$ in the local alternatives) in the proof of Theorem 1, and using the fact that $\hat{p}^+\hat{p}^-\Delta(x,y) = p^+p^-\Delta(x,y) + o_p(1)$, we have

$$
\begin{aligned}
\sqrt{T}S_T(x,y) \;&=\; \sqrt{T}\tilde{S}^1_T(x,y) - F^+_T(x,y) p^- \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left(1(X_t > c)1(Y_t > c) - p^+ \right) \\
&\quad + F^-_T(x,y) p^+ \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left(1(X_t < -c)1(Y_t < -c) - p^- \right) \\
&\quad + p^+p^-\Delta(x,y) + o_p(1) \\
&=\; \sqrt{T}\tilde{S}^1_T(x,y) - F^+_T(x,y)\sqrt{T}\tilde{R}_T \\
&\quad - \Delta(x,y) p^+ \frac{1}{T} \sum_{t=1}^{T} \left(1(X_t < -c)1(Y_t < -c) - p^- \right) + p^+p^-\Delta(x,y) + o_p(1) \\
&=\; \sqrt{T}\tilde{S}^1_T(x,y) - F^+_T(x,y)\sqrt{T}\tilde{R}_T + p^+p^-\Delta(x,y) + o_p(1), \qquad (9)
\end{aligned}
$$

where the second step follows due to $F^+_T(x,y) = F^-_T(x,y) + \Delta(x,y)/\sqrt{T}$ under H_{1T}, and the last step follows from the fact that,

$$
\frac{1}{T} \sum_{t=1}^{T} \left(1(X_t < -c)1(Y_t < -c) - p^- \right) \xrightarrow{p} \mathbb{E}\left[1(X_1 < -c)1(Y_1 < -c) \right] - p^- \equiv 0. \qquad (10)
$$

The above follows by the weak law of large numbers for mixingale random variables [see e.g., McLeish (1975) and Andrews (1988)] and the fact that the function of the mixing process is still mixing.

It remains to prove the weak convergence result $\sqrt{T}\tilde{S}_T^1(\cdot,\cdot) \Rightarrow S_\infty(\cdot,\cdot)$. Similar to the proof of **Lemma 1**, to prove that under the local alternatives H_{1T} in (11), the process $\sqrt{T}\tilde{S}_T^1(\cdot,\cdot)$ in (8) converges weakly to the zero mean Gaussian process $S_\infty(\cdot,\cdot)$ stated in Corollary 1, we only have to note that $\sqrt{T}\tilde{S}_T^1(x,y)$ can be rewritten as

$$\begin{aligned}
\sqrt{T}\tilde{S}_T^1(x,y) =& \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left[1\left(c<X_t\leq x\right)1\left(c<Y_t\leq y\right)p^- - 1\left(c<-X_t\leq x\right)1\left(c<-Y_t\leq y\right)p^+\right] \\
&- p^+p^-\Delta(x,y) \\
=& p^-\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left\{1\left(c<X_t\leq x\right)1\left(c<Y_t\leq y\right) - \mathbb{E}\left[1\left(c<X_t\leq x\right)1\left(c<Y_t\leq y\right)\right]\right\} \\
&- p^+\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left\{1\left(c<-X_t\leq x\right)1\left(c<-Y_t\leq y\right) - \mathbb{E}\left[1\left(c<-X_t\leq x\right)1\left(c<-Y_t\leq y\right)\right]\right\} \\
&+ \sqrt{T}p^+p^-\left(F_T^+(x,y) - F_T^-(x,y) - \frac{\Delta(x,y)}{\sqrt{T}}\right) \\
\equiv& p^-\alpha_T(x,y) - p^+\beta_T(x,y),
\end{aligned}$$

where in the last step we have used $F_T^+(x,y) = F_T^-(x,y) + \Delta(x,y)/\sqrt{T}$ under H_{1T}. Here, $\alpha_T(x,y)$ and $\beta_T(x,y)$ are the empirical processes already defined in the proof of **Lemma 1**. As a result, by the proof of **Lemma 1**, we have $\sqrt{T}\tilde{S}_T^1(\cdot,\cdot) \Rightarrow S_\infty(\cdot,\cdot)$.

Recall that $\sqrt{T}\tilde{R}_T$ converges to a zero mean normal random variable defined in Corollary 1. Therefore, as $T\to\infty$, we have

$$\sqrt{T}\tilde{S}_T^1(x,y) - F_T^+(x,y)\sqrt{T}\tilde{R}_T \Rightarrow S_\infty(x,y) - F^+(x,y)V \equiv \hat{S}_\infty(x,y).$$

As a result, by noting the decomposition in (9), we can readily conclude that

$$\sqrt{T}S_T(x,y) \Rightarrow \hat{S}_\infty(x,y) + p^+p^-\Delta(x,y) \equiv \hat{S}_\infty^1(x,y).$$

This ends the proof of **Theorem 3**. ■

Recall the following notations:

$$\hat{\varepsilon}_s\left(x,y\right) = 1\left(c<X_s\leq x\right)1\left(c<Y_s\leq y\right)\hat{p}^- - 1\left(c<-X_s\leq x\right)1\left(c<-Y_s\leq y\right)\hat{p}^+,$$

$$\hat{\varepsilon}_s\left(\infty,\infty\right) = 1\left(X_s>c\right)1\left(Y_s>c\right)\hat{p}^- - 1\left(X_s<-c\right)1\left(Y_s<-c\right)\hat{p}^+,$$

$$\varepsilon_s\left(x,y\right) = 1\left(c<X_s\leq x\right)1\left(c<Y_s\leq y\right)p^- - 1\left(c<-X_s\leq x\right)1\left(c<-Y_s\leq y\right)p^+,$$

$$\varepsilon_s\left(\infty,\infty\right) = 1\left(X_s>c\right)1\left(Y_s>c\right)p^- - 1\left(X_s<-c\right)1\left(Y_s<-c\right)p^+,$$

$$S_T^*\left(x,y\right) = \frac{1}{T}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}\left[\hat{\varepsilon}_s\left(x,y\right) - \hat{F}^+(x,y)\hat{\varepsilon}_s\left(\infty,\infty\right)\right]. \tag{11}$$

In addition, define

$$\tilde{S}_T^*(x,y) = \frac{1}{T}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}\left[\varepsilon_s(x,y) - F^+(x,y)\varepsilon_s(\infty,\infty)\right]. \tag{12}$$

To theoretically justify the validity of the proposed dependent multiplier bootstrap procedure as stated in **Theorem 4**, in the following we shall introduce and prove two auxiliary lemmas **Lemma A1** and **Lemma A2**. In light of these two lemmas, the proof of **Theorem 4** is immediate.

Lemma A1 establishes that the infeasible empirical process $\sqrt{T}\tilde{S}_T^*(x,y)$ defined in (12) converges weakly to $\hat{S}_\infty(x,y)$, with $\hat{S}_\infty(x,y)$ the same Gaussian process as defined in **Corollary 1**, conditional on the original sample $\{(X_t,Y_t)'\}_{t=1}^T$.

Lemma A1: Suppose Assumptions **A.1-A.2** are satisfied. Then, conditional on the original sample $\{(X_t,Y_t)'\}_{t=1}^T$,

$$\sqrt{T}\tilde{S}_T^*(\cdot,\cdot) \Rightarrow \hat{S}_\infty(\cdot,\cdot), \tag{13}$$

where $\hat{S}_\infty(x,y)$ is the same Gaussian process as defined in **Corollary 1** in the main text.

Proof of Lemma A1: In order to establish the weak convergence of $\sqrt{T}\tilde{S}_T^*(x,y)$ to $\hat{S}_\infty(x,y)$ under the bootstrap law (i.e., conditional on the original sample $\{(X_t,Y_t)'\}_{t=1}^T$), we first rewrite $\sqrt{T}\tilde{S}_T^*(x,y)$ as

$$\sqrt{T}\tilde{S}_T^*(x,y) = \sum_{t=1}^{T-L+1}\phi_{Tt}(\xi_t;x,y),$$

where

$$\phi_{Tt}(\xi_t;x,y) = \frac{1}{\sqrt{T}}\xi_t\sum_{s=t}^{t+L-1}\left[\varepsilon_s(x,y) - F^+(x,y)\varepsilon_s(\infty,\infty)\right].$$

Note that $\phi_{Tt}(\xi_t;x,y)$ has an envelope function given by

$$\bar{\phi}_{Tt}(\xi_t) = \frac{1}{\sqrt{T}}|\xi_t|\sup_{(x,y)}\left|\sum_{s=t}^{t+L-1}\left[\varepsilon_s(x,y) - F^+(x,y)\varepsilon_s(\infty,\infty)\right]\right|.$$

Note also that conditional on the original sample $\{(X_t,Y_t)'\}_{t=1}^T$, the triangular array $\{\phi_{Tt}(\xi_t;x,y)\}$ is independent within rows. Thus, we can apply Theorem 10.6 of Pollard (1990) to show the weak convergence of $\sqrt{T}\tilde{S}_T^*(x,y)$ to $\hat{S}_\infty(x,y)$. Recall that Pollard (1990)'s theorem allows the function $\phi_{Tt}(\xi_t;x,y)$ to depend on both T and t. The following proof is largely adapted from Inoue (2001)'s proof of Theorem 2.3 as well as Su and White (2012)'s proof of Theorem 5.

To start, we define the following pseudo-metric:

$$\rho_T((x,y),(x',y')) = \left\{\sum_{t=1}^T\mathbb{E}\left[\left|\phi_{Tt}(\xi_t;x,y) - \phi_{Tt}(\xi_t;x',y')\right|^2\middle|\{(X_t,Y_t)'\}_{t=1}^T\right]\right\}^{1/2}.$$

Then, to show the weak convergence of $\sqrt{T}\tilde{S}_T^*(x,y)$ conditional on the original sample, according to Theorem 10.6 of Pollard (1990), it suffices for us to verify the following five conditions:

(i) $\{\phi_{Tt}(\xi_t;x,y)\}$ is manageable[1] in the sense of Definition 7.9 of Pollard (1990, p. 38);

(ii) $\mathbb{E}\left[T\tilde{S}^*_T(x,y)\tilde{S}^*_T(x',y')\middle|\{(X_t,Y_t)'\}_{t=1}^T\right]\xrightarrow{p}\hat{\mathbb{K}}((x,y),(x',y'))$ for every (x,y), (x',y');

(iii) $\lim_{T\to\infty}\sum_{t=1}^T\mathbb{E}\left[\bar{\phi}^2_{Tt}(\xi_t)\middle|\{(X_t,Y_t)'\}_{t=1}^T\right]$ is stochastically bounded;

(iv) $\sum_{t=1}^T\mathbb{E}\left[\bar{\phi}^2_{Tt}(\xi_t)1\left(\bar{\phi}_{Tt}(\xi_t)>\epsilon\right)\middle|\{(X_t,Y_t)'\}_{t=1}^T\right]\xrightarrow{p}0$ for each $\epsilon>0$;

(v) the pseudo-metric $\rho((x,y),(x',y'))=\lim_{T\to\infty}\rho_T((x,y),(x',y'))$ is well defined and, for all deterministic sequences (x_T,y_T) and (x'_T,y'_T), if $\rho((x_T,y_T),(x'_T,y'_T))\to 0$, then $\rho_T((x_T,y_T),(x'_T,y'_T))\xrightarrow{p}0$.

Proof of part (i): In order for the triangular array of process $\{\phi_{Tt}(\xi_t;x,y)\}$ to be manageable with respect to the envelope $\bar{\phi}_{Tt}(\xi_t)$, we need to find a deterministic function $\lambda(\epsilon_0)$ that bounds the covering number of $\alpha\bigodot\Phi_T=\{\alpha_t\phi_{Tt}(\xi_t;x,y):(x,y)\in\overline{\mathbb{R}}^2$, where α_t are nonnegative finite constants for all $t=1,\ldots,T\}$ with $\sqrt{\log\lambda(\epsilon_0)}$ integrable. Here, the covering number refers to the smallest number of closed balls with radius $(\epsilon_0/2)\sqrt{\sum_{t=1}^T\alpha_t^2\bar{\phi}^2_{Tt}(\xi_t)}$ whose unions cover $\alpha\bigodot\Phi_T$ (see Pollard, 1990, inequality (10.7), p. 54). It follows that within each closed ball,

$$\begin{aligned}&\sum_{t=1}^T\alpha_t^2\mathbb{E}\left[\left|\phi_{Tt}(\xi_t;x,y)-\phi_{Tt}(\xi_t;x',y')\right|^2\middle|\{(X_t,Y_t)'\}_{t=1}^T\right]\\&\le\frac{\epsilon_0^2}{4}\sum_{t=1}^T\alpha_t^2E\left[\bar{\phi}^2_{Tt}(\xi_t)\middle|\{(X_t,Y_t)'\}_{t=1}^T\right],\ \forall\epsilon_0\in(0,1].\end{aligned}\tag{14}$$

For the left-hand side of (14), it follows that

$$\begin{aligned}&\sum_{t=1}^T\alpha_t^2\mathbb{E}\left[\left|\phi_{Tt}(\xi_t;x,y)-\phi_{Tt}(\xi_t;x',y')\right|^2\middle|\{(X_t,Y_t)'\}_{t=1}^T\right]\\&=\frac{1}{T}\sum_{t=1}^{T-L+1}\alpha_t^2\frac{1}{L}\left|\sum_{s=t}^{t+L-1}\left[\varepsilon_s(x,y)-F^+(x,y)\varepsilon_s(\infty,\infty)\right]-\sum_{s=t}^{t+L-1}\left[\varepsilon_s\left(x',y'\right)-F^+(x',y')\varepsilon_s(\infty,\infty)\right]\right|^2\\&\xrightarrow{p}\sum_{t=1}^{\infty}\alpha_t^2\left[\hat{\mathbb{K}}((x,y),(x,y))-2\hat{\mathbb{K}}((x,y),(x',y'))+\hat{\mathbb{K}}((x',y'',y'))\right]\\&\equiv\sum_{t=1}^{\infty}\alpha_t^2\rho^2((x,y),(x',y')),\ \text{say.}\end{aligned}$$

Next, for the right-hand side of (14), we have

$$\begin{aligned}&\sum_{t=1}^T\alpha_t^2E\left[\bar{\phi}^2_{Tt}(\xi_t)\middle|\{(X_t,Y_t)'\}_{t=1}^T\right]\\&=\frac{1}{T}\sum_{t=1}^T\alpha_t^2\sup_{(x,y)}\left|\frac{1}{\sqrt{L}}\sum_{s=t}^{t+L-1}\left[\varepsilon_s(x,y)-F^+(x,y)\varepsilon_s(\infty,\infty)\right]\right|^2\\&=O_p(1),\end{aligned}\tag{15}$$

[1]See also Pollard (1989, p. 348) for the definition of manageable class of functions. Pollard (1989, pp. 350–352) provides several examples for manageable classes as well.

where the last equality follows because $L^{-1/2}\sum_{s=t}^{t+L-1}[\varepsilon_s(x,y)-F^{+}(x,y)\varepsilon_s(\infty,\infty)]$ is an empirical process indexed by (x,y) and weakly converges to a zero mean Gaussian process with a certain (although complicated) long-run covariance kernel. To prove this, using the same arguments as in the proof of **Lemma 1**, we can see that under either the null H_0 or the sequence of local alternatives H_{1T}, we have

$$\begin{aligned}
&\frac{1}{\sqrt{L}}\sum_{s=t}^{t+L-1}[\varepsilon_s(x,y)-F^{+}(x,y)\varepsilon_s(\infty,\infty)]\\
=&p^{-}\frac{1}{\sqrt{L}}\sum_{s=t}^{t+L-1}\{1(c<X_s\le x)\,1(c<Y_s\le y)-\mathbb{E}[1(c<X_1\le x)\,1(c<Y_1\le y)]\}\\
&-p^{+}\frac{1}{\sqrt{L}}\sum_{s=t}^{t+L-1}\{1(c<-X_s\le x)\,1(c<-Y_s\le y)-\mathbb{E}[1(c<-X_1\le x)\,1(c<-Y_1\le y)]\}\\
&-p^{-}F^{+}(x,y)\frac{1}{\sqrt{L}}\sum_{s=t}^{t+L-1}\left\{1(X_s>c)\,1(Y_s>c)-p^{+}\right\}\\
&+p^{+}F^{+}(x,y)\frac{1}{\sqrt{L}}\sum_{s=t}^{t+L-1}\left\{1(X_s<-c)\,1(Y_s<-c)-p^{-}\right\}\\
&+p^{+}p^{-}\sqrt{L}\left(F^{+}(x,y)-F^{-}(x,y)\right)\\
:=&p^{-}\alpha_L(x,y)-p^{+}\beta_L(x,y)-p^{-}F^{+}(x,y)A_{L1}+p^{+}F^{+}(x,y)A_{L2}+p^{+}p^{-}\Delta(x,y)\sqrt{\frac{L}{T}},
\end{aligned}\tag{16}$$

where $\Delta(x,y)\equiv 0$ under H_0, while $\Delta(x,y)$ is uniformly bounded under H_{1T}.

Clearly, in (16), whether it is under H_0 or under H_{1T}, the remainder term $p^{+}p^{-}\Delta(x,y)\sqrt{L/T}$ is $o_p(1)$ uniformly in (x,y) due to $L/T\to 0$. Following the same arguments as in the proof of **Lemma 1**, we can show that $\alpha_L(\cdot,\cdot)\Rightarrow\alpha_\infty(\cdot,\cdot)$ and $\beta_L(\cdot,\cdot)\Rightarrow\beta_\infty(\cdot,\cdot)$.

Furthermore, note that both A_{L1} and A_{L2} are summations of indicator functions of strong mixing random variables. Then, according to a central limit theorem for bounded random variables under strong mixing conditions [see e.g., Corollary 5.1 in Hall and Heyde (1980, p. 132) and de Jong (1997)] and the fact that the indicator function of the mixing process is still mixing, we can prove the asymptotic normality of A_{L1} and A_{L2} with mean zero and long-run variances given respectively by (26) and (27) as $L\to\infty$. The proof of absolute convergence of both long-run variances can be found in the online **Appendix B**.

Thus, we have $A_{L1}=O_p(1)$ and $A_{L2}=O_p(1)$. Consequently, these results together imply that

$$\sup_{(x,y)}\left|\frac{1}{\sqrt{L}}\sum_{s=t}^{t+L-1}[\varepsilon_s(x,y)-F^{+}(x,y)\varepsilon_s(\infty,\infty)]\right|^2=O_p(1),$$

which leads to (15).

Therefore, the above results imply that for any small $\epsilon_1>0$, there exists a large constant $M_1\equiv M_1(\epsilon_1)>0$ such that the following inequality holds:

$$\sum_{t=1}^{\infty}\alpha_t^2\rho^2((x,y),(x',y'))\le\frac{\epsilon_0^2}{4}M_1,\tag{17}$$

for sufficiently large T on a set with probability $1-\epsilon_1$.

Selecting grid points $\{x : -\infty = x_0 < x_1 < \cdots < x_{T1-1} < x_{T1} = \infty$ such that $|x_j - x_{j-1}| < \delta_1\}$ and $\{y : -\infty = y_0 < y_1 < \cdots < y_{T2-1} < y_{T2} = \infty$ such that $|y_k - y_{k-1}| < \delta_2\}$. For $(x,y) \in [x_{j-1}, x_j] \times [y_{j-1}, y_j]$, it is easy to note that

$$\mathbb{E}\left[\varepsilon_t(x,y) - \varepsilon_t(x_j, y_k)\right]^2 \leq C(\delta_1 + \delta_2),$$

and

$$(F^+(x,y) - F^+(x_j,y_k))^2 \leq C(\delta_1 + \delta_2)$$

as well as

$$0 < \lim_{T\to\infty} T^{-1}\mathbb{E}\left[\sum_{t=1}^{T} \varepsilon_t(\infty,\infty)\right]^2 := \sigma^2 < \infty.$$

Let $\delta = \sqrt{\delta_1^2 + \delta_2^2} < 1$. By the Cauchy-Schwartz and Davydov inequalities, it can be seen that

$$\begin{aligned}
&\rho^2((x,y),(x_j,y_k)) \\
=&\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T}[\varepsilon_t(x,y) - F^+(x,y)\varepsilon_t(\infty,\infty)] - \sum_{t=1}^{T}[\varepsilon_t(x_j,y_k) - F^+(x_j,y_k)\varepsilon_t(\infty,\infty)]\right]^2 \\
\leq&2\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T}[\varepsilon_t(x,y) - \varepsilon_t(x_j,y_k)]\right]^2 + 2(F^+(x,y) - F^+(x_j,y_k))^2\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T}\varepsilon_t(\infty,\infty)\right]^2 \\
\leq&C\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T}[1(c < \pm X_t \leq x) - 1(c < \pm X_t \leq x_j)]\right]^2 \\
&+ C\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T}[1(c < \pm Y_t \leq y) - 1(c < \pm Y_t \leq y_k)]\right]^2 \\
&+ 2(F^+(x,y) - F^+(x_j,y_k))^2\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T}\varepsilon_t(\infty,\infty)\right]^2 \\
\leq&C\left[(\delta_1+\delta_2) + \delta_1\sum_{s=1}^{\infty}\alpha(s)^{1/2} + \delta_2\sum_{s=1}^{\infty}\alpha(s)^{1/2}\right] \\
\leq&C\delta
\end{aligned}$$

by **Assumption A.1**, where the exact values of C may vary across lines. If we choose $\delta = \epsilon_0^2$, then

$$\sum_{t=1}^{\infty}\alpha_t^2\rho^2((x,y),(x_j,y_k)) \leq C_1\epsilon_0^2\sum_{t=1}^{\infty}\alpha_t^2,$$

so that (17) can be satisfied for all sufficiently large T and M_1 within each closed ball. Because the capacity bound is $O(\delta^{-2}) = O(\epsilon_0^{-4})$, the integrability condition is also satisfied.

Proof of part (ii): Note that

$$\begin{aligned}
&\mathbb{E}\left[T\tilde{S}_T^*(x,y)\tilde{S}_T^*(x',y')\middle|\left\{(X_t,Y_t)'\right\}_{t=1}^T\right] \\
=&\frac{1}{T}\sum_{t=1}^{T-L+1}\frac{1}{L}\sum_{s=t}^{t+L-1}\sum_{s'=t}^{t+L-1}\left[\varepsilon_s\left(x,y\right)-F^+(x,y)\varepsilon_s\left(\infty,\infty\right)\right]\left[\varepsilon_{s'}\left(x',y'\right)-F^+(x',y')\varepsilon_{s'}\left(\infty,\infty\right)\right] \\
\equiv&\bar{S}_T^*,
\end{aligned}$$

where, for purpose of simplicity, we have suppressed the dependence of the quantity $\bar{S}_T^* \equiv \bar{S}_T^*((x,y),(x',y'))$ on (x,y,x',y').

First,

$$\begin{aligned}
\mathbb{E}[\bar{S}_T^*] =&\frac{1}{T}\sum_{t=1}^{T-L+1}\frac{1}{L}\sum_{s=t}^{t+L-1}\sum_{s'=t}^{t+L-1}\mathbb{E}\left\{\left[\varepsilon_s\left(x,y\right)-F^+(x,y)\varepsilon_s\left(\infty,\infty\right)\right]\left[\varepsilon_{s'}\left(x',y'\right)-F^+(x',y')\varepsilon_{s'}\left(\infty,\infty\right)\right]\right\} \\
\to&\hat{\mathbb{K}}((x,y),(x',y')).
\end{aligned}$$

We next prove that $\mathrm{Var}(\bar{S}_T^*)=o(1)$. Let

$$\vartheta_{Tt}^* \equiv \vartheta_{Tt}^*((x,y),(x',y')) = \frac{1}{L^2}\sum_{s=t}^{t+L-1}\sum_{s'=t}^{t+L-1}\left[\varepsilon_s\left(x,y\right)-F^+(x,y)\varepsilon_s\left(\infty,\infty\right)\right]\left[\varepsilon_{s'}\left(x',y'\right)-F^+(x',y')\varepsilon_{s'}\left(\infty,\infty\right)\right],$$

and let

$$\vartheta_{Tt}(x,y) \equiv \frac{1}{L}\sum_{s=t}^{t+L-1}\left[\varepsilon_s\left(x,y\right)-F^+(x,y)\varepsilon_s\left(\infty,\infty\right)\right].$$

Let $||\cdot||_p = (\mathbb{E}|X|^p)^{1/p}$. By the Cauchy inequality,

$$||\vartheta_{Tt}^*||_8 = ||\vartheta_{Tt}(x,y)\vartheta_{Tt}(x',y')||_8 \leq ||\vartheta_{Tt}(x,y)||_{16}||\vartheta_{Tt}(x',y')||_{16}.$$

By Lemma 3.1 of Andrews and Pollard (1994) with $Q=16$,

$$||\vartheta_{Tt}(x,y)||_{16}^{16} = \mathbb{E}\left|\frac{1}{L}\sum_{s=t}^{t+L-1}\left[\varepsilon_s\left(x,y\right)-F^+(x,y)\varepsilon_s\left(\infty,\infty\right)\right]\right|^{16} = O(L^{-8}).$$

Consequently, $\mathbb{E}|\vartheta_{Tt}^*|^8 = O(L^{-8})$. Let

$$\kappa_{4T} = \sup_{t\leq T}\sup_{(x,y),(x',y')}\mathbb{E}|\vartheta_{Tt}^*|^8 = O(L^{-8}),$$

and

$$\kappa_{2T} = \sup_{t\leq T}\sup_{(x,y),(x',y')}\mathbb{E}|\vartheta_{Tt}^*|^4 = O(L^{-4}).$$

By Lemma A.1(b) of Inoue (2001) with $\delta=2$, see also Lemma 9 of Bühlmann (1994),

$$\mathbb{E}\left|\frac{L}{T}\sum_{t=1}^{T-L+1}\vartheta_{Tt}^*\right|^4 = O(L^4T^{-4}L^2(T^2\kappa_{4T}^{1/2}+T\kappa_{2T})) = O(T^{-2}L^2) = o(1).$$

Hence, $\bar{S}_T^* = \hat{\mathbb{K}}((x,y),(x',y')) + o_p(1)$ by the Chebyshev inequality.

Proof of part (iii): This follows immediately from the proof of part (i) by simply taking $\alpha_t = 1$, $\forall t$.

Proof of part (iv): By the conditional Chebyshev inequality, we have

$$\begin{aligned}
&\mathbb{P}\left(\bar{\phi}_{Tt}(\xi_t) > \epsilon \middle| \left\{(X_t, Y_t)'\right\}_{t=1}^T\right) \\
\leq &\frac{L}{T\epsilon^2}\left\{\sup_{(x,y)}\left|\frac{1}{\sqrt{L}}\sum_{s=t}^{t+L-1}\left[\varepsilon_s(x,y) - F^+(x,y)\varepsilon_s(\infty,\infty)\right]\right|\right\}^2 \\
= &O_p\left(\frac{L}{T}\right).
\end{aligned}$$

Then, by the Cauchy-Schwartz inequality,

$$\begin{aligned}
&\sum_{t=1}^T \mathbb{E}\left[\bar{\phi}_{Tt}^2(\xi_t) 1\left(\bar{\phi}_{Tt}(\xi_t) > \epsilon\right) \middle| \left\{(X_t, Y_t)'\right\}_{t=1}^T\right] \\
= &\frac{1}{T}\sum_{t=1}^T \mathbb{E}\left[\xi_t^2 \sup_{(x,y)}\left|\frac{1}{\sqrt{L}}\sum_{s=t}^{t+L-1}\left[\varepsilon_s(x,y) - F^+(x,y)\varepsilon_s(\infty,\infty)\right]\right|^2 1\left(\bar{\phi}_{Tt}(\xi_t) > \epsilon\right) \mid \left\{(X_t, Y_t)'\right\}_{t=1}^T\right] \\
\leq &\frac{1}{T}\sum_{t=1}^T \left[\frac{1}{L^2}\sup_{(x,y)}\left|\sum_{s=t}^{t+L-1}\left[\varepsilon_s(x,y) - F^+(x,y)\varepsilon_s(\infty,\infty)\right]\right|^4 \mathbb{P}\left(\bar{\phi}_{Tt}(\xi_t) > \epsilon \middle| \left\{(X_t, Y_t)'\right\}_{t=1}^T\right)\right]^{1/2} \\
= &O_p\left(\sqrt{L/T}\right) \\
= &o_p(1).
\end{aligned}$$

Proof of part (v): From part (i), we know that $\rho^2((x,y),(x',y')) = plim_{n\to\infty}\rho_T^2((x,y),(x',y'))$ is well defined. If $\rho((x_T,y_T),(x_T',y_T')) \to 0$, then

$$\rho_T((x_T,y_T),(x_T',y_T')) \leq |\rho_T((x_T,y_T),(x_T',y_T')) - \rho((x_T,y_T),(x_T',y_T'))| + \rho((x_T,y_T),(x_T',y_T')) \xrightarrow{p} 0.$$

The proof of **Lemma A1** is therefore complete. ■

On the other hand, **Lemma A2** establishes the asymptotic uniform equivalence between the feasible empirical process $S_T^*(x,y)$ defined in (11) and the infeasible empirical process $\tilde{S}_T^*(x,y)$ defined in (12), conditional on the original sample $\{(X_t, Y_t)'\}_{t=1}^T$.

Lemma A2: Suppose Assumptions **A.1-A.2** are satisfied. Then, conditional on the original sample $\left\{(X_t, Y_t)'\right\}_{t=1}^T$, uniformly in (x,y),

$$\sqrt{T}S_T^*(x,y) = \sqrt{T}\tilde{S}_T^*(x,y) + o_p(1).$$

Proof of Lemma A2: First of all, observe that

$$\begin{aligned}
&\hat{\varepsilon}_s(x,y) - \hat{F}^{+}(x,y)\hat{\varepsilon}_s(\infty,\infty) \\
=&\left[\varepsilon_s(x,y) - F^{+}(x,y)\varepsilon_s(\infty,\infty)\right] \\
&+\left[\hat{\varepsilon}_s(x,y) - \varepsilon_s(x,y)\right] \\
&- F^{+}(x,y)\left[\hat{\varepsilon}_s(\infty,\infty) - \varepsilon_s(\infty,\infty)\right] \\
&- \left[\hat{F}^{+}(x,y) - F^{+}(x,y)\right]\varepsilon_s(\infty,\infty) \\
&- \left[\hat{F}^{+}(x,y) - F^{+}(x,y)\right]\left[\hat{\varepsilon}_s(\infty,\infty) - \varepsilon_s(\infty,\infty)\right].
\end{aligned}$$

As a consequence, $S_T^*(x,y)$ in (11) can be decomposed as follows:

$$\begin{aligned}
&\sqrt{T}S_T^*(x,y) \\
=&\frac{1}{\sqrt{T}}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}\left[\hat{\varepsilon}_s(x,y) - \hat{F}^{+}(x,y)\hat{\varepsilon}_s(\infty,\infty)\right] \\
=&\frac{1}{\sqrt{T}}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}\left[\varepsilon_s(x,y) - F^{+}(x,y)\varepsilon_s(\infty,\infty)\right] \\
&+\frac{1}{\sqrt{T}}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}\left[\hat{\varepsilon}_s(x,y) - \varepsilon_s(x,y)\right] \\
&- F^{+}(x,y)\frac{1}{\sqrt{T}}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}\left[\hat{\varepsilon}_s(\infty,\infty) - \varepsilon_s(\infty,\infty)\right] \\
&- \left[\hat{F}^{+}(x,y) - F^{+}(x,y)\right]\frac{1}{\sqrt{T}}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}\varepsilon_s(\infty,\infty) \\
&- \left[\hat{F}^{+}(x,y) - F^{+}(x,y)\right]\frac{1}{\sqrt{T}}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}\left[\hat{\varepsilon}_s(\infty,\infty) - \varepsilon_s(\infty,\infty)\right] \\
:=&\sqrt{T}\tilde{S}_T^*(x,y) + D_{T1}^*(x,y) - D_{T2}^*(x,y) - D_{T3}^*(x,y) - D_{T4}^*(x,y).
\end{aligned} \tag{18}$$

In the following, we shall show that, conditional on the original sample $\{(X_t, Y_t)'\}_{t=1}^{T}$, the four terms $D_{T1}^*(x,y)$, $D_{T2}^*(x,y)$, $D_{T3}^*(x,y)$, and $D_{T4}^*(x,y)$ are all asymptotically negligible uniformly in (x,y), when $L \to \infty$ at a suitable rate as $T \to \infty$ as stated in **Assumption A.2 (iii)**.

For the first term $D_{T1}^*(x,y)$ in (18), recalling the definitions of $\hat{\varepsilon}_s(x,y)$ and $\varepsilon_s(x,y)$ mentioned above **Lemma A.1**, we observe that

$$\begin{aligned}
D_{T1}^*(x,y) =&\frac{1}{\sqrt{T}}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}\left[\hat{\varepsilon}_s(x,y) - \varepsilon_s(x,y)\right] \\
=&\left(\hat{p}^{-} - p^{-}\right)\frac{1}{\sqrt{T}}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}1(c < X_s \le x)\,1(c < Y_s \le y) \\
&- \left(\hat{p}^{+} - p^{+}\right)\frac{1}{\sqrt{T}}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}1(c < -X_s \le x)\,1(c < -Y_s \le y).
\end{aligned}$$

Then, $D^*_{T1}(x,y)$ can be further decomposed as

$$\begin{aligned}
&D^*_{T1}(x,y)\\
=&\mathbb{E}\left[1\left(c<X_1\leq x\right)1\left(c<Y_1\leq y\right)\right]\left(\hat{p}^- - p^-\right)\frac{L}{\sqrt{T}}\sum_{t=1}^{T-L+1}\xi_t\\
&-\mathbb{E}\left[1\left(c<-X_1\leq x\right)1\left(c<-Y_1\leq y\right)\right]\left(\hat{p}^+ - p^+\right)\frac{L}{\sqrt{T}}\sum_{t=1}^{T-L+1}\xi_t\\
&+\left(\hat{p}^- - p^-\right)\frac{1}{\sqrt{T}}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}\left\{1\left(c<X_s\leq x\right)1\left(c<Y_s\leq y\right)-\mathbb{E}\left[1\left(c<X_1\leq x\right)1\left(c<Y_1\leq y\right)\right]\right\}\\
&-\left(\hat{p}^+ - p^+\right)\frac{1}{\sqrt{T}}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}\left\{1\left(c<-X_s\leq x\right)1\left(c<-Y_s\leq y\right)-\mathbb{E}\left[1\left(c<-X_1\leq x\right)1\left(c<-Y_1\leq y\right)\right]\right\}\\
:=&D^*_{T11}(x,y)-D^*_{T12}(x,y)+D^*_{T13}(x,y)-D^*_{T14}(x,y).
\end{aligned}$$

Now note that $\mathbb{E}\left[1\left(c<X_1\leq x\right)1\left(c<Y_1\leq y\right)\right]$ and $\mathbb{E}\left[1\left(c<-X_1\leq x\right)1\left(c<-Y_1\leq y\right)\right]$ are simply two uniformly bounded nonnegative constants (in fact, they are uniformly bounded between zero and one). We also note the facts that $\mathbb{E}\left(T^{-1/2}\sum_{t=1}^{T-L+1}\xi_t\right)=0$ and $\mathbb{E}\left(T^{-1/2}\sum_{t=1}^{T-L+1}\xi_t\right)^2\to L^{-1}$ by the properties of $\{\xi_t\}_{t=1}^T$, which implies

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T-L+1}\xi_t=O_p\left(L^{-1/2}\right),$$

as well as $\hat{p}^+ - p^+ = O_p\left(T^{-1/2}\right)$ and $\hat{p}^- - p^- = O_p\left(T^{-1/2}\right)$, which are established in the online **Appendix B**. Then we immediately have $D^*_{T11}(x,y)=O_p\left(T^{-1/2}\right)O_p\left(L^{1/2}\right)=O_p\left((L/T)^{1/2}\right)=o_p(1)$ and $D^*_{T12}(x,y)=O_p\left(T^{-1/2}\right)O_p\left(L^{1/2}\right)=O_p\left((L/T)^{1/2}\right)=o_p(1)$ uniformly in (x,y) conditional on the original sample $\left\{(X_t,Y_t)'\right\}_{t=1}^T$, because of the condition $L/T\to 0$, which is implied by $L/\sqrt{T}\to 0$ as $L\to\infty$ and $T\to\infty$ in **Assumption A.2 (iii)**.

On the other hand, it is easy to see that the double summations in $D^*_{T13}(x,y)$ and $D^*_{T14}(x,y)$ have a similar structure as $\sqrt{T}\tilde{S}^*_T(x,y)$ in (12). Then, following identical arguments as proving the weak convergence in (13) conditional on the original sample $\left\{(X_t,Y_t)'\right\}_{t=1}^T$, we can readily establish the following two weak convergence results conditional on the original sample $\left\{(X_t,Y_t)'\right\}_{t=1}^T$:

$$\begin{aligned}
\gamma_T(\cdot,\cdot):=&\frac{1}{\sqrt{T}}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}\left\{1\left(c<X_s\leq\cdot\right)1\left(c<Y_s\leq\cdot\right)-\mathbb{E}\left[1\left(c<X_1\leq\cdot\right)1\left(c<Y_1\leq\cdot\right)\right]\right\}\\
&\Rightarrow\gamma_\infty(\cdot,\cdot),
\end{aligned}\tag{19}$$

and

$$\begin{aligned}
\upsilon_T(\cdot,\cdot):=&\frac{1}{\sqrt{T}}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}\left\{1\left(c<-X_s\leq\cdot\right)1\left(c<-Y_s\leq\cdot\right)-\mathbb{E}\left[1\left(c<-X_1\leq\cdot\right)1\left(c<-Y_1\leq\cdot\right)\right]\right\}\\
&\Rightarrow\upsilon_\infty(\cdot,\cdot),
\end{aligned}\tag{20}$$

where $\gamma_\infty(\cdot,\cdot)$ and $\upsilon_\infty(\cdot,\cdot)$ are two zero mean Gaussian processes with the respective long-run covariance kernel given by that of $\alpha_\infty(\cdot,\cdot)$ and $\beta_\infty(\cdot,\cdot)$, two zero mean Gaussian processes as defined in **Lemma 1**. As a direct consequence of the weak convergence results in (19) and (20), we conclude that $\gamma_T(x,y) = O_p(1)$ and $\upsilon_T(x,y) = O_p(1)$ uniformly in (x,y) conditional on the original sample $\left\{(X_t,Y_t)'\right\}_{t=1}^T$. Then,

$$D^*_{T13}(x,y) := \left(\hat{p}^- - p^-\right)\gamma_T(x,y) = O_p\left(T^{-1/2}\right)O_p(1) = o_p(1),$$

and

$$D^*_{T14}(x,y) := \left(\hat{p}^+ - p^+\right)\upsilon_T(x,y) = O_p\left(T^{-1/2}\right)O_p(1) = o_p(1),$$

uniformly in (x,y) conditional on the original sample $\left\{(X_t,Y_t)'\right\}_{t=1}^T$.

Therefore, summarizing the previous results, we have shown that $D^*_{T1}(x,y) = o_p(1)$ uniformly in (x,y) conditional on the original sample $\left\{(X_t,Y_t)'\right\}_{t=1}^T$.

For the second term $D^*_{T2}(x,y)$ in (18), recalling the definitions of $\hat{\varepsilon}_s(\infty,\infty)$ and $\varepsilon_s(\infty,\infty)$ mentioned above **Lemma A.1**, we can rewrite $D^*_{T2}(x,y)$ as

$$\begin{aligned}
&D^*_{T2}(x,y)\\
=&F^+(x,y)\sqrt{T}\left(\hat{p}^- - p^-\right)\frac{1}{T}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}1\left(X_s>c\right)1\left(Y_s>c\right)\\
&-F^+(x,y)\sqrt{T}\left(\hat{p}^+ - p^+\right)\frac{1}{T}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}1\left(X_s<-c\right)1\left(Y_s<-c\right)\\
:=&D^*_{T21}(x,y) - D^*_{T22}(x,y).
\end{aligned}$$

Note that conditional on the original sample $\left\{(X_t,Y_t)'\right\}_{t=1}^T$, applying the properties of $\{\xi_t\}_{t=1}^T$, we find

$$\mathbb{E}\left(\frac{1}{T}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}1\left(X_s>c\right)1\left(Y_s>c\right)\right) = 0, \tag{21}$$

and

$$\begin{aligned}
&\mathbb{E}\left(\frac{1}{T}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}1\left(X_s>c\right)1\left(Y_s>c\right)\right)^2\\
=&\frac{1}{T^2}\sum_{t=1}^{T-L+1}\sum_{t'=1}^{T-L+1}\mathbb{E}\left(\xi_t\xi_{t'}\right)\sum_{s=t}^{t+L-1}\sum_{s'=t'}^{t'+L-1}1\left(X_s>c\right)1\left(Y_s>c\right)1\left(X_{s'}>c\right)1\left(Y_{s'}>c\right)\\
=&\frac{1}{T^2}\sum_{t=1}^{T-L+1}\mathbb{E}\left(\xi_t^2\right)\sum_{s=t}^{t+L-1}\sum_{s'=t}^{t+L-1}1\left(X_s>c\right)1\left(Y_s>c\right)1\left(X_{s'}>c\right)1\left(Y_{s'}>c\right)\\
=&\frac{1}{T^2L}\sum_{t=1}^{T-L+1}\sum_{s=t}^{t+L-1}\sum_{s'=t}^{t+L-1}1\left(X_s>c\right)1\left(Y_s>c\right)1\left(X_{s'}>c\right)1\left(Y_{s'}>c\right)\\
\leq&\frac{(T-L+1)L^2}{T^2L} = \frac{(T-L+1)L}{T^2} \leq \frac{L}{T}.
\end{aligned} \tag{22}$$

As a result, under the condition of $L/T \to 0$ as $L \to \infty$ and $T \to \infty$, conditional on the original sample $\left\{(X_t, Y_t)'\right\}_{t=1}^{T}$,

$$\frac{1}{T}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}1\left(X_s > c\right)1\left(Y_s > c\right) = O_p\left(\sqrt{\frac{L}{T}}\right) = o_p(1), \tag{23}$$

Therefore, together with $\sqrt{T}\left(\hat{p}^- - p^-\right) = O_p(1)$ and the uniform boundedness of $F^+(x, y)$, conditional on the original sample $\left\{(X_t, Y_t)'\right\}_{t=1}^{T}$, uniformly in (x, y),

$$D^*_{T21}\left(x, y\right) = O_p(1)o_p(1) = o_p(1).$$

Similar arguments as proving (21) and (22) yield immediately that, conditional on the original sample $\left\{(X_t, Y_t)'\right\}_{t=1}^{T}$,

$$\frac{1}{T}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}1\left(X_s < -c\right)1\left(Y_s < -c\right) = O_p\left(\sqrt{\frac{L}{T}}\right) = o_p(1). \tag{24}$$

Then by $\sqrt{T}\left(\hat{p}^+ - p^+\right) = O_p(1)$ and the uniform boundedness of $F^+(x, y)$, conditional on the original sample $\left\{(X_t, Y_t)'\right\}_{t=1}^{T}$, uniformly in (x, y),

$$D^*_{T22}\left(x, y\right) = O_p(1)o_p(1) = o_p(1).$$

Combing the results of $D^*_{T21}\left(x, y\right) = o_p(1)$ and $D^*_{T22}\left(x, y\right) = o_p(1)$, we have $D^*_{T2}\left(x, y\right) = o_p(1)$ uniformly in (x, y), conditional on the original sample $\left\{(X_t, Y_t)'\right\}_{t=1}^{T}$.

For the third term $D^*_{T3}\left(x, y\right)$ in (18), note that it can be rewritten as

$$\begin{aligned}
&D^*_{T3}\left(x, y\right)\\
=&p^-\sqrt{T}\left[\hat{F}^+(x, y) - F^+(x, y)\right]\frac{1}{T}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}1\left(X_s > c\right)1\left(Y_s > c\right)\\
&-p^+\sqrt{T}\left[\hat{F}^+(x, y) - F^+(x, y)\right]\frac{1}{T}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}1\left(X_s < -c\right)1\left(Y_s < -c\right)\\
:=&D^*_{T31}\left(x, y\right) - D^*_{T32}\left(x, y\right).
\end{aligned}$$

Then in light of (23) and (24), in order to prove the asymptotically uniform negligibility of $D^*_{T31}\left(x, y\right)$ and $D^*_{T32}\left(x, y\right)$ conditional on the original sample $\left\{(X_t, Y_t)'\right\}_{t=1}^{T}$, it suffices to prove that

$$\sqrt{T}\left[\hat{F}^+(x, y) - F^+(x, y)\right] = O_p(1) \tag{25}$$

uniformly in (x, y).

Towards this end, we first recall that $F^{+}(x,y)=\mathbb{E}\left[1\left(c<X_1\leq x\right)1\left(c<Y_1\leq y\right)\right]/p^{+}$ and note that

$$\begin{aligned}
&\sqrt{T}\left[\hat{F}^{+}(x,y)-F^{+}(x,y)\right]\\
=&\sqrt{T}\left[\frac{1}{T}\sum_{t=1}^{T}\frac{1\left(c<X_t\leq x\right)1\left(c<Y_t\leq y\right)}{\hat{p}^{+}}-F^{+}(x,y)\right]\\
=&\frac{1}{\hat{p}^{+}}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left\{1\left(c<X_t\leq x\right)1\left(c<Y_t\leq y\right)-\mathbb{E}\left[1\left(c<X_1\leq x\right)1\left(c<Y_1\leq y\right)\right]\right\}\\
&-\mathbb{E}\left[1\left(c<X_1\leq x\right)1\left(c<Y_1\leq y\right)\right]\frac{1}{p^{+}\hat{p}^{+}}\sqrt{T}\left(\hat{p}^{+}-p^{+}\right)\\
=&\left(\frac{1}{p^{+}}+o_p(1)\right)\alpha_T\left(x,y\right)-\left(\frac{1}{p^{+2}}+o_p(1)\right)\mathbb{E}\left[1\left(c<X_1\leq x\right)1\left(c<Y_1\leq y\right)\right]\sqrt{T}\left(\hat{p}^{+}-p^{+}\right),
\end{aligned}$$

where the last equality holds due to the fact that $\hat{p}^{+}=p^{+}+o_p(1)$ and the Slutsky's theorem, and the empirical process $\alpha_T\left(x,y\right)$ is defined in (1) in the proof of **Lemma 1**. Recalling the weak convergence result $\alpha_T(\cdot,\cdot)\Rightarrow\alpha_\infty(\cdot,\cdot)$ established in the proof of **Lemma 1**. This result immediately implies that $\alpha_T\left(x,y\right)=O_p(1)$ uniformly in (x,y). On the other hand, $\sqrt{T}\left(\hat{p}^{+}-p^{+}\right)=O_p(1)$. As a result, (25) holds uniformly in (x,y). Thus, (25) together with (23) and (24) imply

$$D^{*}_{T31}\left(x,y\right)=O_p(1)o_p(1)=o_p(1),$$

and

$$D^{*}_{T32}\left(x,y\right)=O_p(1)o_p(1)=o_p(1),$$

respectively.

As a consequence, $D^{*}_{T3}\left(x,y\right)=D^{*}_{T31}\left(x,y\right)-D^{*}_{T32}\left(x,y\right)=o_p(1)$ uniformly in (x,y), conditional on the original sample $\left\{(X_t,Y_t)'\right\}_{t=1}^{T}$.

For the last term $D^{*}_{T4}\left(x,y\right)$ in (18), observe it can be decomposed as

$$\begin{aligned}
&D^{*}_{T4}\left(x,y\right)\\
=&\left(\hat{p}^{-}-p^{-}\right)\sqrt{T}\left[\hat{F}^{+}(x,y)-F^{+}(x,y)\right]\frac{1}{T}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}1\left(X_s>c\right)1\left(Y_s>c\right)\\
&-\left(\hat{p}^{+}-p^{+}\right)\sqrt{T}\left[\hat{F}^{+}(x,y)-F^{+}(x,y)\right]\frac{1}{T}\sum_{t=1}^{T-L+1}\xi_t\sum_{s=t}^{t+L-1}1\left(X_s<-c\right)1\left(Y_s<-c\right)\\
:=&D^{*}_{T41}\left(x,y\right)-D^{*}_{T42}\left(x,y\right).
\end{aligned}$$

Then in light of the results (25), (23) and (24) as well as the facts of $\hat{p}^{+}-p^{+}=o_p(1)$ and $\hat{p}^{-}-p^{-}=o_p(1)$, it is immediate that $D^{*}_{T41}\left(x,y\right)=o_p(1)O_p(1)o_p(1)=o_p(1)$ and $D^{*}_{T42}\left(x,y\right)=o_p(1)O_p(1)o_p(1)=o_p(1)$ uniformly in (x,y), conditional on the original sample $\left\{(X_t,Y_t)'\right\}_{t=1}^{T}$. Therefore, $D^{*}_{T4}\left(x,y\right)=o_p(1)$ uniformly in (x,y), conditional on the original sample $\left\{(X_t,Y_t)'\right\}_{t=1}^{T}$.

Finally, based on the decomposition in (18) and combining the previous results concerning $D_{Tj}^{*}(x,y) = o_p(1)$ uniformly in (x,y) for $j=1,2,3,4$, conditional on the original sample $\left\{(X_t,Y_t)'\right\}_{t=1}^{T}$, we have completed the proof of **Lemma A2**. ■

Proof of Theorem 4: First note that **Lemma A2** indicates that, conditional on the sample $\{(X_t,Y_t)'\}_{t=1}^{T}$, it suffices to study the weak convergence of $\sqrt{T}\tilde{S}_T^{*}(x,y)$, as the asymptotic behavior of $\sqrt{T}S_T^{*}(x,y)$ would be the same as that of $\sqrt{T}\tilde{S}_T^{*}(x,y)$.

On the other hand, **Lemma A1** indicates that, conditional on the sample $\{(X_t,Y_t)'\}_{t=1}^{T}$, $\sqrt{T}\tilde{S}_T^{*}(x,y)$ weakly converges to the same zero mean Gaussian process $\hat{S}_{\infty}(x,y)$ as defined in **Corollary 1** in the main text.

Therefore, the combination of **Lemma A1** and **Lemma A2** completes the proof of **Theorem 4**. ■

B Appendix: Testing $p^+(c) = p^-(c)$ vs. $p^+(c) \neq p^-(c)$

In this appendix we provide the asymptotic properties of estimators of $p^+(c)$ and $p^-(c)$ and the details of how to test $p^+(c) = p^-(c)$ vs. $p^+(c) \neq p^-(c)$ using these estimators.

For some fixed non-negative exceedance level c, recall that $p^+(c) := \mathbb{E}[1(X_t > c)1(Y_t > c)]$ and $p^-(c) := \mathbb{E}[1(X_t < -c)1(Y_t < -c)]$. To proceed, recall that

$$\hat{p}^+(c) = \frac{1}{T}\sum_{t=1}^{T} p_t^+(c),$$

$$\hat{p}^-(c) = \frac{1}{T}\sum_{t=1}^{T} p_t^-(c),$$

and let

$$\overline{I}(c) = \frac{1}{T}\sum_{t=1}^{T} I_t(c) \equiv \hat{p}^+(c) - \hat{p}^-(c),$$

where $p_t^+(c) = 1(X_t > c)1(Y_t > c)$, $p_t^-(c) = 1(X_t < -c)1(Y_t < -c)$, and $I_t(c) = p_t^+(c) - p_t^-(c)$.

Note that both $\hat{p}^+(c)$ and $\hat{p}^-(c)$ are summations of indicator functions of strong mixing random variables. Then, according to a central limit theorem for bounded random variables under strong mixing conditions [see e.g., Corollary 5.1 in Hall and Heyde (1980, p. 132) and de Jong (1997)] and the fact that the indicator function of the mixing process is still mixing, we can prove the asymptotic normality of $\sqrt{T}\left(\hat{p}^+(c) - p^+(c)\right)$ and $\sqrt{T}\left(\hat{p}^-(c) - p^-(c)\right)$. In particular, we have that

$$\sqrt{T}\left(\hat{p}^+(c) - p^+(c)\right) \xrightarrow{d} N\left(0, \sigma_+^2(c)\right),$$

where the long-run variance

$$\sigma_+^2(c) = p^+(c)\left(1 - p^+(c)\right) + 2\sum_{i=1}^{\infty} \mathbb{E}\left[\left(p_1^+(c) - p^+(c)\right)\left(p_{1+i}^+(c) - p^+(c)\right)\right], \tag{26}$$

and

$$\sqrt{T}\left(\hat{p}^-(c) - p^-(c)\right) \xrightarrow{d} N\left(0, \sigma_-^2(c)\right),$$

where the long-run variance

$$\sigma_-^2(c) = p^-(c)\left(1 - p^-(c)\right) + 2\sum_{i=1}^{\infty} \mathbb{E}\left[\left(p_1^-(c) - p^-(c)\right)\left(p_{1+i}^-(c) - p^-(c)\right)\right]. \tag{27}$$

Here, the long-run variances $\sigma_+^2(c)$ and $\sigma_-^2(c)$ are absolutely convergent, which follows from the Davydov inequality [see e.g., Bosq (1998)] and the strong mixing condition in **Assumption A.1 (i)**:

$$
\begin{aligned}
\mathbb{E}\left[\sqrt{T}\left(\hat{p}^{+}(c)-p^{+}(c)\right)\right]^{2} =& \frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\mathbb{E}\left[\left(p_t^{+}(c)-p^{+}(c)\right)\left(p_s^{+}(c)-p^{+}(c)\right)\right] \\
=& \left|\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\mathbb{E}\left[\left(p_t^{+}(c)-p^{+}(c)\right)\left(p_s^{+}(c)-p^{+}(c)\right)\right]\right| \\
\leq& \frac{16}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\alpha\left(|t-s|\right) \\
\leq& 16\sum_{j=0}^{\infty}\alpha(j)<\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}\left[\sqrt{T}\left(\hat{p}^{-}(c)-p^{-}(c)\right)\right]^{2} =& \frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\mathbb{E}\left[\left(p_t^{-}(c)-p^{-}(c)\right)\left(p_s^{-}(c)-p^{-}(c)\right)\right] \\
=& \left|\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\mathbb{E}\left[\left(p_t^{-}(c)-p^{-}(c)\right)\left(p_s^{-}(c)-p^{-}(c)\right)\right]\right| \\
\leq& \frac{16}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\alpha\left(|t-s|\right) \\
\leq& 16\sum_{j=0}^{\infty}\alpha(j)<\infty.
\end{aligned}
$$

Clearly, when $\{(X_t, Y_t)\}_{t=1}^{T}$ is a martingale difference sequence (MDS), the long-run variances simply reduce to $\sigma_{+}^{2}(c) = p^{+}(c)\left(1-p^{+}(c)\right)$ and $\sigma_{-}^{2}(c) = p^{-}(c)\left(1-p^{-}(c)\right)$, respectively, and are of course absolutely convergent.

As a direct result of the above asymptotic normality results, we thus obtain that

$$\sqrt{T}\left(\hat{p}^{+}(c)-p^{+}(c)\right) = O_p(1),$$

and

$$\sqrt{T}\left(\hat{p}^{-}(c)-p^{-}(c)\right) = O_p(1).$$

In the following, we outline the testing procedure employed to calculate the asymptotic p-values reported in Table 1 of the online **Appendix C** for testing the null hypothesis $H_0^p : p^{+}(c) = p^{-}(c)$ against the alternative hypothesis $H_1^p : p^{+}(c) \neq p^{-}(c)$. Clearly, testing $H_0^p : p^{+}(c) = p^{-}(c)$ is equivalent to testing $H_0^p : \mathbb{E}[I_t(c)] = 0$.

Motivated by the above observation, a straightforward t-type test statistic can be constructed by

$$\hat{t}_T(c) = \frac{\sqrt{T}\,\overline{I}(c)}{\sqrt{\widehat{\Omega}_T(c)}},$$

where $\widehat{\Omega}_T(c)$ is a consistent estimator for the long-run variance $\Omega(c) = \lim_{T\to\infty} \mathrm{Var}(\sqrt{T}\bar{I}(c)) = \mathrm{Var}(I_t(c)) + 2\sum_{j=1}^{\infty} \mathrm{Cov}\left(I_1(c), I_{1+j}(c)\right) := \Gamma(0) + 2\sum_{j=1}^{\infty} \Gamma(j)$. Under the null hypothesis H_0^p, the test statistic $\hat{t}_T(c)$ is asymptotically distributed as a standard normal random variable.

Note that $\Gamma(0) = \mathrm{Var}\left(p_t^+(c) - p_t^-(c)\right) = p^+(c)\left(1 - p^+(c)\right) + p^-(c)\left(1 - p^-(c)\right) + 2p^+(c)p^-(c) = p^+(c) + p^-(c) - \left(p^+(c) - p^-(c)\right)^2$, where the second equality follows from the facts that $p_t^+(c)$ and $p_t^-(c)$ are Bernoulli random variables with parameters $p^+(c)$ and $p^-(c)$, respectively, and $p_t^+(c)p_t^-(c) = 0$ almost surely because of the binary properties of $p_t^+(c)$ and $p_t^-(c)$. Also note that $\Gamma(0) = p^+(c) + p^-(c) = 2p^+(c) = 2p^-(c)$ under the null hypothesis H_0^p.

The estimation of long-run variance has an extensive literature, see e.g., Newey and West (1987) and Andrews (1991) for the two seminal works. In this paper, following the suggestion in Hong et al. (2007, p. 1553), the long-run variance $\Omega(c)$ is estimated by

$$\widehat{\Omega}_T(c) = k(0)\hat{\Gamma}_T(0) + 2\sum_{j=1}^{T-1} k\left(\frac{j}{m_T}\right)\hat{\Gamma}_T(j),$$

where

$$\hat{\Gamma}_T(0) = \hat{p}^+(c) + \hat{p}^-(c) - \left(\hat{p}^+(c) - \hat{p}^-(c)\right)^2,$$

$$\hat{\Gamma}_T(j) = \frac{1}{T}\sum_{t=1}^{T-j}\left(Z_t(c) - \left(\hat{p}^+(c) - \hat{p}^-(c)\right)\right)\left(Z_{t+j}(c) - \left(\hat{p}^+(c) - \hat{p}^-(c)\right)\right), \text{ for } j = 1, 2, \ldots, T-1.$$

Here, the function $k(\cdot)$ is taken to be the Bartlett kernel and the bandwidth sequence $m_T = m(T)$ satisfies $m_T \to \infty$ and $m_T/T \to 0$ as $T \to \infty$.

Note that the last term $\left(\hat{p}^+(c) - \hat{p}^-(c)\right)^2$ in the expression of $\hat{\Gamma}_T(0)$ can be omitted under the null hypothesis H_0^p. But we keep it in the calculation of the test statistic $\hat{t}_T(c)$ as it is nonzero under the alternative hypothesis H_1^p and thus may enhance $\hat{t}_T(c)$'s testing power.

C Appendix: Empirical results

In this appendix we provide the empirical results discussed in the main paper, which include the estimation and testing for the equality of joint probabilities $p^+(c)$ and $p^-(c)$ for different exceedance levels c, the summary statistics of daily S&P 500 index and 29 daily individual stocks, results of testing symmetric comovements for different exceedance levels c. Detailed discussions for Tables 1–7 can be found in the main paper.

To further illustrate the usefulness of the propose tests, we report the results of testing symmetric comovements using 5 major market indices for different exceedance levels c in Tables 8–12. Specifically, the following market indices are considered: S&P 500, FTSE 100, DAX 30, NIKKEI 225, and HS 300. We remark that Deng (2016) has also considered S&P 500, FTSE 100, DAX 30, and NIKKEI 225 but with a shorter time period. We further include HS 300 to make the second empirical study more systematic and offer a more complete picture. Our daily data are collected from January 2007 to December 2016, with a total number of observations 2180 after deleting all missing values.

As shown in Table 1, test results in Table 8 indicate that the assumption of $p^+(c) = p^-(c)$ does not hold for the market returns for all four exceedance levels c considered, which justify again the importance of adjusting for marginal probabilities $p^+(c)$ and $p^-(c)$ in the testing procedure. In addition, Tables 9–12 show that the J_ρ test fails to reject the null hypothesis of symmetric comovements for all pairwise market comparisons. The C_ρ test only rejects the null for FTSE 100 vs DAX 30, FTSE 100 vs NIKKEI 225 and DAX 30 vs NIKKEI 225 when $c = 0$, while it only reject the null for FTSE 100 vs NIKKEI 225 when $c = 0.5$.

On the contrary, our tests CvM_T and KS_T are able to reject the null for most of the pairwise market comparisons when c is relatively small, indicating that asymmetric comovements among international stock markets may be a common phenomenon. Lastly, the tests $\widetilde{CvM}_T$ and $\widetilde{KS}_T$ that ignore completely the inequality of $p^+(c)$ and $p^-(c)$ demonstrate less reliable testing results; that is, they reject too much when $c = 0$, while they do not reject at all when $c = 1$. This behavior is also confirmed in Tables 2–7 and suggests we should always use CvM_T and KS_T.

Table 1: Estimation and testing for the equality of joint probabilities $p^+(c)$ and $p^-(c)$ for $c \in \{0, 0.5, 1, 1.5\}$

	$c=0$			$c=0.5$			$c=1$			$c=1.5$		
Stocks	$\hat{p}^+(c)$	$\hat{p}^-(c)$	p -value	$\hat{p}^+(c)$	$\hat{p}^-(c)$	p -value	$\hat{p}^+(c)$	$\hat{p}^-(c)$	p -value	$\hat{p}^+(c)$	$\hat{p}^-(c)$	p -value
AXP	0.402	0.364	0.000***	0.131	0.135	0.341	0.049	0.062	0.000***	0.024	0.032	0.000***
BA	0.385	0.349	0.000***	0.138	0.148	0.054*	0.052	0.061	0.009***	0.021	0.033	0.000***
BAC	0.399	0.362	0.000***	0.110	0.125	0.001***	0.040	0.046	0.023**	0.018	0.022	0.019**
CAT	0.395	0.365	0.001***	0.147	0.146	0.881	0.058	0.068	0.005***	0.024	0.034	0.000***
CSCO	0.389	0.360	0.000***	0.149	0.145	0.361	0.051	0.062	0.001***	0.024	0.026	0.272
CVX	0.400	0.360	0.000***	0.140	0.144	0.431	0.053	0.068	0.000***	0.021	0.033	0.000***
DD	0.406	0.365	0.000***	0.147	0.159	0.029**	0.057	0.065	0.017**	0.023	0.035	0.000***
DIS	0.402	0.366	0.000***	0.145	0.151	0.324	0.055	0.066	0.001***	0.026	0.035	0.000***
GE	0.407	0.374	0.000***	0.137	0.141	0.533	0.056	0.062	0.034**	0.024	0.032	0.000***
HD	0.377	0.356	0.014**	0.131	0.138	0.152	0.050	0.063	0.000***	0.025	0.030	0.005***
HPQ	0.387	0.337	0.000***	0.133	0.127	0.198	0.045	0.055	0.003***	0.014	0.024	0.000***
IBM	0.387	0.358	0.000***	0.146	0.147	0.875	0.056	0.062	0.073*	0.022	0.029	0.001***
INTC	0.387	0.346	0.000***	0.139	0.146	0.161	0.053	0.066	0.000***	0.023	0.031	0.000***
JNJ	0.374	0.355	0.018**	0.131	0.132	0.807	0.045	0.059	0.000***	0.019	0.025	0.000***
JPM	0.402	0.376	0.001***	0.130	0.141	0.018**	0.049	0.056	0.012**	0.022	0.031	0.000***
KO	0.371	0.333	0.000***	0.123	0.123	1.000	0.047	0.052	0.071*	0.019	0.023	0.021**
MCD	0.364	0.322	0.000***	0.127	0.129	0.737	0.045	0.058	0.000***	0.020	0.026	0.001***
MMM	0.416	0.374	0.000***	0.155	0.155	0.940	0.061	0.071	0.004***	0.028	0.035	0.001***
MRK	0.366	0.344	0.006***	0.130	0.124	0.222	0.041	0.048	0.025**	0.016	0.022	0.000***
MSFT	0.373	0.357	0.055*	0.138	0.138	1.000	0.052	0.064	0.004***	0.021	0.028	0.000***
PFE	0.373	0.358	0.050*	0.132	0.131	0.934	0.052	0.061	0.007***	0.023	0.026	0.123
PG	0.366	0.340	0.001***	0.124	0.119	0.271	0.046	0.050	0.137	0.019	0.023	0.017**
T	0.375	0.333	0.000***	0.127	0.128	0.935	0.046	0.058	0.000***	0.021	0.028	0.000***
TRV	0.390	0.354	0.000***	0.120	0.130	0.025**	0.043	0.051	0.002***	0.019	0.024	0.002***
UNH	0.366	0.341	0.002***	0.106	0.110	0.280	0.034	0.045	0.000***	0.014	0.018	0.004***
UTX	0.403	0.377	0.002***	0.151	0.155	0.449	0.060	0.074	0.000***	0.029	0.035	0.009***
VZ	0.362	0.325	0.000***	0.124	0.133	0.044**	0.050	0.055	0.063*	0.021	0.025	0.040**
WMT	0.352	0.318	0.000***	0.115	0.117	0.727	0.043	0.049	0.051*	0.019	0.020	0.331
XOM	0.389	0.365	0.004***	0.140	0.143	0.580	0.054	0.062	0.018**	0.021	0.029	0.000***

Note: This table reports results of estimation and testing of joint probabilities $p^+(c)$ $=\Pr(X_t > c, Y_t > c)$ and $p^-(c)$ $=\Pr(X_t < -c, Y_t < -c)$ for different c's, with X_t the S&P 500 daily return and Y_t the daily return on each of the following 29 stocks: AXP, BA, BAC, CAT, CSCO, CVX, DD, DIS, GE, HD, HPQ, IBM, INTC, JNJ, JPM, KO, MCD, MMM, MRK, MSFT, PFE, PG, T, TRV, UNH, UTX, VZ, WMT, and XOM. The p-values correspond to t-test of $H_0^p : p^+(c) = p^-(c)$ versus $H_1^p : p^+(c) \neq p^-(c)$; see the description of the testing procedure in the online **Appendix B**.

Table 2: Summary statistics of daily S&P 500 index and 29 daily individual stocks

Stocks	Mean	S.D.	Skewness	Kurtosis
GSPC	0.018	1.320	-0.327	12.939
AXP	0.015	2.508	0.023	13.825
BA	0.032	1.821	-0.006	7.936
BAC	-0.028	3.702	-0.287	21.957
CAT	0.028	2.091	0.016	7.753
CSCO	0.010	1.909	-0.449	14.569
CVX	0.034	1.750	0.125	16.197
DD	0.032	1.836	-0.280	9.174
DIS	0.051	1.746	0.141	11.295
GE	0.007	1.984	0.043	13.863
HD	0.058	1.728	0.389	8.245
HPQ	-0.002	2.157	-0.571	14.178
IBM	0.030	1.418	-0.187	8.339
INTC	0.035	1.859	-0.144	8.049
JNJ	0.034	1.032	0.467	15.237
JPM	0.033	2.766	0.313	16.942
KO	0.033	1.184	0.420	15.167
MCD	0.053	1.197	0.085	8.948
MMM	0.043	1.410	-0.221	8.816
MRK	0.027	1.663	-0.328	14.150
MSFT	0.039	1.779	0.185	12.384
PFE	0.025	1.436	-0.009	8.980
PG	0.022	1.122	-0.217	10.553
T	0.029	1.389	0.527	15.042
TRV	0.043	1.869	0.294	26.994
UNH	0.049	2.158	0.462	28.632
UTX	0.031	1.535	0.194	10.103
VZ	0.036	1.372	0.279	11.613
WMT	0.024	1.240	-0.035	12.668
XOM	0.018	1.589	0.083	17.618

Note: This table provides the descriptive statistics of stock returns that we consider in our empirical application. These stocks are daily S&P 500 index and 29 daily individual stocks: AXP, BA, BAC, CAT, CSCO, CVX, DD, DIS, GE, HD, HPQ, IBM, INTC, JNJ, JPM, KO, MCD, MMM, MRK, MSFT, PFE, PG, T, TRV, UNH, UTX, VZ, WMT, and XOM. A detailed description of these stocks is provided in Section 6.

Table 3: Testing symmetric comovements in financial markets for $c = 0$

Stocks	J_ρ	C_ρ	$\widetilde{CvM}_T$	$\widetilde{KS}_T$	CvM_T	KS_T
AXP	0.529	0.042**	0.003***	0.002***	0.027**	0.055*
BA	0.624	0.237	0.006***	0.003***	0.106	0.067*
BAC	0.422	0.033**	0.001***	0.003***	0.016**	0.014**
CAT	0.561	0.098*	0.025**	0.021**	0.070*	0.229
CSCO	0.764	0.563	0.015**	0.005***	0.040**	0.020**
CVX	0.867	0.575	0.002***	0.001***	0.035**	0.021**
DD	0.863	0.522	0.001***	0.001***	0.038**	0.030**
DIS	0.831	0.452	0.001***	0.000***	0.046**	0.055*
GE	0.416	0.035**	0.007***	0.003***	0.102	0.135
HD	0.907	0.722	0.087*	0.057*	0.176	0.218
HPQ	0.761	0.664	0.000***	0.000***	0.010***	0.020**
IBM	0.956	0.893	0.030**	0.010***	0.139	0.066*
INTC	0.852	0.685	0.000***	0.001***	0.053*	0.051*
JNJ	0.951	0.852	0.107	0.027**	0.144	0.116
JPM	0.524	0.112	0.016**	0.023**	0.038**	0.045**
KO	0.828	0.665	0.002***	0.001***	0.074*	0.087*
MCD	0.710	0.423	0.000***	0.000***	0.015**	0.013**
MMM	0.892	0.641	0.003***	0.000***	0.096*	0.062*
MRK	0.701	0.464	0.131	0.051*	0.163	0.135
MSFT	0.679	0.350	0.266	0.095*	0.070*	0.004***
PFE	0.681	0.228	0.128	0.105	0.160	0.235
PG	0.794	0.547	0.032**	0.009***	0.093*	0.055*
T	0.947	0.842	0.000***	0.000***	0.011**	0.029**
TRV	0.638	0.215	0.005***	0.003***	0.021**	0.018**
UNH	0.983	0.955	0.041**	0.017**	0.097*	0.095*
UTX	0.905	0.644	0.055*	0.033**	0.092*	0.070*
VZ	0.733	0.353	0.000***	0.001***	0.007***	0.009***
WMT	0.732	0.532	0.002***	0.003***	0.015**	0.033**
XOM	0.861	0.594	0.084*	0.030**	0.132	0.062*

Note: This table provides p-values of testing the symmetric comovements between the S&P 500 daily return and the daily return on each of the following 29 stocks: AXP, BA, BAC, CAT, CSCO, CVX, DD, DIS, GE, HD, HPQ, IBM, INTC, JNJ, JPM, KO, MCD, MMM, MRK, MSFT, PFE, PG, T, TRV, UNH, UTX, VZ, WMT, and XOM. Several tests are compared: J_ρ test of Hong, Tu, and Zhou (2007), C_ρ test of Chen (2016), and the $\widetilde{CvM}_T$, $\widetilde{KS}_T$, CvM_T and KS_T tests. Here the exceedance level $c = 0$ and the block length $L = [T^{1/4}]$. The number of bootstrap resamples is $B = 1000$.

Table 4: Testing symmetric comovements in financial markets for $c = 0.5$

Stocks	J_ρ	C_ρ	$\widetilde{CvM}_T$	$\widetilde{KS}_T$	CvM_T	KS_T
AXP	0.679	0.285	0.144	0.182	0.000***	0.001***
BA	0.922	0.817	0.856	0.698	0.118	0.116
BAC	0.526	0.184	0.350	0.227	0.263	0.328
CAT	0.705	0.344	0.184	0.204	0.019**	0.018**
CSCO	0.996	0.989	0.008***	0.010***	0.000***	0.000***
CVX	0.964	0.874	0.217	0.218	0.005***	0.004***
DD	0.983	0.936	0.756	0.711	0.170	0.123
DIS	0.974	0.920	0.312	0.565	0.011**	0.063*
GE	0.900	0.694	0.219	0.295	0.008***	0.023**
HD	0.898	0.690	0.640	0.613	0.083*	0.067*
HPQ	0.636	0.571	0.013**	0.035**	0.003***	0.003***
IBM	0.949	0.878	0.517	0.558	0.167	0.184
INTC	0.665	0.263	0.554	0.677	0.046**	0.078*
JNJ	0.893	0.681	0.139	0.133	0.009***	0.002***
JPM	0.865	0.637	0.580	0.345	0.118	0.304
KO	0.781	0.464	0.203	0.184	0.020**	0.010***
MCD	0.989	0.977	0.105	0.208	0.003***	0.019**
MMM	0.880	0.636	0.382	0.317	0.093*	0.055*
MRK	0.743	0.547	0.060*	0.040**	0.016**	0.005***
MSFT	0.951	0.891	0.365	0.245	0.085*	0.021**
PFE	0.603	0.162	0.533	0.489	0.251	0.186
PG	0.738	0.385	0.034**	0.075*	0.001***	0.001***
T	0.854	0.477	0.111	0.177	0.003***	0.011**
TRV	0.577	0.203	0.787	0.553	0.218	0.289
UNH	0.617	0.173	0.158	0.254	0.003***	0.007***
UTX	0.990	0.960	0.722	0.613	0.146	0.107
VZ	0.862	0.517	0.714	0.692	0.146	0.100*
WMT	0.930	0.847	0.809	0.715	0.386	0.275
XOM	0.927	0.787	0.270	0.396	0.009***	0.030**

Note: This table provides p-values of testing the symmetric comovements between the S&P 500 daily return and the daily return on each of the following 29 stocks: AXP, BA, BAC, CAT, CSCO, CVX, DD, DIS, GE, HD, HPQ, IBM, INTC, JNJ, JPM, KO, MCD, MMM, MRK, MSFT, PFE, PG, T, TRV, UNH, UTX, VZ, WMT, and XOM. Several tests are compared: J_ρ test of Hong, Tu, and Zhou (2007), C_ρ test of Chen (2016), and the $\widetilde{CvM}_T$, $\widetilde{KS}_T$, CvM_T and KS_T tests. Here the exceedance level $c = 0.5$ and the block length $L = [T^{1/4}]$. The number of bootstrap resamples is $B = 1000$.

Table 5: Testing symmetric comovements in financial markets for $c = 1$

Stocks	J_ρ	C_ρ	$\widetilde{CvM}_T$	$\widetilde{KS}_T$	CvM_T	KS_T
AXP	0.580	0.254	0.117	0.073*	0.431	0.698
BA	0.865	0.588	0.372	0.344	0.017**	0.104
BAC	0.497	0.326	0.381	0.391	0.087*	0.092*
CAT	0.973	0.939	0.410	0.356	0.015**	0.025**
CSCO	0.931	0.805	0.142	0.113	0.827	0.185
CVX	0.968	0.814	0.077*	0.046**	0.225	0.200
DD	0.891	0.681	0.711	0.621	0.163	0.274
DIS	0.820	0.496	0.179	0.140	0.534	0.225
GE	0.860	0.645	0.662	0.738	0.050**	0.210
HD	0.962	0.879	0.106	0.047**	0.824	0.806
HPQ	0.208	0.126	0.653	0.229	0.226	0.280
IBM	0.968	0.930	0.760	0.634	0.180	0.220
INTC	0.537	0.172	0.165	0.093*	0.201	0.295
JNJ	0.887	0.656	0.110	0.041**	0.163	0.373
JPM	0.712	0.439	0.223	0.181	0.236	0.410
KO	0.787	0.451	0.807	0.534	0.736	0.538
MCD	0.857	0.742	0.169	0.067*	0.787	0.928
MMM	0.627	0.227	0.445	0.247	0.020**	0.034**
MRK	0.807	0.547	0.641	0.453	0.280	0.323
MSFT	0.739	0.534	0.234	0.119	0.941	0.960
PFE	0.695	0.259	0.288	0.257	0.880	0.929
PG	0.673	0.238	0.720	0.827	0.068*	0.118
T	0.809	0.268	0.126	0.080*	0.650	0.701
TRV	0.550	0.232	0.417	0.201	0.577	0.616
UNH	0.711	0.372	0.097*	0.092*	0.700	0.643
UTX	0.797	0.304	0.171	0.106	0.386	0.542
VZ	0.886	0.565	0.558	0.696	0.156	0.467
WMT	0.981	0.956	0.654	0.658	0.667	0.405
XOM	0.937	0.758	0.393	0.470	0.042**	0.077*

Note: This table provides p-values of testing the symmetric comovements between the S&P 500 daily return and the daily return on each of the following 29 stocks: AXP, BA, BAC, CAT, CSCO, CVX, DD, DIS, GE, HD, HPQ, IBM, INTC, JNJ, JPM, KO, MCD, MMM, MRK, MSFT, PFE, PG, T, TRV, UNH, UTX, VZ, WMT, and XOM. Several tests are compared: J_ρ test of Hong, Tu, and Zhou (2007), C_ρ test of Chen (2016), and the $\widetilde{CvM}_T$, $\widetilde{KS}_T$, CvM_T and KS_T tests. Here the exceedance level $c = 1$ and the block length $L = [T^{1/4}]$. The number of bootstrap resamples is $B = 1000$.

Table 6: Testing symmetric comovements in financial markets for $c = 1.5$

Stocks	J_ρ	C_ρ	$\widetilde{CvM}_T$	$\widetilde{KS}_T$	CvM_T	KS_T
AXP	0.572	0.169	0.105	0.179	0.616	0.836
BA	0.624	0.208	0.044**	0.015**	0.207	0.382
BAC	0.286	0.289	0.613	0.442	0.493	0.257
CAT	0.662	0.421	0.068*	0.062*	0.950	0.915
CSCO	0.830	0.582	0.648	0.763	0.171	0.387
CVX	0.989	0.931	0.023**	0.022**	0.482	0.537
DD	0.934	0.811	0.043**	0.052*	0.696	0.698
DIS	0.975	0.917	0.091*	0.074*	0.950	0.891
GE	0.915	0.848	0.289	0.190	0.018**	0.084*
HD	0.987	0.966	0.191	0.203	0.299	0.179
HPQ	0.241	0.035**	0.033**	0.008**	0.618	0.646
IBM	0.797	0.630	0.311	0.132	0.691	0.578
INTC	0.532	0.250	0.166	0.141	0.898	0.947
JNJ	0.909	0.682	0.160	0.104	0.950	0.974
JPM	0.607	0.426	0.015**	0.029**	0.036**	0.026**
KO	0.531	0.072*	0.316	0.442	0.022**	0.096*
MCD	0.904	0.757	0.346	0.248	0.977	0.936
MMM	0.572	0.252	0.488	0.191	0.275	0.236
MRK	0.800	0.565	0.159	0.102	0.631	0.390
MSFT	0.807	0.422	0.291	0.136	0.723	0.720
PFE	0.751	0.427	0.955	0.825	0.919	0.919
PG	0.636	0.239	0.542	0.436	0.121	0.019**
T	0.770	0.167	0.172	0.111	0.979	0.870
TRV	0.518	0.285	0.383	0.328	0.302	0.475
UNH	0.534	0.216	0.646	0.402	0.361	0.560
UTX	0.708	0.236	0.620	0.410	0.431	0.376
VZ	0.981	0.902	0.518	0.610	0.803	0.737
WMT	0.810	0.557	0.661	0.938	0.152	0.538
XOM	0.985	0.936	0.111	0.069*	0.551	0.539

Note: This table provides p-values of testing the symmetric comovements between the S&P 500 daily return and the daily return on each of the following 29 stocks: AXP, BA, BAC, CAT, CSCO, CVX, DD, DIS, GE, HD, HPQ, IBM, INTC, JNJ, JPM, KO, MCD, MMM, MRK, MSFT, PFE, PG, T, TRV, UNH, UTX, VZ, WMT, and XOM. Several tests are compared: J_ρ test of Hong, Tu, and Zhou (2007), C_ρ test of Chen (2016), and the $\widetilde{CvM}_T$, $\widetilde{KS}_T$, CvM_T and KS_T tests. Here the exceedance level $c = 1.5$ and the block length $L = [T^{1/4}]$. The number of bootstrap resamples is $B = 1000$.

Table 7: Testing symmetric comovements in financial markets for $c = \{0, 0.5, 1, 1.5\}$

Stocks	J_ρ	C_ρ	$\widetilde{CvM}_T$	$\widetilde{KS}_T$	CvM_T	KS_T
AXP	0.798	0.021**	0.025**	0.049**	0.028**	0.067*
BA	0.358	0.013**	0.106	0.071*	0.113	0.071*
BAC	0.739	0.235	0.014**	0.018**	0.024**	0.028**
CAT	0.405	0.065*	0.056*	0.193	0.053*	0.135
CSCO	0.902	0.898	0.059*	0.025**	0.064*	0.026**
CVX	0.992	0.901	0.023**	0.013**	0.030**	0.018**
DD	0.977	0.685	0.034**	0.024**	0.047**	0.036**
DIS	0.588	0.216	0.045**	0.051*	0.048**	0.057*
GE	0.204	0.061*	0.106	0.153	0.114	0.168
HD	0.926	0.731	0.170	0.210	0.184	0.241
HPQ	0.282	0.135	0.013**	0.025**	0.018**	0.034**
IBM	0.984	0.972	0.173	0.096*	0.168	0.107
INTC	0.943	0.675	0.054*	0.062*	0.067*	0.072*
JNJ	0.960	0.731	0.149	0.123	0.139	0.129
JPM	0.386	0.392	0.046**	0.062*	0.047**	0.061*
KO	0.745	0.258	0.076*	0.088*	0.079*	0.107
MCD	0.854	0.514	0.018**	0.011**	0.021**	0.010***
MMM	0.835	0.489	0.105	0.072*	0.119	0.080*
MRK	0.680	0.647	0.154	0.126	0.154	0.121
MSFT	0.590	0.107	0.071*	0.006***	0.058*	0.005***
PFE	0.968	0.727	0.148	0.212	0.156	0.259
PG	0.993	0.813	0.116	0.068*	0.129	0.070*
T	0.993	0.584	0.010***	0.016**	0.009***	0.022**
TRV	0.980	0.800	0.017**	0.016**	0.022**	0.022**
UNH	0.136	0.015**	0.119	0.095*	0.105	0.097*
UTX	0.912	0.173	0.101	0.091*	0.099*	0.065*
VZ	0.957	0.829	0.017**	0.018**	0.016**	0.020**
WMT	0.933	0.781	0.021**	0.052*	0.027**	0.060*
XOM	0.874	0.302	0.137	0.059*	0.131	0.050*

Note: This table provides p-values of testing the symmetric comovements between the S&P 500 daily return and the daily return on each of the following 29 stocks: AXP, BA, BAC, CAT, CSCO, CVX, DD, DIS, GE, HD, HPQ, IBM, INTC, JNJ, JPM, KO, MCD, MMM, MRK, MSFT, PFE, PG, T, TRV, UNH, UTX, VZ, WMT, and XOM. Several tests are compared: J_ρ test of Hong, Tu, and Zhou (2007), C_ρ test of Chen (2016), and the $\widetilde{CvM}_T$, $\widetilde{KS}_T$, CvM_T, and KS_T tests. Here the exceedance levels $c = \{0,\ 0.5,\ 1,\ 1.5\}$ and the block length $L = [T^{1/4}]$. The number of bootstrap resamples is $B = 1000$.

Table 8: Test results (p-values) of pairwise market index, where the null hypothesis is $p^+(c) = p^-(c)$ for $c \in \{0, 0.5, 1, 1.5\}$

Market index i	Market index j	$p^+(c) = p^-(c)$			
		$c = 0$	$c = 0.5$	$c = 1$	$c = 1.5$
S&P 500	FTSE 100	0.000***	0.018**	0.006***	0.000***
	DAX 30	0.000***	0.246	0.014**	0.023**
	NIKKEI 225	0.000***	0.285	0.004***	0.056*
	HS 300	0.000***	0.033**	0.007***	0.000***
FTSE 100	DAX 30	0.000***	0.012**	0.023**	0.000***
	NIKKEI 225	0.000***	0.266	0.000***	0.000***
	HS 300	0.000***	0.653	0.087*	0.000***
DAX 30	NIKKEI 225	0.000***	0.051*	0.000***	0.000***
	HS 300	0.000***	0.015**	0.040**	0.000***
NIKKEI 225	HS 300	0.000***	0.262	0.000***	0.000***

Note: This table reports results of testing of joint probabilities $p^+(c)$ $=\Pr(X_t > c, Y_t > c)$ and $p^-(c)$ $=\Pr(X_t < -c, Y_t < -c)$ for different c's, with X_t the daily return on one market and Y_t the daily return on other market. Five market indices are considered: S&P 500, FTSE 100, DAX 30, NIKKEI 225, and HS 300. p-values correspond to t-test of $H_0^p : p^+(c) = p^-(c)$ versus $H_1^p : p^+(c) \neq p^-(c)$; see the description of the testing procedure in the online **Appendix B**.

Table 9: Test results (p-values) of pairwise market index, $c = 0$

Market index i	Market index j	Tests					
		J_ρ	C_ρ	$\widetilde{CvM}_T$	$\widetilde{KS}_T$	CvM_T	KS_T
S&P 500	FTSE 100	0.582	0.301	0.000***	0.000***	0.018**	0.000***
	DAX 30	0.958	0.904	0.000***	0.000***	0.000***	0.000***
	NIKKEI 225	0.938	0.915	0.000***	0.000***	0.111	0.010***
	HS 300	0.466	0.424	0.000***	0.000***	0.344	0.065*
FTSE 100	DAX 30	0.522	0.045**	0.000***	0.001***	0.028**	0.058*
	NIKKEI 225	0.236	0.057*	0.000***	0.002***	0.049**	0.017**
	HS300	0.848	0.804	0.003***	0.001***	0.128	0.062*
DAX 30	NIKKEI 225	0.160	0.039**	0.000***	0.000***	0.039**	0.039**
	HS 300	0.388	0.310	0.006***	0.000***	0.473	0.337
NIKKEI 225	HS 300	0.117	0.059*	0.004***	0.001***	0.086*	0.018**

Note: This table provides p-values of testing the symmetric comovements between the daily return on one market and the daily return on other market. Five market indices are considered: S&P 500, FTSE 100, DAX 30, NIKKEI 225, and HS 300. Several tests are compared: J_ρ test of Hong, Tu, and Zhou (2007), C_ρ test of Chen (2016), and the $\widetilde{CvM}_T$, $\widetilde{KS}_T$, CvM_T and KS_T tests. Here the exceedance level $c = 0$ and the block length $L = [T^{1/4}]$. The number of bootstrap resamples is $B = 1000$.

Table 10: Test results (p-values) of pairwise market index, $c = 0.5$

Market index i	Market index j	Tests					
		J_ρ	C_ρ	$\widetilde{CvM}_T$	$\widetilde{KS}_T$	CvM_T	KS_T
S&P 500	FTSE 100	0.833	0.766	0.000***	0.001***	0.000***	0.000***
	DAX 30	0.736	0.504	0.003***	0.006***	0.000***	0.000***
	NIKKEI 225	0.481	0.351	0.679	0.398	0.108	0.040**
	HS 300	0.261	0.248	0.007***	0.026**	0.003***	0.001***
FTSE 100	DAX 30	0.487	0.103	0.007***	0.027**	0.001***	0.008***
	NIKKEI 225	0.242	0.090*	0.508	0.327	0.071*	0.027**
	HS 300	0.824	0.785	0.196	0.261	0.077*	0.058*
DAX 30	NIKKEI 225	0.306	0.169	0.620	0.558	0.031**	0.017**
	HS 300	0.694	0.649	0.061*	0.046**	0.153	0.126
NIKKEI 225	HS 300	0.501	0.467	0.137	0.237	0.002***	0.005***

Note: This table provides p-values of testing the symmetric comovements between the daily return on one market and the daily return on other market. Five market indices are considered: S&P 500, FTSE 100, DAX 30, NIKKEI 225, and HS 300. Several tests are compared: J_ρ test of Hong, Tu, and Zhou (2007), C_ρ test of Chen (2016), and the $\widetilde{CvM}_T$, $\widetilde{KS}_T$, CvM_T and KS_T tests. Here the exceedance level $c = 0.5$ and the block length $L = [T^{1/4}]$. The number of bootstrap resamples is $B = 1000$.

Table 11: Test results (p-values) of pairwise market index, $c = 1$

Market index i	Market index j	Tests					
		J_ρ	C_ρ	$\widetilde{CvM}_T$	$\widetilde{KS}_T$	CvM_T	KS_T
S&P 500	FTSE 100	0.395	0.225	0.262	0.361	0.000***	0.003***
	DAX 30	0.697	0.373	0.243	0.322	0.003***	0.001***
	NIKKEI 225	0.915	0.881	0.663	0.566	0.189	0.280
	HS 300	0.450	0.457	0.572	0.623	0.082*	0.170
FTSE 100	DAX 30	0.407	0.132	0.657	0.697	0.046**	0.200
	NIKKEI 225	0.272	0.216	0.131	0.105	0.889	0.826
	HS 300	0.355	0.209	0.150	0.304	0.000***	0.003***
DAX 30	NIKKEI 225	0.348	0.311	0.133	0.106	0.310	0.293
	HS 300	0.692	0.661	0.322	0.498	0.005***	0.033**
NIKKEI 225	HS 300	0.887	0.889	0.610	0.387	0.033**	0.008***

Note: This table provides p-values of testing the symmetric comovements between the daily return on one market and the daily return on other market. Five market indices are considered: S&P 500, FTSE 100, DAX 30, NIKKEI 225, and HS 300. Several tests are compared: J_ρ test of Hong, Tu, and Zhou (2007), C_ρ test of Chen (2016), and the $\widetilde{CvM}_T$, $\widetilde{KS}_T$, CvM_T and KS_T tests. Here the exceedance level $c = 1$ and the block length $L = [T^{1/4}]$. The number of bootstrap resamples is $B = 1000$.

Table 12: Test results (p-values) of pairwise market index, $c = 1.5$

Market index i	Market index j	Tests					
		J_ρ	C_ρ	$\widetilde{CvM}_T$	$\widetilde{KS}_T$	CvM_T	KS_T
S&P 500	FTSE 100	0.099*	0.107	0.130	0.088*	0.705	0.763
	DAX 30	0.686	0.418	0.539	0.166	0.037**	0.056*
	NIKKEI 225	0.194	0.179	0.480	0.557	0.711	0.547
	HS 300	0.387	0.367	0.788	0.869	0.999	0.994
FTSE 100	DAX 30	0.310	0.182	0.432	0.233	0.080 *	0.197
	NIKKEI 225	0.158	0.186	0.226	0.301	0.513	0.210
	HS 300	0.219	0.113	0.006***	0.000***	0.604	0.791
DAX 30	NIKKEI 225	0.270	0.355	0.488	0.667	0.008***	0.075*
	HS 300	0.858	0.855	0.351	0.283	0.450	0.522
NIKKEI 225	HS 300	0.655	0.654	0.756	0.758	0.069*	0.244

Note: This table provides p-values of testing the symmetric comovements between the daily return on one market and the daily return on other market. Five market indices are considered: S&P 500, FTSE 100, DAX 30, NIKKEI 225, and HS 300. Several tests are compared: J_ρ test of Hong, Tu, and Zhou (2007), C_ρ test of Chen (2016), and the $\widetilde{CvM}_T$, $\widetilde{KS}_T$, CvM_T and KS_T tests. Here the exceedance level $c = 1.5$ and the block length $L = [T^{1/4}]$. The number of bootstrap resamples is $B = 1000$.

D Appendix: Additional Simulation Results

In this appendix we provide additional simulation results, which illustrate the effects of different block length choice L under the GARCH(1,1) process as well as different degrees of dependence displayed in the data (characterized by the autoregressive parameter in the AR(1) process in the main paper) on the finite sample properties of the proposed test statistics CvM_T and KS_T. Specifically, we report results when $L = 2 \times [T^{1/4}]$ and $L = 4 \times [T^{1/4}]$, and $\rho = 0$ (the i.i.d. case), $\rho = 0.5$, $\rho = 0.8$, and $\rho = 0.9$ with ρ the autoregressive parameter in the AR(1) process. But we remark that larger ρ (thus stronger dependence in the data) deteriorates the testing performance very seriously and the associated results are not informative.

In addition, it is reasonable to conjecture that there is a kind of tradeoff between the degree of dependence and choice of block length L, the detailed investigation of which is left for future study. To summarize, dependence hurts the size performance via size distortion (for our simulations, undersize). Less dependence improves the test performance. For example, in the extreme when $\rho = 0$ (no dependence case), size distortion disappears completely; see Tables 17 and 18.

Other observations as discussed in the main paper are similar; for example, KS_T is slightly more powerful than CvM_T for both GARCH(1,1) and AR(1) processes across different L's and different ρ's.

Finally, for the purpose of comparison, the test results of two popular parametric tests for asymmetric comovements (i.e., the J_ρ and C_ρ tests) are also reported in Tables 25 and 26, respectively. From Table 25 it is clear that J_ρ test has serious size distortion (extremely undersized) for all three nominal levels, while from Table 26 we see that C_ρ test preserves the sizes very well. In addition, the two parametric tests are much more powerful (especially for the current type of dependence structures considered) than the two proposed nonparametric tests. Nevertheless, it is worthwhile to emphasize that this is not completely unexpected, given the fully model-free nature of our CvM_T and KS_T tests. To improve the test performance, developing an alternative bootstrap procedure may deserve further attention.

Table 13: Empirical rejection rates of the block multiplier bootstrap-based Cramér-von Mises test statistic $[CvM_T]$ for $c = 0$ under the GARCH(1,1) process when $L = 2 \times [T^{1/4}]$

Nominal Sizes	DGPs					
	DGP S1	DGP P1	DGP P2	DGP P3	DGP P4	DGP P5
			$T = 240$			
$\alpha = 1\%$	0.003	0.006	0.008	0.009	0.017	0.026
$\alpha = 5\%$	0.031	0.045	0.041	0.056	0.069	0.102
$\alpha = 10\%$	0.062	0.078	0.094	0.100	0.146	0.164
			$T = 600$			
$\alpha = 1\%$	0.004	0.006	0.012	0.021	0.035	0.077
$\alpha = 5\%$	0.033	0.051	0.076	0.111	0.156	0.247
$\alpha = 10\%$	0.064	0.103	0.146	0.223	0.265	0.397
			$T = 1600$			
$\alpha = 1\%$	0.002	0.010	0.061	0.121	0.186	0.357
$\alpha = 5\%$	0.022	0.060	0.199	0.337	0.475	0.728
$\alpha = 10\%$	0.058	0.139	0.319	0.483	0.620	0.876
			$T = 3200$			
$\alpha = 1\%$	0.003	0.041	0.179	0.348	0.522	0.858
$\alpha = 5\%$	0.024	0.126	0.441	0.677	0.850	0.987
$\alpha = 10\%$	0.058	0.212	0.620	0.834	0.950	0.999
			$T = 4800$			
$\alpha = 1\%$	0.005	0.050	0.354	0.624	0.817	0.983
$\alpha = 5\%$	0.029	0.177	0.676	0.900	0.980	1.000
$\alpha = 10\%$	0.066	0.302	0.825	0.966	0.998	1.000

Note: This table reports the empirical size and power of block multiplier bootstrap-based test statistic CvM_T for testing the null of symmetric comovements in (1) against the alternative in (2) under the DGPs in Table 1 of the main paper and for $\alpha = 1\%$, 5%, and 10% significance levels. The number of simulations is 1000, the number of bootstrap resamples is $B = 200$, and the block length is $L = 2 \times [T^{1/4}]$. Here we set the exceedance level to be $c = 0$.

Table 14: Empirical rejection rates of the block multiplier bootstrap-based Kolmogorov-Smirnov test statistic $[KS_T]$ for $c = 0$ under the GARCH(1,1) process when $L = 2 \times [T^{1/4}]$

Nominal Sizes	DGPs					
	DGP S1	DGP P1	DGP P2	DGP P3	DGP P4	DGP P5
			$T = 240$			
$\alpha = 1\%$	0.004	0.005	0.009	0.017	0.020	0.037
$\alpha = 5\%$	0.021	0.035	0.045	0.068	0.084	0.140
$\alpha = 10\%$	0.061	0.065	0.093	0.123	0.164	0.235
			$T = 600$			
$\alpha = 1\%$	0.003	0.007	0.017	0.032	0.057	0.168
$\alpha = 5\%$	0.028	0.055	0.091	0.166	0.210	0.392
$\alpha = 10\%$	0.073	0.106	0.175	0.289	0.340	0.529
			$T = 1600$			
$\alpha = 1\%$	0.005	0.014	0.089	0.204	0.344	0.678
$\alpha = 5\%$	0.027	0.069	0.267	0.444	0.626	0.900
$\alpha = 10\%$	0.048	0.140	0.387	0.621	0.769	0.957
			$T = 3200$			
$\alpha = 1\%$	0.001	0.041	0.293	0.547	0.795	0.984
$\alpha = 5\%$	0.025	0.147	0.584	0.831	0.959	0.999
$\alpha = 10\%$	0.061	0.245	0.734	0.923	0.980	1.000
			$T = 4800$			
$\alpha = 1\%$	0.008	0.067	0.555	0.832	0.957	1.000
$\alpha = 5\%$	0.032	0.221	0.824	0.972	0.996	1.000
$\alpha = 10\%$	0.058	0.357	0.908	0.994	1.000	1.000

Note: This table reports the empirical size and power of block multiplier bootstrap-based test statistic KS_T for testing the null of symmetric comovements in (1) against the alternative in (2) under the DGPs in Table 1 of the main paper and for $\alpha = 1\%$, 5%, and 10% significance levels. The number of simulations is 1000, the number of bootstrap resamples is $B = 200$, and the block length is $L = 2 \times [T^{1/4}]$. Here we set the exceedance level to be $c = 0$.

Table 15: Empirical rejection rates of the block multiplier bootstrap-based Cramér-von Mises test statistic $[CvM_T]$ for $c = 0$ under the GARCH(1,1) process when $L = 4 \times [T^{1/4}]$

Nominal Sizes	DGPs					
	DGP S1	DGP P1	DGP P2	DGP P3	DGP P4	DGP P5
			$T = 240$			
$\alpha = 1\%$	0.008	0.005	0.017	0.014	0.018	0.029
$\alpha = 5\%$	0.043	0.040	0.072	0.071	0.086	0.132
$\alpha = 10\%$	0.090	0.090	0.126	0.139	0.150	0.235
			$T = 600$			
$\alpha = 1\%$	0.006	0.006	0.026	0.035	0.055	0.086
$\alpha = 5\%$	0.028	0.044	0.093	0.126	0.194	0.251
$\alpha = 10\%$	0.077	0.108	0.164	0.236	0.306	0.405
			$T = 1600$			
$\alpha = 1\%$	0.003	0.019	0.073	0.136	0.209	0.403
$\alpha = 5\%$	0.030	0.060	0.235	0.362	0.498	0.749
$\alpha = 10\%$	0.072	0.128	0.366	0.519	0.675	0.883
			$T = 3200$			
$\alpha = 1\%$	0.002	0.031	0.212	0.375	0.547	0.855
$\alpha = 5\%$	0.021	0.124	0.477	0.700	0.846	0.988
$\alpha = 10\%$	0.058	0.211	0.630	0.836	0.949	0.999
			$T = 4800$			
$\alpha = 1\%$	0.001	0.060	0.340	0.597	0.821	0.990
$\alpha = 5\%$	0.022	0.188	0.685	0.891	0.976	1.000
$\alpha = 10\%$	0.056	0.311	0.839	0.963	0.997	1.000

Note: This table reports the empirical size and power of block multiplier bootstrap-based test statistic CvM_T for testing the null of symmetric comovements in (1) against the alternative in (2) under the DGPs in Table 1 of the main paper and for $\alpha = 1\%$, 5%, and 10% significance levels. The number of simulations is 1000, the number of bootstrap resamples is $B = 200$, and the block length is $L = 4 \times [T^{1/4}]$. Here we set the exceedance level to be $c = 0$.

Table 16: Empirical rejection rates of the block multiplier bootstrap-based Kolmogorov-Smirnov test statistic $[KS_T]$ for $c = 0$ under the GARCH(1,1) process when $L = 4 \times [T^{1/4}]$

Nominal Sizes	DGPs					
	DGP S1	DGP P1	DGP P2	DGP P3	DGP P4	DGP P5
			$T = 240$			
$\alpha = 1\%$	0.007	0.006	0.019	0.009	0.019	0.055
$\alpha = 5\%$	0.041	0.042	0.067	0.082	0.087	0.168
$\alpha = 10\%$	0.087	0.082	0.132	0.155	0.177	0.287
			$T = 600$			
$\alpha = 1\%$	0.005	0.006	0.030	0.046	0.087	0.160
$\alpha = 5\%$	0.033	0.049	0.108	0.163	0.262	0.389
$\alpha = 10\%$	0.072	0.095	0.183	0.284	0.385	0.562
			$T = 1600$			
$\alpha = 1\%$	0.004	0.027	0.115	0.216	0.365	0.679
$\alpha = 5\%$	0.028	0.076	0.301	0.486	0.676	0.914
$\alpha = 10\%$	0.063	0.149	0.437	0.642	0.807	0.966
			$T = 3200$			
$\alpha = 1\%$	0.002	0.036	0.323	0.568	0.788	0.978
$\alpha = 5\%$	0.024	0.153	0.592	0.832	0.953	1.000
$\alpha = 10\%$	0.057	0.250	0.737	0.927	0.980	1.000
			$T = 4800$			
$\alpha = 1\%$	0.008	0.081	0.515	0.811	0.955	1.000
$\alpha = 5\%$	0.017	0.226	0.820	0.963	0.997	1.000
$\alpha = 10\%$	0.051	0.375	0.922	0.988	1.000	1.000

Note: This table reports the empirical size and power of block multiplier bootstrap-based test statistic KS_T for testing the null of symmetric comovements in (1) against the alternative in (2) under the DGPs in Table 1 of the main paper and for $\alpha = 1\%$, 5%, and 10% significance levels. The number of simulations is 1000, the number of bootstrap resamples is $B = 200$, and the block length is $L = 4 \times [T^{1/4}]$. Here we set the exceedance level to be $c = 0$.

Table 17: Empirical rejection rates of the multiplier bootstrap-based Cramér-von Mises test statistic $[CvM_T]$ for $c = 0$ under the i.i.d. process (i.e., the autoregressive parameter in AR(1) process is set to zero) when $L = 1$

Nominal Sizes	DGPs					
	DGP S1	DGP P1	DGP P2	DGP P3	DGP P4	DGP P5
			$T = 240$			
$\alpha = 1\%$	0.007	0.006	0.020	0.023	0.020	0.043
$\alpha = 5\%$	0.036	0.036	0.091	0.099	0.107	0.146
$\alpha = 10\%$	0.082	0.081	0.156	0.177	0.201	0.248
			$T = 600$			
$\alpha = 1\%$	0.010	0.005	0.047	0.061	0.086	0.138
$\alpha = 5\%$	0.046	0.043	0.146	0.190	0.249	0.348
$\alpha = 10\%$	0.103	0.087	0.239	0.296	0.376	0.502
			$T = 1600$			
$\alpha = 1\%$	0.009	0.011	0.134	0.218	0.311	0.531
$\alpha = 5\%$	0.048	0.090	0.321	0.449	0.601	0.834
$\alpha = 10\%$	0.097	0.148	0.457	0.592	0.736	0.951
			$T = 3200$			
$\alpha = 1\%$	0.010	0.050	0.327	0.518	0.724	0.946
$\alpha = 5\%$	0.045	0.179	0.599	0.786	0.933	0.998
$\alpha = 10\%$	0.097	0.293	0.744	0.904	0.983	1.000
			$T = 4800$			
$\alpha = 1\%$	0.013	0.071	0.527	0.785	0.926	0.999
$\alpha = 5\%$	0.054	0.241	0.787	0.951	0.994	1.000
$\alpha = 10\%$	0.102	0.395	0.894	0.985	1.000	1.000

Note: This table reports the empirical size and power of multiplier bootstrap-based test statistic CvM_T for testing the null of symmetric comovements in (1) against the alternative in (2) under the DGPs in Table 1 of the main paper and for $\alpha = 1\%$, 5%, and 10% significance levels. The number of simulations is 1000, the number of bootstrap resamples is $B = 200$, and the block length is $L = 1$ (for i.i.d. data). Here we set the exceedance level to be $c = 0$.

Table 18: Empirical rejection rates of the multiplier bootstrap-based Kolmogorov-Smirnov test statistic $[KS_T]$ for $c = 0$ under the i.i.d. process (i.e, the autoregressive parameter in AR(1) process is set to zero) when $L = 1$

Nominal Sizes	DGPs					
	DGP S1	DGP P1	DGP P2	DGP P3	DGP P4	DGP P5
			$T = 240$			
$\alpha = 1\%$	0.008	0.009	0.023	0.025	0.028	0.067
$\alpha = 5\%$	0.039	0.034	0.094	0.109	0.133	0.209
$\alpha = 10\%$	0.091	0.078	0.165	0.1900	0.239	0.331
			$T = 600$			
$\alpha = 1\%$	0.010	0.011	0.058	0.094	0.153	0.296
$\alpha = 5\%$	0.044	0.040	0.164	0.248	0.341	0.548
$\alpha = 10\%$	0.096	0.095	0.264	0.355	0.482	0.680
			$T = 1600$			
$\alpha = 1\%$	0.008	0.024	0.192	0.343	0.538	0.857
$\alpha = 5\%$	0.049	0.091	0.407	0.597	0.782	0.977
$\alpha = 10\%$	0.100	0.174	0.542	0.731	0.871	0.995
			$T = 3200$			
$\alpha = 1\%$	0.007	0.066	0.493	0.731	0.925	0.997
$\alpha = 5\%$	0.045	0.208	0.735	0.911	0.989	1.000
$\alpha = 10\%$	0.098	0.316	0.844	0.967	0.999	1.000
			$T = 4800$			
$\alpha = 1\%$	0.013	0.086	0.726	0.936	0.991	1.000
$\alpha = 5\%$	0.053	0.290	0.908	0.989	0.999	1.000
$\alpha = 10\%$	0.099	0.423	0.961	0.999	1.000	1.000

Note: This table reports the empirical size and power of multiplier bootstrap-based test statistic KS_T for testing the null of symmetric comovements in (1) against the alternative in (2) under the DGPs in Table 1 of the main paper and for $\alpha = 1\%$, 5%, and 10% significance levels. The number of simulations is 1000, the number of bootstrap resamples is $B = 200$, and the block length is $L = 1$ (for i.i.d. data). Here we set the exceedance level to be $c = 0$.

Table 19: Empirical rejection rates of the block multiplier bootstrap-based Cramér-von Mises test statistic $[CvM_T]$ for $c = 0$ under the AR(1) process with the autoregressive parameter being 0.8 when $L = [T^{1/4}]$

Nominal Sizes	DGPs					
	DGP S1	DGP P1	DGP P2	DGP P3	DGP P4	DGP P5
			$T = 240$			
$\alpha = 1\%$	0.001	0.001	0.001	0.003	0.001	0.002
$\alpha = 5\%$	0.013	0.023	0.014	0.021	0.026	0.021
$\alpha = 10\%$	0.046	0.047	0.046	0.060	0.053	0.043
			$T = 600$			
$\alpha = 1\%$	0.003	0.001	0.001	0.002	0.002	0.002
$\alpha = 5\%$	0.010	0.016	0.016	0.023	0.020	0.029
$\alpha = 10\%$	0.029	0.045	0.048	0.051	0.054	0.067
			$T = 1600$			
$\alpha = 1\%$	0.001	0.001	0.001	0.001	0.007	0.007
$\alpha = 5\%$	0.007	0.011	0.016	0.022	0.044	0.040
$\alpha = 10\%$	0.021	0.033	0.047	0.070	0.089	0.121
			$T = 3200$			
$\alpha = 1\%$	0.001	0.001	0.003	0.007	0.009	0.015
$\alpha = 5\%$	0.006	0.014	0.029	0.044	0.049	0.085
$\alpha = 10\%$	0.021	0.037	0.086	0.097	0.125	0.192
			$T = 4800$			
$\alpha = 1\%$	0.001	0.002	0.007	0.011	0.013	0.039
$\alpha = 5\%$	0.004	0.016	0.058	0.059	0.087	0.172
$\alpha = 10\%$	0.018	0.044	0.108	0.152	0.197	0.327

Note: This table reports the empirical size and power of block multiplier bootstrap-based test statistic CvM_T for testing the null of symmetric comovements in (1) against the alternative in (2) under the DGPs in Table 1 of the main paper and for $\alpha = 1\%$, 5%, and 10% significance levels. The number of simulations is 1000, the number of bootstrap resamples is $B = 200$, and the block length is $L = [T^{1/4}]$. Here we set the exceedance level to be $c = 0$.

Table 20: Empirical rejection rates of the block multiplier bootstrap-based Kolmogorov-Smirnov test statistic $[KS_T]$ for $c = 0$ under the AR(1) process with the autoregressive parameter being 0.8 when $L = [T^{1/4}]$

Nominal Sizes	DGPs					
	DGP S1	DGP P1	DGP P2	DGP P3	DGP P4	DGP P5
			$T = 240$			
$\alpha = 1\%$	0.002	0.001	0.001	0.001	0.001	0.004
$\alpha = 5\%$	0.013	0.023	0.013	0.021	0.026	0.021
$\alpha = 10\%$	0.034	0.045	0.034	0.054	0.057	0.050
			$T = 600$			
$\alpha = 1\%$	0.004	0.002	0.001	0.002	0.003	0.005
$\alpha = 5\%$	0.014	0.011	0.017	0.026	0.024	0.035
$\alpha = 10\%$	0.032	0.043	0.053	0.047	0.056	0.090
			$T = 1600$			
$\alpha = 1\%$	0.001	0.001	0.002	0.004	0.006	0.015
$\alpha = 5\%$	0.010	0.017	0.028	0.037	0.046	0.080
$\alpha = 10\%$	0.024	0.031	0.060	0.084	0.103	0.152
			$T = 3200$			
$\alpha = 1\%$	0.001	0.001	0.005	0.014	0.018	0.043
$\alpha = 5\%$	0.007	0.013	0.037	0.064	0.099	0.158
$\alpha = 10\%$	0.021	0.043	0.098	0.124	0.189	0.283
			$T = 4800$			
$\alpha = 1\%$	0.001	0.002	0.019	0.027	0.026	0.095
$\alpha = 5\%$	0.003	0.022	0.072	0.109	0.155	0.293
$\alpha = 10\%$	0.015	0.057	0.134	0.200	0.273	0.445

Note: This table reports the empirical size and power of block multiplier bootstrap-based test statistic KS_T for testing the null of symmetric comovements in (1) against the alternative in (2) under the DGPs in Table 1 of the main paper and for $\alpha = 1\%$, 5%, and 10% significance levels. The number of simulations is 1000, the number of bootstrap resamples is $B = 200$, and the block length is $L = [T^{1/4}]$. Here we set the exceedance level to be $c = 0$.

Table 21: Empirical rejection rates of the block multiplier bootstrap-based Cramér-von Mises test statistic $[CvM_T]$ for $c = 0$ under the AR(1) process with the autoregressive parameter being 0.9 when $L = [T^{1/4}]$

Nominal Sizes	DGPs					
	DGP S1	DGP P1	DGP P2	DGP P3	DGP P4	DGP P5
			$T = 240$			
$\alpha = 1\%$	0.007	0.006	0.006	0.011	0.011	0.011
$\alpha = 5\%$	0.043	0.042	0.038	0.053	0.044	0.052
$\alpha = 10\%$	0.098	0.088	0.073	0.099	0.094	0.092
			$T = 600$			
$\alpha = 1\%$	0.001	0.007	0.007	0.007	0.002	0.008
$\alpha = 5\%$	0.031	0.026	0.030	0.031	0.021	0.039
$\alpha = 10\%$	0.067	0.057	0.062	0.061	0.051	0.075
			$T = 1600$			
$\alpha = 1\%$	0.003	0.002	0.008	0.003	0.004	0.007
$\alpha = 5\%$	0.022	0.020	0.031	0.033	0.025	0.039
$\alpha = 10\%$	0.051	0.050	0.065	0.068	0.075	0.079
			$T = 3200$			
$\alpha = 1\%$	0.001	0.003	0.002	0.007	0.006	0.014
$\alpha = 5\%$	0.011	0.013	0.029	0.032	0.032	0.061
$\alpha = 10\%$	0.027	0.036	0.070	0.070	0.076	0.122
			$T = 4800$			
$\alpha = 1\%$	0.001	0.001	0.005	0.007	0.011	0.009
$\alpha = 5\%$	0.008	0.016	0.028	0.036	0.041	0.070
$\alpha = 10\%$	0.025	0.044	0.072	0.086	0.084	0.131

Note: This table reports the empirical size and power of block multiplier bootstrap-based test statistic CvM_T for testing the null of symmetric comovements in (1) against the alternative in (2) under the DGPs in Table 1 of the main paper and for $\alpha = 1\%$, 5%, and 10% significance levels. The number of simulations is 1000, the number of bootstrap resamples is $B = 200$, and the block length is $L = [T^{1/4}]$. Here we set the exceedance level to be $c = 0$.

Table 22: Empirical rejection rates of the block multiplier bootstrap-based Kolmogorov-Smirnov test statistic $[KS_T]$ for $c = 0$ under the AR(1) process with the autoregressive parameter being 0.9 when $L = [T^{1/4}]$

Nominal Sizes	DGPs					
	DGP S1	DGP P1	DGP P2	DGP P3	DGP P4	DGP P5
			$T = 240$			
$\alpha = 1\%$	0.010	0.011	0.008	0.012	0.011	0.013
$\alpha = 5\%$	0.057	0.044	0.037	0.042	0.048	0.053
$\alpha = 10\%$	0.096	0.106	0.087	0.090	0.101	0.111
			$T = 600$			
$\alpha = 1\%$	0.001	0.009	0.007	0.008	0.006	0.012
$\alpha = 5\%$	0.034	0.033	0.037	0.033	0.027	0.048
$\alpha = 10\%$	0.072	0.074	0.073	0.075	0.063	0.088
			$T = 1600$			
$\alpha = 1\%$	0.004	0.004	0.009	0.009	0.007	0.010
$\alpha = 5\%$	0.028	0.022	0.043	0.040	0.035	0.044
$\alpha = 10\%$	0.069	0.055	0.074	0.079	0.097	0.104
			$T = 3200$			
$\alpha = 1\%$	0.001	0.003	0.006	0.010	0.007	0.013
$\alpha = 5\%$	0.015	0.014	0.034	0.045	0.039	0.078
$\alpha = 10\%$	0.034	0.040	0.082	0.092	0.096	0.150
			$T = 4800$			
$\alpha = 1\%$	0.001	0.001	0.002	0.005	0.016	0.022
$\alpha = 5\%$	0.012	0.014	0.036	0.049	0.064	0.095
$\alpha = 10\%$	0.027	0.046	0.074	0.105	0.127	0.170

Note: This table reports the empirical size and power of block multiplier bootstrap-based test statistic KS_T for testing the null of symmetric comovements in (1) against the alternative in (2) under the DGPs in Table 1 of the main paper and for $\alpha = 1\%$, 5%, and 10% significance levels. The number of simulations is 1000, the number of bootstrap resamples is $B = 200$, and the block length is $L = [T^{1/4}]$. Here we set the exceedance level to be $c = 0$.

Table 23: Empirical rejection rates of the block multiplier bootstrap-based Cramér-von Mises test statistic [CvM_T] for $c = 0$ under the AR(1) process with the autoregressive parameter being 0.9 when $L = 2 \times [T^{1/4}]$

Nominal Sizes	DGPs					
	DGP S1	DGP P1	DGP P2	DGP P3	DGP P4	DGP P5
			$T = 240$			
$\alpha = 1\%$	0.004	0.005	0.005	0.002	0.002	0.001
$\alpha = 5\%$	0.018	0.021	0.023	0.019	0.017	0.018
$\alpha = 10\%$	0.048	0.049	0.043	0.047	0.047	0.052
			$T = 600$			
$\alpha = 1\%$	0.001	0.001	0.005	0.001	0.001	0.002
$\alpha = 5\%$	0.007	0.009	0.016	0.014	0.015	0.021
$\alpha = 10\%$	0.031	0.032	0.038	0.040	0.040	0.046
			$T = 1600$			
$\alpha = 1\%$	0.001	0.001	0.001	0.001	0.001	0.001
$\alpha = 5\%$	0.002	0.002	0.009	0.010	0.009	0.017
$\alpha = 10\%$	0.020	0.019	0.024	0.027	0.033	0.034
			$T = 3200$			
$\alpha = 1\%$	0.001	0.001	0.001	0.001	0.001	0.002
$\alpha = 5\%$	0.006	0.006	0.008	0.015	0.005	0.017
$\alpha = 10\%$	0.017	0.014	0.026	0.046	0.037	0.047
			$T = 4800$			
$\alpha = 1\%$	0.001	0.002	0.001	0.001	0.001	0.002
$\alpha = 5\%$	0.003	0.002	0.009	0.010	0.016	0.024
$\alpha = 10\%$	0.008	0.015	0.031	0.029	0.056	0.070

Note: This table reports the empirical size and power of block multiplier bootstrap-based test statistic CvM_T for testing the null of symmetric comovements in (1) against the alternative in (2) under the DGPs in Table 1 of the main paper and for $\alpha = 1\%$, 5%, and 10% significance levels. The number of simulations is 1000, the number of bootstrap resamples is $B = 200$, and the block length is $L = 2 \times [T^{1/4}]$. Here we set the exceedance level to be $c = 0$.

Table 24: Empirical rejection rates of the block multiplier bootstrap-based Kolmogorov-Smirnov test statistic $[KS_T]$ for $c = 0$ under the AR(1) process with the autoregressive parameter being 0.9 when $L = 2 \times [T^{1/4}]$

Nominal Sizes	DGPs					
	DGP S1	DGP P1	DGP P2	DGP P3	DGP P4	DGP P5
			$T = 240$			
$\alpha = 1\%$	0.003	0.003	0.003	0.002	0.002	0.003
$\alpha = 5\%$	0.018	0.024	0.026	0.021	0.018	0.018
$\alpha = 10\%$	0.055	0.054	0.050	0.049	0.052	0.044
			$T = 600$			
$\alpha = 1\%$	0.001	0.001	0.003	0.001	0.006	0.004
$\alpha = 5\%$	0.009	0.008	0.021	0.019	0.017	0.022
$\alpha = 10\%$	0.037	0.028	0.044	0.045	0.049	0.041
			$T = 1600$			
$\alpha = 1\%$	0.001	0.001	0.001	0.003	0.002	0.002
$\alpha = 5\%$	0.006	0.007	0.010	0.011	0.016	0.017
$\alpha = 10\%$	0.020	0.019	0.029	0.029	0.038	0.046
			$T = 3200$			
$\alpha = 1\%$	0.001	0.001	0.001	0.002	0.001	0.004
$\alpha = 5\%$	0.005	0.004	0.011	0.024	0.015	0.025
$\alpha = 10\%$	0.015	0.015	0.029	0.055	0.040	0.068
			$T = 4800$			
$\alpha = 1\%$	0.001	0.001	0.001	0.001	0.002	0.007
$\alpha = 5\%$	0.003	0.007	0.014	0.011	0.029	0.038
$\alpha = 10\%$	0.014	0.021	0.034	0.045	0.069	0.085

Note: This table reports the empirical size and power of block multiplier bootstrap-based test statistic KS_T for testing the null of symmetric comovements in (1) against the alternative in (2) under the DGPs in Table 1 of the main paper and for $\alpha = 1\%$, 5%, and 10% significance levels. The number of simulations is 1000, the number of bootstrap resamples is $B = 200$, and the block length is $L = 2 \times [T^{1/4}]$. Here we set the exceedance level to be $c = 0$.

Table 25: Empirical rejection rates of the J_ρ test statistic for $c = 0$

Nominal Sizes	DGPs					
	DGP S1	DGP P1	DGP P2	DGP P3	DGP P4	DGP P5
			$T = 240$			
$\alpha = 1\%$	0	0.001	0.009	0.030	0.096	0.396
$\alpha = 5\%$	0	0.005	0.051	0.145	0.326	0.755
$\alpha = 10\%$	0	0.008	0.139	0.318	0.532	0.895
			$T = 600$			
$\alpha = 1\%$	0	0	0.041	0.191	0.492	0.916
$\alpha = 5\%$	0	0.002	0.247	0.596	0.861	0.989
$\alpha = 10\%$	0	0.016	0.498	0.787	0.951	0.998
			$T = 1600$			
$\alpha = 1\%$	0	0	0.478	0.879	0.971	1.000
$\alpha = 5\%$	0	0.032	0.874	0.984	0.998	1.000
$\alpha = 10\%$	0	0.105	0.961	0.994	1.000	1.000
			$T = 3200$			
$\alpha = 1\%$	0	0.012	0.955	0.993	0.998	1.000
$\alpha = 5\%$	0	0.251	0.992	1.000	1.000	1.000
$\alpha = 10\%$	0	0.560	0.998	1.000	1.000	1.000
			$T = 4800$			
$\alpha = 1\%$	0	0.058	0.991	1.000	1.000	1.000
$\alpha = 5\%$	0	0.586	0.999	1.000	1.000	1.000
$\alpha = 10\%$	0	0.875	1.000	1.000	1.000	1.000

Note: This table reports the empirical size and power of test statistic J_ρ of Hong, Tu and Zhou (2007) for testing the null of symmetric comovements in (1) against the alternative in (2) under the DGPs in Table 1 of the main paper and for $\alpha = 1\%$, 5%, and 10% significance levels. The number of simulations is 1000. Here we set the exceedance level to be $c = 0$.

Table 26: Empirical rejection rates of the C_ρ test statistic for $c = 0$

Nominal Sizes	DGPs					
	DGP S1	DGP P1	DGP P2	DGP P3	DGP P4	DGP P5
			$T = 240$			
$\alpha = 1\%$	0.006	0.180	0.746	0.904	0.981	1.000
$\alpha = 5\%$	0.054	0.448	0.933	0.980	0.999	1.000
$\alpha = 10\%$	0.104	0.596	0.968	0.989	0.999	1.000
			$T = 600$			
$\alpha = 1\%$	0.009	0.618	0.992	1.000	1.000	1.000
$\alpha = 5\%$	0.062	0.846	1.000	1.000	1.000	1.000
$\alpha = 10\%$	0.122	0.919	1.000	1.000	1.000	1.000
			$T = 1600$			
$\alpha = 1\%$	0.009	0.984	1.000	1.000	1.000	1.000
$\alpha = 5\%$	0.039	0.997	1.000	1.000	1.000	1.000
$\alpha = 10\%$	0.090	0.998	1.000	1.000	1.000	1.000
			$T = 3200$			
$\alpha = 1\%$	0.009	0.999	1.000	1.000	1.000	1.000
$\alpha = 5\%$	0.043	0.999	1.000	1.000	1.000	1.000
$\alpha = 10\%$	0.081	0.999	1.000	1.000	1.000	1.000
			$T = 4800$			
$\alpha = 1\%$	0.005	0.999	1.000	1.000	1.000	1.000
$\alpha = 5\%$	0.047	1.000	1.000	1.000	1.000	1.000
$\alpha = 10\%$	0.094	1.000	1.000	1.000	1.000	1.000

Note: This table reports the empirical size and power of test statistic C_ρ of Chen (2016) for testing the null of symmetric comovements in (1) against the alternative in (2) under the DGPs in Table 1 of the main paper and for $\alpha = 1\%$, 5%, and 10% significance levels. The number of simulations is 1000. Here we set the exceedance level to be $c = 0$.

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